Order Bases: Structured Linear Algebra

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Tuesday Outline

1. Recall from Monday
2. Euclidean Algorithm and Padé Approximation
3. Structured Matrices : Sylvester
4. Structured Matrices : Hankel
5. An Algorithm

Preamble

In lecture 2 we discuss computation of rational approximation. In the case of equations

$$a_1(z)p_1(z) + a_2(z)p_2(z) = O(z^{\sigma})$$

we note the similarity of compuing a sequence of Padé approximants via elimijnating coefficients from below to that of the Euclidean gcd algorithm which elimintes terms from above. The relationship is seen through the process of reversing coefficients. We also provide details about solving rational approximation problems via structured matrices. These are illustrated using Sylvesters matrices but the recursive method is easily extended to Hermite-Padé problem having Sylvester matrices with additional stripes.

Recall from Last Day

- Problem studied : $\mathbf{A}(z)\mathbf{P}(z) = \mathrm{O}(z^\sigma)$.
  - Want to describe all solutions $\mathbf{P}(z)$.
- Lots of different examples from rational approximation
  - Padé, Hermite-Padé, Simultaneous-Padé
  - Matrix versions
- History on how these originated
- Rational approximation leads to structured linear systems
- Implied structured linear systems give efficient algorithms
- Everything good for scalar Padé problem
  - Structure of Padé table
- Not much known for other order problems.

Euclidean Algorithm

and

Padé Approximation

Extnded Euclidean Algorithm

Given $a(z), b(z) \in \mathbb{K}[z]$ find $u(z), v(z)$ such that

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Process:

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$$\deg a = m, \;\; \deg b = n, \;\; \deg r_i = k_i, \;\; \deg q_i = \rho_i = k_{i-1} - k_i$$

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Note : $k_1 > k_2 > \cdots > k_{\ell-1}$

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Note : $m - n = m_i - n_i$ and $\rho_i = k_{i-1} - k_i \geq 1$.

Reversing Order of coefficients

$a(z) \in \mathbb{K}[z]$, $\deg a = m$

$$\begin{aligned} a(z) &= a_0 + a_1 z + \cdots + a_m z^m \\ a(z^{-1}) &= a_0 + a_1 z^{-1} + \cdots + a_m z^{-m} \\ a^*(z) = z^m a(z^{-1}) &= a_0 z^m + a_1 z^{m-1} + \cdots + a_m \end{aligned}$$

Padé Approximation via EEA

- Reverse order of coefficients:

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$$\begin{aligned} a^*(z){u_i}^*(z) + b^*(z){v_i}^*(z) &= z^{m+n_i-k_i}{r_i}^*(z) \\ &= z^{m_i+n_i+\rho_i}{r_i}^*(z). \end{aligned}$$

We have used $m = n_i + k_{i-1}$ and $n = m_i + k_{i-1}$.

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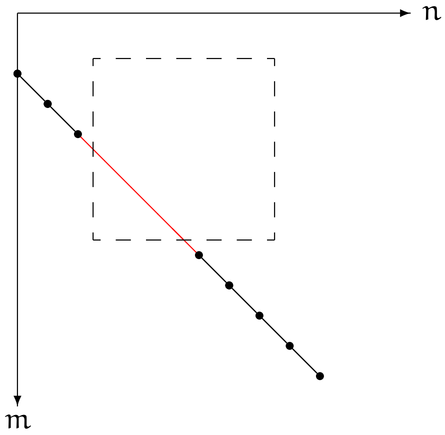
- $({u_i}^*(z), {v_i}^*(z))$ Padé approx. type (m_i, n_i) for $(a^*(z), b^*(z))$.
- Computation follows off-diagonal path
- Note : $\rho_i \geq 1$ and when $\rho_i > 1$ we have a 'jump'

Off-diagonal path of computation in a Padé table:



Additional Points

- Can do EEA superfast: $O(n \log^2 n)$ using Half-gcd.
  - Implies can compute Padé approximants superfast*.
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- Compute Padé approximants fraction-free using subresultants
- EEA gives block structure for Padé table
- Not helpful for matrix Padé approximation
- Not helpful for Hermite-Padé approximation
- Not helpful for Simultaneous-Padé approximation
- etc

Superfast Padé Approximation*

Compute Padé approximant of type (n, n) for $(a(z), b(z))$:

- Compute pairs of Padé approx for $(a(z), b(z))$ for degrees

$$(1,1),\ (2,2),\ (4,4),\ (8,8),\ \ldots,\ (2^{\log n}, 2^{\log n})$$

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- Combine to get pair of approximants of order $(2^{\ell+1}, 2^{\ell+1})$.

Superfast Hankel Inversion**

Inverse of

$$H_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix}$$

given by

$$\begin{bmatrix} v_{n-1} & \cdot\cdot & v_0 \\ \vdots & & \\ v_0 & & \end{bmatrix} \begin{bmatrix} q_{n-1} & \cdots & q_0 \\ & & \vdots \\ & \ddots & \\ & & q_{n-1} \end{bmatrix} - \begin{bmatrix} q_{n-2} & \cdot\cdot\, q_0 & 0 \\ \vdots & & \\ q_0 & & \\ 0 & & \end{bmatrix} \begin{bmatrix} v_n & \cdots & v_1 \\ & \ddots & \vdots \\ & & v_n \end{bmatrix}$$

where

$$H_n \begin{bmatrix} v_n \\ \\ v_2 \\ v_1 \end{bmatrix} = - \begin{bmatrix} a_{n+1} \\ \\ \vdots \\ a_{2n+1} \end{bmatrix} \text{ and } H_n \begin{bmatrix} q_{n-1} \\ \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Structured Linear Algebra

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Given : $a(z), b(z) \in \mathbb{K}[[z]]$ with $b_0 \neq 0$. Then solving

$$a(z)u(z) + b(z)v(z) = z^8 w(z), \qquad \deg u \leq 3, \qquad \deg v \leq 4.$$

is same as:

$$\begin{bmatrix} a_0 & & & & b_0 & & & & \\ a_1 & a_0 & & & b_1 & b_0 & & & \\ a_2 & a_1 & a_0 & & b_2 & b_1 & b_0 & & \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & b_1 & b_0 & \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & b_2 & b_1 & b_0 \\ a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & b_3 & b_2 & b_1 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & b_4 & b_3 & b_2 \\ a_7 & a_6 & a_5 & a_4 & b_7 & b_6 & b_5 & b_4 & b_3 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \hline v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0$$

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- Assume submatrix $S_{2,3}$ is nonsingular.

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- Assume submatrix $S_{2,3}$ is nonsingular.
- Reduce coefficient matrix $S_{4,5}$ making use of nonsingular $S_{2,3}$.

$$\text{Want}: \begin{bmatrix} a_0 & & & & b_0 & & & & \\ a_1 & a_0 & & & b_1 & b_0 & & & \\ a_2 & a_1 & a_0 & & b_2 & b_1 & b_0 & & \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & b_1 & b_0 & \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & b_2 & b_1 & b_0 \\ a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & b_3 & b_2 & b_1 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & b_4 & b_3 & b_2 \\ a_7 & a_6 & a_5 & a_4 & b_7 & b_6 & b_5 & b_4 & b_3 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \hline v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0$$

Solve $\begin{bmatrix} a_0 & & b_0 & & \\ a_1 & a_0 & b_1 & b_0 & \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & b_3 & b_2 & b_1 \\ a_4 & a_3 & b_4 & b_3 & b_2 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ q_0 \\ q_1 \\ q_2 \end{bmatrix} = - \begin{bmatrix} \\ \\ a_0 \\ a_1 \\ a_2 \end{bmatrix}$ i.e. $a(z)p(z) + b(z)q(z) = z^5 r(z)$

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Solve $$\begin{bmatrix} a_0 & & b_0 & & \\ a_1 & a_0 & b_1 & b_0 & \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & b_3 & b_2 & b_1 \\ a_4 & a_3 & b_4 & b_3 & b_2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \\ v_2 \end{bmatrix} = - \begin{bmatrix} \\ \\ \\ b_0 \\ b_1 \end{bmatrix}$$ i.e. $a(z)u(z) + b(z)v(z) = z^5 w(z)$

Solve $\begin{bmatrix} a_0 & & b_0 & & \\ a_1 & a_0 & b_1 & b_0 & \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & b_3 & b_2 & b_1 \\ a_4 & a_3 & b_4 & b_3 & b_2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \\ v_2 \end{bmatrix} = - \begin{bmatrix} \\ \\ \\ b_0 \\ b_1 \end{bmatrix}$ i.e. $a(z)u(z) + b(z)v(z) = z^5 w(z)$

$$\left[\begin{array}{cccccccccc} a_0 & & & & b_0 & & & & \\ a_1 & a_0 & & & b_1 & b_0 & & & \\ a_2 & a_1 & a_0 & & b_2 & b_1 & b_0 & & \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & b_1 & b_0 & \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & b_2 & b_1 & b_0 \\ \hline a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & b_3 & b_2 & b_1 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & b_4 & b_3 & b_2 \\ a_7 & a_6 & a_5 & a_4 & b_7 & b_6 & b_5 & b_4 & b_3 \end{array}\right] \left[\begin{array}{cc} u_0 & \\ u_1 & u_0 \\ & u_1 \\ & \\ \hline v_0 & \\ v_1 & v_0 \\ v_2 & v_1 \\ 1 & v_2 \\ & 1 \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline w_0 & 0 \\ w_1 & w_0 \\ w_2 & w_1 \end{array}\right]$$

$$\left[\begin{array}{ccccccccc} a_0 & & & & b_0 & & & & \\ a_1 & a_0 & & & b_1 & b_0 & & & \\ a_2 & a_1 & a_0 & & b_2 & b_1 & b_0 & & \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & b_1 & b_0 & \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & b_2 & b_1 & b_0 \\ \hline a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & b_3 & b_2 & b_1 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & b_4 & b_3 & b_2 \\ a_7 & a_6 & a_5 & a_4 & b_7 & b_6 & b_5 & b_4 & b_3 \end{array}\right] \left[\begin{array}{cc|ccc|cccc} 1 & 0 & 0 & 0 & 0 & p_0 & 0 & u_0 & 0 \\ 0 & 1 & 0 & 0 & 0 & p_1 & p_0 & u_1 & u_0 \\ 0 & 0 & 0 & 0 & 0 & 1 & p_1 & 0 & u_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & u_0 \\ \hline 0 & 0 & 1 & 0 & 0 & q_0 & 0 & v_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & q_1 & q_0 & v_1 & v_0 \\ 0 & 0 & 0 & 0 & 1 & q_2 & q_1 & v_2 & v_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_2 & 1 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$
\left[\begin{array}{ccccccccc}
a_0 & & & & b_0 & & & & \\
a_1 & a_0 & & & b_1 & b_0 & & & \\
a_2 & a_1 & a_0 & & b_2 & b_1 & b_0 & & \\
a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & b_1 & b_0 & \\
a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & b_2 & b_1 & b_0 \\
\hline
a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & b_3 & b_2 & b_1 \\
a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & b_4 & b_3 & b_2 \\
a_7 & a_6 & a_5 & a_4 & b_7 & b_6 & b_5 & b_4 & b_3
\end{array}\right]
\left[\begin{array}{cc|ccc|cccc}
1 & 0 & 0 & 0 & 0 & p_0 & 0 & u_0 & 0 \\
0 & 1 & 0 & 0 & 0 & p_1 & p_0 & u_1 & u_0 \\
0 & 0 & 0 & 0 & 0 & 1 & p_1 & 0 & u_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & u_0 \\
\hline
0 & 0 & 1 & 0 & 0 & q_0 & 0 & v_0 & 0 \\
0 & 0 & 0 & 1 & 0 & q_1 & q_0 & v_1 & v_0 \\
0 & 0 & 0 & 0 & 1 & q_2 & q_1 & v_2 & v_1 \\
0 & 0 & 0 & 0 & 0 & 0 & q_2 & 1 & v_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
= \left[\begin{array}{ccccc|cccc}
a_0 & & b_0 & & & & & & \\
a_1 & a_0 & b_1 & b_0 & & & & & \\
a_2 & a_1 & b_2 & b_1 & b_0 & & & & \\
a_3 & a_2 & b_3 & b_2 & b_1 & & & & \\
a_4 & a_3 & b_4 & b_3 & b_2 & & & & \\
\hline
a_5 & a_4 & b_5 & b_4 & b_3 & r_0 & & w_0 & \\
a_6 & a_5 & b_6 & b_5 & b_4 & r_1 & r_0 & w_1 & w_0 \\
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- Rational approximation problems all seem to to have linear systems
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- Our approach : March forward using invertible submatrices
- Variation for block and Hermite-Padé, etc cases
- More issues in block case
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Invertible Hankel Matrices

Recall:

Theorem

$$H_n = \begin{bmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ a_{n-1} & & a_{2n-2} \\ a_n & \cdots & a_{2n-1} \end{bmatrix}$$

is nonsingular iff can solve two equations

$$H_n \begin{bmatrix} q_{n-1} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \textit{and} \quad H_n \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = -\begin{bmatrix} a_{n+1} \\ \vdots \\ a_{2n-1} \\ a_{2n} \end{bmatrix}$$

Invertible Hankel Matrices

Theorem

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is nonsingular iff have

$$\begin{aligned} a(z)q(z) - p(z) &= z^{2n-1}r(z) \text{ with } r(0) = 1 \\ a(z)v(z) - u(z) &= z^{2n+1}w(z) \text{ with } v(0) = 1 \end{aligned}$$

$(p(z), q(z))$ *and* $(u(z), v(z))$ *are Padé Approximants for* $a(z)$ *of type* $(n-1, n-1)$ *and* (n, n), *respectively.*

An Algorithm for Padé Approximation

- Recursive step: Assume H_{n_i} is nonsingular. Then

$$
\begin{aligned}
a(z)q_i(z) - p_i(z) &= z^{m_i+n_i-1} r_i(z), \quad r_i(0) = 1 \\
a(z)v_i(z) - u_i(z) &= z^{m_i+n_i+1} w_i(z), \quad v_i(0) = 1
\end{aligned}
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- Form $\hat{a}_i(z) = r_i(z)^{-1} \cdot w_i(z)$. Find first nonsingular matrix $\hat{H}_t$ coming from $\hat{a}(z)$. Then we have

$$\begin{aligned} \hat{a}_i(z)\hat{q}(z) - \hat{p}(z) &= z^{2t_i-1} \hat{r}(z), \quad \hat{r}_i(0) = 1 \\ \hat{a}_i(z)\hat{v}(z) - \hat{u}(z) &= z^{2t_i+1} \hat{w}(z), \quad \hat{v}_i(0) = 1 \end{aligned}$$

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- Then $H_{n_i+t_i}$ is nonsingular and

$$\begin{aligned} p_{i+1}(z) &= u_i(z)\hat{q}(z) - z^2 p_i(z)\hat{p}(z) \\ q_{i+1}(z) &= v_i(z)\hat{q}(z) - z^2 q_i(z)\hat{p}(z) \\ u_{i+1}(z) &= u_i(z)\hat{v}(z) - z^2 p_i(z)\hat{u}(z) \\ v_{i+1}(z) &= v_i(z)\hat{v}(z) - z^2 q_i(z)\hat{u}(z) \end{aligned}$$

Off-diagonal path of computation in a Padé table: