Integrating massive Feynman Diagrams with operator insertions at 3 loops

Fabian Wissbrock, DESY

in collaboration with Johannes Blümlein and Francis Brown

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Contents

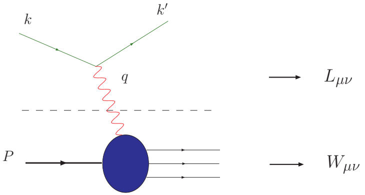
Introduction

There are many challenging questions in elementary particle physics these days. Two of them are:

- Hadrons as neutron and protons are composite objects consisting of quarks and gluons. But how exactly are quarks and gluons distributed inside the hadrons?
- The strength of the strong interaction is determined by the strong coupling constant α_s. What is the exact value for $\alpha_s(M_Z)$?

The answers for these questions are of vital importance for the interpretation of experimental data from many particle accelerators, including the Tevatron and the LHC.

- The most important way to study these questions is the analysis of scattering proccesses:



- kinematic quantities: $Q^2 := -q^2, \quad x := \frac{Q^2}{2pq}, \quad \nu := \frac{Pq}{M}$

- differential cross-section: $\frac{d\sigma}{dQ^2\ dx} \sim W_{\mu\nu} L^{\mu\nu}$

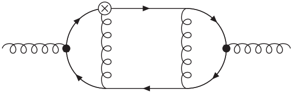
$$
\begin{aligned}
W_{\mu\nu}(q,P,s) &= \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J^{em}_{\mu}(\xi), J^{em}_{\nu}(0)] \mid P,s\rangle \\
&= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)
\end{aligned}
$$

.

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions

Until now, the light flavor contributions to the structure functions are known to a higher precions than the massive ones.

- The massive contributions are given by the massive operator matrix elements which can be computed in terms of Feynman diagrams with an operator insertion.



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- Each Feynman diagram consists of a set of internal edges $\{e_1, \cdots, e_m\}$ and a set of vertices $\{v_1, \cdots, v_r\}$.
- The loop number (or first Betti number) of a graph is defined by

$$l = m - r + 1 \ .$$

- A power n_i and an integration parameter α_i attributed to each edge e_i. We denote $n = \sum n_i$.
- A mass m_j may be attributed to an edge. In our case there is at least one massive line and massive lines are always closed.

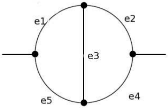
- Every Feynman graph G encodes an integral represensation:

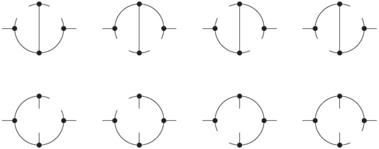
$$I_G(n_1, \cdots, n_L, N) = \frac{\pi^{l} D/2\Gamma(n - l\, D/2)}{\prod_i \Gamma(n_i)}$$
$$\int_{\alpha_\lambda=1} \int_0^\infty d\alpha 1 \cdots d\alpha_{\lambda-1} d\alpha_{\lambda+1} \cdots d\alpha_m \; Op(\alpha_i, N) \, \frac{\prod_i \alpha_i^{n_i-1}}{U_G^{D/2}} \left(\frac{U_G}{M_G}\right)^{n-l\, D/2}$$

,where $H_\lambda = \alpha_i : \alpha_\lambda = 1$ and $D = 4 + \varepsilon$.

- In this talk I will only consider example where all $n_i = 1$.
- M_G denotes a polynomial which describes the mass distribution of the diagram: $V = \sum_j \sum_{l \in m_j} \alpha_l m_j$.
- U_G is known as the graph polynomial (or Kirchoff polynomial) of G. It is given by $U_G = \sum_T \prod_{l \notin T} \alpha_l$, where T denotes the spanning trees of G
- A spanning tree T of G is a subgraph of G which satisfies the following conditions:
  1. T contains all the vertices of G.
  2. The loop number of T is zero.
  3. T is connected.

- For Example:





- $\rightarrow U_G = (\alpha_1 + \alpha_5)(\alpha_2 + \alpha_4) + \alpha_3(\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)$
- The functions $Op(\alpha_i, N)$ depend on the operator insertion. Examples are

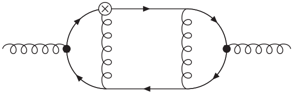
$$Op(\alpha_i, N) = P_1(\alpha_i)^N$$

or

$$Op(\alpha_i, N) = \sum_{j=0}^{N} P_1(\alpha_i)^{N-j} P_2(\alpha_i)^j \ .$$

Here P_1 and P_2 are further polynomials, which are specific Dodgson polynomials of the graph where the external lines are closed.

Example: Diagram 4



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$$
\begin{aligned}
I_4(N) &= \int \cdots \int d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4\, d\alpha_5\, d\alpha_6\, d\alpha_7\, d\alpha_8\, \frac{\sum_{j=0}^{N} T_{4a}^{N-j} T_{4b}^{j}}{U_G^{2+\varepsilon/2} M_G^{2-3/2\,\varepsilon}} \\
T_{4a} &= \alpha_5\alpha_7\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_2\alpha_5\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_5\alpha_7\alpha_8 + \alpha_2\alpha_3\alpha_8 \\
&\quad + \alpha_7\alpha_2\alpha_8 + \alpha_6\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_2\alpha_3\alpha_6 + \alpha_4\alpha_2\alpha_8 + \alpha_2\alpha_6\alpha_4 + \alpha_4\alpha_7\alpha_2 \\
T_{4b} &= +\alpha_2\alpha_5\alpha_4 + \alpha_4\alpha_2\alpha_8 + \alpha_4\alpha_7\alpha_2 + \alpha_2\alpha_5\alpha_8 + \alpha_2\alpha_3\alpha_5 + \alpha_7\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_8\alpha_5\alpha_4 \\
&\quad + \alpha_5\alpha_7\alpha_4 + \alpha_4\alpha_1\alpha_8 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_1\alpha_7 + \alpha_1\alpha_3\alpha_7 \\
U_G &= \alpha_2\alpha_5\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_3\alpha_5 + \alpha_5\alpha_7\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_6 + \alpha_2\alpha_6\alpha_4 \\
&\quad + \alpha_5\alpha_6\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_7 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_7\alpha_2 + \alpha_4\alpha_7\alpha_2 + \alpha_3\alpha_5\alpha_6 \\
&\quad + \alpha_2\alpha_3\alpha_8 + \alpha_2\alpha_5\alpha_8 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_8\alpha_5\alpha_6 + \alpha_5\alpha_3\alpha_8 + \alpha_1\alpha_8\alpha_5 + \alpha_1\alpha_8\alpha_6 \\
&\quad + \alpha_6\alpha_2\alpha_8 + \alpha_1\alpha_8\alpha_3 + \alpha_4\alpha_1\alpha_8 + \alpha_4\alpha_2\alpha_8 + \alpha_7\alpha_2\alpha_8 + \alpha_8\alpha_1\alpha_7 \\
M_G &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7
\end{aligned}
$$

- The integral above is a projective integral: one α-parameter may be set 1, the others have to be integrated from 0 to ∞
- We are interested in the Laurent expansion in ε up to $O(\varepsilon^0)$. In this case the integral is finite for $\varepsilon \longrightarrow 0$, so we may evaluate it at $\varepsilon = 0$.

Open questions:

- How can the Laurent expansion be performed before the integration?
- Is there an algorithmic way to determine a priori if a multiple parametric integral without singularities inside the integration domain is divergent ?

Hyperlogarithms

- Feynman parameter integrals are peformed in terms of hyperlogarithmic functions $L(\overrightarrow{w}, z) : \mathbb{C}\backslash\Sigma \to \mathbb{C}$ [Brown 2008, Brown 2009], where
  - $\Sigma = \{\sigma_0, \sigma_1, ..., \sigma_N\}$ are distinct points in $\mathbb{C}$ which may contain variables
  - $\overrightarrow{w}$ is a word over the alphabet $\mathfrak{A} = \{a_0, a_1, ..., a_N\}$ where each letter a_i corresponds to a point σ_i
- $L\left(\overrightarrow{w}, z\right)$ is uniquely defined by the following properties

  1. $L\left(\{\}, z\right) = 1$, and $L\left(0^n, z\right) = \frac{1}{n!}\log^n(z)$ for $n \geq 1$
  2. $\frac{\partial}{\partial z} L\left(\left\{a_i, \overrightarrow{w}\right\}, z\right) = \frac{1}{z-\sigma_i} L_{\overrightarrow{w}}(z)$ for $z \in \mathbb{C}\backslash\Sigma$
  3. If $\overrightarrow{w}$ is not of the form $w = (0, 0, \cdots, 0)$, then $\lim_{z\to 0} L_{\overrightarrow{w}}(z) = 0$.

- e.g. $L\left(\{a_i\}, z\right) = \log(z - \sigma_i) - \log(\sigma_i)$
- The weight of $L\left(\overrightarrow{w}, z\right)$ is given by the number of letters in $\overrightarrow{w}$

- The Hyperlogarithms satisfy shuffle relations $L_{\vec{w_1}}(z)L_{\vec{w_2}}(z) = L_{\vec{w_1} \sqcup\!\sqcup \vec{w_2}}(z)$, e.g.: $L_{a_1,a_2}(z)L_{a_3}(z) = L_{a_3,a_1,a_2}(z) + L_{a_1,a_3,a_2}(z) + L_{a_1,a_2,a_3}(z)$
- These properties allow to express a primitive in z for expressions consisting of rational and hyperlogarithmic functions in z in terms of different hyperlogarthmic functions, when the denominator factors into linear polynomials in z.
- These primitives have then have to be evaluated at the respective integration limits. Individual terms can diverge at $z \to 0$,$z \to \infty$. We need an (asymptotic) expansion in z.
    - The expansion around $z \to 0$ is trivially obtained by computing a taylor series for the hyperlogarithmic functions.
    - The asymptotic expansion is more sophisticated. General idea:
        1. Fix an integration order before performing the first integral.
        2. Compute the derivative with respect to the next integration variable x, (this lowers the weight by one).
        3. Perform the series expansion of the derivative.
        4. Perform the indefinite integration with respect to x.
        5. Determine the respective integration constant.

Example: $L\left(\{-x-1,0\},y\right) \underset{y\to\infty}{\simeq} ?$

- Compute $\frac{\partial}{\partial x} L\left(\{-x-1,0\},y\right)$:
  - $\frac{\partial}{\partial x}\frac{\partial}{\partial y} L\left(\{-x-1,0\},y\right) = -\frac{L(\{0\},y)}{(y+1+x)^2}$

$$\begin{aligned}\frac{\partial}{\partial x} L\left(\{-x-1,0\},y\right) &= \int dy \frac{\partial}{\partial x}\frac{\partial}{\partial y} L\left(\{-x-1,0\},y\right) + Const \\ &= \left(\frac{1}{y+x+1} - \frac{1}{x+1}\right) L\left(\{0\},y\right) + \frac{L\left(\{-x-1\},y\right)}{x+1} + Const\end{aligned}$$

  - Fix $Const$ such that $\lim_{y\to 0} \frac{\partial}{\partial x} L\left(\{-x-1,0\},y\right) = 0$, in this case $Const = 0$
- The asymptotic expansion of this derivative is

$$asympt\left(\frac{\partial}{\partial x} L\left(\{-x-1,0\},y\right)\right) = -\frac{L\left(\{-1\},x\right)}{x+1}$$

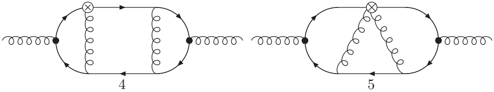
- $\int asympt\left(\frac{\partial}{\partial x} L\left(\{-x-1,0\},y\right)\right) = L\left(\{-1,-1\},x\right)$
- We have to add the respective integration constant , which is given by $asympt\left(L\left(\{-1,0\},y\right)\right) = L\left(\{0,0\},y\right) - \zeta_2$
- Thus $lim_{y\to\infty} L\left(\{-x-1,0\},y\right) = L\left(\{-1,-1\},x\right) + L\left(\{0,0\},y\right) - \zeta_2$.

- This method only works as long as all denominators factor into linear expressions in the integration variable at every integration step.
- This can be tested for any rational integrand with Stembridge's reduction algorithm [Stembridge 1998, Brown 2008, Brown 2009.]
- Basic idea:
- If we integrate expressions with different linear polynomials $p_i(x) = A_i(x_i) + B_i(x_i)\ x$ in the denominator from 0 to ∞ we get expressions whose denominators are all included in the set

$$\left\{ p_i(x)|_{x=0}, \frac{\partial}{\partial x} p_i(x), res(p_i, p_j) \right\} = \quad \{B_i, A_i, A_i\ B_j - A_j\ B_i\} \ .$$

- Now: Do this iteratively for all possible integration orders, and skip those, where I run into quadratic expressions in the respective present integration variable.
- Finally: Take the intersection of the results for all integration orders. Usually many terms cancel here.
- The denominators at a specific integration step do not only depend on the denominators of the previous step. The whole integration history matters.
- This reduction algorithm also allows to determine which class of transcendental numbers and which alphabets of Hyperlogarithms or Harmonic Sums will appear in the final result.
- E.g. Denominators in the last step $\{\frac{dx}{x+1}\} \longrightarrow$ multiple zeta values.
- Denominators in the last step $\{\frac{dx}{x+1}, \frac{dx}{x-1}\} \longrightarrow$ alternating multiple zeta values.

- Using this method we have computed a number of fixed Mellin-Moments from $N = 0..19$ e.g.:



N	Diag 4	Diag 5_a	Diag 5_b
0	$2 - 2\zeta_3$	$2\zeta_3$	$2\zeta_3$
1	$-2 + 2\zeta_3$	$-\frac{5}{2} - \zeta_3$	$-2 - 2\zeta_3$
2	$\frac{29}{12} - \frac{83}{36}\zeta_3$	$\frac{133}{72} + \frac{41}{8}\zeta_3$	$\frac{71}{24} + \frac{5}{2}\zeta_3$
3	$-\frac{17}{6} + \frac{47}{18}\zeta_3$	$-\frac{1735}{432} - \frac{35}{36}\zeta_3$	$-\frac{905}{216} - \frac{5}{2}\zeta_3$
...	...	...	...
19	$-\frac{58251582368792530944134896585691981}{2503562235895708381108915200000} - \frac{104899807174743864253}{54192375991353600}\zeta_3$	$-\frac{128090266890628029062643215783549}{133523319247771113659142144000} + \frac{238388793949217497}{301068755507520}\zeta_3$	$-\frac{254116903575797385411050257769}{25288507433289983647564800000} - \frac{1968329}{635040}\zeta_3$

[Ablinger, Blümlein, Hasselhuhn, Klein, Schneider, Wißbrock 2012.]

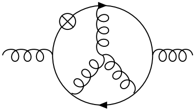
General N-representation

- Due to the power function the integrals do not fit directly into the framwork of the algorithm for general values of N
- In order to use the algorithm also on integrals with general values of N, a generating function is constructed by the mapping

$$p(\alpha_1, \cdots, \alpha_n)^N \rightarrow \frac{1}{1 - x\, p(\alpha_1, \cdots, \alpha_n)} \ ,$$

  which defines a formal power series in x
- Performing the α-parameter integrations then leads to an expression which contains Hyperlogarithms $L_w(x)$ in the variable x
- Finally the Nth coefficient of this expression in x has to be extracted symbolically. This has been done with the package HarmonicSums by J.Ablinger [Ablinger, Blümlein, Schneider 2012.]

- For example, Benz-topology:



$$
\begin{aligned}
I(x) \quad = \quad & \frac{1}{(1+N)(2+N)x}\Big\{\zeta_3\Big[2L(\{-1\},x) - 2(-1+2x)L(\{1\},x) - 4L(\{1,1\},x)\Big) \\
& -3L(\{-1,0,0,1\},x) + 2L(\{-1,0,1,1\},x) - 2xL(\{0,0,1,1\},x) \\
& +3xL(\{0,1,0,1\},x) - xL(\{0,1,1,1\},x) + (-3+2x)L(\{1,0,0,1\},x) \\
& +2xL(\{1,0,1,1\},x) - (-1+5x)L(\{1,1,0,1\},x) + xL(\{1,1,1,1\},x) \\
& -2L(\{1,0,0,1,1\},x) + 3L(\{1,0,1,0,1\},x) - L(\{1,0,1,1,1\},x) \\
& +2L(\{1,1,0,0,1\},x) + 2L(\{1,1,0,1,1\},x) - 5L(\{1,1,1,0,1\},x) \\
& +L(\{1,1,1,1,1\},x)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
I(N) &= \frac{1}{(N+1)(N+2)(N+3)}\Bigg\{\frac{648+1512N+1458N^2+744N^3+212N^4+32N^5+2N^6}{(1+N)^3(2+N)^3(3+N)^3} \\
&-\frac{2\left(-1+(-1)^N+N+(-1)^N N\right)}{(1+N)}\zeta_3-(-1)^N S_{-3}-\frac{N}{6(1+N)}S_1^3+\frac{1}{24}S_1^4 \\
&-\frac{(7+22N+10N^2)}{2(1+N)^2(2+N)}S_2-\frac{19}{8}S_2^2-\frac{1+4N+2N^2}{2(1+N)^2(2+N)}S_1^2+\frac{9}{4}S_2-\frac{(-9+4N)}{3(1+N)}S_3 \\
&-\frac{1}{4}S_4-2(-1)^N S_{-2,1}+\frac{(-1+6N)}{(1+N)}S_{2,1}+\frac{54+207N+246N^2+130N^3+32N^4+3N^5}{(1+N)^3(2+N)^2(3+N)^2}S_1 \\
&+4\zeta_3 S_1-\frac{(-2+7N)}{2(1+N)}S_2S_1+\frac{13}{3}S_3S_1-7S_{2,1}S_1-7S_{3,1}+10S_{2,1,1}\Bigg\}
\end{aligned}
$$

- Harmonic Sums and there generalizations are defined by

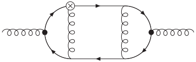
$$
S_{a_1,\ldots,a_m}(N)=\sum_{n_1=1}^{N}\sum_{n_2=1}^{n_1}\cdots\sum_{n_m=1}^{n_{m-1}}\frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}}\frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}}\cdots\frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}}
$$

[Vermaseren 1998, Blümlein, Kurth, 1998]

$$
S_{a_1,\ldots,a_m}(x_1,\cdots,x_m)(N)=\sum_{n_1=1}^{N}\sum_{n_2=1}^{n_1}\cdots\sum_{n_m=1}^{n_{m-1}}\frac{(\mathrm{sign}(a_1))^{n_1}x_1^{n_1}}{n_1^{|a_1|}}\frac{(\mathrm{sign}(a_2))^{n_2}x_2^{n_2}}{n_2^{|a_2|}}\cdots\frac{(\mathrm{sign}(a_m))^{n_m}x_m^{n_m}}{n_m^{|a_m|}}
$$

[Moch, Uwer, Weinzierl, 2002]

- Further example: Diagram 4
- In intermediate steps around 60.000 different hyperlogarithms have been observed. Almost all of them canceled as predicted by the Stembridge reduction.
- The computation took around 18h on one machine.



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$$
\begin{aligned}
\hat{I}_4 = {} & \frac{Q_1(N)}{2(1+N)^5(2+N)^5(3+N)^5} + \frac{Q_2(N)}{(1+N)^2(2+N)^2(3+N)^2}\zeta_3 + \frac{(-1)^N(65+101N+56N^2+13N^3+N^4)}{2(1+N)^2(2+N)^2(3+N)^2}S_{-3} \\
& + \frac{(-24-5N+2N^2)}{12(2+N)^2(3+N)^2}S_1^3 - \frac{1}{2(1+N)(2+N)(3+N)}S_2^2 + \frac{1}{(2+N)(3+N)}S_1^2S_2 \\
& + \frac{Q_4(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_1^2 - \frac{3}{2}S_5 - \frac{Q_5(N)}{6(1+N)^2(2+N)^2(3+N)^2}S_3 - 2S_{-2,-3} - 2\zeta_3S_{-2} - S_{-2,1}S_{-2} \\
& + \frac{(-1)^N(65+101N+56N^2+13N^3+N^4)}{(1+N)^2(2+N)^2(3+N)^2}S_{-2,1} + \frac{(59+42N+6N^2)}{2(1+N)(2+N)(3+N)}S_4 + \frac{(5+N)}{(1+N)(3+N)}\zeta_3S_1(2) \\
& - \frac{Q_6(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_2 - \zeta_3S_2 - \frac{3}{2}S_3S_2 - 2S_{2,1}S_2 + \frac{(99+225N+190N^2+65N^3+7N^4)}{2(1+N)^2(2+N)^2(3+N)}S_{2,1} \\
& + \frac{Q_3(N)}{(1+N)^4(2+N)^4(3+N)^4}S_1 - \frac{(11+5N)}{(1+N)(2+N)(3+N)}\zeta_3S_1 - \frac{Q_7(N)}{4(1+N)^2(2+N)^2(3+N)^2}S_2S_1 - S_{2,3} \\
& + \frac{(53+29N)}{2(1+N)(2+N)(3+N)}S_3S_1 - \frac{3(3+2N)}{(1+N)(2+N)(3+N)}S_1S_{2,1} + \frac{(-79-40N+N^2)}{2(1+N)(2+N)(3+N)}S_{3,1} - 3S_{4,1} \\
& + S_{-2,1,-2} + \frac{\mathbf{2^{N+1}}(-28-25N-4N^2+N^3)}{(1+N)^2(2+N)(3+N)^2}S_{1,2}\left(\frac{1}{2},1\right) - \frac{(-7+2N^2)}{(1+N)(2+N)(3+N)}S_{2,1,1} \\
& + 5S_{2,2,1} + 6S_{3,1,1} + \frac{\mathbf{2^{N}}(-28-25N-4N^2+N^3)}{(1+N)^2(2+N)(3+N)^2}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
& - \frac{(5+N)}{(1+N)(3+N)}S_{1,1,2}\left(2,\frac{1}{2},1\right) - \frac{(5+N)}{2(1+N)(3+N)}S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)
\end{aligned}
$$

- One may ask: Is does this result diverge as $N \to \infty$?
- Asymptotic Expansion using HarmonicSums.m gives:

$$
\begin{aligned}
\hat{I}_4 \approx\ & \zeta_2^2 \left[\frac{1115231}{20N^{10}} - \frac{74121}{4N^9} + \frac{122951}{20N^8} - \frac{40677}{20N^7} + \frac{13391}{20N^6} - \frac{873}{4N^5} + \frac{1391}{20N^4} - \frac{417}{20N^3} + \frac{101}{20\,N^2}\right] \\
& + \zeta_3 \left[\left(-\frac{95855}{2N^{10}} + \frac{31525}{2N^9} - \frac{10295}{2N^8} + \frac{3325}{2N^7} - \frac{1055}{2N^6} + \frac{325}{2N^5} - \frac{95}{2N^4} + \frac{25}{2N^3} - \frac{5}{2\,N^2}\right)\ln(N)\right. \\
& \left. - \frac{23280115}{2016N^{10}} + \frac{2093041}{1008N^9} - \frac{177251}{1008N^8} - \frac{25843}{336N^7} + \frac{2569}{48N^6} - \frac{155}{8N^5} + \frac{91}{24N^4} + \frac{2}{3N^3} - \frac{11}{12\,N^2}\right] \\
& + \zeta_2 \left[\left(\frac{19171}{N^{10}} - \frac{6305}{N^9} + \frac{2059}{N^8} - \frac{665}{N^7} + \frac{211}{N^6} - \frac{65}{N^5} + \frac{19}{N^4} - \frac{5}{N^3} + \frac{1}{N^2}\right)\ln^2(N)\right. \\
& + \left(\frac{103016863}{2520N^{10}} - \frac{3091261}{315N^9} + \frac{2571839}{1260N^8} - \frac{6215}{21N^7} - \frac{293}{20N^6} + \frac{2071}{60N^5} - \frac{103}{6N^4} + \frac{67}{12N^3} - \frac{1}{N^2}\right)\ln(N) \\
& \left. + \frac{292993001621}{302400N^{10}} - \frac{4402272031}{30240N^9} + \frac{22261739}{840N^8} - \frac{78507473}{14112N^7} + \frac{180961}{144N^6} - \frac{111807}{400N^5} + \frac{629}{12N^4} - \frac{319}{72N^3} - \frac{7}{4\,N^2}\right] \\
& + \left(\frac{249223}{6N^{10}} - \frac{145015}{12N^9} + \frac{10295}{3N^8} - \frac{11305}{12N^7} + \frac{1477}{6N^6} - \frac{715}{12N^5} + \frac{38}{3N^4} - \frac{25}{12N^3} + \frac{1}{6\,N^2}\right)\mathbf{ln^3(N)} \\
& + \left(\frac{193493767}{10080N^{10}} + \frac{210658237}{10080N^9} - \frac{21541697}{2520N^8} + \frac{243269}{96N^7} - \frac{30539}{48N^6} + \frac{2123}{16N^5} - \frac{59}{3N^4} + \frac{5}{8N^3} + \frac{1}{2\,N^2}\right)\ln^2(N) \\
& + \left(-\frac{2207364771673}{4233600N^{10}} + \frac{1390655509}{352800N^9} + \frac{285594061}{22050N^8} - \frac{67234111}{14400N^7} + \frac{8617073}{7200N^6} - \frac{35209}{144N^5} + \frac{116}{3N^4} - \frac{119}{24N^3} + \frac{1}{N^2}\right)\ln(N) \\
& + \frac{1344226725047831}{889056000N^{10}} - \frac{165849841805771}{889056000N^9} + \frac{808151260279}{27783000N^8} - \frac{708430537}{120960N^7} + \frac{304474703}{216000N^6} \\
& - \frac{606811}{1728N^5} + \frac{1867}{24N^4} - \frac{1813}{144N^3} + \frac{1}{N^2} + O(N^{-11})
\end{aligned}
$$

Conclusion

- Massive operator matrix elements (OMEs) are essential to understand the heavy quark behaviour of different processes, e.g. deeply inelastic scattering.
- OMEs are evaluated in terms of Feynman diagrams with additional Feynman rules representing the operator insertion.
- We present a method based on hyperlogarithmic functions and their asymptotic representations to evaluate non-singular diagrams with operator insertions in $d = 4$ dimensions.
- The method allows a relativly fast evaluation of fixed Mellin-Moments.
- Furthermore it has been extended to allow also for the computation of Feynman diagrams with operator insertions at general values of N. It is relatively fast, yields compact results and allows to compute diagrams, which could not be evaluated with different methods so far.