Factorization, reduction and decomposition problems

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Factorization, reduction and decomposition problems

- Let D be an Ore algebra.
- Let $R \in D^{q \times p}$ be a matrix of functional operators.
- Questions:

1. $\exists \; R_1 \in D^{r \times p}, \; R_2 \in D^{q \times r} : \quad R = R_2 \, R_1 \, ?$

2. $\exists \; W \in \mathrm{GL}_p(D), \; V \in \mathrm{GL}_q(D)$ s.t. $V \, R \, W = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}?$

3. $\exists \; W \in \mathrm{GL}_p(D), \; V \in \mathrm{GL}_q(D)$ s.t. $V \, R \, W = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}?$

Outline

• Type of systems: OD and PD/difference/differential time-delay... linear systems: linear functional systems.

• General topic: algebraic study of linear functional systems coming from mathematical physics, engineering sciences, control theory...

• Techniques: module theory and homological algebra.

• Applications: equivalences of systems, symmetries, quadratic first integrals/conservation laws, decoupling problems...

• Implementation: package OREMORPHISMS:

http://www-sop.inria.fr/members/Alban.Quadrat/OreMorphisms/index.html.

Jacobson/Smith normal form

- Let D be a principal ideal domain.

- Theorem: $\forall\, R \in D^{q \times p}, \quad \exists\, V \in \mathrm{GL}_q(D), \quad \exists\, U \in \mathrm{GL}_p(D)$:

$$\overline{R} = V\,R\,U = \begin{pmatrix} \alpha_1 & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \alpha_2 & \ddots & & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & \alpha_r & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 \\ \vdots & & & & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 \end{pmatrix},$$

where $\alpha_1 \,||\, \alpha_2 \,||\, \dots \,||\, \alpha_r \neq 0$.

Example: Smith normal form

- Let us consider 2 pendulum of the same length mounted on a car:

$$\begin{cases} \ddot{x}_1(t) + \alpha\, x_1(t) - \alpha\, u(t) = 0, \\ \ddot{x}_2(t) + \alpha\, x_2(t) - \alpha\, u(t) = 0, \end{cases} \qquad \alpha = \frac{g}{l}.$$

- Let us consider the principal ideal domain $D = \mathbb{Q}(\alpha)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$.

$$P = \sum_{i=0}^{n} a_i\, \partial^i \in D, \quad a_i \in \mathbb{Q}(\alpha).$$

$$\begin{pmatrix} -\alpha & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \partial^2 + \alpha & 0 & -\alpha \\ 0 & \partial^2 + \alpha & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & \partial^2 + \alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial^2 + \alpha & 0 \end{pmatrix}.$$

Example: Jacobson normal form

- Let us consider the time-varying linear system:

$$\begin{cases} t\,\dot{y}_1(t) - y_1(t) - t^2\,\dot{y}_2(t) + u_1(t) = 0, \\ \dot{y}_1(t) + t\,\dot{y}_2(t) - y_2(t) + u_2(t) = 0. \end{cases}$$

- We consider the left principal ideal domain $D = \mathbb{Q}(t)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$.

$$\begin{pmatrix} t\,\partial - 1 & -t^2\,\partial & 1 & 0 \\ \partial & t\,\partial - 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -t\,\partial + 1 & t^2\,\partial \\ 0 & 1 & -\partial & -t\,\partial + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

- Implementation in the package JACOBSON.

Eigenring

- Let us consider the first order OD system:

$$\partial\, y = E(t)\, y \quad (\star)$$

- Does it exist an invertible change of variables $y = P(t)\, z$ s.t.

$$(\star) \quad \Leftrightarrow \quad \partial\, z = F(t)\, z, \quad F = P^{-1}\,(E\, P - \partial P),$$

is either of the form:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} \quad \text{or} \quad F = \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}?$$

$$\partial\, I - F(t) = \begin{pmatrix} \partial\, I - F_{11} & -F_{12} \\ 0 & \partial\, I - F_{22} \end{pmatrix} \text{ or } \begin{pmatrix} \partial\, I - F_{11} & 0 \\ 0 & \partial\, I - F_{22} \end{pmatrix}.$$

- If $E(t) = E \in \mathbb{R}^{n \times n}$, then $F = P^{-1}\, E\, P$: Jordan normal form.

Eigenring: example

- Let us consider the system $\dot{y}(t) = E(t)\, y(t)$, where:

$$E(t) = \begin{pmatrix} t\,(2\,t+1) & -2\,t^3 - 2\,t^2 + 1 \\ 2\,t & -t\,(2\,t+1) \end{pmatrix}.$$

- The eigenring of the system $\partial y(t) = E(t)\, y(t)$ is:

$$\mathcal{E} = \{P \in \mathbb{Q}(t)^{2\times 2} \mid \dot{P}(t) = E(t)\, P(t) - P(t)\, E(t)\}.$$

- Computing the rational solutions of $\dot{P} = [E, P]$, we then get:

$$\mathcal{E} = \left\{ P = \begin{pmatrix} a_1 - a_2\,(t+1) & a_2\,t\,(t+1) \\ -a_2 & a_2\,t + a_1 \end{pmatrix} \mid a_1,\ a_2 \in \mathbb{Q} \right\}.$$

- P is isospectral because (E, P) is a Lax pair:

$$\det(P - \lambda\, I_2) = (\lambda - a_1)\,(\lambda - a_1 + a_2).$$

Eigenring: example

- Computing a Jordan normal form of P, we obtain

$$J = V^{-1}\,P\,V = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 - a_2 \end{pmatrix},$$

$$V = \begin{pmatrix} -t & 1+t \\ -1 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -(t+1) \\ 1 & -t \end{pmatrix}.$$

- Let us denote by $z = V^{-1}\,y = (y_1 - (t+1)\,y_2 \quad y_1 - t\,y_2)^T$:

$$\dot{y}(t) = E(t)\,y(t) \quad \Leftrightarrow \quad \dot{z}(t) = \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} z(t)$$

$$\Rightarrow \begin{cases} z_1(t) = C_1\,e^{-t^2/2}, \\ z_2(t) = C_2\,e^{t^2/2}, \end{cases} \Rightarrow \begin{cases} y_1(t) = -C_1\,t\,e^{-t^2/2} + C_2\,(t+1)\,e^{t^2/2}, \\ y_2(t) = -C_1\,e^{-t^2/2} + C_2\,e^{t^2/2}. \end{cases}$$

Finitely presented left D-modules

- Let D be an Ore algebra, $R \in D^{q \times p}$ and a left D-module $\mathcal{F}$.
- Let us consider the system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$.
- Let us consider the left D-homomorphism:

$$\begin{array}{ccc} D^{1\times q} & \longrightarrow & D^{1\times p} \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda\, R. \end{array}$$

- As in number theory or algebraic geometry, we associate with the system $\ker_{\mathcal{F}}(R.)$ the finitely presented left D-module:

$$M = D^{1\times p}/(D^{1\times q}\, R).$$

- Theorem: (Malgrange) We have the following isomorphism:

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}) = \{f : M \to \mathcal{F} \mid f \text{ left } D\text{-linear}\}.$$

Homomorphims of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- Let us consider the finitely presented left D-modules:

$$M = D^{1 \times p}/(D^{1 \times q}\, R), \quad M' = D^{1 \times p'}/(D^{1 \times q'}\, R').$$

- $\hom_D(M, M')$: abelian group of D-morphisms from M to M':

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ & & & & \downarrow f & \\ D^{1 \times q'} & \xrightarrow{.R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0. \end{array}$$

Homomorphims of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- We have the following commutative exact diagram:

$$\begin{array}{ccccccc}
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow .Q & & \downarrow .P & & \downarrow f & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
\end{array}$$

$\exists\, f : M \rightarrow M' \Leftrightarrow \exists\, P \in D^{p \times p'}$, $Q \in D^{q \times q'}$ such that:

$$R\,P = Q\,R'.$$

Moreover, we have $f(\pi(\lambda)) = \pi'(\lambda\,P)$, for all $\lambda \in D^{1\times p}$.

Eigenring: $\partial y = E\, y$ & $\partial z = F\, z$

- $D = A[\partial; \alpha, \beta]$, $E,\ F \in A^{p \times p}$, $R = \partial\, I_p - E$, $R' = \partial\, I_p - F$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial_p\, I - E)} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\
0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - F)} & D^{1\times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
\end{array}$$

$$(\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \Leftrightarrow \left\{ \begin{array}{l} \alpha(P) = Q \in A^{p \times p}, \\ \beta(P) = E\, P - \alpha(P)\, F. \end{array} \right.$$

If $P \in A^{p \times p}$ is invertible, we then have:

$$F = -\alpha(P)^{-1}(\beta(P) - E\, P).$$

Eigenring: $\partial y = E\, y$ & $\partial z = F\, z$

- $D = A[\partial; \alpha, \beta]$, $\quad E,\, F \in A^{p \times p}$, $R = \partial\, I_p - E$, $R' = \partial\, I_p - F$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial_p\, I - E)} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - F)} & D^{1\times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

$$(\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \;\Leftrightarrow\; \left\{ \begin{array}{l} \alpha(P) = Q \in A^{p \times p}, \\ \beta(P) = E\, P - \alpha(P)\, F. \end{array} \right.$$

If $P \in A^{p \times p}$ is invertible, we then have:

$$F = -\alpha(P)^{-1} (\beta(P) - E\, P).$$

- Differential case: $\beta = \frac{d}{dt}$, $\alpha = \mathrm{id}$:

$$\dot{P} = E\, P - P\, F, \quad F = -P^{-1} (\dot{P} - E\, P).$$

Eigenring: $\partial y = E\, y$ & $\partial z = F\, z$

- $D = A[\partial; \alpha, \beta]$, $\quad E,\, F \in A^{p \times p}$, $R = \partial\, I_p - E$, $R' = \partial\, I_p - F$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial_p\, I - E)} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ 0 & \longrightarrow & D^{1\times p} & \xrightarrow{.(\partial\, I_p - F)} & D^{1\times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

$$(\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \;\Leftrightarrow\; \left\{ \begin{array}{l} \alpha(P) = Q \in A^{p \times p}, \\ \beta(P) = E\, P - \alpha(P)\, F. \end{array} \right.$$

If $P \in A^{p \times p}$ is invertible, we then have:

$$F = -\alpha(P)^{-1} (\beta(P) - E\, P).$$

- Discrete case: $\beta = 0$, $\alpha(k) = k + 1$:

$$E_k\, P_k - P_{k+1}\, F_k = 0, \quad F_k = P_{k+1}^{-1}\, E_k\, P_k.$$

Example: Lax pairs for the KdV equation

- Let us consider the differential ring $\mathbb{Q}\{u\}$ formed by differential polynomials in u, the prime differential ideal of $\mathbb{Q}\{u\}$ defined by

$$\mathfrak{p} = \left\{ \frac{\partial u}{\partial t} - 6\, u \left(\frac{\partial u}{\partial x} \right) + \frac{\partial^3 u}{\partial x^3} \right\},$$

the differential ring $L = \mathbb{Q}\{u\}/\mathfrak{p}$ and $K = \{n/d \mid 0 \neq d,\, n \in L\}$ the differential field defined by the KdV equation.

- Let us consider the rings $A = K\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]$, $D = A\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]$,

$$\left\{ \begin{array}{l} E = -4\, \partial_x^3 + 6\, u\, \partial_x + 3 \left(\dfrac{\partial\, u}{\partial\, x} \right) \in D, \\ R = \partial_t - E \in D, \end{array} \right. \qquad M = D/(D\, R).$$

- The Schrödinger operator $P = -\partial_x^2 + u$ with potential u satisfies:

$$R\, P - P\, R = \partial_t\, P - E\, P + P\, E = \frac{\partial u}{\partial t} - 6\, u \left(\frac{\partial u}{\partial x} \right) - \frac{\partial^3 u}{\partial x^3} = 0.$$

Example: Lax pairs for the KdV equation

In the inverse scattering theory, a key point is that the smooth one-parameter family of differential operators

$$t \longmapsto -\partial_x^2 + u(x,t)$$

defines an isospectral flow on the solutions of $\partial_t \eta = E \eta$:

$$(-\partial_x^2 + u(x,0))\,\psi(x) = \lambda\,\psi(x),$$

$$\begin{cases} \partial_t \eta(x,t) = E\,\eta(x,t), \quad E = -4\,\partial_x^3 + 6\,u\,\partial_x + 3\left(\dfrac{\partial\,u}{\partial\,x}\right), \\ \eta(x,0) = \psi(x), \end{cases}$$

$$\Rightarrow (-\partial_x^2 + u(x,t))\,\eta(x,t) = \lambda\,\eta(x,t),$$

$\Rightarrow$ the inverse scattering method proves that the KdV equation is completely integrable.

Computation of $\hom_D(M, M')$

- Problem: Given $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, find $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying the relation $R\, P = Q\, R'$.

- If D is a commutative ring, then $\hom_D(M, M')$ is a D-module.

- The Kronecker product of $E \in D^{q \times p}$ and $F \in D^{r \times s}$ is:

$$E \otimes F = \begin{pmatrix} E_{11}\, F & \dots & E_{1p}\, F \\ \vdots & \vdots & \vdots \\ E_{q1}\, F & \dots & E_{qp}\, F \end{pmatrix} \in D^{(q\, r) \times (p\, s)}.$$

Lemma: If $U \in D^{a \times b}$, $V \in D^{b \times c}$ and $W \in D^{c \times d}$, then we have:

$$\mathrm{row}(U\, V\, W) = \mathrm{row}(V)\, (U^T \otimes W).$$

$$\mathrm{row}(R\, P\, I_{p'}) = \mathrm{row}(P)\, (R^T \otimes I_{p'}),\ \mathrm{row}(I_q\, Q\, R') = \mathrm{row}(Q)\, (I_q \otimes R'),$$

$$\Rightarrow (\mathrm{row}(P) \ \ -\mathrm{row}(Q)) \begin{pmatrix} R^T \otimes I_{p'} \\ -I_q \otimes R' \end{pmatrix} = 0.$$

Example: Tank model (Dubois-Petit-Rouchon, ECC99)

- Let $D = \mathbb{Q}[\partial, \delta]$ be the commutative polynomial ring and $M = D^{1\times 3}/(D^{1\times 2}\, R)$ the D-module finitely presented by:

$$R = \begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix} \in D^{2\times 3}.$$

- The D-module $\mathrm{end}_D(M)$ is defined by:

$$\begin{aligned} P_\alpha \;&=\; \left(\begin{matrix} \alpha_1 \\ \alpha_2 + 2\,\alpha_4\,\partial + 2\,\alpha_5\,\partial\,\delta \\ \alpha_4\,\delta + \alpha_5 \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} \alpha_2 & 2\,\alpha_3\,\partial\,\delta \\ \alpha_1 - 2\,\alpha_4\,\partial - 2\,\alpha_5\,\partial\,\delta & 2\,\alpha_3\,\partial\,\delta \\ -\alpha_4\,\delta - \alpha_5 & \alpha_1 + \alpha_2 + \alpha_3\,(\delta^2 + 1) \end{matrix}\right), \end{aligned}$$

$$Q_\alpha = \begin{pmatrix} \alpha_1 - 2\,\alpha_4\,\partial & \alpha_2 + 2\,\alpha_4\,\partial \\ \alpha_2 + 2\,\alpha_5\,\partial\,\delta & \alpha_1 - 2\,\alpha_5\,\partial\,\delta \end{pmatrix}, \quad \forall\, \alpha_1, \ldots, \alpha_5 \in D.$$

Computation of $\hom_D(M, M')$

• If D is a non-commutative ring, then $\hom_D(M, M')$ is an abelian group and generally an infinite-dimensional k-vector space.

$\Rightarrow$ Find a k-basis of morphisms with given degrees in x_i and in ∂_j:

1. Take an ansatz for P of fixed degrees.
2. Compute $R\,P$ and a Gröbner basis G of the rows of R'.
3. Reduce the rows of $R\,P$ w.r.t. G.
4. Solve the system on the coefficients of the ansatz so that all the normal forms vanish.
5. Substitute the solutions in P and compute Q by means of a factorization.

Example: OD system

- Let $D = \mathbb{Q}[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ and $M = D^{1\times 4}/(D^{1\times 4}\,R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4\times 4}.$$

- $f \in \mathrm{end}_D(M)$ is defined by (P, P) where $P \in \mathbb{Q}[t]^{4\times 4}$ satisfies

$$P = \begin{pmatrix} a_4 - 2\,a_2\,t^2 & a_1 + a_5\,t^2 + a_3\,t^4 & 0 & 0 \\ -4\,a_3 & a_4 + 2\;a_5 + 2\;a_3\,t^2 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix},$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Q}$, i.e., $R\,P = P\,R$.

Euler-Tricomi equation

- Let us consider the Euler-Tricomi equation (transonic flow):

$$\partial_1^2\, u(x_1, x_2) - x_1\, \partial_2^2\, u(x_1, x_2) = 0.$$

- Let $D = A_2(\mathbb{Q})$, $R = (\partial_1^2 - x_1\, \partial_2^2) \in D$ and $M = D/(D\, R)$.
  - $\mathrm{end}_D(M)_{1,1}$ is defined by:

$$\begin{cases} P = a_1 + a_2\, \partial_2 + \frac{3}{2}\, a_3\, x_2\, \partial_2 + a_3\, x_1\, \partial_1, \\ Q = (a_1 + 2\, a_3) + a_2\, \partial_2 + \frac{3}{2}\, a_3\, x_2\, \partial_2 + a_3\, x_1\, \partial_1, \end{cases}$$

  - $\mathrm{end}_D(M)_{2,0}$ is defined by $P = Q = a_1 + a_2\, \partial_2 + a_3\, \partial_2^2$.
  - $\mathrm{end}_D(M)_{2,1}$ is defined by:

$$\begin{cases} P = & a_1 + a_2\, \partial_2 + \frac{3}{2}\, a_3\, x_2\, \partial_2 + a_3\, x_1\, \partial_1 \\ & + a_4\, \partial_2^2 + \frac{3}{2}\, a_5\, x_2\, \partial_2^2 + a_5\, x_1\, \partial_1\, \partial_2, \\ Q = & (a_1 + 2\, a_3) + a_2\, \partial_2 + \frac{3}{2}\, a_3\, x_2\, \partial_2 + a_3\, x_1\, \partial_1 \\ & + a_4\, \partial_2^2 + a_5\, x_1\, \partial_1\, \partial_2 + 2\, a_5\, \partial_2 + \frac{3}{2}\, a_5\, x_2\, \partial_2^2. \end{cases}$$

Galois-like transformations

We have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 & \\ \downarrow .Q & & \downarrow .P & & \downarrow f & & (\star) \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0. & \end{array}$$

If $\mathcal{F}$ is a left D-module, then, applying the functor $\hom_D(\cdot, \mathcal{F})$ to $(\star)$, we obtain the following commutative exact diagram:

$$\begin{array}{ccccccc} 0 = Q\,(R'\,y) = R\,(P\,y) & \longleftarrow\!\shortmid & P\,y & & & & \\ \mathcal{F}^{q} & \xleftarrow{R.} & \mathcal{F}^{p} & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0 \\ \uparrow Q. & & \uparrow P. & & \uparrow f^\star & & \\ \mathcal{F}^{q'} & \xleftarrow{R'.} & \mathcal{F}^{p'} & \longleftarrow & \ker_{\mathcal{F}}(R'.) & \longleftarrow & 0. \\ 0 = R'\,y & \longleftarrow\!\shortmid & y & & & & \end{array}$$

$\Rightarrow$ $f^\star$ sends $\ker_{\mathcal{F}}(R.')$ to $\ker_{\mathcal{F}}(R.)$.

$(R' = R$: Galois-like transformations$)$.

Example: Linear elasticity

- Consider the Killing operator for the euclidian metric defined by:

$$R = \begin{pmatrix} \partial_1 & 0 \\ \partial_2/2 & \partial_1/2 \\ 0 & \partial_2 \end{pmatrix}.$$

- The system $R\,y = 0$ admits the following general solution:

$$y = \begin{pmatrix} c_1\,x_2 + c_2 \\ -c_1\,x_1 + c_3 \end{pmatrix}, \quad c_1,\, c_2,\, c_3 \in \mathbb{R}. \quad (\star)$$

- $\mathrm{end}_D(M)$, where $M = D^{1\times 2}/(D^{1\times 3}\,R)$, is defined by:

$$P = \begin{pmatrix} \alpha_1 & \alpha_2\,\partial_1 \\ 0 & 2\,\alpha_3\,\partial_1 + \alpha_1 \end{pmatrix}, \quad \alpha_1,\, \alpha_2,\, \alpha_3 \in D.$$

- Applying P to $(\star)$, we get the new solution:

$$\overline{y} = P\,y = \begin{pmatrix} \alpha_1\,c_1\,x_2 + \alpha_1\,c_2 - \alpha_2\,c_1 \\ -\alpha_1\,c_1\,x_1 + \alpha_1\,c_3 - 2\,\alpha_3\,c_1 \end{pmatrix}, \text{ i.e., } R\,\overline{y} = 0.$$

Formal adjoint

- Let $D = A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$ be the ring of differential operators with coefficients in A (e.g., $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$).

- The formal adjoint $\widetilde{R} \in D^{p \times q}$ of $R \in D^{q \times p}$ is defined by:

$$(\lambda, R\,\eta) = (\widetilde{R}\,\lambda, \eta) + \sum_{i=1}^{n} \partial_i\, \Phi_i(\lambda, \eta).$$

- The formal adjoint $\widetilde{R}$ can be defined by $\widetilde{R} = (\theta(R_{ij}))^T \in D^{p \times q}$, where $\theta : D \longrightarrow D$ is the involution defined by:

1. $\forall\, a \in A, \quad \theta(a) = a.$
2. $\theta(\partial_i) = -\partial_i, \quad i = 1, \ldots, n.$

Involution: $\theta^2 = \mathrm{id}_D$, $\forall\, P_1, P_2 \in D$: $\theta(P_1\, P_2) = \theta(P_2)\, \theta(P_1)$.

Quadratic conservation laws

- Let us consider the left D-modules:

$$M = D^{1\times p}/(D^{1\times q}\, R), \quad \widetilde{N} = D^{1\times q}/(D^{1\times p}\, \widetilde{R}).$$

- Let $f : \widetilde{N} \longrightarrow M$ be a homomorphism defined by P and Q.
- Let $\mathcal{F}$ be a left D-module and the commutative exact diagram:

$$\begin{array}{ccccccc}
\mathcal{F}^p & \xleftarrow{\widetilde{R}.} & \mathcal{F}^q & \longleftarrow & \ker_{\mathcal{F}}(\widetilde{R}.) & \longleftarrow & 0 \\
\uparrow Q. & & \uparrow P. & & \uparrow f^\star & & \\
\mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0.
\end{array}$$

- $\eta \in \mathcal{F}^p$ solution of $R\,\eta = 0 \;\Rightarrow\; \lambda = P\,\eta$ is a solution of $\widetilde{R}\,\lambda = 0$.

$$\Rightarrow\; (P\,\eta, R\,\eta) - (\widetilde{R}\,(P\,\eta), \eta) \;=\; \sum_{i=1}^{n} \partial_i\, \Phi_i(P\,\eta, \eta) = 0,$$

i.e., $\Phi = (\Phi_1(P\,\eta, \eta), \ldots, \Phi_n(P\,\eta, \eta))^T$ satisfies $\mathrm{div}\,\Phi = 0$.

Example: Hydrodynamics

- The movement of an incompressible rotating fluid with a rotation axis lies along the x_3 axis and a small velocity is defined by:

$$\left\{\begin{array}{l} \rho_0\,\partial_t u_1 - 2\,\rho_0\,\Omega_0\,u_2 + \partial_1 p = 0, \\ \rho_0\,\partial_t u_2 + 2\,\rho_0\,\Omega_0\,u_1 + \partial_2 p = 0, \\ \rho_0\,\partial_t u_3 + \partial_3 p = 0, \\ \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0, \end{array}\right.$$

$u = (u_1 \quad u_2 \quad u_3)^T$: local rate of velocity, p: pressure, ρ_0: constant fluid density, Ω_0: constant angle speed.

- We have: $R = \begin{pmatrix} \rho_0\,\partial_t & -2\,\rho_0\,\Omega_0 & 0 & \partial_1 \\ 2\,\rho_0\,\Omega_0 & \rho_0\,\partial_t & 0 & \partial_2 \\ 0 & 0 & \rho_0\,\partial_t & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix} = -\widetilde{R}.$

Example: Hydrodynamics

- $\widetilde{R} = -R$ implies that if $(\vec{u}, p)$ is a solution of the system, so is:

$$\lambda_1 = u_1, \quad \lambda_2 = u_2, \quad \lambda_3 = u_3, \quad \lambda_4 = p.$$

- Denote by $\xi = (u_1 \quad u_2 \quad u_2 \quad p)^T$. We have the identity:

$$(\lambda, R\,\xi) = (\xi, \widetilde{R}\,\lambda) + (\partial_t \quad \partial_1 \quad \partial_2 \quad \partial_3) \begin{pmatrix} \rho_0\,(\lambda_1\,u_1 + \lambda_2\,u_2 + \lambda_3\,u_3) \\ \lambda_1\,p + \lambda_4\,u_1 \\ \lambda_2\,p + \lambda_4\,u_2 \\ \lambda_3\,p + \lambda_4\,u_3 \end{pmatrix}$$

- If we take $\lambda = \xi$, then we get $\widetilde{R}\,\lambda = 0$ and

$$\partial_t\,(\rho_0\,(u_1^2 + u_2^2 + u_3^2)) + \partial_1\,(2\,p\,u_1) + \partial_2\,(2\,p\,u_2) + \partial_3\,(2\,p\,u_3) = 0,$$

i.e., we obtain the quadratic conservation of law:

$$\partial_t\,\left(\frac{1}{2}\,\rho_0\,\parallel \vec{u} \parallel^2\right) + \mathrm{div}\,(p\,\vec{u}) = 0.$$

Example: Electromagnetism

- Let us consider the Maxwell equations in the vaccum:

$$\begin{cases} \partial_t \vec{B} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \dfrac{1}{\mu_0} \vec{\nabla} \wedge \vec{B} - \epsilon_0 \, \partial_t \vec{E} = \vec{0}, \end{cases}$$

where $\vec{B}$ (resp., $\vec{E}$): magnetic (resp., electric) field, μ_0 (resp., ϵ_0): magnetic (resp., electric) constant.

- Let us consider $D = \mathbb{Q}(\mu_0, \epsilon_0)[\partial_t, \partial_1, \partial_2, \partial_3]$ and the matrix:

$$R = \begin{pmatrix} \partial_t & 0 & 0 & 0 & -\partial_3 & \partial_2 \\ 0 & \partial_t & 0 & \partial_3 & 0 & -\partial_1 \\ 0 & 0 & \partial_t & -\partial_2 & \partial_1 & 0 \\ 0 & -\partial_3/\mu_0 & \partial_2/\mu_0 & -\epsilon_0 \, \partial_t & 0 & 0 \\ \partial_3/\mu_0 & 0 & -\partial_1/\mu_0 & 0 & -\epsilon_0 \, \partial_t & 0 \\ -\partial_2/\mu_0 & \partial_1/\mu_0 & 0 & 0 & 0 & -\epsilon_0 \, \partial_t \end{pmatrix}.$$

Example: Electromagnetism

$$\widetilde{R} = \begin{pmatrix} -\partial_t & 0 & 0 & 0 & -\partial_3/\mu_0 & \partial_2/\mu_0 \\ 0 & -\partial_t & 0 & \partial_3/\mu_0 & 0 & -\partial_1/\mu_0 \\ 0 & 0 & -\partial_t & -\partial_2/\mu_0 & \partial_1/\mu_0 & 0 \\ 0 & -\partial_3 & \partial_2 & \epsilon_0\,\partial_t & 0 & 0 \\ \partial_3 & 0 & -\partial_1 & 0 & \epsilon_0\,\partial_t & 0 \\ -\partial_2 & \partial_1 & 0 & 0 & 0 & \epsilon_0\,\partial_t \end{pmatrix}.$$

- $\xi = (B_1 \;\; B_2 \;\; B_3 \;\; E_1 \;\; E_2 \;\; E_3)^T$, $\lambda = (C_1 \;\; C_2 \;\; C_3 \;\; F_1 \;\; F_2 \;\; F_3)^T$.
- We have the differential relation:

$$(\lambda, R\,\xi) = (\xi, \widetilde{R}\,\lambda) \; + \partial_t \left(\sum_{i=1}^3 C_i\,B_i - \epsilon_0\,\sum_{i=1}^3 F_i\,E_i\right)$$
$$+\vec{\nabla}.\begin{pmatrix} C_3\,E_2 - C_2\,E_3 + (F_3\,B_2 - F_2\,B_3)/\mu_0 \\ C_1\,E_3 - C_3\,E_1 + (F_1\,B_3 - F_3\,B_1)/\mu_0 \\ C_2\,E_1 - C_1\,E_2 + (F_2\,B_1 - F_1\,B_2)/\mu_0 \end{pmatrix}.$$

Example: Electromagnetism

- Let us consider $M = D^{1\times 6}/(D^{1\times 6}\, R)$ and $\widetilde{N} = D^{1\times 6}/(D^{1\times 6}\, \widetilde{R})$.
- A homomorphism $f \in \hom_D(\widetilde{N}, M)$ is defined by:

$$P = \begin{pmatrix} 1/\mu_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\mu_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q = -P.$$

- If ξ is a solution of the system, then $\lambda = P\,\xi$, i.e.,

$$C_i = B_i/\mu_0, \quad F_i = -E_i, \quad i = 1, 2, 3,$$

is a solution of $\widetilde{R}\,\lambda = 0$. Then, we obtain the conservation law:

$$\partial_t \underbrace{\left(\frac{1}{\mu_0} \parallel \vec{B} \parallel^2 + \epsilon_0 \parallel \vec{E} \parallel^2\right)}_{\text{electromagnetic energy}} + \mathrm{div} \underbrace{\left(\frac{1}{\mu_0}(\vec{E} \wedge \vec{B})\right)}_{\text{Poynting vector}} = 0.$$

Kernel and factorization

$$\begin{array}{ccccccc}
 & & \color{red}{\lambda} & \longmapsto & \color{red}{y} & & \\
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow .Q & & \downarrow .P & & \downarrow f & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \\
\color{red}{\exists\, \mu} & \longmapsto & \color{red}{\mu\, R' = \lambda\, P} & \longmapsto & \color{red}{0} & &
\end{array}$$

- $\ker_D\left(.\left(\begin{array}{c} P \\ R' \end{array}\right)\right) = D^{1\times r}\,(S \quad -T)$

$$\Rightarrow \{\lambda \in D^{1\times p} \,|\, \lambda\, P \in D^{1\times q}\, R'\} = D^{1\times r}\, S$$

$$\Rightarrow \color{red}{\ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R)}.$$

- $(D^{1\times q}\,(R \quad -Q)) \in \ker_D\left(.\left(\begin{array}{c} P \\ R' \end{array}\right)\right) \Rightarrow (D^{1\times q}\, R) \subseteq (D^{1\times r}\, S).$

$$\exists\, V \in D^{q\times r} : \color{red}{R = V\, S}.$$

Example: Linearized Euler equations

- Let $R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix}$ over $D = \mathbb{Q}[\partial_t, \partial_1, \partial_2, \partial_3]$.
- Let us consider $f \in \mathrm{end}_D(M)$ defined by:

$$P = \begin{pmatrix} 0 & \partial_3 & -\partial_2 & 0 \\ -\partial_3 & 0 & \partial_1 & 0 \\ \partial_2 & -\partial_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_3 & -\partial_2 \\ 0 & -\partial_3 & 0 & \partial_1 \\ 0 & \partial_2 & -\partial_1 & 0 \end{pmatrix}.$$

- Computing $\ker_D\left(.\begin{pmatrix} P \\ -R \end{pmatrix}\right)$ and factorizing R by S, we get:

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & -1 & \partial_3 \end{pmatrix}, \quad S = \begin{pmatrix} -\partial_t & 0 & 0 & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & -\partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example: Linearized Euler equations

- We have $R = V\,S$ where:

$$\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & -1 & \partial_3 \end{pmatrix} \begin{pmatrix} -\partial_t & 0 & 0 & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 \\ 0 & \partial_t & 0 & 0 \\ 0 & 0 & -\partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The solutions of $S\,y = 0$ are particular solutions of $R\,y = 0$.

- Integrating S, we obtain the following solutions of $R\,y = 0$:

$$\left\{ \begin{array}{l} \vec{v}(x,t) = \mathrm{curl}\,\vec{\psi}(x), \\ p(x,t) = 0, \end{array} \right. \qquad \forall\; \vec{\psi} = (\psi_1,\,\psi_2,\,\psi_3)^T \in C^{\infty}(\Omega)^3.$$

Ker f, im f, coim f and coker f

• Proposition: Let $M = D^{1\times p}/(D^{1\times q}\, R)$, $M' = D^{1\times p'}/(D^{1\times q'}\, R')$ and $f : M \longrightarrow M'$ be a homomorphism defined by $R\,P = Q\,R'$.

Let us consider the matrices $S \in D^{r\times p}$, $T \in D^{r\times q'}$, $U \in D^{s\times r}$ and $V \in D^{q\times r}$ satisfying $R = V\,S$, $\ker_D(.S) = D^{1\times s}\,U$ and:

$$\ker_D\left(.\begin{pmatrix} P \\ R' \end{pmatrix}\right) = D^{1\times r}\,(S \quad -T).$$

Then, we have:

- $\ker f = (D^{1\times r}\,S)/(D^{1\times q}\,R) \cong D^{1\times l}/\left(D^{1\times(q+s)}\begin{pmatrix} U \\ V \end{pmatrix}\right)$,
- $\mathrm{coim}\, f = M/\ker f = D^{1\times p}/(D^{1\times r}\,S)$,
- $\mathrm{im}\, f = D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}/(D^{1\times q}\,R) \cong D^{1\times p}/(D^{1\times r}\,S)$,
- $\mathrm{coker}\, f = M'/\mathrm{im}\, f = D^{1\times p'}/\left(D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}\right)$.

- Corollary: Let us consider $f \in \hom_D(M, M')$. Then:

1. f is injective iff one of the assertions holds:
   - There exists $L \in D^{r \times q}$ such that $S = L\,R$.
   - $\left(\begin{array}{c} U \\ V \end{array}\right)$ admits a left-inverse over D.
2. f is surjective iff $\left(\begin{array}{c} P \\ R' \end{array}\right)$ admits a left-inverse over D.
3. f is an isomorphism, i.e., $M \cong M'$, iff 1 and 2 are satisfied.

Example

- Equivalence of the systems defined by the following R and R'?

$$R = \begin{pmatrix} \partial_1^2\,\partial_2^2 - 1 & -\partial_1\,\partial_2^3 - \partial_2^2 \\ \partial_1^3\,\partial_2 - \partial_1^2 & -\partial_1^2\,\partial_2^2 \end{pmatrix}, \quad R' = (\partial_1\,\partial_2 - 1 \quad -\partial_2^2).$$

- We find a homomorphism $f \in \hom_D(M, M')$ defined by:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 + \partial_1\,\partial_2 \\ \partial_1^2 \end{pmatrix}.$$

- $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 + \partial_1\,\partial_2 \\ \partial_1^2 \end{pmatrix}$ admits the left-inverse $(1 - \partial_1\,\partial_2 \quad \partial_2^2)$.
- $\begin{pmatrix} P \\ R' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1\,\partial_2 - 1 & -\partial_2^2 \end{pmatrix}$ admits the left-inverse $(I_2 \quad 0)$.

$$\Rightarrow M = D^{1\times 2}/(D^{1\times 2}\,R) \cong M' = D^{1\times 2}/(D\,R').$$

Block triangular decomposition

- Theorem: Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $f \in \mathrm{end}_D(M)$ defined by P and Q satisfying $R\, P = Q\, R$.

If the left D-modules

$$\ker_D(.P), \quad \mathrm{coim}_D(.P) = D^{1 \times p}/\ker_D(.P),$$
$$\ker_D(.Q), \quad \mathrm{coim}_D(.Q) = D^{1 \times q}/\ker_D(.Q),$$

are free of rank m, $p - m$, l, $q - l$, then there exist two matrices

$$U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D), \quad V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D),$$

such that

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} V_1\, R\, W_1 & 0 \\ V_2\, R\, W_1 & V_2\, R\, W_2 \end{pmatrix} \in D^{q \times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p \times m}$, $W_2 \in D^{p \times (p-m)}$ and:

$$U_1 \in D^{m \times p}, \quad U_2 \in D^{(p-m) \times p}, \quad V_1 \in D^{l \times q}, \quad V_2 \in D^{(q-l) \times q}.$$

Exemple: Electromagnetism

$$\sigma\,\partial_t\,\vec{A} + \frac{1}{\mu}\,\vec{\nabla}\wedge\vec{\nabla}\,\vec{A} - \sigma\,\vec{\nabla}\,V = 0$$

$$\Rightarrow\ R = \begin{pmatrix} \sigma\,\partial_t - \frac{1}{\mu}\,(\partial_2^2+\partial_3^2) & \frac{1}{\mu}\,\partial_1\,\partial_2 & \frac{1}{\mu}\,\partial_1\,\partial_3 & -\sigma\,\partial_1 \\ \frac{1}{\mu}\,\partial_1\,\partial_2 & \sigma\,\partial_t - \frac{1}{\mu}\,(\partial_1^2+\partial_3^2) & \frac{1}{\mu}\,\partial_2\,\partial_3 & -\sigma\,\partial_2 \\ \frac{1}{\mu}\,\partial_1\,\partial_3 & \frac{1}{\mu}\,\partial_2\,\partial_3 & \sigma\,\partial_t - \frac{1}{\mu}\,(\partial_1^2+\partial_2^2) & -\sigma\,\partial_3 \end{pmatrix}.$$

- Let $D = \mathbb{Q}[\partial_t, \partial_1, \partial_2, \partial_3]$ and $M = D^{1\times 4}/(D^{1\times 3}\,R)$.

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma\,\mu\,\partial_t & 0 & -\sigma\,\mu\,\partial_2 \\ 0 & 0 & \sigma\,\mu\,\partial_t & -\sigma\,\mu\,\partial_3 \\ 0 & \partial_t\,\partial_2 & \partial_t\,\partial_3 & -(\partial_2^2+\partial_3^2) \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ -\partial_1\,\partial_2 & \sigma\,\mu\,\partial_t - \partial_2^2 & -\partial_2\,\partial_3 \\ -\partial_1\,\partial_3 & -\partial_2\,\partial_3 & \sigma\,\mu\,\partial_t - \partial_3^2 \end{pmatrix},$$

satisfy $R\,P = Q\,R$ and define a endomorphism $f \in \mathrm{end}_D(M)$.

Exemple: Electromagnetism

- The modules $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$, $\mathrm{coim}_D(.Q)$ are free D-modules (Quillen-Suslin theorem) and:

$$\begin{cases} \ker_D(.P) = D^{1\times 2}\, U_1, \quad U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & -\sigma\,\mu \end{pmatrix}, \\ \mathrm{coim}_D(.P) = D^{1\times 2}\, U_2, \quad U_2 = \dfrac{1}{\sigma\,\mu} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \ker_D(.Q) = D^{1\times 2}\, V_1, \quad V_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \\ \mathrm{coim}_D(.Q) = D^{1\times 2}\, V_2, \quad V_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{cases}$$

- The matrix R is then equivalent to $\overline{R} = V\, R\, U^{-1}$ defined by:

$$\overline{R} = \begin{pmatrix} \sigma\,\partial_t - \frac{1}{\mu}\,(\partial_2^2 + \partial_3^2) & \frac{1}{\mu}\,\partial_1 & 0 & 0 \\ \frac{1}{\mu}\,\partial_1\,\partial_2 & \frac{1}{\mu}\,\partial_2 & \sigma\,(\sigma\,\mu\,\partial_t - (\partial_1^2 + \partial_2^2 + \partial_3^2)) & 0 \\ \frac{1}{\mu}\,\partial_1\,\partial_3 & \frac{1}{\mu}\,\partial_3 & 0 & \sigma\,(\sigma\,\mu\,\partial_t - (\partial_1^2 + \partial_2^2 + \partial_3^2)) \end{pmatrix}.$$

Idempotents of $\mathrm{end}_D(M)$

- Lemma: An endomorphism f of $M = D^{1\times p}/(D^{1\times q}\, R)$, defined by the matrices P and Q, is a idempotent, i.e., $f^2 = f$, iff there exist $Z \in D^{p\times q}$ and $Z' \in D^{q\times t}$ such that

$$\begin{cases} P^2 = P + Z\, R, \\ Q^2 = Q + R\, Z + Z'\, R_2, \end{cases}$$

where $R_2 \in D^{t\times q}$ satisfies $\ker_D(.R) = D^{1\times t}\, R_2$.

- Example: $D = A_1(\mathbb{Q})$, $R = (\partial^2 \quad -t\,\partial - 1)$, $M = D^{1\times 2}/(D\, R)$.

$$P = \left(\begin{array}{cc} -(t+a)\,\partial + 1 & t^2 + a\,t \\ 0 & 1 \end{array}\right), \quad P^2 = P + \left(\begin{array}{c} (t+a)^2 \\ 0 \end{array}\right) R.$$

Idempotents of $\mathrm{end}_D(M)$

- Proposition: f is a idempotents of $\mathrm{end}_D(M)$, i.e., $f^2 = f$, iff there exists a matrix $X \in D^{p \times s}$ such that $P = I_p - X\, S$ and we have the following commutative exact diagram:

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \ker f \arrow[d, "i"] & \\
 & D^{1\times q} \arrow[r, ".R"] \arrow[d, shift left, ".V"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, shift left, ".I_p"] & M \arrow[r] \arrow[d, shift left, "\kappa"] & 0 \\
D^{1\times s} \arrow[r, ".U"] & D^{1\times r} \arrow[r, ".S"] \arrow[u, shift left, ".T"] & D^{1\times p} \arrow[r, "\pi'"] \arrow[u, shift left, ".P"] \arrow[l, shift left, ".X"] & M/\ker f \arrow[r] \arrow[u, shift left, "f"] \arrow[d] & 0. \\
 & & & 0 &
\end{tikzcd}
$$

$$\Rightarrow \; M \cong \ker f \oplus \mathrm{im}\, f \quad \& \quad S - S\, X\, S = T\, R. \quad (\star)$$

- Corollary: If $\ker_D(.S) = 0$, then $R = V\, S$ satisfies:

$$S\, X - T\, V = I_r.$$

Decomposition of solutions

• Corollary: Let us suppose that $\mathcal{F}$ is an injective left D-module. Then, we have the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & V\,z = 0 = R\,y & \longleftarrow\!\!\!\shortmid & y & & & & \\
 & & \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0 \\
 & & \uparrow V. & & \| & & \uparrow f^\star & & \\
\mathcal{F}^s & \xleftarrow{U.} & \mathcal{F}^r & \underset{\xrightarrow{X.}}{\xleftarrow{S.}} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(S.) & \longleftarrow & 0. \\
0 = U\,z & \longleftarrow\!\!\!\shortmid & z = S\,y & \longleftarrow\!\!\!\shortmid & y & & & &
\end{array}$$

Moreover, we have: $\quad R\,y = 0 \quad \Leftrightarrow \quad \begin{pmatrix} U \\ V \end{pmatrix} z = 0, \quad S\,y = z.$

General solution: $y = u + X\,z$ where $S\,u = 0$ and $\begin{pmatrix} U \\ V \end{pmatrix} z = 0.$

Example: OD system

- Let $D = \mathbb{Q}[t]\,[\partial; \mathrm{id}, \partial]$ and $M = D^{1\times 4}/(D^{1\times 4}\, R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4\times 4}.$$

- An idempotent of $\mathrm{end}_D(M)$ is defined by:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{Q}^{4\times 4}: \quad P^2 = P.$$

- We obtain the factorization $R = V\, S$, where:

$$S = \begin{pmatrix} \partial & -t & 0 & 0 \\ 0 & \partial & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & t & \partial \\ 1 & t & \partial & -1 \\ 1 & 0 & \partial + t & \partial - 1 \\ 1 & 1 & t & \partial \end{pmatrix}.$$

Example

- Using the identity $I_p - P = X\,S$, we obtain:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R\,y = 0 \;\Leftrightarrow\; y = u + X\,z: \;\; V\,z = 0, \; S\,u = 0.$$

- The solution of $S\,u = 0$ is defined by:

$$u_1 = \frac{1}{2}\,C_1\,t^2 + C_2, \quad u_2 = C_1, \quad u_3 = 0, \quad u_4 = 0.$$

- The solution of $V\,z = 0$ is defined by: $z_1 = 0$, $z_2 = 0$ and

$$z_3(t) = C_3\,\mathrm{Ai}(t) + C_4\,\mathrm{Bi}(t), \; z_4(t) = C_3\,\partial\,\mathrm{Ai}(t) + C_4\,\partial\,\mathrm{Bi}(t).$$

- The general solution of $R\,y = 0$ is then defined by:

$$y = u + X\,z = \begin{pmatrix} \frac{1}{2}\,C_1\,t^2 + C_2 & C_1 & z_3(t) & z_4(t) \end{pmatrix}^T.$$

- Lemma: Let us suppose that $\ker_D(.R) = 0$ and $P^2 = P + Z\,R$. If there exists a solution $\Lambda \in D^{p\times q}$ of the algebraic Riccati equation

$$\Lambda\,R\,\Lambda + (P - I_p)\,\Lambda + \Lambda\,Q + Z = 0, \quad (\star)$$

then the matrices $\overline{P} = P + \Lambda\,R$ and $\overline{Q} = Q + R\,\Lambda$ satisfy:

$$R\,\overline{P} = \overline{Q}\,R, \quad \overline{P}^2 = \overline{P}, \quad \overline{Q}^2 = \overline{Q}.$$

- Example: $\Lambda = (a\,t \quad a\,\partial - 1)^T$ is a solution of $(\star)$

$$\Rightarrow\ \overline{P} = \begin{pmatrix} a\,t\,\partial^2 - (t+a)\,\partial + 1 & t^2\,(1 - a\,\partial) \\ (a\,\partial - 1)\,\partial^2 & -a\,t\,\partial^2 + (t - 2\,a)\,\partial + 2 \end{pmatrix}, \overline{Q} = 0,$$

then satisfy $\overline{P}^2 = \overline{P}$ and $\overline{Q}^2 = \overline{Q}$.

Block diagonal decomposition

- Theorem: Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q} R)$ and $f \in \mathrm{end}_D(M)$ defined by P and Q satisfying:

$$P^2 = P, \quad Q^2 = Q \quad \text{(idempotents)} \quad \Rightarrow \quad f^2 = f.$$

If the left D-modules

$$\ker_D(.P), \quad \mathrm{im}_D(.P) = \ker_D(.(I_p - P)),$$
$$\ker_D(.Q), \quad \mathrm{im}_D(.Q) = \ker_D(.(I_q - Q)),$$

are free of rank m, $p - m$, l, $q - l$, then there exist two matrices

$$U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D), \quad V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D),$$

such that

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} V_1\, R\, W_1 & 0 \\ 0 & V_2\, R\, W_2 \end{pmatrix} \in D^{q \times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p \times m}$, $W_2 \in D^{p \times (p-m)}$ and:

$$U_1 \in D^{m \times p}, \quad U_2 \in D^{(p-m) \times p}, \quad V_1 \in D^{l \times q}, \quad V_2 \in D^{(q-l) \times q}.$$

Example: OD system

- Let us consider the matrix again:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix}.$$

- An idempotent $f \in \mathrm{end}_D(M)$ is defined by the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix}.$$

where P and Q satisfy:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$$

Example: OD system

- Computing bases of the left D-modules

$$\ker_D(.P), \quad \mathrm{im}_D(.P), \quad \ker_D(.Q), \quad \mathrm{im}_D(.Q),$$

we obtain the unimodular matrices:

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -t & -1 & 1 & t \\ t+1 & 1 & -1 & -t \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

- R is then equivalent to the following block diagonal matrix:

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} \partial & -1 & 0 & 0 \\ t & \partial & 0 & 0 \\ 0 & 0 & \partial & -t \\ 0 & 0 & 0 & \partial \end{pmatrix}.$$

Example: Cauchy-Riemann equations

- Let us consider the Cauchy-Riemann equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0, \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0. \end{cases}$$

- $D = \mathbb{Q}(i)[\partial_x, \partial_y], \quad R = \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix}, \quad M = D^{1\times 2}/(D^{1\times 2}\, R).$

- The matrices P and Q defined by $P = Q = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

satisfy $R\,P = P\,R$ and $P^2 = P$, i.e., define an idempotent.

$$\begin{cases} \ker_{\mathbb{Q}(i)}(.P) = \mathbb{Q}(i)\,(1 \quad -i), \\ \mathrm{im}_{\mathbb{Q}(i)}(.P) = \mathbb{Q}(i)\,(1 \quad i), \end{cases} \Rightarrow U = V = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in \mathrm{GL}_2(D).$$

$$\Rightarrow \overline{R} = U\,R\,U^{-1} = \begin{pmatrix} \partial_x - i\,\partial_y & 0 \\ 0 & \partial_x + i\,\partial_y \end{pmatrix} = 2\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}.$$

Example: Wave equation

- Let us consider the following system of PDEs:

$$\left\{ \begin{array}{l} \dfrac{\partial y_1}{\partial x} + a\, \dfrac{\partial y_2}{\partial t} = 0, \\ \dfrac{\partial y_1}{\partial t} + b\, \dfrac{\partial y_2}{\partial x} = 0. \end{array} \right.$$

- Acoustic wave: $y_1 = u$, $y_2 = p$, $a = 1/\rho$, $b = \rho\, c^2$.
- LC transmission line: $y_1 = v$, $y_2 = i$, $a = L$, $b = 1/C$.
- $D = \mathbb{Q}(a, b)[\partial_x, \partial_t]$, $R = \begin{pmatrix} \partial_x & a\, \partial_t \\ \partial_t & b\, \partial_x \end{pmatrix}$, $M = D^{1 \times 2}/(D^{1 \times 2}\, R)$.
- An idempotent $f \in \mathrm{end}_D(M)$ is defined by the idempotents

$$P = \frac{1}{2} \begin{pmatrix} 1 & 2\, a\, b\, \alpha \\ 2\, \alpha & 1 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 1 & 2\, a\, \alpha \\ 2\, b\, \alpha & 1 \end{pmatrix},$$

where α satisfies $4\, a\, b\, \alpha^2 - 1 = 0$.

Example: Wave equation

- Let us denote by $D' = \mathbb{Q}(a, b, \alpha)/(4\, a\, b\, \alpha^2 - 1)[\partial_x, \partial_t]$.
- $\ker_{D'}(.P)$, $\mathrm{im}_{D'}(.P)$, $\ker_{D'}(.Q)$ and $\mathrm{im}_{D'}(.Q)$ are free with bases:

$$\left\{ \begin{array}{ll} \ker_{D'}(.P) = D'\, U_1, & U_1 = (-2\,\alpha \quad 1)\,, \\ \mathrm{im}_{D'}(.P) = D'\, U_2, & U_2 = (2\,\alpha \quad 1)\,. \end{array} \right.$$

$$\left\{ \begin{array}{ll} \ker_{D'}(.Q) = D'\, V_1, & V_1 = (2\, b\, \alpha \quad -1)\,, \\ \mathrm{im}_{D'}(.Q) = D'\, V_2, & V_2 = (2\, b\, \alpha \quad 1)\,. \end{array} \right.$$

- $U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_2(D')$, $V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_2(D')$.
- The matrix R is equivalent to $(1/(2\,\alpha) = \sqrt{a\, b})$:

$$\overline{R} = V\, R\, U^{-1} = \left(\begin{array}{cc} -b\,\partial_x + \frac{1}{2\,\alpha}\,\partial_t & 0 \\ 0 & b\,\partial_x + \frac{1}{2\,\alpha}\partial_t \end{array} \right).$$

Example: Dirac equation

- Let us consider the following complex matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The Dirac equation has the form $\sum_{i=1}^{4} \gamma^i \, \partial\, y / \partial x_i = 0$:

$$\begin{cases} \partial_4\, y_1 - i\, \partial_3\, y_3 - (i\, \partial_1 + \partial_2)\, y_4 = 0, \\ \partial_4\, y_2 - (i\, \partial_1 - \partial_2)\, y_3 + i\, \partial_3\, y_4 = 0, \\ i\, \partial_3\, y_1 + (i\, \partial_1 + \partial_2)\, y_2 - \partial_4\, y_3 = 0, \\ (i\, \partial_1 - \partial_2)\, y_1 - i\, \partial_3\, y_2 - \partial_4\, y_4 = 0, \end{cases}$$

Example: Dirac equation

- Let us consider $D = \mathbb{Q}(i)[\partial_1, \partial_2, \partial_3, \partial_4]$, the matrix

$$R = \left(\begin{array}{cccc} \partial_4 & 0 & -i\,\partial_3 & -(i\,\partial_1 + \partial_2) \\ 0 & \partial_4 & -i\,\partial_1 + \partial_2 & i\,\partial_3 \\ i\,\partial_3 & i\,\partial_1 + \partial_2 & -\partial_4 & 0 \\ i\,\partial_1 - \partial_2 & -i\,\partial_3 & 0 & -\partial_4 \end{array}\right) \in D^{4\times 4},$$

and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 4}\,R)$.

- Computing idempotents of $\mathrm{end}_D(M)$, we obtain a idempotent f defined by the pair of matrices:

$$P = \frac{1}{2}\left(\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right), \quad Q = \frac{1}{2}\left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right).$$

- We have $P^2 = P$ and $Q^2 = Q$, i.e., the D-modules $\ker_D(.P)$, $\mathrm{im}\,(.P)$, $\ker_D(.Q)$ and $\mathrm{im}\,(.Q)$ are free.

Example: Dirac equation

- Computing bases for these modules, we then get:

$$\begin{cases} \ker_D(.P) = D^{1\times 2}\, U_1, & U_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \\ \mathrm{im}\,(.P) = D^{1\times 2}\, U_2, & U_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \\ \ker_D(.Q) = D^{1\times 2}\, V_1, & V_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \\ \mathrm{im}\,(.Q) = D^{1\times 2}\, V_2, & V_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}. \end{cases}$$

- Let us form the unimodular matrices:

$$U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_4(D), \quad V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_4(D).$$

Example: Dirac equation

- The matrix R is then equivalent to the block-diagonal one:

$$V\,R\,U^{-1} = \begin{pmatrix} i\,\partial_3 - \partial_4 & -i\,\partial_1 - \partial_2 & 0 & 0 \\ i\,\partial_1 - \partial_2 & i\,\partial_3 + \partial_4 & 0 & 0 \\ 0 & 0 & i\,\partial_3 + \partial_4 & i\,\partial_1 + \partial_2 \\ 0 & 0 & i\,\partial_1 - \partial_2 & -i\,\partial_3 + \partial_4 \end{pmatrix}.$$

- If we denote by $z = U\,y$, we obtain that the Dirac equation is then equivalent to the decoupled system of PDEs:

$$\begin{cases} (i\,\partial_3 - \partial_4)\,z_1 - (i\,\partial_1 + \partial_2)\,z_2 = 0, \\ (i\,\partial_1 - \partial_2)\,z_1 + (i\,\partial_3 + \partial_4)\,z_2 = 0, \\ (i\,\partial_3 + \partial_4)\,z_3 + (i\,\partial_1 + \partial_2)\,z_4 = 0, \\ (i\,\partial_1 - \partial_2)\,z_3 + (-i\,\partial_3 + \partial_4)\,z_4 = 0. \end{cases}$$

Example: 2-D rotational isentropic flow

- We consider the linearized approximation of the steady two-dimensional rotational isentropic flow (Courant-Hilbert)

$$\begin{cases} u\,\rho\,\dfrac{\partial\,\omega}{\partial x} + c^2\,\dfrac{\partial\,\sigma}{\partial x} = 0, \\ u\,\rho\,\dfrac{\partial\,\lambda}{\partial x} + c^2\,\dfrac{\partial\,\sigma}{\partial y} = 0, \\ \rho\,\dfrac{\partial\,\omega}{\partial x} + \rho\,\dfrac{\partial\,\lambda}{\partial y} + u\,\dfrac{\partial\sigma}{\partial x} = 0, \end{cases}$$

where u is a constant velocity parallel to the x-axis, ρ a constant density and c the sound speed.

- Let us consider $D = \mathbb{Q}(u, \rho, c)[\partial_x, \partial_y]$, the system matrix

$$R = \begin{pmatrix} u\,\rho\,\partial_x & c^2\,\partial_x & 0 \\ 0 & c^2\,\partial_y & u\,\rho\,\partial_x \\ \rho\,\partial_x & u\,\partial_x & \rho\,\partial_y \end{pmatrix} \in D^{3\times 3},$$

and the D-module $M = D^{1\times 3}/(D^{1\times 3}\,R)$.

Example: 2-D rotational isentropic flow

- If α satisfies $1+4\,(c^2-u^2)\,\alpha^2=0$ and we denote by

$$D' = (\mathbb{Q}(u,\rho,c,\alpha)/(1+4\,(c^2-u^2)\,\alpha^2))[\partial_x,\partial_y],$$

$$U = \begin{pmatrix} 0 & 2\,\alpha\,c\,(c^2-u^2) & u\,\rho \\ 0 & 2\,\alpha\,c\,(c^2-u^2) & -u\,\rho \\ u\,\rho & c^2 & 0 \end{pmatrix} \in \mathrm{GL}_3(D'),$$

$$V = \begin{pmatrix} 2\,\alpha\,c & 1 & -2\,\alpha\,c\,u \\ 2\,\alpha\,c & -1 & -2\,\alpha\,c\,u \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(D'),$$

$$\Rightarrow \overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial_x - 2\,\alpha\,c\,\partial_y & 0 & 0 \\ 0 & \partial_x + 2\,\alpha\,c\,\partial_y & 0 \\ 0 & 0 & \partial_x \end{pmatrix}.$$

- We have $M \cong M_1 \oplus M_2 \oplus M_3$, where $M_3 = D'/(D'\,\partial_x)$ and:

$$M_1 = D'/(D'\,(\partial_x - 2\,\alpha\,c\,\partial_y)),\ \ M_2 = D'/(D'\,(\partial_x + 2\,\alpha\,c\,\partial_y)).$$

Example: Tank model I

- We consider $D = \mathbb{Q}\,[\partial, \delta]$ and the system matrix

$$R = \left(\begin{array}{ccc} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{array} \right)$$

considered in Dubois, Petit, Rouchon, ECC99.

- An idempotent $f \in \mathrm{end}_D(M)$ is defined by the matrices

$$P = \frac{1}{2} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right), \quad Q = \frac{1}{2} \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right),$$

i.e., P and Q satisfy:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$$

Example: Tank model I

$$\begin{cases} U_1 = \ker_D(.P) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}, \\ U_2 = \mathrm{im}_D(.P) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ V_1 = \ker_D(.Q) = \begin{pmatrix} 1 & -1 \end{pmatrix}, \\ V_2 = \mathrm{im}_D(.Q) = \begin{pmatrix} 1 & 1 \end{pmatrix}, \end{cases}$$

and we obtain the following two unimodular matrices:

$$U = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

- We easily check that we have the following block diagonal matrix:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \delta^2 - 1 & 0 & 0 \\ 0 & 1 + \delta^2 & -4\,\partial\,\delta \end{pmatrix}.$$

Example: Tank model I

$$\overline{R} = \begin{pmatrix} \delta^2 - 1 & 0 & 0 \\ 0 & 1 + \delta^2 & -4\,\partial\,\delta \end{pmatrix}.$$

- If $\mathcal{F} = C^\infty(\mathbb{R})$ and ψ is any smooth $2\,h$-periodic function, then

$$\forall\, \xi \in \mathcal{F}, \quad \begin{cases} z_1(t) = \psi(t), \\ z_2(t) = 4\,\partial\,\delta\,\xi(t) = 4\,\dot{\xi}(t-h), \\ v(t) = (\delta^2+1)\,\xi(t) = \xi(t-2\,h) + \xi(t), \end{cases}$$

is a solution of $\overline{R}\,z = 0$. Hence, the solution of $R\,y = 0$ are:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = U^{-1} \begin{pmatrix} z_1(t) \\ z_2(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,\psi(t) + 2\,\dot{\xi}(t-h) \\ -\frac{1}{2}\,\psi(t) + 2\,\dot{\xi}(t-h) \\ \xi(t-2\,h) + \xi(t) \end{pmatrix}$$

Example: Tank model II

- Model of a one-dimensional tank containing a fluid subjected to an horizontal move (Petit, Rouchon, IEEE TAC, 2002):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-2\,h) + \alpha\,\ddot{y}_3(t-h) = 0, \\ \dot{y}_1(t-2\,h) - \dot{y}_2(t) + \alpha\,\ddot{y}_3(t-h) = 0, \end{cases} \qquad \alpha \in \mathbb{R}, \quad h \in \mathbb{R}_+.$$

- Let us consider $D = \mathbb{Q}(\alpha)\,[\partial, \delta]$, the system matrix

$$R = \begin{pmatrix} \partial & -\partial\,\delta^2 & \alpha\,\partial^2\,\delta \\ \partial\,\delta^2 & -\partial & \alpha\,\partial^2\,\delta \end{pmatrix} \in D^{2\times 3},$$

and the D-module $M = D^{1\times 3}/(D^{1\times 2}\,R)$.

- The matrices $P = \begin{pmatrix} 1 & 0 & 0 \\ \delta^2 & 0 & \alpha\,\partial\,\delta \\ 0 & 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -\delta^2 \\ 0 & 0 \end{pmatrix}$

satisfy $\quad R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$

Example: Tank model II

- $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free with bases:

$$\begin{cases} \ker_D(.P) = D\,\begin{pmatrix} \delta^2 & -1 & \alpha\,\partial\,\delta \end{pmatrix}, & \ker_D(.Q) = D\,(0 \quad 1), \\ \mathrm{im}_D(.P) = D^{1\times 2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \mathrm{im}_D(.Q) = D\,(-1 \quad \delta^2). \end{cases}$$

- If we denote by

$$U = \begin{pmatrix} \delta^2 & -1 & \alpha\,\partial\,\delta \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = \begin{pmatrix} 0 & 1 \\ -1 & \delta^2 \end{pmatrix} \in \mathrm{GL}_2(D),$$

then R is equivalent to the following block-diagonal matrix:

$$V\,R\,U^{-1} = \begin{pmatrix} \partial & 0 & 0 \\ 0 & \partial\,(\delta-1)\,(\delta+1)\,(\delta^2+1) & \alpha\,\partial^2\,\delta\,(\delta-1)\,(\delta+1) \end{pmatrix}.$$

Example: Tank model II

- Another idempotent of $\mathrm{end}_D(M)$ is defined by the idempotentmatrices P' and Q' defined by:

$$P' = \frac{1}{2}\left(\begin{array}{ccc} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right), \quad Q' = \frac{1}{2}\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right).$$

- Using linear algebraic techniques, we obtain

$$U' = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right) \in \mathrm{GL}_3(D), \quad V' = \left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right) \in \mathrm{GL}_2(D),$$

and R is equivalent to the following block-diagonal matrix:

$$V'\,R\,U'^{-1} = \left(\begin{array}{ccc} \partial\,(1-\delta)\,(\delta+1) & 0 & 0 \\ 0 & \partial\,(\delta^2+1) & 2\,\alpha\,\partial^2\,\delta \end{array}\right).$$

Example: Flexible rod

- Flexible rod (Mounier, Rudolph, Petitot, Fliess ECC95):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases}$$

$$\Rightarrow \; R = \begin{pmatrix} \partial & -\partial\,\delta & -1 \\ 2\,\partial\,\delta & -\partial\,\delta^2 - \partial & 0 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1+\delta^2 & -\frac{1}{2}\,\delta\,(1+\delta^2) & 0 \\ 2\,\delta & -\delta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -\frac{1}{2}\,\delta \\ 0 & 0 \end{pmatrix},$$

$$\Rightarrow U = \begin{pmatrix} -2\,\delta & \delta^2+1 & 0 \\ 2\,\partial\,(1-\delta^2) & \partial\,\delta\,(\delta^2-1) & -2 \\ -1 & \frac{1}{2}\,\delta & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 2 & -\delta \end{pmatrix},$$

$$\Rightarrow \overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example: Flexible rod

$$\overline{R} = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- All the smooth solutions of the differential time-delay system

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0, \end{cases}$$

are of the form

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = U^{-1} \begin{pmatrix} c \\ 0 \\ z_3(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,c - z_3(t-2) - z_3(t) \\ c - 2\,z_3(t-1) \\ \dot{z}_3(t-2) - \dot{z}_3(t) \end{pmatrix},$$

where c (resp., z_3) is an arbitrary constant (resp., smooth function).

Corollary

- Corollary: Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ and $f \in \mathrm{end}_D(M)$ be defined by P and Q and satisfying $P^2 = P$ and $Q^2 = Q$. Let us suppose that one of the conditions holds:

1. $D = A[\partial]$, where A is a field,
2. $D = k[\partial_1, \ldots, \partial_n]$ is a commutative Ore algebra,
3. $D = A[\partial_1, \ldots, \partial_n]$, where $A = k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$ and k is a field of characteristic 0, and:

$$\begin{array}{ll} \mathrm{rank}_D(\ker_D(.P)) \geq 2, & \mathrm{rank}_D(\mathrm{im}_D\,(.P)) \geq 2, \\ \mathrm{rank}_D(\ker_D(.Q)) \geq 2, & \mathrm{rank}_D(\mathrm{im}_D\,(.Q)) \geq 2. \end{array}$$

Then, there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that $\overline{R} = V\, R\, U^{-1}$ is a block diagonal matrix.

Simplification problem

• Theorem: Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^q$ such that there exists $U \in \mathrm{GL}_{p+1}(D)$ satisfying:

$$(R \quad -\Lambda)\, U = (I_q \quad 0).$$

Let us denote by

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \in \mathrm{GL}_{p+1}(D),$$

where:

$$S_1 \in D^{p \times q}, \; S_2 \in D^{1 \times q}, \; Q_1 \in D^{p \times (p+1-q)}, \; Q_2 \in D^{1 \times (p+1-q)}.$$

Then, we have:

$$M = D^{1 \times p}/(D^{1 \times q}\, R) \;\cong\; L = D^{1 \times (p+1-q)}/(D\, Q_2).$$

The converse result also holds. These results only depend on:

$$\rho(\Lambda) \in \mathrm{ext}_D^1(M, D) = D^q/(R\, D^p), \; \rho : D^q \longrightarrow D^q/(R\, D^p).$$

Corollaries

- Corollary: We have the following isomorphism:

$$\begin{array}{rcl} \psi : M = D^{1\times p}/(D^{1\times q}\, R) & \longrightarrow & L = D^{1\times (p+1-q)}/(D\, Q_2) \\ \pi(\lambda) & \longmapsto & \kappa(\lambda\, Q_1). \end{array}$$

Its inverse $\psi^{-1} : L \longrightarrow M$ is defined by $\psi^{-1}(\kappa(\mu)) = \pi(\mu\, T_1)$:

$$U^{-1} = \begin{pmatrix} R & -\Lambda \\ T_1 & T_2 \end{pmatrix}, \quad T_1 \in D^{(p+1-q)\times p}, \quad T_2 \in D^{(p+1-q)}.$$

- Corollary: Let $\mathcal{F}$ be a left D-module and the linear systems:

$$\left\{ \begin{array}{l} \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}, \\ \ker_{\mathcal{F}}(Q_2.) = \{\zeta \in \mathcal{F}^{p+1-q} \mid Q_2\, \zeta = 0\}. \end{array} \right.$$

Then, we have the isomorphism $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.)$ and:

$$\ker_{\mathcal{F}}(R.) = Q_1\, \ker_{\mathcal{F}}(Q_2.), \quad \ker_{\mathcal{F}}(Q_2.) = T_1\, \ker_{\mathcal{F}}(R.).$$

Ring conditions

- Proposition: Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^q$ such that $P = (R \quad -\Lambda) \in D^{q \times (p+1)}$ admits a right-inverse over D. Moreover, if D is either a

 1. principal left ideal domain,
 2. commutative polynomial ring with coefficients in a field,
 3. Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $p - q \geq 1$,

 then there exists $U \in \mathrm{GL}_{p+1}(D)$ satisfying that $P\,U = (I_q \quad 0)$.

- The matrix U can be obtained by means of:

 1. a Jacobson form (Jacobson),
 2. the Quillen-Suslin theorem (QuillenSuslin),
 3. Stafford's theorem (Stafford).

Example: Wind tunnel model

- The wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a\, x_1(t) - k\, a\, x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\, x_2(t) + 2\,\zeta\,\omega\, x_3(t) - \omega^2\, u(t) = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(a, k, \omega, \zeta)[\partial, \delta]$, the system matrix

$$R = \begin{pmatrix} \partial + a & -k\, a\, \delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \in D^{3\times 4},$$

and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 3}\, R)$.

- The D-module $\mathrm{ext}^1_D(M, D) = D^3/(R\, D^4)$ is a $\mathbb{Q}(a, k, \omega, \zeta)$-vector space of dimension 1 and $\rho((1 \quad 0 \quad 0)^T)$ is a basis.

Example: Wind tunnel model

- Let us consider $\Lambda = (1 \quad 0 \quad 0)^T$ and $P = (R \quad -\Lambda)$.
- The matrix P admits the following right-inverse S:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\frac{\partial + 2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} \\ -1 & 0 & 0 \end{pmatrix} \in D^{5 \times 3}.$$

- According to Quillen-Suslin theorem, $E = D^{1\times 5}/(D^{1\times 3}\,P)$ is free D-module of rank 2.

Example: Wind tunnel model

- Computing a basis of E, we obtain that $U \in \mathrm{GL}_5(D)$,

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & 0 & \omega^2\,\partial \\ 0 & -\frac{\partial+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} & 0 & \partial^2+2\,\zeta\,\omega\,\partial+\omega^2 \\ -1 & 0 & 0 & -(\partial+a) & -\omega^2\,k\,a\,\delta \end{pmatrix},$$

satisfies that $P\,U = (I_3 \quad 0)$ (OreModules, QuillenSuslin).

- The wind tunnel model is equivalent to the sole equation:

$$(\partial+a)\,\zeta_1+\omega^2\,k\,a\,\delta\,\zeta_2=0$$

$$\Leftrightarrow \quad \dot{\zeta}_1(t)+a\,\zeta_1(t)+\omega^2\,k\,a\,\zeta_2(t-h)=0.$$

Algorithmic issue

1. Consider an ansatz $\Lambda \in D^q$ of a given order.
2. Compute a Gröbner basis of $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$.
3. Compute the normal form $\overline{\Lambda} \in D^q$ of $\rho(\Lambda)$.
4. Compute the obstructions to freeness of the left D-module $\overline{E} = D^{1\times(p+1)}/(D^{1\times q}\,(R \quad -\overline{\Lambda}))$ (π-polynomials).
5. Solve the systems in the arbitrary coefficients obtained by making the obstructions vanish.
6. If a solution $\Lambda_\star$ exists, then compute $U \in \mathrm{GL}_{p+1}(D)$ satisfying that $(R \quad -\Lambda)\,U = (I_q \quad 0)$ and return $Q_2 \in D^{1\times(p+1-q)}$.

• Remark: If $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$ is 0-dimensional, then we take $\overline{\Lambda}$ to be a generic combination of a basis of $\mathrm{ext}^1_D(M, D)$.

Example: Transmission line

- Let us consider a general transmission line:

$$\begin{cases} \dfrac{\partial V}{\partial x} + L\,\dfrac{\partial I}{\partial t} + R'\,I = 0, \\[2ex] C\,\dfrac{\partial V}{\partial t} + G\,V + \dfrac{\partial I}{\partial x} = 0. \end{cases}$$

- Let $D = \mathbb{Q}(L, R', C, G)[\partial_t, \partial_x]$ and $M = D^{1\times 2}/(D^{1\times 2}\,R)$, where:

$$R = \begin{pmatrix} \partial_x & L\,\partial_t + R' \\ C\,\partial_t + G & \partial_x \end{pmatrix} \in D^{2\times 2}.$$

- We consider $A = D[\alpha, \beta]$, $\Lambda = (\alpha \quad \beta)^T$, $P = (R \quad -\Lambda) \in A^{2\times 3}$.
- If we denote by $N = A^{1\times 2}/(A^{1\times 3}\,P^T)$, then we have:

$$\mathrm{ext}^1_A(N, A) = 0, \quad \mathrm{ext}^2_A(N, A) = A/(L_1, L_2),$$

$$\begin{cases} L_1 = (C\,\alpha^2 - L\,\beta^2)\,\partial_t + G\,\alpha^2 - R'\,\beta^2, \\ L_2 = (C\,\alpha^2 - L\,\beta^2)\,\partial_x + (L\,G - R'\,C)\,\alpha\,\beta. \end{cases}$$

Example: Transmission line

- We consider $\beta = C \neq 0$, $\alpha^2 = L\,C \neq 0$ and $R'\,C - L\,G \neq 0$.
- Over $B = D[\alpha]/(\alpha^2 - L\,C)$, we have $\mathrm{ext}^2_B(B \otimes_D N, B) = 0$, i.e., $E = B^{1\times 3}/(B^{1\times 2}\,P)$ is a projective B-module, and thus, is free.
- Then, we have:

$$
\begin{array}{l}
S = \frac{1}{R'\,C - L\,G} \left(\begin{array}{cc} -\alpha & L \\ -C & \alpha \\ -\frac{(\alpha\,\partial_x + C\,L\,\partial_t + L\,G)}{\alpha} & \frac{(\alpha\,\partial_x + L\,C\,\partial_t + R'\,C)}{C} \end{array} \right), \\
Q_1 = \alpha\,\partial_x - L\,C\,\partial_t - R'\,C \quad C\,\partial_x - \alpha\,C\,\partial_t - \alpha\,G, \\
Q_2 = \partial_x^2 - L\,C\,\partial_t^2 - (L\,C + R'\,C)\,\partial_t - R'\,G.
\end{array}
$$

- The transmission line is equivalent to the sole equation:

$$(\partial_x^2 - L\,C\,\partial_t^2 - (L\,C + R'\,C)\,\partial_t - R'\,G)\,Z(t,x) = 0.$$

Torsion-free degree

- Theorem: $\mathrm{ext}^1_D(M, D)$ is 0-dimensional iff the torsion-free degree of M is $n-1$ (the last but one step before projectiveness).

  1. $n = 2$, M is torsion-free,
  2. $n = 3$, M is reflexive, ...

Then, we can constructively check whether or not M ($\ker_{\mathcal{F}}(R.)$) can be generated by 1 relation (1 equation)!

- If $M = D^{1\times p}/(D^{1\times q}\, R)$ is free of rank $p-q$, i.e., there exists $V \in \mathrm{GL}_p(D)$ satisfying that $R\, V = (I_q \quad 0)$, then we have:

$$(R \quad 0) \left(\begin{array}{cc} V & 0 \\ 0 & 1 \end{array} \right) = (I_q \quad 0),$$

$$\Rightarrow \; M \cong D^{1\times (p+1-q)}/(D\,(0 \; \ldots \; 1)) \cong D^{1\times (p-q)} \quad \text{(0 equation!)}.$$

Example: String with an interior mass

- Model of a string with an interior mass (Fliess et al, COCV 98):

$$\begin{cases} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1\, \phi_1(t) - \eta_1\, \psi_1(t) - \eta_2\, \phi_2(t) + \eta_2\, \psi_2(t) = 0, \\ \phi_1(t - 2\, h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2\, h_2) - v(t - h_2) = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(\eta_1, \eta_2)[\partial, \sigma_1, \sigma_2]$, the system matrix

$$R = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \partial + \eta_1 & \partial - \eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} \in D^{4\times 6},$$

and the finitely presented D-module $M = D^{1\times 6}/(D^{1\times 4}\, R)$.

Example: String with an interior mass

- We can prove that M is a reflexive D-module (OREMODULES)

$\Rightarrow$ the D-module $\mathrm{ext}^1_D(M, D) = D^4/(R\, D^6)$ is a $\mathbb{Q}(\eta_1, \eta_2)$-vector space of dimension 1 and $\rho((0 \quad 1 \quad 0 \quad 0)^T)$ is a basis.

- Let us consider $\Lambda = (0 \quad 1 \quad 0 \quad 0)^T$ and $P = (R \quad -\Lambda)$.
- The matrix P admits the following right-inverse S:

$$S = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & -\sigma_1 & 0 \\ -\sigma_2 & 0 & 0 & -\sigma_2 \\ -\eta_2 & -1 & -2\,\eta_1 & -2\,\eta_2 \end{pmatrix} \in D^{7\times 4}.$$

$\Rightarrow$ the D-module $E = D^{1\times 7}/(D^{1\times 4}\, P)$ is free of rank 3.

Example: String with an interior mass

- Computing a basis of N, we obtain that $U \in \mathrm{GL}_7(D)$,

$$U = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & \sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\sigma_2 \\ -1 & 0 & 0 & -1 & -1 & 0 & \sigma_2 \\ 0 & 0 & -\sigma_1 & 0 & -\sigma_1 & \sigma_1^2 - 1 & 0 \\ -\sigma_2 & 0 & 0 & -\sigma_2 & -\sigma_2 & 0 & \sigma_2^2 - 1 \\ -\eta_2 & -1 & -2\,\eta_1 & -2\,\eta_2 & -(\partial + \eta_1 + \eta_2) & 2\,\eta_1\,\sigma_1 & 2\,\eta_2\,\sigma_2 \end{pmatrix},$$

satisfies that $P\,U = (I_4 \quad 0)$ (OreModules, QuillenSuslin).

- The string model is then equivalent to the sole equation:

$$(\partial + \eta_1 + \eta_2)\,\zeta_1 - 2\,\eta_1\,\sigma_1\,\zeta_2 - 2\,\eta_2\,\sigma_2\,\zeta_3 = 0$$

$$\Leftrightarrow \; \dot{\zeta}_1(t) + (\eta_1 + \eta_2)\,\zeta_1(t) - 2\,\eta_1\,\zeta_2(t - h_1) - 2\,\eta_2\,\zeta_3(t - h_2) = 0.$$

Example: Stress tensor (elasticity)

- Let $D = \mathbb{Q}[\partial_x, \partial_y]$ and $M = D^{1\times 3}/(D^{1\times 2}\, R)$, where:

$$R = \left(\begin{array}{ccc} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{array}\right) \in D^{2\times 3}.$$

- The D-module $\mathrm{ext}^1_D(M, D) = D^2/(R\, D^3)$ is a $\mathbb{Q}$-vector space of dimension 3 with basis $\{\rho((1 \quad 0)^T), \rho((0 \quad 1)^T), \rho((0 \quad \partial_x)^T)\}$.

- Let us consider $\Lambda = (a \quad b + c\,\partial_x)^T$, $P = (R \quad -\Lambda)$.

- If we denote by $A = D[a, b, c]$ and $N = A^2/(P\, A^4)$, then we get:

$$\mathrm{ext}^1_A(N, A) = 0, \quad \mathrm{ext}^2_A(N, A) = A/(\partial_x, \partial_y).$$

- Hence, $E = A^{1\times 4}/(A^{1\times 2}\, P)$ is never a projective A-module and

$$\left\{\begin{array}{l} \partial_x\, \sigma^{11} + \partial_y\, \sigma^{12} = 0, \\ \partial_x\, \sigma^{12} + \partial_y\, \sigma^{22} = 0, \end{array}\right.$$

cannot be defined by a sole equation! $(\mu(M) = 3)$.

Equivalence

• Theorem: If $\Lambda \in D^q$ admits a left-inverse $\Gamma \in D^{1\times q}$, i.e., $\Gamma\,\Lambda = 1$, then Q_1 admits the left-inverse $T_1 + T_2\,\Gamma\,R \in D^{(p+1-q)\times p}$ and the left D-module $\ker_D(.Q_1)$ is stably free of rank $q-1$.

If the left D-module $\ker_D(.Q_1)$ is free, then $\exists\, Q_3 \in D^{p\times(q-1)}$ s.t.:

$$V = (Q_3^T \quad Q_1^T)^T \in \mathrm{GL}_p(D).$$

Then, we have $W = (R\,Q_3 \quad \Lambda) \in \mathrm{GL}_q(D)$,

$$W^{-1} = \begin{pmatrix} Y_3\,S_1 \\ -S_2 + Q_2\,Y_1\,S_1 \end{pmatrix},$$

with $V^{-1} = (Y_3^T \quad Y_1^T)^T$, $Y_3 \in D^{(q-1)\times p}$, $Y_1 \in D^{(p-q+1)\times p}$ and:

$$W^{-1}\,R\,V = \begin{pmatrix} I_{q-1} & 0 \\ 0 & Q_2 \end{pmatrix}.$$

- The vector $\Lambda = (1 \quad 0 \quad 0)^T$ admits the left-inverse $\Gamma = \Lambda^T$.
- We compute $Q_3 \in D^{2\times 2}$ such that $V = (Q_3^T \quad Q_1^T) \in \mathrm{GL}_4(D)$:

$$V = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & \omega^2\,\partial \\ -\frac{1}{\omega^2} & -\frac{\partial+2\,\zeta\,\omega}{\omega^2} & 0 & \partial^2+2\,\zeta\,\omega\,\partial+\omega^2 \end{pmatrix}.$$

- We have $W = (R\,Q_3 \quad \Lambda) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(D)$ and:

$$W^{-1}\,R\,V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -(\partial+a) & -\omega^2\,k\,a\,\delta \end{pmatrix}.$$

Example: String with an interior mass

- The vector $\Lambda = (0 \quad 1 \quad 0 \quad 0)^T$ admits the left-inverse $\Gamma = \Lambda^T$.
- We compute $Q_3 \in D^{6\times 3}$ such that $V = (Q_3^T \quad Q_1^T) \in \mathrm{GL}_6(D)$:

$$V = \begin{pmatrix} 1 & 0 & 0 & -1 & \sigma_1 & 0 \\ 0 & -1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma_2 \\ 0 & -1 & -1 & -1 & 0 & \sigma_2 \\ 0 & 0 & 0 & -\sigma_1 & \sigma_1^2 - 1 & 0 \\ 0 & -\sigma_2 & -\sigma_2 & -\sigma_2 & 0 & \sigma_2^2 - 1 \end{pmatrix}.$$

$$W = (R\,Q_3 \quad \Lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial + \eta_1 & -\partial + \eta_1 - \eta_2 & -2\,\eta_2 & 1 \\ \sigma_1^2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}_4(D).$$

$$\Rightarrow W^{-1}\,R\,V = \mathrm{diag}(1, 1, 1, (-(\partial + \eta_1 + \eta_2) \quad 2\,\eta_1\,\sigma_1 \quad 2\,\eta_2\,\sigma_2)).$$

Further extensions

- The previous results can be extended to the cases

$$M \cong L = D^{1\times(p-m)}/(D^{1\times(q-m)}\, Q_2), \;\; Q_2 \in D^{(q-m)\times(p-m)},$$
$$W^{-1}\, R\, V = \mathrm{diag}(I_m, \star),$$

using the homological algebraic classical result:

$$\mathrm{ext}^1_D\left(M, D^{1\times(q-m)}\right) \cong \mathrm{ext}^1_D(M, D) \otimes_D D^{1\times(q-m)}.$$

- We then consider $\Lambda \in D^{q\times(q-m)}$, $P = (R \quad -\Lambda) \in D^{q\times(p+q-m)}$.
- The results only depend on the residue classes of the columns of Λ in the right D-module $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$.
- If $\mathrm{ext}^1_D(M, D)$ is 0-dimensional, then a minimal presentation of M, i.e., a minimal representation of $\ker_{\mathcal{F}}(R.)$, can be computed

(constellations (Levandovskyy-Zerz 07)).

The OreMorphisms package

- The algorithms have been implemented in the package called OreMorphisms based on the library OreModules:

http://www-sop.inria.fr/members/Alban.Quadrat/OreMorphisms/index.html.

T. Cluzeau, A. Quadrat, "Factoring and decomposing a class of linear functional systems", *Linear Algebra and Its Applications*, 428 (2008), 324-381.

T. Cluzeau, A. Quadrat, "OreMorphisms: A homological algebraic package for factoring, reducing and decomposing linear functional systems", in *Topics in Time-Delay Systems: Analysis, Algorithms and Control*, Lecture Notes in Control and Information Sciences (LNCIS), Springer, to appear, 2009.

M. S. Boudellioua, A. Quadrat, "Reduction of linear systems based on Serre's theorem", proceedings of MTNS 2008, Blacksburg, Virginia (USA) (28/07-01/08/08).

Conclusion

- Contributions:
  - Based on constructive homological algebra, we have studied the factorization, reduction, decomposition and simplification problems.
  - Computation of quadratic first integrals and conservative laws.
- Computation of bases of free left D-modules:
  - If D is a left principal ideal domain, then we can use Smith or Jacobson normal forms (Culianez-Q.).
  - If D is the Weyl algebra $A_n(\mathbb{Q})$ or $B_n(\mathbb{Q})$, then we use the implementation of the Stafford theorem (Q.-Robertz).
  - If D is a commutative polynomial ring, then we use the implementation of the Quillen-Suslin theorem (Fabiańska-Q.).