Symbolic Computation of Linear Relations of Parameter Integrals

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Parameter Integrals

Integrals depending on parameters

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,dx$$

$$\mathrm{Li}_2(z) = \int_0^\infty \frac{zx}{e^x - z}\,dx$$

$$J_n(y) = \int_0^\pi \frac{\cos(nx - y\sin(x))}{\pi}\,dx$$

$$\int_0^\infty e^{-sx}\gamma(a,x)\,dx = \frac{\Gamma(a)}{s(s+1)^a}$$

$$\int_0^1 e^{-2n\pi ix}\ln(\sin(\tfrac{\pi}{2}x))\,dx = -\frac{1}{4n} + \frac{i}{n\pi}\sum_{k=1}^{n}\frac{1}{2k-1}$$

Example: Gamma function

$$\Gamma(z) := \int_0^\infty \underbrace{x^{z-1}e^{-x}}_{=:f(z,x)}\,dx \qquad \text{for } z > 0$$

We compute

$$zf(z,x) - f(z+1,x) = \frac{d}{dx}x^z e^{-x}$$

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In other words, we proved

$$z\Gamma(z) - \Gamma(z+1) = 0$$

Linear Relations

Integrals depending on one parameter

- $c_0(y)f(x,y) + \dots + c_m(y)\frac{\partial^m f}{\partial y^m}(x,y) = \frac{d}{dx}g(x,y)$

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- $c_0(n)f(x,n) + \cdots + c_m(n)f(x,n+m) = \frac{d}{dx}g(x,n)$ yields a recurrence for

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Parametric integration

Compute linear relation of integrals

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Transfer this to a relation of corresponding integrals

$$c_0 \int_a^b f_0(x)dx + \dots + c_m \int_a^b f_m(x)dx = g(b) - g(a)$$

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Certificate

$g(x)$ is a certificate for the relation

$$c_0 \int_a^b f_0(x)dx + \cdots + c_m \int_a^b f_m(x)dx = r$$

It is easy to verify

$$c_0 f_0(x) + \cdots + c_m f_m(x) = g'(x) \quad \text{and} \quad r = g(b) - g(a)$$

Differential Algebra

Differential field

(F, D) such that for any $f, g \in F$

$$D(f+g) = Df + Dg \quad \text{and} \quad D(fg) = (Df)g + f(Dg)$$

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- Given (F, D) and $f_0, \dots, f_m \in F$

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Parametric elementary integration

- Given (F, D) and $f_0, \ldots, f_m \in F$
- Find elementary extension (E, D) of (F, D) and $g \in E$, $c_0, \ldots, c_m \in \mathrm{Const}(F)$ s.t.

$$c_0 f_0 + \cdots + c_m f_m = Dg$$

Risch's Algorithm

Risch 1969, Mack 1976

complete algorithm for elementary (F, D)

Singer et al. 1985

complete algorithm for regular Liouvillian (F, D)

Bronstein 1990, 1997

partial results for (F, D) generated by monomials

CGR 2012

complete algorithm for (F, D) generated by monomials subject to some technical conditions

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Generalization to 2-dim systems

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} b_1(x) \\ b_2(x) \end{pmatrix}$$

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orthogonal polynomials, associated Legendre functions, complete elliptic integrals, Bessel/Struve/Anger/Weber/Lommel functions, Airy/Scorer functions, Whittaker functions, hypergeometric functions, Heun functions, Mathieu functions, etc.

Elementary Integrals

Elementary extension

Any (E, D) generated from (F, D) by adjoining

- algebraics: $y(x)^m + a_{m-1}(x)y(x)^{m-1} + \cdots + a_0(x) = 0$
- logarithms: $y(x) = \log(a(x))$
- exponentials: $y(x) = \exp(a(x))$

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Liouville's Theorem

If $f \in F$ has an integral in an elementary extension of (F, D), then there exist $c_1, \ldots, c_j \in \overline{\text{Const}(F)}$ and $u_0, \ldots, u_j \in F(c_1, \ldots, c_j)$ s.t.

$$\int f = u_0 + \sum_{i=1}^{j} c_i \log(u_i)$$

Example

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Our algorithm finds

$$f(n+1,x) - \tfrac{n}{n+1} f(n,x) =$$
$$\tfrac{d}{dx} \tfrac{e^{-2(n+1)\pi ix}}{2(n+1)\pi i} \left(\tfrac{1}{4(n+1)} + \tfrac{e^{\pi ix}}{2n+1} + \tfrac{e^{2\pi ix}}{4n} + (e^{2\pi ix}-1) \ln(\sin(\tfrac{\pi}{2}x)) \right)$$

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Solution:

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Integrating again implies

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Example: Evaluate Double Integral

$$I_2 := \int_0^1 \int_0^{1-x_1} \frac{x_1^{m_1-1} x_2^{m_2-1} (1 - x_1 - x_2)^{m_3-1}}{(a_1 x_1 + a_2 x_2 + a_3(1 - x_1 - x_2))^{m_1+m_2+m_3}} \, dx_2 dx_1$$

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Differential equations for I_1:

$$\frac{\partial^2 I_1}{\partial x_1^2} - \left(\frac{2(m_1-1)}{x_1} - \frac{m_2+m_3-2}{1-x_1} + \frac{(m_1+m_2+1)(a_2-a_1)}{a_1x_1+a_2(1-x_1)} + \frac{(m_1+m_3+1)(a_3-a_1)}{a_1x_1+a_3(1-x_1)}\right)\frac{\partial I_1}{\partial x_1} + \left(\frac{\dots}{x_1} + \frac{m_1(m_1-1)}{x_1^2} + \frac{\dots}{1-x_1} + \frac{\dots}{a_1x_1+a_2(1-x_1)} + \frac{\dots}{a_1x_1+a_3(1-x_1)}\right) I_1 = 0$$

$$\frac{\partial I_1}{\partial a_1} - \frac{x_1(1-x_1)}{a_1}\frac{\partial I_1}{\partial x_1} + \frac{m_1-1+2x_1}{a_1} I_1 = 0$$

$$\frac{\partial I_1}{\partial a_2} + \frac{(1-x_1)(a_1x_1+a_2(1-x_1))}{a_1(a_2-a_3)}\frac{\partial I_1}{\partial x_1} + \frac{2(a_3-a_1)x_1^2+(\dots)x_1-(m_1-1)a_3}{a_1(a_2-a_3)x_1} I_1 = 0$$

$$\frac{\partial I_1}{\partial a_3} + \frac{(1-x_1)(a_1x_1+a_2(1-x_1))}{a_1(a_3-a_2)}\frac{\partial I_1}{\partial x_1} + \frac{2(a_2-a_1)x_1^2+(\dots)x_1-(m_1-1)a_2}{a_1(a_3-a_2)x_1} I_1 = 0$$

Example: Evaluate Double Integral

$$I_2 := \int_0^1 \int_0^{1-x_1} \frac{x_1^{m_1-1} x_2^{m_2-1} (1-x_1-x_2)^{m_3-1}}{(a_1x_1+a_2x_2+a_3(1-x_1-x_2))^{m_1+m_2+m_3}}\, dx_2 dx_1$$

$$I_1 := \int_0^{1-x_1} \frac{x_1^{m_1-1} x_2^{m_2-1} (1-x_1-x_2)^{m_3-1}}{(a_1x_1+a_2x_2+a_3(1-x_1-x_2))^{m_1+m_2+m_3}}\, dx_2$$

General solution:

$$I_1 = c_1 \frac{x_1^{m_1-1}(1-x_1)^{m_2+m_3-1}}{(a_1x_1+a_2(1-x_1))^{m_2}(a_1x_1+a_3(1-x_1))^{m_1+m_3}} {}_2F_1\left(\begin{matrix} -m_1 \quad m_2 \\ m_2+m_3 \end{matrix} \middle| \frac{(a_2-a_3)(1-x_1)}{a_1x_1+a_2(1-x_1)}\right) +$$
$$c_2 \frac{x_1^{m_1-1}(a_1x_1+a_2(1-x_1))^{m_3-1}}{(a_2-a_3)^{m_2+m_3-1}(a_1x_1+a_3(1-x_1))^{m_1+m_3}} {}_2F_1\left(\begin{matrix} 1-m_3 \quad 1-m_1-m_2-m_3 \\ 2-m_2-m_3 \end{matrix} \middle| \frac{(a_2-a_3)(1-x_1)}{a_1x_1+a_2(1-x_1)}\right)$$

Example: Evaluate Double Integral

$$I_2 := \int_0^1 \int_0^{1-x_1} \frac{x_1^{m_1-1} x_2^{m_2-1} (1-x_1-x_2)^{m_3-1}}{(a_1x_1 + a_2x_2 + a_3(1-x_1-x_2))^{m_1+m_2+m_3}} \, dx_2 dx_1$$

$$I_1 := \int_0^{1-x_1} \frac{x_1^{m_1-1} x_2^{m_2-1} (1-x_1-x_2)^{m_3-1}}{(a_1x_1 + a_2x_2 + a_3(1-x_1-x_2))^{m_1+m_2+m_3}} \, dx_2$$

General solution:

$$I_1 = c_1 \frac{x_1^{m_1-1}(1-x_1)^{m_2+m_3-1}}{(a_1x_1+a_2(1-x_1))^{m_2}(a_1x_1+a_3(1-x_1))^{m_1+m_3}} \, {}_2F_1\left(\begin{matrix} -m_1 \quad m_2 \\ m_2+m_3 \end{matrix} \,\middle|\, \frac{(a_2-a_3)(1-x_1)}{a_1x_1+a_2(1-x_1)}\right) +$$
$$c_2 \frac{x_1^{m_1-1}(a_1x_1+a_2(1-x_1))^{m_3-1}}{(a_2-a_3)^{m_2+m_3-1}(a_1x_1+a_3(1-x_1))^{m_1+m_3}} \, {}_2F_1\left(\begin{matrix} 1-m_3 \quad 1-m_1-m_2-m_3 \\ 2-m_2-m_3 \end{matrix} \,\middle|\, \frac{(a_2-a_3)(1-x_1)}{a_1x_1+a_2(1-x_1)}\right)$$

Initial value at $a_1 = a_2 = a_3 = 1$:

$$I_2 = \frac{\Gamma(m_2)\Gamma(m_3)}{\Gamma(m_2+m_3)} x_1^{m_1-1}(1-x_1)^{m_2+m_3-1} \quad \Rightarrow \quad c_1 = \frac{\Gamma(m_2)\Gamma(m_3)}{\Gamma(m_2+m_3)}, \, c_2 = 0$$

Example: Evaluate Double Integral (cont.)

$$I_2 = \int_0^1 \frac{\Gamma(m_2)\Gamma(m_3)x_1^{m_1-1}(1-x_1)^{m_2+m_3-1}}{\Gamma(m_2+m_3)(a_1x_1+a_2(1-x_1))^{m_2}(a_1x_1+a_3(1-x_1))^{m_1+m_3}} \cdot {}_2F_1\left(\begin{matrix}-m_1 \quad m_2 \\ m_2+m_3\end{matrix} \;\middle|\; \frac{(a_2-a_3)(1-x_1)}{a_1x_1+a_2(1-x_1)}\right) dx_1$$

Example: Evaluate Double Integral (cont.)

$$I_2 = \int_0^1 \frac{\Gamma(m_2)\Gamma(m_3)x_1^{m_1-1}(1-x_1)^{m_2+m_3-1}}{\Gamma(m_2+m_3)(a_1x_1+a_2(1-x_1))^{m_2}(a_1x_1+a_3(1-x_1))^{m_1+m_3}} \cdot {}_2F_1\left(\begin{matrix}-m_1 & m_2 \\ m_2+m_3 & \end{matrix} \middle| \frac{(a_2-a_3)(1-x_1)}{a_1x_1+a_2(1-x_1)}\right) dx_1$$

Differential equations for I_2:

$$\begin{aligned}
\frac{\partial I_2}{\partial a_1} + \frac{m_1}{a_1} I_2 &= 0 \\
\frac{\partial I_2}{\partial a_2} + \frac{m_2}{a_2} I_2 &= 0 \\
\frac{\partial I_2}{\partial a_3} + \frac{m_3}{a_3} I_2 &= 0
\end{aligned}$$

Example: Evaluate Double Integral (cont.)

$$I_2 = \int_0^1 \frac{\Gamma(m_2)\Gamma(m_3)x_1^{m_1-1}(1-x_1)^{m_2+m_3-1}}{\Gamma(m_2+m_3)(a_1x_1+a_2(1-x_1))^{m_2}(a_1x_1+a_3(1-x_1))^{m_1+m_3}} \cdot {}_2F_1\left(\begin{matrix}-m_1 & m_2\\ & m_2+m_3\end{matrix} \middle| \frac{(a_2-a_3)(1-x_1)}{a_1x_1+a_2(1-x_1)}\right) dx_1$$

Differential equations for I_2:

$$\begin{aligned}\frac{\partial I_2}{\partial a_1} + \frac{m_1}{a_1} I_2 &= 0\\ \frac{\partial I_2}{\partial a_2} + \frac{m_2}{a_2} I_2 &= 0\\ \frac{\partial I_2}{\partial a_3} + \frac{m_3}{a_3} I_2 &= 0\end{aligned}$$

Initial value at $a_1 = a_2 = a_3 = 1$: $\qquad I_2 = \frac{\Gamma(m_1)\Gamma(m_2)\Gamma(m_3)}{\Gamma(m_1+m_2+m_3)}$

Example: Evaluate Double Integral (cont.)

$$I_2 = \int_0^1 \frac{\Gamma(m_2)\Gamma(m_3)x_1^{m_1-1}(1-x_1)^{m_2+m_3-1}}{\Gamma(m_2+m_3)(a_1x_1+a_2(1-x_1))^{m_2}(a_1x_1+a_3(1-x_1))^{m_1+m_3}} \cdot {}_2F_1\left(\begin{matrix} -m_1 & m_2 \\ & m_2+m_3 \end{matrix} \,\middle|\, \frac{(a_2-a_3)(1-x_1)}{a_1x_1+a_2(1-x_1)}\right) dx_1$$

Differential equations for I_2:

$$\begin{aligned} \frac{\partial I_2}{\partial a_1} + \frac{m_1}{a_1} I_2 &= 0 \\ \frac{\partial I_2}{\partial a_2} + \frac{m_2}{a_2} I_2 &= 0 \\ \frac{\partial I_2}{\partial a_3} + \frac{m_3}{a_3} I_2 &= 0 \end{aligned}$$

Initial value at $a_1 = a_2 = a_3 = 1$: $\quad I_2 = \frac{\Gamma(m_1)\Gamma(m_2)\Gamma(m_3)}{\Gamma(m_1+m_2+m_3)}$

Solution:

$$I_2 = \frac{\Gamma(m_1)\Gamma(m_2)\Gamma(m_3)}{\Gamma(m_1 + m_2 + m_3)a_1^{m_1}a_2^{m_2}a_3^{m_3}}$$