Sharp bounds on the least eigenvalue of a graph determined from edge clique partitions

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Extended Abstract

1 Introduction

In this talk, motivated by edge clique partitions [4], sharp lower and upper bounds on the least eigenvalue of a graph are presented as well as a necessary and sufficient (just sufficient) condition for the lower (upper) bound be attained. As an application, we consider the Queens' graph $\mathcal{Q}(n)$, which is obtained from the $n \times n$ chessboard where its squares are the vertices of the graph and two of them are adjacent if and only if they are in the same row, column or diagonal of the chessboard. We conclude that the least eigenvalue of $\mathcal{Q}(n)$ is equal to -4 for every $n \geq 4$ and its multiplicity is $(n-3)^2$. Additionally, some results on the edge clique partition graph parameters are presented.

2 Edge clique partitions

Edge clique partitions (ECP for short) were introduced in [4], where the *content* of a graph G, denoted by $C(G)$, was defined as the minimum number of edge disjoint cliques whose union includes all the edges of G. Such minimum ECP is called in [4] *content decomposition* of G. As proved in [4], in general, the determination of $C(G)$ is **NP**-Complete.

Definition 2.1. *(Clique degree and maximum clique degree) Consider a graph G and an ECP, $P = \{E_i \mid i \in I\}$. Then $V_i = V(G[E_i])$ is a clique of G for every $i \in I$. For any $v \in V(G)$, the clique degree of v relative to P, denoted $m_v(P)$, is the number of cliques V_i containing the vertex v, and the maximum clique degree of G relative to P, denoted $m_G(P)$, is the maximum of clique degrees of the vertices of G relative to P.*

Remark 2.2. *It is clear that if P is an ECP of G, then $m_G(P)$ is not greater than $|P|$. In particular, if P is a content decomposition of G, then $m_G(P) \leq C(G)$.*

The next theorem allows the construction of families of connected graphs $\mathcal{G}(H) = \{G_k \mid k \geq m_H(P)\}$, obtained from an arbitrary connected graph H with an ECP, P, where each graph $G_k \in \mathcal{G}(H)$ has H as a subgraph and admits an ECP, P_k, such that $m_{G_k}(P_k) = k$.

Theorem 2.3. *Let H be a connected graph with an ECP, P. Then for every $k \geq m_H(P)$ there exists a connected graph G_k which has H as a subgraph and admits an ECP, P_k, such that $m_{G_k}(P_k) = k$.*

The above defined family of graphs

$$\mathcal{G}(H) = \{G_k \mid k \geq m_H(P)\} \tag{1}$$

depends from the initial graph $G_{m_H(P)} = H$ and from the permutations π_k. If the chosen graph H admits an ECP, P, which is a content decomposition, it is immediate that for every $k \geq m_H(P)$, independently of the chosen permutations π_k, P_k is a content decomposition of G_k. So this property is invariant to the permutations π_k.

3 Main results

Using the above defined graph parameters, the next theorem states a lower bound on the least eigenvalue of a graph and a necessary and sufficient condition for to be attained in a particular ECP.

Theorem 3.1. *Let $P = \{E_i \mid i \in I\}$ be an ECP of a graph G, $m = m_G(P)$ and $m_v = m_v(P)$ for every $v \in V(G)$. Then*

1. *If μ is an eigenvalue of G, then $\mu \geq -m$.*
2. *$-m$ is an eigenvalue of G if and only if there exists a vector $X \neq \mathbf{0}$ such that*
   (a) *$\sum_{j \in V(G[E_i])} x_j = 0$, for every $i \in I$ and*
   (b) *$\forall v \in V(G)\ \ x_v = 0$ whenever $m_v \neq m$.*

   In the positive case, X is an eigenvector associated with $-m$.

The best lower bound (obtained from Theorem 3.1) for the least eigenvalue of a graph G is the one associated to an ECP, P, such that $m_G(P) \leq m_G(P')$ for every ECP P' of G.

Corollary 3.2. *Let μ be the least eigenvalue of a graph G. Then $-\mu \leq C(G)$.*

The following corollaries are also direct consequences of Theorem 3.1.

Corollary 3.3. *Let G be a graph of order n. Then $X \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ is an eigenvector associated with the eigenvalue $-m$ if and only if the conditions 2a and 2b of Theorem 3.1 hold.*

Corollary 3.4. *Let P be an ECP of a graph G. If $-m_G(P)$ is an eigenvalue of G, then it is the least eigenvalue of G and for every ECP of G, P', $m_G(P') \geq m_G(P)$.*

Now, as a corollary of Theorem 3.1, we state a sharp upper bound on the least eigenvalue of a graph.

Corollary 3.5. *Let G be a graph with least eigenvalue μ. Assume that H is an induced subgraph of G for which there exists an ECP, P', fulfilling the conditions 2a and 2b of Theorem 3.1. Then*

1. $\mu \leq -m_H(P')$.

2. If G admits an ECP, P, such that $m_G(P) = m_H(P')$, then $\mu = -m_H(P')$.

Remark 3.6. *Item 2 of Corollary 3.5 states a sufficient condition for $\mu = -m_H(P')$ when μ is the least eigenvalue of a graph G, H is a subgraph of G and P' is an ECP of H fulfilling the conditions 2a and 2b of Theorem 3.1. However, this condition is not a necessary condition for $\mu = -m_H(P')$.*

We may conclude that the best upper bound (obtained from Corollary 3.5) for the least eigenvalue of a graph G is the one associated to an induced subgraph H having an ECP, P, fulfilling the conditions 2a and 2b of Theorem 3.1, such that $m_H(P) \geq m_{H'}(P')$ for every induced subgraph H' and every ECP P' of H', fulfilling the same conditions.

4 The least eigenvalue of $\mathcal{Q}(n)$, for every $n \geq 4$

As an application of the main results, we can determine the least eigenvalue of the n-Queens' graph, $\mathcal{Q}(n)$, which is a graph associated to the $n \times n$ chessboard $\mathcal{T}_n$, with n^2 vertices, each one corresponding to a square of the chessboard. Two vertices of $\mathcal{Q}(n)$ are adjacent if and only if the corresponding squares in $\mathcal{T}_n$ are in the same row or in the same column or in the same diagonal.

The rows and columns of the chessboard are numbered from the top to the bottom and from the left to the right, respectively. We use the $(i, j) \in [n]^2$ coordinates as labels of the chessboard squares belonging to the i^{th} row and j^{th} column as well as labels of the corresponding vertices in $\mathcal{Q}(n)$.

Theorem 4.1. *Let $n \in \mathbb{N}$ such that $n \geq 4$.*

1. *The least eigenvalue of $\mathcal{Q}(n)$ is -4.*

2. *$X \neq \mathbf{0}$ is a eigenvector associated to -4 if and only if*

   (a) $\sum_{j=1}^{n} x_{(k,j)} = 0$ *and* $\sum_{i=1}^{n} x_{(i,k)} = 0$, *for every* $k \in [n]$,

   (b) $\sum_{i+j=k+2} x_{(i,j)} = 0$, *for every* $k \in [2n-3]$,

   (c) $\sum_{i-j=k+1-n} x_{(i,j)} = 0$, *for every* $k \in [2n-3]$,

   (d) $x_{(1,1)} = x_{(1,n)} = x_{(n,1)} = x_{(n,n)} = 0$.

The proof of this theorem can be obtained as an application of Theorem 3.1 and Corollary 3.5. Indeed, the proof of the first item follows taking into account that an induced subgraph $\mathcal{Q}(4)$ of $\mathcal{Q}(n)$ admits an ECP, P', such that $m_{\mathcal{Q}(4)}(P') = 4$ and a vector X with coordinates $(i, j) \in [4]^2$ fulfilling the necessary and sufficient conditions of Theorem 3.1 and thus -4 is the least eigenvalue of $\mathcal{Q}(4)$. Since $\mathcal{Q}(n)$ admits an ECP, P, such that $m_{\mathcal{Q}(n)}(P) = m_{\mathcal{Q}(4)}(P')$, applying Corollary 3.5 the result follows. The proof of the second item follows taking into account that the summations 2a–2c correspond to the summations 2a in Theorem 3.1. Here, the cliques obtained from the ECP, P, of $\mathcal{Q}(n)$ are the cliques with vertices associated with each of the n columns, n rows, $2n - 3$ left to right diagonals and $2n - 3$ right to left diagonals.

We may conclude that the multiplicity of -4 as an eigenvalue of $\mathcal{Q}(n)$ coincides with the corank of the coefficient matrix of the system of $6n-2$ linear equations 2a–2d. Therefore, to say that the multiplicity of -4 is $(n-3)^2$ is equivalent to say that the rank of the coefficient matrix of the system of $6n$ linear equations 2a–2d is $6n-9$ (since $n^2-6n+9=(n-3)^2$).

For an easier representation of the vectors, they are displayed over the chessboard. So the ℓ^{th} coordinate of a vector X is displayed at the entry of the chessboard corresponding to the vertex ℓ, i.e. at the entry $(i,j)=(\lceil \frac{\ell}{n}\rceil, \ell+n-n\lceil \frac{\ell}{n}\rceil)$. Then, the ℓ^{th} coordinate of X can be denoted by X_ℓ or $X_{(i,j)}$.

0	**1**	**-1**	0
-1	0	0	**1**
1	0	0	**-1**
0	**-1**	**1**	0

Table 1: X_4.

Consider the family of vectors

$$\mathcal{F}_n = \{X_n^{(a,b)} \in \mathbb{R}^{n^2} \mid (a,b) \in [n-3]^2\}$$

where $X_n^{(a,b)}$ is the vector defined by

$$\left[X_n^{(a,b)}\right]_{(i,j)} = \begin{cases} \left[X_4\right]_{(i-a+1,j-b+1)}, & \text{if } (i,j) \in A \times B; \\ 0, & \text{otherwise,} \end{cases}$$

with $A=\{a,a+1,a+2,a+3\}$, $B=\{b,b+1,b+2,b+3\}$ and X_4 is the vector presented in Table 1.

Theorem 4.2. *-4 is an eigenvalue of $\mathcal{Q}(n)$ with multiplicity $(n-3)^2$ and $\mathcal{F}_n$ is a basis for $\mathcal{E}_{\mathcal{Q}(n)}(-4)$.*

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