Order Bases

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The Plan for this week:

- Monday : Introduction and History
- Tuesday : Structured Linear Algebra
- Wednesday : The Sigma Basis Algorithm
- Thursday: Fraction-Free Computation
- Friday : Working with Alternate Bases

Monday Outline

1. Motivation
2. Order Bases and Rational Approximation
3. History
4. Padé Approximation
5. Associated Structured Linear Systems

Preamble

In this lecture we introduce our problem, that of finding all solutions to

$$a_1(z)p_1(z) + \cdots a_m(z)p_m(z) \approx 0$$

for given power series $a_i(z)$ and certain degree constraints on the polynomials $p_i(z)$. We give some classic examples from rational approximation and show how they have been solved using linear algebra on the unknown coefficients of the $p_i(z)$. We discuss the history of sch problems and show why linear algebra is fine for the scalar case of Paé approximation. The linear algebra encounters structured matrices such as Hankel or Sylvester matrices.

Our Topic : Order Bases

Problem: Find solutions (polynomials) of equations of the form

$$a_1(z)p_1(z) + \cdots + a_m(z)p_m(z) = z^{\sigma} r_0 + z^{\sigma+1} r_1 + \cdots$$

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$a_i(z) \in \mathbb{K}[[z]]$ can be scalar power series ($\mathbb{K}$ a field)

or

$a_i(z) \in \mathbb{K}^{c \times d}[[z]]$ can be matrices of power series

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Order Bases describes all solutions.

Our Topic : Order Bases

Problem: Find solutions (polynomials) of equations of the form

$$a_1(z)p_1(z) + \cdots + a_m(z)p_m(z) = 0 \quad \text{(sometimes)}$$

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- Each solution $\mathbf{P}(z)$ can be written as

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  for a unique set of polynomials $\alpha_i(z)$.

- problem degrees bounds implies degree bounds for $\alpha_i(z)$.

Problem from Algebraic Combinatorics

Given a generating function $y(z)$ determine if it is D-Finite.

Find polynomials $p_0(z), \ldots, p_m(z)$ such that

$$p_0(z)y(z) + p_1(z)y'(z) + \cdots + p_m(z)y^{(m)}(z) = 0$$

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Problem from Creative Telescoping

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Recall : Hermite reduction

$$\int \frac{(1+tx^7)}{(t^2+x^2)^3(1+x^3)^4}\,dx$$

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$$\int \frac{(1+tx^7)}{(t^2+x^2)^3(1+x^3)^4}dx = \quad h(x,t) + \int \frac{r(x,t)}{(t^2+x^2)(1+x^3)}dx$$

Problem from Creative Telescoping

Recall : Hermite reduction

$$\frac{(1+tx^7)}{(t^2+x^2)^3(1+x^3)^4} = \partial_x h(x,t) + \frac{r(x,t)}{(t^2+x^2)(1+x^3)}$$

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$$F = \partial_x\Big(\cdots\Big) + \frac{r_0}{d}$$

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$$\partial_t(F) = \partial_x\Big(\cdots\Big) + \frac{r_1}{d}$$

Problem from Creative Telescoping

$$
\begin{aligned}
F &= \partial_x\Big(\cdots\Big) + \frac{r_0}{d} \\
\partial_t(F) &= \partial_x\Big(\cdots\Big) + \frac{r_1}{d} \\
&\vdots \\
\partial_t^{\rho}(F) &= \partial_x\Big(\cdots\Big) + \frac{r_\rho}{d}
\end{aligned}
$$

Problem from Creative Telescoping

$$c_0(t)\, F = \partial_x\Big(\cdots\Big) + c_0(t)\, \frac{r_0}{d}$$

$$c_1(t)\, \partial_t(F) = \partial_x\Big(\cdots\Big) + c_1(t)\, \frac{r_1}{d}$$

$$\vdots$$

$$c_\rho(t)\, \partial_t^\rho(F) = \partial_x\Big(\cdots\Big) + c_\rho(t)\, \frac{r_\rho}{d}$$

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\right.
$$

$$
\left(\sum_{i=0}^{\rho} c_i(t)\partial_t^i\right) F = \partial_x\Big(\cdots\Big) + \left(\sum_{i=0}^{\rho} c_i(t) r_i\right)\frac{1}{d}
$$

Problem from Creative Telescoping

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telescoper?

$$c_\rho(t)\,\partial_t^\rho(F) = \partial_x\Big(\cdots\Big) + c_\rho(t)\,\frac{r_\rho}{d}$$

$\stackrel{?}{=} 0$

$$\left(\sum_{i=0}^{\rho} c_i(t)\partial_t^i\right)F = \partial_x\Big(\cdots\Big) + \left(\sum_{i=0}^{\rho} c_i(t) r_i\right)\frac{1}{d}$$

Problem from Creative Telescoping

$$c_0(t)r_0 + c_1(t)r_1 + \cdots + c_\rho(t)r_\rho \stackrel{?}{=} 0$$

$$\Downarrow$$

a linear system with unknowns $c_i(t)$

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a telescoper $c_0(t) + c_1(t)\partial_t + \cdots + c_\rho(t)\partial_t^\rho$

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Remark.

- The first linear depend. leads to a minimal telescoper.

Rational Approximation

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Want to find all solutions of equations of the form

$$a_1(z)p_1(z) + \cdots + a_m(z)p_m(z) = O(z^{\vec{\sigma}})$$

Typically also have degree constraints $\vec{n} = (n_1, \ldots, n_m)$ and want

$$\deg\, p_i(z) \le n_i.$$

where $a_i(z)$ can be scalar power series

or

$a_i(z)$ can be matrices of power series

Example : Padé Approximation

Given integers m and n and a power series

$$a(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

Find polynomials $p(z), q(z)$ with

(a) deg $p(z) \leq m$, deg $q(z) \leq n$

(b) $a(z) \approx \frac{p(z)}{q(z)}$.

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$$a(z)q(z) - p(z) = O(z^{m+n+1})$$

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Observe: Problem has $m + n + 2$ unknowns and $m + n + 1$ conditions.

Associated Linear System

e.g. $m = 1$, $n = 3$

$$a(z)q(z) - p(z) = O(z^5)$$

Unknown coefficients from $p(z)$ and $q(z)$ solved via :

$$\begin{aligned}
a_0 q_0 &= p_0 \\
a_1 q_0 + a_0 q_1 &= p_1 \\
a_2 q_0 + a_1 q_1 + a_0 q_2 &= 0 \\
a_3 q_0 + a_2 q_1 + a_1 q_2 + a_0 q_3 &= 0 \\
a_4 q_0 + a_3 q_1 + a_2 q_2 + a_1 q_3 &= 0
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\end{aligned}$$

Notice one free choice. Typically specify q_0

Always has a solution.

Example : e^z

$$
\begin{aligned}
e^z &= \frac{1 + 1/2z}{1 - 1/2z} + O(z^3) \\
&= \frac{1 + 1/2z + 1/12z^2}{1 - 1/2z - 1/12z^2} + O(z^5) \\
&= \frac{1 + 1/2z + 1/12z^2 + 1/120z^3}{1 - 1/2z - 1/12z^2 + 1/120z^3} + O(z^7)
\end{aligned}
$$

Example : Hermite-Padé Approximation

Given power series $a_1(z), \ldots, a_m(z)$ and integers $n_1, \ldots, n_m$

Find $p_1(z), ..., p_m(z)$ with

(a) $a_1(z)p_1(z) + \cdots + a_m(z)p_m(z) = O(z^N)$

with $N = n_1 + \cdots + n_m + m - 1.$

(b) deg $p_i(z) \leq n_i$

Models: $a_1(z)p_1(z) + \cdots + a_m(z)p_m(z) \approx 0$

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Models: $\quad a_1(z)p_1(z) + \cdots + a_m(z)p_m(z) \approx 0$

Problem has $N + 1$ unknowns and N conditions.

Example : Hermite-Padé

Associated linear system:

$$a(z) \cdot p(z) + b(z) \cdot q(z) + c(z) \cdot s(z) = O(z^8)$$

with deg $p(z) \leq 2$, deg $q(z) \leq 3$, deg $s(z) \leq 1$

$$\left[\begin{array}{ccc|cccc|cc} a_0 & 0 & 0 & b_0 & 0 & 0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & 0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & 0 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & b_0 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & b_1 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & b_2 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & b_3 & c_6 & c_5 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & b_4 & c_7 & c_6 \end{array}\right] \cdot \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ \hline q_0 \\ q_1 \\ q_2 \\ q_3 \\ \hline s_0 \\ s_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

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$$\left[\begin{array}{ccc|cccc|cc} a_0 & 0 & 0 & b_0 & 0 & 0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & 0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & 0 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & b_0 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & b_1 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & b_2 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & b_3 & c_6 & c_5 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & b_4 & c_7 & c_6 \end{array}\right] \cdot \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ \hline q_0 \\ q_1 \\ q_2 \\ q_3 \\ \hline s_0 \\ s_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

One more unknown than equation. Solution always exists.

Simultaneous Padé Approximants

Given power series $a_1(z), \ldots, a_m(z)$ and integers $n_0, \ldots, n_m$

Find $p_1(z), ..., p_m(z)$ and $p_0(z)$ with

(a) $$a_i(z)p_0(z) - p_i(z) = O(z^{N+1})$$

with $$N = n_0 + \cdots + n_m.$$

(b) deg $p_i(x) \leq N - n_i$

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(b) deg $p_i(x) \leq N - n_i$

Models :

$$a_1(z) \approx \frac{p_1(z)}{p_0(z)}, \quad a_2(z) \approx \frac{p_2(z)}{p_0(z)}, \quad \ldots, \quad a_m(z) \approx \frac{p_m(z)}{p_0(z)}.$$

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Given power series $a_1(z), \ldots, a_m(z)$ and integers $n_0, \ldots, n_m$

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Note : $mN + m + 1$ unknowns and $mN + m$ conditions.

Example

Associated linear system:

$$a(z) \cdot p(z) - q(z) = b(z) \cdot p(z) - r(z) = c(z) \cdot p(z) - s(z) = O(z^4)$$

with deg $p(z) \leq 3$, deg $q(z) \leq 2$, deg $r(z) \leq 2$, deg $s(z) \leq 2$

$$\left[\begin{array}{cccc|ccc|ccc|ccc}
-1 & & & & a_0 & & & & & & & & \\
 & -1 & & & a_1 & a_0 & & & & & & & \\
 & & -1 & & a_2 & a_1 & a_0 & & & & & & \\
 & & & -1 & a_3 & a_2 & a_1 & & & & & & \\
-1 & & & & & & & b_0 & & & & & \\
 & -1 & & & & & & b_1 & b_0 & & & & \\
 & & -1 & & & & & b_2 & b_1 & b_0 & & & \\
 & & & -1 & & & & b_3 & b_2 & b_1 & & & \\
-1 & & & & & & & & & & c_0 & & \\
 & -1 & & & & & & & & & c_1 & c_0 & \\
 & & -1 & & & & & & & & c_2 & c_1 & c_0 \\
 & & & -1 & & & & & & & c_3 & c_2 & c_1
\end{array}\right]
\left[\begin{array}{c}
p_0 \\ p_1 \\ p_2 \\ p_3 \\ \hline q_0 \\ q_1 \\ q_2 \\ \hline r_0 \\ r_1 \\ r_2 \\ \hline s_0 \\ s_1 \\ s_2
\end{array}\right] = 0$$

Example

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 & -1 & & & & & & b_1 & b_0 & & & & \\
 & & -1 & & & & & b_2 & b_1 & b_0 & & & \\
 & & & -1 & & & & b_3 & b_2 & b_1 & & & \\
-1 & & & & & & & & & & c_0 & & \\
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\left[\begin{array}{c}
p_0 \\ p_1 \\ p_2 \\ p_3 \\ \hline q_0 \\ q_1 \\ q_2 \\ \hline r_0 \\ r_1 \\ r_2 \\ \hline s_0 \\ s_1 \\ s_2
\end{array}\right] = 0$$

One more unknown than equation. Solution always exists.

Example : Matrix Padé Approximation

Given $s \times s$ matrix of power series and $m, n \in \mathbb{Z}_{geq0}$:

$$a(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

Find matrix polynomials $p(z), q(z)$ with

(a) deg $p(z) \leq m$, deg $q(z) \leq n$

(b) $a(z)q(z) - p(z) = O(z^{m+n+1})$

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Models $a(z) \approx p(z)q(z)^{-1}$.

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Models $a(z) \approx p(z)q(z)^{-1}$. Can also specify on left.

Example : Matrix Padé Approximation

Given $s \times s$ matrix of power series and $m, n \in \mathbb{Z}_{geq0}$:

$$a(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

Find matrix polynomials $p(z), q(z)$ with

(a) deg $p(z) \leq m$, deg $q(z) \leq n$

(b) $a(z)q(z) - p(z) = O(z^{m+n+1})$

Models $a(z) \approx p(z)q(z)^{-1}$. Can also specify on left.

Problem has $s(m+n+2)$ unknowns and $s(m+n+1)$ conditions.

Associated Linear System

e.g. $m = 1$, $n = 3$

$$a(z)q(z) - p(z) = O(z^5)$$

Unknown column coefficients from $p(z)$ and $q(z)$ solved via :

$$\begin{aligned}
a_0 q_0 &= p_0 \\
a_1 q_0 + a_0 q_1 &= p_1 \\
a_2 q_0 + a_1 q_1 + a_0 q_2 &= 0 \\
a_3 q_0 + a_2 q_1 + a_1 q_2 + a_0 q_3 &= 0 \\
a_4 q_0 + a_3 q_1 + a_2 q_2 + a_1 q_3 &= 0
\end{aligned}$$

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$$\begin{array}{lcr} a_0 q_0 & = & p_0 \\ a_1 q_0 + a_0 q_1 & = & p_1 \\ a_2 q_0 + a_1 q_1 + a_0 q_2 & = & 0 \\ a_3 q_0 + a_2 q_1 + a_1 q_2 + a_0 q_3 & = & 0 \\ a_4 q_0 + a_3 q_1 + a_2 q_2 + a_1 q_3 & = & 0 \end{array}$$

Now s free choices. Can always build a (matrix) solution.

Not clear $q(z)^{-1}$ can exist (unfortunately).

Order Bases

Gives all solutions of all rational approximation problems.

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Rational approximation problems appear in:

- Transcendence of e and other famous numbers
- Inversion formulae for structured matrices
- Linear diophantine equations (and hence to GCDs)
- Guessing recurrence formulae (e.g. Gfun)
- Reconstruction of power series to polynomial problems
  (e.g. DFactor)
- Fast polynomial matrix arithmetic
  (e.g. Determinant, Inverse, Nullspace, etc)
- Matrix normal forms (Hermite, Popov, etc)
- ...

History

People

- 1730-1870: Lambert, Lagrange, Hankel, Frobenius, ...
  - connection to continued fractions, etc
- 1850-1890: Hermite
  - used these (informally) to prove transcendence of e (1873)
- 1890s: Padé
  - student of Hermite
  - first systematic study of rational approximation (1893)
  - Padé table

Early Construction by Hermite

For any polynomial $f(x)$ of degree ℓ, integration by parts gives:

$$\int_0^1 f(x)e^{-zx}dx = \frac{p(z) - e^{-z}q(z)}{z^{\ell+1}}$$

with

$$\begin{aligned} p(z) &= f(0)z^{\ell} + f'(0)z^{\ell-1} + \cdots + f^{(\ell)}(0), \\ q(z) &= f(1)z^{\ell} + f'(1)z^{\ell-1} + \cdots + f^{(\ell)}(1). \end{aligned}$$

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Suppose : $\quad f(x) = x^n(x-1)^m \quad$ with $\; m + n = \ell$. Then we get:

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Suppose : $f(x) = x^n(x-1)^m$ with $m + n = \ell$. Then we get:

- $\deg p(z) \leq m$ & $\deg q(z) \leq n$
- $e^z \cdot p(z) - q(z) = O(z^{m+n+1})$
- Construction of Padé Approximant of type (m, n) for e^z

Another early construction by Hermite

For any polynomial $f(x)$ of degree ℓ, integration by parts gives:

$$\int_0^k f(x)e^{-zx}dx = \frac{p(z) - e^{-kz}q_k(z)}{z^{\ell+1}}$$

with

$$\begin{aligned} p(z) &= f(0)z^\ell + f'(0)z^{\ell-1} + \cdots + f^{(\ell)}(0), \\ q_k(z) &= f(k)z^\ell + f'(k)z^{\ell-1} + \cdots + f^{(\ell)}(k). \end{aligned}$$

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If $f(x) = x^{n-1}(x-1)^n \cdots (x-m)^n$ and $N = mn + n - 1$ then

- deg $p(z) \leq N - (n-1)$
- deg $q_k(z) \leq N - n$
- $e^{kz}p(z) - q_k(z) = O(z^{N+1})$

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If $f(x) = x^{n-1}(x-1)^n \cdots (x-m)^n$ and $N = mn + n - 1$ then

- deg $p(z) \leq N - (n-1)$
- deg $q_k(z) \leq N - n$
- $e^{kz}p(z) - q_k(z) = O(z^{N+1})$
- Simultaneous Padé: type $(n-1, n, \ldots, n)$ for $(1, e^z, \ldots, e^{mz})$.

Example : e^z and e^{2z} and e^{3z}

e.g. when $N = 3$ and $n = 1$:

$$e^z = \frac{2\,z^2 - 6\,z + 6}{-6\,z^3 + 11\,z^2 - 12\,z + 6} + O(z^4)$$

$$e^{2z} = \frac{-z^2 + 6}{-6\,z^3 + 11\,z^2 - 12\,z + 6} + O(z^4)$$

$$e^{3z} = \frac{2z^2 + 6\,z + 6}{-6\,z^3 + 11\,z^2 - 12\,z + 6} + O(z^4)$$

Simultaneous Padé of type $(0, 1, 1, 1)$ for $(1, e^z, e^{2z}, e^{3z})$.

Transcendence of e

Suppose we have integers $a_0, a_1, \ldots, a_m$:

$$a_m e^m + \cdots + a_1 e + a_0 = 0$$

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Find integers d_0, $d_1, \ldots, d_m$ such that $\quad e^k \approx \frac{d_k}{d_0}$

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$$\text{i.e.} \qquad d_0 e^k - d_k = \eta_k \qquad \eta_k \text{ small !}$$

Transcendence of e

Suppose we have integers $a_0, a_1, \ldots, a_m$:

$$a_m e^m + \cdots + a_1 e + a_0 = 0$$

Find integers $d_0,\ d_1, \ldots, d_m$ such that $\quad e^k \approx \frac{d_k}{d_0}$

$$\text{i.e.} \qquad d_0 e^k - d_k = \eta_k \qquad \eta_k \text{ small !}$$

This gives:

$$a_m d_m + \cdots + a_1 d_1 + a_0 d_0 \;=\; -a_m \eta_m - \cdots - a_1 \eta_1$$

$$\textit{nonzero integer} \;=\; \textit{small}$$

Transcendence of e

Recall : $p_k(z) - q_k(z)e^{-kz} = z^{\ell+1}\int_0^k f(x)e^{-zx}dx$ with

$$\begin{aligned} f(x) &= \text{degree } \ell, \\ p_k(z) &= f(0)z^\ell + f'(0)z^{\ell-1} + \cdots + f^{(\ell)}(0), \\ q_k(z) &= f(k)z^\ell + f'(k)z^{\ell-1} + \cdots + f^{(\ell)}(k). \end{aligned}$$

$f(x) = x^{n-1}(x-1)^n \cdots (x-m)^n$ gives Simultaneous Padé Approximant for $(1, e^z, \ldots, e^{mz})$ of type $(n-1, n, \ldots, n)$

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- $\eta_k = e^k p_k(1) - q_k(1) = e^k F(0) - F(k)$.

- η_k can be made small (via controlling n)

Charles Hermite



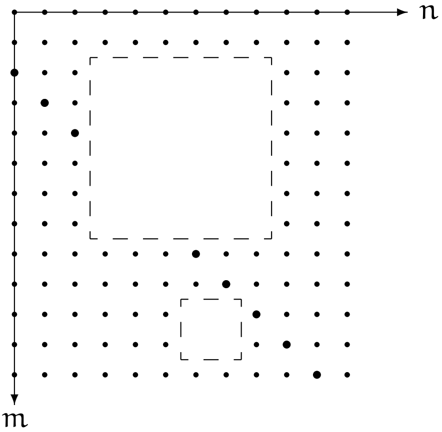

Henri Padé

Padé Approximation

Padé Table

- Table of (m, n) Padé approximants $\frac{p_m(z)}{q_n(z)}$.
- Table has a specific rectangular structure
- Common approximants inside rectangles
- Rectangles correspond to singular linear systems

Padé Table



Linear System : $a(z)v(z) - u(z) = z^{m+n+1}w(z)$

$$(a_0 + a_1 z + \cdots)(v_0 + \cdots + v_n z^n) - (u_0 + \cdots + u_m z^m) = z^{m+n+1} w_0 + \cdots$$

Same as

$$\left[\begin{array}{ccccc}
a_0 & & & & \\
a_1 & a_0 & & & \\
\vdots & & & & \\
a_m & \cdots & \cdots & & a_{m-n} \\
\hline
a_{m+1} & \cdots & \cdots & a_{m-n} & a_{m-n+1} \\
a_{m+2} & \cdots & \cdots & a_{m-n+1} & a_{m-n+2} \\
\vdots & & & & \vdots \\
\vdots & & & & \vdots \\
a_{m+n-1} & \cdots & \cdots & a_m & a_{m-1} \\
a_{m+n} & \cdots & \cdots & a_{m+1} & a_m
\end{array}\right] \cdot \left[\begin{array}{c} v_0 \\ v_1 \\ \vdots \\ \vdots \\ v_{n-1} \\ v_n \end{array}\right] = \left[\begin{array}{c} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \\ u_m \\ \hline 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{array}\right]$$

Note the structured coefficient matrix.

Linear System : $a(z)v(z) - u(z) = z^{m+n+1}w(z)$

$$(a_0 + a_1 z + \cdots)(v_0 + \cdots + v_n z^n) - (u_0 + \cdots + u_m z^m) = z^{m+n+1} w_0 + \cdots$$

Focus on Hankel system:

$$\begin{bmatrix} a_{m-n+1} & \cdots & \cdots & a_{n-1} & a_n \\ a_{m-n+2} & \cdots & \cdots & a_n & a_{n+1} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n-1} & \cdots & \cdots & & a_{m+n-2} \\ a_n & \cdots & \cdots & a_{m+n-2} & a_{m+n-1} \end{bmatrix} \cdot \begin{bmatrix} v_n \\ v_{n-1} \\ \vdots \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = -v_0 \begin{bmatrix} a_{m+1} \\ a_{m+2} \\ \vdots \\ \vdots \\ a_{m+n-1} \\ a_{m+n} \end{bmatrix}$$

So find $v(z)$ first, then $u(z)$.

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So find $v(z)$ first, then $u(z)$.

Similarly for a_i square matrices (get Block Hankel system).

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$$\begin{bmatrix} a_{m-n+1} & \cdots & \cdots & a_{n-1} & a_n \\ a_{m-n+2} & \cdots & \cdots & a_n & a_{n+1} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n-1} & \cdots & \cdots & & a_{m+n-2} \\ a_n & \cdots & \cdots & a_{m+n-2} & a_{m+n-1} \end{bmatrix} \cdot \begin{bmatrix} v_n \\ v_{n-1} \\ \vdots \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = -v_0 \begin{bmatrix} a_{m+1} \\ a_{m+2} \\ \vdots \\ \vdots \\ a_{m+n-1} \\ a_{m+n} \end{bmatrix}$$

So find $v(z)$ first, then $u(z)$.

Similarly for a_i square matrices (get Block Hankel system).

Nice when LHS coefficient matrix is nonsingular.

Invertible Hankel Matrices

Theorem

$$H_{m,n} = \begin{bmatrix} a_{m-n+1} & \cdots & a_n \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_n & \cdots & a_{m+n-1} \end{bmatrix}$$

is nonsingular iff can solve two equations

$$H_{m,n} \begin{bmatrix} q_{n-1} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \textit{and} \quad H_{m,n} \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = - \begin{bmatrix} a_{m+1} \\ \vdots \\ a_{m+n-1} \\ a_{m+n} \end{bmatrix}$$

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Same is true for block case where a_i are square matrices.

Invertible Hankel Matrices (easier if $m = n$)

Theorem

$$H_n = \begin{bmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ a_{n-1} & & a_{2n-2} \\ a_n & \cdots & a_{2n-1} \end{bmatrix}$$

is nonsingular iff can solve two equations

$$H_n \begin{bmatrix} q_{n-1} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \textit{and} \quad H_n \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = -\begin{bmatrix} a_{n+1} \\ \vdots \\ a_{2n-1} \\ a_{2n} \end{bmatrix}$$

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Invertible Hankel Matrices

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is nonsingular iff have

$$\begin{aligned} a(z)q(z) - p(z) &= z^{2n-1} r(z) \text{ with } r(0) = 1 \\ a(z)v(z) - u(z) &= z^{2n+1} w(z) \text{ with } v(0) = 1 \end{aligned}$$

$(p(z), q(z))$ *and* $(u(z), v(z))$ *are two Padé Approximants for* $a(z)$ *of type* $(n-1, n-1)$ *and* (n, n), *respectively.*

Computation

Modified Schur Complements

Suppose H_m is nonsingular and

$$H_m[v_m, \dots, v_1]^T = -[a_{m+1}, \dots, a_{2m}]^T$$

Also

$$a(z)v(z) - u(z) = z^{2m+1+k}w(z) \quad \text{with } v(0) = 1 \quad \text{and } k \geq 0.$$

Then

$$\left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ 0 & & 1 & & & \\ v_m & \cdots & v_1 & v_0 & & \\ & \ddots & & & \ddots & \\ & & v_m & \cdots & v_1 & v_0 \end{array}\right] \left[\begin{array}{cccccc} a_1 & \cdots & a_m & a_{m+1} & \cdots & a_{m+k} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_m & \cdots & a_{2m-1} & a_{2m} & \cdots & a_{2m+k-1} \\ a_{m+1} & \cdots & a_{2m} & a_{2m+1} & \cdots & a_{2m+k} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m+k} & \cdots & a_{2m+2k-1} & a_{2m+2k} & \cdots & a_{2m+2k-1} \end{array}\right]$$

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Also

$$a(z)v(z) - u(z) = z^{2m+1+k} w(z) \quad \text{with } v(0) = 1 \quad \text{and } k \geq 0.$$

Then

$$L \cdot H_{m+k} = \left[\begin{array}{c|c} H_m & * \\ \hline 0 & \hat{H}_k \end{array}\right] = \left[\begin{array}{ccc|ccc} a_1 & \cdots & a_m & a_{m+1} & \cdots & a_{m+k} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_m & \cdots & a_{2m-1} & a_{2m} & \cdots & a_{2m+k-1} \\ \hline 0 & \cdots & 0 & 0 & & w_0 \\ \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 0 & w_0 & \cdots & w_{k-1} \end{array}\right]$$

Padé Table

Note: $a(z)v(z) - u(z) = z^{2n+1+k}w(z)$ with $v(0) \neq 0$ and $w(0) \neq 0$:

(i) $H_{n+\ell}$ singular for all $0 \leq \ell < k$.

(ii) $(u(z)), v(z))$ and Padé approximant of type (n, n) then

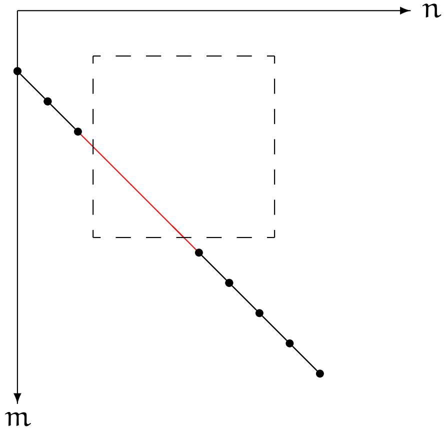
$$(u(z)) \cdot \alpha(z), v(z) \cdot \alpha(z))$$

$\alpha(z)$ arbitrary polynomial of degree at most ℓ is Padé approximant of type $(n + \ell, n + \ell)$

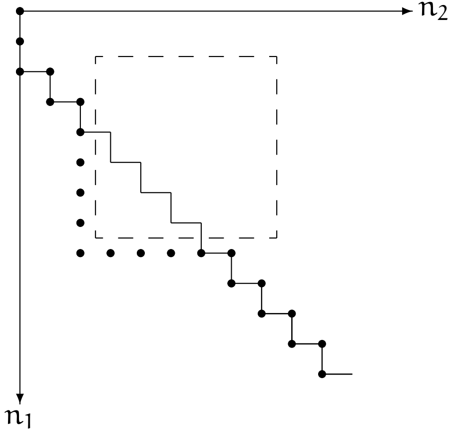
(iii) $(u(z)), v(z))$ and Padé approximant of type (n, n) then $(u(z), v(z))$ Padé approximant of type $(n + s, n + t)$ for $s, t < k$.

(iv) Algorithms need to go around or jump these square blocks.

Off-diagonal path of computation in a Padé table:



Staircase path of computation in a Padé table:



Padé Table

Suppose that H_n is a singular Hankel matrix and that

$$H_m = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ a_2 & a_3 & & \vdots \\ \vdots & & \ddots & \vdots \\ a_m & \cdots & \cdots & a_{2m-1} \end{bmatrix}$$

is the largest principal nonsingular submatrix of H_n with $m < n$. As H_m is nonsingular we can solve the linear system

$$H_m \cdot [v_m, \ldots, v_1]^t = -[a_{m+1}, \ldots, a_{2m}]^t$$

and use this in the matrix identity

$$L \cdot H_{m+k} = \left[\begin{array}{c|c} H_m & * \\ \hline 0 & \hat{H}_k \end{array}\right] = \left[\begin{array}{ccc|ccc} a_1 & \cdots & a_m & a_{m+1} & \cdots & a_{m+k} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_m & \cdots & a_{2m-1} & a_{2m} & \cdots & a_{2m+k-1} \\ \hline 0 & \cdots & 0 & 0 & & w_0 \\ \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 0 & w_0 & \cdots & w_{k-1} \end{array}\right]$$

Note

- All Padé approximants known in scalar case
  - Padé table in scalar case has a type of block structure
- Padé approximants related to diophantine equations
  - There are algorithms corresponding to Euclidean algorithm
  - Fast way to compute Padé approximants in scalar case
- Nothing known about structure of matrix Padé case or Hermite-Padé case by 1990
- Use in inversion formulas for Hankel and Toeplitz matrices known in scalar and block cases.