```
> restart:
> with(OreModules):
> with(linalg):
>
> A:=DefineOreAlgebra(diff=[d,t],polynom=[t],comm=[g,l1,l2]):
>
> R:=evalm([[d^2,l1*d^2+g,0],[d^2,0,l2*d^2+g]]);
```

$$R := \begin{bmatrix} d^2 & l1\, d^2+g & 0 \\ d^2 & 0 & l2\, d^2+g \end{bmatrix} \tag{1}$$

```
> Eqs:=ApplyMatrix(R,[x(t),theta1(t),theta2(t)],A);
```

$$Eqs := \begin{bmatrix} \frac{d^2}{dt^2} x(t) + g\, \theta1(t) + l1 \left(\frac{d^2}{dt^2} \theta1(t) \right) \\ \frac{d^2}{dt^2} x(t) + g\, \theta2(t) + l2 \left(\frac{d^2}{dt^2} \theta2(t) \right) \end{bmatrix} \tag{2}$$

```
> Integrability(R,A,[x,theta1,theta2,Phi1,Phi2],A);
```

$$[\theta1\, l1\, d^2 + \theta1\, g - \Phi1 - \theta2\, l2\, d^2 - \theta2\, g + \Phi2, x\, d^2 + \theta2\, l2\, d^2 + \theta2\, g - \Phi2] \tag{3}$$

```
> SyzygyModule(R,A);
```

$$INJ(2) \tag{4}$$

```
> OreRank(R,A);
```

$$1 \tag{5}$$

```
> DimensionRat(R,A);
```

$$1 \tag{6}$$

```
> R_adj:=Involution(R,A);
```

$$R\_adj := \begin{bmatrix} d^2 & d^2 \\ l1\, d^2+g & 0 \\ 0 & l2\, d^2+g \end{bmatrix} \tag{7}$$

```
> GB:=Integrability(R_adj,A,[lambda1,lambda2,mu1,mu2,mu3],A);
```

$$\begin{aligned} GB &:= [l2\, \mu1\, g\, d^2 - l2\, \mu2\, d^4 + l2\, d^4\, l1\, \mu1 - \mu3\, g\, d^2 + \mu1\, g^2 \\ &+ g\, d^2\, l1\, \mu1 - \mu2\, d^2\, g - d^4\, l1\, \mu3, -l2^2\, g\, \mu1 - d^2\, l2^2\, l1\, \mu1 + d^2\, l2^2\, \mu2 + l2\, d^2\, l1\, \mu3 - l2\, g^2\, \lambda2 \\ &+ l2\, g\, \mu3 + l1\, g^2\, \lambda2 - l1\, g\, \mu3, l2\, d^2\, l1^2\, \mu1 - g^2\, l2\, \lambda1 + l2\, \mu2\, g - l2\, l1\, d^2\, \mu2 - d^2\, l1^2\, \mu3 \\ &+ g^2\, \lambda1\, l1 - g\, l1\, \mu2 + g\, l1^2\, \mu1] \end{aligned} \tag{8}$$

```
> CC:=remove(has,GB,{lambda1,lambda2});
```

$$CC := [l2\, \mu1\, g\, d^2 - l2\, \mu2\, d^4 + l2\, d^4\, l1\, \mu1 - \mu3\, g\, d^2 + \mu1\, g^2 + g\, d^2\, l1\, \mu1 - d^4\, l1\, \mu3 - \mu2\, d^2\, g] \tag{9}$$

```
> Q_adj:=SyzygyModule(R_adj,A);
```

$$Q\_adj := \begin{bmatrix} l2\, d^4\, l1 + l2\, g\, d^2 + g\, l1\, d^2 + g^2 & -l2\, d^4 - d^2\, g & -d^2\, g - d^4\, l1 \end{bmatrix} \tag{10}$$

```
> FreeResolution(R_adj,A);
```

$$\begin{array}{ccc} & d^2 & d^2 \\ table & l1\, d^2+g & 0 \\ & 0 & l2\, d^2+g \end{array} \qquad (11)$$

```
> Q:=Involution(Q_adj,A);
```

$$Q := \begin{bmatrix} l2\, g\, d^2+l2\, d^4\, l1+g^2+g\, d^2\, l1 \\ -l2\, d^4-d^2\, g \\ -d^2\, g-d^4\, l1 \end{bmatrix} \qquad (12)$$

```
> Rp:=SyzygyModule(Q,A);
```

$$Rp := \begin{bmatrix} d^2 & 0 & l2\, d^2+g \\ 0 & l1\, d^2+g & -l2\, d^2-g \end{bmatrix} \qquad (13)$$

```
> Quotient(Rp,R,A);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (14)$$

```
> ext1:=Exti(R_adj,A,1);
```

$$ext1 := \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} d^2 & 0 & l2\, d^2+g \\ 0 & l1\, d^2+g & -l2\, d^2-g \end{bmatrix}, \begin{bmatrix} l2\, g\, d^2+l2\, d^4\, l1+g^2+g\, d^2\, l1 \\ -l2\, d^4-d^2\, g \\ -d^2\, g-d^4\, l1 \end{bmatrix} \right] \qquad (15)$$

```
> Para:=ApplyMatrix(ext1[3],[xi(t)],A);
```

$$Para := \begin{bmatrix} g^2\, \xi(t)+\left(\frac{d^2}{dt^2}\xi(t)\right) l2\, g+\left(\frac{d^2}{dt^2}\xi(t)\right) g\, l1+l2\, l1\left(\frac{d^4}{dt^4}\xi(t)\right) \\ -g\left(\frac{d^2}{dt^2}\xi(t)\right)-l2\left(\frac{d^4}{dt^4}\xi(t)\right) \\ -g\left(\frac{d^2}{dt^2}\xi(t)\right)-l1\left(\frac{d^4}{dt^4}\xi(t)\right) \end{bmatrix} \qquad (16)$$

```
> ApplyMatrix(R,Para,A);
```

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (17)$$

```
> S:=RightInverse(R,A);
```

(18)

$$S := \begin{bmatrix} -\frac{l1^2 (l2\, d^2 + g)}{g^2 (l1 - l2)} & \frac{l2^2 (l1\, d^2 + g)}{g^2 (l1 - l2)} \\ \frac{l1\, l2\, d^2 - l2\, g + l1\, g}{g^2 (l1 - l2)} & -\frac{d^2\, l2^2}{g^2 (l1 - l2)} \\ \frac{l1^2\, d^2}{g^2 (l1 - l2)} & \frac{-l1\, l2\, d^2 + l1\, g - l2\, g}{g^2 (l1 - l2)} \end{bmatrix} \tag{18}$$

```
> T:=LeftInverse(ext1[3],A);
```

$$T := \begin{bmatrix} \frac{1}{g^2} & \frac{l1^2}{g^2 (l1 - l2)} & -\frac{l2^2}{g^2 (l1 - l2)} \end{bmatrix} \tag{19}$$

```
> ApplyMatrix(T,[x(t),theta1(t),theta2(t)],A);
```

$$\begin{bmatrix} \frac{x(t)\, l1 - x(t)\, l2 + l1^2\, \theta 1(t) - l2^2\, \theta 2(t)}{g^2 (l1 - l2)} \end{bmatrix} \tag{20}$$

```
> U:=stackmatrix(R,T);
```

$$U := \begin{bmatrix} d^2 & l1\, d^2 + g & 0 \\ d^2 & 0 & l2\, d^2 + g \\ \frac{1}{g^2} & \frac{l1^2}{g^2 (l1 - l2)} & -\frac{l2^2}{g^2 (l1 - l2)} \end{bmatrix} \tag{21}$$

```
> U_inv:=LeftInverse(U,A);
```

$$U\_inv := \begin{bmatrix} -\frac{l1^2 (l2\, d^2 + g)}{g^2 (l1 - l2)} & \frac{l2^2 (l1\, d^2 + g)}{g^2 (l1 - l2)} & l2\, g\, d^2 + l2\, d^4\, l1 + g^2 + g\, d^2\, l1 \\ \frac{l1\, l2\, d^2 - l2\, g + l1\, g}{g^2 (l1 - l2)} & -\frac{d^2\, l2^2}{g^2 (l1 - l2)} & -(l2\, d^2 + g)\, d^2 \\ \frac{l1^2\, d^2}{g^2 (l1 - l2)} & \frac{-l1\, l2\, d^2 + l1\, g - l2\, g}{g^2 (l1 - l2)} & -(l1\, d^2 + g)\, d^2 \end{bmatrix} \tag{22}$$

```
>
> B:=Brunovsky(R,A);
```

(23)

$$B := \begin{bmatrix} \frac{1}{g^2} & \frac{l1^2}{g^2\,(l1-l2)} & -\frac{l2^2}{g^2\,(l1-l2)} \\ \frac{d}{g^2} & \frac{d\,l1^2}{g^2\,(l1-l2)} & -\frac{d\,l2^2}{g^2\,(l1-l2)} \\ 0 & -\frac{l1}{g\,(l1-l2)} & \frac{l2}{g\,(l1-l2)} \\ 0 & -\frac{d\,l1}{g\,(l1-l2)} & \frac{d\,l2}{g\,(l1-l2)} \\ 0 & \frac{1}{l1-l2} & -\frac{1}{l1-l2} \end{bmatrix} \tag{23}$$

```
> L:=stackmatrix(R,B);
```

$$L := \begin{bmatrix} d^2 & l1\,d^2+g & 0 \\ d^2 & 0 & l2\,d^2+g \\ \frac{1}{g^2} & \frac{l1^2}{g^2\,(l1-l2)} & -\frac{l2^2}{g^2\,(l1-l2)} \\ \frac{d}{g^2} & \frac{d\,l1^2}{g^2\,(l1-l2)} & -\frac{d\,l2^2}{g^2\,(l1-l2)} \\ 0 & -\frac{l1}{g\,(l1-l2)} & \frac{l2}{g\,(l1-l2)} \\ 0 & -\frac{d\,l1}{g\,(l1-l2)} & \frac{d\,l2}{g\,(l1-l2)} \\ 0 & \frac{1}{l1-l2} & -\frac{1}{l1-l2} \end{bmatrix} \tag{24}$$

```
> ApplyMatrix(L,[x1(t),x2(t),u(t)],A)=evalm([[0]$2,seq([z[i]],i=1..4)]);
```

(25)

$$\left[\begin{array}{c} \frac{d^2}{dt^2} x1(t) + g\, x2(t) + l1 \left(\frac{d^2}{dt^2} x2(t)\right) \\ \frac{d^2}{dt^2} x1(t) + g\, u(t) + l2 \left(\frac{d^2}{dt^2} u(t)\right) \\ \frac{x1(t)\, l1 - x1(t)\, l2 + l1^2\, x2(t) - l2^2\, u(t)}{g^2\, (l1 - l2)} \\ \frac{\left(\frac{d}{dt} x1(t)\right) l1 - \left(\frac{d}{dt} x1(t)\right) l2 + l1^2 \left(\frac{d}{dt} x2(t)\right) - l2^2 \left(\frac{d}{dt} u(t)\right)}{g^2\, (l1 - l2)} \\ -\frac{l1\, x2(t) - l2\, u(t)}{g\, (l1 - l2)} \\ -\frac{l1 \left(\frac{d}{dt} x2(t)\right) - l2 \left(\frac{d}{dt} u(t)\right)}{g\, (l1 - l2)} \\ -\frac{-x2(t) + u(t)}{l1 - l2} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{array}\right] \tag{25}$$

```
> E:=Elimination(L,[x1,x2,u],[0,0,seq(z[i],i=1..4),v],A);
```

$$E := table\left(\left[1 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right], 2 = \left[\begin{array}{ccccc} 0 & 0 & 0 & -d & 1 \\ 0 & 0 & -d & 1 & 0 \\ 0 & -d & 1 & 0 & 0 \\ -d & 1 & 0 & 0 & 0 \\ 0 & 0 & -g & 0 & -l1 \\ 0 & 0 & -g & 0 & -l2 \\ g^2 & 0 & l1\, g + l2\, g & 0 & l1\, l2 \end{array}\right]\right]\right) \tag{26}$$

```
> ApplyMatrix(E[1],[x1(t),x2(t),u(t)],A)=ApplyMatrix(E[2],[seq(z[i]
  (t),i=1..4),v(t)],A);
```

(27)

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u(t) \\ x2(t) \\ x1(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{d}{dt} z_4(t)\right) + v(t) \\ -\left(\frac{d}{dt} z_3(t)\right) + z_4(t) \\ -\left(\frac{d}{dt} z_2(t)\right) + z_3(t) \\ -\left(\frac{d}{dt} z_1(t)\right) + z_2(t) \\ -g\, z_3(t) - l1\, v(t) \\ -g\, z_3(t) - l2\, v(t) \\ g^2 z_1(t) + z_3(t)\, l1\, g + z_3(t)\, l2\, g + l1\, l2\, v(t) \end{bmatrix} \tag{27}$$

\>

```
> R2:=subs(l2=l1,evalm(R));
```

$$R2 := \begin{bmatrix} d^2 & l1\, d^2 + g & 0 \\ d^2 & 0 & l1\, d^2 + g \end{bmatrix} \tag{28}$$

```
> R2_adj:=Involution(R2,A);
```

$$R2\_adj := \begin{bmatrix} d^2 & d^2 \\ l1\, d^2 + g & 0 \\ 0 & l1\, d^2 + g \end{bmatrix} \tag{29}$$

```
> Q2_adj:=SyzygyModule(R2_adj,A);
```

$$Q2\_adj := \begin{bmatrix} l1\, d^2 + g & -d^2 & -d^2 \end{bmatrix} \tag{30}$$

```
> Q2:=Involution(Q2_adj,A);
```

$$Q2 := \begin{bmatrix} l1\, d^2 + g \\ -d^2 \\ -d^2 \end{bmatrix} \tag{31}$$

```
> R2p:=SyzygyModule(Q2,A);
```

$$R2p := \begin{bmatrix} d^2 & 0 & l1\, d^2 + g \\ 0 & 1 & -1 \end{bmatrix} \tag{32}$$

```
> Quotient(R2p,R2,A);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & l1\, d^2 + g \end{bmatrix} \tag{33}$$

```
> Ext1:=Exti(R2_adj,A,1);
```

$$Ext1 := \left[\begin{bmatrix} 1 & 0 \\ 0 & l1\, d^2+g \end{bmatrix}, \begin{bmatrix} d^2 & 0 & l1\, d^2+g \\ 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} l1\, d^2+g \\ -d^2 \\ -d^2 \end{bmatrix}\right] \tag{34}$$

```
> Exti(R2_adj,A,2);
```

$$\left[\left[\ 1\ \right], \left[\ 1\ \right], SURJ(1)\right] \tag{35}$$

```
> TorsionElements(R2,[x,theta1,theta2],A);
```

$$\left[\left[g\,\theta_2(t) + l1 \left(\frac{d^2}{dt^2}\,\theta_2(t)\right) = 0 \right], \left[\theta_2(t) = \theta 1 - \theta 2 \right]\right] \tag{36}$$

```
> AutonomousElements(R2,[x(t),theta1(t),theta2(t)],A);
```

$$\left[\left[g\,\theta_1(t) + l1 \left(\frac{d^2}{dt^2}\,\theta_1(t)\right) = 0 \right], \left[\theta_1 = \_C1 \sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) + \_C2 \cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \right], \left[\theta_1 = \theta 1(t) - \theta 2(t) \right]\right] \tag{37}$$

```
> V:=FirstIntegral(R2,[theta1(t),theta2(t),u(t)],A);
```

$$\begin{aligned} V := {} & l1 \left(\frac{d}{dt}\,\theta 2(t)\right) \_C1 \sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \\ & + l1 \left(\frac{d}{dt}\,\theta 2(t)\right) \_C2 \cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) - \sqrt{l1}\,\theta 2(t)\,\_C1 \cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)\sqrt{g} \\ & + \sqrt{l1}\,\theta 2(t)\,\_C2 \sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)\sqrt{g} - l1 \left(\frac{d}{dt}\,u(t)\right) \_C1 \sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \\ & - l1 \left(\frac{d}{dt}\,u(t)\right) \_C2 \cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \\ & + \sqrt{l1}\,u(t)\,\_C1 \cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)\sqrt{g} - \sqrt{l1}\,u(t)\,\_C2 \sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)\sqrt{g} \end{aligned} \tag{38}$$

```
> dotV:=diff(V,t);
```

$$\begin{aligned} dotV := {} & l1 \left(\frac{d^2}{dt^2}\,\theta 2(t)\right) \_C1 \sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) + l1 \left(\frac{d^2}{dt^2}\,\theta 2(t)\right) \_C2 \cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \\ & + \theta 2(t)\,\_C1 \sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) g \\ & + \theta 2(t)\,\_C2 \cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) g - l1 \left(\frac{d^2}{dt^2}\,u(t)\right) \_C1 \sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \end{aligned} \tag{39}$$

$$-l1\left(\frac{d^2}{dt^2}u(t)\right)\_C2\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)-u(t)\,\_C1\sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)g-u(t)\,\_C2\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)g$$

```
> a:=coeff(dotV,diff(theta2(t),`$`(t,2)));
```

$$a := l1\,\_C1\sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)+l1\,\_C2\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \tag{40}$$

```
> b:=coeff(dotV,diff(u(t),`$`(t,2)));
```

$$b := -l1\,\_C1\sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)-l1\,\_C2\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \tag{41}$$

```
>
> Sys2 := ApplyMatrix(R2,[theta1(t),theta2(t),u(t)],A);
```

$$Sys2 := \begin{bmatrix} \frac{d^2}{dt^2}\theta1(t)+g\,\theta2(t)+l1\left(\frac{d^2}{dt^2}\theta2(t)\right) \\ \frac{d^2}{dt^2}\theta1(t)+g\,u(t)+l1\left(\frac{d^2}{dt^2}u(t)\right) \end{bmatrix} \tag{42}$$

```
> L :=simplify(expand(evalm([a/l1,b/l1] &* Sys2)[1]));
```

$$\begin{aligned} L := {} & l1\left(\frac{d^2}{dt^2}\theta2(t)\right)\_C1\sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)+l1\left(\frac{d^2}{dt^2}\theta2(t)\right)\_C2\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \\ & +\theta2(t)\,\_C1\sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)g \\ & +\theta2(t)\,\_C2\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)g-l1\left(\frac{d^2}{dt^2}u(t)\right)\_C1\sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right) \\ & -l1\left(\frac{d^2}{dt^2}u(t)\right)\_C2\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)-u(t)\,\_C1\sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)g-u(t)\,\_C2\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)g \end{aligned} \tag{43}$$

```
> simplify(diff(V,t)-L);
```

$$0 \tag{44}$$

```
>
> P2:=Parametrization(R2,A);
```

$$P2 := \begin{bmatrix} g\,\xi_1(t)+l1\left(\frac{d^2}{dt^2}\xi_1(t)\right) \\ \_C1\sin\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)+\_C2\cos\left(\frac{\sqrt{g}\,t}{\sqrt{l1}}\right)-\left(\frac{d^2}{dt^2}\xi_1(t)\right) \\ -\frac{d^2}{dt^2}\xi_1(t) \end{bmatrix} \tag{45}$$

```
> ApplyMatrix(R2,P2,A);
```

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{46}$$

>

>

>