RISC–DESY Workshop, RISC, Hagenberg, May 7–8 2012

Computer algebra and elementary particle physics: a successful story

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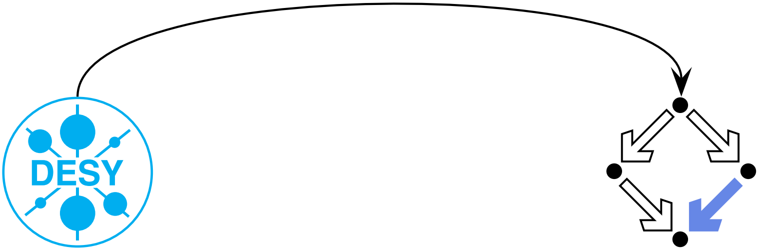
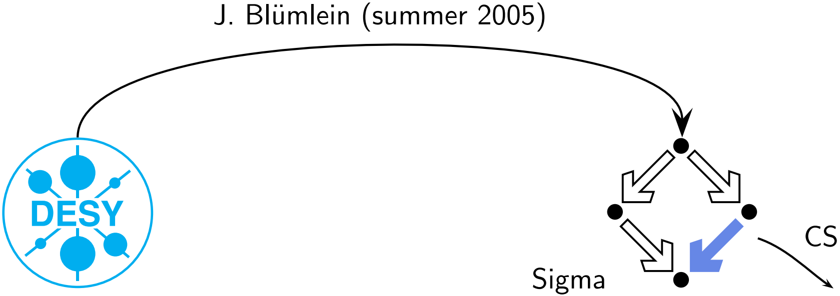
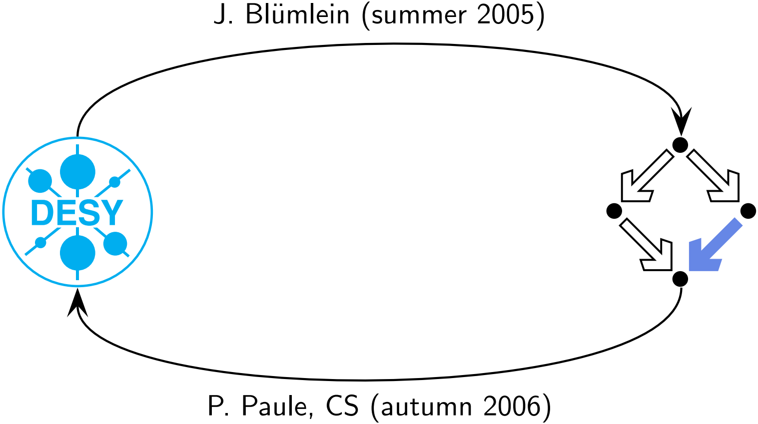
involved actors:
J. Blümlein, I. Bierenbaum, S. Klein, F. Wissbrock, A. Hasselhuhn,
A. de Freitas, T. Riemann (DESY)
J. Ablinger, M. Kauers, F. Stan, M. Round (RISC)

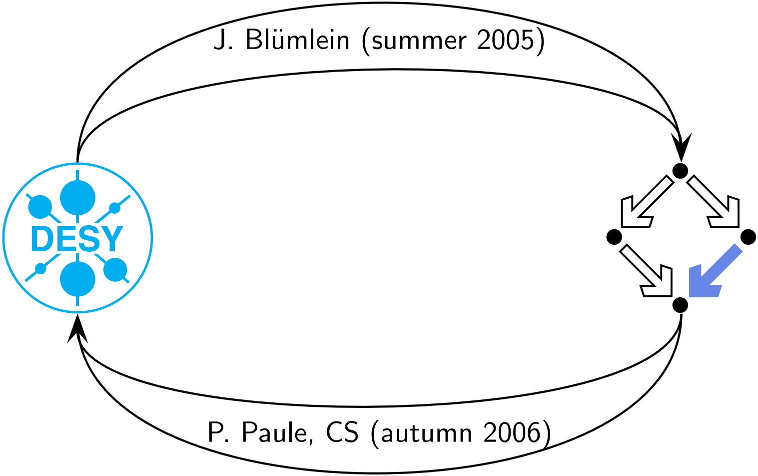
May 7, 2012

J. Blümlein (summer 2005)




J. Blümlein (summer 2005)
DESY
Sigma
CS


J. Blümlein (summer 2005)
DESY
P. Paule, CS (autumn 2006)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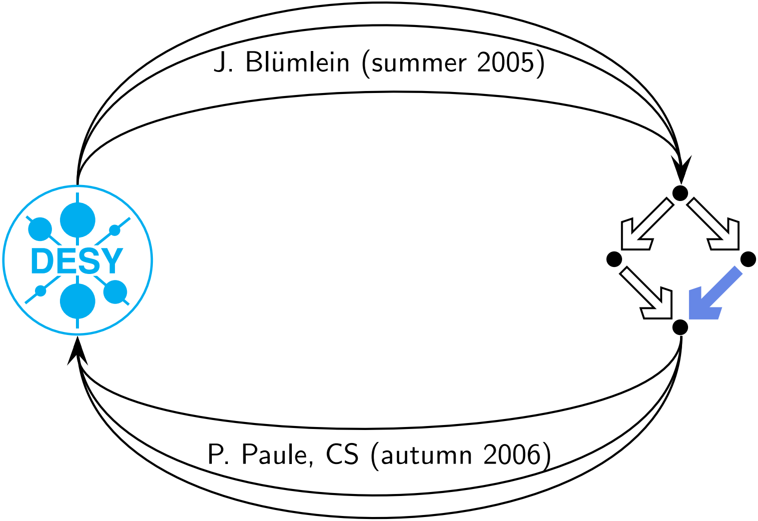
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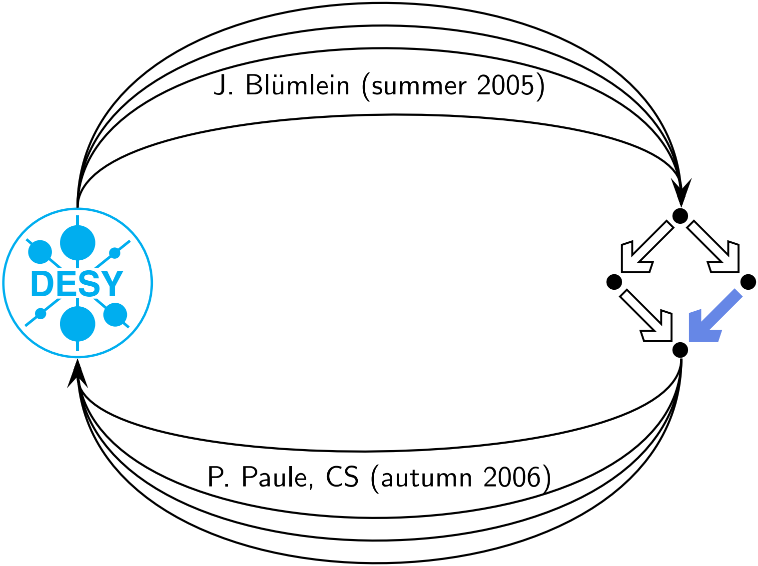
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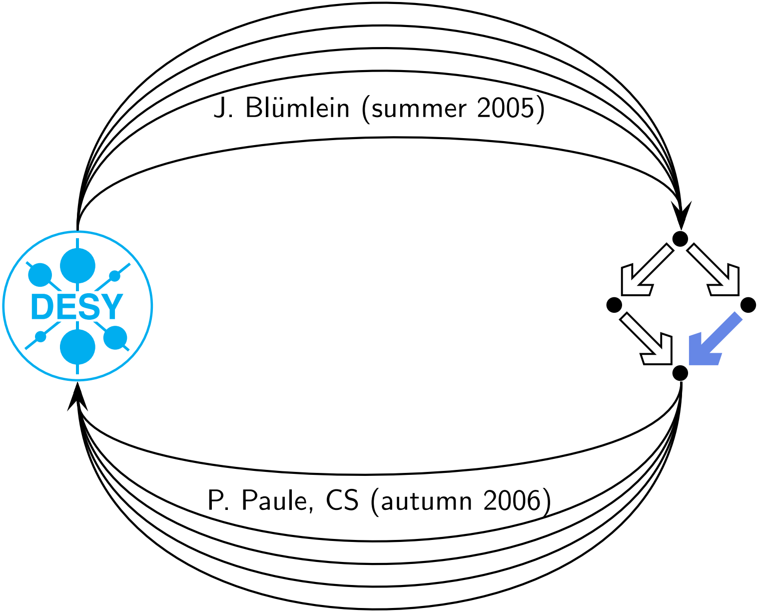
J. Blümlein (summer 2005)
DESY
P. Paule, CS (autumn 2006)

A First Attack

One of the first sums:

$$\text{GIVEN } F(n) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)} \times$$

$$\times \underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right)}_{f(n,k,j)}.$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

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FIND the first coefficients of the ϵ-expansion

$$F(N) = F_0(n) + \varepsilon F_1(n) + \dots$$

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Step 1: Compute the first coefficients of the ϵ-expansion

$$f(n,k,j) = \qquad f_0(n,k,j) + \qquad \varepsilon f_1(n,k,j) + \dots$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

One of the first sums:

$$\text{GIVEN } F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}\right.
+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(n,k,j)}.$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

Arose in the context of

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Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(n+1)!}{(k+1)(n+1)(k+n+1)!}\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}}^{f(n,k,j)}$$
$$\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}(=H_n)$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(n+1)!}{(k+1)(n+1)(k+n+1)!}\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}}^{f(n,k,j)}$$

$$\sum_{j=0}^{a} f(n,k,j) = ?$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(n+1)!}{(k+1)(n+1)(k+n+1)!}\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}}^{f(n,k,j)}$$
$$\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

$$\sum_{j=0}^{a} f(n,k,j) = \text{?}$$

FIND $g(j)$:

$$\boxed{f(n,k,j)=g(j+1)-g(j)}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(n+1)!}{(k+1)(n+1)(k+n+1)!}\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}}^{f(n,k,j)}$$
$$\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

$$\sum_{j=0}^{a} f(n,k,j) = \textbf{?}$$

FIND $g(j)$:

$$\boxed{f(n,k,j)=g(j+1)-g(j)}$$

Sigma (based on a refined version of M. Karr's difference fields (1981)) computes

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(n+1)!}{(k+1)(n+1)(k+n+1)!}\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}}^{f(n,k,j)}$$
$$\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

$$\sum_{j=0}^{a} f(n,k,j) = ?$$

FIND $g(j)$:

$$\boxed{f(n,k,j)=g(j+1)-g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\boxed{\sum_{j=0}^{a} f(n,k,j)=g(a+1)-g(0)}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(n+1)!}{(k+1)(n+1)(k+n+1)!}\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}}^{f(n,k,j)}$$
$$\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

$$\sum_{j=0}^{a} f(n,k,j)=\underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} +\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}+\frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(n+1)!}{(k+1)(n+1)(k+n+1)!}\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}}^{f(n,k,j)}$$

$$\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(n+1)!}{(k+1)(n+1)(k+n+1)!}\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}}^{f(n,k,j)}$$
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$$\sum_{k=1}^{a}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=1}^{a}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$

Telescoping

GIVEN

$$\text{SUM}(n) := \underbrace{\sum_{k=1}^{a} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}}_{=: f(n,k)}.$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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$$\text{SUM}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+1)(n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)n!(n+1)^2}$$

Zeilberger's creative telescoping paradigm

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 0 to a gives:

$$\boxed{g(n,a+1)-g(n,0)} = \boxed{c_0(n)\,\mathsf{SUM}(n)+c_1(n)\,\mathsf{SUM}(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{SUM}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)n!}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

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$$\boxed{g(n,a+1) - g(n,0)} = \boxed{c_0(n)\,\text{SUM}(n) + c_1(n)\,\text{SUM}(n+1)}$$

$$\| \qquad\qquad \|$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!} \qquad -n\text{SUM}(n) + (1+n)(2+n)\text{SUM}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} | c \in \mathbb{R}\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = \frac{1}{2} \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

Simplify the constant term:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\overbrace{\frac{(k+1)!(n+1)!}{(k+1)(n+1)(k+n+1)!}\frac{(j+1)!(j+k+n+1)!}{(j+k+1)!(j+n+1)!}}^{f(n,k,j)}$$
$$\times\frac{\frac{2j+k+n+2}{(j+k+1)(j+n+1)}-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)}{(j+1)(j+k+n+1)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!}$$
$$=\frac{S_1(n)^2+S_2(n)}{2n(n+1)!}$$

where

$$S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

1. Creative telescoping (for the original version for hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

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FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products and sums
(Abramov/Bronstein/Petkovšek/Schneider, in preparation)

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NOTE: By construction, the solutions are highly nested.

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3. Indefinite summation (by a new summation theory of $\Pi\Sigma^*$-fields)

Simplify the solutions:

- The sums have minimal nested depth.
- No algebraic relations occur among the sums.

1. Creative telescoping (for the original version for hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=0}^{n} f(n,k);$$

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FIND all solutions expressible by indefinite nested products and sums
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4. Find a “closed form”

A(n)=combined solutions.

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\,\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$
$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)=\frac{-S_1(n)^3-3S_2(n)S_1(n)-8S_3(n)}{6n(n+1)!}.$$

Partnerschaftsabkommen zwischen der JKU und dem DESY

[15.02.2007] Am 12. Februar 2007 wurde ein Partnerschaftsabkommen zwischen der JKU und dem DESY unterzeichnet. Das Deutsche Elektronen-Synchrotron DESY in der Helmholtz-Gemeinschaft ist eines der weltweit führenden Zentren für die Forschung an Teilchenbeschleunigern.

DESY ist ein mit öffentlichen Mitteln finanziertes nationales Forschungszentrum und hat zwei Standorte: Hamburg und Zeuthen (bei Berlin). Der wissenschaftliche Partner auf JKU Seite ist die RISC Arbeitsgruppe von Univ.-Prof. Dr. Peter Paule, welche zu den international führenden Teams bezüglich der algorithmischen Behandlung von speziellen Funktionen zählt.

Der Partner auf DESY Seite ist die Abteilung fuer theoretische Physik von Dr. habil. Johannes Blümlein am DESY Zeuthen. Der Gegenstand künftiger gemeinsamer Forschung betrifft die Entwicklung und den Einsatz von Methoden der Computeralgebra im Zusammenhang mit Problemstellungen der Quantenfeldtheorie, angewandt auf die

RISC - DESY Workshop
on the occasion of the 5th year jubilee of the RISC-DESY cooperation

Feb 7, 2012: RISC (JKU) and DESY prolong their cooperation until 2017

Highlight 1:

Efficient difference field algorithm

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_1(n)}{2n(n+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)=\frac{-S_1(n)^3-3S_2(n)S_1(n)-8S_3(n)}{6n(n+1)!}.$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)=\frac{1}{96n(n+1)}\Big(S_1(n)^4+(12\zeta_2+54S_2(n))S_1(n)^2$$

$$+104S_3(n)S_1(n)-48S_{2,1}(n)S_1(n)+51S_2(n)^2+36\zeta_2S_2(n)$$

$$+126S_4(n)-48S_{3,1}(n)-96S_{1,1,2}(n)\Big)$$

GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$
$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^3\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

Sigma computes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(n,k,j)=\frac{1}{960n(n+1)}\Big(S_1(n)^5+(20\zeta_2+130S_2(n))S_1(n)^3+$$
$$(40\zeta_3+380S_3(n))S_1(n)^2+\left(135S_2(n)^2+60\zeta_2S_2(n)+510S_4(n)\right)S_1(n)$$
$$-240S_{3,1}(n)S_1(n)-240S_{1,1,2}(n)S_1(n)+160\zeta_2S_3(n)+S_2(n)(120\zeta_3$$
$$+380S_3(n))+624S_5(n)+\left(-120S_1(n)^2-120S_2(n)\right)S_{2,1}(n)$$
$$-240S_{4,1}(n)-240S_{1,1,3}(n)+240S_{2,2,1}(n)\Big)$$

Completion of massive 2–loop calculations: I. Bierenbaum, J. Blümlein, S. Klein, CS, Nucl.Phys. **B803** (2008)

$$\sum_{i,j=1}^{\infty}\frac{S_1(i)S_1(i+j+n)}{i(i+j)(j+n)}$$
$$=6\frac{S_1(n)}{n}\zeta_3+\zeta_2\left(2\frac{S_1^2(n)}{n}+\frac{S_2(n)}{n}\right)+\frac{S_1^4(n)}{6n}+\frac{S_1^2(n)S_2(n)}{n}$$
$$-\frac{S_2^2(n)}{n}+4\frac{S_{2,1,1}(n)}{n}+S_1(n)\left(-3\frac{S_{2,1}(n)}{n}+4\frac{S_3(n)}{3n}\right)-2\frac{S_{3,1}(n)}{n}-\frac{S_4(n)}{2n},$$

$$\sum_{i=1}^{\infty}\frac{B(n,i)}{i+n+2}S_1(i)S_1(n+i) \qquad B(n,i)=\frac{\Gamma(n)\Gamma(i)}{\Gamma(n+i)}$$
$$=\frac{(-1)^n}{n(n+1)(n+2)}\Bigg(4S_{-2,1}(n)-6S_{-3}(n)-4S_{-2}(n)S_1(n)$$
$$-2\zeta_2S_1(n)-2\zeta_3-2\frac{\zeta_2}{(n+1)}-4\frac{S_{-2}(n)}{(n+1)}\Bigg)-2\frac{S_3(n)}{n+2}$$
$$-\frac{S_1(n)S_2(n)}{n+2}+\frac{\zeta_2S_1(n)}{n+2}+\frac{2\zeta_3}{n+2}+\frac{2+7n+7n^2+5n^3+n^4}{n^3(n+1)^3(n+2)}S_1(n)$$

$+$ **60 similar sums**

Highlight 2: Guessing and Finding

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\dots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

where $K\in\mathbb{N}$, $r_i,s_i\in\mathbb{Q}$, and p_i,q_i are polynomials in $x_1,\dots,x_7$.

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

The 3-loop anomalous dimensions can be derived from the single pole part of $F(n,\varepsilon)$. The other poles are needed for the renormalization.

Vermaseren, Moch: 3-5 CPU years (2004)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon)=\int_0^1 dx_1\dots\int_0^1 dx_7\sum_{i=1}^{K}\frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+F_{-1}(n)\varepsilon^{-1}+\boxed{F_0(n)}\varepsilon^0+\dots$$

$$\downarrow$$

Initial values $F_0(i)$, $i=1,\dots,5114$

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

$$\downarrow$$

Initial values $F_0(i)$, $i = 1,\dots,5114$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}\,F_0(n+35) = 0$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}\,F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1 n + A_2 n^2 + \cdots + A_{938} n^{983} \in \mathbb{Z}[n]$$

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \cdots + \boxed{a_{35}(n)}\,F_0(n+35) = 0$$

$$a_{35}(n) = \boxed{A_0} + A_1 n + A_2 n^2 + \cdots + A_{938} n^{983} \in \mathbb{Z}[n]$$

$A_0 =$ 4640944309211313672503980223716264124200407085993854002412460315194
9576502126934497104844629972221629340528573833320076715019401639150166 6
2795021380735610971095204560396627338875778269758860220127798356053201 7
3748759267144591132576514527194521425546215314730842059721076159532936 5
5156345299861313538471891130525329905319889360640146402160891162097419 2
0900166802995162078018294725826293945080115451177452783250387434166189 8
8916752210737846879797981026538551064393704386755756346752374040609465 8
9910046793335373195964562497752442467299065442773230988168534648377112 8
6902083714745202440152816907940693366534447618126024334417209769163670 6
6280305967553580902716969306447414771961021984962848689607964231297513 6
2077687686774188348836384694485449648262937243682969905539136917885039 7
0038163801161230267958089748807664772131193063473531678777962075765995 1
520280997829905375390143206735962615 1

(885 decimal digits)

In the non-singlet (3-loop) case ~ 360 diagrams contribute. The integrals are of the form:

$$F(n,\varepsilon) = \int_0^1 dx_1 \dots \int_0^1 dx_7 \sum_{i=1}^{K} \frac{p_i(x_1,x_2,\dots,x_7)^{n+\dots+r_i\varepsilon+\dots}}{q_i(x_1,x_2,\dots,x_7)^{\dots+s_i\varepsilon+\dots}}$$

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}\varepsilon^0 + \dots$$

$\downarrow$

Initial values $F_0(i)$, $i = 1,\dots,5114$

$\downarrow$ Recurrence finder (M. Kauers)

$$a_0(n)F_0(n) + a_1(n)F_0(n+1) + \dots + a_{35}(n)F_0(n+35) = 0$$

$\downarrow$ Sigma

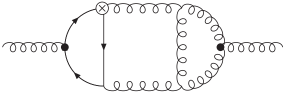
CLOSED FORM

Highlight 3: Automatization

Example: All n-Results for 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY), A. Hasselhuhn (DESY), S. Klein (RWTH)

In total around 50 diagrams (for this class) have been calculated, like e.g.



(containing three massive fermion propagators)

$\Downarrow$

Around 1000 sums have to be calculated for this diagram

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\|$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+ \frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$

$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}.$

Mathematica Session:

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $$\mathbf{EvaluateMultiSum}\Big[\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!},$$

$$\{\{s,0,n\text{-}j+r\text{-}2\},\{r,0,j+1\},\{j,0,n\text{-}2\}\}\Big]$$

Out[4]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

$$=\frac{\left(2n^2+6n+5\right)S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$

$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(n)=\sum_{i=1}^{n}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98/Blümlein/Kurth 99

A typical sum

$$\sum_{j=0}^{n-2}\sum_{s=1}^{j+1}\sum_{r=0}^{n+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+n+s-2}{r}(n-j)!(s-1)!\sigma!S_1(r+2)}{(n-r)(r+1)(r+2)(-j+n+\sigma+1)(-j+n+\sigma+2)(-j+n+s+\sigma)!}$$

$$=\frac{\left(2n^2+6n+5\right)S_{-2}(n)^2}{2(n+1)(n+2)}+S_{-2,-1,2}(n)+S_{-2,1,-2}(n)$$

$$+\dots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;n)+S_{2,1,1,1}(1,\frac{1}{2},1,2;n)$$

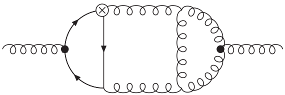
$$+\dots$$

where, e.g.,

145 S–sums occur

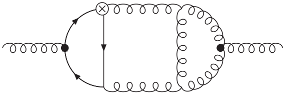
$$S_{2,1,1,1}(1,\frac{1}{2},1,2;n)=\sum_{i=1}^{n}\frac{\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^j\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



$\Downarrow$ Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums



$\Downarrow$ Sigma.m

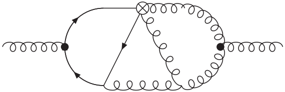
Around 1000 sums are calculated containing in total 533 S–sums

$\Downarrow$ J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

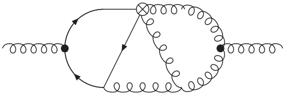
$$S_{-4}(n), S_{-3}(n), S_{-2}(n), S_1(n), S_2(n), S_3(n), S_4(n), S_{-3,1}(n),$$
$$S_{-2,1}(n), S_{2,-2}(n), S_{2,1}(n), S_{3,1}(n), S_{-2,1,1}(n), S_{2,1,1}(n)$$

So far, the most complicated 3–loop ladder graph:



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

So far, the most complicated 3–loop ladder graph:



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

$$\|$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

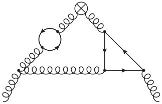
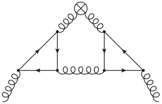
$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4+\frac{(17n+5)S_1(n)^3}{3n(n+1)}+\Big(\frac{35n^2-2n-5}{2n^2(n+1)^2}+\frac{13S_2(n)}{2}+\frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+\Big(-\frac{4(13n+5)}{n^2(n+1)^2}+\Big(\frac{4(-1)^n(2n+1)}{n(n+1)}-\frac{13}{n}\Big)S_2(n)+\Big(\frac{29}{3}-(-1)^n\Big)S_3(n)$$
$$+\big(2+2(-1)^n\big)S_{2,1}(n)-28S_{-2,1}(n)+\frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n)+\Big(\frac{3}{4}+(-1)^n\Big)S_2(n)^2$$
$$-2(-1)^nS_{-2}(n)^2+S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)}+\big(26+4(-1)^n\big)S_1(n)+\frac{4(-1)^n}{n+1}\Big)$$
$$+\Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)}-\frac{5}{2n^2}\Big)S_2(n)+S_{-2}(n)\Big(10S_1(n)^2+\Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+\frac{4(3n-1)}{n(n+1)}\Big)S_1(n)+\frac{8(-1)^n(3n+1)}{n(n+1)^2}+\big(-22+6(-1)^n\big)S_2(n)-\frac{16}{n(n+1)}\Big)$$
$$+\Big(\frac{(-1)^n(9n+5)}{n(n+1)}-\frac{29}{3n}\Big)S_3(n)+\Big(\frac{19}{2}-2(-1)^n\Big)S_4(n)+\big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+\Big(-\frac{2(-1)^n(9n+5)}{n(n+1)}-\frac{2}{n}\Big)S_{2,1}(n)+\big(20+2(-1)^n\big)S_{2,-2}(n)+\big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$-\frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n)-\big(24+4(-1)^n\big)S_{-3,1}(n)+\big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+32S_{-2,1,1}(n)+\left(\frac{3}{2}S_1(n)^2-\frac{3S_1(n)}{n}+\frac{3}{2}(-1)^nS_{-2}(n)\right)\zeta(2)$$

Highlight 4:

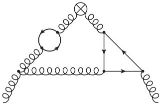
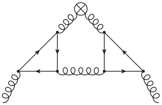
Packages for Large Scale Problems

Example: 3-loop topologies of gluonic massive operator matrix elements with two fermion lines (unpolarized case)

$D_\varepsilon(n) =$  $+$  $+ \sim$**80 further diagrams**

[by axodraw (J. Vermaseren)]

Example: 3-loop topologies of gluonic massive operator matrix elements with two fermion lines (unpolarized case)

$D_\varepsilon(n) =$ 
[by axodraw (J. Vermaseren)]
$+$ 
$+ \sim$**80 further diagrams**

$\downarrow$ J. Blümlein, A. Hasselhuhn

$$D_\varepsilon(n) = \sum_{j_1=0}^{n-5} \sum_{j_2=0}^{n-j_1-6} \pi 2^{\varepsilon+3} e^{-\frac{3\gamma\varepsilon}{2}} \Gamma(2-\varepsilon)\Gamma\left(\frac{\varepsilon}{2}+2\right)\Gamma\left(-\frac{3\varepsilon}{2}\right) \qquad (\sim 2\text{GB})$$

$$\times \frac{(-1)^{j_1}(j_2+1)\Gamma\left(-\frac{\varepsilon}{2}+j_1+4\right)\Gamma(-j_1+n-2)\Gamma(\varepsilon-j_1-j_2+n-5)}{(\varepsilon-10)(\varepsilon-8)(\varepsilon-2)\varepsilon\Gamma\left(\frac{5}{2}-\varepsilon\right)\Gamma\left(\frac{\varepsilon+5}{2}\right)\Gamma\left(\frac{\varepsilon}{2}+n+1\right)\Gamma(-j_1-j_2+n-4)}$$

$+ \sim$**2400 further multi-sums**

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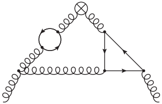
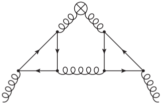
$D_\varepsilon(n) =$  [by axodraw (J. Vermaseren)] $+$  $+\sim$**80 further diagrams**

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$$\times \frac{(-1)^{j_1}(j_2+1)\Gamma\left(-\frac{\varepsilon}{2}+j_1+4\right)\Gamma(-j_1+n-2)\Gamma(\varepsilon-j_1-j_2+n-5)}{(\varepsilon-10)(\varepsilon-8)(\varepsilon-2)\varepsilon\Gamma\left(\frac{5}{2}-\varepsilon\right)\Gamma\left(\frac{\varepsilon+5}{2}\right)\Gamma\left(\frac{\varepsilon}{2}+n+1\right)\Gamma(-j_1-j_2+n-4)}$$

$+\sim$**2400 further multi-sums**

$\downarrow$ **time: $\sim$8 month**

$$D_\varepsilon(n) = \varepsilon^{-3}F_{-3}(n) + \varepsilon^{-2}F_{-2}(n) + \varepsilon^{-1}F_{-1}(n) + \varepsilon^{0}F_{0}(n) + \dots$$

Efficient calculation

1. Reduction to key sums

- Synchronize 2400 sums to 4 sums:

$$\sum_{i_2=5}^{n-5}\sum_{i_1=0}^{i_2} h_1(\varepsilon,n,i_2,i_1) \qquad \sum_{i_2=0}^{n-5}\sum_{i_1=0}^{n-i_2-5} h_2(\varepsilon,n,i_2,i_1)$$

$$\sum_{i_1=5}^{n-5} h_3(\varepsilon,n,i_1) \qquad \sum_{i_1=0}^{\infty} h_4(\varepsilon,n,i_1)$$

Note: 4 sums plus sum-free expression $>$ 2 GB

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- Eliminate algebraic relations among Γ/Pochhammer/binomial symbols

Efficient calculation

1. Reduction to key sums

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Note: 4 sums plus sum-free expression $>$ 2 GB

- Eliminate algebraic relations among Γ/Pochhammer/binomial symbols

- Write the sums in the form

$\sum$ (product of Γ/Pochhammer/binomial symbols)*(rational function)

Note: 29 sums, total size: 7.6 MB

Efficient calculation

1. Reduction to key sums

In[5]:= $<<$ **SumProduction.m**

SumProduction - A summation package by Carsten Schneider © RISC-Linz

In[6]:= **expr** $=<<$ **DESYInput.txt;**

In[7]:= **compactExpr** $=$

ReduceMultiSums[expr, {n}, {5}];

7hours
2GB $\rightarrow$ **7.6**MB

Efficient calculation

1. Reduction to key sums

In[5]:= << **SumProduction.m**

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In[6]:= **expr =<< DESYInput.txt;**

In[7]:= **compactExpr =**

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7hours
2GB → **7.6**MB

2. (Parallel) calculation of the ε–expansion for each sum:

In[8]:= **ProcessEachSum[compactExpr, {n}, {6},**

ExpandIn → {ep, −3, 0}]

90minutes

Efficient calculation

1. Reduction to key sums

In[5]:= $<<$ **SumProduction.m**

SumProduction - A summation package by Carsten Schneider © RISC-Linz

In[6]:= $\mathbf{expr} =<< \mathbf{DESYInput.txt;}$

In[7]:= $\mathbf{compactExpr} =$

$$\mathbf{ReduceMultiSums[expr, \{n\}, \{5\}];}$$

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2GB $\rightarrow$ **7.6**MB

2. (Parallel) calculation of the ε–expansion for each sum:

In[8]:= $\mathbf{ProcessEachSum[compactExpr, \{n\}, \{6\},}$

$$\mathbf{ExpandIn \rightarrow \{ep, -3, 0\}]}$$

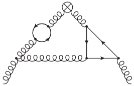
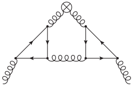
90minutes

3. Combine expansions (+eliminate algebraic relations):

In[9]:= $\mathbf{CombineExpression[compactExpr, \{n\}, \{6\}];}$

20seconds
7.6MB $\rightarrow$ **0.1**MB

Example: 3-loop topologies of gluonic massive operator matrix elements with two fermion lines (unpolarized case)

$D_\varepsilon(n) =$  [by axodraw (J. Vermaseren)] $+$  $+$ ∼80 further diagrams

$$\varepsilon^{-3}F_{-3}(n)+\varepsilon^{-2}F_{-2}(n)+\varepsilon^{-1}F_{-1}(n)+\varepsilon^{0}F_{0}(n)+\dots$$

where

$$F_{-3} = C_A\Big(\tfrac{512}{27}S_1(n) - \frac{64(3n^4+6n^3+19n^2+28n+28)}{27(n-1)n(n+1)(n+2)}\Big) - \frac{512C_F(n^2+n+2)^2}{9(n-1)n^2(n+1)^2(n+2)}$$

$$F_{-2} = C_A\Big(\frac{64(20n^2-20n+9)S_1(n)}{81(n-1)n} - \frac{16(3n^6+9n^5+367n^4+839n^3+1046n^2+568n+96)}{81(n-1)n^2(n+1)^2(n+2)}\Big)$$
$$+C_F\Big(\frac{128(n^2+n+2)^2S_1(n)}{9(n-1)n^2(n+1)^2(n+2)} - \frac{128(14n^6+33n^5+59n^4+39n^3+55n^2+20n-12)}{27(n-1)n^3(n+1)^3(n+2)}\Big)$$

$$F_{-1} = C_A\Big(\zeta_2\Big(\tfrac{64}{9}S_1(n) - \frac{8(3n^4+6n^3+19n^2+28n+28)}{9(n-1)n(n+1)(n+2)}\Big) + \frac{64(20n^6+57n^5+12n^4-56n^3-61n^2-30n-16)S_1(n)}{27(n-1)n^2(n+1)^2(n+2)} -$$
$$-\frac{4(57n^8+228n^7+4044n^6+12486n^5+17787n^4+12342n^3+1952n^2-2368n-960)}{81(n-1)n^3(n+1)^3(n+2)}\Big)$$
$$+C_F\Big(-\frac{160(n^2+n+2)^2S_1(n)^2}{9(n-1)n^2(n+1)^2(n+2)} + \frac{32(n^2+n+2)^2S_2(n)}{3(n-1)n^2(n+1)^2(n+2)} + \frac{64(16n^6+57n^5+268n^4+465n^3+410n^2+208n+4}{27(n-1)n^3(n+1)^3(n+2)}$$
$$-\frac{64(n^2+n+2)^2\zeta_2}{3(n-1)n^2(n+1)^2(n+2)} - \frac{64(185n^8+704n^7+1712n^6+2699n^5+3694n^4+3801n^3+2249n^2+744n+180)}{81(n-1)n^4(n+1)^4(n+2)}\Big)$$

$$
\begin{aligned}
F_0 = C_A\Big(&\frac{8S_1(n)^3}{27(n-1)n} + \frac{4\big(16n^5+31n^4-38n^3-3n^2+50n+32\big)S_1(n)^2}{27(n-1)n^2(n+1)^2(n+2)} + \Big(\frac{8\big(6944n^8+26480n^7+24941n^6-7003n^5-247\ldots}{729(n-1)n^3(n+\ldots}\\
&+\frac{8S_2(n)}{9(n-1)n}\Big)S_1(n) + \frac{4809n^{10}+24045n^9-224384n^8-1104398n^7-2105327n^6-2139551n^5-1133210n^4-209408n^3\ldots}{729(n-1)n^4(n+1)^4(n+2)}\\
&+\zeta_3\Big(\frac{56\big(3n^4+6n^3+19n^2+28n+28\big)}{27(n-1)n(n+1)(n+2)} - \frac{448}{27}S_1(n)\Big) + \zeta_2\Big(\frac{8\big(20n^2-20n+9\big)S_1(n)}{27(n-1)n}\\
&-\frac{2\big(3n^6+9n^5+367n^4+839n^3+1046n^2+568n+96\big)}{27(n-1)n^2(n+1)^2(n+2)}\Big) - \frac{4\big(40n^6+112n^5+7n^4-126n^3-251n^2-190n-96\big)S_2(n)}{27(n-1)n^2(n+1)^2(n+2)} + \ldots\\
&+C_F\Big(\frac{112\big(n^2+n+2\big)^2S_1(n)^3}{27(n-1)n^2(n+1)^2(n+2)} - \frac{16\big(44n^6+123n^5+386n^4+543n^3+520n^2+248n+24\big)S_1(n)^2}{27(n-1)n^3(n+1)^3(n+2)}\\
&+\Big(\frac{16S_2(n)\big(n^2+n+2\big)^2}{3(n-1)n^2(n+1)^2(n+2)} + \frac{32\big(205n^8+856n^7+3169n^6+6484n^5+7310n^4+4722n^3+1534n^2+48n-72\big)}{81(n-1)n^4(n+1)^4(n+2)}\Big)S_1(n)\\
&-\frac{32\big(1976n^{10}+9385n^9+24088n^8+38989n^7+50214n^6+53872n^5+35219n^4+6890n^3-4233n^2-2844n-756\big)}{243(n-1)n^5(n+1)^5(n+2)} + \frac{44\ldots}{9(n-\ldots}\\
&-\frac{16\big(14n^6+33n^5+59n^4+39n^3+55n^2+20n-12\big)}{9(n-1)n^3(n+1)^3(n+2)}\Big) + \frac{16\big(4n^6+3n^5-50n^4-129n^3-100n^2-56n-24\big)S_2(n)}{9(n-1)n^3(n+1)^3(n+2)} - \frac{160\ldots}{27(n\ldots}
\end{aligned}
$$

SUMMARY:

- Required time: 9 1/2 hours.
- Occuring sums (after reduction):

$$\zeta_2, \zeta_3, (-1)^n, S_1(n), S_2(n), S_3(n), S_{2,1}(n), S_{3,1}(n), S_{2,1,1}(n).$$

$$
\begin{aligned}
F_0 = {} & C_A\Big(\frac{8S_1(n)^3}{27(n-1)n} + \frac{4\big(16n^5+31n^4-38n^3-3n^2+50n+32\big)S_1(n)^2}{27(n-1)n^2(n+1)^2(n+2)} + \Big(\frac{8\big(6944n^8+26480n^7+24941n^6-7003n^5-247\dots}{729(n-1)n^3(n+\dots}\\
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- Required time: 9 1/2 hours.
- Occuring sums (after reduction):

$$\zeta_2, \zeta_3, (-1)^n, S_1(n), S_2(n), S_3(n), S_{2,1}(n), S_{3,1}(n), S_{2,1,1}(n).$$

- For another problem about 100000 triple sums have been reduced to 40 key sums and have been calculated afterwards.

Highlight 5:

Algorithms for Special Functions

Computer algebra and special functions:

Harmonic sums (J. Vermaseren, J. Blümlein; M. Hoffman, D. Broadhurst,...)

$$S_{2,1}(n) = \boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

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$$S_{2,1}(n) = \boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1-x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

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$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)$$

$$- \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta(3) + O(\frac{\ln(n)}{n^6}).$$

limit computations **numerical evaluation**

Computer algebra and special functions:

Generalization to cyclotomic harmonic sums (J. Ablinger, J. Blümlein, CS)

$$\boxed{\sum_{k=1}^{n} \frac{(-1)^k}{2k+1}} =$$

Integral representation:

$$=-(-1)^n \int_0^1 \frac{x^{2n}}{x^2+1} dx + \frac{(-1)^n}{2n+1} - 1 + \frac{\pi}{4},$$

Asymptotic expansion:

$$=(-1)^n \left(-\frac{3}{64n^5} - \frac{1}{16n^4} + \frac{3}{16n^3} - \frac{1}{4n^2} + \frac{1}{4n}\right) + \frac{\pi}{4} - 1 + O\left(\frac{1}{n^6}\right)$$

limit computations **numerical evaluation**

$$\sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{j^2}}{(2i+1)^2} = \text{asymptotic expansion?}$$

In[10]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[11]:= **SExpansion[S[{{2, 1, 2}, {1, 0, 2}}, n], n, 10]**

$$\sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{j^2}}{(2i+1)^2} = \text{asymptotic expansion?}$$

In[10]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[11]:= **SExpansion[S[{{2, 1, 2}, {1, 0, 2}}, n], n, 10]**

Out[11]=
$$\left(-\frac{16\ln 2^2}{3} + \frac{3}{128n^{10}} - \frac{367}{5760n^9} + \frac{7}{96n^8} - \frac{221}{2016n^7} + \frac{5}{24n^6} - \frac{127}{360n^5} + \frac{1}{2n^4} - \frac{11}{18n^3} + \frac{2}{3n^2} - \frac{2}{3n} - \frac{1936}{15}\right)\frac{1}{4}(\pi-4)^2 + \left(-\frac{32\ln 2^2}{3} + \frac{3}{64n^{10}} - \frac{367}{2880n^9} + \frac{7}{48n^8} - \frac{221}{1008n^7} + \frac{5}{12n^6} - \frac{127}{180n^5} + \frac{1}{n^4} - \frac{11}{9n^3} + \frac{4}{3n^2} - \frac{4}{3n} - \frac{3872}{45}\right)\frac{1}{4}(\pi-4) - \frac{968}{45}\frac{1}{4}(\pi-4)^4 - \frac{3872}{45}\frac{1}{4}(\pi-4)^3 + 8\text{li4half} + \frac{\ln 2^4}{3} - \frac{16\ln 2^2}{3} + 7\ln 2\ z3 + \frac{125891}{1075200n^{10}} - \frac{10259}{80640n^9} + \frac{92257}{645120n^8} - \frac{5507}{20160n^7} + \frac{2837}{5760n^6} - \frac{509}{720n^5} + \frac{161}{192n^4} - \frac{31}{36n^3} + \frac{19}{24n^2} - \frac{2}{3n} - \frac{968}{45}$$

Highlight 6: New Strategies

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n + d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

$$\sum_{i_1} \dots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

MultiSum Package
(Flavia Stan)

Holonomic/difference field Approach
(Mark Round)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n + d) = h(\varepsilon, n)$$

Find a recurrence for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \dots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \dots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \dots, i_7)$$

MultiSum Package
(Flavia Stan)

Holonomic/difference field Approach
(Mark Round)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \dots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

given

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(n)F_0(n+1)+\dots+a_d(n)F_0(n+d)=h_0(n)}$$

Ansatz (for power series)

$$\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(n)F_0(n+1)+\dots+a_d(n)F_0(n+d)=h_0(n)}$$

If $F_0(n)$ (with required initial values) is not expressible in terms of indefinite nested sums and products: **game over**

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(n)F_0(n+1)+\dots+a_d(n)F_0(n+d)=h_0(n)}
$$

Ansatz (for power series)

$$a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]$$

$$+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]$$

$$+$$

$$\vdots$$

$$+a_d(\varepsilon,n)\Big[F_0(n+d)+F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]$$

$$=h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(n)F_0(n+1)+\dots+a_d(n)F_0(n+d)=h_0(n)}$$

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
\vdots&\\
+&a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon+F_2(n+d)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h'_0(n)+h'_1(n)\varepsilon+h'_2(n)\varepsilon^2+\dots
\end{aligned}
$$

$$
\begin{aligned}
& a_0(\varepsilon,n)\Big[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots\Big] \\
+& a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon + F_2(n+1)\varepsilon^2 + \dots\Big] \\
+& \\
& \vdots \\
+& a_d(\varepsilon,n)\Big[F_1(n+d)\varepsilon + F_2(n+d)\varepsilon^2 + \dots\Big] \\
& \qquad = \underbrace{h'_0(n)}_{=0} + h'_1(n)\varepsilon + h'_2(n)\varepsilon^2 + \dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,n)\Big[F_1(n+d)+F_2(n+d)\varepsilon+\dots\Big]=h_1'(n)+h_2'(n)\varepsilon+\dots
\end{aligned}
$$

Now repeat for $F_1(n), F_2(n), \dots$

Remark: Works the same for Laurent series.

Joint publications (RISC–DESY)

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3. I. Bierenbaum, J. Blümlein, S. Klein, and C. Schneider. Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. *Nucl.Phys. B*, 803(1-2):1–41. 2008.
4. J. Blümlein, M. Kauers, S. Klein, and C. Schneider. From moments to functions in higher order QCD (*International Workshop on Advanced Computing and Analysis Techniques in Physics Research 2008*). In *Proc. of Science*, volume PoS(ACAT08)106: 1–7, 2009.
5. J. Blümlein, M. Kauers, S. Klein, and C. Schneider. Determining the closed forms of the $O(a_s^3)$ anomalous dimensions and Wilson coefficients from Mellin moments by means of computer algebra. *Comput. Phys. Comm.*, 180: 2143–2165. 2009.
6. J. Ablinger, J. Blümlein, S. Klein, C. Schneider. Modern Summation Methods and the Computation of 2- and 3-loop Feynman Diagrams. Nucl. Phys. B (Proc. Suppl.) 205-206, pp. 110-115. 2010.
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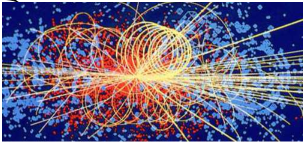
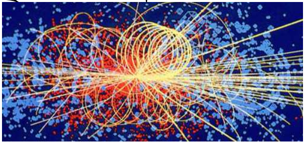
8. J. Ablinger, J. Blümlein, S. Klein, C. Schneider, F. Wissbrock. The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity. Nucl. Phys. B 844, pp. 26-54. 2011.
9. J. Ablinger, J. Blümlein, C. Schneider. Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials. J. Math. Phys. 52(10), pp. 1-52. 2011.
10. J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, F. Wissbrock. New Heavy Flavor Contributions to the DIS Structure Function $F_2(x, Q^2)$ at $O(\alpha_s^3)$. In: Proceedings of 10th International Symposium on Radiative Corrections (RADCOR), PoS(RADCOR2011)31, pp1–8. 2012.
11. J. Blümlein, A. Hasselhuhn, C. Schneider. Evaluation of Multi-Sums for Large Scale Problems. In: Proceedings of 10th International Symposium on Radiative Corrections, PoS(RADCOR2012)32, pp1–9. 2012.
12. J. Ablinger, J. Blümlein, S. Klein, C. Schneider, and F. Wißbrock. 3-loop heavy flavor corrections to dis with two massive fermion lines. To appear in the proceedings of 19th International Workshop on Deep-Inelastic Scattering and Related Subjects (DIS 2011), 2011.
13. J. Blümlein, S. Klein, C. Schneider, F. Stan. A Symbolic Summation Approach to Feynman Integral Calculus. Deutsches Elektronen–Synchrotron. To appear in J. Symbolic Comput. 2012.

DESY



DESY

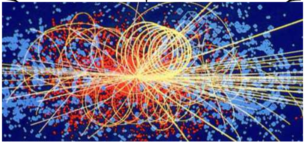
Sigma



DESY

Sigma

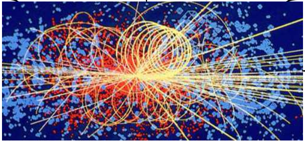
HarmonicSums
(J. Ablinger)



DESY

Sigma

HarmonicSums
(J. Ablinger)

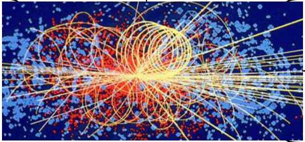


EvaluateMultiSums

DESY

Sigma

HarmonicSums
(J. Ablinger)



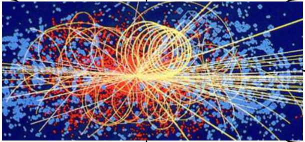
EvaluateMultiSums

Guessing
(M. Kauers)

DESY

Sigma

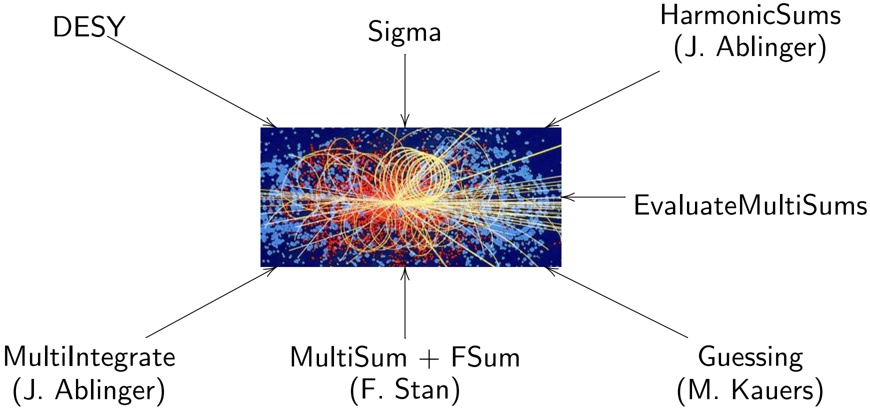
HarmonicSums
(J. Ablinger)



EvaluateMultiSums

MultiSum + FSum
(F. Stan)

Guessing
(M. Kauers)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HarmonicSums
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EvaluateMultiSums
MultiIntegrate
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MultiSum + FSum
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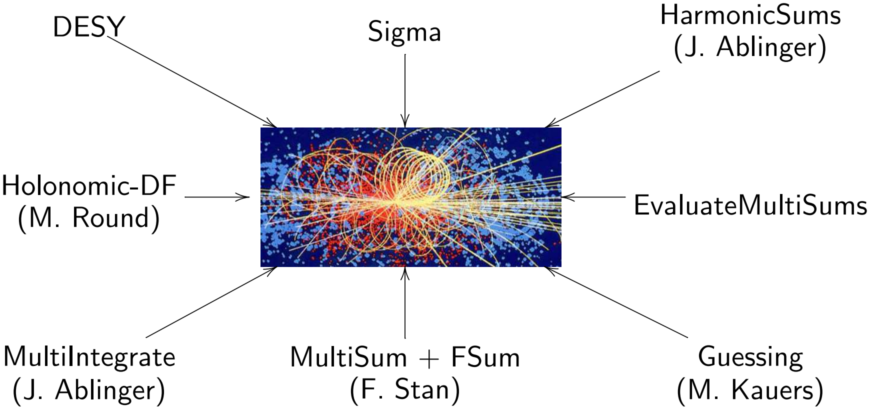
Guessing
(M. Kauers)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(F. Stan)
Guessing
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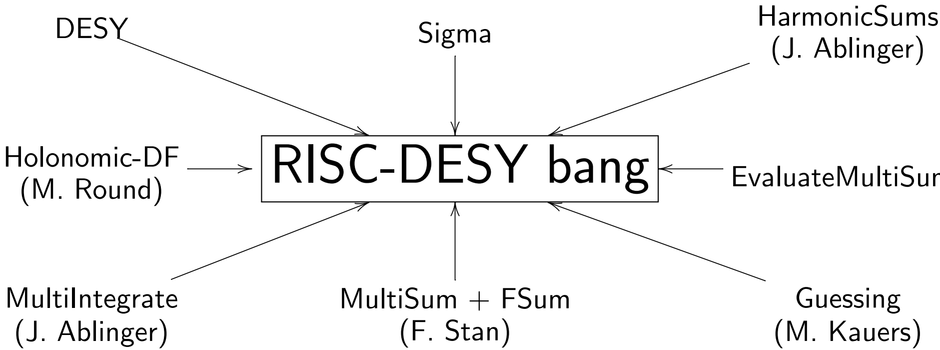
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