```
> restart:
> with(OreModules):
> with(linalg):
>
> A:=DefineOreAlgebra(diff=[d1,x1],diff=[d2,x2],diff=[d3,x3],
  polynom=[x1,x2,x3]):
>
> R:=Mult(1/2,evalm([[x2*d1,2*(x2*d2+1),2*x2*d3+d1],[-x2*d2-3,0,d2],
  [-2*d1-x2*d3,-2*d2,-d3]]),A);
```

$$R := \begin{bmatrix} \frac{1}{2}\, x2\, d1 & x2\, d2 + 1 & x2\, d3 + \frac{1}{2}\, d1 \\ -\frac{1}{2}\, x2\, d2 - \frac{3}{2} & 0 & \frac{1}{2}\, d2 \\ -d1 - \frac{1}{2}\, x2\, d3 & -d2 & -\frac{1}{2}\, d3 \end{bmatrix} \tag{1}$$

```
> R_adj:=Involution(R,A);
```

$$R\_adj := \begin{bmatrix} -\frac{1}{2}\, x2\, d1 & -1 + \frac{1}{2}\, x2\, d2 & d1 + \frac{1}{2}\, x2\, d3 \\ -x2\, d2 & 0 & d2 \\ -x2\, d3 - \frac{1}{2}\, d1 & -\frac{1}{2}\, d2 & \frac{1}{2}\, d3 \end{bmatrix} \tag{2}$$

```
> Ext1:=Exti(R_adj,A,1);
```

$$Ext1 := \left[\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x2\, d3 + 2\, d1 & 2\, d2 & d3 \\ 3 + x2\, d2 & 0 & -d2 \\ x2\, d1 & 2 + 2\, x2\, d2 & d1 + 2\, x2\, d3 \\ -d3 + d2\, d1 & d2^2 & d2\, d3 \\ -d1^2 & -d2\, d1 + d3 + x2\, d2\, d3 & x2\, d3^2 \\ -d1 & 2\, d2 + x2\, d2^2 & d2\, d1 + x2\, d2\, d3 + d3 \end{bmatrix}, [\right.$$

$$\left. -d2], [x2\, d3 + d1], [-2 - x2\, d2] \right] \tag{3}$$

```
> Ext2:=Exti(R_adj,A,2);
```

$$Ext2 := [[\ 1\ ], [\ 1\ ], SURJ(1)] \tag{4}$$

```
> Ext3:=Exti(R_adj,A,3);
```

$$Ext3 := [undefined, ZERO, ZERO] \tag{5}$$

```
> Q:=Ext1[3];
```

$$Q := \begin{bmatrix} -d2 \\ x2\,d3 + d1 \\ -2 - x2\,d2 \end{bmatrix} \tag{6}$$

```
> R2:=SyzygyModule(Q,A);
```

$$R2 := \begin{bmatrix} x2\,d3 + 2\,d1 & 2\,d2 & d3 \\ 3 + x2\,d2 & 0 & -d2 \\ x2\,d1 & 2 + 2\,x2\,d2 & d1 + 2\,x2\,d3 \\ -d3 + d2\,d1 & d2^2 & d2\,d3 \\ -d1^2 & -d2\,d1 + d3 + x2\,d2\,d3 & x2\,d3^2 \\ -d1 & 2\,d2 + x2\,d2^2 & d2\,d1 + x2\,d2\,d3 + d3 \end{bmatrix} \tag{7}$$

```
> Quotient(R2,R,A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

```
> Quotient(R,R2,A);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

```
> Factorize(R2,R,A);
```

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & d3 & -d2 \\ d3 & 0 & d1 \\ d2 & d1 & 0 \end{bmatrix} \tag{10}$$

```
> Factorize(R,R2,A);
```

(11)

$$\begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

```
> T:=LeftInverse(Q,A);
```

$$T := \begin{bmatrix} \frac{1}{2}\, x2 & 0 & -\frac{1}{2} \end{bmatrix} \tag{12}$$

```
> P:=Parametrization(R,A);
```

$$P := \begin{bmatrix} -\frac{\partial}{\partial x2}\, \xi_1(x1, x2, x3) \\ \frac{\partial}{\partial x1}\, \xi_1(x1, x2, x3) + x2 \left(\frac{\partial}{\partial x3}\, \xi_1(x1, x2, x3) \right) \\ -2\, \xi_1(x1, x2, x3) - x2 \left(\frac{\partial}{\partial x2}\, \xi_1(x1, x2, x3) \right) \end{bmatrix} \tag{13}$$

```
> ApplyMatrix(R,P,A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

```
> x:=x1,x2,x3:
> evalm([seq([y[i](x)],i=1..3)])=evalm(P);
```

$$\begin{bmatrix} y_1(x1, x2, x3) \\ y_2(x1, x2, x3) \\ y_3(x1, x2, x3) \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial x2}\, \xi_1(x1, x2, x3) \\ \frac{\partial}{\partial x1}\, \xi_1(x1, x2, x3) + x2 \left(\frac{\partial}{\partial x3}\, \xi_1(x1, x2, x3) \right) \\ -2\, \xi_1(x1, x2, x3) - x2 \left(\frac{\partial}{\partial x2}\, \xi_1(x1, x2, x3) \right) \end{bmatrix} \tag{15}$$

```
> xi[1](x)=ApplyMatrix(T,evalm([seq([y[i](x)],i=1..3)]),A)[1,1];
```

$$\xi_1(x1, x2, x3) = \frac{1}{2}\, x2\, y_1(x1, x2, x3) - \frac{1}{2}\, y_3(x1, x2, x3) \tag{16}$$

```
>
>
```