Hopf algebras of homomorphisms, restrictions and application to well-know Hopf algebras

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1 General context

Let us begin this summary with a classic result (see [3], [4]):

Lemma 1. *If $(\mathcal{C}, \Delta, \varepsilon)$ is a cogebra over a field k and $(\mathcal{A}, m, \eta)$ is an algebra over k, then the set $\mathcal{H}om_k(\mathcal{C}, \mathcal{A})$ of linear maps from $\mathcal{C}$ to $\mathcal{A}$ sending $1_{\mathcal{C}}$ to $1_{\mathcal{A}}$ has an algebra structure given by the so-called* convolution product $\star$*:*

If $\alpha,\ \beta \in \mathcal{H}om(\mathcal{C}, \mathcal{A})$, $\alpha \star \beta \in \mathcal{H}om(\mathcal{C}, \mathcal{A})$ is defined by:

$$\alpha \star \beta := m \circ j_{\mathcal{C},\mathcal{A}}(\alpha \otimes \beta) \circ \Delta \ . \qquad (1)$$

The unit of this algebra is $\eta \circ \varepsilon$.

Usually, $\begin{array}{rcl} j_{\mathcal{C},\mathcal{A}} : \mathcal{H}om(E_1, E_2) \otimes \mathcal{H}om(E_1, E_2) & \longrightarrow & \mathcal{H}om(E_1 \otimes E_1, E_2 \otimes E_2) \\ f \otimes g & \longmapsto & \Big(a \otimes b \longmapsto f(a) \otimes g(b) \Big) \ . \end{array}$ is a map hidden under the rock because it consists of an easy identification. Nevertheless, to type correctly the objects we will deal with, we will continue to use it explicitely.

This result can of course be applied when $\mathcal{C}$ and $\mathcal{A}$ are Hopf algebras $\mathcal{H}_1$ and $\mathcal{H}_2$ over k. Therefore, the natural question is:

Does $\mathcal{H} = \mathcal{H}om_k(\mathcal{H}_1, \mathcal{H}_2)$ has a natural Hopf algebra structure?

2 The Hopf algebra $\mathcal{H}om_k(\mathcal{H}_1, \mathcal{H}_2)$, if $\mathcal{H}_1$ is a finite dimensional Hopf algebra

When $(\mathcal{H}_1, m_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ and $(\mathcal{H}_2, m_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ are two Hopf algebra over a field k, $\mathcal{H}_1$ being a finite dimensional algebra, we know that $\mathcal{H}om_k(\mathcal{H}_1, \mathcal{H}_2)$ is a Hopf algebra over k, according to $\mathcal{H}om_k(\mathcal{H}_1, \mathcal{H}_2) \simeq \mathcal{H}_2 \otimes \mathcal{H}_1^\star$. Pulling back the Hopf structure of $\mathcal{H}_2 \otimes \mathcal{H}_1^\star$ gives us **explicit** formulas:

Theorem 1. *Let $(\mathcal{H}_1, m_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ and $(\mathcal{H}_2, m_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ be two Hopf algebra over a field k, $\mathcal{H}_1$ being a finite dimensional algebra.*

Then, the vector space $\mathcal{H} = \mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$ of linear maps from $\mathcal{H}_1$ to $\mathcal{H}_2$ has a Hopf algebra structure ***explicitely*** *defined for all $f,\ g \in \mathcal{H}$ by:*

$$f \star g = m_2 \circ j_{\mathcal{H}_1,\mathcal{H}_2}(f \otimes g) \circ \Delta_1 \ . \qquad (2)$$

$$\eta(1_k) = \eta_2 \circ \varepsilon_1 \ . \qquad (3)$$

$$\mathbb{\Delta}(f) = j^{-1}_{\mathcal{H}_1,\mathcal{H}_2}\big(\Delta_2 \circ f \circ m_1\big) \ . \qquad (4)$$

$$\boldsymbol{\varepsilon}(f) = \varepsilon_2 \circ f \circ \eta_1(1_k) \ . \qquad (5)$$

$$S(f) = S_2 \circ f \circ S_1 \ . \qquad (6)$$

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This result can be applied to locally finite graded Hopf algebras $E = \bigoplus_{n\geq 0} \mathcal{E}_n$ and $F = \bigoplus_{n\geq 0} \mathcal{F}_n$, all dual vector spaces that have be considered are to be understood as a graded dual.

3 Applications

3.1 Application to the Malvenuto-Reutenauer Hopf algebra

Let us consider a k-vector space V and its tensor algebra $T(V)$ defined by:

$$T(V) = \bigoplus_{n\geq 0} V^{\otimes n} \ , \tag{7}$$

where $V^{\otimes} = k$. (See [3] and [4]) It is well-known that $T(V)$ has a Hopf algebra structure which imply a Hopf algebra on $\Big(End_k\big(T(V)\big), \star, \eta\circ\varepsilon, \boldsymbol{\Delta}, \boldsymbol{\varepsilon}, S\Big)$.

Let us now define the subspace $End_k^{\mathrm{st}}\big(T(V)\big)$ of $End_k\big(T(V)\big)$ by:

$$End_k^{\mathrm{st}}\big(T(V)\big) = \bigcap_{W \text{ linear subspace of } V} \big\{\varphi \in End_k\big(T(V)\big)\ ,\ \forall n \in \mathbb{N}\ ,\ \varphi(V^{\otimes n}) \subset V^{\otimes n}\big\}\ , \tag{8}$$

It is now possible to show that we can send the natural Hopf structure alpgebra of $End_k\big(T(V)\big)$ to $End_k^{\mathrm{st}}\big(T(V)\big)$:

Theorem 2. *The Hopf algebra constructed on the subspace $End_k^{st}\big(T(V)\big)$ is the Malvenuto-Reutenauer Hopf algebra (see [5])*

The main point, here, is that this application to Malvenuto-Reutenauer algebra opens ways to understand how many coproducts on combinatorial Hopf algebra could be constructed.

3.2 Application to (co)mould calculus

Mould calculus has been interpreted in the noncommutative series world (see for example [1]). Nevertheless, comould calculus has not yet such an interpretation.

Let us remind that a mould can be seen as an application defined over the set of nocommutative polynomial constructed over an alphabet Ω. Therefore, a comould is a sort of dual object to moulds and are linear applications This notion is useful to understand the origin of the mould symmetries from an analytical point of view.

However, it would be nice to have interpretations of comould operations (see [2]) as well as its cosymmetries as we have some for moulds. The construction of the Hopf algebra $\mathcal{H}om_k(\mathcal{H}_1, \mathcal{H}_2)$, with $\mathcal{H}_1$ of finite dimension, suggests to continue this work in this direction.

References

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