Integrals in Two-Loop Massive Operator Matrix Elements with External Massive Legs

Abilio De Freitas

DESY–Zeuthen

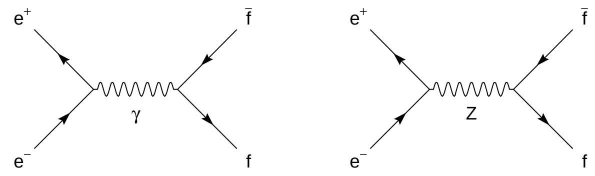
in collaboration with
J. Blümlein (DESY) and W.L. van Neerven† (U. Leiden)

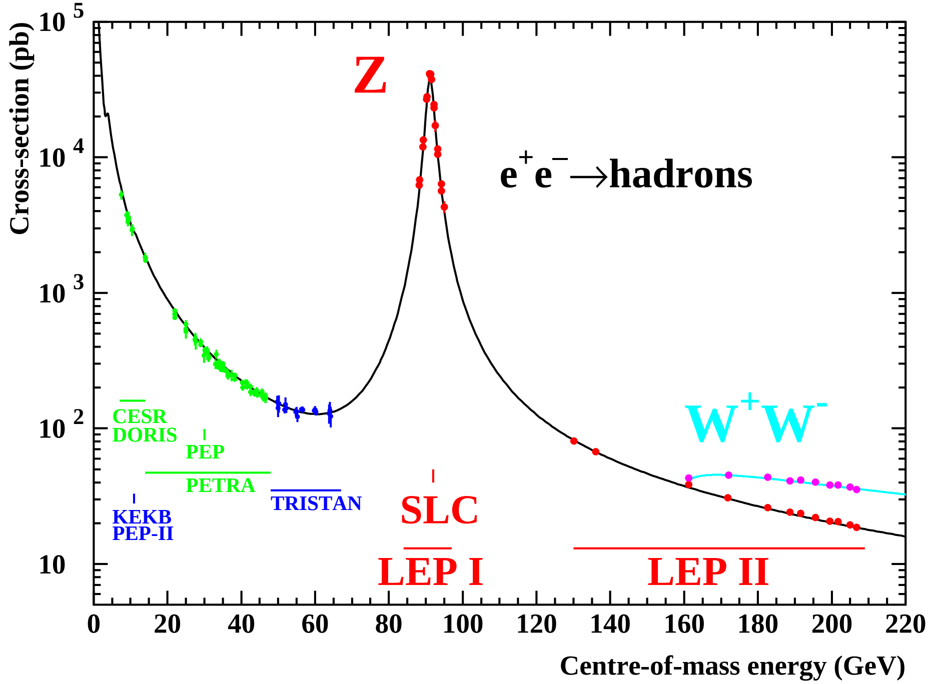
Outline

1. Introduction.
2. Calculation of the two-loop OME's.
3. Integration by parts.
4. Mellin-Barnes representation.
5. Sample Calculations.
6. Polylogarithmic integrals and analytic continuations.
7. Results.
8. Conclusions.

1. Introduction

We want to revisit the calculation of the two-loop order initial state radiative corrections to electron-positron annihilation into heavy fermions:



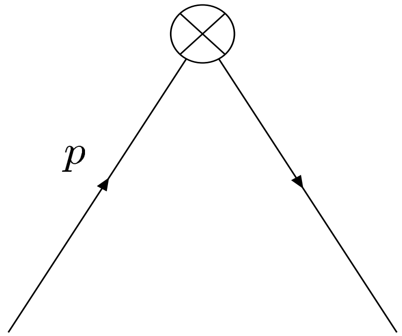


Some time ago, Berends, van Neerven and Burgers (Nucl. Phys. **B297** 1988) 429; E: B304 (1988) 921.) calculated the corrections due to initial state radiation directly (for massive electrons), including soft and virtual photons, hard bremsstrahlung, as well as fermion pair production.

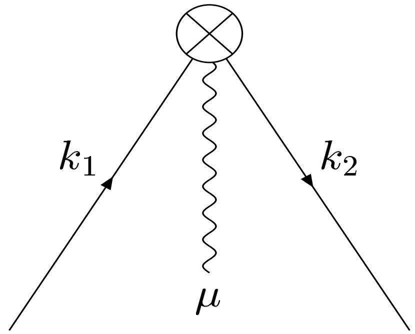
- The process is of central importance at LEP, Giga-Z, and the ILC.
- Recalculate these corrections by a method based on the renormalization group.

2. Calculation of the Two-Loop Operator Matrix Elements

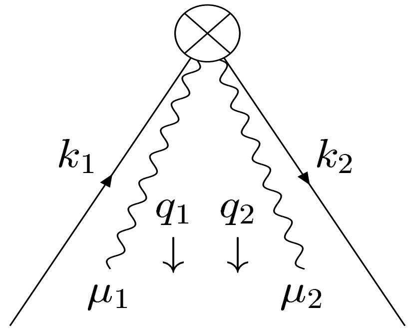
The main "ingredients" required for the calculation of the two loop oder initial state radiative corrections to $e^+e^- \to \gamma^*, Z^*$ are the two-loop massive operator matrix elements $A_{ee}^{(2)}$ in QED. These are calculated using the following Feynman rules:



$$\not\Delta(\Delta.p)^{N-1}$$

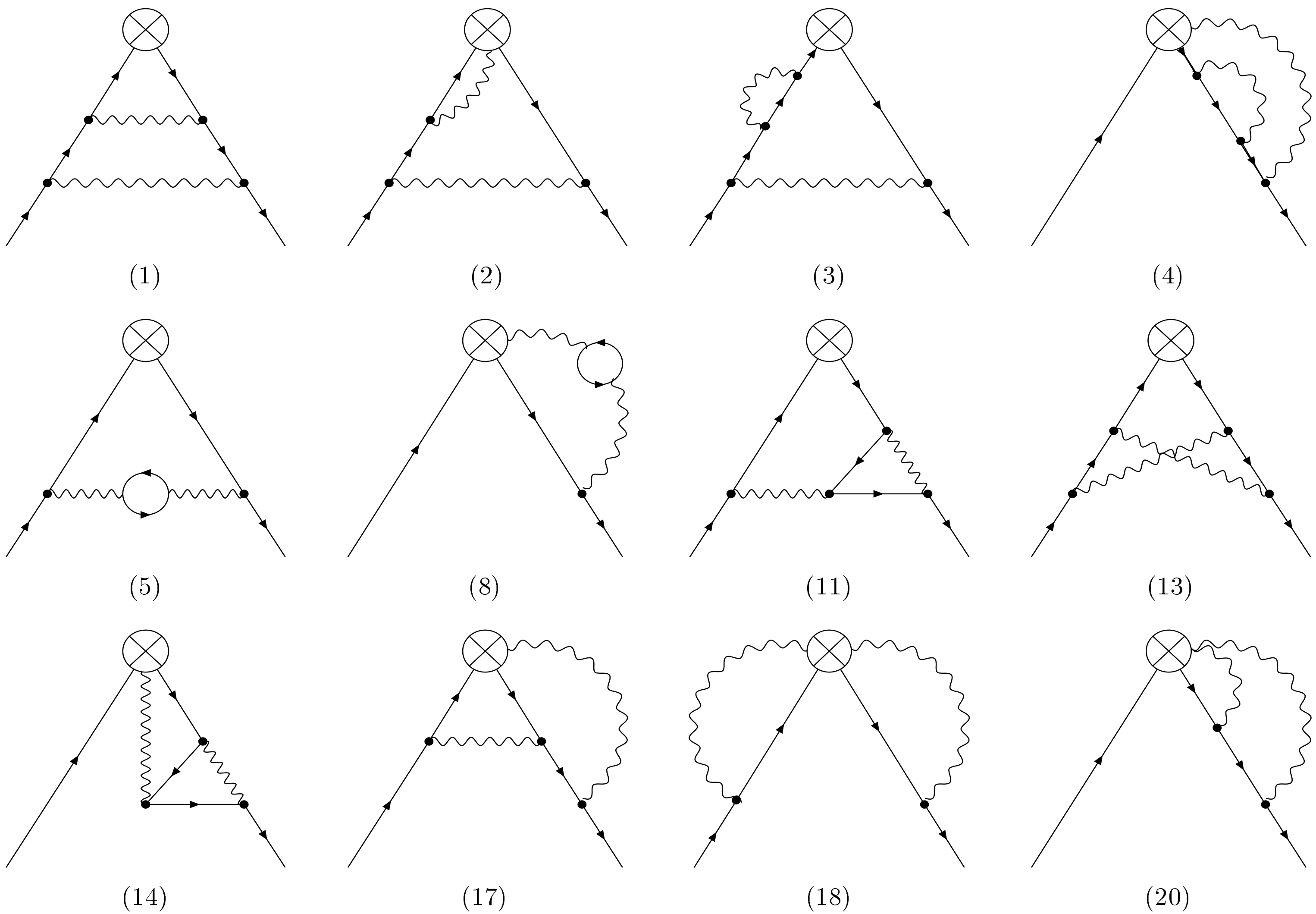


$$-e\Delta_\mu \sum_{j=1}^{N-1}(\Delta.k_1)^{j-1}(\Delta.k_2)^{N-1-j}$$



$$e^2\not\Delta\Delta_{\mu_1}\Delta_{\mu_2}\sum_{j=1}^{N-2}\sum_{i=1}^{j}(\Delta.k_1)^{i-1}(\Delta.k_2)^{N-2-j}\left\{[\Delta.\,(k_1-q_2)]^{j-i}+[\Delta.\,(k_2+q_2)]^{j-i}\right\}$$

where Δ is light-like vector (with $\Delta\cdot\Delta = 0$).



Two-loop diagrams contributing to the massive operator matrix element $A_{ee}(N, \alpha)$.
The antisymmetric diagrams count twice.

All the diagrams shown can be written in terms of integrals of these type:

$$A_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5}^{a,b} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

$$B_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5}^{a,b} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{k_2 \cdot p(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

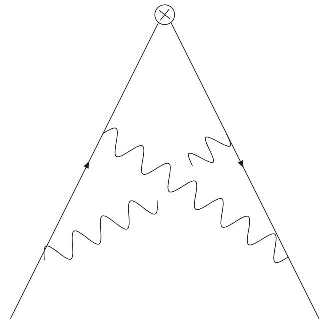
$$E_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5}^{a,b} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta \cdot k_1)^j (\Delta \cdot k_2)^{n-1-j}$$

$$F_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5}^{a,b} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta \cdot p)^j (\Delta \cdot k_1)^{n-1-j}$$

where,

$$D_1 = k_1^2 - m^2 \ ; \quad D_2 = k_2^2 - m^2 \ ; \quad D_3 = (k_1 - p)^2 \ ;$$
$$D_4 = (k_1 - k_2)^2 \ ; \quad D_5 = (k_2 - k_1 + p)^2 - m^2 \ ; \quad D_6 = (k_2 - p)^2$$

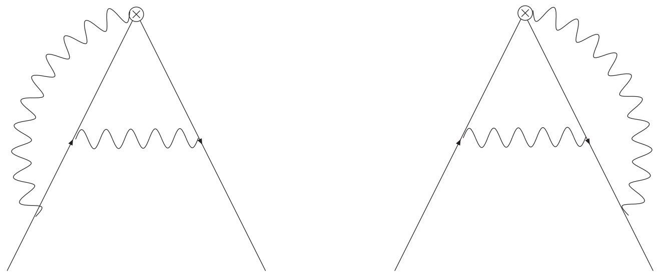
For example, in terms of these integrals, the crossed box gives



$$= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\mathrm{Tr}[\gamma^\mu (\not{p} - \not{k}_1 + \not{k}_2 + m)\gamma^\nu (\not{k}_2 + m) \not{\Delta} (\not{k}_2 + m)\gamma_\mu (\not{k}_1 + m)\gamma_\nu (\not{p} + m)]}{D_1 D_2^2 D_3 D_4 D_5} (\Delta \cdot k_2)^n =$$

$$= (D-4)(D-2)\left[2A^{0,n+1}_{12001} - 2A^{0,n+1}_{02110} - A^{1,n}_{01111} + A^{1,n}_{11011} - A^{1,n}_{11101} + A^{1,n}_{11110} + (\Delta\cdot p)A^{0,n}_{01111}\right]$$
$$+(D-8)(D-2)\left[A^{0,n+1}_{01111} - A^{0,n+1}_{11011} - (\Delta\cdot p)A^{0,n}_{10111} + (\Delta\cdot p)A^{0,n}_{11101}\right] + 16(D-3)m^2\left[A^{0,n+1}_{12011} + A^{0,n+1}_{12101}\right]$$
$$+4(D-4)m^2\left[(\Delta\cdot p)A^{0,n}_{11111} - A^{0,n+1}_{11111}\right] + 8m^2 A^{0,n+1}_{02111} + 8m^2 A^{0,n+1}_{12110} + 32m^4 A^{0,n+1}_{12111}$$
$$4(D-2)\left[A^{0,n+1}_{11101} - A^{0,n+1}_{11110} - 2B^{0,n+1}_{12011} - 2B^{0,n+1}_{12101} + (\Delta\cdot p)A^{0,n}_{11011}\right] \quad ,$$

Another example:



$$= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\mathrm{Tr}[\gamma^\mu (\not{k}_1 + m) \not{\Delta} (\not{k}_2 + m)(\not{p} - \not{k}_1 + \not{k}_2 + m) \not{\Delta} (\not{p} + m)]}{D_1 D_2 D_3 D_4 D_5} \sum_{j=0}^{n-1} (\Delta{\cdot}k_1)^j (\Delta{\cdot}k_2)^{n-1-j} =$$

$$= 2(D-4)\left[E^{2,0}_{01111} - E^{2,0}_{11011} - E^{2,0}_{11101} + E^{2,0}_{11110} + (\Delta\cdot p)E^{0,0}_{10111}\right] - 4E^{0,2}_{01111} + 4E^{0,2}_{11011} + 4(\Delta\cdot p)E^{0,1}_{10111}$$

$$+2(D-2)\left[E^{0,1}_{11011} - E^{1,0}_{11011} - E^{1,0}_{11110} - (\Delta\cdot p)E^{0,0}_{01111} + (\Delta\cdot p)E^{0,0}_{10111} - (\Delta\cdot p)E^{0,0}_{11101}\right]$$

$$+2(D-6)m^2\left[E^{1,1}_{11011} - E^{1,1}_{01111}\right] + 4(D-3)(\Delta\cdot p)\left[E^{1,0}_{11101} + E^{1,0}_{01111}\right] + 4E^{1,1}_{11101} + 4E^{1,1}_{11110}$$

$$+8m^2 E^{1,1}_{11111} - 8m^2 E^{2,0}_{11111} - 4(\Delta\cdot p)E^{0,1}_{11101} + 8m^2(\Delta\cdot p)E^{0,1}_{11111} - 2D(\Delta\cdot p)E^{1,0}_{10111} + 8m^2(\Delta\cdot p)E^{1,0}_{11111}$$

Type A integrals

ν_1	ν_2	ν_3	ν_4	ν_5	$(a\ ,\ b)$
1	1	1	1	0	
1	1	1	0	1	
1	1	0	1	1	
1	0	1	1	1	0,n
0	1	1	1	1	1,n
2	1	1	1	0	n,0
1	2	1	1	0	n,1
1	2	1	0	1	
2	1	0	1	1	
0	2	1	1	1	
2	0	1	1	1	n,0
1	1	1	0	2	n,1
1	1	0	1	2	
1	1	1	2	0	0,n
0	1	1	2	1	1,n
1	0	1	1	2	
1	0	1	2	1	
2	1	1	0	2	0,n
1	2	1	0	2	n,0
3	1	1	1	0	
0	3	1	1	1	n,0
1	3	1	0	1	
2	2	1	1	0	
2	1	2	1	0	
1	1	3	1	0	
1	2	2	1	0	
0	1	3	1	1	
0	2	2	1	1	0,n
3	1	1	0	1	
1	1	2	0	2	
2	1	2	0	1	
1	0	3	1	1	
3	0	1	1	1	
2	0	2	1	1	

Type B integrals

ν_1	ν_2	ν_3	ν_4	ν_5	$(a\ ,\ b)$
2	1	1	1	0	
2	1	1	0	1	n,0
2	1	0	1	1	
1	2	1	1	0	
1	2	1	0	1	0,n
1	2	0	1	1	

Type E and F integrals

1	1	1	1	0	
1	1	1	0	1	1,0
1	1	0	1	1	0,1
0	1	1	1	1	1,1
1	2	1	0	1	2,0
1	1	1	0	2	
0	2	1	1	1	
					0,0
1	0	1	1	1	1,0
					0,1
0	1	1	1	1	0,0
1	1	0	1	1	0,2
2	1	1	1	0	1,1

In total there are 155+ integrals

The integrals are calculated in $D = 4 + \epsilon$ dimensions using dimensional regularization:

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - R^2)^\nu} = (-1)^\nu \frac{\Gamma(\nu - D/2)}{\Gamma(\nu)} \frac{1}{(R^2)^{\nu - D/2}}$$

and introducing Feynman parameters to combine the propagators:

$$\frac{1}{A_1^{\nu_1} A_2^{\nu_2} \cdots A_k^{\nu_k}} = \frac{\Gamma(\sum_{i=1}^k \nu_i)}{\prod_{j=1}^k \Gamma(\nu_j)} \int_0^1 dx_1 \cdots \int_0^1 dx_k \frac{\delta\left(1 - \sum_{i=1}^k x_i\right) \prod_{j=1}^k x_j^{\nu_j - 1}}{(x_1 A_1 + x_2 A_2 + \cdots + x_k A_k)}$$

However, if we apply this to the 5-propagator integrals directly the resulting expressions turn out to be rather cumbersome, for example,

$$A_{12111}^{0,n} \propto \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \frac{w(1-w) y^{1-\epsilon/2} z^{2-\epsilon/2} (xyz + (1-y)z - x)^n w^n}{\left((xy + 1 - y)^2 z^2 + yz\left(\frac{x^2 w}{1-w} + \frac{1}{w}\right) - (1-y)z + 1 - z\right)^{1-\epsilon}}$$

For this reason, we express the 5-propagator integrals as linear combinations of 4-propagator integrals, using integration by parts identities (IBP):

$$\mathcal{E}^{a,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5}(q,l) = \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{\partial}{\partial q^\mu}\left(l^\mu \frac{(\Delta\cdot k_1)^a(\Delta\cdot k_2)^b}{D_1^{\nu_1}D_2^{\nu_2}D_3^{\nu_3}D_4^{\nu_4}D_5^{\nu_5}}\right) = 0 \ ,$$

where $q^\mu = k_1^\mu$, k_2^μ and $l^\mu = p^\mu$, k_1^μ, k_2^μ.

The derivatives with respect to k_1^μ or k_2^μ will rise the powers of some of the propagators, so that the equation given above will lead to linear equations relating integrals with different powers of propagators. These equations can be solved to express some of the integrals in terms of others that might be easier to calculate.

For example, subtracting $\mathcal{E}^{n,0}_{11111}(k_2,k_1)$ from $\mathcal{E}^{n,0}_{11111}(k_2,k_2)$, we obtain

$$A^{n,0}_{11111} = \frac{1}{\epsilon}\left(A^{n,0}_{12101} - A^{n,0}_{02111} + A^{n,0}_{11102}\right)$$

And doing

$$\mathcal{E}^{0,n}_{21111}(k_1,k_2)-\mathcal{E}^{0,n}_{21111}(k_1,k_1)+\mathcal{E}^{0,n}_{11211}(k_1,k_2)-\mathcal{E}^{0,n}_{11211}(k_1,k_1)$$
$$-(D-5)\left[\mathcal{E}^{0,n}_{12111}(k_1,k_2)-\mathcal{E}^{0,n}_{12111}(k_1,k_1)\right]=0\ .$$

it can be seen that

$$\begin{aligned} A^{0,n}_{12111} \quad = \quad -\frac{1}{\epsilon}\Bigg[& -A^{0,n}_{12210}-A^{0,n}_{02211}-A^{0,n}_{12102}-A^{0,n}_{22101} \\ & +\frac{1}{1-\epsilon}\Big(-2A^{0,n}_{31101}+2A^{0,n}_{30111}-A^{0,n}_{21210}+2A^{0,n}_{20211}-A^{0,n}_{21102} \\ & -A^{0,n}_{21201}-2A^{0,n}_{11310}-2A^{0,n}_{10311}-A^{0,n}_{01311}-A^{0,n}_{11202}\Big)\Bigg] \end{aligned}$$

Similarly,

$$A^{0,n}_{11111} \quad = \quad \frac{1}{\epsilon}\left(A^{0,n}_{21011}+A^{0,n}_{11120}+A^{0,n}_{10121}-A^{0,n}_{10121}+A^{0,n}_{11012}-A^{0,n}_{10112}\right)$$

$$A^{n,0}_{21111} \quad = \quad -\frac{1}{\epsilon}\left(A^{n,0}_{22101}+A^{n,0}_{21102}\right)+\frac{1}{(1-\epsilon)}\left(2A^{n,0}_{03111}-2A^{n,0}_{13101}-A^{n,0}_{12102}\right)$$

The 4-propagator integrals can be represented in terms of up to three Feynman parameter integrals over the unit cube. In some cases, a direct calculation will give integrals with the following structure

$$I(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz\; x^n f(x, y, z; \varepsilon),$$

while in other cases they will be of the form

$$I(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz\; x^n y^n f(x, y, z; \varepsilon).$$

In the first case, the integrals turn out to be expressed directly as Mellin transforms after integrating in y and z. In the second case, the Feynman parameters can be rearranged into the first form using the following change of variables

$$x' = xy, \quad y' = \frac{x(1-y)}{1-xy},$$

This change of variables leaves the integration volume (unit square) untouched.

$$
\begin{aligned}
A_{12111}^{0,n} \quad = \quad \int_0^1 dx\, x^n \Bigg\{ & \frac{2}{3}\mathrm{Li}_2(1-x) + \frac{1}{3}\ln^2(x) + \frac{2}{3(1-x)}\ln(x) + \frac{2}{3}\zeta_2 \\
& - \frac{1}{3}(1-x)^{-3+2\epsilon}\ln^2(x) - \frac{2}{3}(1-x)^{-2+2\epsilon}\ln(x) \\
& + \frac{2}{3}(1-x)^{-1+2\epsilon} - \epsilon(1-x)^{-1+2\epsilon}(2\zeta_2 - 1) \\
& + (-1)^n \Bigg[\frac{4}{3}\mathrm{Li}_2(-x) - \frac{2}{3}\left(1 - \frac{2}{(1+x)^3}\right)\mathrm{Li}_2(1-x) \\
& \qquad - \left(1 - \frac{1}{(1+x)^3}\right)\ln^2(x) + \frac{4}{3}\ln(x)\ln(1+x) \\
& \qquad + \frac{2}{3(1+x)}\left(\frac{2}{(1+x)^2} - \frac{1}{(1+x)} - 1\right)\ln(x) \\
& \qquad - \frac{2}{3(1+x)} + \frac{4}{3(1+x)^2} + \frac{2}{3}\zeta_2 \Bigg] \Bigg\}
\end{aligned}
$$

Let's consider now the E-type integrals. Using

$$\sum_{j=0}^{n-1} (\Delta \cdot k_2)^j (\Delta \cdot k_1)^{n-1-j} = \frac{(\Delta \cdot k_2)^n - (\Delta \cdot k_1)^n}{\Delta \cdot k_2 - \Delta \cdot k_1} \ ,$$

we can rewrite the E-type integrals as

$$E^{a,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = J^{a,n+b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} - J^{n+a,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} \ ,$$

where

$$J^{a,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_\triangle D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \ ,$$

with $D_\triangle = \Delta \cdot k_2 - \Delta \cdot k_1$. We define also

$$K^{a,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_\triangle^2 D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \ .$$

Then, from

$$\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\partial}{\partial k_2^\mu} \left((k_2 - k_1)^\mu \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_\Delta D_1 D_2 D_3 D_4 D_5} \right) = 0 \ ,$$

and using

$$K^{a+1,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} - K^{a,b+1}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = J^{a,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} \ ,$$

and

$$J^{a+1,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} - J^{a,b+1}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = A^{a,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} \ ,$$

we obtain,

$$E^{a,b}_{11111} = \frac{1}{1-\varepsilon} \left(-bA^{n+a,b-1}_{11111} + (n+b) A^{a,n+b-1}_{11111} + E^{a,b}_{12101} - E^{a,b}_{02111} + E^{a,b}_{11102} \right) .$$

Similar manipulations lead to

$$F^{a,b}_{11111} = \frac{1}{b-\varepsilon} \left(bF^{a+1,b-1}_{11111} + F^{a,b}_{12101} - F^{a,b}_{02111} + F^{a,b}_{11102} \right) .$$

Additional IBP identities:

$$A^{n,0}_{31110} = \frac{1}{1-\epsilon}\left(A^{n,0}_{32100} + \frac{2}{\epsilon}\left(A^{n,0}_{23100} + \frac{3}{1+\epsilon}\left(A^{n,0}_{14100} - A^{n,0}_{04110}\right)\right)\right)$$

$$0 = n(\Delta\cdot p)A^{0,n-1}_{11011} - A^{0,n}_{11020} + A^{0,n}_{11002} - 2B^{0,n}_{12011} - 2m^2A^{0,n}_{11012}$$

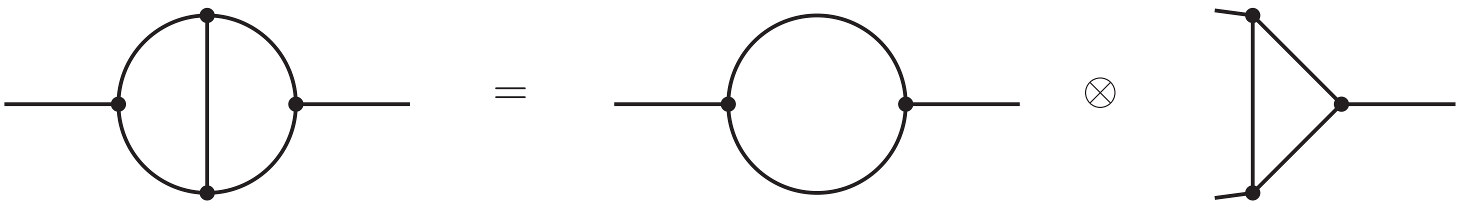
$$0 = -\epsilon A^{0,n}_{11101} + n(\Delta\cdot p)A^{0,n-1}_{11101} - 2B^{0,n}_{12101} - 2m^2A^{0,n}_{21101} - A^{0,n}_{01201} + A^{0,n}_{10102} - A^{0,n}_{11002}$$

$$\begin{aligned}0 = {} & nA^{n-1,1}_{11101} - A^{n,0}_{21001} + A^{n,0}_{21100} - A^{n,0}_{20101} - 2B^{n,0}_{21101} + A^{n,0}_{11200} - A^{n,0}_{10201} \\ & -A^{n,0}_{11002} + A^{n,0}_{10102} + 2m^2A^{n,0}_{11102}\end{aligned}$$

$$0 = n(\Delta\cdot p)A^{0,n-1}_{11110} - 2m^2A^{0,n}_{21110} - 2B^{0,n}_{12110} - A^{0,n}_{01210} + A^{0,n}_{21010}$$

4. Mellin-Barnes Representation

Using the decomposition of propagator-type integrals as a product (convolution) of two one-loop integrals (I. Bierenbaum and S. Weinzierl, hep-ph/0308311):



we can write our integrals as Mellin-Barnes ones in just two variables:

$$A^{a,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} \propto \int_{-i\infty}^{+i\infty} d\sigma \int_{-i\infty}^{+i\infty} d\tau \sum_{k=0}^{b} (-1)^k \frac{b!}{(b-k)!k!} \Gamma(-\sigma)\Gamma(-\tau)\Gamma(\sigma+\tau+\nu_{245}-2-\epsilon/2)$$
$$\times \frac{\Gamma(\sigma+k+\nu_5)\Gamma(\sigma+\tau+\nu_2)}{\Gamma(2\sigma+\tau+k+\nu_{25})} \frac{\Gamma(-\sigma-\tau+\nu_{13}-2-\epsilon/2)}{\Gamma(\nu_3-\sigma)\Gamma(\nu_1-\tau)}$$
$$\times \frac{\Gamma(2\sigma+\tau+k-\nu_1-2\nu_3+4+\epsilon)\Gamma(-\sigma+a+b-k+\nu_3)}{\Gamma(\sigma+\tau+a+b-\nu_{13}+4+\epsilon)}$$
$$\times \frac{\Gamma(\tau+b-k+\nu_4)\Gamma(-\tau-\nu_{25}-2\nu_4+k+4+\epsilon)}{\Gamma(b-\nu_{245}+4+\epsilon)}$$

Although IBPs allow to obtain 5-propagator integrals in terms of simpler 4-propagator integrals, this simplicity is somewhat spoiled by the high powers of the propagators appearing in the resulting expressions. For example,

$$A_{11310}^{0,n} \quad \propto \quad \int_0^1 dx \int_0^1 dy \int_0^1 dz \; x^{n+2}(1-x)^{-3-2\varepsilon}(x+y-xy)^{1+\varepsilon}$$

$$\times\, y^{-2-\varepsilon/2}(1-y)^{-\varepsilon/2} z^{-1+\varepsilon/2}(z+y-zy)^{-2-\varepsilon} \; .$$

It is not easy to disentangle the singularity structure in ϵ shared by the three Feynman parameters.

We decomposed the integral as

$$A_{11310}^{0,n} \propto I_1 + I_2 - I_3 \; ,$$

where

$$I_1 = \frac{2}{\varepsilon}\int_0^1 dx \int_0^1 dy\, x^{n+2}(1-x)^{-3-2\varepsilon}(x+y-xy)^{1+\varepsilon}y^{-4-\varepsilon/2}(1-y)^{-\varepsilon/2} \, ,$$

$$I_2 = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, x^{n+2}(1-x)^{-3-2\varepsilon}(x+y-xy)^{1+\varepsilon} \times y^{-4-\varepsilon/2}(1-y)^{1-\varepsilon/2}z^{\varepsilon/2}(z+y-zy)^{-1-\varepsilon} \, ,$$

$$I_3 = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, x^{n+2}(1-x)^{-3-2\varepsilon}(x+y-xy)^{1+\varepsilon} \times y^{-3-\varepsilon/2}(1-y)^{1-\varepsilon/2}z^{\varepsilon/2}(z+y-zy)^{-2-\varepsilon} \, .$$

I_1 was obtained doing integration by parts in z. The change of variables

$$y = y'z' \, , \quad z = \frac{y'(1-z')}{1-z'y'} \, ,$$

which was used before to write the integrals as Mellin transforms, can be used now to obtain expressions that can be cleanly expanded in ε.

For example, for integrals I_2 and I_3, we do this change which leads to

$$
\begin{aligned}
I_2 &= \int_0^1 dx \int_0^1 dy' \int_0^1 dz' \; x^{n+3+\varepsilon}(1-x)^{-3-2\varepsilon}\left(1+\frac{1-x}{x}y'z'\right)^{1+\varepsilon} \\
&\qquad \times y'^{-4-\varepsilon} z'^{-4-\varepsilon/2}(1-z')^{\varepsilon/2}(1-y'z')^{-\varepsilon} \; , \\
I_3 &= \int_0^1 dx \int_0^1 dy' \int_0^1 dz' \; x^{n+3+\varepsilon}(1-x)^{-3-2\varepsilon}\left(1+\frac{1-x}{x}y'z'\right)^{1+\varepsilon} \\
&\qquad \times y'^{-4-\varepsilon} z'^{-3-\varepsilon/2}(1-z')^{\varepsilon/2}(1-y'z')^{-\varepsilon} \; .
\end{aligned}
$$

Now, the expansion in ε will produce only logarithms that are regular at every limit of integration, and can be calculated using

$$
\begin{aligned}
\int_0^1 dy \; y^{-a+b\varepsilon}(1-y)^{c\varepsilon}\ln^k(1+\chi y) &= \frac{1}{1-a+b\varepsilon}\left[-k\chi\int_0^1 dy \frac{y^{-a+1+b\varepsilon}(1-y)^{c\varepsilon}}{1+\chi y}\ln^{k-1}(1+\chi y)\right. \\
&\qquad \left. + c\varepsilon\int_0^1 dy \; y^{-a+1+b\varepsilon}(1-y)^{-1+c\varepsilon}\ln^k(1+\chi y)\right] \; ,
\end{aligned}
$$

recursively to analytically continue the expressions by bringing down the highly singular powers in the integration variables. A few other integrals can be done the same way.

Another 4-propagator integral example:

$$E^{1,1}_{21110} \propto \int_0^1 dx \int_0^1 dy \int_0^1 dz \sum_{j=0}^{n-1} x^{n+1-j+\varepsilon/2}(1-x)^{-\varepsilon/2} y^{-\varepsilon/2}(1-y)^{n+1} \times z^{-1+\varepsilon/2}(1-z)[z(1-x)+xy]^{-1-\varepsilon} = I_a - I_b \ ,$$

where

$$I_a = \int_0^1 dx \int_0^1 dy \int_0^1 dz\ x^{n+1} \frac{y^{-1-\varepsilon/2}(1-y)^{2+\varepsilon/2} z^{-1+\varepsilon/2}(1-z)}{(1-x)^{\varepsilon/2}[zy+(1-y)(1-x)]^{1+\varepsilon}} \ ,$$

and

$$I_b = \int_0^1 dx \int_0^1 dy \int_0^1 dz\ x^{n+1} \frac{y^{-\varepsilon/2}(1-y)^{-1-\varepsilon/2} z^{-1+\varepsilon/2}(1-z)}{(1-x)^{1+2\varepsilon}(x+y-xy)^{-\varepsilon}(y+z-yz)^{1+\varepsilon}} \ ,$$

Integral I_a is particularly difficult. One way to do it is to use the integral representation of the hypergeometric function

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 dt\, t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha} \ ,$$

for $\mathrm{Re}\ \gamma > \mathrm{Re}\ \beta > 0$.

And then use the following analytic continuation

$$ {}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{-\beta}{}_2F_1\left(\beta,\gamma-\alpha;\gamma;\frac{z}{1-z}\right) \ , $$

to write

$$ I_a = I_{a1} + I_{a2} - I_{a3} \ , $$

with

$$ \begin{aligned} I_{a1} &= K\int_0^1 dx \int_0^1 dy \int_0^x dz \; x^{n+1}\frac{(1-x)^{-1-\varepsilon}y^{-1-\varepsilon/2}z^{-\varepsilon/2}(1-z)^{\varepsilon}}{[1-x+(x-z)y]^{\varepsilon/2}} \ , \\ I_{a2} &= K\int_0^1 dx \int_0^1 dy \int_x^1 dz \; x^{n+1}\frac{(1-x)^{-1-\varepsilon}y^{-1-\varepsilon/2}z^{-\varepsilon/2}(1-z)^{\varepsilon}}{[1-x+(x-z)y]^{\varepsilon/2}} \ , \\ I_{a3} &= K\int_0^1 dx \int_0^1 dy \int_0^1 dz \; x^{n+1}\frac{(1-x)^{-1-\varepsilon}y^{-\varepsilon/2}z^{-\varepsilon/2}(1-z)^{\varepsilon}}{[1-x(1-y)-yz]^{\varepsilon/2}} \ , \end{aligned} $$

where $K = \frac{2}{\varepsilon} - \varepsilon\zeta_2 + \frac{\varepsilon^2}{2}\zeta_3 + O(\varepsilon^3)$. The integral I_{a3} can be done without problems.

For integral I_{a1} we do the change of variables $z = xz'$, while for I_{a2} we do $z = (1-x)z' + x$, which leads to

$$I_{a1} = K \int_0^1 dx \int_0^1 dy \int_0^1 dz' \; x^{n+2-\varepsilon/2} \frac{(1-x)^{-1-\varepsilon} y^{-1-\varepsilon/2} z'^{-\varepsilon/2} (1-xz')^{\varepsilon}}{[1-x+(1-z')xy]^{\varepsilon/2}} \,,$$

$$I_{a2} = K \int_0^1 dx \int_0^1 dy \int_0^1 dz' \; x^{n+1} \frac{(1-x)^{-\varepsilon/2} y^{-1-\varepsilon/2} (1-z')^{\varepsilon}}{[(1-x)z'+x]^{\varepsilon/2}[1-z'y]^{\varepsilon/2}} \,.$$

Now integral I_{a2} is easy to obtain, and integral I_{a1} can be done using another analytic continuation of the hypergeometric function, namely

$$\begin{aligned} {}_2F_1(\alpha,\beta;\gamma;z) = {} & \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha}{}_2F_1\left(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{1}{z}\right) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}{}_2F_1\left(\beta,\beta+1-\gamma;\beta+1-\alpha;\frac{1}{z}\right) \end{aligned}$$

we get

$$
\begin{aligned}
I_{a1} \quad = \quad K \int_0^1 dx \int_0^1 dy \int_0^1 dz \Bigg[& - x^{n+2-\varepsilon}(1-x)^{-1-\varepsilon} y^{-1+\varepsilon} z^{-\varepsilon/2}(1-z)^{-\varepsilon/2} \\
& \times (1-xz)^{\varepsilon} \left(1 + \frac{1-x}{(1-z)x} y\right)^{-\varepsilon/2} \\
& + \left(-\frac{1}{\varepsilon} - \frac{\varepsilon}{2}\zeta_2 + \frac{\varepsilon^2}{4}\zeta_3\right) x^{n+2}(1-x)^{-1-2\varepsilon} \\
& \times z^{-\varepsilon/2}(1-z)^{\varepsilon/2}(1-xz)^{\varepsilon} \Bigg] \; ,
\end{aligned}
$$

which is now doable.

In the course of calculation, we ecountered several polylogarithmic integrals not found in popular integral tables, and which could not be done with `Maple` or `Mathematica` directly, e.g.

$$\int_0^1 dy\ \frac{x}{(1-(1-x)y)^2}\mathrm{Li}_2\left(-\frac{1-y}{x^2y}\right) = -\mathrm{Li}_2(1-x) - \frac{1}{2}\ln^2(x) - \zeta_2\ ,$$

$$\int_0^1 dy\ \frac{x}{1-xy}\mathrm{Li}_2\left(-\frac{1-x}{x^2y(1-y)}\right) = 2\mathrm{S}_{1,2}(x) + 4\mathrm{S}_{1,2}(1-x) - 4\zeta_3 - \ln(x)\ln^2(1-x)\ ,$$

$$\begin{aligned}\int_0^1 dy\ \mathrm{Li}_2\left(\frac{x}{(1-y+xy)(x+y-xy)}\right) &= \frac{1+x}{1-x}\left[\mathrm{Li}_2(x^2) + 2\ln(x)\ln(1+x)\right] \\ &\quad - 4\ln(1-x) - \frac{2x}{1-x}\left[\zeta_2 + 2\ln(x)\right]\ .\end{aligned}$$

$$\begin{aligned}\int_0^1 dy\ y(1-y)\mathrm{Li}_2\left(\frac{x}{(1-x)^2y(1-y)}\right) &= -\frac{4}{9}\frac{x}{(1-x)^2} + \frac{5}{9}\ln(1-x) - \frac{1}{3}\ln^2(1-x) \\ &\quad + \frac{1}{3}\ln(x)\ln(1-x) + \frac{x(5x^2-12x+3)}{9(1-x)^3}\ln(x)\ .\end{aligned}$$

We also needed to calculate many double integrals, like for example,

$$\int_0^1 dy \int_0^1 dz \; \frac{z\ln(1-zy)\ln(1+uyz)}{1-zy} \;=\; 2+\frac{1+u}{u}\left[\frac{1}{2}\ln^2(1+u)-\ln(1+u)+\mathrm{Li}_2(-u)\right]$$

$$\int_0^1 dy \int_0^1 dz \; \frac{z\ln(1-z)\ln(1+uyz)}{1-zy} \;=\; 3+\frac{1+u}{u}\left[\frac{1}{2}\ln^2(1+u)-2\ln(1+u)+\mathrm{Li}_2(-u)\right]$$

$$\int_0^1 dy \int_0^1 dz \; \frac{z\ln(z)\ln(1+uyz)}{1-zy} \;=\; 3-\ln(1+u)\mathrm{Li}_2(-u)+\ln(1+u)\zeta_2+\frac{1}{u}\mathrm{Li}_2(-u)$$
$$-2\left(1+\frac{1}{u}\right)\ln(1+u)+\mathrm{Li}_3(-u)-2S_{1,2}(-u)$$

$$\int_0^1 dy \int_0^1 dz \; \frac{z\ln(y)\ln(1+uyz)}{1-zy} \;=\; -1+\left(1+\frac{1}{u}-\zeta_2\right)\ln(1+u)+\ln(1+u)\mathrm{Li}_2(-u)$$
$$-\mathrm{Li}_3(-u)+2S_{1,2}(-u)$$

we calculated 44 integrals of this type.

Also, some analytic continuations in ϵ were required, for example:

$$\int_0^1 dy\ y^{-2-\varepsilon}\ln(1+uy) = -\frac{u}{\varepsilon} + u - (1+u)\ln(1+u) + \varepsilon\left[u\mathrm{Li}_2(-u) + (1+u)\ln(1+u) - u\right]$$

$$\int_0^1 dy\ y^{-3-\varepsilon}\ln^2(1+uy) = -\frac{u^2}{\varepsilon} + \frac{3}{2}u^2 - u(1+u)\ln(1+u) + u^2\mathrm{Li}_2(-u) - \frac{1}{2}(1-u^2)\ln^2(1+u)$$

$$\begin{aligned}\int_0^1 dy\ y^{-4-\varepsilon}\ln(1+u_1y)\ln(1+u_2y) \quad = \quad & \frac{1}{2\varepsilon}u_1^2u_2 - \frac{1}{6}u_2u_1^2 - \frac{1}{6}\ln(1+u_1)\ln(1+u_2) \\ & -\frac{u_1^2}{3}\left[u_2 - (1+u_2)\ln(1+u_2)\right] - \frac{u_1^3}{3}\mathrm{Li}_2(-u_2) \\ & -\frac{u_1}{6}\left[u_2 + (1-u_2^2)\ln(1+u_2) + \frac{u_2^2}{2}\right] \\ & +\frac{u_1^3}{3}\mathrm{Li}_2\left(\frac{u_1}{u_1-u_2}\right) - \frac{u_1^3}{3}\mathrm{Li}_2\left(\frac{u_1(1+u_2)}{u_1-u_2}\right) \\ & -\frac{u_1^3}{3}\ln(1+u_2)\ln\left(\frac{u_2(1+u_1)}{u_2-u_1}\right) + \{u_1 \leftrightarrow u_2\}\end{aligned}$$

The result for constant part of the operator matrix element is

$$
\begin{aligned}
&\frac{1+3x^2}{1-x}\left[6\zeta_2\ln(x)-8\ln(x)\mathrm{Li}_2(1-x)-4\ln^2(x)\ln(1-x)\right]+\left(\frac{122}{3}x+22+\frac{32}{1-x}\right)\zeta_2+(8-112\zeta_2)\,\mathcal{D}_1(x)\\
&+16\frac{1+x^2}{1-x}\left[2\mathrm{Li}_3(-x)-\ln(x)\mathrm{Li}_2(-x)\right]+\frac{80}{3(1-x)}+56(1+x)\zeta_2\ln(1-x)+(16-52\zeta_2+128\zeta_3)\,\mathcal{D}_0(x)\\
&+\left(\frac{22}{3}x+32+\frac{64}{3(1-x)^2}-\frac{51}{1-x}-\frac{16}{3(1-x)^3}\right)\ln^2(x)-(92+20x)\ln^2(1-x)+14(x-2)\ln(1-x)+120\mathcal{D}_2(x)\\
&+\left(\frac{178}{3}-36x+\frac{64}{3(1-x)^2}-\frac{140}{3(1-x)}-\frac{48}{1+x}\right)\ln(x)-\frac{1}{3}(1+x)\ln^3(x)+4\frac{x^2-8x-6}{1-x}\ln(x)\ln(1-x)\\
&-2\frac{1+17x^2}{1-x}\ln(x)\ln^2(1-x)-\frac{112}{3}(1+x)\ln^3(1-x)+32\frac{1+x}{1-x}\left[\ln(x)\ln(1+x)+\mathrm{Li}_2(-x)\right]-22x-\frac{62}{3}\\
&-4\frac{13x^2+9}{1-x}\mathrm{S}_{1,2}(1-x)+4\frac{5-11x^2}{1-x}\left[\ln(1-x)\mathrm{Li}_2(1-x)-\mathrm{Li}_3(1-x)-2\zeta_3\right]+\frac{4(16x^2-10x-27)}{3(1-x)}\mathrm{Li}_2(1-x)\\
&+\frac{224}{3}\mathcal{D}_3(x)+\left[\frac{433}{8}-\frac{67}{45}\pi^4+\left(\frac{37}{2}-48\ln(2)\right)\zeta_2+58\zeta_3\right]\delta(1-x)+(-1)^n\left\{\frac{2(1-x)(45x^2+74x+45)}{3(1+x)^2}\right.\\
&+\frac{2(9+12x+30x^2-20x^3-15x^4)}{3(1+x)^3}\ln(x)+\frac{4(x^2+10x-3)}{3(1+x)}\left(\zeta_2+2\mathrm{Li}_2(-x)+2\ln(x)\ln(1+x)\right)\\
&+\frac{1+x^2}{1+x}\Big[36\zeta_3-24\zeta_2\ln(1+x)+8\zeta_2\ln(x)-\frac{2}{3}\ln^3(x)+40\mathrm{Li}_3(-x)-4\ln^2(x)\ln(1+x)-24\ln(x)\ln^2(1+x)\\
&-24\ln(x)\mathrm{Li}_2(-x)-48\ln(1+x)\mathrm{Li}_2(-x)-8\ln(x)\mathrm{Li}_2(1-x)-16\mathrm{S}_{1,2}(1-x)-48\mathrm{S}_{1,2}(-x)\Big]\\
&\left.-\frac{16(x^4+12x^3+12x^2+8x+3)}{3(1+x)^3}\mathrm{Li}_2(1-x)+4x\frac{1-x-5x^2+x^3}{(1+x)^3}\ln^2(x)\right\}
\end{aligned}
$$

8. Conclusions

- We were able to calculate the two-loop operator matrix elements required to obtain the $O(\alpha^2)$ initial state radiative corrections to e^+e^- annihilations into a virtual boson, using renormalization group methods.
- All the required integrals were calculated using a variety of techniques.
- Integration by parts identities proved to be useful in order to reduce the complexity of the calculation of the 5-propagator integrals by allowing to express them in terms of 4-propagator ones.
- Mellin-Barnes integral representations were used to check our results (also other checks were performed, like the use of additional IBP identities of the `Mathematica` program `TARCER`).
- At various stages of the calculation, polylogarithmic integrals were encountered, which needed to be calculated, and analytic continuations in ϵ needed to be performed.