Exercise Sheet 1: Generating Functions

> Everything should be as simple as it can be, but not simpler.
>
> Roger Sessions, paraphrasing Albert Einstein

> All men have eyes but few have the gift of penetration. Every one sees your exterior, but few can discern what you have in your heart.
>
> Machiavelli, *The Prince*

Exercise 1: Stanley's Conjecture

In class we saw that the generating function

$$\sum_{n\geq 0} \binom{2n}{n}^2 z^n$$

is transcendental. In 1980, Richard Stanley conjectured that

$$\sum_{n\geq 0} \binom{2n}{n}^k z^n$$

is transcendental for all $k \geq 2$. This was proven by Philippe Flajolet in 1987.

1. (Easy) Prove Stanley's conjecture for all even $k \geq 2$.
2. (Harder) Prove Stanley's conjecture for all $k \geq 2$.

Exercise 2: Trivariate Diagonals

Find a trivariate rational function with transcendental diagonal.
(There are some very simple ones)

Exercise 3: Binary Strings (Credit to Alin Bostan)

Solve the following problem, posed in the December 2011 Mathematical Monthly.

> **11610**. *Proposed by Richard P. Stanley, Massachussetts Institute of Technology, Cambridge, MA.* Let $f(n)$ be the number of binary words $a_1 \cdots a_n$ of length n that have the same number of pairs $a_i a_{i+1}$ equal to 00 as equal to 01. Show that
>
> $$\sum_{n=0}^{\infty} f(n)t^n = \frac{1}{2}\left(\frac{1}{1-t} + \frac{1+2t}{\sqrt{(1-t)(1-2t)(1+t+2t^2)}}\right).$$

Hint: Let $Z(a,b,c)$ and $O(a,b,c)$ be the generating functions for the number of binary words ending in a 0 or 1 (respectively) where a counts the length of the word, b counts the number of 00 patterns, and c counts the number of 01 patterns. Set up and solve a system of equations involving Z and O to obtain the generating function $W(a,b,c)$ of words ending in either a 0 or a 1. Use a residue computation to find an (incomplete) diagonal.

Exercise 4: Hadamard Products

The *Hadamard product* of two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ is the sequence defined by their pairwise product $(a_n \cdot b_n)_{n\geq 0}$.

1. Show that the class of algebraic functions is not closed under the Hadamard product.
2. Show that the Hadamard product of a rational function and an algebraic function is algebraic.

Exercise 5: Algebraic Function Coefficients

Prove the following statement, from Flajolet and Sedgewick's Analytic Combinatorics.

> ▷ **VII.34.** *Multinomial sums and algebraic coefficients*. Let $F(z)$ be an algebraic function. Then $F_n = [z^n]F(z)$ is a (finite) linear combination of "multinomial forms" defined as
>
> $$S_n(\mathbf{C}; h; c_1, \ldots, c_r) := \sum_{\mathbf{C}} \binom{n_0 + h}{n_1, \ldots, n_r} c_1^{n_1} \cdots c_r^{n_r},$$
>
> where the summation is over all values of $n_0, n_1, \ldots, n_r$ satisfying a collection of linear inequalities $\mathbf{C}$ involving n.