```
> restart:
>
> with(OreModules):
> with(linalg):
>
> A:=DefineOreAlgebra(diff=[d,t],dual_shift=[delta,s],polynom=[t,s],
  comm=[zeta,k,a,omega],shift_action=[delta,t,h]):
>
> R:=matrix(3,4,[d+a,k*a*delta,0,0,0,d,-1,0,0,omega^2,d+2*zeta*
  omega,-omega^2]);
```

$$R := \begin{bmatrix} d+a & k\,a\,\delta & 0 & 0 \\ 0 & d & -1 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 \end{bmatrix} \tag{1}$$

```
> R_adj:=Involution(R,A):
> Ext1:=Exti(R_adj,A,1);
```

$$Ext1 := \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} d+a & k\,a\,\delta & 0 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 \\ 0 & d & -1 & 0 \end{bmatrix}, [\omega^2\,k\,a\,\delta], [-d\,\omega^2-a\,\omega^2], [-\omega^2\,d^2-\omega^2\,a\,d], [-d^3-2\,d^2\,\zeta\,\omega-a\,d^2-d\,\omega^2-2\,a\,d\,\zeta\,\omega-a\,\omega^2] \right] \tag{2}$$

```
> Ext2:=Exti(R_adj,A,2);
```

$$Ext2 := \left[\begin{bmatrix} \delta \\ d+a \end{bmatrix}, [\,1\,], SURJ(1) \right] \tag{3}$$

```
> Ext3:=Exti(R_adj,A,3);
```

$$Ext3 := [undefined, ZERO, ZERO] \tag{4}$$

```
> Q:=Ext1[3];
```

$$Q := \begin{bmatrix} \omega^2\,k\,a\,\delta \\ -d\,\omega^2-a\,\omega^2 \\ -\omega^2\,d^2-\omega^2\,a\,d \\ -d^3-2\,d^2\,\zeta\,\omega-a\,d^2-d\,\omega^2-2\,a\,d\,\zeta\,\omega-a\,\omega^2 \end{bmatrix} \tag{5}$$

```
> SyzygyModule(Q,A);
```

(6)

$$\begin{bmatrix} d+a & k\,a\,\delta & 0 & 0 \\ 0 & \omega^2 & d+2\,\zeta\,\omega & -\omega^2 \\ 0 & d & -1 & 0 \end{bmatrix} \quad (6)$$

```
> P:=ApplyMatrix(Q,[xi(t)],A);
```

$$P := [\omega^2\,k\,a\,\xi(t-h)], [-\omega^2\,\mathrm{D}(\xi)(t) - a\,\omega^2\,\xi(t)], [-\omega^2\,\mathrm{D}^{(2)}(\xi)(t) - a\,\omega^2\,\mathrm{D}(\xi)(t)], [-\mathrm{D}^{(3)}(\xi)(t) - 2\,\zeta\,\omega\,\mathrm{D}^{(2)}(\xi)(t) - a\,\mathrm{D}^{(2)}(\xi)(t) - \omega^2\,\mathrm{D}(\xi)(t) - 2\,a\,\zeta\,\omega\,\mathrm{D}(\xi)(t) - a\,\omega^2\,\xi(t)] \quad (7)$$

```
> ApplyMatrix(R,P,A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

```
> LeftInverse(Q,A);
```

$$[\;] \quad (9)$$

```
> PiPolynomial(R,A);
```

$$[\delta, d+a] \quad (10)$$

```
> pi:=delta;
```

$$\pi := \delta \quad (11)$$

```
> T:=LocalLeftInverse(Q,[pi],A);
```

$$T := \begin{bmatrix} \dfrac{1}{\delta\,\omega^2\,k\,a} & 0 & 0 & 0 \end{bmatrix} \quad (12)$$

```
> basisofMod:=ApplyMatrix(T,[x1(t),x2(t),x3(t),u(t)],A)[1,1];
```

$$basisofMod := \frac{x1(t+h)}{\omega^2\,k\,a} \quad (13)$$

```
> P2:=submatrix(evalm(Q&*T),1..4,1..1);
```

$$P2 := \begin{bmatrix} 1 \\ \dfrac{-d\,\omega^2 - a\,\omega^2}{\delta\,\omega^2\,k\,a} \\ \dfrac{-\omega^2\,d^2 - \omega^2\,a\,d}{\delta\,\omega^2\,k\,a} \\ \dfrac{-d^3 - 2\,d^2\,\zeta\,\omega - a\,d^2 - d\,\omega^2 - 2\,a\,d\,\zeta\,\omega - a\,\omega^2}{\delta\,\omega^2\,k\,a} \end{bmatrix} \quad (14)$$

```
> eta:=ApplyMatrix(P2,[xi(t)],A);
```

$$\eta := \left[\frac{x1(t+h)}{\omega^2\,k\,a}\right], \left[-\frac{x1(t+2\,h)\,a + \mathrm{D}(x1)(t+2\,h)}{k^2\,a^2\,\omega^2}\right], \left[-\frac{\mathrm{D}(x1)(t+2\,h)\,a + \mathrm{D}^{(2)}(x1)(t+2\,h)}{k^2\,a^2\,\omega^2}\right], [\quad (15)$$

$$-\frac{1}{\omega^4 k^2 a^2}\left(x1(t+2h)\,\omega^2 a+2\,\omega\,D(x1)(t+2h)\,a\,\zeta+\omega^2 D(x1)(t+2h)\right.$$
$$\left.+D^{(3)}(x1)(t+2h)+D^{(2)}(x1)(t+2h)\,a+2\,D^{(2)}(x1)(t+2h)\,\zeta\,\omega\right)]$$

```
> ApplyMatrix(R,eta,A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

```
>
> restart:
>
> with(OreModules):
> with(linalg):
>
> A:=DefineOreAlgebra(diff=[d,t],dual_shift=[sigma[1],s[1]],
  dual_shift=[sigma[2],s[2]],polynom=[t,s[1],s[2]],comm=[eta[1],eta
  [2]],shift_action=[sigma[1],t,h1],shift_action=[sigma[2],t,h2]):
>
> R:=evalm([[1,1,-1,-1,0,0],[d+eta[1],d-eta[1],-eta[2],eta[2],0,0],
             [sigma[1]^2,1,0,0,-sigma[1],0],[0, 0,1,sigma[2]^2,0,-
  sigma[2]]]);
```

$$R := \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ d+\eta_1 & d-\eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{bmatrix} \tag{17}$$

```
>
> R_adj:=Involution(R,A):
> Ext1:=Exti(R_adj,A,1);
```

$$Ext1 := \left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -2\,\eta_1 & d-\eta_2+\eta_1 & \eta_2+\eta_1+d & 0 & 0 \\ 0 & -1+\sigma_1^2 & -\sigma_1^2 & -\sigma_1^2 & \sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{bmatrix} \right. \tag{18}$$

$$, \left[\begin{array}{ccc} 2\,\sigma_2\,\eta_2 & -\sigma_2\,\eta_2\,\sigma_1 & -\sigma_1\,\eta_2+\sigma_1\,\eta_1-\sigma_1\,d \\ 0 & \sigma_2\,\eta_2\,\sigma_1 & \sigma_1\,\eta_2+\sigma_1\,d+\sigma_1\,\eta_1 \\ \sigma_2\,d+\sigma_2\,\eta_2+\sigma_2\,\eta_1 & -\sigma_2\,\sigma_1\,\eta_1 & 0 \\ -\sigma_2\,d+\sigma_2\,\eta_2-\sigma_2\,\eta_1 & \sigma_2\,\sigma_1\,\eta_1 & 2\,\sigma_1\,\eta_1 \\ 2\,\sigma_2\,\eta_2\,\sigma_1 & \sigma_2\,\eta_2-\sigma_2\,\eta_2\,\sigma_1^2 & -\eta_2\,\sigma_1^2+\eta_2+d+\eta_1\,\sigma_1^2-\sigma_1^2\,d+\eta_1 \\ -d\,\sigma_2^2+\eta_2\,\sigma_2^2-\eta_1\,\sigma_2^2+d+\eta_2+\eta_1 & -\sigma_1\,\eta_1+\sigma_1\,\eta_1\,\sigma_2^2 & 2\,\sigma_2\,\sigma_1\,\eta_1 \end{array}\right]\Bigg]$$

```
> Ext2:=Exti(R_adj,A,2);
```

$$Ext2 := \left[\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \left[\begin{array}{ccc} \sigma_2\,\eta_2 & 0 & \sigma_1\,\eta_1 \\ \eta_2+\eta_1+d & -\sigma_1\,\eta_1 & 0 \\ 0 & \sigma_2\,\eta_2 & \eta_2+\eta_1+d \end{array}\right], \left[\begin{array}{c} -\sigma_1\,\eta_1 \\ -d-\eta_1-\eta_2 \\ \sigma_2\,\eta_2 \end{array}\right]\right] \tag{19}$$

```
> Ext3:=Exti(R_adj,A,3);
```

$$Ext3 := \left[\left[\begin{array}{c} \sigma_2 \\ \sigma_1 \\ \eta_2+\eta_1+d \end{array}\right], \left[\begin{array}{c} 1 \end{array}\right], SURJ(1)\right] \tag{20}$$

```
> Ext4:=Exti(R_adj,A,4);
```

$$Ext4 := [undefined, ZERO, ZERO] \tag{21}$$

```
> Q:=Ext1[3];
```

$$Q := \left[\begin{array}{ccc} 2\,\sigma_2\,\eta_2 & -\sigma_2\,\eta_2\,\sigma_1 & -\sigma_1\,\eta_2+\sigma_1\,\eta_1-\sigma_1\,d \\ 0 & \sigma_2\,\eta_2\,\sigma_1 & \sigma_1\,\eta_2+\sigma_1\,d+\sigma_1\,\eta_1 \\ \sigma_2\,d+\sigma_2\,\eta_2+\sigma_2\,\eta_1 & -\sigma_2\,\sigma_1\,\eta_1 & 0 \\ -\sigma_2\,d+\sigma_2\,\eta_2-\sigma_2\,\eta_1 & \sigma_2\,\sigma_1\,\eta_1 & 2\,\sigma_1\,\eta_1 \\ 2\,\sigma_2\,\eta_2\,\sigma_1 & \sigma_2\,\eta_2-\sigma_2\,\eta_2\,\sigma_1^2 & -\eta_2\,\sigma_1^2+\eta_2+d+\eta_1\,\sigma_1^2-\sigma_1^2\,d+\eta_1 \\ -d\,\sigma_2^2+\eta_2\,\sigma_2^2-\eta_1\,\sigma_2^2+d+\eta_2+\eta_1 & -\sigma_1\,\eta_1+\sigma_1\,\eta_1\,\sigma_2^2 & 2\,\sigma_2\,\sigma_1\,\eta_1 \end{array}\right] \tag{22}$$

```
> SyzygyModule(Q,A);
```

$$\left[\begin{array}{cccccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -2\,\eta_1 & d-\eta_2+\eta_1 & \eta_2+\eta_1+d & 0 & 0 \\ 0 & -1+\sigma_1^2 & -\sigma_1^2 & -\sigma_1^2 & \sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{array}\right] \tag{23}$$

```
> P:=ApplyMatrix(Q,[xi1(t),xi2(t),xi3(t)],A);
```

$$P := \begin{bmatrix} 2\,\eta_2\,\xi 1(t) - \eta_2\,\xi 2(t) - \xi 3(t)\,\eta_2 + \eta_1\,\xi 3(t) - \left(\frac{d}{dt}\,\xi 3(t)\right) \\ \eta_2\,\xi 2(t) + \xi 3(t)\,\eta_2 + \eta_1\,\xi 3(t) + \frac{d}{dt}\,\xi 3(t) \\ \eta_2\,\xi 1(t) + \xi 1(t)\,\eta_1 + \frac{d}{dt}\,\xi 1(t) - \eta_1\,\xi 2(t) \\ \eta_2\,\xi 1(t) - \xi 1(t)\,\eta_1 - \left(\frac{d}{dt}\,\xi 1(t)\right) + \eta_1\,\xi 2(t) + 2\,\eta_1\,\xi 3(t) \\ 2\,\eta_2\,\xi 1(t) + 2\,\eta_1\,\xi 3(t) \\ 2\,\eta_2\,\xi 1(t) + 2\,\eta_1\,\xi 3(t) \end{bmatrix} \tag{24}$$

```
> ApplyMatrix(R,P,A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

```
> LeftInverse(Q,A);
```

$$[\;] \tag{26}$$

```
> PiPolynomial(R,A);
```

$$[\sigma_2, \sigma_1, \eta_2 + \eta_1 + d] \tag{27}$$

```
> pi:=sigma[1];
```

$$\pi := \sigma_1 \tag{28}$$

```
> Min:=MinimalParametrizations(R,A);
```

Min (29)

$$:= \left[\begin{bmatrix} 2\,\sigma_2\,\eta_2 & -\sigma_2\,\eta_2\,\sigma_1 \\ 0 & \sigma_2\,\eta_2\,\sigma_1 \\ \sigma_2\,d + \sigma_2\,\eta_2 + \sigma_2\,\eta_1 & -\sigma_2\,\sigma_1\,\eta_1 \\ -\sigma_2\,d + \sigma_2\,\eta_2 - \sigma_2\,\eta_1 & \sigma_2\,\sigma_1\,\eta_1 \\ 2\,\sigma_2\,\eta_2\,\sigma_1 & \sigma_2\,\eta_2 - \sigma_2\,\eta_2\,\sigma_1^2 \\ -d\,\sigma_2^2 + \eta_2\,\sigma_2^2 - \eta_1\,\sigma_2^2 + d + \eta_2 + \eta_1 & -\sigma_1\,\eta_1 + \sigma_1\,\eta_1\,\sigma_2^2 \end{bmatrix}, [2\,\sigma_2\,\eta_2, -\sigma_1\,\eta_2 + \sigma_1\,\eta_1 - \sigma_1\,d], \right.$$

$[0, \sigma_1\,\eta_2 + \sigma_1\,d + \sigma_1\,\eta_1], [\sigma_2\,d + \sigma_2\,\eta_2 + \sigma_2\,\eta_1, 0], [-\sigma_2\,d + \sigma_2\,\eta_2 - \sigma_2\,\eta_1, 2\,\sigma_1\,\eta_1], [2\,\sigma_2\,\eta_2\,\sigma_1,$
$-\eta_2\,\sigma_1^2 + \eta_2 + d + \eta_1\,\sigma_1^2 - \sigma_1^2\,d + \eta_1], [-d\,\sigma_2^2 + \eta_2\,\sigma_2^2 - \eta_1\,\sigma_2^2 + d + \eta_2 + \eta_1, 2\,\sigma_2\,\sigma_1\,\eta_1], [-\sigma_2\,\eta_2\,\sigma_1,$
$-\sigma_1\,\eta_2 + \sigma_1\,\eta_1 - \sigma_1\,d], [\sigma_2\,\eta_2\,\sigma_1, \sigma_1\,\eta_2 + \sigma_1\,d + \sigma_1\,\eta_1], [-\sigma_2\,\sigma_1\,\eta_1, 0], [\sigma_2\,\sigma_1\,\eta_1, 2\,\sigma_1\,\eta_1], [$

$$\sigma_2\,\eta_2-\sigma_2\,\eta_2\,\sigma_1^2,\ -\eta_2\,\sigma_1^2+\eta_2+d+\eta_1\,\sigma_1^2-\sigma_1^2\,d+\eta_1],\ [-\sigma_1\,\eta_1+\sigma_1\,\eta_1\,\sigma_2^2,\ 2\,\sigma_2\,\sigma_1\,\eta_1]\Bigg]$$

```
> T:=LocalLeftInverse(Min[3],[pi],A);
```

$$T := \begin{bmatrix} 0 & 0 & 0 & \frac{\sigma_2}{\sigma_1\,\eta_1} & 0 & -\frac{1}{\sigma_1\,\eta_1} \\ 0 & 0 & \frac{1}{2\,\sigma_1\,\eta_1} & \frac{1}{2\,\sigma_1\,\eta_1} & 0 & 0 \end{bmatrix} \tag{30}$$

```
> basisofMod:=ApplyMatrix(T,[phi1(t),psi1(t),phi2(t),psi2(t),u(t),v
  (t)],A)[1,1];
```

$$basisofMod := -\frac{-\psi 2(t-h2+h1)+v(t+h1)}{\eta_1} \tag{31}$$

```
>
>
> A:=DefineOreAlgebra(diff=[d,t],dual_shift=[delta,s],polynom=[t,s],
  shift_action=[delta,t]):
>
> R:=evalm([[d,-d*delta,-1],[2*d*delta,-d*delta^2-d,0]]);
```

$$R := \begin{bmatrix} d & -d\,\delta & -1 \\ 2\,d\,\delta & -d\,\delta^2-d & 0 \end{bmatrix} \tag{32}$$

```
> R_adj:=Involution(R,A):
> Ext1:=Exti(R_adj,A,1);
```

$$Ext1 := \left[\begin{bmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -2\,\delta & 1+\delta^2 & 0 \\ -d & d\,\delta & 1 \\ d\,\delta & -d & \delta \end{bmatrix}, \begin{bmatrix} 1+\delta^2 \\ 2\,\delta \\ -d\,\delta^2+d \end{bmatrix} \right] \tag{33}$$

```
> Ext2:=Exti(R_adj,A,2);
```

$$Ext2 := \left[\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, SURJ(1) \right] \tag{34}$$

```
> Ext3:=Exti(R_adj,A,3);
```

$$Ext3 := [undefined, ZERO, ZERO] \tag{35}$$

```
> Q:=Ext1[3];
```

(36)

$$Q := \begin{bmatrix} 1+\delta^2 \\ 2\,\delta \\ -d\,\delta^2+d \end{bmatrix} \tag{36}$$

```
> SyzygyModule(Q,A);
```

$$\begin{bmatrix} -2\,\delta & 1+\delta^2 & 0 \\ -d & d\,\delta & 1 \\ d\,\delta & -d & \delta \end{bmatrix} \tag{37}$$

```
> P:=ApplyMatrix(Q,[xi(t)],A);
```

$$P := \begin{bmatrix} \xi(t)+\xi(t-2) \\ 2\,\xi(t-1) \\ -\mathrm{D}(\xi)(t-2)+\mathrm{D}(\xi)(t) \end{bmatrix} \tag{38}$$

```
> ApplyMatrix(R,P,A);
```

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{39}$$

```
> T:=LeftInverse(Q,A);
```

$$T := \begin{bmatrix} 1 & -\frac{1}{2}\,\delta & 0 \end{bmatrix} \tag{40}$$

```
> basisofM_over_t(M):=ApplyMatrix(T,[y1(t),y2(t),u(t)],A)[1,1];
```

$$basisofM\_over\_t(M) := y1(t) - \frac{1}{2}\,y2(t-1) \tag{41}$$

```
>
>
```