Ladder and Gluonic Contributions at 3 Loops

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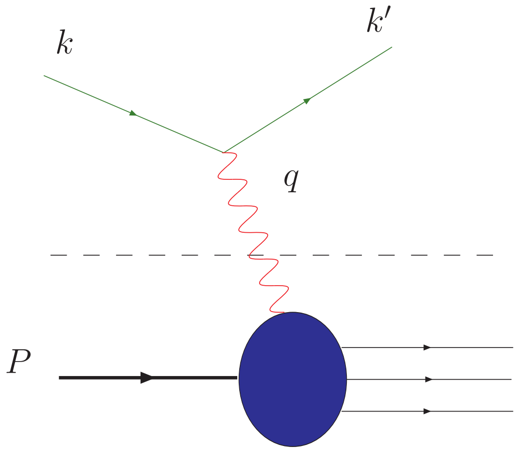
- Introduction → Massive OMEs
- Calculating Massive OMEs in Perturbative QCD
- 3-Loop Contributions to the Gluonic OME's
- 3-Loop Ladder Diagrams
- Conclusions

Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s \rangle$$

$$= \frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x,Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x,Q^2) \ .$$

Structure Functions: $F_{2,L}$

in QCD they contain light and heavy quark contributions.

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) := \int_0^1 dx \, x^{N-1} f(x) \ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

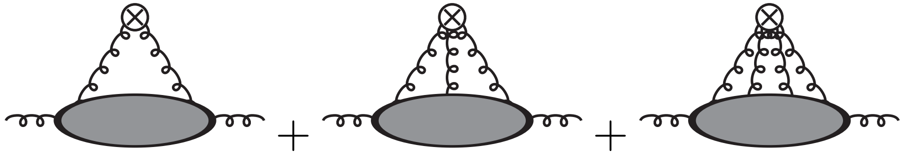
$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

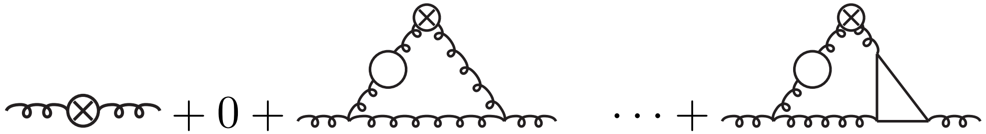
The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

$A_{gg} \to \hat{\hat{A}}_{gg} =$  $+ \ldots$

$= \text{“tree”} + \text{“1-loop”} + \text{“2-loop”} + \text{“3-loop”} + \ldots$

$=$  $\ldots$

graph generation: QGRAF [Nogueira 1993 J. Comput. Phys]

Feynman rules: graphs $\to$ momentum integrals (generic space time dimension $D = 4 + \varepsilon$)

“dimensional regularization”

numerator structure: polynomial of scalar products of momenta $k_i.k_j$ $\to$ lin. comb. of scalar integrals

Feynman parametrization:
$$\frac{1}{\prod A_i^{a_i}} = \frac{\Gamma\left(\sum a_i\right)}{\prod \Gamma(a_i)} \int_{[0,1]^n} dx_1 \ldots dx_n \, \delta\left(1 - \sum x_i\right) \frac{\prod x_i^{a_i - 1}}{\left(\sum A_i x_i\right)^{\sum a_i}},$$
$$(i = 1 \ldots n)$$

$\to$ carry out the momentum integrals

$\rightarrow$ Feynman parameter integrals $\displaystyle\int_{[0,1]^n} dx_1 \ldots dx_n \prod_j \Big(\text{polynomial}_j(x_1,\ldots,x_n)\Big)^{b_j}$

to study integration methods: set numerator structure to 1 ($\rightarrow$ "the scalar graph")
$\rightarrow$ full (physical) diagram = sum of similar integrals differing by integer shifts in the b_j

structure of the exponents: the $b_j =$ "$\{\pm N, 0\} \pm \{1,2,3,...\}\frac{\varepsilon}{2} \pm \{1,2,3,...\}$"

general structure of the result: $\hat{\hat{A}}_{ij} = \delta_{ij} + \displaystyle\sum_{l=1}^{\infty} \hat{a}_s^l \sum_{k=0}^{\infty} \frac{a_{ij}^{(l,k)}}{\varepsilon^{l-k}}$ ($\hat{a}_s$ – strong coupling constant)

where $a_{ij}^{(l,k)} = (\text{factors}) \times \sum_i (\text{color\_factor})_i \times f_i(N)$

core task: expand the functions of (ε, N) defined by Feynman parameter integrals in terms of a Laurent series in ε with coefficients depending on N. Represent the coefficients via functions which

- have an analytic continuation to complex N
- may be efficiently evaluated at complex N
- are represented in a linear independent basis, [Blümlein, Klein, Schneider, Stan 2010]
  $\rightarrow$ compact expressions & algebraic check for equality [Blümlein 2009 Comput.Phys.Commun.]

The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gg,Q}$

calculation of the 1PI part of $A_{gg,Q}^{(3),n_f T_F^2}$

- generation of Diagrams with QGRAF [Nogueira 1993 J. Comput. Phys] $\rightarrow$ 76 Diagrams
- momentum integrals (regularized in $D = 4 + \varepsilon$ dimensions) $\rightarrow$ Feynman parametrization $\rightarrow$ binomial sums and hypergeometric functions
- All-ε representation: maximum nestedness 2, hypergeometric functions ${}_3F_2$

Moments were tested using earlier calculations based on MATAD by [M. Steinhauser, 2000 CPC].
Then the packages SIGMA, EvaluateMultiSums, SumProduction [C. Schneider, 2005-] are used for:

- reducing the sums to a small number of key sums
- expanding the summands in ε
- simplifying by symbolic summation algorithms based on $\Pi\Sigma$-fields [Karr 1981 J. ACM, Schneider 2005–]
- harmonic sums are algebraically reduced using the package HarmonicSums (Ablinger) [Ablinger, Blümlein, Schneider 2011]

$\longrightarrow$ single harmonic sums and ζ-values of max. weight 3 $\quad S_i \equiv \sum_{j=1}^{N} \frac{1}{j^i}$

The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gg,Q}$

The unrenormalized 1PI contribution to $O(\alpha_s^3 n_f T_F^2 C_{A/F})$ to $A_{gg,Q}$:

$$
\begin{aligned}
\hat{\hat{A}}_{gg,Q}^{(3),n_f T_F^2,\mathrm{1PI}} &= S_\varepsilon^3 a_s^3 n_f T_F^2 \frac{1+(-1)^N}{2}\left(\frac{m^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon}\Biggl\{\frac{1}{\varepsilon^3}\Biggl(\mathbf{C_A}\left[\frac{512}{27}S_1 - \frac{64\,(3N^4+6N^3+13N^2+10N+16)}{27(N-1)N(N+1)(N+2)}\right] \\
&- \mathbf{C_F}\frac{512\,(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)}\Biggr) + \frac{1}{\varepsilon^2}\Biggl(\mathbf{C_A}\left[\frac{1280}{81}S_1 - \frac{16P_1}{81(N-1)N^2(N+1)^2(N+2)}\right] \\
&+ \mathbf{C_F}\frac{1}{(N-1)(N+2)}\left[\frac{128\,(N^2+N+2)^2}{9N^2(N+1)^2}S_1 - \frac{128P_2}{27N^3(N+1)^3}\right]\Biggr) + \mathbf{C_A}\frac{1}{(N-1)(N+2)}\Biggl[\frac{4P_3}{27N^2(N+1)^2}S_1^2 \\
&+ \frac{8P_4}{729N^3(N+1)^3}S_1 + \frac{160}{27}(N-1)(N+2)\zeta_2 S_1 - \frac{448}{27}(N-1)(N+2)\zeta_3 S_1 + \frac{P_5}{729N^4(N+1)^4} \\
&- \frac{2P_6}{27N^2(N+1)^2}\zeta_2 + \frac{56\,(3N^4+6N^3+13N^2+10N+16)}{27N(N+1)}\zeta_3 - \frac{4P_7}{27N^2(N+1)^2}S_2\Biggr] \\
&+ \frac{1}{\varepsilon}\Biggl(\mathbf{C_A}\frac{1}{(N-1)(N+2)}\Biggl[-\frac{4P_8}{81N^3(N+1)^3} - \frac{8\,(3N^4+6N^3+13N^2+10N+16)}{9N(N+1)}\zeta_2 + \frac{32P_9}{27N^2(N+1)^2}S_1 \\
&+ \frac{64}{9}(N-1)(N+2)\zeta_2 S_1\Biggr] + \mathbf{C_F}\frac{1}{(N-1)(N+2)}\Biggl[-\frac{160\,(N^2+N+2)^2}{9N^2(N+1)^2}S_1^2 - \frac{64\,(N^2+N+2)^2}{3N^2(N+1)^2}\zeta_2 \\
&+ \frac{32\,(N^2+N+2)^2}{3N^2(N+1)^2}S_2 - \frac{64P_{10}}{81N^4(N+1)^4} + \frac{64P_{11}}{27N^3(N+1)^3}S_1\Biggr]\Biggr) + \mathbf{C_F}\frac{1}{(N-1)(N+2)}\Biggl[\frac{112\,(N^2+N+2)^2}{27N^2(N+1)^2}S_1^3 \\
&- \frac{16P_{12}}{27N^3(N+1)^3}S_1^2 + \frac{32P_{13}}{81N^4(N+1)^4}S_1 + \frac{16\,(N^2+N+2)^2}{3N^2(N+1)^2}\zeta_2 S_1 + \frac{16\,(N^2+N+2)^2}{3N^2(N+1)^2}S_2 S_1 - \frac{32P_{14}}{243N^5(N+1)^5} \\
&- \frac{16P_2}{9N^3(N+1)^3}\zeta_2 + \frac{448\,(N^2+N+2)^2}{9N^2(N+1)^2}\zeta_3 + \frac{16P_{15}}{9N^3(N+1)^3}S_2 - \frac{160\,(N^2+N+2)^2}{27N^2(N+1)^2}S_3\Biggr]\Biggr\}
\end{aligned}
$$

Renormalization of the OME: organize the systematic elimination of poles to remove artifacts of perturbation theory.

[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B]

1. include contributions from reducible diagrams $\rightarrow \hat{\hat{A}}_{gg,Q}^{(3)} = \frac{a_{gg,Q}^{(3,0)}}{\varepsilon^3} + \frac{a_{gg,Q}^{(3,1)}}{\varepsilon^2} + \frac{a_{gg,Q}^{(3,2)}}{\varepsilon} + a_{gg,Q}^{(3)}$
2. perform on-shell mass renormalization $\rightarrow \delta m_i^{(k)}$
3. renormalize the coupling in a MOM-scheme, using the background field method $\rightarrow \beta_i, \beta_{i,Q}$
4. remove remaining UV singularities defining operator Z-factors
   $\rightarrow$ anomalous dimensions: $\quad \gamma_{ij} \equiv \sum_{l=1}^{\infty} a_s^{\overline{\mathsf{MS}}\,l} \gamma_{ij}^{(l)} \quad$ with $\quad \hat{\gamma}_{ij}^{(k)} \equiv \gamma_{ij}^{(k)}(n_f+1) - \gamma_{ij}^{(k)}(n_f)$
5. remove collinear singularities via coll. factorization
   $(\rightarrow \beta_i, \beta_{i,Q}, \gamma_{ij})$
6. transform coupling constant to $\overline{\mathsf{MS}}$
7. choice: m on-shell or $m_{\overline{\mathsf{MS}}}$

The structure of the unrenormalized OME: [Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B]

$$
\begin{aligned}
\hat{\hat{A}}_{gg,Q}^{(3)} = & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2}\Biggl[\frac{1}{\varepsilon^3}\Biggl(-\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}}{6}\Bigl[\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+6\beta_0+4n_f\beta_{0,Q}+10\beta_{0,Q}\Bigr]-\frac{2\gamma_{gg}^{(0)}\beta_{0,Q}}{3}\Bigl[2\beta_0+7\beta_{0,Q}\Bigr] \\
& -\frac{4\beta_{0,Q}}{3}\Bigl[2\beta_0^2+7\beta_{0,Q}\beta_0+6\beta_{0,Q}^2\Bigr]\Biggr)+\frac{1}{\varepsilon^2}\Biggl(\frac{\hat{\gamma}_{qg}^{(0)}}{6}\Bigl[\gamma_{gq}^{(1)}-(2n_f-1)\hat{\gamma}_{gq}^{(1)}\Bigr]+\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}}{3}-\frac{\hat{\gamma}_{gg}^{(1)}}{3}\Bigl[4\beta_0+7\beta_{0,Q}\Bigr] \\
& +\frac{2\beta_{0,Q}}{3}\Bigl[\gamma_{gg}^{(1)}+\beta_1+\beta_{1,Q}\Bigr]+\frac{2\gamma_{gg}^{(0)}\beta_{1,Q}}{3}+\delta m_1^{(-1)}\Bigl[-\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}-2\beta_{0,Q}\gamma_{gg}^{(0)}-10\beta_{0,Q}^2-6\beta_{0,Q}\beta_0\Bigr]\Biggr) \\
& +\frac{1}{\varepsilon}\Biggl(\frac{\hat{\gamma}_{gg}^{(2)}}{3}-2(2\beta_0+3\beta_{0,Q})\mathbf{a_{gg,Q}^{(2)}}-n_f\hat{\gamma}_{qg}^{(0)}\mathbf{a_{gq,Q}^{(2)}}+\gamma_{gq}^{(0)}\mathbf{a_{Qg}^{(2)}}+\beta_{1,Q}^{(1)}\gamma_{gg}^{(0)}+\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Bigl[\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)} \\
& +2(2n_f+1)\beta_{0,Q}+6\beta_0\Bigr]+\frac{\beta_{0,Q}\zeta_2}{4}\Bigl[\gamma_{gg}^{(0)}\{2\beta_0-\beta_{0,Q}\}+4\beta_0^2-2\beta_{0,Q}\beta_0-12\beta_{0,Q}^2\Bigr] \\
& +\delta m_1^{(-1)}\Bigl[-3\delta m_1^{(-1)}\beta_{0,Q}-2\delta m_1^{(0)}\beta_{0,Q}-\hat{\gamma}_{gg}^{(1)}\Bigr]+\delta m_1^{(0)}\Bigl[-\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}-2\gamma_{gg}^{(0)}\beta_{0,Q}-4\beta_{0,Q}\beta_0-8\beta_{0,Q}^2\Bigr] \\
& +2\delta m_2^{(-1)}\beta_{0,Q}\Biggr)+a_{gg,Q}^{(3)}\Biggr]\,.
\end{aligned}
$$

$\longrightarrow$ use for checking the ε singular parts

We confirm the $n_f T_F^2$ part of the 3-Loop anomalous dimension:

[Moch, Vermaseren, Vogt 2004 Nucl.Phys.B]

$$
\begin{aligned}
\hat{\gamma}_{gg}^{(2)} \;=\; & n_f T_F^2 \mathbf{C_A} \Bigg[-\frac{32\,(8N^6+24N^5-19N^4-78N^3-253N^2-210N-96)}{27(N-1)N^2(N+1)^2(N+2)} S_1 \\
& -\frac{8\,(87N^8+348N^7+848N^6+1326N^5+2609N^4+3414N^3+2632N^2+1088N+192)}{27(N-1)N^3(N+1)^3(N+2)} \Bigg] \\
& + n_f T_F^2 \mathbf{C_F} \Bigg[\frac{64\,(N^2+N+2)^2}{3(N-1)N^2(N+1)^2(N+2)} \left(S_1^2 - 3S_2\right) - \frac{16 P_1}{27(N-1)N^4(N+1)^4(N+2)} \\
& + \frac{128\,(4N^6+3N^5-50N^4-129N^3-100N^2-56N-24)}{9(N-1)N^3(N+1)^3(N+2)} S_1 \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
P_1 \;=\; & 33N^{10}+165N^9+256N^8-542N^7-3287N^6-8783N^5-11074N^4-9624N^3 \\
& -5960N^2-2112N-288
\end{aligned}
$$

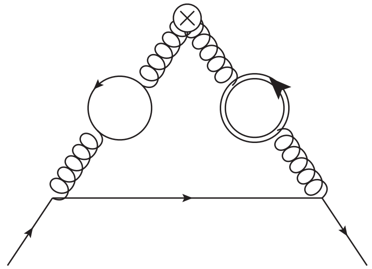
First diagrammatic recalculation

$$
\begin{aligned}
A_{gg,Q}^{(3),\overline{\mathsf{MS}}} = \; & \frac{1}{48}\Biggl\{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\left(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-6\beta_0-4n_f\beta_{0,Q}-10\beta_{0,Q}\right)-4\left(\gamma_{gg}^{(0)}\left[2\beta_0+7\beta_{0,Q}\right]+4\beta_0^2+14\beta_{0,Q}\beta_0\right. \\
& \left.+12\beta_{0,Q}^2\right)\beta_{0,Q}\Biggr\}\ln^3\left(\frac{\mathbf{m^2}}{\mu^2}\right)+\frac{1}{8}\Biggl\{\hat{\gamma}_{qg}^{(0)}\left(\gamma_{gq}^{(1)}+(1-n_f)\hat{\gamma}_{gq}^{(1)}\right)+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}-4\hat{\gamma}_{gg}^{(1)}[\beta_0+2\beta_{0,Q}] \\
& +4\gamma_{gg}^{(1)}\beta_{0,Q}+4[\beta_1+\beta_{1,Q}]\beta_{0,Q}+2\gamma_{gg}^{(0)}\beta_{1,Q}\Biggr\}\ln^2\left(\frac{\mathbf{m^2}}{\mu^2}\right)+\frac{1}{16}\Biggl\{8\hat{\gamma}_{gg}^{(2)}-8n_f a_{gq,Q}^{(2)}\hat{\gamma}_{qg}^{(0)}+8\gamma_{gq}^{(0)}a_{Qg}^{(2)} \\
& -16a_{gg,Q}^{(2)}(2\beta_0+3\beta_{0,Q})+8\gamma_{gg}^{(0)}\beta_{1,Q}^{(1)}+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_2\left(\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+6\beta_0+4n_f\beta_{0,Q}+6\beta_{0,Q}\right) \\
& +4\beta_{0,Q}\zeta_2\left(\gamma_{gg}^{(0)}+2\beta_0\right)\left(2\beta_0+3\beta_{0,Q}\right)\Biggr\}\ln\left(\frac{\mathbf{m^2}}{\mu^2}\right)+2(2\beta_0+3\beta_{0,Q})\overline{\mathbf{a}}_{\mathbf{gg,Q}}^{(\mathbf{2})}+n_f\hat{\gamma}_{qg}^{(0)}\overline{\mathbf{a}}_{\mathbf{gq,Q}}^{(\mathbf{2})}-\gamma_{gq}^{(0)}\overline{\mathbf{a}}_{\mathbf{Qg}}^{(\mathbf{2})} \\
& -\beta_{1,Q}^{(2)}\gamma_{gg}^{(0)}+\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_3}{48}\left(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-2[2n_f+1]\beta_{0,Q}-6\beta_0\right)+\frac{\beta_{0,Q}\zeta_3}{12}\left([\beta_{0,Q}-2\beta_0]\gamma_{gg}^{(0)}\right. \\
& \left.+2[\beta_0+6\beta_{0,Q}]\beta_{0,Q}-4\beta_0^2\right)-\frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\left(\gamma_{gq}^{(1)}+\hat{\gamma}_{gq}^{(1)}\right)+\frac{\beta_{0,Q}\zeta_2}{8}\left(\hat{\gamma}_{gg}^{(1)}-2\gamma_{gg}^{(1)}-2\beta_1-2\beta_{1,Q}\right) \\
& +\frac{\delta m_1^{(-1)}}{4}\left(8a_{gg,Q}^{(2)}+24\delta m_1^{(0)}\beta_{0,Q}+8\delta m_1^{(1)}\beta_{0,Q}+\zeta_2\beta_{0,Q}\beta_0+9\zeta_2\beta_{0,Q}^2\right)+\delta m_1^{(0)}\left(\beta_{0,Q}\delta m_1^{(0)}+\hat{\gamma}_{gg}^{(1)}\right) \\
& +\delta m_1^{(1)}\left(\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}+2\beta_{0,Q}\gamma_{gg}^{(0)}+4\beta_{0,Q}\beta_0+8\beta_{0,Q}^2\right)-2\delta m_2^{(0)}\beta_{0,Q}+a_{gg,Q}^{(3)}
\end{aligned}
$$

The final renormalized contribution with the $\overline{\mathsf{MS}}$-mass $\bar{m}$:

$$
\begin{aligned}
A_{gg,Q}^{(3),n_f T_F^2,\overline{\mathsf{MS}}} = n_f T_F^2 \Bigg\{ \Bigg(& \mathbf{C_F} \frac{64\,(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} + \mathbf{C_A}\left[\frac{128\,(N^2+N+1)}{27(N-1)N(N+1)(N+2)} - \frac{64}{27}S_1\right]\Bigg) \mathbf{ln^3}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right) \\
& - \mathbf{C_F}\frac{16}{3}\mathbf{ln^2}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right) + \Bigg(\mathbf{C_A}\frac{1}{(N-1)(N+2)}\left[-\frac{4P_1}{81N^3(N+1)^3} - \frac{16P_2}{81N^2(N+1)^2}S_1\right] \\
& + \mathbf{C_F}\frac{1}{(N-1)(N+2)}\left[\frac{16\,(N^2+N+2)^2}{N^2(N+1)^2}\left(S_1^2 - \frac{5}{3}S_2\right) - \frac{4P_3}{9N^4(N+1)^4} - \frac{32P_4}{3N^3(N+1)^3}S_1\right]\Bigg)\mathbf{ln}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right) \\
& + \mathbf{C_A}\frac{1}{(N-1)(N+2)}\Bigg[-\frac{4P_5}{27N^2(N+1)^2}S_1^2 - \frac{8P_6}{729N^3(N+1)^3}S_1 + \frac{512}{27}(N-1)(N+2)\zeta_3 S_1 \\
& - \frac{2P_7}{729N^4(N+1)^4} - \frac{1024\,(N^2+N+1)}{27N(N+1)}\zeta_3 + \frac{4P_8}{27N^2(N+1)^2}S_2\Bigg] \\
& + \mathbf{C_F}\frac{1}{(N-1)(N+2)}\Bigg[\frac{64\,(N^2+N+2)^2}{9N^2(N+1)^2}\left(-\frac{1}{3}S_1^3 - 8\zeta_3 + \frac{4}{3}S_3\right) + \frac{32P_9}{27N^3(N+1)^3}S_1^2 \\
& - \frac{64P_{10}}{81N^4(N+1)^4}S_1 - \frac{32P_{11}}{243N^5(N+1)^5} - \frac{32P_{12}}{3N^3(N+1)^3}S_2\Bigg]\Bigg\}
\end{aligned}
$$

The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gq,Q}$



The all-ε result constituting the color factor $T_F^2 n_f C_F$

$$\hat{\hat{A}}^{(3)}_{gq,T_F^2 n_f} = -96 a_s^3 T_F^2 n_f C_F \left(\frac{m^2}{\mu^2}\right)^{\frac{3\varepsilon}{2}} S_\varepsilon^3 \frac{1+(-1)^N}{2} e^{-\frac{3\varepsilon}{2}\gamma} \frac{(\varepsilon-1)^2(\varepsilon+2)(\varepsilon+N^2+N+2)}{\varepsilon(\varepsilon+1)(\varepsilon+3)}$$
$$\times \Gamma(1-\varepsilon)^2 \Gamma\left(-\frac{\varepsilon}{2}-4\right)\Gamma\left(\frac{\varepsilon}{2}+2\right)\frac{\Gamma\left(\frac{\varepsilon}{2}+5\right)\Gamma\left(-\frac{3\varepsilon}{2}\right)\Gamma(N-1)}{\Gamma(4-2\varepsilon)\Gamma\left(\frac{\varepsilon}{2}+N+2\right)}$$

yields the renormalized contribution

$$A_{gq,Q}^{(3),n_f T_F^2,\overline{\mathsf{MS}}} = n_f T_F^2 \frac{1+(-1)^N}{2}\Biggl\{ \mathbf{C_F} \frac{32(N^2+N+2)}{9(N-1)N(N+1)} \mathbf{ln^3}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right) + \mathbf{C_F}\Biggl[-\frac{16(N^2+N+2)}{3(N-1)N(N+1)}\left(S_1^2+S_2\right)$$
$$+\frac{32(8N^3+13N^2+27N+16)}{9(N-1)N(N+1)^2}S_1 + \frac{32(19N^4+81N^3+86N^2+80N+38)}{27(N-1)N(N+1)^3}\Biggr]\mathbf{ln}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right)$$
$$+\mathbf{C_F}\Biggl[\frac{32(N^2+N+2)}{27(N-1)N(N+1)}\left(S_1^3+3S_2S_1+2S_3-24\zeta_3\right) - \frac{32(8N^3+13N^2+27N+16)}{27(N-1)N(N+1)^2}\left(S_1^2+S_2\right)$$
$$+\frac{64(4N^4+4N^3+23N^2+25N+8)}{27(N-1)N(N+1)^3}S_1 + \frac{64(197N^5+824N^4+1540N^3+1961N^2+1388N+394)}{243(N-1)N(N+1)^4}\Biggr]\Biggr\}$$

Here we confirm the n_f contribution to the anomalous dimension:
[Moch, Vermaseren, Vogt 2004 Nucl.Phys.B]

$$\hat{\gamma}_{gq}^{(2),n_f} = n_f T_F^2 C_F \left(\frac{64\,(N^2+N+2)}{3(N-1)N(N+1)} - \left(S_1^2 + S_2\right) + \frac{128\,(8N^3+13N^2+27N+16)}{9(N-1)N(N+1)^2} S_1 \right.$$
$$\left. - \frac{128\,(4N^4+4N^3+23N^2+25N+8)}{9(N-1)N(N+1)^3} \right)$$

in an independent calculation.

Furthermore we are able to check a result for the combination

$$\tilde{\gamma}_{gg}^{(2)} + \frac{\tilde{\gamma}_{gq}^{(2)} \gamma_{qg}^{(0)}}{\tilde{\gamma}_{gg}^{(0)} n_f}$$

of 3-loop anomalous dimensions, derived from the large n_f expansion in QCD by [Bennett, Gracey 1997]; where we denote with $\tilde{\gamma}_{ij}^{(k)}$ the leading n_f coefficient of $\gamma_{ij}^{(k)}$.

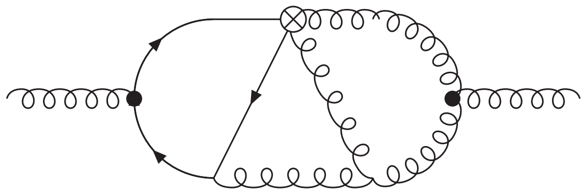
Polarized OMEs

For polarized processes the Feynman rules change slightly, but the OMEs are obtained analogously to the unpolarized case. We obtain:

$$\begin{aligned}
\hat{\hat{A}}_{gq,Q}^{(3),n_f T_F^2,1\mathrm{PI}} &= i a_s^3 \frac{1-(-1)^N}{2}\left(\frac{m^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} S_\varepsilon^3\, n_f T_F^2 C_F \Biggl\{ -\frac{\mathbf{1}}{\boldsymbol{\varepsilon^3}}\frac{128(N+2)}{9N(N+1)} + \frac{\mathbf{1}}{\boldsymbol{\varepsilon^2}}\left(\frac{64(N+2)}{9N(N+1)}S_1 - \frac{64\left(N^2+4N+2\right)}{9N(N+1)^2}\right) \\
&+\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}\left(\frac{32\left(N^2+4N+2\right)}{9N(N+1)^2}S_1 - \frac{16(N+2)}{9N(N+1)}S_1^2 - \frac{16(N+2)}{9N(N+1)}S_2 - \frac{32\left(28N^3+115N^2+140N+56\right)}{27N(N+1)^3}\right. \\
&\left.-\frac{16(N+2)}{3N(N+1)}\zeta_2\right) - \frac{8\left(N^2+4N+2\right)}{9N(N+1)^2}S_1^2 - \frac{8\left(N^2+4N+2\right)}{9N(N+1)^2}S_2 + \frac{16\left(28N^3+115N^2+140N+56\right)}{27N(N+1)^3}S_1 \\
&+\frac{8(N+2)}{3N(N+1)}\zeta_2 S_1 + \frac{8(N+2)}{27N(N+1)}S_1^3 + \frac{8(N+2)}{9N(N+1)}S_2 S_1 + \frac{16(N+2)}{27N(N+1)}S_3 \\
&-\frac{8\left(N^2+4N+2\right)}{3N(N+1)^2}\zeta_2 - \frac{16\left(700N^4+3752N^3+6777N^2+5152N+1400\right)}{243N(N+1)^4} + \frac{112(N+2)}{9N(N+1)}\zeta_3 \Biggr\}
\end{aligned}$$

$$
\begin{aligned}
\hat{\hat{A}}_{gg,Q}^{(3),n_f T_F^2,1\mathrm{PI}} = \; & a_s^3 \frac{1-(-1)^N}{2} \left(\frac{m^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} S_\varepsilon^3 \, n_f T_F^2 \Biggl\{ \frac{\mathbf{1}}{\boldsymbol{\varepsilon}^{\mathbf{2}}} \Biggl(C_A \left[\frac{512}{27} S_1 - \frac{64\left(3N^2+3N+16\right)}{27N(N+1)} \right] - C_F \frac{512(N-1)(N+2)}{9N^2(N+1)^2} \Biggr) \\
& + \frac{\mathbf{1}}{\boldsymbol{\varepsilon}} \Biggl(C_A \left[\frac{2816}{81} S_1 - \frac{16\left(39N^4+78N^3+439N^2+400N-48\right)}{81N^2(N+1)^2} \right] \\
& + C_F \left[\frac{128(N-1)(N+2)}{9N^2(N+1)^2} S_1 - \frac{128(N-1)(N+2)(4N+3)(5N-1)}{27N^3(N+1)^3} \right] \Biggr) \\
& + C_A \left[-\frac{8\left(3N^2+3N+16\right)}{9N(N+1)} \zeta_2 + \frac{32\left(160N^4+302N^3+115N^2-27N+48\right)}{81N^2(N+1)^2} S_1 - \frac{4P_1}{81N^3(N+1)^3} + \frac{64}{9} \zeta_2 S_1 \right] \\
& + C_F \Biggl[-\frac{160(N-1)(N+2)}{9N^2(N+1)^2} S_1^2 + \frac{32(N-1)(N+2)}{3N^2(N+1)^2} S_2 - \frac{64(N-1)(N+2)}{3N^2(N+1)^2} \zeta_2 \\
& + \frac{64(N-1)(N+2)\left(19N^2+28N+12\right)}{27N^3(N+1)^3} S_1 - \frac{64(N-1)(N+2)\left(218N^4+373N^3+224N^2+96N+45\right)}{81N^4(N+1)^4} \Biggr] \Biggr\}
\end{aligned}
$$

Ladder Graphs: 3 Massive Propagators



The Feynman parametrization for the scalar graph is

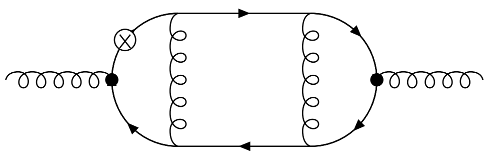
$$\hat{I}_{10a} = \exp\left(-3\frac{\varepsilon}{2}\gamma_E\right)\Gamma\left(1-3\frac{\varepsilon}{2}\right)\sum_{j=0}^{N}\sum_{k=0}^{N-j}\int_{[0,1]^6} dx\, dy\, du\, dw\, ds\, dt\, \theta(1-s-t)$$
$$y^{-\varepsilon}(1-y)^{-\varepsilon}w^{\frac{\varepsilon}{2}-1}(1-w)^{\frac{\varepsilon}{2}-1}s^{\varepsilon-1}t^{-\frac{\varepsilon}{2}}(1-y)^j$$
$$[x-(1-s-t)-sx-tu]^j\ [x(1-y)+y((1-s-t)+sx+tu)]^{N-j-k}$$
$$\Big([x(1-y)-u(1-w)+(y-w)(1-s-t+sx+tu)]^k$$
$$+[x(1-y)+u(1-w)-(1-y-w)(1-s-t+sx+tu)]^k\Big)$$

Operator → 2 (physical) sums, additionally introduce binomial sums

$$\hat{I}_{10a} = \exp\left(-3\frac{\varepsilon}{2}\gamma_E\right)\Gamma\left(1-3\frac{\varepsilon}{2}\right)\sum_{j=0}^{N}\binom{N+2}{j+2}\sum_{k=0}^{j}\binom{j+1}{k+1}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\sum_{q=0}^{N-j}\binom{N-j}{q}$$
$$(-1)^{N-j-q}\sum_{r_2=0}^{N-l-q}\binom{N-l-q}{r_2}\sum_{r_1=0}^{N-l-q-r_2}\binom{N-l-q-r_2}{r_1}\frac{B(1-\varepsilon,N+2-j-\varepsilon)B(\frac{\varepsilon}{2},k+1+\frac{\varepsilon}{2})}{(N+1-q-r_1-r_2)(q+r_2+1)}$$
$$B(r_2+\varepsilon,r_1+1)B\left(N+1-l-q-r_1-r_2-\frac{\varepsilon}{2},r_1+r_2+1+\varepsilon\right)$$

$$
\begin{aligned}
\hat{I}_{10a} \;=\; & \frac{1}{(N+3)(N+4)}\Biggl\{\frac{\mathbf{1}}{\boldsymbol{\varepsilon}^{\mathbf{2}}}\Biggl[-\frac{4\left(N^3+3N^2-N-5\right)}{(N+1)(N+2)(N+3)}S_1+2S_1^2+\frac{4(-1)^N}{N+3}S_1+4S_{-2}+2(2N+5)S_2+\frac{4(-1)^N\left(2N^3+7N^2+4N-3\right)}{(N+1)^2(N+2)^2(N+3)} \\
& +\frac{4\left(6N^3+34N^2+63N+39\right)}{(N+1)^2(N+2)^2(N+3)}\Biggr]+\frac{\mathbf{1}}{\boldsymbol{\varepsilon}}\Biggl[\frac{\left(-4N^4-25N^3-30N^2+49N+76\right)}{(N+1)(N+2)(N+3)(N+4)}S_1^2-\frac{4\left(2N^4+14N^3+27N^2+5N-16\right)}{(N+1)(N+2)(N+3)(N+4)}S_{-2} \\
& +\frac{\left(10N^4+73N^3+158N^2+73N-52\right)}{(N+1)(N+2)(N+3)(N+4)}S_2+\frac{2(-1)^N P_{29}}{(N+1)^2(N+2)^2(N+3)^2(N+4)}S_1-\frac{2P_{30}}{(N+1)^2(N+2)^2(N+3)^2(N+4)}S_1 \\
& +S_1^3+\frac{(-1)^N}{N+3}\left(S_1^2-S_2\right)+4S_{-2}S_1-5S_2S_1+2(4N+15)S_{-3}+2(N-1)S_3-12S_{-2,1}+8(N+4)S_{2,1} \\
& +\frac{2(-1)^N P_{31}}{(N+1)^3(N+2)^3(N+3)^2(N+4)}+\frac{2P_{32}}{(N+1)^3(N+2)^3(N+3)^2(N+4)}\Biggr]+\frac{7}{24}S_1^4+\frac{\left(-10N^4-61N^3-68N^2+129N+188\right)}{6(N+1)(N+2)(N+3)(N+4)}S_1^3 \\
& +\frac{(-1)^N P_{33}}{2(N+1)^2(N+2)^2(N+3)^2(N+4)}S_1^2+\frac{P_{22}}{2(N+1)^2(N+2)^2(N+3)^2(N+4)^2}S_1^2+\frac{3}{4}\zeta_2S_1^2-4S_{-2}S_1^2-\frac{13}{4}S_2S_1^2 \\
& +\frac{(-1)^N P_{23}}{(N+1)^3(N+2)^3(N+3)^3(N+4)^2}S_1+\frac{P_{24}}{(N+1)^3(N+2)^3(N+3)^3(N+4)^2}S_1-\frac{3\left(N^3+3N^2-N-5\right)}{2(N+1)(N+2)(N+3)}\zeta_2S_1-2S_{-3}S_1 \\
& -\frac{4\left(4N^4+41N^3+155N^2+254N+148\right)}{(N+1)(N+2)(N+3)(N+4)}S_{-2}S_1+\frac{(-1)^N}{N+3}\left(-4S_{-2}S_1+\frac{9}{2}S_2S_1+\frac{3}{2}\zeta_2S_1+\frac{1}{6}S_1^3-2S_{-3}+\frac{10}{3}S_3+2S_{2,1}+12S_{-2,1}\right) \\
& +\frac{\left(-14N^4-201N^3-936N^2-1715N-1044\right)}{2(N+1)(N+2)(N+3)(N+4)}S_2S_1-\frac{119}{3}S_3S_1-12S_{-2,1}S_1+22S_{2,1}S_1-2S_{-2}^2+\frac{1}{8}(32N+119)S_2^2 \\
& +\frac{(-1)^N P_{25}}{(N+1)^4(N+2)^4(N+3)^3(N+4)^2}+\frac{P_{26}}{(N+1)^4(N+2)^4(N+3)^3(N+4)^2}+\frac{3(-1)^N\left(2N^3+7N^2+4N-3\right)}{2(N+1)^2(N+2)^2(N+3)}\zeta_2 \\
& +\frac{3\left(6N^3+34N^2+63N+39\right)}{2(N+1)^2(N+2)^2(N+3)}\zeta_2+(8N+39)S_{-4}+\frac{2P_{34}}{(N+1)(N+2)(N+3)(N+4)}S_{-3}-\frac{4(-1)^N\left(2N^3+7N^2+4N-3\right)}{(N+1)^2(N+2)^2(N+3)}S_{-2} \\
& -\frac{4P_{27}}{(N+1)^2(N+2)^2(N+3)^2(N+4)^2}S_{-2}+\frac{3}{2}\zeta_2S_{-2}+\frac{(-1)^N P_{35}}{2(N+1)^2(N+2)^2(N+3)^2(N+4)}S_2+\frac{P_{28}}{2(N+1)^2(N+2)^2(N+3)^2(N+4)^2}S_2 \\
& +\frac{3}{4}(2N+5)\zeta_2S_2+8S_{-2}S_2+\frac{P_{36}}{3(N+1)(N+2)(N+3)(N+4)}S_3+\frac{1}{4}(20N-29)S_4-14S_{-3,1}+\frac{4\left(4N^4+22N^3+11N^2-85N-96\right)}{(N+1)(N+2)(N+3)(N+4)}S_{-2,1} \\
& -14S_{-2,2}+\frac{2\left(11N^4+107N^3+397N^2+640N+361\right)}{(N+1)(N+2)(N+3)}S_{2,1}+2(N+36)S_{3,1}+28S_{-2,1,1}+2(2N-7)S_{2,1,1}\Biggr\}+O(\varepsilon)
\end{aligned}
$$

Ladder Graphs: 6 Massive Propagators



The Feynman rules provide us with the following integral ($\hat{d}k \equiv \frac{d^D k}{(2\pi)^D}$ and $D = 4 + \epsilon$):

$$I_{2a} := \iiint \frac{\hat{d}k\ \hat{d}r\ \hat{d}s \qquad (\Delta.k)^{N-1}}{((k-p)^2-m^2)((r-p)^2-m^2)((s-p)^2-m^2)(s^2-m^2)(r^2-m^2)(k^2-m^2)(k-r)^2(s-r)^2}$$

Apply Feynman parametrization proceeding from outer to inner loops

$$I_{2a} = (\text{const.})\,\Gamma\left(2-\frac{3\varepsilon}{2}\right)\int_0^1 dx\,dz\,du\,dw\,da\,ds\,dt\ z^{\frac{\varepsilon}{2}-1}(1-z)^{\frac{\varepsilon}{2}}(1-u)w^{\frac{\varepsilon}{2}-1}(1-w)^{\frac{\varepsilon}{2}}\times$$
$$\times s^{-\frac{\varepsilon}{2}}t^{-\frac{\varepsilon}{2}}\theta(1-s-t)(1-s-t)\left(1-s\frac{z-1}{z}-t\frac{w-1}{w}\right)^{-2+3\varepsilon/2}\times$$
$$\times(u(1-w)+wa(1-s-t)+wsx+wtu)^{N-1}$$

The topology of massive lines leads to characteristic terms of an integral representation of Appell's function F_1

$$F_1\left[a;b,b';c;X,Y\right] = \int_0^1 dsdt\ \frac{\theta(1-s-t)\ s^{b-1}\ t^{b'-1}\ (1-s-t)^{c-b-b'-1}}{(1-sX-tY)^a}$$

F_1 occurs due to the diagram's topology and mass distribution, independently from the operator insertion.

$$F_1\left[a;b,b';c;\frac{w-1}{w},\frac{z-1}{z}\right]$$

An analytic continuation formula is needed:

$$F_1\left[a;b,b';c;\frac{w-1}{w},\frac{z-1}{z}\right]=w^b z^{b'} F_1\left[c-a;b,b';c;(1-w),(1-z)\right] \ .$$

$\rightarrow$ convergent series representation for $w,z\in[0,1]$.

$$F_1\left[c-a;b,b';c;(1-w),(1-z)\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(c-a)_{m+n}(b)_m(b')_n}{m!\;n!\;(c)_{m+n}}(1-w)^m(1-z)^n$$

with the Pochhammer symbol

$$(a)_m\equiv a(a+1)...(a+m-1) \ .$$

The operator insertion contributes as an integer power of a polynomial linear in each Feynman parameter.

- Application of the binomial theorem leads to an integral for F_1.
- Apply the series representation for F_1 & perform Beta-type integrals.

$\rightarrow$ binomial sums over a hypergeometric series:

$$\begin{aligned}
I_{2a} &= iS_\varepsilon^3 \Gamma\left(2-\frac{3}{2}\varepsilon\right) \frac{1}{(N+1)(N+2)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=2}^{N+2} \binom{N+2}{l} \sum_{j=2}^{l} \binom{l}{j} \Bigg\{ \\
&\times \sum_{k=1}^{j} \binom{j}{k} \sum_{r=0}^{l-k} \binom{l-k}{r} (-1)^{l+j+k+r} B\left(k, m+1+\frac{\varepsilon}{2}\right) \\
&\times \Gamma\left[\begin{array}{c} k+r+j+m+n+\frac{\varepsilon}{2} \\ m+1, n+1, k+r+\frac{\varepsilon}{2} \end{array}\right] \frac{B\left(k+m-\frac{\varepsilon}{2}, r+1+n-\frac{\varepsilon}{2}\right) B\left(r+l-1, n+1+\frac{\varepsilon}{2}\right)}{(k+r+1+m+n-\varepsilon)(N+3-j)} \\
&+ \sum_{r=0}^{l-j} \binom{l-j}{r} (-1)^{l+j+r} B\left(j, m+1+\frac{\varepsilon}{2}\right) \\
&\times \Gamma\left[\begin{array}{c} j+r+m+n+\frac{\varepsilon}{2} \\ m+1, n+1, j+r+\frac{\varepsilon}{2} \end{array}\right] \frac{B\left(j+m-\frac{\varepsilon}{2}, r+1+n-\frac{\varepsilon}{2}\right) B\left(r+l-1, n+1-\frac{\varepsilon}{2}\right)}{(j+r+1+m+n-\varepsilon)(N+3-j)} \Bigg\}
\end{aligned}$$

This expression is finite as $\varepsilon \rightarrow 0$.

Sums over hypergeometric expressions are solved using the packages Sigma and EvaluateMultiSums by C.Schneider [C. Schneider, 2005-] applying symbolic summation techniques. The result can be written in terms of rational expressions, harmonic sums and generalizations thereof in N:

$$
\begin{aligned}
I_{2a} \;=\; & \frac{1}{(N+1)(N+2)(N+3)}\Biggl\{\frac{1}{6}S_1^3+\frac{N^2+12N+16}{2(N+1)(N+2)}S_1^2+\frac{4(2N+3)}{(N+1)^2(N+2)}S_1 \\
& +\frac{8(2N+3)}{(N+1)^3(N+2)}+2\Bigl[-2^{N+3}+3-(-1)^N\Bigr]\zeta_3-(-1)^N S_{-3}+\left[\frac{3N^2+40N+56}{2(N+1)(N+2)}-2S_1\right]S_2 \\
& -\frac{3N+17}{3}S_3-2(-1)^N S_{-2,1}-(N+3)S_{2,1}+2^{N+4}S_{1,2}\left(\frac{1}{2},1\right)+2^{N+3}S_{1,1,1}\left(\frac{1}{2},1,1\right)\Biggr\}+O(\varepsilon)\ .
\end{aligned}
$$

Where we used the shorthand for harmonic sums and generalizations defined resp.:

$$S_{b,\vec{a}}(N)=\sum_{k=1}^{N}\frac{\mathrm{sign}(b)^k}{k^{|b|}}S_{\vec{a}}(k) \quad \rightarrow S_{b,\vec{a}}\ , \quad b,a_i\in\mathbb{Z}\backslash\{0\}$$

[Blümlein, Kurth 1998 Phys.Rev.D]
[Vermaseren 1998 Int.J.Mod.Phys.A]

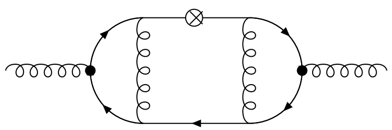
$$S_{b,\vec{a}}(\xi',\vec{\xi};N)=\sum_{k=1}^{N}\frac{\mathrm{sign}(b)^k\xi'^k}{k^{|b|}}S_{\vec{a}}(\vec{\xi};k)\ \rightarrow S_{b,\vec{a}}(\xi',\vec{\xi})\ ,\ \xi',\xi_i\in[0,1]$$

[Moch, Uwer, Weinzierl 2002 J.Math.Phys.]
[Ablinger, Blümlein, Schneider 2012].

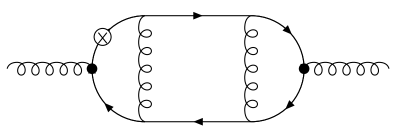
The powers 2^N drop out in the asymptotic expansion in N.

Further diagrams

Using the methods described above the corresponding expressions for the following topologies were calculated:



1



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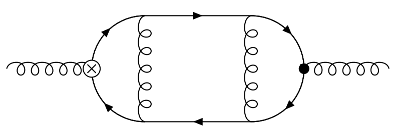

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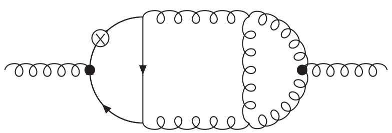

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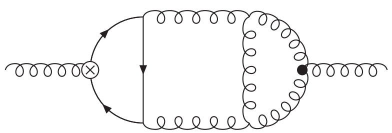

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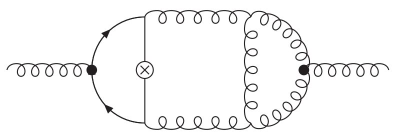

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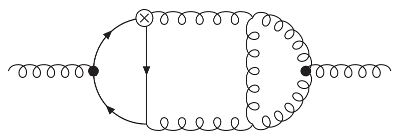

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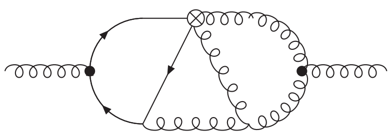

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10

Characteristics of Associated Recursions

Diagram	rational			ζ_3		
	# Moments	Degree	Order	# Moments	Degree	Order
I_{1a}	203	26	8	15	3	2
I_{1b}	269	36	9	15	3	2
I_{2a}	215	31	8	19	3	3
I_{2b}	269	42	9	35	6	3
I_4	623	90	13	131	24	6

Diagram	ε^{-2}			ε^{-1}			ε^0 rat.			$\varepsilon^0\zeta_2$		
	#	Deg.	Ord.	#	Deg.	Ord.	#	Deg.	Ord.	#	Deg.	Ord.
I_{6a}	15	3	2	55	11	3	142	25	5	15	3	2
I_{6b}	15	3	2	55	12	3	142	27	5	15	3	2
I_{8a}	19	4	2	69	14	3	164	30	5	19	4	2
I_{8b}	19	4	2	79	16	3	175	34	5	19	4	2
I_9	142	26	9	463	83	10	1199	210	16	142	26	5
I_{10a}	47	6	4	341	57	10	949	156	16	109	17	6
I_{10b}	109	17	6	323	53	10	911	152	16	47	6	4

Conclusions

- Massive OMEs at 3-loop order are needed for the DIS structure functions at $O(\alpha_s^3)$ as well as for the definition of variable flavor number schemes to the same order.
- A series of moments ($N \leq 14$) for the transition matrix elements A_{ij} at 3-loop order were given in [Bierenbaum, Blümlein, Klein 2009 Nucl. Phys. B].
- We calculated the 3-loop contributions to A_{gg} and A_{gq} in $O(n_f T_F^2 C_{A,F})$.
- The corresponding quarkonic 3-loop contributions of $O(n_f T_F^2 C_{A,F})$ to A_{qq} and A_{qg} were calculated in [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl. Phys. B].
- similarly the 2 and 3-loop contributions to the polarized OMEs $A_{gg,Q}^{\text{pol}}$, $A_{gq,Q}^{\text{pol}}$ were calculated
- In order to pave the way for more complicated topologies, we calculated scalar ladder graphs, which may be mapped onto a summation problem using hypergeometric functions ${}_2F_1, {}_3F_2$ and Appell's F_1.
- The chain of computer algebra packages SIGMA, EvaluateMultiSums, HarmonicSums allows for an ε expansion and simplification of the coefficients onto a basis of (generalized) harmonic sums.
- once the calculation of scalar graphs is understood, the calculation of corresponding physical graphs can be automated ($\rightarrow$ efficiency issues, c.f. the package SumProduction)