Supporting proving and discovering geometric inequalities in GeoGebra by using Tarski

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² Author is supported by the grant EFOP-3.6.2-16-2017-00015.
We introduce *GeoGebra Discovery* that can automatically prove or discover geometric inequalities. It consists of

- an extended version of *GeoGebra*,
- a controller web service *realgeom*,
- and the computational tool *Tarski* (with the extensive help of *QEPCAD B*).

We successfully solve several non-trivial problems in Euclidean planar geometry via a simple graphical user interface.
Supporting inequalities in GeoGebra by using Tarski


<table>
<thead>
<tr>
<th>Feature</th>
<th>GeoGebra</th>
<th>GeoGebra Discovery</th>
<th>Next step</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discover tool/command</td>
<td>no</td>
<td>yes</td>
<td>Scheduled for merging into GeoGebra</td>
</tr>
<tr>
<td>Compare command</td>
<td>no</td>
<td>yes</td>
<td>GeoGebra Team: approve/update</td>
</tr>
<tr>
<td>IncircleCenter command</td>
<td>no</td>
<td>yes (with prover support)</td>
<td>GeoGebra Team: approve (discuss Center(Incircle) first)</td>
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<tr>
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<tr>
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<td>no</td>
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<td>2.2</td>
<td>2.4</td>
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<td>ApplyMap command</td>
<td>no</td>
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### GeoGebra Discovery: an experimental version of GeoGebra

[github.com/kovzol/geogebra-discovery](https://github.com/kovzol/geogebra-discovery)

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Supporting inequalities in GeoGebra by using Tarski
Implementation: System layout of GeoGebra Discovery

Planned, on-going work

Supporting inequalities in GeoGebra by using Tarski
Motivation
A generalization of the Pythagorean Theorem

1.

Equational hypotheses:

\[
\begin{align*}
A(v_1, v_2) & : v_1^2 - v_4^2 - v_3^2 + 2v_4v_2 - v_2^2 + 2v_3v_1 - v_1^2 = 0 \\
b & : v_5^2 - v_6^2 - v_5^2 + 2v_6v_2 - v_2^2 + 2v_5v_1 - v_1^2 = 0 \\
a & : v_8^2 - v_9^2 - v_5^2 + 2v_6v_4 - v_4^2 + 2v_5v_3 - v_3^2 = 0
\end{align*}
\]
Motivation
A generalization of the Pythagorean Theorem

1 Equational hypotheses:

\[ A(v_1, v_2) \]

\[ c: v_7^2 - v_4^2 - v_3^2 + 2v_4v_2 - v_2^2 + 2v_3v_1 - v_1^2 = 0 \]

\[ b: v_5^2 - v_6^2 - v_9^2 + 2v_6v_2 - v_2^2 + 2v_5v_1 - v_1^2 = 0 \]

\[ a: v_8^2 - v_9^2 - v_5^2 + 2v_5v_4 - v_4^2 + 2v_5v_3 - v_3^2 = 0 \]

2 Non-degeneracy condition:

\[ v_{10} \cdot (v_5 \cdot v_4 - v_6 \cdot v_3 - v_5 \cdot v_2 + v_3 \cdot v_2 + v_6 \cdot v_1 - v_4 \cdot v_1) = 1 \]

3 Exploration related equation: \( \mu \cdot v_7^2 = v_8^2 + v_9^2 \)

4 Non-equational assumptions: \( v_7 > 0 \land v_8 > 0 \land v_9 > 0 \)
Motivation
A generalization of the Pythagorean Theorem

1. Equational hypotheses:

\[ a : v_8^2 - v_5^2 - v_6^2 + 2v_5v_4 - v_4^2 + 2v_5v_3 - v_3^2 = 0 \]
\[ b : v_9^2 - v_6^2 - v_7^2 + 2v_6v_2 - v_2^2 + 2v_5v_1 - v_1^2 = 0 \]
\[ c : v_7^2 - v_4^2 - v_3^2 + 2v_4v_2 - v_2^2 + 2v_3v_1 - v_1^2 = 0 \]

2. Non-degeneracy condition:

\[ v_{10} \cdot (v_5 \cdot v_4 - v_6 \cdot v_3 - v_5 \cdot v_2 + v_3 \cdot v_2 + v_6 \cdot v_1 - v_4 \cdot v_1) = 1 \]

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\[ \mu \cdot v_7^2 = v_8^2 + v_9^2 \]

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\[ v_7 > 0 \land v_8 > 0 \land v_9 > 0 \]

⇒
Motivation
A generalization of the Pythagorean Theorem

1. **Equational hypotheses:**

   \[ A(v_1, v_2) \]

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2. **Non-degeneracy condition:**

   \[ v_{10} \cdot \left( v_5 \cdot v_4 - v_6 \cdot v_3 - v_5 \cdot v_2 + v_3 \cdot v_2 + v_6 \cdot v_1 - v_4 \cdot v_1 \right) = 1 \]

3. **Exploration related equation:**

   \[ \mu \cdot v_7^2 = v_8^2 + v_9^2 \]

4. **Non-equational assumptions:**

   \[ v_7 > 0 \land v_8 > 0 \land v_9 > 0 \]

   \[ \Rightarrow \mu > 1/2 \]
Symbolic check in GeoGebra (via $\text{Relation}(a^2 + b^2, c^2)$):

It is generally true that:
- $(a^2 + b^2) \geq (1/2) \cdot (c^2)$

under the condition:
- the construction is not degenerate
1 Exploration related equation:

\[ Q_1 = \mu \cdot Q_2 \]

where \( Q_1 \) and \( Q_2 \) are the geometric quantities to compare and \( \mu \in \mathbb{R} \) is a new variable ("proportion" or "ratio").

2 Derivation of an equivalent form of the (semi-)algebraic system:

1 elimination via Gröbner bases, for algebraic systems,
2 cylindrical algebraic decomposition (CAD) and real quantifier elimination (RQE), for semi-algebraic systems.
Exploration related equation:

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Derivation of an equivalent form of the (semi-)algebraic system:

1. Elimination via Gröbner bases, for algebraic systems,
2. Cylindrical algebraic decomposition (CAD) and real quantifier elimination (RQE), for semi-algebraic systems.
Applied methods

1. Exploration related equation:

\[ Q_1 = \mu \cdot Q_2 \]

where \( Q_1 \) and \( Q_2 \) are the geometric quantities to compare and \( \mu \in \mathbb{R} \) is a new variable (“proportion” or “ratio”).

2. Derivation of an equivalent form of the (semi-)algebraic system:

- elimination via Gröbner bases, for algebraic systems,
- cylindrical algebraic decomposition (CAD) and real quantifier elimination (RQE), for semi-algebraic systems.

\[ \Rightarrow m \cdot Q_2 \leq Q_1 \leq M \cdot Q_2 \]

where \( m, M \in \mathbb{R}_0^+ \) are sharp constants.
A semi-algebraic technique
Cylindrical Algebraic Decomposition (CAD) and Real Quantifier Elimination (RQE)

**Definition**

Given a set \( S \) of polynomials in \( \mathbb{Z}[x_1, x_2, \ldots, x_n] \), a CAD is a decomposition of \( \mathbb{R}^n \) into special connected semi-algebraic sets, on which each polynomial has constant sign, either +, − or 0.

Example: \( S = \{x_1^2 + x_2^2 - 1\} \) and a CAD of it. Here \( \mathbb{R}^2 \) can be decomposed into 13 semi-algebraic sets (\( 13 = 1 + 3 + 5 + 3 + 1 \)).
Reformulating the problem as input for RQE (via CAD)

Generalization of the Pythagorean theorem

The quantified formula (after simplifying):

$$\exists \nu_{10}, \nu_5, \nu_6, \nu_7, \nu_8, \nu_9 \in \mathbb{R}$$

$$\nu_7 > 0 \land \nu_8 > 0 \land \nu_9 > 0 \land$$

$$\nu_{10} \nu_6 = 1 \land -\nu_5^2 + 2\nu_5 - \nu_6^2 + \nu_8^2 = 1 \land \nu_5^2 + \nu_6^2 = \nu_9^2 \land$$

$$\nu_7 = 1 \land \mu = \nu_8^2 + \nu_9^2.$$
Reformulating the problem as input for RQE (via CAD)

Generalization of the Pythagorean theorem

The quantified formula (after simplifying):

\[ \exists v_10, v_5, v_6, v_7, v_8, v_9 \in \mathbb{R} \]
\[ v_7 > 0 \land v_8 > 0 \land v_9 > 0 \land \]
\[ v_{10} v_6 = 1 \land -v_5^2 + 2v_5 - v_6^2 + v_8^2 = 1 \land v_5^2 + v_6^2 = v_9^2 \land \]
\[ v_7 = 1 \land \mu = v_8^2 + v_9^2. \]

\[ \Rightarrow \mu > 1/2 \text{ (a quantifier-free formula).} \]
Direct proof by typing \( \text{Prove}(a^2 + b^2 > c^2/2) \), or by trial-and-error:
- e.g. \( \text{Prove}(a^2 + b^2 > c^2) \)
Additional ways for users to enter input
...instead of using Relation($a^2 + b^2, c^2$)

Direct proof by typing Prove($a^2 + b^2 > c^2/2$), or by trial-and-error:
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   - e.g. $\text{Prove}(a^2 + b^2 > c^2) \rightarrow \text{false}$,
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   - e.g. \( \text{Prove}(a^2 + b^2 > c^2/3) \)
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- e.g. `Prove(a^2 + b^2 > c^2) → false`
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- ...
1. **Direct proof by typing** $\text{Prove}(a^2 + b^2 > c^2/2)$, or by trial-and-error:
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   - e.g. $\text{Prove}(a^2 + b^2 > c^2/3) \rightarrow \text{true}$,
   - ... 

2. **Low-level command** $\text{Compare}(a^2 + b^2, c^2)$ to get direct result ($\rightarrow$ JavaScript API)
Additional ways for users to enter input
...instead of using \( \text{Relation}(a^2 + b^2, c^2) \)

1. Direct proof by typing \( \text{Prove}(a^2 + b^2 > c^2/2) \), or by trial-and-error:
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   - e.g. \( \text{Prove}(a^2 + b^2 > c^2/3) \rightarrow \text{true} \),
   - ...

2. Low-level command \( \text{Compare}(a^2 + b^2, c^2) \) to get direct result (\( \rightarrow \) JavaScript API)

3. In simpler cases: point-and-click (via the Relation tool)
Shortest path between two sides of a regular pentagon?

Quick answer by using the Relation tool

Brown, Kovács and Vajda
Supporting inequalities in GeoGebra by using Tarski
Shortest path between two sides of a regular pentagon?
First attempt: a numerical comparison (no result)

**Relation**

k does not have the same length as f
(checked numerically)
Shortest path between two sides of a regular pentagon?
Second-third attempts: symbolic comparisons with proportions

It is generally true that:
\[ f \leq k \leq \left( \frac{\sqrt{5}+1}{2} \right) \cdot f \]
under the condition:
\[ \text{the construction is not degenerate} \]
Shortest path between two sides of a regular pentagon?
The (semi-)algebraic translation of the geometric setup

\[ E(v_{11}, v_{12}) \]

\[ C(v_7, v_8) \]

\[ F(v_{13}, v_{14}) \]

\[ A(v_1, v_2) \]

\[ G(v_{15}, v_{16}) \]

\[ B(v_3, v_4) \]

- \( v_7v_4 + v_8v_3 + v_7v_2 - v_3v_2 - v_8v_1 + v_4v_1 > 0 \)
- \( v_8v_1 - v_4^2 + v_7v_3 - v_3^2 - v_8v_2 + v_4v_2 - v_7v_1 + v_3v_1 > 0 \)
- \( -1 + 2v_5 + 4v_5^2 = 0 \)
- \( -1 + v_6^2 + v_6^2 = 0 \)
- \( -v_7 - v_6v_4 + v_3 + v_5v_3 + v_6v_2 - v_3v_1 = 0 \)
- \( -v_8 + v_4 + v_5v_4 + v_6v_3 - v_5v_2 - v_6v_1 = 0 \)
- \( -v_9 + v_7 - v_8v_5 + v_7v_5 + v_6v_4 - v_3v_3 = 0 \)
- \( -v_{10} + v_8 + v_7v_6 + v_8v_5 - v_7v_4 - v_3v_3 = 0 \)
- \( -v_{11} + v_9 - v_10v_6 + v_8v_6 + v_9v_5 - v_7v_5 = 0 \)
- \( -v_{12} + v_{10} + v_9v_6 - v_7v_6 + v_{10}v_5 - v_8v_5 = 0 \)
- \( v_{20} - v_{10}^2 - v_{15}^2 + 2v_{10}v_{14} - v_{14}^2 + 2v_{15}v_{13} - v_{13}^2 = 0 \)

- \( v_{19} - v_4^2 - v_3^2 + 2v_4v_2 - v_2^2 + 2v_3v_1 - v_1^2 = 0 \)

\( -v_{16}^2 - v_{15}^2 + v_{16}v_8 + v_{15}v_7 + v_{16}v_4 - v_8v_4 + v_{15}v_3 - v_7v_3 \geq 0 \)

\( -v_{15}v_8 + v_{16}v_7 + v_{15}v_4 - v_7v_4 - v_{16}v_3 + v_8v_3 = 0 \)
Shortest path between two sides of a regular pentagon?

Final input for Tarski (after delineraization) ⇒ output

Brown, Kovács and Vajda
Supporting inequalities in GeoGebra by using Tarski
Euler’s Inequality

Theorem (Euler 1765, Chapple 1746)

In all triangle it holds that $R \geq 2 \cdot r$ where $R$ is the circumradius and $r$ is the inradius of the triangle.
Euler’s Inequality in an isosceles triangle
(Semi-)algebraic translation

\[ 2v_{15} - v_9 - v_4 = 0 \]
\[ 2v_{14} - v_{10} - v_2 = 0 \]
\[ v_{10} - v_4 + v_1 + 0 = 0 \]
\[ v_{15} + v_{14} - v_3 - v_2 = 0 \]
Euler’s Inequality in an isosceles triangle
Output in GeoGebra Discovery

R and \( r \) are parallel (checked numerically)

It is generally true that:

- \( R \geq (2) \cdot r \)

under the condition:

- the construction is not degenerate
### Euler’s Inequality

Benchmarking (outputs in seconds, timeout: 30 secs, Intel Xeon CPU X5675 @ 3.07GHz)

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Euler’s Inequality

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Benchmarks

- 131 simple/moderate tests
  - 117/116 can be successfully solved (Mathematica/Tarski) within 30 seconds
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- 117/116 can be successfully solved (Mathematica/Tarski) within 30 seconds

Density estimate on 103 tests that work uniformly (timing in ms),

\[ \mu_M = 1361, \mu_T = 2841, \sigma_M = 3379, \sigma_T = 4616 \]
**Benchmarks**

- 131 simple/moderate tests
  - 117/116 can be successfully solved (Mathematica/Tarski) within 30 seconds

Density estimate on 103 tests that work uniformly (timing in ms),

\[ \mu_M = 1361, \mu_T = 2841, \sigma_M = 3379, \sigma_T = 4616 \]

- 46 additional tests to prove a given conjecture
  - 33/35 can be successfully proven (Mathematica/Tarski) within 40 seconds
The yellow region corresponds to a semi-algebraic set!