Exercises: Serre's reduction

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Exercise 1 We study Serre's reduction of the following linear OD time-delay system

$$\left\{\begin{array}{l}\dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0,\\ \dot{x}_2(t) - x_3(t) = 0,\\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0,\end{array}\right. \qquad (1)$$

where $x = (x_1 \quad x_2 \quad x_3)^T$ denotes the state, u the input, a, k, ω and ζ the constant parameters and $h \in \mathbb{R}_+$. This system corresponds to a wind tunnel model studied in A. Manitius, "Feedback controllers for a wind tunnel model involving a delay: analytical design and numerical simulations", *IEEE Trans. Autom. Contr.*, 29 (1984), 1058-1068.

1. Let $D = \mathbb{Q}(a, k, \omega, \zeta)\left[\partial; \mathrm{id}, \frac{d}{dt}\right][\delta; \alpha, 0]$ be the commutative polynomial ring of OD time-delay operator ($\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t-h)$),

$$R = \begin{pmatrix} \partial + a & -k\,a\,\delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \in D^{3\times 4},$$

   the presentation matrix of (1) and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 3}\,R)$ associated with (1).

2. Prove that R has full row rank, i.e., $\ker_D(.R) = 0$. Deduce a finite free resolution of M.

3. Deduce that $\mathrm{ext}^1_D(M, D) = D^3/(R\,D^4) = D^{1\times 3}/\left(D^{1\times 4}\,R^T\right)$.

4. Describe the D-module $\mathrm{ext}^1_D(M, D)$ in terms of generators and relations.

5. Prove that $\mathrm{ext}^1_D(M, D)$ is a 1-dimensional $\mathbb{Q}(a, k, \omega, \zeta)$-vector space. Give a basis $\rho(\Lambda)$ of $\mathrm{ext}^1_D(M, D)$, where $\rho : D^3 \longrightarrow D^3/(R\,D^4)$ denotes the canonical projection.

6. Deduce that $\rho(\Lambda)$ generates $\mathrm{ext}^1_D(M, D)$, i.e., $\mathrm{ext}^1_D(M, D)$ is a cyclic D-module.

7. Consider the matrix $P = (R \quad -\Lambda) \in D^{3\times 5}$. Show that P admits a left-inverse over D defined by:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\dfrac{\partial + 2\,\zeta\,\omega}{\omega^2} & -\dfrac{1}{\omega^2} \\ -1 & 0 & 0 \end{pmatrix} \in D^{5\times 3}.$$

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Deduce that the D-module $E = D^{1\times 5}/(D^{1\times 3}\, P)$ is a free D-module of rank 2.

8. Describe the D-module E in terms of generators and relations.
9. Compute an injective parametrization of E, namely, a matrix $Q \in D^{5\times 2}$ which admits a left-inverse over D and satisfies $\ker_D(.Q) = D^{1\times 3}\, P$. Deduce a basis of E.
10. Write $Q = (Q_1^T \quad Q_2^T)^T$, where $Q_1 \in D^{4\times 2}$ and $Q_2 \in D^{1\times 2}$. Deduce that:
$$M \cong L = D^{1\times 2}/(D\, Q_2).$$
11. Does Λ admit a left-inverse over D? Is then R equivalent to the diagonal matrix $\mathrm{diag}(I_2, Q_2)$?
12. Compute $\ker_D(.Q_1)$ and find a matrix $F \in D^{2\times 4}$ such that $\ker_D(.Q_1) = D^{1\times 2}\, F$. Deduce that $\ker_D(.Q_1)$ is a free D-module of rank 2.
13. Compute a right-inverse Q_3 of F over D and check that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_4(D)$.
14. Prove that $U = (R\, Q_3 \quad \Lambda) \in \mathrm{GL}_3(D)$ and compute $V = U^{-1}$.
15. Show that $\overline{R} = V\, R\, W = \mathrm{diag}(I_2, Q_2)$ and thus (1) is equivalent to the following OD time-delay equation:
$$\dot{z}(t) + a\, z(t) - \omega^2\, k\, a\, v(t-h) = 0. \tag{2}$$
16. Characterize the module properties of L and thus those of (1).
17. Using the commands KBasis, LeftInverse, RightInverse, MinimalParametrizations and SyzygyModule of OreModules, redo the previous computations.

Exercise 2 Let us consider the following linear OD time-delay system
$$\begin{cases} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1\, \phi_1(t) - \eta_1\, \psi_1(t) - \eta_2\, \phi_2(t) + \eta_2\, \psi_2(t) = 0, \\ \phi_1(t - 2\, h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2\, h_2) - v(t - h_2) = 0, \end{cases} \tag{3}$$
which describes the movement of a vibrating string with an interior mass (see H. Mounier, J. Rudolph, M. Fliess, P. Rouchon, "Tracking control of a vibrating string with an interior mass viewed as delay system", *ESAIM COCV*, 3 (1998), 315-321). In (3), h_1 and $h_2 \in \mathbb{R}_+$ are such that $\mathbb{Q}\, h_1 + \mathbb{Q}\, h_2$ is a two-dimensional $\mathbb{Q}$-vector space (i.e., there exists no relation of the form $m\, h_1 + n\, h_2 = 0$, where $m, n \in \mathbb{Z}$), and η_1 and η_2 are two constant parameters of the system. The condition on h_1 and h_2 implies that σ_1 and σ_2 are two non-commensurate time-delay operators, i.e., define two independent variables.

1. Let $D = \mathbb{Q}(\eta_1, \eta_2)\left[\partial; \mathrm{id}, \frac{d}{dt}\right][\sigma_1; \alpha_1, 0][\sigma_2; \alpha_2, 0]$ be the commutative polynomial ring of OD time-delay operators with coefficients in the field $\mathbb{Q}(\eta_1, \eta_2)$, where $\sigma_i(y(t)) = y(t - h_i)$,
$$R = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \partial + \eta_1 & \partial - \eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} \in D^{4\times 6},$$
the presentation matrix of (3) and $M = D^{1\times 6}/(D^{1\times 4}\, R)$ the finitely presented D-module associated with (3).

2. Following the same lines as in Example 1, prove $M \cong L = D^{1\times 3}/(D\, Q_2)$, where:

$$Q_2 = (-\partial - \eta_1 - \eta_2 \quad \eta_1\, \sigma_1 \quad -\sigma_2).$$

3. Check that Λ admits a left-inverse over D and deduce that R is equivalent to the diagonal matrix $\mathrm{diag}(I_3, Q_2)$.

4. Following the same lines as in Example 1, prove that the matrices defined by

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\partial - \eta_1 & 1 & 0 & 0 \\ \sigma_1^2 & 0 & -1 & 0 \\ \dfrac{(\sigma_2^2-1)\,(\partial + \eta_1 + 2\,\eta_1\,\sigma_1^2)}{2\,\eta_2} & -\dfrac{\sigma_2^2-1}{2\,\eta_2} & -\dfrac{\eta_1\,(\sigma_2^2-1)}{\eta_2} & 1 \end{pmatrix} \in \mathrm{GL}_4(D),$$

$$W = \begin{pmatrix} 1 & 0 & 1 & -2\,\eta_2 & \eta_2\,\sigma_1 & 0 \\ 0 & 0 & -1 & 0 & -\eta_2\,\sigma_1 & 0 \\ 0 & -\dfrac{1}{2\,\eta_2} & \dfrac{\eta_1}{\eta_2} & -\partial - \eta_1 - \eta_2 & \eta_1\,\sigma_1 & 0 \\ 0 & \dfrac{1}{2\,\eta_2} & -\dfrac{\eta_1}{\eta_2} & \partial + \eta_1 - \eta_2 & -\eta_1\,\sigma_1 & 0 \\ 0 & 0 & \sigma_1 & -2\,\eta_2\,\sigma_1 & \eta_2\,(\sigma_1-1)\,(\sigma_1+1) & 0 \\ 0 & 0 & 0 & \sigma_2\,(\partial + \eta_1 - \eta_2) & -\eta_1\,\sigma_1\,\sigma_2 & 1 \end{pmatrix} \in \mathrm{GL}_6(D)$$

satisfy $V\, R\, W = \mathrm{diag}(I_3, Q_2)$.

5. Post-multiplying W by the following unimodular matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_2 \end{pmatrix} \in \mathrm{GL}_6(D),$$

show that (3) is then equivalent to the sole linear OD time-delay equation:

$$\dot{x}_1(t) + (\eta_1 + \eta_2)\, x_1(t) - \eta_1\, x_2(t - h_1) - \eta_2\, x_3(t - h_2) = 0.$$

6. Study the module properties of L and deduce those of L.

7. Deduce the properties of the linear OD time-delay system (3).

Exercise 3 We consider the following time-varying OD linear system

$$\left\{\begin{array}{l} \rho\,\dfrac{\partial\theta}{\partial\rho}+\dfrac{1}{2}\,(\theta+K)+\dfrac{(\lambda+\mu)}{2}\,\left(\rho\,\dfrac{\partial\sigma}{\partial\rho}-\sigma\right)=0,\\ 2\,\rho\,\dfrac{\partial\theta}{\partial\rho}+\rho\,\dfrac{\partial K}{\partial\rho}+3\,K-(3\,\lambda+2\,\mu)\,\sigma=0,\\ \lambda\,\sigma+2\,\mu\,\left(G+\rho\,\dfrac{\partial G}{\partial\rho}\right)-\rho\,\dfrac{\partial\theta}{\partial\rho}-K=0, \end{array}\right. \tag{4}$$

where σ, G, θ and K are functions of $\rho=\sqrt{x^2+y^2+z^2}$ and λ and μ are real parameters, considered in J. Hadamard, “Sur l'équilibre des plaques élastiques circulaires libres ou appuyées et celui de la sphère isotrope”, *Annales Scientifiques de l'Ecole Normale Supérieure*, 18 (1901), 313-324.

1. Let $D=A_1(\mathbb{Q}(\lambda,\mu))=\mathbb{Q}(\lambda,\mu)[\rho]\left[\partial;\mathrm{id},\frac{d}{d\rho}\right]$ be the first Weyl algebra over the field $\mathbb{Q}(\lambda,\mu)$
$$R=\begin{pmatrix} \rho\,\partial+\frac{1}{2} & \frac{1}{2}\,(\lambda+\mu)\,(\rho\,\partial-1) & \frac{1}{2} & 0\\ 2\,\rho\,\partial & -3\,\lambda-2\,\mu & \rho\,\partial+3 & 0\\ -\rho\,\partial & \lambda & -1 & 2\,\mu\,(\rho\,\partial+1)\end{pmatrix}\in D^{3\times 4},$$
the presentation matrix of (4) and the left D-module $M=D^{1\times 4}/(D^{1\times 3}\,R)$.
2. Using the command Involution of OreModules, compute the formal adjoint $\widetilde{R}$ of R.
3. Using the command Dimension, check that $\dim_D(\widetilde{N})=1$ and deduce that $\widetilde{N}$ is a holonomic left D-module.
4. Denote by $E=B_1(\mathbb{Q}(\lambda,\mu))=\mathbb{Q}(\lambda,\mu)(\rho)\left[\partial;\mathrm{id},\frac{d}{d\rho}\right]$ the second Weyl algebra. Using the command KBasis, show that $\widetilde{N}=E^{1\times 3}/(E^{1\times 4}\,\widetilde{R})$ is a 1-dimensional $\mathbb{Q}(\lambda,\mu)(\rho)$-vector space and compute a basis. Find a row vector $\widetilde{\Lambda}\in E^{1\times 3}$ such that the matrix
$$\widetilde{P}=\begin{pmatrix}\widetilde{R}\\ \widetilde{\Lambda}\end{pmatrix}\in D^{5\times 3}$$
admits a left-inverse over E using the command LeftInverseRat.
5. Check that $\widetilde{\Lambda}\in D^{1\times 3}$ and $\widetilde{P}$ admits a left-inverse over D using the command LeftInverse. Deduce that $\widetilde{N}$ is a cyclic left D-module.
6. Deduce that the matrix $P=(R\quad -\Lambda)$ admits a right-inverse over D and check again the last result using the command RightInverse.
7. Using Stafford's theorem, show that the left D-module $E=D^{1\times 5}/(D^{1\times 3}\,P)$ is free of rank 2.
8. Using the command MinimalParametrizations, compute an injective parametrization $Q\in D^{5\times 2}$ of M and a basis of the free left D-module E.
9. Write $Q=(Q_1^T\quad Q_2^T)^T$, where $Q_1\in D^{4\times 2}$ and $Q_2\in D^{1\times 2}$, and deduce that:
$$M\cong L=D^{1\times 2}/(D\,Q_2).$$

10. Check that Λ admits a left-inverse over D and deduce that R is equivalent to the diagonal matrix $\mathrm{diag}(I_2, Q_2)$.

11. Compute a matrix $F \in D^{2\times 4}$ such that $\ker_D(.Q_1) = D^{1\times 2}\, F$ by means of the command SyzygyModule and check that $\ker_D(.Q_1)$ is a free left D-module of rank 2.

12. Compute a right-inverse $Q_3 \in D^{4\times 2}$ of F and deduce that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_4(D)$.

13. Form the matrix $U = (R\, Q_3 \quad \Lambda) \in D^{3\times 3}$ and, using the command LeftInverse, check that $V = U^{-1} \in D^{3\times 3}$, i.e., $V \in \mathrm{GL}_3(D)$.

14. Compute the block diagonal matrix $\overline{R} = V\, R\, W$ and check that $\overline{R} = \mathrm{diag}(I_2, Q_2)$.

15. If $\mathcal{F}$ is a left D-module, then deduce that (4) is equivalent to the following OD system:

$$z_1 \in \mathcal{F} : \; \rho\, \frac{dz_1}{d\rho} + z_1 = 0, \quad \forall\, z_2 \in \mathcal{F}.$$

16. Study the module properties of L. Deduce those of M and the properties of the linear OD system (4).

Exercise 4 Let $D = \mathbb{Q}(K_1, K_2, K_c, K_e, T_p)\left[\partial; \mathrm{id}, \frac{d}{dt}\right][\delta; \alpha, 0]$ be the commutative polynomial ring of OD time-delay operators ($\partial\, y(t) = \dot{y}(t)$, $\delta\, y(t) = y(t-1)$), where K_1, K_2, K_c, K_e and K_p denote constant parameters, and the matrix of OD time-delay operators defined by:

$$R = \begin{pmatrix} \partial & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial + \dfrac{K_2}{T_e} & 0 & 0 & 0 & 0 & -\dfrac{K_p}{T_e}\,\delta & -\dfrac{K_c}{T_e}\,\delta & -\dfrac{K_c}{T_e}\,\delta \\ 0 & 0 & \partial & -K_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial + \dfrac{K_2}{T_e} & 0 & 0 & -\dfrac{K_c}{T_e}\,\delta & -\dfrac{K_p}{T_e}\,\delta & -\dfrac{K_c}{T_e}\,\delta \\ 0 & 0 & 0 & 0 & \partial & -K_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial + \dfrac{K_2}{T_e} & -\dfrac{K_c}{T_e}\,\delta & -\dfrac{K_c}{T_e}\,\delta & -\dfrac{K_p}{T_e}\,\delta \end{pmatrix} \in D^{6\times 9}.$$

The matrix R is the presentation matrix of a model of a two reflector antenna studied in V. Kolmanovskii, V. Nosov, *Stability of Functional Differential Equations*, Academic Press, 1986. The purpose of this exercise is to use OreModules to find an equivalent system defined by fewer equations and unknowns.

1. Prove that R has full row rank, i.e., $\ker_D(.R) = 0$. Deduce a finite free resolution of M.

2. Show that $\mathrm{ext}^1_D(M, D) = D^6/(R\, D^9) = D^{1\times 6}/(D^{1\times 9}\, R^T)$.

3. Prove that $\mathrm{ext}^1_D(M, D)$ is a 6-dimensional $\mathbb{Q}(K_1, K_2, K_c, K_e, T_p)$-vector space.

4. Define the matrix

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and prove that the matrix $P = (R \quad -\Lambda) \in D^{6\times 12}$ admits a right-inverse over D.

5. Deduce that the D-module $E = D^{1\times 12}/(D^{1\times 6}\, P)$ is free of rank 6.
6. Compute an injective parametrization $Q \in D^{12\times 6}$ of E and a basis of E.
7. Write $Q = (Q_1^T \quad Q_2^T)^T$, where $Q_1 \in D^{9\times 6}$ and $Q_2 \in D^{3\times 6}$. Deduce that:

$$M \cong L = D^{1\times 6}/(D^{1\times 3}\, Q_2).$$

8. Check that Λ admits a left-inverse over D. Deduce that R is equivalent to the diagonal matrix $\operatorname{diag}(I_3, Q_2)$.
9. Compute a matrix $F \in D^{3\times 9}$ satisfying $\ker_D(.Q_1) = D^{1\times 3}\, F$. Check that F admits a right-inverse $Q_3 \in D^{9\times 3}$. Deduce that $\ker_D(.Q_1)$ is a free D-module of rank 3.
10. Form the matrices $W = (Q_3 \quad Q_1)$ and $U = (R\, Q_3 \quad \Lambda)$ and check that $U \in \mathrm{GL}_6(D)$ and $W \in \mathrm{GL}_9(D)$.
11. Deduce that $U^{-1}\, R\, W = \operatorname{diag}(I_3, Q_2)$ and thus:

$$R\,\eta = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{l} T_e\, \ddot{\zeta}_1(t) + K_2\, \dot{\zeta}_1(t) + (K_p + 2\, K_c)\,(K_c - K_p)\,\zeta_2(t-1) = 0, \\ T_e\, \ddot{\zeta}_3(t) + K_2\, \dot{\zeta}_3(t) + (K_p + 2\, K_c)\,(K_c - K_p)\,\zeta_4(t-1) = 0, \\ T_e\, \ddot{\zeta}_5(t) + K_2\, \dot{\zeta}_5(t) + (K_p + 2\, K_c)\,(K_c - K_p)\,\zeta_6(t-1) = 0. \end{array} \right.$$

12. Study the module properties of L and deduce those of M.

Exercise 5 We consider the general transmission line defined by

$$\left\{ \begin{array}{l} \dfrac{\partial V}{\partial x} + L\,\dfrac{\partial I}{\partial t} + R\, I = 0, \\ C\,\dfrac{\partial V}{\partial t} + G\, V + \dfrac{\partial I}{\partial x} = 0, \end{array} \right. \tag{5}$$

where I denotes the current, V the voltage, L the self-inductance, R the resistance, C the capacitor and G the conductance. Let $D = \mathbb{Q}(L,R,C,G)\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]$ be the commutative polynomial ring of PD operators with coefficients in the field $\mathbb{Q}(L,R,C,G)$,

$$S = \begin{pmatrix} \partial_x & L\,\partial_t + R \\ C\,\partial_t + G & \partial_x \end{pmatrix} \in D^{2\times 2}$$

the presentation matrix of (5) and the D-module $M = D^{1\times 2}/(D^{1\times 2}\, S)$.

1. Check that S has full row rank, i.e., $\ker_D(.S) = 0$. Give a finite free resolution of M.
2. Deduce that $\mathrm{ext}^1_D(M,D) = D^2/(R\, D^2) = D^{1\times 2}/(D^{1\times 2}\, R^T)$.
3. Compute $\dim_D(\mathrm{ext}^1_D(M,D))$. What is the dimension of $\mathrm{ext}^1_D(M,D)$ as a $\mathbb{Q}(L,R,C,G)$-vector space?
4. Consider $\Lambda = (a \quad b)^T$, where a and b are two arbitrary constants and form the matrix $P = (R \quad -\Lambda)$. Does P admit a right-inverse over D?

5. Using the command PiPolynomial of OreModules, compute the obstructions for the D-module $E = D^{1\times 3}/(D^{1\times 2}\, P)$ to be projective, i.e., free by the Quillen-Suslin theorem.

6. Prove that if we take $b = C$ and $a^2 = L\,C$, then one of the obstructions becomes 1, i.e., $A \otimes_D E$ is a free $A = K\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]$-module and $K = \mathbb{Q}(L, R, C, G)[a]/(a^2 - L\,C)$.

7. Deduce an injective parametrization $Q \in A^3$ of $A \otimes_D E$.

8. Write $Q = (Q_1^T \quad Q_2^T)$, where $Q_1 \in A^2$ and $Q_2 \in A$, and deduce that $A \otimes_D M \cong L = A/(A\,Q_2)$, i.e., M is a cyclic A-module.

9. Check that Λ admits a left-inverse over A. Deduce that R is equivalent to the diagonal matrix $\mathrm{diag}(1, Q_2)$.

10. Compute a matrix $F \in A^{1\times 2}$ such that $\ker_A(.Q_1) = A\,F$. Show that $\ker_A(.Q_1)$ is a free A-module.

11. Compute a right-inverse $Q_3 \in A^2$ of F over A and prove that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_2(A)$.

12. Form the matrix $U = (R\,Q_3 \quad \Lambda) \in A^{2\times 2}$ and check that $U \in \mathrm{GL}_2(A)$.

13. Finally, check that $U^{-1}\,R\,W = \mathrm{diag}(1, Q_2)$ and (5) is equivalent to the following DE:

$$\left(\partial_x^2 - L\,C\,\partial_t^2 - (R\,C + G\,L)\,\partial_t - G\,R\right)\,Z(x,t) = 0. \tag{6}$$

14. Note that (6) corresponds to the determinant of R, and thus V and I also satisfy (6).

Exercise 6 We consider the so-called *conjugated Beltrami equation* with $\sigma(x, y) = x$:

$$\begin{cases} \dfrac{\partial z_1(x,y)}{\partial x} - x\,\dfrac{\partial z_2(x,y)}{\partial y} = 0, \\[2mm] \dfrac{\partial z_1(x,y)}{\partial y} + x\,\dfrac{\partial z_2(x,y)}{\partial x} = 0. \end{cases} \tag{7}$$

Let $D = A_2(\mathbb{Q}) = \mathbb{Q}[x, y]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right]$ be the first Weyl algebra over $\mathbb{Q}$,

$$R = \begin{pmatrix} \partial_x & -x\,\partial_y \\ \partial_y & x\,\partial_y \end{pmatrix} \in D^{2\times 2}$$

the presentation matrix of (7) and the left D-module $M = D^{1\times 2}/(D^{1\times 2}\,R)$.

1. Compute the formal adjoint $\widetilde{R}$ of R and compute $\dim_D(\widetilde{N})$, where $\widetilde{N} = D^{1\times 2}/(D^{1\times 2}\,\widetilde{R})$ is the left D-module finitely presented by $\widetilde{R}$. Deduce that $\widetilde{N}$ is not a holonomic left D-module.

2. Consider $\Lambda = (a \quad b)^T$, where a and b are two arbitrary constants and form the matrix $P = (R \quad -\Lambda)$. Check that P admits a right-inverse over D when $a \neq 0$ and $b \neq 0$. Deduce that $E = D^{1\times 3}/(D^{1\times 2}\,P)$ is a stably free left D-module of rank 1. Does Stafford's theorem hold in this case?

3. Compute minimal parametrizations of the left D-module E. Do they admit a left-inverse over D?

4. Compute the annihilator of the minimal parametrizations.

5. Prove that if we take $a = i$ and $b = 1$, then one of the annihilators reduces to D, i.e., the corresponding minimal parametrization Q admits a left-inverse over the new ring $A = A_2(\mathbb{Q}[a]/(a^2+1))$.

6. Write $Q = (Q_1^T \quad Q_2^T)^T$, where $Q_1 \in A^2$ and $Q_2 \in A$, and deduce that $A \otimes_D M \cong L = A/(A\, Q_2)$, i.e., $A \otimes_D M$ is a cyclic left A-module.

7. Check that Λ admits a left-inverse over A. Deduce that R is equivalent to the diagonal matrix $\mathrm{diag}(1, Q_2)$.

8. Compute $F \in A^{1\times 2}$ such that $\ker_A(.Q_1) = A\, F$. Deduce that $\ker_A(.Q_1)$ is a free left A-module of rank 1.

9. Prove that F admits a right-inverse $Q_3 \in A^2$ and show that $W = (Q_3 \quad Q_1) \in \mathrm{GL}_2(A)$.

10. Form the matrix $U = (R\, Q_3 \quad \Lambda)$ and prove that $U \in \mathrm{GL}_2(A)$.

11. Prove that $U^{-1}\, R\, W = \mathrm{diag}(1, Q_2)$ and deduce that (7) is equivalent to the following PDE

$$(i\, x\, \Delta + \partial_y)\, u(x, y) = 0,$$

where $\Delta = \partial_x^2 + \partial_y^2$ denotes the Laplacian operator.

12. Using the command Exti of OreModules, check that z_1 and z_2 respectively satisfy:

$$(x\, \Delta - \partial_x)\, z_1 = 0, \quad (x\, \Delta + \partial_x)\, z_2 = 0.$$

Contrary to the previous exercise (transmission line), note that z_1 and z_2 do not satisfy the same equation. Does the determinant of R make sense over the noncommutative polynomial D ring?

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