Hopf Algebra Structures on Quasi-Shuffle Algebras

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Outline

Introduction

The quasi-shuffle product I introduced in [J. Algebraic Combin. 2000] has been useful in understanding multiple zeta values, harmonic sums, and multiple polylogarithms. Recently the definition was generalized by K. Ihara, J. Kajikawa, Y. Ohno, and J. Okuda [J. Algebra 2010] (henceforth IKOO), who removed some significant limitations of the original construction. Nevertheless, IKOO did not make use of some features of my construction, particularly the coalgebra structure and the representation of formal power series by linear isomorphisms of the quasi-shuffle algebra. At the beginning of this year I started collaborating with Ihara in extending these features to the more general setting.
Ref.: M. E. Hoffman and K. Ihara, "Quasi-shuffle products revisited," MPIM preprint 2012-16.

Shuffle Products

Let k be a field, A a countable set of "letters", and let $k\langle A\rangle$ be the k-vector space of noncommutative monomials in elements of A. A basis for $k\langle A\rangle$ is the set of "words" (monomials) in elements of A, including the empty word 1. We define $\ell(w)$ to be the number of letters in the word w.

Of course $k\langle A\rangle$ is a noncommutative algebra with the obvious product (juxtaposition). But there is a commutative product on $k\langle A\rangle$ given by shuffle product ⧢: for example, if $a, b, c \in A$, then

$$a ⧢ bc = abc + bac + bca.$$

More precisely, the shuffle product can be defined by $v ⧢ 1 = 1 ⧢ v = v$ for all words w, together with the inductive rule

$$av ⧢ bw = a(v ⧢ bw) + b(av ⧢ w)$$

for $a, b \in A$ and words v, w.

Quasi-Shuffle Products

Now suppose there is a product $\diamond$ on kA (the vector space with A as basis) which is commutative and associative. We define the quasi-shuffle product $*$ by $v * 1 = 1 * v = 1$ for any word v, and

$$av * bw = a(v * bw) + b(av * w) + (a \diamond b)(v * w)$$

for $a, b \in A$ and words v, w. By induction on $\ell(v) + \ell(w)$ one can show that the product so defined is commutative and associative.
We shall denote by $\star$ the quasi-shuffle product obtained by replacing $\diamond$ with its negative $\bar{\diamond}$ (i.e., $a \bar{\diamond} b = -a \diamond b$). It is often useful to treat $*$ and $\star$ in parallel.

Quasi-Shuffle Products Cont'd

For example, if $a, b, c, d \in A$,

$$a * b = ab + ba + a \diamond b$$
$$a \star b = ab + ba - a \diamond b$$

$$a * bc = abc + bac + bca + a \diamond bc + ab \diamond c$$
$$a \star bc = abc + bac + bca - a \diamond bc - ab \diamond c$$

$$ab \star cd = abcd + acbd + acdb + cabd + cadb + cdab - a \diamond cbd$$
$$- a \diamond cdb - ca \diamond db - acb \diamond d - cab \diamond d - ab \diamond cd + a \diamond cb \diamond d,$$

and $ab * cd$ looks the same without minus signs.

Quasi-Shuffle Products Cont'd

To summarize, we have at this point four products on the vector space $k\langle A\rangle$:

1. The original noncommutative product (juxtaposition);
2. The commutative shuffle product $\sqcup\!\sqcup$;
3. The commutative quasi-shuffle product $*$;
4. The commutative quasi-shuffle product $\star$.

We shall also define a fifth product $\diamond$ by

$$u \diamond v = a_1 \cdots a_{n-1}(a_n \diamond b_1)b_2 \cdots b_m$$

for $u = a_1 \cdots a_n$ and $v = b_1 \cdots b_m$. The product $\diamond$ is noncommutative on $k\langle A\rangle$, though its restriction to the subalgebra $kA + k1 \subset k\langle A\rangle$ is commutative.

Linear maps induced by formal power series

Let

$$f = c_1 t + c_2 t^2 + \cdots$$

be a formal power series with $c_1 \neq 0$. The set $\mathfrak{P}$ of such formal power series evidently forms a group under the composition operation: i.e., if

$$g = d_1 t + d_2 t^2 + \dots$$

is also in $\mathfrak{P}$, then

$$\begin{aligned} g \circ f &= d_1 f + d_2 f^2 + \dots \\ &= d_1 c_1 t + (d_1 c_2 + d_2 c_1^2) t^2 + \cdots \end{aligned}$$

Now define a linear map Ψ_f from $k\langle A \rangle$ to itself as follows.

Induced Linear Maps Cont'd

For words $w = a_1 a_2 \cdots a_n$ in $k\langle A\rangle$, let

$$\Psi_f(w) = \sum_{I \in \mathcal{C}(n)} c_I I[w]$$

where the sum is over all compositions $I = (i_1, \ldots, i_k)$ of n (finite sequences of positive integers with $i_1 + \cdots + i_k = n$), where

$$c_{(i_1,\ldots,i_k)} = c_{i_1} \cdots c_{i_k}$$

and

$$I[a_1 \ldots a_n] =$$
$$a_1 \diamond \cdots \diamond a_{i_1} a_{i_1+1} \diamond \cdots \diamond a_{i_1+i_2} \cdots a_{i_1+\cdots+i_{k-1}+1} \diamond \cdots \diamond a_n.$$

Examples of Induced Linear Maps

Some important examples of power series in $\mathfrak{P}$ are

$$e^t - 1 = t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots$$
$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots$$
$$\frac{t}{1-t} = t + t^2 + t^3 + \cdots$$

leading to

$$\exp := \Psi_{e^t-1}, \quad \log := \Psi_{\log(1+t)}, \quad \Sigma := \Psi_{\frac{t}{1-t}}$$

We also define $T := \Psi_{-t}$.

Induced Maps of Composed Functions

The key result about induced maps is the following.

Theorem

Let $f, g \in \mathfrak{P}$. Then $\Psi_{f \circ g} = \Psi_f \Psi_g$.

Corollary

If $f \circ g = t$, then $\Psi_f \Psi_g = \mathrm{id}$.

In particular, $\exp^{-1} = \log$, $T^{-1} = T$, and $\Sigma^{-1} = \Psi_{\frac{t}{1+t}}$.

Corollary

$T \Sigma T = \Sigma^{-1}$.

Corollary

$\Sigma = \exp T \log T$.

Algebraic Properties of Induced Maps

In general the induced maps Ψ_f are not homomorphisms of the quasi-shuffle products. But we do have several results for special cases.

Theorem (Hoffman, J. Algebraic Combin. 2000)

For all words v, w, $\exp(v ⧢ w) = \exp(v) * \exp(w)$.

In fact, exp is an algebra isomorphism from $(k\langle A \rangle, ⧢)$ to $(k\langle A \rangle, *)$, and $\exp^{-1} = \log$ is an isomorphism from $(k\langle A \rangle, *)$ to $(k\langle A \rangle, ⧢)$.
We note that $T(w) = (-1)^{\ell(w)} w$. Using this fact, it is easy to prove the following.

Proposition

For any words v, w, $T(v * w) = T(v) \star T(w)$ and $T(v \star w) = T(v) * T(w)$.

Algebraic Properties Cont'd

Since $\Sigma = \exp T \log T$, we have the following result, which is proved in IKOO with considerably more effort.

Proposition

For words v, w, $\Sigma(v \star w) = \Sigma(v) * \Sigma(w)$ (so $\Sigma^{-1}(v * w) = \Sigma^{-1}(v) \star \Sigma^{-1}(w)$).

Corollary

ΣT is a homomorphism from $(k\langle A\rangle, *)$ to itself.

Note that in fact ΣT is an involution (i.e., a self-inverse automorphism) of the algebra $(k\langle A\rangle, *)$ since $\Sigma T \Sigma T = \Sigma\Sigma^{-1} = \mathrm{id}$. Similarly, $T\Sigma$ is an involution of $(k\langle A\rangle, \star)$.

What is the Significance of Σ?

In the paper of S. Moch, P. Uwer and S. Weinzierl [J. Math. Phys. 2002], "S-sums" and "Z-sums" are given by

$$S(N; i_1, \dots, i_k) = \sum_{N \ge n_1 \ge \cdots \ge n_k \ge 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}$$

$$Z(N; i_1, \dots, i_k) = \sum_{N \ge n_1 > \cdots > n_k \ge 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}.$$

For the quasi-shuffle algebra $(k\langle A\rangle, *)$ with $A = \{z_1, z_2, \dots\}$ and $z_i \diamond z_j = z_{i+j}$, there is a homomorphism $Z : (k\langle A\rangle, *) \to \mathbb{R}$ sending $z_{i_1} \cdots z_{i_k}$ to $Z(N; i_1, \dots, i_k)$; then $Z\Sigma : (k\langle A\rangle, \star) \to \mathbb{R}$ sends $z_{i_1} \cdots z_{i_k}$ to $S(N; i_1, \dots, i_k)$.

Significance of Σ Cont'd

If $i_1 > 1$, the limit

$$\lim_{N\to\infty} Z(N; i_1, \dots, i_k)$$

exists, and is the multiple zeta value $\zeta(i_1, \dots, i_k)$. The corresponding limits of the S-sums are usually called "multiple zeta-star values" in the literature. So if $\zeta : (\mathfrak{H}^0, *) \to \mathbb{R}$ (where $\mathfrak{H}^0$ is the subalgebra of $k\langle A\rangle$ consisting of words that don't begin with z_1) is defined by $\zeta(z_{i_1} \cdots z_{i_k}) = \zeta(i_1, \dots, i_k)$, then $\zeta^\star(i_1, \dots i_k) = \zeta\Sigma(z_{i_1} \cdots z_{i_k})$. The study of multiple zeta-star values was a principal objective of IKOO.

The Coalgebra Structure

There is a coalgebra structure on $k\langle A\rangle$ given by having the counit ϵ send 1 to $1 \in k$ and all nonempty words to 0, and coproduct Δ given by the "deconcatenation"

$$\Delta(w) = \sum_{uv=w} u \otimes v,$$

where the sum is over all decompositions of w into subwords u and v (including the cases $u = 1$, $v = w$ and $u = w$, $v = 1$). This coproduct defines a convolution product $\odot$ on the set $\mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ of linear maps from $k\langle A\rangle$ to itself: for $L_1, L_2 \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$, let

$$L_1 \odot L_2(w) = \sum_{uv=w} L_1(u)L_2(v).$$

Coalgebra Structure Cont'd

The identity element of the convolution ring is the element $\eta\epsilon \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ that sends 1 to 1 and all words of positive length to zero. Any $L \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ with $L(1) = 1$ has a convolutional inverse $L^{\odot(-1)}$.

We call $C \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ a contraction if $C(1) = 0$ and $C(w)$ is primitive (i.e., $\Delta C(w) = C(w) \otimes 1 + 1 \otimes C(w)$) for all words w, and $E \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ an expansion if $E(1) = 1$ and E is a coalgebra map (i.e., $E(\Delta(w)) = \Delta(E(w))$).

If E is an expansion and C is a contraction, (E, C) is called an inverse pair if

$$E = (\eta\epsilon - C)^{\odot(-1)} = \eta\epsilon + C + C \odot C + \cdots$$

or equivalently

$$C = \eta\epsilon - E^{\odot(-1)}.$$

Coalgebra Structure Cont'd

Proposition

Suppose $E, C \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$. If C is a contraction and $E = (\eta\epsilon - C)^{\odot(-1)}$, then (E, C) is an inverse pair. If E is an expansion and $C = \eta\epsilon - E^{\odot(-1)}$, then (E, C) is an inverse pair.

For $f = c_1 t + c_2 t^2 + \cdots \in \mathfrak{P}$, let $C_f \in \mathrm{Hom}_k(k\langle A\rangle, k\langle A\rangle)$ be defined by $C_f(1) = 0$ and $C_f(a_1 a_2 \cdots a_n) = c_n a_1 \diamond a_2 \diamond \cdots \diamond a_n$ for all nonempty words $a_1 \cdots a_n$. It is easy to check that C is a contraction and $E = (\eta\epsilon - C)^{\odot(-1)}$, so the proposition above implies

Proposition

For any $f \in \mathfrak{P}$, (Ψ_f, C_f) is an inverse pair.

Hopf Algebra Structures

It is routine to check that the coproduct Δ is a homomorphism for the quasi-shuffle product $*$, i.e., that

$$\Delta(u * v) = \Delta(u) * \Delta(v)$$

for all words u, v, where $(a \otimes b) * (c \otimes d) = (a * c) \otimes (b * d)$. Thus $(k\langle A\rangle, *, \Delta)$ is a bialgebra. Now $k\langle A\rangle$ is filtered by word length, and this is enough to insure that $(k\langle A\rangle, *, \Delta)$ has an antipode, i.e., a function S_* such that

$$\sum_{uv=w} S_*(u) * v = \sum_{uv=w} u * S_*(v) = \epsilon(w)$$

for all words w. Thus $(k\langle A\rangle, *, \Delta)$ is a Hopf algebra. In fact, an inductive argument shows that S_* is given explicitly by

$$S_*(w) = \Sigma T R(w)$$

where R is the "reverse" map, i.e., $R(a_1 a_2 \cdots a_n) = a_n \cdots a_2 a_1$.

Hopf Algebra Structures Cont'd

It is immediate that $R^2 = \mathrm{id}$ and that R commutes with T and Σ. With a little work R can be shown to be a homomorphism of $(k\langle A\rangle, *)$. Now from the general theory S_* is an involution of $(k\langle A\rangle, *)$, so it follows that $\Sigma T = RS_*$ is an involution of $(k\langle A\rangle, *)$: of course we have already proved this above. Similarly, Δ is a homomorphism for $\star$, so that $(k\langle A\rangle, \star, \Delta)$ is a Hopf algebra. In this case the antipode is

$$S_\star = \Sigma^{-1} TR = T\Sigma R.$$

So $T\Sigma R$ and thus $T\Sigma$ are involutions of $(k\langle A\rangle, \star)$; we have already proved the latter.

Infinitesimal Hopf Algebra

An infinitesimal Hopf algebra is a vector space with algebra and coalgebra structures such that the coproduct is a derivation for the algebra, and which has an antipode. In our case the algebra is $(k\langle A\rangle, \diamond)$ and the coalgebra is $(k\langle A\rangle, \tilde{\Delta})$, where $\tilde{\Delta}$ is the reduced deconcatenation, i.e., $\tilde{\Delta}(1) = 0$ and

$$\begin{aligned}\tilde{\Delta}(w) &= \sum_{uv=w, u,v\neq 1} u \otimes v \\ &= \Delta(w) - w \otimes 1 - 1 \otimes w\end{aligned}$$

nonempty words w. The condition that $\tilde{\Delta}$ be a derivation is

$$\tilde{\Delta}(u \diamond v) = \tilde{\Delta}(u) \diamond v + u \diamond \tilde{\Delta}(v),$$

where $(w_1 \otimes w_2) \diamond v = w_1 \otimes (w_2 \diamond v)$ and $u \diamond (w_1 \otimes w_2) = (u \diamond w_1) \otimes w_2$. This is straightforward to check.

Infinitesimal Hopf Algebra Cont'd

The infinitesimal version of an antipode is a function $S_\diamond$ such that

$$\sum_{uv=w,u,v\neq 1} S_\diamond(u)\diamond v + S_\diamond(w) + w = 0$$

$$= \sum_{uv=w,u,v,\neq 1} u \diamond S_\diamond(v) + S_\diamond(w) + w$$

for all words w. In fact this equation can be shown to hold for

$$S_\diamond(w) = -\Sigma^{-1}(w),$$

so we have the following.

Theorem

$(k\langle A\rangle, \diamond, \tilde{\Delta})$ *is an infinitesimal Hopf algebra with antipode* $S_\diamond = -\Sigma^{-1}$.

Infinitesimal Hopf Algebra Cont'd

From the general theory of infinitesimal Hopf algebras $S_\diamond = -e^{-D}$, where $D = \diamond\tilde{\Delta}$ is the "canonical derivation" of $k\langle A\rangle$. More concretely, $D(w) = 0$ for $\ell(w) \leq 1$ and

$$D(a_1 a_2 \cdots a_n) = \sum_{i=1}^{n-1} a_1 \cdots a_i \diamond a_{i+1} \cdots a_n$$

for $n \geq 2$. Here

$$e^{-D} = \mathrm{id} - D + \frac{D^2}{2!} - \frac{D^3}{3!} + \cdots$$

which makes sense because $D^n w = 0$ for $n \geq \ell(w)$.

Infinitesimal Hopf Algebra Cont'd

Since $S_\diamond = -\Sigma^{-1}$, we have $\Sigma^{-1} = e^{-D}$. This can be improved to the following representation of arbitrary iterates Σ^r of Σ.

Proposition

For all $r \in k$, $\Psi_{\frac{t}{1-rt}} = \Sigma^r = e^{rD}$.

Since the exponential of a derivation is an automorphism, it follows that Σ^r is an automorphism of $(k\langle A\rangle, \diamond)$ for all r, i.e., $\Sigma^r(u \diamond v) = \Sigma^r(u) \diamond \Sigma^r(v)$.

Algebraic Formulas

For $f = c_1 t + c_2 t^2 + c_3 t^3 + \cdots \in \mathfrak{P}$, define

$$f_\bullet(\lambda z) = c_1 \lambda z + c_2 \lambda^2 z \bullet z + c_3 \lambda^3 z \bullet z \bullet z + \cdots \tag{1}$$

where $z \in kA$, λ is a formal parameter, and $\bullet$ is any of the symbols $*$, $\star$, $⧢$, or $\diamond$. (The expression (1) is in the formal power series ring $kA[[\lambda]]$ if $\bullet = \diamond$, and in $k\langle A\rangle[[\lambda]]$ otherwise.) We define $\exp_\bullet(\lambda z)$ to be $1 + g_\bullet(\lambda z)$ and $\log_\bullet(1 + \lambda z)$ to be $f_\bullet(\lambda z)$, where

$$g = t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots = e^t - 1 \in \mathfrak{P}$$
$$f = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots = \log(1 + t) \in \mathfrak{P}$$

The Master Theorem

Our "master theorem" here is

Theorem

For any $f \in \mathfrak{P}$, $z \in kA[[\lambda]]$,

$$\Psi_f\left(\frac{1}{1-\lambda z}\right) = \frac{1}{1 - f_\diamond(\lambda z)},$$

where Ψ_f is extended to $k\langle A\rangle[[\lambda]]$ by setting $\Psi_f(\lambda) = 1$.

which can be deduced by taking $(E, C) = (\Psi_f, C_f)$ in

Proposition

Let (E, C) be an inverse pair. Then for $z \in kA[[\lambda]]$,

$$E\left(\frac{1}{1-\lambda z}\right) = \frac{1}{1 - C(\lambda z + \lambda^2 z^2 + \cdots)}.$$

Master Theorem Cont'd

As an example of the use of our "master theorem", take $f = \log(1+t)$. This gives

$$\log\left(\frac{1}{1-\lambda z}\right) = \frac{1}{1-\log_\diamond(1+\lambda z)}$$

or

$$\frac{1}{1-\lambda z} = \exp\left(\frac{1}{1-\log_\diamond(1+\lambda z)}\right)$$

for $z \in kA[[\lambda]]$. On the other hand, since $a^{\sqcup\!\sqcup n} = n!a$ for any letter a, it follows that

$$\exp_{\sqcup\!\sqcup}(\lambda u) = \frac{1}{1-\lambda u}$$

for all $u \in kA[[\lambda]]$.

Exponential Formula

Further, since exp is a homomorphism from $(k\langle A\rangle, ⧢)$ to $(k\langle A\rangle, *)$ we have

$$\exp_*(\lambda u) = \exp(\exp_{⧢}(\lambda u)) = \exp\left(\frac{1}{1-\lambda u}\right).$$

Set $\lambda u = \log_\diamond(1+\lambda z)$ and use the previous slide to get

Theorem

For $z \in kA[[\lambda]]$,

$$\frac{1}{1-\lambda z} = \exp_*(\log_\diamond(1+\lambda z)).$$

This was proved in the MZV setting by K. Ihara, M. Kaneko, and D. Zagier [Compos. Math. 2006].

Exponential Formula Cont'd

This result allows one to express MZVs, finite multiple harmonic sums, multiple polylogarithms, etc. with repeated values in terms of the non-multiple version. For example, for multiple polylogarithms it implies

$$Li_{\underbrace{i,\dots,i}_{k}}(\underbrace{x,\dots,x}_{k}) =$$

$$\text{coefficient of } \lambda^k \text{ in } \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1}\lambda^j}{j} Li_{ij}(x^j)\right).$$

Further Results

Another result that follows from the "master theorem" is

$$\Sigma^s\left(\frac{1}{1-\lambda z}\right) * \Sigma^{1-s}\left(\frac{1}{1+\lambda z}\right) = 1 \qquad (2)$$

for any $s \in k$, $z \in kA[[\lambda]]$. This generalizes IKOO, which gave the result for $s = 1$:

$$\Sigma\left(\frac{1}{1-\lambda z}\right) * \frac{1}{1+\lambda z} = 1.$$

We haven't yet found a nice use for the additional generality of equation (2), but recent work by S. Yamamoto (`arXiv 1203.1118[NT]`) indicates that having results for arbitrary iterates of Σ could be useful.

Further Results Cont'd

The next result, which appears in IKOO, is considerably harder to prove (though I think the proof in Ihara's and my preprint is easier to understand than the proof in IKOO).

Theorem

For $a, b \in A$,

$$\Sigma\left(\frac{1}{1-\lambda ab}\right) = \frac{1}{1-\lambda ab} * \Sigma\left(\frac{1}{1-\lambda a \diamond b}\right)$$

The point of this formula is to treat MZVs, finite multiple harmonic sums, etc. where there is a repeated pattern of length two.

Further Results Cont'd

Translated into MZVs, this says, for example, that

$$\sum_{n=0}^{\infty} \lambda^n \zeta^\star((z_3 z_1)^n) = \left(\sum_{i=0}^{\infty} \lambda^i \zeta((z_3 z_1)^i) \right) \left(\sum_{j=0}^{\infty} \lambda^j \zeta^*(z_4^j) \right),$$

which in view of the Zagier-Broadhurst result

$$\zeta((z_3 z_1)^n) = 4^{-n} \zeta((z_4)^n)$$

makes it clear that

$$\zeta^\star((z_3 z_1)^n) = \zeta^\star(\underbrace{3, 1, \dots, 3, 1}_{n \text{ repetitions}})$$

is a rational multiple of π^{4n}.