Exercises: Factorization, reduction and decomposition problems

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Exercise 1 Let A be a domain, α an injective endomorphism of A, $D = A[\partial;\alpha,\beta]$ a skew polynomial ring, $E \in A^{n\times n}$ and $F \in A^{n\times n}$, $R = \partial\, I_n - E \in D^{n\times n}$ and $R' = \partial\, I_n - F$ two matrices with entries in D, $M = D^{1\times n}/(D^{1\times n}\, R)$ and $M' = D^{1\times n}/(D^{1\times n}\, R')$ two left D-modules respectively finitely presented by R and R'.

1. Describe the left D-modules M and M' in terms of generators and relations.

2. Prove that any $f \in \hom_D(M, M')$ can be defined by means of a matrix $P \in A^{n\times n}$ satisfying the relation $R\,P = Q\,R'$.

3. Deduce that $Q \in A^{n\times n}$ and prove that $R\,P = Q\,R'$ is then equivalent to:

$$\begin{cases} Q = \alpha(P), \\ \beta(P) = E\,P - \alpha(P)\,F. \end{cases} \tag{1}$$

   If $F = E$, then the ring $\mathcal{E} = \{P \in A^{n\times n} \mid \beta(P) = E\,P - \alpha(P)\,F\}$ is called the *eigenring*[1] of the linear system:

$$\partial\, y = E\, y.$$

4. Let $\mathcal{F}$ be a left D-module and $\eta \in \ker_{\mathcal{F}}(R'.)$, i.e., $(\partial\, I_n - F)\,\eta = 0$. Then, prove that $\overline{\eta} = P\,\eta \in \ker_{\mathcal{F}}(R.)$, i.e., $(\partial\, I_n - E)\,\overline{\eta} = 0$.

5. If $\alpha(P)$ is an invertible matrix, i.e., $\alpha(P) \in \mathrm{GL}_n(A)$, then (1) is equivalent to:

$$F = \alpha(P)^{-1}\,E\,P - \alpha(P)^{-1}\,\beta(P).$$

   If $P \in \mathrm{GL}_n(A)$ and $S = P^{-1}$, then prove that (1) is equivalent to:

$$F = \alpha(S)\,E\,S^{-1} + \beta(S)\,S^{-1}.$$

6. Simplify (1) in the case of a skew polynomial ring $D = A\,[\partial;\alpha,0]$ of shift operators.

7. Simplify (1) in the case of $D = A\left[\partial;\mathrm{id},\frac{d}{dt}\right]$. We now suppose that A is a field. Taking $F = E$ and using the following identities of the trace,

$$\mathrm{tr}(P_1 + P_2) = \mathrm{tr}(P_1) + \mathrm{tr}(P_2), \quad \mathrm{tr}(P_1\,P_2) = \mathrm{tr}(P_2\,P_1),$$

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[1] M. F. Singer, "Testing reducibility of linear differential operators: a group theoretic perspective", *Appl. Algebra Engrg. Comm. Comput.*, 7 (1996), 77-104.

prove:

$$\forall\, P \in \mathcal{E}, \quad \forall\, k \in \mathbb{N}, \quad \frac{d\mathrm{tr}(P^k)}{dt} = 0.$$

Since the coefficients of the characteristic polynomial $p(\lambda) = \det(\lambda\, I_n - P)$ are symmetric functions of the eigenvalues of P, prove that they are constant, i.e., they are first integrals of the linear OD system $\partial\, y = E\, y$.

8. Consider the integral domain $A = \mathbb{Q}[t]$ and the following matrix:

$$E = \begin{pmatrix} t\,(2\,t+1) & -2\,t^3 - 2\,t^2 + 1 \\ 2\,t & -t\,(2\,t+1) \end{pmatrix} \in \mathbb{Q}[t]^{2\times 2}.$$

Let us suppose that the eigenring of $\partial\, \eta = E\, \eta$ is defined by:

$$\mathcal{E} = \left\{ P = \begin{pmatrix} a_1 - a_2\,(t+1) & a_2\, t\,(t+1) \\ -a_2 & a_2\, t + a_1 \end{pmatrix} \mid a_1,\ a_2 \in \mathbb{Q} \right\}.$$

Compute the characteristic polynomial of a generic element P of $\mathcal{E}$ and show that its eigenvalues are constant. Then, compute a Jordan form $J = U^{-1}\, P\, U$ of P and prove that $\overline{\eta} = U^{-1}\, \eta$ satisfies the following linear OD system:

$$\dot{\overline{\eta}} = \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} \overline{\eta}.$$

Finally, integrating this last linear OD system, determine the solution η of $\partial\, \eta = E\, \eta$.

9. Let $\mathbb{Q}\{u\}$ be the differential ring formed by differential polynomials in u, namely, polynomials in a finite number of derivatives of u with respect to x and t and

$$\mathfrak{p} = \left\{ \frac{\partial u}{\partial t} - 6\, u \left(\frac{\partial u}{\partial x} \right) + \frac{\partial^3 u}{\partial x^3} \right\}$$

a prime differential ideal of $\mathbb{Q}\{u\}$, $L = \mathbb{Q}\{u\}/\mathfrak{p}$ the differential ring and

$$K = \{n/d \mid 0 \neq d,\, n \in L\}$$

its quotient field, i.e., the differential field defined by the *Korteweg-de Vries (KdV) equation*:

$$\frac{\partial u}{\partial t} - 6\, u \left(\frac{\partial u}{\partial x} \right) + \frac{\partial^3 u}{\partial x^3} = 0. \tag{2}$$

Let us also consider the rings of PD operators $A = K\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]$ and $D = A\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]$, the two following PD operators

$$\begin{cases} E = -4\, \partial_x^3 + 6\, u\, \partial_x + 3 \left(\dfrac{\partial\, u}{\partial\, x} \right) \in A, \\ R = \partial_t - E \in D, \end{cases}$$

and the finitely presented left D-module $M = D/(D\, R)$. Prove that the Schrödinger operator $P = -\partial_x^2 + u$ with the potential u satisfies

$$R\, P - P\, R = \partial_t\, P - E\, P + P\, E = \frac{\partial u}{\partial t} - 6\, u \left(\frac{\partial u}{\partial x} \right) + \frac{\partial^3 u}{\partial x^3} = 0,$$

i.e., P defines $f \in \mathrm{end}_D(M)$. In the inverse scattering theory, the last result implies that the smooth one-parameter family of differential operators $t \longmapsto -\partial_x^2 + u(x,t)$ defines an *isospectral flow* on the solutions of the evolution equation $\partial_t \, \eta = E \, \eta$, namely, if $\psi(x)$ is an eigenvector of the differential operator $-\partial_x^2 + u(x,0)$ with eigenvalue λ, then the solution $\eta(x,t)$ of the equation $\partial_t \, \eta(x,t) = E \, \eta(x,t)$ with the initial value $\eta(x,0) = \psi(x)$ is an eigenvector of the differential operator $-\partial_x^2 + u(x,t)$ with the same eigenvalue λ.

Exercise 2 Let us consider the linear OD time-delay system

$$\left\{ \begin{array}{l} y_1(t - 2\,h) + y_2(t) - 2\,\dot{u}(t-h) = 0, \\ y_1(t) + y_2(t - 2\,h) - 2\,\dot{u}(t-h) = 0, \end{array} \right. \tag{3}$$

describing a model of a tank containing a fluid and subjected to a one-dimensional horizontal (F. Dubois, N. Petit, P. Rouchon, "Motion planning and nonlinear simulations for a tank containing a fluid", in the proceedings of the 5^{th} *European Control Conference*, Karlsruhe (Germany), 1999.). Let $D = \mathbb{Q}\left[\partial; \mathrm{id}, \frac{d}{dt}\right][\delta; \alpha, 0]$ be the commutative polynomial ring of OD time-delay operators with rational coefficients (i.e., $\alpha(y(t)) = y(t-h)$), the presentation matrix of (3)

$$R = \begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix} \in D^{2\times 3}. \tag{4}$$

and the corresponding finitely presented D-module $M = D^{1\times 3}/(D^{1\times 2}\,R)$.

1. Using the command MorphismsConstCoeff of the package OreMorphisms, find a family of generators $\{f_i\}_{i=1,\ldots,r}$ of the D-module $\mathrm{end}_D(M)$ and their D-relations.

2. If $g_1, \ldots, g_s \in \mathrm{end}_D(M)$ and $p \in D[x_1, \ldots, x_s]$, show that $p(f_1, \ldots, f_r)$ can be expressed as a D-linear combination of the f_i's. Deduce that we only need to know the expressions of the products $f_j \circ f_i$ in terms of the f_k's (multiplication table) to compute $p(f_1, \ldots, f_r)$.

3. Add a third argument to the command MorphismsConstCoeff to get the previous multiplication table and the "structure polynomials".

4. Give a description of $\mathrm{end}_D(M)$ as a quotient of a free associative D-algebra by a certain two-sided ideal. Do the generators of this ideal form a noncommutative Gröbner basis for the total order?

Exercise 3 We consider the so-called *conjugated Beltrami equation* with $\sigma(x,y) = x$:

$$\left\{ \begin{array}{l} \dfrac{\partial z_1(x,y)}{\partial x} - x\,\dfrac{\partial z_2(x,y)}{\partial y} = 0, \\[2ex] \dfrac{\partial z_1(x,y)}{\partial y} + x\,\dfrac{\partial z_2(x,y)}{\partial x} = 0. \end{array} \right. \tag{5}$$

Let $D = A_2(\mathbb{Q}) = \mathbb{Q}[x,y]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right]$ be the first Weyl algebra over $\mathbb{Q}$,

$$R = \begin{pmatrix} \partial_x & -x\,\partial_y \\ \partial_y & x\,\partial_x \end{pmatrix} \in D^{2\times 2}$$

the presentation matrix of (5) and the left D-module $M = D^{1\times 2}/(D^{1\times 2}\,R)$.

1. Using the command DIMENSION of OREMODULES, compute $\dim_D(M)$. Deduce that M is an infinite-dimensional $\mathbb{Q}$-vector space.

2. Let $E = B_2(\mathbb{Q}) = \mathbb{Q}[x, y]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right]$ be the second Weyl algebra over $\mathbb{Q}$. Compute $\dim_E(E \otimes_D M)$ using the command DIMENSIONRAT.

3. Using the Maple command `pdsolve`, compute one explicit solution of (5).

4. Write this solution under the form of a column vector Z and using the command APPLYMATRIX of OREMODULES, check again that Z satisfies $R\,Z = 0$.

5. Using the command MORPHISMS of the package OREMORPHISMS, compute $\mathrm{end}_D(M)_{(0,0)}$, $\mathrm{end}_D(M)_{(1,0)}$, $\mathrm{end}_D(M)_{(0,1)}$, $\mathrm{end}_D(M)_{(1,1)}$ and $\mathrm{end}_D(M)_{(2,2)}$.

6. For each case, show that we can obtain a new solution of (5) by considering the new vector $Z_{(i,j)} = P_{(i,j)}\,Z$, where $P_{(i,j)}$ denotes the first output of $\mathrm{end}_D(M)_{(i,j)}$.

Exercise 4 Let A be a differential domain, $D = A\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ the skew polynomial ring of OD operators with coefficients in A, $E \in A^{n\times n}$, $R = \partial\, I_n - E \in D^{n\times n}$ and $M = D^{1\times n}/(D^{1\times n}\,R)$.

1. Compute the formal adjoint $\widetilde{R}$ of R.

2. Let $\widetilde{N} = D^{1\times n}/(D^{1\times n}\,\widetilde{N})$ be the adjoint left D-module of M. Using Exercise 1, prove that any $f \in \hom_D(\widetilde{N}, M)$ can be defined by $P \in A^{n\times n}$ satisfying the Lyapunov equation:
$$\frac{dP}{dt} + E^T\,P + P\,E = 0.$$

3. Let $\mathcal{F}$ be a left D-module, $\ker_{\mathcal{F}}(R.)$ and $\eta \in \ker_{\mathcal{F}}(R.)$. Considering the quadratic form $V = \eta^T\,P\,\eta$, prove that $\frac{dV}{dt} = 0$, i.e., V is a quadratic first integral of $\ker_{\mathcal{F}}(R.)$.

4. Compute a generic quadratic first integral of the linear OD system $\ker_{\mathcal{F}}(R.)$, where
$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & \alpha \end{pmatrix} \in \mathbb{Q}(\omega, \alpha)^{4\times 4},$$
where ω and α are two real constants and $\mathcal{F} = C^\infty(\mathbb{R}_+)$.

Exercise 5 The purpose of this exercise is to study by algebraic means the quadratic first integrals of the following simple mechanical system $\ddot{x}(t) + \alpha\,\dot{x}(t) + \beta\,x(t) = 0$, where α and β are two real parameters. Let $D = \mathbb{Q}(\alpha, \beta)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$, $R = (\partial^2 + \alpha\,\partial + \beta)$ and $M = D/(D\,R)$.

1. Compute the formal adjoint $\widetilde{R}$ of R and prove the following identity:
$$\lambda\,(R\,x) = (\widetilde{R}\,\lambda)\,x + \frac{d}{dt}\,(\lambda\,\dot{x} - (\dot{\lambda} - \alpha\,\lambda)\,x). \qquad (6)$$

2. Let $\widetilde{N} = D/(D\,\widetilde{R})$. Using the commutativity of D and
$$\gcd(\partial^2 - \alpha\,\partial + \beta, \partial^2 + \alpha\,\partial + \beta) = \begin{cases} 1, & \text{if } \alpha \neq 0,\ \beta \neq 0, \\ \partial, & \text{if } \alpha \neq 0,\ \beta = 0, \\ \partial^2 + \beta, & \text{if } \alpha = 0, \end{cases}$$

prove that $f \in \hom_D(\widetilde{N}, M)$ can be defined by:

$$(P \quad -Q) = \begin{cases} T\,(R \quad -\widetilde{R}), \quad T \in D, & \text{if } \alpha \neq 0,\ \beta \neq 0, \\ T\,(\partial + \alpha \quad \partial - \alpha), \quad T \in D, & \text{if } \alpha \neq 0,\ \beta = 0, \\ T\,(1 \quad -1), \quad T \in D, & \text{if } \alpha = 0. \end{cases}$$

3. If $\mathcal{F} = C^{\infty}(\mathbb{R}_+)$ and $x \in \ker_{\mathcal{F}}(R.)$, then show that

$$\lambda = \begin{cases} R\,x = 0, & \text{if} \quad \alpha \neq 0,\ \beta \neq 0, \\ T\,(\dot{x} + \alpha\, x), \quad T \in D, & \text{if} \quad \alpha \neq 0,\ \beta = 0, \\ T\,x, \quad T \in D, & \text{if} \quad \alpha = 0, \end{cases}$$

is an element of $\ker_{\mathcal{F}}(\widetilde{R}.)$ and, using (6), prove that

$$V(x) = \begin{cases} 0, & \text{if} \quad \alpha \neq 0,\ \beta \neq 0, \\ (T\,(\dot{x} + \alpha\, x))\,\dot{x} - (T\,(\ddot{x} + \alpha\,\dot{x} - \alpha\,(\dot{x} + \alpha\, x)))\,x = T\,(\dot{x} + \alpha\, x)^2, & \text{if} \quad \alpha \neq 0,\ \beta = 0, \\ (T\,x)\,\dot{x} - (T\,\dot{x})\,x, & \text{if} \quad \alpha = 0, \end{cases}$$

are quadratic first integrals of $\ker_{\mathcal{F}}(\widetilde{R}.)$.

4. In the last case, since the system is a second order ODE, we can take $T = a_1\,\partial + a_2$, where $a_1,\ a_2 \in \mathbb{Q}(\beta, m)$. Prove that we get the following first integral of $\ker_{\mathcal{F}}(R.)$:

$$V(x) = (a_1\,\dot{x} + a_2\,x)\,\dot{x} + (-a_1\,\ddot{x} - a_2\,\dot{x})\,x = a_1\,\left(\dot{x}^2 + \beta\,x^2\right).$$

Exercise 6 The movement of an incompressible fluid rotating with a small velocity around the axis lying along the x_3 axis can be defined by

$$\begin{cases} \rho_0\,\dfrac{\partial u_1}{\partial t} - 2\,\rho_0\,\Omega_0\,u_2 + \dfrac{\partial p}{\partial x_1} = 0, \\ \rho_0\,\dfrac{\partial u_2}{\partial t} + 2\,\rho_0\,\Omega_0\,u_1 + \dfrac{\partial p}{\partial x_2} = 0, \\ \rho_0\,\dfrac{\partial u_3}{\partial t} + \dfrac{\partial p}{\partial x_3} = 0, \\ \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3} = 0, \end{cases} \tag{7}$$

where $u = (u_1,\ u_2,\ u_3)^T$ denotes the local rate of velocity, p the pressure, ρ_0 the constant fluid density and Ω_0 the constant angle speed.

Let $D = \mathbb{Q}(\rho_0, \Omega_0)\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right]$ be the commutative polynomial ring of differential operators,

$$R = \begin{pmatrix} \rho_0\,\partial_t & -2\,\rho_0\,\Omega_0 & 0 & \partial_1 \\ 2\,\rho_0\,\Omega_0 & \rho_0\,\partial_t & 0 & \partial_2 \\ 0 & 0 & \rho_0\,\partial_t & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix} \in D^{4\times 4}$$

the presentation matrix of (7) and the D-module $M = D^{1\times 4}/(D^{1\times 4}\,R)$ associated with (7).

1. Compute the formal adjoint $\widetilde{R}$ of R. What is the relation between R and $\widetilde{R}$.

2. If we denote by $\eta = (u_1 \quad u_2 \quad u_2 \quad p)^T$, then check the following identity:

$$(\lambda, R\,\eta) = (\eta, \widetilde{R}\,\lambda) + (\partial_t \quad \partial_1 \quad \partial_2 \quad \partial_3) \begin{pmatrix} \rho_0\,(\lambda_1\,u_1 + \lambda_2\,u_2 + \lambda_3\,u_3) \\ \lambda_1\,p + \lambda_4\,u_1 \\ \lambda_2\,p + \lambda_4\,u_2 \\ \lambda_3\,p + \lambda_4\,u_3 \end{pmatrix}.$$

3. Compare $\widetilde{N} = D^{1\times 4}/(D^{1\times 4}\,\widetilde{R})$ and $M = D^{1\times 4}/(D^{1\times 4}\,R)$. Compare $\hom_D(\widetilde{N}, M)$ and $\mathrm{end}_D(M)$. What are the simplest elements of $\hom_D(\widetilde{N}, M)$?

4. If $(\vec{u},\ p)$ is a solution of (7), then find a simple solution of $\widetilde{R}\,\lambda = 0$ and prove that (7) admits the following quadratic conservation law:

$$\partial_t\,(\rho_0\,(u_1^2 + u_2^2 + u_3^2)) + \partial_1\,(2\,p\,u_1) + \partial_2\,(2\,p\,u_2) + \partial_3\,(2\,p\,u_3) = 0,$$

$$\Leftrightarrow \quad \partial_t\left(\frac{\rho_0}{2}\,\parallel \vec{u} \parallel^2\right) + \vec{\nabla}\,.\,(p\,\vec{u}) = 0.$$

Exercise 7 Let $D = \mathbb{Z}$, $R = 4$, $M = D/(D\,R) = \mathbb{Z}/(4\,\mathbb{Z})$ and $f \in \mathrm{end}_D(M)$ defined by $f(m) = 2\,m$, for all $m \in M$.

1. Find a finite free resolution of the D-module M.

2. Compute $\mathrm{coker}\, f$ and $\ker f$.

3. Deduce a free resolution of $\mathbb{Z}_2 = \mathbb{Z}/(2\,\mathbb{Z})$ as $\mathbb{Z}/(4\,\mathbb{Z})$-module. Conclude on its length and on the projective dimension of $\mathbb{Z}_2 = \mathbb{Z}/(2\,\mathbb{Z})$ as $\mathbb{Z}/(4\,\mathbb{Z})$-module.

Exercise 8 Let us consider the so-called *Oseen equations* defined by

$$\begin{cases} -\nu\,\Delta\,\vec{u} + (\vec{b}\,.\,\vec{\nabla})\,\vec{u} + c\,\vec{u} + \vec{\nabla}\,p = 0, \\ \vec{\nabla}\,.\,\vec{u} = 0, \end{cases} \tag{8}$$

where $\vec{u}$ denotes the velocity, p the pressure, ν the viscosity and $\vec{b}$ a steady velocity and c a constant reaction coefficient, which describe the flow of a viscous and incompressible fluid at small Reynolds numbers (linearization of the incompressible Navier-Stokes equations at a steady state). Let $D = \mathbb{Q}(\nu, b_1, b_2, c)\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right]$ be the ring of PD operators with coefficients in the field $\mathbb{Q}(\nu, b_1, b_2, c)$, the presentation matrix of (8)

$$R = \begin{pmatrix} -\nu\,\Delta + b_1\,\partial_x + b_2\,\partial_y + c & 0 & \partial_x \\ 0 & -\nu\,\Delta + b_1\,\partial_x + b_2\,\partial_y + c & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \in D^{3\times 3},$$

and the finitely presented D-module $M = D^{1\times 3}/(D^{1\times 3}\,R)$.

1. Check that $P = Q = \Delta\,I_3$ defines a D-endomorphism f_1 of M, where $\Delta = \partial_x^2 + \partial_y^2$.

2. Compute $\ker f_1$. Is f_1 injective? You can use the command TestInj of OreMorphisms. If not, compute a factorization $R = L_1\,S_1$ of R.

3. Find directly the matrix S_1 using the command CoimMorphism of OreMorphisms.

4. Check that $P = Q = (\nu\,\Delta - b_1\,\partial_x - b_2\,\partial_y - c)\,I_3$ defines a D-endomorphism f_2 of M.

5. Compute $\ker f_2$. Is f_2 injective? If not, compute a factorization $R = L_2\,S_2$ of R.

6. Find directly the matrix S_2 using the command CoimMorphism of OreMorphisms.

Exercise 9 Let $D = \mathbb{Q}\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$, $R = (\partial \quad -1)$, $R' = (\partial^2 \quad -1)$ and$M = D^{1\times 2}/(D\,R)$ and $M' = D^{1\times 2}/(D\,R')$ two D-modules which, in terms of generators and relations, are respectively defined by $\partial\,x = u$ and $\partial^2\,y = v$. Using the commands MorphismsConstCoeff and TestIso of OreMorphisms, prove that $M \cong M'$. In particular, compute an isomorphism $f : M \longrightarrow M'$ and its inverse f^{-1}.

Exercise 10 Using the commands MorphismsConstCoeff and TestIso of OreMorphisms, prove that the following systems are equivalent:

$$\left\{\begin{array}{l} \partial_1\,\xi_1 = 0, \\ \frac{1}{2}\,(\partial_2\,\xi_1 + \partial_1\,\xi_2) = 0, \\ \partial_2\,\xi_2 = 0, \end{array}\right. \qquad \left\{\begin{array}{l} \partial_1\,\zeta_1 = 0, \\ \partial_2\,\zeta_1 - \zeta_2 = 0, \\ \partial_1\,\zeta_2 = 0, \\ \partial_1\,\zeta_3 + \zeta_2 = 0, \\ \partial_2\,\zeta_3 = 0, \\ \partial_2\,\zeta_2 = 0. \end{array}\right.$$

In particular, exhibit an explicit isomorphism between the two underlying differential modules and its inverse.

Exercise 11 Let $D = A_1(\mathbb{Q})$ be the first Weyl algebra, R the following matrix:

$$R = \left(\begin{array}{cccc} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{array}\right) \in D^{4\times 4},$$

and $M = D^{1\times 4}/(D^{1\times 4}\,R)$ the left D-module finitely presented by R.

1. Using the command Morphisms of OreMorphisms, compute $\mathrm{end}_D(M)_{(0,0)}$.

2. Check if the generic element P obtained in 1 admits a non-trivial left kernel.

3. Compute the determinant of P and deduce values of the arbitrary parameters for which P becomes singular. Compute the corresponding Q's.

4. For each of those values:

   (a) Check that the left D-modules $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}_D(.Q)$ are free and compute a basis for each of them. Denote the corresponding matrices respectively by U_1, U_2, V_1 and V_2.

   (b) Form the matrices $U = (U_1^T \quad U_2^T)^T \in D^{4\times 4}$ and $V = (V_1^T \quad V_2^T)^T \in D^{4\times 4}$ and check that $U \in \mathrm{GL}_4(D)$ and $V \in \mathrm{GL}_4(D)$.

(c) Conclude that R is equivalent the the block-triangular matrix $\overline{R} = V\, R\, U^{-1}$.

(d) Check this last result using HEURISTICREDUCTION.

Exercise 12 Let us consider the following four complex matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The Dirac equation for a massless particle has the form

$$\sum_{j=1}^{4} \gamma^j \, \frac{\partial \psi(x)}{\partial x_j} = 0, \tag{9}$$

where $\psi = (\psi_1 \quad \psi_2 \quad \psi_3 \quad \psi_4)^T$ and $x = (x_1, x_2, x_3, x_4)$ are the space-time coordinates.

Let $D = \mathbb{Q}(i) \left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right] \left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right] \left[\partial_4; \mathrm{id}, \frac{\partial}{\partial x_4}\right]$ be the commutative polynomial ring of PD operators ($\partial_4 = -i\, \partial_t$),

$$R = \begin{pmatrix} \partial_4 & 0 & -i\, \partial_3 & -(i\, \partial_1 + \partial_2) \\ 0 & \partial_4 & -i\, \partial_1 + \partial_2 & i\, \partial_3 \\ i\, \partial_3 & i\, \partial_1 + \partial_2 & -\partial_4 & 0 \\ i\, \partial_1 - \partial_2 & -i\, \partial_3 & 0 & -\partial_4 \end{pmatrix} \in D^{4\times 4}$$

the presentation matrix of (9) and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 4}\, R)$.

Redo Exercise 11 with the ring D and the matrix R.

Exercise 13 We consider the following linear PD system

$$\sigma\, \partial_t\, \vec{A} + \frac{1}{\mu}\, \vec{\nabla} \wedge \vec{\nabla}\, \vec{A} - \sigma\, \vec{\nabla}\, V = 0,$$

where $(\vec{A}, V)$ denotes the electromagnetic quadri-potential, σ the electric conductivity and μ the magnetic permeability. It corresponds to the equations satisfied by quadri-potential $(\vec{A}, V)$ when it is assumed that the term $\partial_t\, \vec{D}$ can be neglected in the Maxwell equations, i.e., the electric displacement $\vec{D}$ is constant in time. It seems that Maxwell was led to introduce the term $\partial_t\, \vec{D}$ in his famous equations for pure mathematical reasons.

Redo Exercise 12 with now $D = \mathbb{Q}\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right] \left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right] \left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right]$ and the following matrices:

$$R = \frac{1}{\mu} \begin{pmatrix} \sigma\, \mu\, \partial_t - (\partial_2^2 + \partial_3^2) & \partial_1\, \partial_2 & \partial_1\, \partial_3 & -\sigma\, \mu\, \partial_1 \\ \partial_1\, \partial_2 & \sigma\, \mu\, \partial_t - (\partial_1^2 + \partial_3^2) & \partial_2\, \partial_3 & -\sigma\, \mu\, \partial_2 \\ \partial_1\, \partial_3 & \partial_2\, \partial_3 & \sigma\, \mu\, \partial_t - (\partial_1^2 + \partial_2^2) & -\sigma\, \mu\, \partial_3 \end{pmatrix},$$

$$P=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma\,\mu\,\partial_t & 0 & -\sigma\,\mu\,\partial_2 \\ 0 & 0 & \sigma\,\mu\,\partial_t & -\sigma\,\mu\,\partial_3 \\ 0 & \partial_t\,\partial_2 & \partial_t\,\partial_3 & -(\partial_2^2+\partial_3^2) \end{pmatrix}, \quad Q=\begin{pmatrix} 0 & 0 & 0 \\ -\partial_1\,\partial_2 & \sigma\,\mu\,\partial_t-\partial_2^2 & -\partial_2\,\partial_3 \\ -\partial_1\,\partial_3 & -\partial_2\,\partial_3 & \sigma\,\mu\,\partial_t-\partial_3^2 \end{pmatrix}.$$

Exercise 14 We consider again Exercise 11 and $f \in \mathrm{end}_D(M)$ defined by the matrices:

$$P=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{Q}^{4\times 4}, \quad Q=\begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix} \in \mathbb{Q}[t]^{4\times 4}.$$

The purpose of this exercise is to study the decomposition of the left D-module M as

$$M = M_1 \oplus M_2,$$

where M_1 and M_2 are two left D-submodules of M, and integrate this decomposition in terms of linear OD systems.

1. Check that $P^2 = P$ and $Q^2 = Q$. Deduce that $f^2 = f$, i.e., f is an idempotent of $\mathrm{end}_D(M)$.

2. Prove that $g = \mathrm{id}_M - f$ is a left D-homomorphism from M to $\ker f$.

3. If $i : \ker f \longrightarrow M$ denotes the canonical injection, then show that $g \circ i = \mathrm{id}_{\ker f}$. Deduce that the following short exact sequence

$$0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{\rho} \mathrm{coim}\, f = M/\ker f \longrightarrow 0$$

   splits, i.e., $M \cong \ker f \oplus \mathrm{coim}\, f$.

4. Prove that $f^\sharp : \mathrm{coim}\, f \longrightarrow M$ defined by $f^\sharp(\rho(m)) = f(m)$ is well-defined by considering two elements m and $m' \in M$ satisfying $\rho(m) = \rho(m')$. Check that $\rho \circ f^\sharp \circ \rho = \rho$, i.e., $f^\sharp \circ \rho = \mathrm{id}_{\mathrm{coim}\, f}$ and conclude that $M = \ker f \oplus f^\sharp(\mathrm{coim}\, f)$.

5. Using the command CoimMorphism, compute a presentation matrix $S \in D^{4\times 4}$ of $\mathrm{coim}\, f$, i.e., $\mathrm{coim}\, f = D^{1\times 4}/(D^{1\times 4}\, S)$.

6. Using the command Factorize of OreModules, compute $L \in D^{4\times 4}$ satisfying $R = L\, S$.

7. Using the command SyzygyModule of OreModules, check that $\ker_D(.S) = 0$. Deduce that L is a presentation matrix of the left D-module $\ker f$, i.e., $\ker f \cong D^{1\times 4}/(D^{1\times 4}\, L)$.

8. Using the command Factorize, compute $X \in D^{4\times 4}$ satisfying $P = I_4 - X\, S$.

9. Using the commands ApplyMatrix of OreModules and `dsolve` of Maple, compute the solution $S\,\zeta = 0$.

10. Similarly, compute the solution of $L\,\tau = 0$.

11. Form $\eta = \zeta + X\,\tau$ and check that $R\,\eta = 0$.

Exercise 15 Let $D = \mathbb{Q}\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]$ be the commutative polynomial ring of PD operators with rational constant coefficients, $R = (\partial_t - \partial_x \quad \partial_t - \partial_x^2)^T$, $I = D^{1\times 2}\, R = (\partial_t - \partial_x, \partial_t - \partial_x^2)$ the ideal of D formed by the transport and the heat operators, the D-module $M = D/I$ and $\pi : D \longrightarrow M$ the canonical projection onto M.

1. Prove that $e : M \longrightarrow M$ defined by $e(\pi(\lambda)) = \lambda\, \partial_t$, for all $\lambda \in D$, is a D-endomorphism of M, i.e., $e \in \mathrm{end}_D(M)$. Deduce that e can be defined by $P = \partial_t$, $Q = \partial_t\, I_2$.

2. Prove that e is an idempotent of the endomorphism ring $E = \mathrm{end}_D(M)$ of M, i.e., $e^2 = e$.

3. Prove that the matrices defined by

$$S = \begin{pmatrix} \partial_t - 1 \\ \partial_x - \partial_t \end{pmatrix}, \quad L = \begin{pmatrix} 0 & -1 \\ -\partial_t & -\partial_t - \partial_x \end{pmatrix}, \quad S_2 = (\partial_x - \partial_t \quad 1 - \partial_t),$$

   and $X = (-1 \quad 0)$ satisfy $\mathrm{coim}\, f = D/(D^{1\times 2}\, S)$, $R = L\, S$, $\ker_D(.S) = D\, S_2$ and $P = 1 - X\, S$.

4. Compute the general solution of the linear PD system $S\, \zeta = 0$.

5. Compute the general solution of the linear PD system $(L^T \quad S_2^T)^T\, \tau = 0$.

6. Finally, check that $\eta = \zeta + X\, \tau = c_1\, e^{x+t} + c_2$, where c_1 and c_2 are two arbitrary constants, is the general solution of $R\, \eta = 0$.

Exercise 16 Let $D = \mathbb{Q}\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]$ be the commutative polynomial ring of PD operators with rational constant coefficients, $R = (\partial_t^2 - \partial_x^2 \quad \partial_t - \partial_x^2)^T$, $I = D^{1\times 2}\, R = (\partial_t - \partial_x, \partial_t - \partial_x^2)$ the ideal of D formed by the wave and the heat operators, the D-module $M = D/I$ and $\pi : D \longrightarrow M$ the canonical projection onto M.

1. Prove that $e : M \longrightarrow M$ defined by $e(\pi(\lambda)) = \lambda\, \partial_t$, for all $\lambda \in D$, is a D-endomorphism of M, i.e., $e \in \mathrm{end}_D(M)$. Deduce that e can be defined by $P = \partial_t$, $Q = \partial_t\, I_2$.

2. Prove that e is an idempotent of the endomorphism ring $E = \mathrm{end}_D(M)$ of M, i.e., $e^2 = e$.

3. Prove that the matrices defined by

$$S = \begin{pmatrix} \partial_t - 1 \\ \partial_x^2 - 1 \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} \partial_t + 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \partial_x^2 - 1 & -\partial_t + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

   and $X = (-1 \quad 0 \quad 0)$ satisfy $\mathrm{coim}\, f = D/(D^{1\times 3}\, S)$, $R = L\, S$, $\ker_D(.S) = D^{1\times 2}\, S_2$ and $P = 1 - X\, S$.

4. Let $\mathcal{F} = C^\infty(\mathbb{R}^2)$. Prove that $\ker_{\mathcal{F}}(S.) = \{\zeta = c_1\, e^{t-x} + c_2\, e^{t+x} \mid c_1,\, c_2 \in \mathbb{R}\}$.

5. Prove $\ker_{\mathcal{F}}((L^T \quad S_2^T)^T.) = \{\tau = (c_3\, x + c_4 \quad c_3\, x + c_4 \quad 0)^T \mid c_3,\, c_4 \in \mathbb{R}\}$.

6. Finally, deduce that $\ker_{\mathcal{F}}(R.) = \{\eta = c_1\, e^{t-x} + c_2\, e^{t+x} - c_3\, x - c_4 \mid c_i \in \mathbb{R},\, i = 1, \ldots, 4\}$.

Exercise 17 Let $R \in D^{q \times p}$ be a full row rank matrix, i.e., $\ker_D(.R) = 0$, and $P \in D^{p \times p}$, $Q \in D^{q \times q}$ two matrices satisfying the relation $R\,P = Q\,R$. Prove that if P is an idempotent of $D^{p \times p}$, i.e., $P^2 = P$, then so is Q, i.e., $Q^2 = Q$.

Considering again Exercise 1, prove that an idempotent e of $\mathrm{end}_D(M)$ can always be defined by $P \in A^{n \times n}$ and $Q \in A^{n \times n}$ satisfying $P^2 = P$, i.e., P is an idempotent matrix of $A^{n \times n}$. Conclude that $Q^2 = Q$, i.e., Q is also an idempotent matrix of $A^{n \times n}$.

Exercise 18 Let $R \in D^{q \times p}$ be a full row rank matrix, $M = D^{1 \times p}/(D^{1 \times q}\,R)$ and $f \in \mathrm{end}_D(M)$ be defined by two matrices $P \in D^{p \times p}$ and $Q \in D^{q \times q}$ satisfying the relations:

$$R\,P = Q\,R, \quad P^2 = P + Z\,R, \quad Q^2 = Q + R\,Z.$$

Prove that if there exists a solution $\Lambda \in D^{p \times q}$ of the following algebraic Riccati equation

$$\Lambda\,R\,\Lambda + (P - I_p)\,\Lambda + \Lambda\,Q + Z = 0, \tag{10}$$

then the matrices defined by

$$\begin{cases} \overline{P} = P + \Lambda\,R, \\ \overline{Q} = Q + R\,\Lambda, \end{cases} \tag{11}$$

satisfy the following relations:

$$R\,\overline{P} = \overline{Q}\,R, \quad \overline{P}^2 = \overline{P}, \quad \overline{Q}^2 = \overline{Q}.$$

Exercise 19 We consider $D = A_1(\mathbb{Q})$, $R = \begin{pmatrix} \partial^2 & -t\,\partial - 1 \end{pmatrix}$ and $M = D^{1 \times 2}/(D\,R)$.

1. Using Morphisms, compute the endomorphisms of M defined by matrices P of degree 1 in ∂ and 2 in t.

2. Using the command Idempotents of OreMorphisms, search for idempotents of $\mathrm{end}_D(M)$ in the previous endomorphisms.

3. For the non trivials ones (i.e., different from 0 and id_M), compute the corresponding matrices Q's.

4. Compute the matrices $Z \in D^2$ satisfying $P^2 = P + Z\,R$ and check that the P's are not idempotents of $D^{2 \times 2}$.

5. Using the command Riccati of OreMorphisms, solve the algebraic Riccati equation (10) for different orders and degrees till you reach non trivial solutions.

6. Using (11), compute the corresponding $\overline{P}$'s and $\overline{Q}$'s and check that they are respectively idempotents of $D^{2 \times 2}$ and D.

Exercise 20 We consider again Exercise 14.

1. Using SyzygyModule, compute $\ker_D(.P)$ and conclude that $\ker_D(.P)$ is a free left D-module. Denote the corresponding matrix by U_1.

2. Using SyzygyModule, compute $\mathrm{im}_D(.P) = \ker_D(.(I_4 - P))$ and conclude that $\mathrm{im}_D(.P)$ is a free left D-module. Denote the corresponding matrix by U_2.

3. Similarly:

(a) Compute $\ker_D(.Q)$ and conclude that $\ker_D(.Q)$ is a free left D-module. Denote the corresponding matrix by V_1.

(b) Compute $\mathrm{im}_D(.Q) = \ker_D(.(I_4 - Q))$ and conclude that $\mathrm{im}_D(.Q)$ is a free left D-module. Denote the corresponding matrix by V_2.

4. Form the matrices $U = (U_1^T \quad U_2^T)^T$ and $V = (V_1^T \quad V_2^T)^T$ and show that $U \in \mathrm{GL}_4(D)$ and $V \in \mathrm{GL}_4(D)$.

5. Check that R is equivalent to the block-diagonal matrix $\overline{R} = V\,R\,U^{-1}$ and find again the solution η of the linear OD system $R\,\eta = 0$ computed in Exercise 14.

6. Check that last result using the command HeuristicDecomposition.

Exercise 21 We consider again Exercise 12. Using the methodology explained in Exercise 20, prove that the matrix R is equivalent to the following matrix

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} -\partial_4 + i\,\partial_3 & i\,\partial_1 + \partial_2 & 0 & 0 \\ i\,\partial_1 - \partial_2 & -\partial_t - i\,\partial_3 & 0 & 0 \\ 0 & 0 & \partial_4 + i\,\partial_3 & i\,\partial_1 + \partial_2 \\ 0 & 0 & i\,\partial_1 - \partial_2 & \partial_4 - i\,\partial_3 \end{pmatrix},$$

where the unimodular matrices U and V are defined by:

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_4(D), \quad V = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \in \mathrm{GL}_4(D).$$

Note that the computations with OreMorphisms of the set of generators of $\mathrm{end}_D(M)$ and some of its idempotents took me respectively 134 and 18 CPU time on my Mac OS X (Maple 10, 2.8 GHz, 4 GB Ram).

Exercise 22 We consider the Cauchy-Riemann equations defined by:

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0, \\[2mm] \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0. \end{cases}$$

Let $D = \mathbb{Q}\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right]$, $R = \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix} \in D^{2\times 2}$ and $M = D^{1\times 2}/(D^{1\times 2}\,R)$.

1. Using MorphismsConstCoeff, compute a family of generators of $\mathrm{end}_D(M)$, their relations and the corresponding multiplication table.

2. Using the command IdempotentsConstCoeff of OreMorphisms, show that the $E = \mathbb{Q}(i)\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right]$-module $N = E^{1\times 2}/(E\,R)$ is decomposable. Deduce that the matrices P and Q defined by

$$P = Q = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

satisfy $R\,P = P\,R$ and $P^2 = P$, i.e., define an idempotent of $\mathrm{end}_E(M)$.

3. Using the command SyzygyModule, compute basis of the free E-modules $\ker_E(.P)$ and $\mathrm{im}_E(.P) = \ker_E(.(I_2 - P))$.

4. Forming the matrix $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_2(E)$, check that R is then equivalent to the following block-diagonal matrix:

$$\overline{R} = U\,R\,U^{-1} = \begin{pmatrix} \partial_x - i\,\partial_y & 0 \\ 0 & \partial_x + i\,\partial_y \end{pmatrix} = 2\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix},$$

with the following notations:

$$U = \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \in \mathrm{GL}_2(E), \quad \overline{\partial} = \frac{1}{2}\,(\partial_x - i\,\partial_y), \quad \partial = \frac{1}{2}\,(\partial_x + i\,\partial_y).$$

Exercise 23 We consider a wave equation defined by the following linear PD system:

$$\begin{cases} \dfrac{\partial y_1}{\partial x} + a\,\dfrac{\partial y_2}{\partial t} = 0, \\ \dfrac{\partial y_1}{\partial t} + b\,\dfrac{\partial y_2}{\partial x} = 0. \end{cases}$$

Acoustic wave: $y_1 = u$, $y_2 = p$, $a = 1/\rho$, $b = \rho\,c^2$.

LC transmission line: $y_1 = v$, $y_2 = i$, $a = L$, $b = 1/C$.

Let $D = \mathbb{Q}(a,b)\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]$, $R = \begin{pmatrix} \partial_x & a\,\partial_t \\ \partial_t & b\,\partial_x \end{pmatrix} \in D^{2\times 2}$ and $M = D^{1\times 2}/(D^{1\times 2}\,R)$.

1. Using MorphismsConstCoeff, compute a family of generators of $\mathrm{end}_D(M)$, their relations and the corresponding multiplication table.

2. Using the command IdempotentsConstCoeff of OreMorphisms, show that the $E = \mathbb{Q}(a,b)[\alpha]/(4\,a\,b\,\alpha^2 - 1)\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right]$-module $N = E^{1\times 2}/(E\,R)$ is decomposable. Deduce that the matrices P and Q defined by

$$P = \frac{1}{2}\begin{pmatrix} 1 & 2\,a\,b\,\alpha \\ 2\,\alpha & 1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 1 & 2\,a\,\alpha \\ 2\,b\,\alpha & 1 \end{pmatrix},$$

satisfy $R\,P = P\,R$ and $P^2 = P$, i.e., define an idempotent of $\mathrm{end}_E(M)$.

3. Using the command SyzygyModule, compute basis of the free E-modules $\ker_E(.P)$ and $\mathrm{im}_D(.P) = \ker_E(.(I_2 - P))$, $\ker_E(.Q)$ and $\mathrm{im}_D(.Q) = \ker_E(.(I_2 - Q))$.

4. Forming the matrices $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_2(E)$ and $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_2(E)$, check that R is then equivalent to the following block-diagonal matrix:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} b\,\partial_x - \dfrac{1}{2\,\alpha}\,\partial_t & 0 \\ 0 & b\,\partial_x + \dfrac{1}{2\,\alpha}\partial_t \end{pmatrix},$$

with the following notations:

$$U = \begin{pmatrix} -2\,\alpha & 1 \\ 2\,\alpha & 1 \end{pmatrix} \in \mathrm{GL}_2(E), \quad V = \begin{pmatrix} -2\,b\,\alpha & 1 \\ 2\,b\,\alpha & 1 \end{pmatrix} \in \mathrm{GL}_2(E).$$

5. Explain that the previous decomposition proves the D'Alembert theorem stating that the solution of a wave equation can be decomposed into two transport equations with opposite speed directions, i.e., the solution of $(\partial_t^2 - c\,\partial_x^2)\,u(t,x) = 0$ can be decomposed as follows:

$$u(t,x) = f\left(x - \sqrt{c}\,t\right) + g\left(x + \sqrt{c}\,t\right).$$

Exercise 24 We consider the linearized approximation of the steady two-dimensional rotational isentropic flow

$$\begin{cases} u\,\rho \dfrac{\partial\,\omega}{\partial x} + c^2\,\dfrac{\partial\,\sigma}{\partial x} = 0, \\ u\,\rho\,\dfrac{\partial\,\lambda}{\partial x} + c^2\,\dfrac{\partial\,\sigma}{\partial y} = 0, \\ \rho\,\dfrac{\partial\,\omega}{\partial x} + \rho\,\dfrac{\partial\,\lambda}{\partial y} + u\,\dfrac{\partial \sigma}{\partial x} = 0, \end{cases} \tag{12}$$

where u is a constant velocity parallel to the x-axis, ρ a constant density and c the speed of sound. See R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Wiley Classics Library, Wiley, 1989. Using OreMorphisms, prove that if α satisfies $1 + 4\,(c^2 - u^2)\,\alpha^2 = 0$ and

$$E = \mathbb{Q}(u,\rho,c)[\alpha]/(1 + 4\,(c^2-u^2)\,\alpha^2)\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right],$$

then the presentation matrix of (12) defined by

$$R = \begin{pmatrix} u\,\rho\,\partial_x & c^2\,\partial_x & 0 \\ 0 & c^2\,\partial_y & u\,\rho\,\partial_x \\ \rho\,\partial_x & u\,\partial_x & \rho\,\partial_y \end{pmatrix} \in E^{3\times 3},$$

is equivalent to the following block-diagonal matrix

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial_x - 2\,\alpha\,c\,\partial_y & 0 & 0 \\ 0 & \partial_x + 2\,\alpha\,c\,\partial_y & 0 \\ 0 & 0 & \partial_x \end{pmatrix},$$

where:

$$U = \begin{pmatrix} 0 & 2\,\alpha\,c\,(c^2 - u^2) & u\,\rho \\ 0 & 2\,\alpha\,c\,(c^2-u^2) & -u\,\rho \\ u\,\rho & c^2 & 0 \end{pmatrix} \in \mathrm{GL}_3(E),\ V = \begin{pmatrix} 2\,\alpha\,c & 1 & -2\,\alpha\,c\,u \\ 2\,\alpha\,c & -1 & -2\,\alpha\,c\,u \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(E).$$

Exercise 25 We consider again Exercise 2.

1. Using the multiplication table, prove that $f = \frac{1}{2}\,(f_1 + f_2) \in \mathrm{end}_D(M)$, defined by

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

   is an idempotent $f \in \mathrm{end}_D(M)$. Deduce that M can be decomposed.

2. Using OreMorphisms, check that result.

3. Following the method explained in Exercise 20, prove that R is equivalent to the following block-diagonal matrix

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} \delta^2 - 1 & 0 & 0 \\ 0 & \delta^2 + 1 & -4\,\partial\,\delta \end{pmatrix},$$

where:

$$U = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

4. Let $\mathcal{F} = C^\infty(\mathbb{R})$. Check that $\ker_{\mathcal{F}}((\delta^2 - 1).)$ is exactly formed by the $2h$-periodic smooth functions.

5. Deduce that $\ker_{\mathcal{F}}(\overline{R}.)$ is defined by

$$\forall\, \xi \in \mathcal{F}, \quad \begin{cases} z_1(t) = \psi(t), \\ z_2(t) = 4\,\dot{\xi}(t-h), \\ v(t) = \xi(t - 2\,h) + \xi(t), \end{cases}$$

where ψ is an arbitrary $2h$-periodic smooth function.

6. Deduce that the $\mathcal{F}$-solutions of (3) are defined by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = U^{-1} \begin{pmatrix} z_1(t) \\ z_2(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,\psi(t) + 2\,\dot{\xi}(t-h) \\ -\frac{1}{2}\,\psi(t) + 2\,\dot{\xi}(t-h) \\ \xi(t-2\,h) + \xi(t) \end{pmatrix},$$

where ψ (resp., ξ) is an arbitrary $2\,h$-periodic smooth (resp., smooth) function.

Exercise 26 We consider the model of a flexible rod with a torque

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0, \end{cases} \tag{13}$$

studied in H. Mounier, J. Rudolph, M. Petitot, M. Fliess, "A flexible rod as a linear delay system", in *Proceedings of* 3^{rd} *European Control Conference*, Rome (Italy), 1995.

Let $D = \mathbb{Q}\left[\partial; \mathrm{id}, \frac{d}{dt}\right][\delta; \alpha, 0]$ be the commutative polynomial ring of OD time-delay operators with rational constant coefficients,

$$R = \begin{pmatrix} \partial & -\partial\,\delta & -1 \\ 2\,\partial\,\delta & -\partial\,\delta^2 - \partial & 0 \end{pmatrix} \in D^{2\times 3}$$

the presentation matrix of (13) and $M = D^{1\times 3}/(D^{1\times 2}\, R)$ the D-module finitely presented by R.

1. Using OreMorphims, prove that M can be decomposed.

2. Using OreMorphims, prove that R is equivalent to the following block-diagonal matrix

$$\overline{R} = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where:

$$U = \begin{pmatrix} -2\,\delta & 1+\delta^2 & 0 \\ 1 & -\dfrac{\delta}{2} & 0 \\ \partial & -\partial\,\delta & -1 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(D).$$

3. Integrating the trivial linear OD system $\overline{R}\,\overline{\eta} = 0$, prove that the general solution of the linear OD time-delay system $R\,\eta = 0$ is defined by

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,c - z_3(t-2) - z_3(t) \\ c - 2\,z_3(t-1) \\ \dot{z}_3(t-2) - \dot{z}_3(t) \end{pmatrix},$$

where c is an arbitrary real constant and z_3 an arbitrary smooth function,

Exercise 27 We consider again Example 19.

1. Using the command SyzygyModule, compute $\ker_D(.\overline{P})$, $\mathrm{im}_D(.\overline{P}) = \ker_D(.(I_2 - \overline{P}))$, $\ker_D(.\overline{Q})$ and $\mathrm{im}_D(.\overline{Q}) = \ker_D(.(1 - \overline{Q}))$.

2. Check that depending on the $\overline{P}$'s, either $\ker_D(.\overline{P})$ or $\mathrm{im}_D(.\overline{P})$ is not a free left D-module.

   Hint. We recall that we can prove that the left $D = A_1(\mathbb{Q})$-module $D^{1\times 2}/(D\,(\partial \quad -t))$ is not free.

3. Conclude that R is not equivalent to a matrix of the form $\overline{R} = (\alpha \quad 0)$, where $\alpha \in D$, over D.

4. Consider the $E = B_1(\mathbb{Q}) = \mathbb{Q}(t)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$-module $N = E^{1\times 2}/(E\,R) = E \otimes_D M$. Show that $\overline{P}$ and $\overline{Q}$ define an idempotent of the ring $\mathrm{end}_E(N)$, i.e., N is a decomposable left E-module.

5. Using the command SyzygyModuleRat, compute $\ker_E(.\overline{P})$, $\mathrm{im}_E(.\overline{P}) = \ker_E(.(I_2 - \overline{P}))$, $\ker_E(.\overline{Q})$ and $\mathrm{im}_E(.\overline{Q}) = \ker_E(.(1 - \overline{Q}))$ and prove that they are free left E-modules.

6. Conclude that R is equivalent to $\overline{R} = R\,U^{-1} = (\partial \quad 0)$ over E, where:

$$U^{-1} = \begin{pmatrix} t & 1 \\ \partial & \dfrac{1}{t}\,\partial \end{pmatrix}.$$

   Note the singularity of U^{-1} at $t = 0$.

7. However, since M can be decomposed over D, following Exercise 14, prove that the general solution $\eta \in \mathcal{F}^2$ of $R\,\eta = 0$, where $\mathcal{F} = C^\infty(\mathbb{R}_+)$, is defined by:

$$\forall\, \xi_1,\, \xi_2 \in \mathcal{F},\ \forall\, c \in \mathbb{R}, \quad \begin{cases} \eta_1(t) = c\,t + t^2\,\xi_1(t) + t\,\dot{\xi}_2(t) - \xi_2(t), \\ \eta_2(t) = t\,\dot{\xi}_1(t) + 2\,\xi_1(t) + \ddot{\xi}_2(t). \end{cases}$$

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