Quantitative Equational Rewriting

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Abstract. Rewriting logic is a logical framework for expressing both concurrent computation and logical deduction using equations and rewrite rules. Quantitative equational reasoning, in contrast, enriches equations with quantitative measures, expressing concepts such as similarity or proximity rather than mere equality of terms. In this article, we bring these two approaches together and propose a quantitative extension of rewriting logic. The resulting formalism is flexible and well-suited to quantitative deduction and computation.

Keywords: Quantitative rewriting · Quantitative equational reasoning · Quantitative matching

1 Introduction

In many modern applications, approximate and quantitative techniques play an increasingly important role. A quantitative notion of correctness is essential in settings where exact equality is either unattainable or too restrictive to be practically meaningful. In such contexts, approximate forms of equivalence have proved to be more adequate and robust.

Prominent examples arise in the analysis of probabilistic and nondeterministic systems, where classical notions of equivalence have been replaced by metric-based approximations that capture behavioural distance, see, e.g., [14, 15, 21, 23, 29, 45]. These kinds of quantitative notions enable a refined understanding of system behaviour, supporting reasoning about correctness in terms of degrees rather than exact notions as, e.g., in [2,22,26]. They also facilitate the integration of quantitative and symbolic techniques, an approach increasingly motivated by developments in neuro-symbolic methods [47, 48].

Recent theoretical advances have led to significant progress in this direction. Methods for quantitative synthesis, modelling, and verification have been developed, and foundational frameworks such as quantitative automata, quantitative algebras, quantitative logics, and quantitative rewriting systems have been systematically investigated, see, e.g., [1, 3, 5–7, 11, 12, 16, 17, 24, 25, 34, 35, 40].

Modelling and reasoning about dynamic systems requires a framework in which both computation and deduction can be treated in a uniform and mathematically precise way. In the classical setting, rewriting logic [9, 36, 37] is one of such prominent frameworks, where system states are represented as algebraic terms, structural properties of states are described by equations, and state transitions are specified by rewrite rules. A rewriting step could be understood as a concurrent computational transformation or as an inference step in a logical system. Due to its expressive power and logical foundation, rewriting logic (and its executable realisation, the Maude system [13, 19]) has been successfully applied in a wide range of areas, including the formal semantics of programming languages, the specification and verification of concurrent and distributed systems, security protocol analysis and model checking [13, 37, 41].

In this paper, we propose a quantitative extension of rewriting logic aimed at making its expressive and deductive power available for modelling systems that exhibit probabilistic, resource-sensitive, approximate, or otherwise graded behaviour. The resulting framework offers a unified perspective on quantitative deduction and computation. Quantitative information is represented at an abstract level by means of quantales, providing a uniform algebraic structure for combining and comparing degrees. Both equations and rewrite rules are equipped with quantitative information, and deduction as well as computation are carried out relative to this quantitative structure. The main features of the obtained quantitative rewriting logic framework can be summarised as follows:

- *Abstract quantitative foundation.* Following [25], quantitative information is expressed at a high level of abstraction using Lawverean quantales. This accommodates a wide range of special cases, including the standard exact (or *crisp*) setting (the Boolean quantale), fuzzy similarity frameworks (e.g., fuzzy quantales with minimum or product t-norms), structures equipped with abstract notions of distance such as Euclidean, Hamming, or Levenshtein distance (the Lawvere quantale), probabilistic metric spaces, etc.
- *Quantitative rewriting modulo.* We introduce a notion of rewriting modulo quantitative relations, denoted by $\rightarrow_{R^\epsilon/E^\epsilon}$, allowing both the rewrite relation R^ϵ and the background equational theory E^ϵ to be quantitative (with exact relations as a special case). Depending on whether R^ϵ and/or E^ϵ is empty, exact, or quantitative, one obtains nine distinct relations as special instances of $\rightarrow_{R^\epsilon/E^\epsilon}$. These include some known ones such as syntactic equality, standard equational reasoning, ordinary rewriting, rewriting modulo, quantitative rewriting, as well as new combinations that, to our knowledge, have not been studied previously.
- *Deduction and operational correspondence.* We investigate deduction in (unsorted, unconditional) quantitative rewriting logic, relating it on the one hand to (the reflexive-transitive closure of) $\rightarrow_{R^\epsilon/E^\epsilon}$, and on the other hand to its operational counterpart $\rightarrow_{R^\epsilon,E^\epsilon}$ (under certain conditions), which performs rewriting via matching modulo E^ϵ. This extends in a natural way the well-known correspondence between proof-theoretic and operational semantics of rewriting logic from the classical setting to the quantitative one.

– *Matching modulo quantitative theories.* We develop a matching algorithm modulo E^ϵ as a generic combination of matching algorithms for exact and approximate components of E^ϵ. We identify conditions on E^ϵ that ensure termination, completeness, and the finitary property of matching, making the equational rewriting relation $\to_{R^\epsilon/E^\epsilon}$ executable via $\to_{R^\epsilon,E^\epsilon}$.

The paper is organised as follows: We introduce the main notions in Sect. 2 and their quantitative counterparts in Sect. 3. In Sect. 4, the quantitative equational rewriting relation $\to_{R^\epsilon/E^\epsilon}$ is defined. In Sect. 5, a quantitative variant of rewriting logic (QRL) is introduced and its provability relation $\longrightarrow_{R^\epsilon/E^\epsilon}$ is characterised in terms of $\to_{R^\epsilon/E^\epsilon}$. In Sect. 6, we consider computational aspects of QRL, introduce an operational counterpart $\to_{R^\epsilon,E^\epsilon}$ of $\to_{R^\epsilon/E^\epsilon}$, and use its strictly coherent version to characterise provability in QRL. The core computational mechanism of $\to_{R^\epsilon,E^\epsilon}$, quantitative equational matching, is discussed in Sect. 7. Related work and concluding remarks can be found in Sect. 8 and 9, respectively.

2 Preliminaries

We assume the reader is familiar with basic notions of term rewriting. For more details we refer to [4].

Terms and Substitutions. Given a signature $\mathcal{F}$, i.e., a set of function symbols with fixed arities, and a set $\mathcal{V}$ of variables, we denote by $\mathcal{T}(\mathcal{F},\mathcal{V})$ the set of terms over $\mathcal{F}$ and $\mathcal{V}$. The set of variables occurring in a term t is denoted by $\mathcal{V}(t)$.

A substitution σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ which maps all except finitely many elements to themselves. The application of the substitution σ to term t is written $t\sigma$.

The set of positions in a term t is denoted by $Pos(t)$. For a position $p \in Pos(t)$, we write $t|_p$ for the subterm of t at the position p, and $t[s]_p$ for the term obtained from t by replacing the subterm at the position p by the term s.

Rewriting Logic. An $\mathcal{F}$-equation (or just equation) is an unoriented pair $t = t'$ of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$. For a given set E of $\mathcal{F}$-equations, equational logic induces a congruence relation $=_E$ on terms [28]. The E-equivalence class of a term t is denoted by $[t]_E$. An equational theory $(\mathcal{F}, E)$ is a pair with $\mathcal{F}$ a signature and E a set of $\mathcal{F}$-equations.

A *rewrite rule* is an oriented pair $l \to r$ of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$, where l is not a variable and $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. The relation $\to_R$ generated by a set R of rewrite rules is defined as $t \to_R t'$ iff there exists a non-variable position p in t, a rule $l \to r$ in R, and a substitution σ such that $t|_p = l\sigma$ and $t' = t[r\sigma]_p$. The transitive (resp. transitive and reflexive) closure of $\to_R$ is denoted by $\to_R^+$ (resp. $\to_R^*$).

A rewrite theory $\mathcal{R} = (\mathcal{F}, E, R)$ in rewriting logic consists of an equational theory $(\mathcal{F}, E)$ and a set of (possibly conditional) rewrite rules R, which define transition rules over equivalences classes $[t]_E$. The rewriting relation generated

$$\text{(ReflRL)}\ \frac{}{t \longrightarrow_{R/E} t} \qquad \text{(EqRL)}\ \frac{t \longrightarrow_{R/E} s \quad t =_E t' \quad s =_E s'}{t' \longrightarrow_{R/E} s'}$$

$$\text{(CongRL)}\ \frac{t_1 \longrightarrow_{R/E} s_1 \quad \cdots \quad t_n \longrightarrow_{R/E} s_n}{f(t_1, \ldots, t_n) \longrightarrow_{R/E} f(s_1, \ldots, s_n)}$$

$$\text{(ReplRL)}\ \frac{\ell : t \to s \in R}{t\sigma \longrightarrow_{R/E} s\sigma} \qquad \text{(TransRL)}\ \frac{t \longrightarrow_{R/E} s \quad s \longrightarrow_{R/E} r}{t \longrightarrow_{R/E} r}$$

Fig. 1: Deduction rules for rewriting logic [37]

by the rewrite rules R *modulo* an equational theory E, called *equational rewriting* relation and written $\to_{R/E}$, is defined as the composition $(=_E; \to_R; =_E)$. Figure 1 provides an alternative definition[6] of the transitive and reflexive closure of $\to_{R/E}$, denoted $\longrightarrow_{R/E}$, which is useful to reason about the equational rewriting relation. Although $\to_{R/E}$ is in general undecidable, *equational rewriting* techniques going back to [43] and including, e.g., [8, 27, 30, 38, 39], provide conditions on R and E that make $\to_{R/E}$ decidable. A key requirement is to have an *E-matching algorithm*: For executability purposes, the relation $\to_{R,E}$ is defined as $t \to_{R,E} t'$ iff there exist a non-variable position p in t, a rule $l \to r$ in R, and a substitution σ such that $t|_p =_E l\sigma$ and $t' = t[r\sigma]_p$. It is a weaker relation, but under certain conditions it can be shown to be equivalent to $\to_{R/E}$. We say that the rewrite relation $\to_{R,E}$ is *strictly* E-coherent iff $\forall u, v, u' : u \to_{R,E} v \wedge u =_E u' \implies \exists v' : u' \to_{R,E} v' \wedge v' =_E v$. If $\to_{R,E}$ is E-coherent, then we have $u \to_{R/E} v \iff \exists w : u \to_{R,E} w \wedge w =_E v$ [38].

3 Quantitative Relations

Before providing our quantitative equational rewriting relation $\to_{R^\epsilon/E^\epsilon}$ in Definition 6 below, we first introduce basic notions for quantitative equational theories and quantitative rewriting, following [24, 25].

A (unital) *quantale* $\mho = (\Omega, \precsim, \otimes, \kappa)$ consists of a monoid $(\Omega, \kappa, \otimes)$ and a complete lattice $(\Omega, \precsim)$ (with join $\vee$ and meet $\wedge$) satisfying the following distributivity laws: $\delta \otimes \left(\bigvee_{i\in I} \varepsilon_i\right) = \bigvee_{i\in I}(\delta \otimes \varepsilon_i)$ and $\left(\bigvee_{i\in I} \varepsilon_i\right) \otimes \delta = \bigvee_{i\in I}(\varepsilon_i \otimes \delta)$. We use Greek letters to denote elements of Ω.

The element κ is called the *unit* of the quantale, and $\otimes$ is called its *tensor* (or multiplication). The *top* and *bottom* elements of a quantale are denoted by $\top$ and $\bot$, respectively. Table 1 recalls some well-known quantales.

Quantales in which the unit κ coincides with $\top$ are called *integral quantales*. A quantale is *commutative* if its underlying monoid is. It is *non-trivial* if $\kappa \neq \bot$. It is *cointegral* if $\varepsilon \otimes \delta = \bot$ implies either $\varepsilon = \bot$ or $\delta = \bot$.

[6] In [37], the replacement inference rule allows rewrites in the matching substitution, which is called "parallelism under one's feet", but it is not considered in this paper.

Table 1: Quantales

	Generic Ω	Boolean $\mathbb{2}$	Lawvere $\mathbb{L}$	Strong Lawvere $\mathbb{L}^{\max}$	Fuzzy $\mathbb{I}$
Carrier	Ω	$\{0,1\}$	$[0,\infty]$	$[0,\infty]$	$[0,1]$
Order	$\precsim$	$\leqslant$	$\geqslant$	$\geqslant$	$\leqslant$
Join	$\vee$	$\exists$	inf	inf	sup
Meet	$\wedge$	$\forall$	sup	sup	inf
Tensor	$\otimes$	$\wedge$	$+$	max	left-continuous T-norm
Unit	κ	1	0	0	1

For $n \in \mathbb{N}$ and $\varepsilon \in \Omega$, we denote $\bigotimes_{i=0}^{n} \varepsilon$ by ε^n. When $n = 0$, we have $\varepsilon^n = \kappa$.

We assume our quantales are commutative, integral, cointegral, and nontrivial. Such quantales are called *Lawverean*. Note that the fuzzy quantale $\mathbb{I}$ is Lawverean for the Gödel and product T-norms, but not for the Łukasiewicz T-norm.

Definition 1 (Quantitative relation[7] [24,25]). *Let Ω be a Lawverean quantale, and let A be a set. A* quantitative relation *on A over Ω is a ternary relation $R \subseteq A \times \Omega \times A$, which is closed under quantitative weakening, i.e., $(a, \varepsilon, c) \in R$ implies $(a, \zeta, c) \in R$ for every $\zeta \precsim \varepsilon$. The composition $R; S$ of two such quantitative relations R and S is the quantitative relation defined by $(R; S)(a, \varepsilon, b)$ iff there exist $\zeta, \eta \in \Omega$ and $c \in A$ such that $(a, \zeta, c) \in R$, $(c, \eta, b) \in S$, and $\varepsilon \precsim \zeta \otimes \eta$. Powers of R are defined inductively by $R^0 := \{(a, \kappa, a) \mid a \in A\}$ and $R^{n+1} := R^n; R$. The* transitive closure *of R is defined by $R^+ := \bigcup_{n>0} R^n$, and its* reflexive transitive closure *by $R^* := \bigcup_{n\geqslant 0} R^n$. The* transpose *of R is defined by $R^- := \{(t, \varepsilon, s) \mid (s, \varepsilon, t) \in R\}$, and the* symmetric closure *of R by $R \cup R^-$.*

We say that $a \in A$ is in normal form *w.r.t. a quantitative relation R, iff $\bigvee\{\varepsilon \mid (a, \varepsilon, b) \in R \text{ for some } b \in A\} = \bot$.*

Note that the composition of quantitative relations is associative and monotonous with respect to $\subseteq$. In the rest of the paper, a signature $\mathcal{F}$, a set of variables $\mathcal{V}$, and a Lawverean quantale Ω are assumed to be fixed. For brevity, below Ω is omitted when it is clear from the context.

Let E^ϵ be a set of triples (t, ε, s), where $\varepsilon \in \Omega$ and t and s are terms. We write them as $\varepsilon \Vdash t \approx_{E^\epsilon} s$ and call them Ω-*quantitative equations*, or just quantitative equations for short. Informally, we read $\varepsilon \Vdash t =_{E^\epsilon} s$ as "t and s are at most ε-apart modulo E^ϵ" or "t and s are equal modulo E^ϵ with degree ε". The Ω-*quantitative equational theory* (abbreviated as QET) generated by E^ϵ, denoted by $=_{E^\epsilon}$, is the relation generated from E^ϵ by the rules in Fig. 2, introduced in [24].[8] We call E^ϵ a *presentation* of $=_{E^\epsilon}$. When there is no confusion, we use

[7] In [25], the term "Ω-ternary relation" is used.

[8] Note that $=_{E^\epsilon}$ is indeed a quantitative relation: rule $(\mathsf{Ord}^\epsilon_=)$ ensures closure under quantitative weakening.

E^ϵ to refer to $=_{E^\epsilon}$. The *signature* of $=_{E^\epsilon}$ is the set of function symbols that appear in E^ϵ. It is denoted by $Sig(E^\epsilon)$.

$$(\mathsf{Ax}^\epsilon_=)\ \frac{\varepsilon \Vdash t \approx s \in E^\epsilon}{\varepsilon \Vdash t =_{E^\epsilon} s} \qquad (\mathsf{Refl}^\epsilon_=)\ \frac{}{\kappa \Vdash t =_{E^\epsilon} t} \qquad (\mathsf{Symm}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s}{\varepsilon \Vdash s =_{E^\epsilon} t}$$

$$(\mathsf{Trans}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s \quad \delta \Vdash s =_{E^\epsilon} r}{\varepsilon \otimes \delta \Vdash t =_{E^\epsilon} r} \qquad (\mathsf{Subst}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s}{\varepsilon \Vdash t\sigma =_{E^\epsilon} s\sigma}$$

$$(\mathsf{NExp}^\epsilon_=)\ \frac{\varepsilon_1 \Vdash t_1 =_{E^\epsilon} s_1 \quad \cdots \quad \varepsilon_n \Vdash t_n =_{E^\epsilon} s_n}{\varepsilon_1 \otimes \cdots \otimes \varepsilon_n \Vdash f(t_1, \ldots, t_n) =_{E^\epsilon} f(s_1, \ldots, s_n)}$$

$$(\mathsf{Ord}^\epsilon_=)\ \frac{\varepsilon \Vdash t =_{E^\epsilon} s \quad \varepsilon \succsim \delta}{\delta \Vdash t =_{E^\epsilon} s} \qquad (\mathsf{Join}^\epsilon_=)\ \frac{\varepsilon_1 \Vdash t =_{E^\epsilon} s \quad \cdots \quad \varepsilon_n \Vdash t =_{E^\epsilon} s}{\varepsilon_1 \vee \cdots \vee \varepsilon_n \Vdash t =_{E^\epsilon} s}$$

Fig. 2: Quantitative equational theory [24]

The $(\mathsf{Join}^\epsilon_=)$ rule also applies to an empty hypothesis, hence $\bot \Vdash t =_{E^\epsilon} s$ holds for any t and s.[9]

A *quantitative rewrite rule* is an oriented quantitative equation, written $\varepsilon \Vdash t \mapsto s$. The *quantitative rewrite relation* $\to_{R^\epsilon}$ generated by a set R^ϵ of quantitative rewrite rules is defined in Fig. 3, following [25].

We denote the transpose of $\to_{R^\epsilon}$ by $\leftarrow_{R^\epsilon}$, and its symmetric closure by $\leftrightarrow_{R^\epsilon}$.

$$\frac{\varepsilon \Vdash t \mapsto_{R^\epsilon} s}{\varepsilon \Vdash C[t\sigma] \to_{R^\epsilon} C[s\sigma]} \qquad \frac{\varepsilon \Vdash t \to_{R^\epsilon} s \quad \varepsilon \succsim \delta}{\delta \Vdash t \to_{R^\epsilon} s}$$

Fig. 3: Quantitative rewrite relation [25]

Note that the context and substitution used to generate a rewrite step preserve the degree ε associated with the rewrite rule (see Fig. 3). Such relation is called *non-expansive* [25]. A more general relation, called graded rewriting in [25], allows the context and substitution to modify the degree.

Lemma 2. *Let R^ϵ be a finite set of triples (l, δ, r), where l and r are terms and δ is an element of the Lawverean quantale Ω, and let $\varepsilon \succ \bot$ be some element of Ω. Then $\varepsilon \Vdash t =_{R^\epsilon} s$ iff there exist $\varepsilon_1, \ldots, \varepsilon_n \in \Omega$ such that $\bigvee_{i=1}^n \varepsilon_i = \varepsilon$ and $\varepsilon_i \Vdash t \leftrightarrow^*_{R^\epsilon} s$ for each i.*

Proof. For the "if"-direction, recall that $=_{R^\epsilon}$ is closed under symmetry, transitivity, join and quantitative weakening; thus, it suffices to show that $\varepsilon \Vdash t \to_{R^\epsilon} s$

[9] In [24] an infinitary Archimedean rule is included, which has no effect on $=_{E^\epsilon}$ whenever the presentation E^ϵ is finite. Therefore, we do not consider it here.

implies $\varepsilon \Vdash t =_{R^\epsilon} s$. If $\varepsilon \Vdash t \to_{R^\epsilon} s$, then there exists a triple $(l, \varepsilon', r) \in R$ with $\varepsilon' \gtrsim \varepsilon$ as well as a position $p \in Pos(t)$ and a substitution σ such that $t|_p = l\sigma$ and $s = t[r\sigma]_p$. By $(\mathsf{Ax}^\epsilon_=)$ and $(\mathsf{Subst}^\epsilon_=)$, we have $\varepsilon' \Vdash l\sigma =_{R^\epsilon} r\sigma$, and by closure under contexts (which is a consequence of $(\mathsf{NExp}^\epsilon_=)$) and quantitative weakening, we obtain $\varepsilon \Vdash t[l\sigma]_p =_{R^\epsilon} t[r\sigma]_p$.

For the converse, we proceed by induction on the length of the proof of $\varepsilon \Vdash t =_{R^\epsilon} s$ and distinguish cases based on the last rule applied.

- If the last rule applied was $(\mathsf{Ax}^\epsilon_=)$, then $(t, \varepsilon, s) \in R$, whence $\varepsilon \Vdash t \to_{R^\epsilon} s$.
- If $\varepsilon \Vdash t =_{R^\epsilon} s$ was obtained via $(\mathsf{Refl}^\epsilon_=)$, then $\varepsilon = \kappa$ and $t = s$, so we have $\varepsilon \Vdash t \to^0_{R^\epsilon} s$.
- In case of a $(\mathsf{Symm}^\epsilon_=)$ step, we apply the induction hypothesis to $\varepsilon \Vdash s =_{R^\epsilon} t$ to obtain $\varepsilon_1, \ldots, \varepsilon_n$ with $\bigvee_{i=1}^n \varepsilon_i = \varepsilon$ and $\varepsilon_i \Vdash s \leftrightarrow^*_{R^\epsilon} t$ for each i; then also $\varepsilon_i \Vdash t \leftrightarrow^*_{R^\epsilon} s$ holds for each i.
- In case of a $(\mathsf{Trans}^\epsilon_=)$ step, there are degrees ζ, η with $\zeta \otimes \eta = \varepsilon$ and a term u such that $\zeta \Vdash t =_{R^\epsilon} u$ and $\eta \Vdash u =_{R^\epsilon} s$; thus, we obtain $\zeta_1, \ldots, \zeta_n$ and $\eta_1, \ldots, \eta_m$ such that $\bigvee_{i=1}^n \zeta_i = \zeta$ and $\bigvee_{i=1}^m \eta_i = \eta$ such that $\zeta_i \Vdash t \leftrightarrow^*_R u$ and $\eta_j \Vdash u \leftrightarrow^*_R s$ for all $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$. Consequently, we have $\zeta_i \otimes \eta_j \Vdash t \leftrightarrow^*_R s$ for all such (i, j). We conclude as

$$\bigvee_{\substack{1 \leqslant i \leqslant n,\\ 1 \leqslant j \leqslant m}} \zeta_i \otimes \eta_j = \bigvee_{i=1}^n \left(\zeta_i \otimes \left(\bigvee_{i=1}^m \eta_j \right) \right) = \left(\bigvee_{i=1}^n \zeta_i \right) \otimes \left(\bigvee_{j=1}^m \eta_j \right) = \zeta \otimes \eta = \varepsilon$$

  using distributivity.
- $(\mathsf{NExp}^\epsilon_=)$ and $(\mathsf{Subst}^\epsilon_=)$ steps are covered as $\leftrightarrow^*_{R^\epsilon}$ is closed under contexts and substitutions.
- In case of a $(\mathsf{Join}^\epsilon_=)$ step, there exist $\varepsilon_1, \ldots, \varepsilon_n$ such that $\bigvee_{i=1}^n \varepsilon_i = \varepsilon$ and $\varepsilon_i \Vdash t =_{R^\epsilon} s$ for each i. By the induction hypothesis, for each i, we obtain $\varepsilon_{i,1}, \ldots, \varepsilon_{i,n_i}$ such that $\bigvee_{j=1}^{n_i} \varepsilon_{i,j} = \varepsilon_i$ and $\varepsilon_{i,j} \Vdash t \leftrightarrow^*_{R^\epsilon} s$ holds for all j. Then also $\bigvee_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n_i} \varepsilon_{i,j} = \bigvee_{i=1}^n \varepsilon_i = \varepsilon$, as desired.
- $(\mathsf{Ord}^\epsilon_=)$ steps are covered as $\leftrightarrow^*_{R^\epsilon}$ is closed under quantitative weakening. □

In particular, over a totally ordered quantale, there is a direct correspondence between $=_{R^\epsilon}$ and $\leftrightarrow^*_{R^\epsilon}$.

Corollary 3. *Let R^ϵ be a finite set of triples (l, δ, r), where l and r are terms and δ is an element of the Lawverean quantale Ω whose order $\precsim$ is total, and let $\varepsilon \succ \bot$ be some element of Ω. Then $\varepsilon \Vdash t =_{R^\epsilon} s$ if and only if $\varepsilon \Vdash t \leftrightarrow^*_{R^\epsilon} s$.*

Proof. By Lemma 2, $\varepsilon \Vdash t =_{R^\epsilon} s$ is equivalent to the existence of $\varepsilon_1, \ldots, \varepsilon_n \in \Omega$ such that $\bigvee_{i=1}^n \varepsilon_i = \varepsilon$ and $\varepsilon_i \Vdash t \leftrightarrow^*_R s$ holds for each i. Since $\precsim$ is total, we have $\bigvee_{i=1}^n \varepsilon_i \in \{\varepsilon_1, \ldots, \varepsilon_n\}$, so the previous statement is equivalent to $\bigvee_{i=1}^n \varepsilon_i \Vdash t \leftrightarrow^*_{R^\epsilon} s$. □

The notion of distance between terms can be seen as a particular case of quantitative equational theories.

Definition 4 (Ω-metric space [31]). *Let Ω be a quantale, X a set. An Ω-*metric *over X is a map $d\colon X \times X \to \Omega$ satisfying (i) $d(x,x) = \kappa$ (reflexivity) and (ii) $d(x,y) \succsim d(x,z) \otimes d(z,y)$ (triangle inequality) for all elements $x, y, z \in X$.*

The pair (X,d) is called an Ω-metric space. *An Ω-metric d is* symmetric *if it also satisfies $d(x,y) = d(y,x)$ for all $x, y \in X$.*

In particular, a symmetric $\mathbb{L}$-metric d is just an extended pseudo-metric – that is, d is a metric except that it can take the value ∞ and $d(x,y) = 0$ does not necessarily imply $x = y$. Thus, Ω-metric spaces can be viewed as generalised metric spaces. Lemma 5 shows that a distance can be defined as a QET.

Lemma 5. *Let Ω be a Lawverean quantale, and let d be a symmetric Ω-metric over $T(\mathcal{F},\mathcal{V})$. Assume that d is closed under contexts and substitutions, i.e. $d(C[t\sigma], C[s\sigma]) \succsim d(t,s)$ holds for all terms t, s, contexts C and substitutions σ. Then d can be viewed as a QET, i.e., there exists some QET E_d such that $d(t,s) = \bigvee\{\varepsilon \mid \varepsilon \Vdash t =_{E_d} s\}$ for all terms t and s.*

Proof. We define $E_d \coloneqq \{d(t,s) \Vdash t \approx s \mid t, s \in T(\mathcal{F},\mathcal{V})\}$. It suffices to show that $\varepsilon \Vdash t =_{E_d} s$ iff $\varepsilon \precsim d(t,s)$. The "if"-direction is granted by the $(\mathsf{Ord}^{\epsilon}_{=})$ rule. For the other direction, we proceed by induction on the length of the proof of $\varepsilon \Vdash t =_{E_d} s$. We distinguish cases based on the last rule applied.

– The case of an $(\mathsf{Ax}^{\epsilon}_{=})$ application holds by definition.
– The cases of $(\mathsf{Refl}^{\epsilon}_{=})$, $(\mathsf{Symm}^{\epsilon}_{=})$, and $(\mathsf{Trans}^{\epsilon}_{=})$ steps are covered as d is symmetric, reflexive, and satisfies the triangle inequality.
– The cases of $(\mathsf{NExp}^{\epsilon}_{=})$ and $(\mathsf{Subst}^{\epsilon}_{=})$ steps are covered by closure under contexts.
– In case of an $(\mathsf{Ord}^{\epsilon}_{=})$ step, we have $\delta \precsim d(t,s)$ for some $\delta \succsim \varepsilon$, so $\varepsilon \precsim d(t,s)$ follows by transitivity of $\precsim$.
– In case of a $(\mathsf{Join}^{\epsilon}_{=})$ step, we have $\varepsilon_i \precsim d(t,s)$ for $i = 1, \ldots, n$, implying $\bigvee_i \varepsilon \precsim d(t,s)$ by definition of the join. □

Similarly, any quantitative equational theory E induces an Ω-metric over the term algebra: $d_E(t,s) = \bigvee\{\varepsilon \mid \varepsilon \Vdash t =_E s\}$ for all terms t and s.

4 Quantitative Rewriting Modulo Quantitative Equations

In this section we define a notion of quantitative equational rewriting that generalises the relation $\to_{R/E}$ to take into account that both R and E could be quantitative. This relation is denoted by $\to_{R^\epsilon/E^\epsilon}$.

Definition 6 (Quantitative equational rewriting $\to_{R^\epsilon/E^\epsilon}$). *Given a set R^ϵ of quantitative rewrite rules and a (possibly empty) set E^ϵ of quantitative equations, the relation $\to_{R^\epsilon/E^\epsilon}$ consists of triples (t, δ, s) defined as follows:*

$\delta \Vdash t \to_{R^\epsilon/E^\epsilon} s$ *if*

$$\delta \precsim \bigvee \left\{ \varepsilon_1 \otimes \rho \otimes \varepsilon_2 \;\middle|\; \begin{array}{l} \textit{there exist terms } t' \textit{ and } s' \textit{ such that} \\ \varepsilon_1 \Vdash t =_{E^\epsilon} t',\ \rho \Vdash t' \to_{R^\epsilon} s',\ \ \varepsilon_2 \Vdash s' =_{E^\epsilon} s \end{array} \right\}.$$

Note that this is a quantitative relation: the use of $\precsim$ in the definition ensures closure by quantitative weakening.

The general definition above can be restricted along different dimensions to obtain some existing and some new notions of quantitative rewriting and quantitative equational reasoning. Below we provide precise definitions of the notions of trivial/crisp R^ϵ and E^ϵ using quantitative ternary relations, which will be needed to formalise the correspondence between $\to_{R^\epsilon/E^\epsilon}$ and previous notions of rewriting and equational reasoning. Crisp relations are sometimes called *exact*.

Definition 7 (Trivial/crisp relation). *A quantitative equational theory E^ϵ is called* trivial *if $=_{E^\epsilon}$ coincides with $=_\emptyset$ (the theory induced by an empty set of quantitative equations), and* crisp *if $=_{E^\epsilon}$ coincides with a theory induced by a set of quantitative equations with degree κ. Similarly, a quantitative term rewrite relation R^ϵ is trivial if $\to_{R^\epsilon} = \to_\emptyset$, and crisp if it coincides with a rewrite relation induced by a set of quantitative rewrite rules with degree κ.*

Remark 8. Let E^ϵ be a crisp equational theory. If $\varepsilon \Vdash t =_{E^\epsilon} s$ for some $\varepsilon \succ \bot$ then $\kappa \Vdash t =_{E^\epsilon} s$. The other direction holds trivially for all ε since quantitative relations are closed under weakening. Let E be the non-quantitative theory obtained as the image of E^ϵ under the projection $\pi\colon (t, \kappa, s) \mapsto (t, s)$. Then $\varepsilon \Vdash t =_{E^\epsilon} s$ for $\varepsilon \succ \bot$ iff $t =_E s$.

Similarly, a crisp rewrite relation R^ϵ satisfies $\varepsilon \Vdash t \to_{R^\epsilon}$ for $\varepsilon \succ \bot$ iff $\kappa \Vdash t \to_{R^\epsilon} s$ iff $t \to_{\pi(R^\epsilon)} s$.

Thus, crisp quantitative rewrite relations (resp. equational theories) and non-quantitative (i.e., standard) rewrite relations (resp. equational theories) can be considered the same. When we want to emphasise that we are talking about crisp R^ϵ and E^ϵ, we will use the notation R and E, respectively.

Table 2 describes the various relations obtained as particular cases of $\to_{R^\epsilon/E^\epsilon}$. References to well-known standard notions are omitted.

Table 2: Quantitative Equational Rewrite Relations

Relation	**Trivial** R	**Crisp** R	**Arbitrary** R^ϵ
Trivial E	Syntactic equality $=$	Standard rewriting $\to_R$	Quantitative rewriting $\to_{R^\epsilon}$ [25, 33]
Crisp E	Standard equational reasoning $=_E$	Standard rewriting modulo $\to_{R/E}$	Quantitative rewriting modulo $\to_{R^\epsilon/E}$
Arbitrary E^ϵ	Quantitative equational reasoning $=_{E^\epsilon}$ [6]	Rewriting modulo quantitative theory $\to_{R/E^\epsilon}$ [18]	Quantitative rewriting modulo quantitative theory $\to_{R^\epsilon/E^\epsilon}$

5 Quantitative Rewriting Logic (QRL)

In this section, we introduce a quantitative counterpart of the version of rewriting logic from Section 2.

Definition 9 (Quantitative rewrite theory). *A* quantitative rewrite theory *$\mathcal{R}$ over $\mathcal{F}$, $\mathcal{V}$ and Ω, QRT for short, is a pair (R^ϵ, E^ϵ), where*

- *R^ϵ is a finite set of labelled quantitative rewrite rules $\ell : t \to_\rho s$, where l is a label, $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $\mathcal{V}(s) \subseteq \mathcal{V}(t)$, and $\rho \in \Omega$ (called the* weight *of the rule),*
- *E^ϵ is a finite set of Ω-quantitative equations over terms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$.*

Given $\mathcal{R} = (R^\epsilon, E^\epsilon)$, the sentences that $\mathcal{R}$ proves are triples of the form $(t, \varepsilon, s) \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \Omega \times \mathcal{T}(\mathcal{F}, \mathcal{V})$, written as $\varepsilon \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$, which are obtained by finite application of the QRL deduction rules in Fig. 4.

$$(\mathsf{Refl}^\epsilon_{\mathsf{RL}})\ \frac{}{\kappa \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} t}$$

$$(\mathsf{Eq}^\epsilon_{\mathsf{RL}})\ \frac{\delta \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s \quad \varepsilon_1 \Vdash t =_{E^\epsilon} t' \quad \varepsilon_2 \Vdash s =_{E^\epsilon} s'}{\delta \otimes \varepsilon_1 \otimes \varepsilon_2 \Vdash t' \longrightarrow_{R^\epsilon/E^\epsilon} s'}$$

$$(\mathsf{NExp}^\epsilon_{\mathsf{RL}})\ \frac{\delta_1 \Vdash t_1 \longrightarrow_{R^\epsilon/E^\epsilon} s_1 \quad \cdots \quad \delta_n \Vdash t_n \longrightarrow_{R^\epsilon/E^\epsilon} s_n}{\delta_1 \otimes \cdots \otimes \delta_n \Vdash f(t_1, \ldots, t_n) \longrightarrow_{R^\epsilon/E^\epsilon} f(s_1, \ldots, s_n)}$$

$$(\mathsf{Repl}^\epsilon_{\mathsf{RL}})\ \frac{\ell : t \to_\rho s \in R}{\rho \Vdash t\sigma \longrightarrow_{R^\epsilon/E^\epsilon} s\sigma} \qquad (\mathsf{Trans}^\epsilon_{\mathsf{RL}})\ \frac{\delta_1 \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s \qquad \delta_2 \Vdash s \longrightarrow_{R^\epsilon/E^\epsilon} r}{\delta_1 \otimes \delta_2 \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} r}$$

$$(\mathsf{Ord}^\epsilon_{\mathsf{RL}})\ \frac{\delta \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s \quad \delta \succsim \delta'}{\delta' \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s} \qquad (\mathsf{Join}^\epsilon_{\mathsf{RL}})\ \frac{\delta_1 \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s \ \cdots \ \delta_n \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s}{\delta_1 \vee \cdots \vee \delta_n \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s}$$

Fig. 4: Deduction rules for Quantitative Rewriting Logic.

These rules extend those of rewriting logic in Fig. 1 with quantitative information and include new rules $(\mathsf{Ord}^\epsilon_{\mathsf{RL}})$ and $(\mathsf{Join}^\epsilon_{\mathsf{RL}})$, which are the counterpart of the rules $(\mathsf{Ord}^\epsilon_{=})$ and $(\mathsf{Join}^\epsilon_{=})$ in Figure 2. When $\delta \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$, we say that δ is the *cost* of reduction.

The resulting framework is general and flexible. The standard rewriting logic is recovered by taking crisp rules and equations, which would correspond to $\longrightarrow_{R/E}$ (see Remark 8). A generic weighted variant of rewriting logic is obtained by taking equations crisp but keeping nontrivial rule weights (i.e, $\longrightarrow_{R^\epsilon/E}$). Varying the underlying quantale structure introduces an additional dimension of generality, enabling the framework to capture a wide range of rewriting-based systems.

To avoid confusion, below we use $\equiv$ for the meta-equality between relations.

Theorem 10. $\to^*_{R^\epsilon/E^\epsilon} \equiv \longrightarrow_{R^\epsilon/E^\epsilon}$.

Proof. We prove equality by showing both inclusions.

($\subseteq$) We first show $\to^*_{R^\epsilon/E^\epsilon} \subseteq \longrightarrow_{R^\epsilon/E^\epsilon}$. For this, it suffices to show $\to_{R^\epsilon/E^\epsilon} \subseteq \longrightarrow_{R^\epsilon/E^\epsilon}$, because then we get $\to^*_{R^\epsilon/E^\epsilon} \subseteq \longrightarrow^*_{R^\epsilon/E^\epsilon}$, from which, by the fact that $\longrightarrow^*_{R^\epsilon/E^\epsilon} = \longrightarrow_{R^\epsilon/E^\epsilon}$, we can conclude that $\to^*_{R^\epsilon/E^\epsilon} \subseteq \longrightarrow_{R^\epsilon/E^\epsilon}$.
Assume $\delta \Vdash t \to_{R^\epsilon/E^\epsilon} s$. By Def. 6, there exist terms t', s' and degrees $\varepsilon_1, \rho, \varepsilon_2$ such that $\varepsilon_1 \Vdash t =_{E^\epsilon} t'$, $\rho \Vdash t' \to_{R^\epsilon} s'$, $\varepsilon_2 \Vdash s =_{E^\epsilon} s'$, and $\delta \precsim \varepsilon_1 \otimes \rho \otimes \varepsilon_2$. From $\rho \Vdash t' \to_{R^\epsilon} s'$, by Definition of $\to_{R^\epsilon}$ in Fig. 3, there should exist a rule $\ell : l \to_\beta r \in R$, a context C, and a substitution σ such that $t' = C[l\sigma]$, $s' = C[r\sigma]$, and $\rho \precsim \beta$. From $\ell : l \to_\beta r \in R$, by ($\mathsf{Repl}^\epsilon_{\mathsf{RL}}$) we get $\beta \Vdash l\sigma \longrightarrow_{R^\epsilon/E^\epsilon} r\sigma$, from which, by ($\mathsf{NExp}^\epsilon_{\mathsf{RL}}$) (and ($\mathsf{Refl}^\epsilon_{\mathsf{RL}}$) whenever needed), moving along the context C, we obtain $\beta \Vdash t' \longrightarrow_{R^\epsilon/E^\epsilon} s'$. Since $\beta \succsim \rho$, the rule ($\mathsf{Ord}^\epsilon_{\mathsf{RL}}$) gives $\rho \Vdash t' \longrightarrow_{R^\epsilon/E^\epsilon} s'$.
Hence, we got $\varepsilon_1 \Vdash t =_{E^\epsilon} t'$, $\rho \Vdash t' \to_{R^\epsilon/E^\epsilon} s'$, and $\varepsilon_2 \Vdash s =_{E^\epsilon} s'$. Rule ($\mathsf{Eq}^\epsilon_{\mathsf{RL}}$) yields $\varepsilon_1 \otimes \rho \otimes \varepsilon_2 \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$. Since $\delta \precsim \varepsilon_1 \otimes \rho \otimes \varepsilon_2$, rule ($\mathsf{Ord}^\epsilon_{\mathsf{RL}}$) gives $\delta \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$. Therefore $\to_{R^\epsilon/E^\epsilon} \subseteq \longrightarrow_{R^\epsilon/E^\epsilon}$, and hence $\to^*_{R^\epsilon/E^\epsilon} \subseteq \longrightarrow^*_{R^\epsilon/E^\epsilon} = \longrightarrow_{R^\epsilon/E^\epsilon}$ by closure of $\longrightarrow_{R^\epsilon/E^\epsilon}$ under ($\mathsf{Refl}^\epsilon_{\mathsf{RL}}$) and ($\mathsf{Trans}^\epsilon_{\mathsf{RL}}$).

($\supseteq$) We now show $\longrightarrow_{R^\epsilon/E^\epsilon} \subseteq \to^*_{R^\epsilon/E^\epsilon}$. Fix a derivation of $\delta \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$. We proceed by induction on the structure of that derivation (i.e., on the last inference rule used in Fig. 4).

- ($\mathsf{Refl}^\epsilon_{\mathsf{RL}}$) Then $s = t$, hence $t \to^*_{R^\epsilon/E^\epsilon} t$ by reflexivity.
- ($\mathsf{Trans}^\epsilon_{\mathsf{RL}}$) Then there exist u and degrees δ_1, δ_2 such that $\delta_1 \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} u$ and $\delta_2 \Vdash u \longrightarrow_{R^\epsilon/E^\epsilon} s$ and $\delta = \delta_1 \otimes \delta_2$. By the induction hypothesis, we get $t \to^*_{R^\epsilon/E^\epsilon} u$ and $u \to^*_{R^\epsilon/E^\epsilon} s$. Concatenating them yields $t \to^*_{R^\epsilon/E^\epsilon} s$.
- ($\mathsf{Ord}^\epsilon_{\mathsf{RL}}$) Then $\delta' \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$ and $\delta \precsim \delta'$. By the induction hypothesis, we have $\delta' \Vdash t \to^*_{R^\epsilon/E^\epsilon} s$. By Def. 1, the composition of quantitative relations is closed by quantitative weakening, therefore, from $\delta \precsim \delta'$ we obtain $\delta \Vdash t \to^*_{R^\epsilon/E^\epsilon} s$.
- ($\mathsf{Join}^\epsilon_{\mathsf{RL}}$) Then for each $i \in I$ we have $\delta_i \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$ and $\delta = \bigvee_{i\in I} \delta_i$. By the induction hypothesis, for every $i \in I$ we have $\delta_i \Vdash t \to^*_{R^\epsilon/E^\epsilon} s$. Since $\delta_i \precsim \bigvee_{i\in I} \delta_i$ for all $i \in I$ and by Def. 1, in $\to^*_{R^\epsilon/E^\epsilon}$ we may weaken the degree and obtain $\delta \Vdash t \to^*_{R^\epsilon/E^\epsilon} s$.
- ($\mathsf{Repl}^\epsilon_{\mathsf{RL}}$) Then there is a rule $\ell : l \to_\rho r \in R^\epsilon$ and a substitution σ such that $t = l\sigma$, $s = r\sigma$, and the conclusion is $\rho \Vdash l\sigma \longrightarrow_{R^\epsilon/E^\epsilon} r\sigma$. From $\ell : l \to_\rho r \in R^\epsilon$, by Definition of $\to_{R^\epsilon}$, we get $\rho \Vdash t \to_{R^\epsilon} s$. Then by Def. 6 (take $\varepsilon_1 = \varepsilon_2 = \kappa$ and $t' = t$, $s' = s$), the one-step rewrite $\rho \Vdash t \to_{R^\epsilon/E^\epsilon} s$ holds, and therefore $\rho \Vdash t \to^*_{R^\epsilon/E^\epsilon} s$.
- ($\mathsf{NExp}^\epsilon_{\mathsf{RL}}$) Then $t = f(t_1, \ldots, t_n)$, $s = f(s_1, \ldots, s_n)$, and there are degrees $\delta_1, \ldots, \delta_n$ such that $\delta_i \Vdash t_i \longrightarrow_{R^\epsilon/E^\epsilon} s_i$ for all i, and $\delta = \delta_1 \otimes \cdots \otimes \delta_n$. By the induction hypothesis, for each i we have $t_i \to^*_{R^\epsilon/E^\epsilon} s_i$. Using the congruence/context closure of $\to_{R^\epsilon}$ (hence of $\to_{R^\epsilon/E^\epsilon}$ and its reflexive-transitive closure), we can rewrite inside f argument by argument, obtaining $\delta_1 \otimes \cdots \otimes \delta_n \Vdash f(t_1, \ldots, t_n) \to^*_{R^\epsilon/E^\epsilon} f(s_1, \ldots, s_n)$. Thus, $\delta \Vdash t \to^*_{R^\epsilon/E^\epsilon} s$.

($\mathsf{Eq}^{\epsilon}_{\mathsf{RL}}$) Then there exist terms t', s' and degrees $\delta', \varepsilon_1, \varepsilon_2$ such that $\delta' \Vdash t' \longrightarrow_{R^\epsilon/E^\epsilon} s'$, $\varepsilon_1 \Vdash t =_{E^\epsilon} t'$, $\varepsilon_2 \Vdash s' =_{E^\epsilon} s$, and $\delta = \varepsilon_1 \otimes \delta' \otimes \varepsilon_2$. By the induction hypothesis, $t' \to^*_{R^\epsilon/E^\epsilon} s'$. Moreover, by Def. 6, any equational step $\varepsilon \Vdash u =_{E^\epsilon} v$ yields a one-step rewrite $\varepsilon \Vdash u \to_{R^\epsilon/E^\epsilon} v$ (take $\rho = \kappa$ and $u' = v$, $v' = v$), hence also $\varepsilon \Vdash u \to^*_{R^\epsilon/E^\epsilon} v$. Therefore, from $\varepsilon_1 \Vdash t =_{E^\epsilon} t'$ and $\varepsilon_2 \Vdash s' =_{E^\epsilon} s$ we obtain $t \to^*_{R^\epsilon/E^\epsilon} t'$ and $s' \to^*_{R^\epsilon/E^\epsilon} s$. Composing these with $t' \to^*_{R^\epsilon/E^\epsilon} s'$ gives $t \to^*_{R^\epsilon/E^\epsilon} s$ as required.

This completes the induction and proves $\longrightarrow_{R^\epsilon/E^\epsilon} \subseteq \to^*_{R^\epsilon/E^\epsilon}$. □

6 Computing with Quantitative Rewriting Logic

The relation $\longrightarrow_{R^\epsilon/E^\epsilon}$ captures the logical, deductive aspect of QRL. From a computational perspective, however, it is impractical, just as its standard counterpart $\longrightarrow_{R/E}$ is. For this reason, in the classical setting, the operationally better-behaved relation $\to_{R,E}$ was introduced as an executable counterpart of $\to_{R/E}$. As mentioned in Section 2, under some conditions, the reflexive-transitive closure of $\to_{R,E}$ can be related to $\longrightarrow_{R/E}$. Following the same idea, we introduce a quantitative relation $\to_{R^\epsilon,E^\epsilon}$ and investigate its relationship with $\longrightarrow_{R^\epsilon/E^\epsilon}$.

Definition 11 (Quantitative equational rewriting $\to_{R^\epsilon,E^\epsilon}$). *Given a QRT (R^ϵ, E^ϵ), the relation $\to_{R^\epsilon,E^\epsilon}$ is the quantitative relation defined using the following inference rules, where C is an arbitrary context:*

$$\frac{\ell : t \to_{\rho} s \in R^\epsilon \qquad \varepsilon \Vdash t\sigma =_{E^\epsilon} r}{\rho \otimes \varepsilon \Vdash C[r] \to_{R^\epsilon,E^\epsilon} C[s\sigma]} \qquad\qquad \frac{\varepsilon \Vdash t \to_{R^\epsilon,E^\epsilon} s \quad \varepsilon \succsim \delta}{\delta \Vdash t \to_{R^\epsilon,E^\epsilon} s}$$

We call the first rule the main rule, and the second one the weakening rule.

In the proofs and examples, for brevity, we use the notation $t_0 \xrightarrow{\varepsilon_1}_{R^\epsilon,E^\epsilon} t_1 \xrightarrow{\varepsilon_2}_{R^\epsilon,E^\epsilon} t_2 \cdots$ for $\varepsilon_1 \Vdash t_0 \to_{R^\epsilon,E^\epsilon} t_1$ and $\varepsilon_2 \Vdash t_1 \to_{R^\epsilon,E^\epsilon} t_2$, etc.

Lemma 12. *For all terms u, v and all $\alpha \in \Omega$, if $\alpha \Vdash u \to_{R^\epsilon,E^\epsilon} v$, then $\alpha \Vdash u \to_{R^\epsilon/E^\epsilon} v$.*

Proof. Assume $\alpha \Vdash u \to_{R^\epsilon,E^\epsilon} v$. By Definition 11, there are two ways to make this step: either with the main rule, or with the weakening rule.

First, assume that the main rule was used. Then there exist a rule $\ell : t \to_{\rho} s \in R^\epsilon$, a context C, a substitution σ, and $\varepsilon \in \Omega$ such that $u = C[r]$, $\varepsilon \Vdash t\sigma =_{E^\epsilon} r$, $v = C[s\sigma]$, $\alpha = \rho \otimes \varepsilon$. We need to show $\alpha \Vdash C[r] \to_{R^\epsilon/E^\epsilon} C[s\sigma]$. Starting from $\varepsilon \Vdash t\sigma =_{E^\epsilon} r$, by repeatedly applying ($\mathsf{NExp}^{\epsilon}_{\mathsf{RL}}$) (and ($\mathsf{Refl}^{\epsilon}_{\mathsf{RL}}$) on sibling subterms) from Fig. 3, we lift the equality to the surrounding context and get $\varepsilon \Vdash C[t\sigma] =_{E^\epsilon} C[r]$. From the rule $\ell : t \to_{\rho} s \in R^\epsilon$, by definition of $\to_{R^\epsilon}$ in Fig. 4, we get $\rho \Vdash C[t\sigma] \to_{R^\epsilon} C[s\sigma]$. Finally, by ($\mathsf{Refl}^{\epsilon}_{\mathsf{RL}}$) we have $\kappa \Vdash C[s\sigma] =_{E^\epsilon} C[s\sigma]$. Therefore, by Definition 6, we get $\rho \otimes \varepsilon \otimes \kappa \Vdash C[t\sigma] \to_{R^\epsilon/E^\epsilon} C[s\sigma]$. Since $\alpha = \rho \otimes \varepsilon \otimes \kappa$, it gives $\alpha \Vdash C[r] \to_{R^\epsilon/E^\epsilon} C[s\sigma]$. This concludes the proof of the case when the main rule was applied.

Now assume the weakening rule was used. Then there exists $\beta \in \Omega$ such that $\alpha \precsim \beta$ and $\beta \Vdash u \rightarrow_{R^\epsilon,E^\epsilon} v$ using the main rule. From the latter, we get $\beta \Vdash u \rightarrow_{R^\epsilon,E^\epsilon} v$ which, by Definition 6 rule, implies $\alpha \Vdash u \rightarrow_{R^\epsilon/E^\epsilon} v$. □

The following theorem states that $\rightarrow^*_{R^\epsilon,E^\epsilon}$ is included into $\longrightarrow_{R^\epsilon/E^\epsilon}$:

Theorem 13 (Operational rewriting implies provability). *Let (R^ϵ, E^ϵ) be a QRT. For all terms t, s and all $\delta \in \Omega$, if $\delta \Vdash t \rightarrow^*_{R^\epsilon,E^\epsilon} s$, then $\delta \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$.*

Proof. We use induction on the length of the operational multistep.

Base case (length 0). Then $t = s$ and without losing generality, we can assume $\delta = \kappa$. Hence $\kappa \Vdash t \rightarrow_{R^\epsilon/E^\epsilon} t$ by $(\mathsf{Refl}^\epsilon_{\mathsf{RL}})$.

Induction step. Assume $\delta \Vdash t \rightarrow^*_{R^\epsilon,E^\epsilon} s$ is witnessed by

$$t = t_0 \xrightarrow{\varepsilon_1}_{R^\epsilon,E^\epsilon} t_1 \xrightarrow{\varepsilon_2}_{R^\epsilon,E^\epsilon} \cdots \xrightarrow{\varepsilon_n}_{R^\epsilon,E^\epsilon} t_n = s \quad \text{with} \quad \delta \precsim \varepsilon_1 \otimes \cdots \otimes \varepsilon_n.$$

By the induction hypothesis we have $\varepsilon_1 \otimes \cdots \otimes \varepsilon_{n-1} \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} t_{n-1}$. The remaining step, that $\varepsilon_n \Vdash t_{n-1} \rightarrow_{R^\epsilon,E^\epsilon} s$ implies $\varepsilon_n \Vdash t_{n-1} \longrightarrow_{R^\epsilon/E^\epsilon} s$, follows from Lemma 12. Combining single-step soundness along the chain with $(\mathsf{Trans}^\epsilon_{\mathsf{RL}})$ and then applying $(\mathsf{Ord}^\epsilon_{\mathsf{RL}})$ to weaken the cost from $\varepsilon_1 \otimes \cdots \otimes \varepsilon_n$ down to δ yields $\delta \Vdash t \longrightarrow_{R^\epsilon/E^\epsilon} s$. □

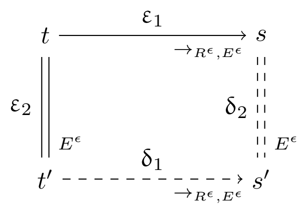
The converse of Lemma 12 does not hold, as shown by the following example (for simplicity, we omit labels in rules).

Example 14. Consider the Lawvere quantale and a signature with a binary function symbol f and two constants a, b. Let $R^\epsilon = \{f(a,b) \rightarrow_1 a\}$ and $E^\epsilon = \{2 \Vdash f(b, f(a,a)) \approx f(a, f(a,b))\}$. By Definition 6, we have $3 \Vdash f(b, f(a,a)) \rightarrow_{R^\epsilon/E^\epsilon} f(a,a)$. However, $f(b, f(a,a))$ is in normal form with respect to $\rightarrow_{R^\epsilon,E^\epsilon}$ and, hence, $\varepsilon \Vdash f(b, f(a,a)) \rightarrow_{R^\epsilon,E^\epsilon} f(a,a)$ does not hold for any $\varepsilon \in [0,\infty)$.

Extending R^ϵ with the rule $\varepsilon \Vdash f(b, f(a,a)) \mapsto f(a,a)$ for $\varepsilon \leqslant 3$ (i.e., $\varepsilon \succsim 3$) makes the step $3 \Vdash f(b, f(a,a)) \rightarrow_{R^\epsilon,E^\epsilon} f(a,a)$ possible. Using $\varepsilon < 3$, however, would change the $\rightarrow_{R^\epsilon/E^\epsilon}$ relation; $\varepsilon = 3$ is the only value that retains $\rightarrow_{R^\epsilon/E^\epsilon}$ while also enabling the desired $\rightarrow_{R^\epsilon,E^\epsilon}$ step.

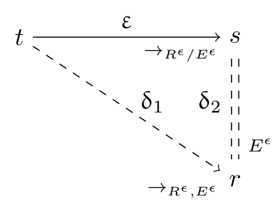
To further relate the $\rightarrow_{R^\epsilon/E^\epsilon}$ and $\rightarrow_{R^\epsilon,E^\epsilon}$ relations, we introduce a notion of quantitative strict coherence, which is a counterpart of strict coherence, defined in [38]:

Definition 15 (Quantitative strict coherence). *We say that the rewrite relation $\rightarrow_{R^\epsilon,E^\epsilon}$ is* strictly E^ϵ-coherent *iff for all terms t, s, t' and for all $\varepsilon_1, \varepsilon_2 \in \Omega$, if $\varepsilon_1 \Vdash t \rightarrow_{R^\epsilon,E^\epsilon} s$ and $\varepsilon_2 \Vdash t =_{E^\epsilon} t'$, then there exist a term s' and $\delta_1, \delta_2 \in \Omega$ such that $\delta_1 \Vdash t' \rightarrow_{R^\epsilon,E^\epsilon} s'$, $\delta_2 \Vdash s' =_{E^\epsilon} s$, and $\varepsilon_1 \otimes \varepsilon_2 = \delta_1 \otimes \delta_2$. Graphically:*



$$\varepsilon_1 \otimes \varepsilon_2 = \delta_1 \otimes \delta_2$$

Theorem 16. *If $\to_{R^\epsilon,E^\epsilon}$ is strictly E^ϵ-coherent, then for all terms t and s and $\varepsilon \in \Omega$, if $\varepsilon \Vdash t \to_{R^\epsilon/E^\epsilon} s$, then there exists a term r and $\delta_1, \delta_2 \in \Omega$ such that $\delta_1 \Vdash t \to_{R^\epsilon,E^\epsilon} r$, $\delta_2 \Vdash r =_{E^\epsilon} s$, and $\varepsilon = \delta_1 \otimes \delta_2$. Graphically:*



$$\varepsilon = \delta_1 \otimes \delta_2$$

Proof. Assume $\varepsilon \Vdash t \to_{R^\epsilon/E^\epsilon} s$. By Def. 6, there exist terms t', s' and degrees $\varepsilon_1, \rho, \varepsilon_2$ such that $\varepsilon_1 \Vdash t =_{E^\epsilon} t'$, $\rho \Vdash t' \to_{R^\epsilon} s'$, $\varepsilon_2 \Vdash s =_{E^\epsilon} s'$, and $\varepsilon \precsim \varepsilon_1 \otimes \rho \otimes \varepsilon_2$.

Step 1: Embedding $\to_{R^\epsilon}$ into $\to_{R^\epsilon,E^\epsilon}$. From $\rho \Vdash t' \to_{R^\epsilon} s'$, by the definition of $\to_{R^\epsilon}$ (Fig. 3), there exist a rule $\ell : l \to_\beta r \in R^\epsilon$, a context C, and a substitution σ such that $t' = C[l\sigma]$, $s' = C[r\sigma]$, and $\rho \precsim \beta$. By reflexivity of $=_{E^\epsilon}$, we have $\kappa \Vdash l\sigma =_{E^\epsilon} l\sigma$. Hence, by the main rule defining $\to_{R^\epsilon,E^\epsilon}$, we obtain $\beta \Vdash C[l\sigma] \to_{R^\epsilon,E^\epsilon} C[r\sigma]$, that is, $\beta \Vdash t' \to_{R^\epsilon,E^\epsilon} s'$. Since $\rho \precsim \beta$, weakening yields $\rho \Vdash t' \to_{R^\epsilon,E^\epsilon} s'$.

Step 2: Strict E^ϵ-coherence. From $\varepsilon_1 \Vdash t =_{E^\epsilon} t'$, by symmetry we obtain $\varepsilon_1 \Vdash t' =_{E^\epsilon} t$. Applying strict E^ϵ-coherence (Def. 15) to $\rho \Vdash t' \to_{R^\epsilon,E^\epsilon} s'$ and $\varepsilon_1 \Vdash t' =_{E^\epsilon} t$, there exist a term r and degrees δ_1, δ' such that $\delta_1 \Vdash t \to_{R^\epsilon,E^\epsilon} r$, $\delta' \Vdash r =_{E^\epsilon} s'$, and $\rho \otimes \varepsilon_1 = \delta_1 \otimes \delta'$.

Step 3: Final composition. From $\varepsilon_2 \Vdash s =_{E^\epsilon} s'$, by symmetry we obtain $\varepsilon_2 \Vdash s' =_{E^\epsilon} s$. By transitivity of $=_{E^\epsilon}$, from $\delta' \Vdash r =_{E^\epsilon} s'$ and $\varepsilon_2 \Vdash s' =_{E^\epsilon} s$ we get $\delta' \otimes \varepsilon_2 \Vdash r =_{E^\epsilon} s$. Let $\delta_2 := \delta' \otimes \varepsilon_2$. Then we obtain $\delta_1 \Vdash t \to_{R^\epsilon,E^\epsilon} r$ and $\delta_2 \Vdash r =_{E^\epsilon} s$. Moreover, $\delta_1 \otimes \delta_2 = \delta_1 \otimes (\delta' \otimes \varepsilon_2) = (\delta_1 \otimes \delta') \otimes \varepsilon_2 = (\rho \otimes \varepsilon_1) \otimes \varepsilon_2 = \varepsilon_1 \otimes \rho \otimes \varepsilon_2$. Together with $\varepsilon \precsim \varepsilon_1 \otimes \rho \otimes \varepsilon_2$, we conclude $\varepsilon \precsim \delta_1 \otimes \delta_2$, which proves the required factorisation. □

Theorem 17. *If $\to_{R^\epsilon,E^\epsilon}$ is strictly E^ϵ-coherent, then $\to_{R^\epsilon/E^\epsilon} \equiv \to_{R^\epsilon,E^\epsilon};=_{E^\epsilon}$.*

Proof. The direction ($\subseteq$) is proved in Theorem 16. The direction ($\supseteq$) follows from the definitions of $\to_{R^\epsilon/E^\epsilon}$ and $\to_{R^\epsilon,E^\epsilon}$. □

Example 18. Let Ω be the Lawvere quantale, $E^\epsilon = \{1 \Vdash x + y \approx y + x,\ 2 \Vdash (x + y) + z \approx x + (y + z)\}$, and $R^\epsilon = \{3 \Vdash a + b \mapsto a\}$. Then we have

$$4 \Vdash a + (a + b) \to_{R^\epsilon/E^\epsilon} a + a \quad \text{and} \quad 4 \Vdash a + (a + b) \to_{R^\epsilon,E^\epsilon} a + a,$$

$$6 \Vdash b + (a + a) \rightarrow_{R^\epsilon/E^\epsilon} a + a \quad \text{but } b + (a + a) \text{ is in } \rightarrow_{R^\epsilon,E^\epsilon}\text{-normal form.}$$

However, if we extend R^ϵ with the rule $\delta \Vdash (a+b)+X \mapsto a+X$ for an arbitrary $\delta \leq 3$, then $\rightarrow_{R^\epsilon,E^\epsilon}$ will become E^ϵ-coherent and we get $3+\delta \Vdash b+(a+a) \rightarrow_{R^\epsilon,E^\epsilon} a + a$, which, by weakening, gives $6 \Vdash b + (a + a) \rightarrow_{R^\epsilon,E^\epsilon} a + a$.

Theorem 19. *If rewrite rules are permitted to apply only on the top position (i.e., the context C in Fig. 3 and Def. 11 is empty), then* $\rightarrow_{R^\epsilon/E^\epsilon} \equiv \rightarrow_{R^\epsilon,E^\epsilon}; =_{E^\epsilon}$.

Proof. Under the top-position restriction, the relation $\rightarrow_{R^\epsilon,E^\epsilon}$ becomes strictly E^ϵ-coherent. Then the theorem follows from Theorem 17. □

This theorem is also interesting from the practical point of view because most concurrent systems are specified as topmost rewrite theories.

The next theorem is the main result in this section: it indicates that deduction in QRL can be modelled by quantitative rewriting via the $\rightarrow_{R^\epsilon,E^\epsilon}$ relation:

Theorem 20 (Modelling deduction via operational rewriting). *If the relation $\rightarrow_{R^\epsilon,E^\epsilon}$ is strictly E^ϵ-coherent, then* $\longrightarrow_{R^\epsilon/E^\epsilon} \equiv \rightarrow^*_{R^\epsilon,E^\epsilon}; =_{E^\epsilon}$.

Proof. By theorems 10 and 17, $\longrightarrow_{R^\epsilon/E^\epsilon} \equiv \rightarrow^*_{R^\epsilon/E^\epsilon} \equiv (\rightarrow_{R^\epsilon,E^\epsilon}; =_{E^\epsilon})^*$. We therefore need to show that $(\rightarrow_{R^\epsilon,E^\epsilon}; =_{E^\epsilon})^* \equiv \rightarrow^*_{R^\epsilon,E^\epsilon}; =_{E^\epsilon}$. The inclusion $(\rightarrow_{R^\epsilon,E^\epsilon}; =_{E^\epsilon})^* \supseteq \rightarrow^*_{R^\epsilon,E^\epsilon}; =_{E^\epsilon}$ is clear as $=_{E^\epsilon}$ is reflexive. For the other inclusion, it suffices to show that $(\rightarrow_{R^\epsilon,E^\epsilon}; =_{E^\epsilon})^n \subseteq \rightarrow^n_{R^\epsilon,E^\epsilon}; =_{E^\epsilon}$ for every n; we proceed by induction on n. The base case $n = 1$ is trivial. For the induction step, suppose that the statement holds up until n. Taking into account the associativity of the composition and its monotonicity w.r.t. inclusion, we have

$$\begin{aligned}
(\rightarrow_{R^\epsilon,E^\epsilon}; =_{E^\epsilon})^{n+1} &\equiv (\rightarrow_{R^\epsilon,E^\epsilon}; =_{E^\epsilon})^n; \rightarrow_{R^\epsilon,E^\epsilon}; =_{E^\epsilon} \\
\text{(by the IH and monotonicity)} &\subseteq \rightarrow^n_{R^\epsilon,E^\epsilon}; =_{E^\epsilon}; \rightarrow_{R^\epsilon,E^\epsilon}; =_{E^\epsilon} \\
\text{(by coherence and monotonicity)} &\subseteq \rightarrow^{n+1}_{R^\epsilon,E^\epsilon}; =^2_{E^\epsilon} \\
\text{(by transitivity of } =_{E^\epsilon}) &\equiv \rightarrow^{n+1}_{R^\epsilon,E^\epsilon}; =_{E^\epsilon} .
\end{aligned}$$

□

Example 21. Consider the Lawvere quantale $\mathbb{L}$ and quantitative rewrite theory (R^ϵ, E^ϵ), where

$$\begin{aligned}
R^\epsilon &= \{\ell_1 : a \rightarrow_2 c,\ \ell_2 : f(h(x), h(b)) \rightarrow_2 g(x, x)\}, \\
E^\epsilon &= \{0 \Vdash f(x, y) \approx f(y, x),\ 1 \Vdash a \approx b,\ 2 \Vdash h(x) \approx p(x)\}.
\end{aligned}$$

Below we show

(a) a proof of $5 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, b)$ and its counterpart via $\rightarrow^*_{R^\epsilon,E^\epsilon}$,
(b) a proof of $6 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, c)$ and its counterpart via $\rightarrow^*_{R^\epsilon,E^\epsilon}$.

(a) Proving $5 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, b)$**:**

$$\dfrac{\dfrac{\begin{array}{c}\ell_2 : f(h(x), h(b)) \rightarrow_2 g(x, x) \in R^\epsilon \\ \sigma = \{x \mapsto a\}\end{array}}{2 \Vdash f(h(a), h(b)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, a)}\,(\mathsf{Repl}^\epsilon_{\mathsf{RL}}) \qquad eq_1 \qquad eq_2}{5 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, b)}\,(\mathsf{Eq}^\epsilon_{\mathsf{RL}})$$

where $eq_1 = 2 \Vdash f(h(a), h(b)) =_{E^\epsilon} f(p(b), h(a))$ and $eq_2 = 1 \Vdash g(a, a) =_{E^\epsilon} g(a, b)$. They can be proved using rules of QET in Fig. 2.

(a) Rewriting using $\rightarrow_{R^\epsilon, E^\epsilon}$**.** Applying $\ell_2 : f(h(x), h(b)) \rightarrow_2 g(x, x))$ with $\sigma = \{x \mapsto a\}$ and $2 \Vdash f(h(x), h(b))\sigma =_{E^\epsilon} f(p(b), h(a))$ gives

$$4 \Vdash f(p(b), h(a)) \rightarrow_{R^\epsilon, E^\epsilon} f(a, a).$$

For $f(a, a)$, we have

$$1 \Vdash f(a, a) =_{E^\epsilon} f(a, b),$$

which can be proved using rules of QET in Fig. 2. Then we get

$$4 \Vdash f(p(b), h(a)) \rightarrow_{R^\epsilon, E^\epsilon} f(a, a), \quad 1 \Vdash f(a, a) =_{E^\epsilon} f(a, b).$$

(b) Proving $6 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, c)$**:**

$$\dfrac{T_1 \qquad T_2}{6 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, c)}\,(\mathsf{Trans}^\epsilon_{\mathsf{RL}})$$

where T_1 denotes the proof tree for $4 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, a)$:

$$\dfrac{\dfrac{\begin{array}{c}\ell_2 : f(h(x), h(b)) \rightarrow_2 g(x, x) \in R^\epsilon \\ \sigma = \{x \mapsto a\}\end{array}}{2 \Vdash f(h(a), h(b)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, a)}\,(\mathsf{Repl}^\epsilon_{\mathsf{RL}}) \qquad eq_1 \qquad eq_2}{4 \Vdash f(p(b), h(a)) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, a)}\,(\mathsf{Eq}^\epsilon_{\mathsf{RL}})$$

with $eq_1 = 2 \Vdash f(h(a), h(b)) =_{E^\epsilon} f(p(b), h(a))$ and $eq_2 = 0 \Vdash g(a, a) =_{E^\epsilon} g(a, a)$ (provable in QET using rules in Fig. 2.

The tree T_2 is the proof tree for $2 \Vdash g(a, a) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, c)$:

$$\dfrac{\dfrac{}{0 \Vdash a \longrightarrow_{R^\epsilon/E^\epsilon} a}\,(\mathsf{Refl}^\epsilon_{\mathsf{RL}}) \qquad \dfrac{\ell_1 : a \rightarrow_2 c \in R}{2 \Vdash a \longrightarrow_{R^\epsilon/E^\epsilon} c}\,(\mathsf{Repl}^\epsilon_{\mathsf{RL}})}{2 \Vdash g(a, a) \longrightarrow_{R^\epsilon/E^\epsilon} g(a, c)}\,(\mathsf{NExp}^\epsilon_{\mathsf{RL}})$$

(b) Rewriting using $\to_{R^\epsilon,E^\epsilon}$. Applying $\ell_2 : f(h(x), h(b)) \to_2 g(x, x))$ with $\sigma = \{x \mapsto a\}$ and $2 \Vdash f(h(x), h(b))\sigma =_{E^\epsilon} f(p(b), h(a))$ gives

$$4 \Vdash f(p(b), h(a)) \to_{R^\epsilon,E^\epsilon} f(a, a).$$

Next step: application of $\ell_1 : a \to_2 c$ to $f(a, a)$ gives

$$2 \Vdash f(a, a) \to_{R^\epsilon,E^\epsilon} f(a, c).$$

Then we get

$$f(p(b), h(a)) \to^4_{R^\epsilon,E^\epsilon} f(a, a) \to^2_{R^\epsilon,E^\epsilon} f(a, c).$$

The central computational mechanism of $\to_{R^\epsilon,E^\epsilon}$ is matching modulo E^ϵ, which is discussed in the next section.

7 Matching Modulo E^ϵ

Performing a single step in $\to_{R^\epsilon,E^\epsilon}$, as defined in Definition 11, requires solving a quantitative matching problem between t and r modulo E^ϵ.

Definition 22 (Quantitative matching problem). *For a quantitative equational theory E^ϵ, an E^ϵ-*matching problem *is a pair (t, s), usually written as $t \ll_{E^\epsilon} s$ (or just $t \ll s$, when E^ϵ is clear from the context).*

A solution *of an E^ϵ-matching problem (t, s) is a pair (σ, ε), where σ is a substitution and $\varepsilon \in \Omega$ is a quantale element such that $\varepsilon \Vdash t\sigma =_{E^\epsilon} s$ holds.*

The quantitative matching problem is formulated as follows:

Given: A quantitative equational theory E^ϵ and a multiset of E^ϵ-matching problems $P = \{\!\{t_1 \ll_{E^\epsilon} r_1, \dots, t_n \ll_{E^\epsilon} r_n\}\!\}$ (P is called an E^ϵ-matching constraint)

Find: A substitution-degree pair (σ, ε) such that there exist $\varepsilon_1, \dots, \varepsilon_n$ satisfying $\varepsilon_1 \otimes \cdots \otimes \varepsilon_n \succsim \varepsilon$ and $\varepsilon_i \Vdash t_i\sigma =_{E^\epsilon} r_i$ for all $i \in \{1, \dots n\}$.
Such a pair (σ, ε) is called a *solution* of P.

7.1 Quantitative Matching as a Combination Problem

First, we note that it is useful to split the theory E^ϵ into its *exact* and *approximate* parts: $E^\epsilon = Ex \uplus App$, where

$$Ex = \{\varepsilon \Vdash s \approx t \in E^\epsilon \mid \varepsilon = \kappa\} \quad \text{and} \quad App = \{\varepsilon \Vdash s \approx t \in E^\epsilon \mid \varepsilon \prec \kappa\}.$$

It is therefore natural to view E^ϵ-matching as a combination problem: Given matching algorithms for Ex and App, how can they be combined to solve E^ϵ-matching problems? Since Ex is a crisp theory, Ex-matching can be viewed as a non-quantitative matching problem, which is well-studied and complete algorithms are available for many important theories. As for App, we will present a complete *App*-matching algorithm for one particular class of theories later in this section.

We provide a rule-based calculus to solve the combination problem for quantitative matching, inspired by combination techniques in the non-quantitative setting [46]. This calculus is subsequently shown to be terminating, sound, and complete under suitable conditions.

We use some quantitative analogues of common terminology in equational reasoning. A quantitative equation $\varepsilon \Vdash t \approx s$ with $\varepsilon \succ \bot$ is called

- *regular*, if $\mathcal{V}(t) = \mathcal{V}(s)$,
- *left-linear*, if t is a linear term,
- *linear*, if both t and s are linear terms,
- *collapse-free*, if neither t nor s is a variable.

We also extend this terminology to sets of equations: E is regular if all its elements are regular, and so on. In the classical case, it is known that an equational theory generated by regular (resp. collapse-free) equations consists of regular (resp. collapse-free) equations only [10]. For the quantitative case, when the equations in presentation E^ϵ have degree $\varepsilon \succ \bot$, then the equations in the theory $=_{E^\epsilon}$ retain this property: regularity (resp. collapse-freeness) in E^ϵ implies regularity (resp. collapse-freeness) in $=_{E^\epsilon}$.

Definition 23. *Let E be an exact theory, and let σ and τ be two substitutions. We say that σ and τ are equal modulo E, denoted $\sigma =_E \tau$, if $x\sigma =_E x\tau$ holds for every variable x. We say that σ is more general than τ modulo E, denoted $\sigma \leqq_E \tau$, if there exists a substitution ρ such that $\sigma\rho =_E \tau$.*

Remark 24. For two substitutions σ and τ, where σ is idempotent, $\sigma \leqq_E \tau$ is equivalent to $\sigma\tau =_E \tau$.

Indeed, suppose that $\tau =_E \sigma\varphi$; then we have $\sigma\tau =_E \sigma\sigma\varphi = \sigma\varphi =_E \tau$ by idempotence. The converse is trivial.

As in the non-quantitative case, we do not aim to compute each matcher explicitly, but want to find a set of matchers from which any other matcher can be obtained.

Definition 25 (Complete set of matchers). *Let P be an E^ϵ-matching constraint. A set C of substitution-degree pairs is a* complete set of matchers *for P if*

- *every element of C is a solution of P, and*
- *for every solution (σ, ε) of P, there exists $(\tau, \zeta) \in C$ such that $\tau \leqq_{Ex} \sigma$ and $\zeta \succsim \varepsilon$.*

Note that this definition will yield the usual notion of a complete set of matchers if $E^\epsilon = Ex$ is an exact theory.

We assume henceforth that we are given algorithms for computing complete sets of matchers modulo *Ex* and *App*, respectively. We write $\text{CSM}_{Ex}(t \ll s)$

and $\textsc{csm}_{App}(t \ll s)$ to denote the complete sets of matchers computed by these algorithms.

Moreover, we will assume that no function symbol appears in both Ex and App, i.e., that the corresponding signatures $\mathcal{F}_{Ex}$ and $\mathcal{F}_{App}$ are disjoint. This assumption will allow us to reduce a matching problem over the combined theory to smaller problems that can be solved in one of the theories. This is achieved by *purifying* the problem, i.e. by iteratively eliminating subterms that are rooted in a different signature than the term itself.

Definition 26 (Alien positions). *Let t be a term over $\mathcal{F}_1 \cup \mathcal{F}_2$. A position $p \in Pos(t)$ is called an* alien position *of t if*

- *p is not the root position,*
- *$t|_p$ is a non-variable term rooted in $\mathcal{F}_i$*
- *for any position q above p, $t|_q$ is rooted in $\mathcal{F}_j$, where $i \neq j$.*

If p is an alien position of t, then $t|_p$ is called an alien subterm *of t.*

We use a rule-based calculus to solve the problem of combining matching algorithms for Ex and App. The rules operate on configurations of the form $P; \delta; \sigma$, where

- P is an E^ϵ-matching constraint (the current state of the problem);
- δ is an element of the quantale domain Ω (the degree consumed so far);
- σ is a substitution (the solution computed so far).

We also include a configuration **F**, indicating failure.

To solve an E^ϵ-matching constraint $\{\!\!\{t_1 \ll_{E^\epsilon} r_1, \dots, t_n \ll_{E^\epsilon} r_n\}\!\!\}$, we create the initial configuration

$$\{\!\!\{t_1 \ll r_1, \dots, t_n \ll r_n\}\!\!\}; \kappa; Id$$

and apply the rules in Figure 5 as long as possible. If the same configuration can be transformed by several rules, they are applied in parallel (*don't know* nondeterminism), generating branches in the derivation tree. A branch is *successful* if it stops with a configuration $\emptyset; \varepsilon; \sigma$, in which case we call (σ, ε) a *computed pair*. A branch is *failed* if it ends with **F**. We refer to this calculus for the combination of quantitative matching algorithms by QCombine.

We introduce some terminology to facilitate the upcoming proofs. We say that a substitution-degree pair (τ, ε), where $\varepsilon \succ \bot$, *solves* a configuration $\{\!\!\{l_1 \ll r_1, \dots, l_n \ll r_n\}\!\!\}; \delta; \sigma$ if $\sigma \leqq_{Ex} \tau$ and there exist $\varepsilon_1, \dots, \varepsilon_n \in \Omega$ such that $\varepsilon_1 \otimes \cdots \otimes \varepsilon_n \succsim \varepsilon$ and $\varepsilon_i \Vdash l_i\tau =_{E^\epsilon} r_i\tau$ holds for $i = 1, \dots, n$.

In particular, solutions of an initial configuration $(P; \kappa; Id)$ are just solutions of P, and solutions of a terminal configuration $\emptyset; \delta; \sigma$ are pairs (τ, ε) such that $\varepsilon \succ \bot$ and $\sigma \leqq_{Ex} \tau$.

As the next lemma shows, matching constraints may be assumed left-linear without loss of generality.

ASA: **Alien subterm abstraction**

$\{\!\{t \ll r\}\!\} \uplus P; \delta; \sigma \Longrightarrow \{\!\{t[x]_p \ll r,\ \ t|_p \ll x\}\!\} \cup P; \delta; \sigma,$

where the position p is an alien position of t and x is a fresh variable.

EM: **Exact matching**

$\{\!\{t \ll r\}\!\} \uplus P; \delta; \sigma \Longrightarrow P \cup \{\!\{x_1 \ll s_1, \ldots, x_m \ll s_m\}\!\}; \delta; \sigma,$

where $t \notin \mathcal{V}$ is a pure *Ex*-term, r is ground, and $\{x_1 \mapsto s_1, \ldots, x_m \mapsto s_m\} \in \mathrm{CSM}_{Ex}(t \ll r)$.

AM: **Approximate matching**

$\{\!\{t \ll r\}\!\} \uplus P; \delta; \sigma \Longrightarrow P\vartheta; \delta \otimes \varepsilon; \sigma\vartheta,$

where t is a pure *App*-term or a variable, r is ground, and $(\vartheta, \varepsilon) \in \mathrm{CSM}_{App}(t \ll r)$.

Fail: **Failure**

$\{\!\{t \ll r\}\!\} \uplus P; \delta; \sigma \Longrightarrow \mathbf{F},$

if t is a pure term, r is ground, and $t \ll r$ does not have a solution.

Fig. 5: QCombine rules

Lemma 27. *Any E^ϵ-matching constraint P can be* left-linearised *by exhaustive application of the rule*

$$P_0 \uplus \{\!\{s \ll t\}\!\} \quad \Longrightarrow \quad P_0 \cup \{\!\{s' \ll t\}\!\} \cup \{\!\{x \ll x_i \mid 1 \leqslant i \leqslant n\}\!\},$$

where x is a variable that occurs n times in s (for $n > 1$) and s' is the term obtained by replacing the first occurrence of x in s by the fresh variable x_1, the second occurrence by the fresh variable x_2, and so on.

The obtained constraint (which is reached after finitely many applications of the rule) has the same set of solutions as the original one when restricted to the original variables, but its elements are left-linear.

Proof. It is clear that the rule can only be applied a finite number of times as each application decreases the total number $\sum_{s \ll t \in P} \mathrm{dvar}(s)$, where $\mathrm{dvar}(s)$ is the number of variables that occur more than once in s.

Moreover, it is obvious that the constraint obtained after exhaustive application of the rule is left-linear as otherwise, the rule could still be applied.

For statement concerning solutions, it suffices to show that solutions are preserved under a single rule application. Suppose that (τ, ε) solves $P_0 \uplus \{\!\{s \ll t\}\!\}$. Then there exist degrees ζ and η such that (τ, ζ) solves P_0, (τ, η) solves $\{\!\{s \ll t\}\!\}$, and $\zeta \otimes \eta \succsim \varepsilon$. Now let s' be the term obtained from s by replacing the n occurrences of the variable x with the fresh variables $x_1, \ldots, x_n$, as described in the rule above. Extending τ to the substitution $\tau' := \tau \cup \{x_1 \mapsto x\tau, \ldots, x_n \mapsto x\tau\}$, we get that (τ', η) solves $\{\!\{s' \ll t\}\!\} \cup \{\!\{x \ll x_i \mid 1 \leqslant i \leqslant n\}\!\}$. Hence, (τ', ε) solves

$P_0 \cup \{\!\{s' \ll t\}\!\} \cup \{\!\{x \ll x_i \mid 1 \leqslant i \leqslant n\}\!\}$, and clearly, τ' and τ coincide when restricted to the variables of $P_0 \uplus \{\!\{s \ll t\}\!\}$.

For the converse, suppose that (ρ, ε) solves $P_0 \uplus \{\!\{s' \ll t\}\!\} \cup \{\!\{x \ll x_i \mid 1 \leqslant i \leqslant n\}\!\}$. As before, there exist degrees ζ and η with $\zeta \otimes \eta$ such that (ρ, ζ) solves P_0 and (ρ, η) solves $\{\!\{s' \ll t\}\!\} \cup \{\!\{x \ll x_i \mid 1 \leqslant i \leqslant n\}\!\}$. Then there exist $\eta_0, \eta_1, \ldots, \eta_n$ with $\bigotimes_{i=0}^{n} \eta_i \gtrsim \eta$ such that $\eta_0 \Vdash s'\rho =_{E^\epsilon} t$ and $\eta_i \Vdash x\rho =_{E^\epsilon} x_i\rho$ for $1 \leqslant i \leqslant n$. Now for each i, let p_i the position of x_i in s', so that $s' = s[x_1]_{p_1} \ldots [x_n]_{p_n}$. Then have

$$
\begin{aligned}
&\eta_1 \Vdash s\rho =_{E^\epsilon} s[x_1]_{p_1}\rho, \\
&\eta_2 \Vdash s[x_1]_{p_1}\rho =_{E^\epsilon} s[x_1]_{p_1}[x_2]_{p_2}\rho, \\
&\quad \vdots \\
&\eta_n \Vdash s[x_1]_{p_1} \ldots [x_{n-1}]_{p_{n-1}}\rho =_{E^\epsilon} s'\rho,
\end{aligned}
$$

where each equation is obtained by stability under contexts. Thus we get $\bigotimes_{i=1}^{n} \eta_i \Vdash s\rho = s'\rho$, and as a consequence, $\bigotimes_{i=1}^{n} \eta_i \Vdash s =_{E^\epsilon} t$, by (Trans). Thus, (ρ, η) solves $\{\!\{s \ll t\}\!\}$, and as a consequence, (ρ, ε) solves $P_0 \uplus \{\!\{s \ll t\}\!\}$, as desired. □

As in the non-quantitative setting, regularity of E^ϵ guarantees that matchers map every variable to a ground term. Therefore, the variables introduced by the ASA rule as the right-hand side of matching equations are eventually instantiated by ground terms (or the algorithm fails before), because they appear in left-hand sides of other equations as well.

Definition 28. *Let P be an E^ϵ-matching constraint. A variable x is said to be* grounded in P *if P contains an element of the form $t \ll s$, where $x \in \mathcal{V}(t)$, and s is either ground or only contains variables which are grounded in P. The constraint P is called* well-moded *if all variables appearing in P are grounded in P.*

Lemma 29. *Suppose that Ex and App are regular, and let $\mathfrak{C} = P; \delta; \sigma$ be some configuration. Then left-linearity of P, well-modedness of P, and groundness of σ are preserved under the application of any of the non-failing rules in Figure 5 to $\mathfrak{C}$, respectively.*

Proof. Left-linearity of P is preserved under ASA and EM as the introduced variables x and $x_1, \ldots, x_m$ are fresh, respectively. Moreover, it is preserved under AM as the substitution ϑ is ground since it solves a right-ground matching problem over a regular theory.

As for well-modedness, we begin with ASA. Write $P = \{\!\{t \ll r\}\!\} \uplus P_0$, and let y be a variable in $P' = \{\!\{t[x]_p \ll r,\ \ t|_p \ll x\}\!\} \cup P_0$. First note that any variable in $t[x]_p$ is grounded in P' as r is ground; in particular, x is grounded in P'. Thus, any variable in $t|_p$ is also grounded in P', i.e. any variable in t is grounded in P'. Therefore any variable $y \in \mathcal{V}(P_0)$ is grounded in P', since it is grounded in $P = \{\!\{t \ll r\}\!\} \uplus P_0$. In case of an EM step, we must have $\mathcal{V}(t) = \{x_1, \ldots, x_n\}$ as

r is ground and Ex is regular. Thus, each x_i is grounded in P', and consequently also every variable from P_0. As for an AM step, well-modedness is preserved as the substitution ϑ is ground (as R is ground and App is regular).

Finally, groundness of σ is preserved because the substitution ϑ in AM is ground, as mentioned before. □

Lemma 30. *If $\mathfrak{C}$ is well-moded and left-linear and $\mathfrak{C}'$ is a terminal configuration of* QCombine *such that $\mathfrak{C} \Longrightarrow^* \mathfrak{C}'$, then $\mathfrak{C}'$ is either $\boldsymbol{F}$, or it has the form $\emptyset; \sigma; \delta$.*

Proof. If $\mathfrak{C}'$ is not $\mathbf{F}$, we can write $\mathfrak{C}'$ as $P; \sigma; \delta$. Assume toward a contradiction that P is non-empty. By Lemma 29, P is well-moded and left-linear, so it contains an element of the form $t \ll s$ where s is ground. We will show that $\mathfrak{C}'$ is not terminal. Indeed, if t is not a pure term, then ASA can be applied. Otherwise, if t is Ex-pure and not a variable, then either EM or Fail can be applied. Finally, if t is App-pure, then either AM or Fail can be applied. Thus, we have shown that $\mathfrak{C}'$ is not terminal, concluding the proof. □

Termination. We prove termination of QCombine based on the assumption that the matcher ϑ of t to r in EM and AM steps is ground and satisfies $dom(\vartheta) = \mathcal{V}(t)$.

Proposition 31 (Termination). *Let $E = App \uplus Ex$ be regular, and let P_0 be a finite set of matching problems, $\delta_0 \in \Omega \setminus \{\bot\}$, and σ_0 a substitution. Every derivation $P_0; \delta_0; \sigma_0 \Longrightarrow \ldots$ terminates after a finite number of steps.*

Proof. We define four different measures for a configuration $\mathfrak{C}$ of the form $P; \delta; \sigma$:

- $\lambda(\mathfrak{C}) \coloneqq |\mathrm{DVars}(P)|$, where $\mathrm{DVars}(P)$ is the set of variables that appear at least twice in the set of left hand sides of equations in P.
- $\mu(\mathfrak{C}) \coloneqq \sum_{t \ll_{\mathfrak{z}} s \in P} \iota(t)$, where the *impurity* $\iota(t)$ of a term $t \in T(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ is defined inductively by $\iota(t) \coloneqq \sum_{s \in \mathrm{AST}(t)} (1 + \iota(s))$ (empty sums are 0 by convention).
- $\nu(\mathfrak{C}) \coloneqq |\mathcal{V}(P)|$.
- $\xi(\mathfrak{C}) \coloneqq \sum_{t \ll_{\mathfrak{z}} s \in P} |t|$.

We will now show that the lexicographic combination (λ, μ, ν, ξ) of these measures decreases with every rule application. First, an ASA step leaves λ invariant (since the introduced variable is fresh), but strictly decreases μ.

Moreover, an EM step does not increase λ, leaves μ and ν invariant, but decreases ξ.

For AM steps, we distinguish three cases:

(i) If the domain of the matcher τ contains a variable that appears more than once in the set of left hand sides of equations in P, then λ decreases as this variable is eliminated.

(ii) If $dom(\tau)$ is non-empty and each variable in $dom(\tau)$ occurs only once in a left hand side of an equation in P, then this occurrence must be the one in the term s (since $dom(\tau) = \mathcal{V}(s)$); thus, μ is non-increasing. Moreover, λ is invariant and ν is decreasing.

(iii) Finally, if $dom(\tau)$ is empty, i.e. τ is the identity substitution, then λ and ν are invariant, μ is non-increasing and ξ decreases. □

Theorem 32 (Soundness). *If $P; \kappa; Id \Longrightarrow^* \emptyset; \delta; \sigma$ is a derivation in* QCOMBINE*, then (σ, δ) solves P.*

Proof. It suffices to show that whenever (τ, ε) solves $\mathfrak{C}'$ and $\mathfrak{C} \Longrightarrow \mathfrak{C}'$ is an instance of a non-failing step of the derivation, (τ, ε) also solves $\mathfrak{C}$. For the first part of the definition of a solution of a configuration – the one stating that $\sigma \leqq_{Ex} \tau$ – this is clear, as all of the rules only instantiate σ or leave it unchanged.

The second part needs to be verified separately for each of the non-failure rules.

- In the case of an ASA step, (τ, ε) solves $\{\!\{t[x]_p \ll r,\ t|_p \ll x\}\!\} \cup P; \delta; \sigma$. Thus, there are degrees ζ, η, ϑ with $\zeta \otimes \eta \otimes \vartheta \succsim \varepsilon$ such that (τ, ζ) solves $P; \delta; \sigma$ and
$$\eta \Vdash t[x]_p\tau =_{E^\epsilon} r\tau \quad \text{and} \quad \vartheta \Vdash t|_p\tau =_{E^\epsilon} x\tau,$$
whence $\eta \otimes \vartheta \Vdash t\tau =_{E^\epsilon} r\tau$ follows, as x does not occur in t. Thus, (τ, ε) solves $\{\!\{t \ll r\}\!\} \uplus P; \delta; \sigma$.
- In the case of an EM step, (τ, ε) solves $P \cup \{\!\{x_1 \ll s_1, \ldots, x_m \ll s_m\}\!\}; \delta; \sigma$, so there are degrees $\zeta, \eta_1, \ldots, \eta_m$ with $\zeta \otimes \eta_1 \otimes \cdots \otimes \eta_m \succsim \varepsilon$ such that (τ, ζ) solves $P; \delta; \sigma$ and $\eta_i \Vdash x_i\tau =_{E^\epsilon} s_i\tau$ holds for $i = 1, \ldots, m$. Since t is linear by Lemma 29, this means that $\bigotimes_{i=1}^n \eta_i \Vdash t\tau =_{E^\epsilon} t\{x_1 \mapsto s_1, \ldots, x_m \mapsto s_m\}\tau$. As we also have $\kappa \Vdash t\{x_1 \mapsto s_1, \ldots, x_m \mapsto s_m\} =_{Ex} r$, we obtain $\bigotimes_{i=1}^n \eta_i \Vdash t\tau =_{E^\epsilon} r\tau$; thus, (τ, ε) solves $\{\!\{t \ll r\}\!\} \uplus P; \delta; \sigma$.
- In the case of AM, (τ, ε) solves $P\vartheta; \zeta \otimes \eta; \sigma\vartheta$. Writing $P = \{\!\{l_1 \ll r_1, \ldots, l_n \ll r_n\}\!\}$, this means that there exist $\varepsilon_1, \ldots, \varepsilon_n$ with $\varepsilon_1 \otimes \cdots \otimes \varepsilon_n \succsim \varepsilon$ such that $\varepsilon_i \Vdash l_i\vartheta\tau =_{E^\epsilon} r_i\vartheta\tau$ holds for $i = 1, \ldots, n$. Since both σ and ϑ are ground substitutions (cf. Lemma 29), $\sigma\vartheta \leqq_{Ex} \tau$ means that $\vartheta\tau =_{Ex} \tau$; thus, we have $\varepsilon_i \Vdash l_i\tau =_{E^\epsilon} r_i\tau$ for $i = 1, \ldots, n$. Since $\sigma\vartheta \leqq_{Ex} \tau$ implies $\sigma \leqq_{Ex} \tau$, we obtain that (τ, ε) solves $\{\!\{t \ll r\}\!\} \uplus P; \delta; \sigma$. □

Completeness. We now address the completeness of the combination algorithm. A central concept used in the completeness proof is the notion of commuting quantitative relations.

Definition 33 (Commuting relations). *Let R_1, R_2 be Ω-ternary relations. We say that R_1 and R_2* commute *if $R_1; R_2 = R_2; R_1$.*

Lemma 34. *If R_1 and R_2 commute, then so do R_1^* and R_2^*.*

Proof. It suffices to show that R_1^i and R_2^j commute for all $i, j \geqslant 0$. We proceed by induction on $n = \max(i, j)$. The base case $n = 1$ is satisfied by assumption. Assume that the statement holds up until m, and let i, j be given with $\max(i, j) = m + 1$. We may assume without loss of generality that $\min(i, j) \geqslant 1$. Since the composition of Ω-ternary relations is associative, we obtain
$$R_1^i; R_2^j = R_1^{i-1}; R_1; R_2; R_2^{j-1} = R_1^{i-1}; R_2; R_1; R_2^{j-1} =$$

$$= R_2; R_1^{i-1}; R_2^{j-1}; R_1 = R_2^i; R_1^j$$

using the induction hypothesis multiple times. □

For now, we will show completeness under the assumption that $=_{Ex}$ and $=_{App}$ commute. The question under which conditions this assumption can be granted will then be addressed at a later point.

First, we need to introduce some auxiliary notation.

Notation. Let E be some quantitative equational theory, let σ and τ be two substitutions, and let $\{x_1, \ldots, x_n\} = dom(\sigma) \cup dom(\tau)$ be the union of their domains. We write $\sigma =_{E,\varepsilon} \tau$ if there exist degrees $\varepsilon_1, \ldots, \varepsilon_n$ such that $\varepsilon_i \Vdash x_i\sigma =_E x_i\tau$ for each i and $\bigotimes_{i=1}^n \varepsilon_i \succsim \varepsilon$.

The next two results prepare the completeness proofs.

Lemma 35. *If $\varepsilon \Vdash t\tau =_E s$, where t is a linear term that does not contain any function symbol appearing in any axiom of E, then there exists a substitution τ' such that $s = t\tau'$ and $\tau =_{E,\varepsilon} \tau'$.*

Proof. We prove the equivalent statement obtained by replacing the assumption "$\varepsilon \Vdash t\tau =_E s$" by "$\varepsilon \Vdash t\tau =_E s$ or $\varepsilon \Vdash s =_E t\tau$" in the lemma. For short, we denote this assumption as $\varepsilon \Vdash t\tau \doteq s$. Moreover, note that it suffices to find a substitution τ'' satisfying $s = t\tau''$ and $\tau|_{\mathcal{V}(t)} =_{E,\varepsilon} \tau''|_{\mathcal{V}(t)}$, since we can extend it to the desired τ' by setting $\tau' \coloneqq \tau''|_{\mathcal{V}(t)} \cup \tau|_{\mathcal{V}\setminus\mathcal{V}(t)}$.

We proceed by induction on the length of the derivation of $\varepsilon \Vdash t\tau \doteq_E s$ in the rules for quantitative equational theories from Figure 2. For the induction step, we distinguish cases based on the last rule applied.

- If the last rule applied was $(\mathsf{Ax}^{\epsilon}_{=})$, then $\varepsilon \Vdash t\tau \approx s \in E$ (or $\varepsilon \Vdash s \approx t\tau \in E$). Since t does not contain any symbol appearing in E, this means that $t = x \in \mathcal{V}$ is a variable, and we can define $\tau' \coloneqq \{x \mapsto s\}$; then $s = t\tau'$ and $\tau =_{E,\varepsilon} \tau'$, as desired.
- If $(\mathsf{Refl}^{\epsilon}_{=})$ was the last rule applied, then $s = t\tau$, $\varepsilon = \kappa$, and we can set $\tau' = \tau$.
- In case of a $(\mathsf{Symm}^{\epsilon}_{=})$ step, say that $\varepsilon \Vdash t\tau = s$ was obtained from $\varepsilon \Vdash s = t\tau$; then we obtain the desired τ' by the induction hypothesis. The case where $\varepsilon \Vdash s = t\tau$ was obtained from $\varepsilon \Vdash t\tau = s$ is treated in the same way.
- In case of a $(\mathsf{Trans}^{\epsilon}_{=})$ step, say that $\varepsilon \Vdash t\tau =_E s$ was obtained from $\zeta \Vdash t\tau =_E u$ and $\eta \Vdash u =_E s$, where $\zeta \otimes \eta = \varepsilon$. Applying the induction hypothesis to $\zeta \Vdash t\tau =_E u$, we obtain a substitution ρ such that $u = t\rho$ and $\tau =_{E,\zeta} \rho$. Thus, we can apply inductiuon hypothesis to $\eta \Vdash t\rho = s$ to obtain τ' such that $t\tau' = s$ and $\rho =_{E,\eta} \tau'$, whence $\tau =_{E,\varepsilon} \tau'$, as desired. The symmetric case (where we have $\varepsilon \Vdash s =_E t\tau$) is treated similarly.
- If the last rule applied was $(\mathsf{NExp}^{\epsilon}_{=})$, then we can write $t\tau = f(t_1, \ldots, t_n)$, $s = f(s_1, \ldots, s_n)$, and $\varepsilon = \bigotimes_{i=1}^n \varepsilon_i$ with $\varepsilon_i \Vdash t_i \doteq s_i$ for $i = 1, \ldots, n$. If $t = x$ is a variable, then we can set $\tau' \coloneqq \{x \mapsto s\}$, as in the $(\mathsf{Ax}^{\epsilon}_{=})$ case. If t is not a variable, then we can write $tf(t'_1, \ldots, t'_n)$, where $t'_i\tau = t_i$ for

each i. Applying the induction hypothesis to $\varepsilon_i \Vdash t'_i\tau \doteq_E s_i$, we obtain τ'_i such that $t'_i\tau'_i = s_i$ and $\tau =_{E,\varepsilon_i} \tau'_i$. Since t is linear, the τ'_i can be assumed domain-disjoint, whence we can combine them to a substitution $\tau' = \bigcup_{i=1}^n \tau'_i$ satisfying $f(t'_1,\dots,t'_n)\tau' = f(s_1,\dots,s_n)$ and $\tau =_{E,\varepsilon} \tau'$, as desired.

– If $(\mathsf{Ord}^{\epsilon}_{=})$ was the last rule applied, then we apply the induction hypothesis to $\delta \Vdash t\tau \doteq_E s$ (where $\delta \succsim \varepsilon$) to obtain τ' with $t\tau' = s$ and $\tau =_{E,\delta} \tau'$, and the latter implies $\tau =_{E,\varepsilon} \tau'$.
– In case of a $(\mathsf{Join}^{\epsilon}_{=})$ step, say we have $\varepsilon = \bigvee_{i=1}^n \varepsilon_i$, and we can apply the induction hypothesis to $\varepsilon_i \Vdash t\tau \doteq_E s$ for each i. Thus, we obtain τ'_i such that $t\tau'_i = s$ and $\tau =_{E,\varepsilon_i} \tau'_i$ for each i. Since each τ'_i is a solution to the syntactic matching problem $s \ll t$ (and solutions to syntactic matching problems are unique), they all coincide, so we just write τ' for them. Then τ' satisfies $t\tau' = s$ and $\tau =_{E,\varepsilon_i} \tau'$ for $i = 1,\dots,n$. Writing $\mathit{dom}(\tau) \cup \mathit{dom}(\tau') =: \{x_1,\dots,x_N\}$, this means that for every j, there exists a degree ε_i^j such that $\varepsilon_i^j \Vdash x_j\tau =_E x_j\tau'$, where $\bigotimes_{j=1}^n \varepsilon_i^j = \varepsilon_i$, for each i. Thus, by $(\mathsf{Join}^{\epsilon}_{=})$, we have $\bigvee_{i=1}^n \varepsilon_i^j \Vdash x_j\tau =_E x_j\tau'$, whence $\tau =_{E,\bigotimes_{j=1}^N(\bigvee_{i=1}^n \varepsilon_i^j)} \tau'$.
Finally, note that

$$\bigotimes_{j=1}^N(\bigvee_{i=1}^n \varepsilon_i^j) = \bigvee_{1\leqslant i_1,\dots,i_N\leqslant n} \bigotimes_{j=1}^N \varepsilon_{i_j}^j \succsim \bigvee_{i=1}^n(\bigotimes_{j=1}^N \varepsilon_i^j) = \bigvee_{i=1}^n \varepsilon_i = \varepsilon$$

by distributivity; thus, $\tau =_{E,\varepsilon} \tau'$ follows. □

Lemma 36. *Let E_1, E_2 be disjoint, commuting quantitative equational theories. Moreover, let t and s be terms, where t is linear and E_1-pure and s is ground, and let τ be a substitution. If $\varepsilon \Vdash t\tau =_{E_1\cup E_2} s$, then there exist degrees ζ, η with $\zeta \otimes \eta = \varepsilon$ and a substitution τ' such that $\tau =_{E_2,\zeta} \tau'$ and $\eta \Vdash t\tau' =_{E_1} s$.*

Proof. As E_1 and E_2 commute, there exist degrees ζ, η with $\zeta \otimes \eta = \varepsilon$ and a term u such that $\zeta \Vdash t\tau =_{E_2} u$ and $\eta \Vdash u =_{E_1} s$. Now we apply Lemma 35 to write $u = t\tau'$, where $\tau =_{E_2,\zeta} \tau'$, as desired. □

We now proceed to prove completeness with respect to the individual QCombine rules, provided that *Ex* and *App* commute.

Lemma 37. *Suppose that Ex and App commute, and let τ be a substitution, $\varepsilon \succ \bot$ some degree. If EM can be applied to the configuration $\mathfrak{C}$ and (τ, ε) solves $\mathfrak{C}$, then there exists a step $\mathfrak{C} \Longrightarrow_{\mathsf{EM}} \mathfrak{C}'$ such that (τ, ε) solves $\mathfrak{C}'$.*

Proof. As in the definition of EM, we write $\mathfrak{C}$ as $P \uplus \{\!\{t \ll s\}\!\}; \delta; \sigma$, where t is a non-variable, *Ex*-pure term and s is ground.

Since (τ, ε) solves $\mathfrak{C}$, can write $\varepsilon = \beta \otimes \gamma$, where (τ, β) solves $P; \delta; \sigma$ and $\gamma \Vdash t\tau =_{E^\epsilon} s\tau$ (i.e. $\gamma \Vdash t\tau =_{E^\epsilon} s$ as s is ground). Thus, by Lemma 36, there exist a substitution τ' and degrees ζ, η with $\zeta \otimes \eta = \gamma$ such that $\tau =_{App,\zeta} \tau'$ and $\eta \Vdash t\tau' =_{Ex} s$.

Since Ex is crisp and $\eta \succsim \gamma \succsim \varepsilon \succ \bot$, we have $\kappa \Vdash t\tau' =_{Ex} s$, whence $\tau' \geqslant_{Ex,\mathcal{V}(t)} \rho$ for some $\rho \in \mathrm{CSM}_{Ex}(t, s)$. We may assume that ρ is ground (as s is ground) and $dom(\rho) = \mathcal{V}(t)$. We obtain the step

$$\mathfrak{C} \Longrightarrow_{\mathsf{EM}} P \cup \{\!\{x \ll x\rho \mid x \in dom(\rho)\}\!\}; \delta; \sigma.$$

We claim that (τ, ε) solves the obtained configuration. Write $dom(\rho) = \{x_1, \ldots, x_n\}$. Since $\tau =_{App,\zeta} \tau'$ and $dom(\rho) \subseteq dom(\tau) \cup dom(\tau')$, there exist $\zeta_1, \ldots, \zeta_n$ such that $\bigotimes_{i=1}^{n} \zeta_i \succsim \zeta$ and $\zeta_i \Vdash x_i\tau =_{App} x_i\tau'$ for $i = 1, \ldots, n$. Since $\tau' =_{Ex,\mathcal{V}(t)} \rho$, we obtain $\zeta_i \Vdash x_i\tau =_{E^\epsilon} x_i\rho$. As ρ is ground, we can also write $\zeta_i \Vdash x_i\tau =_{E^\epsilon} x_i\rho\tau$.

Thus, (τ, ζ) solves the configuration $\{\!\{x \ll x\rho \mid x \in dom(\rho)\}\!\}; \delta; \sigma$, yielding the claim. □

Lemma 38. *Suppose that Ex and App commute, and let τ be a substitution, $\varepsilon \succ \bot$ some degree. If (τ, ε) solves the configuration $P; \delta; \sigma$ to which an* AM *step can be applied, then there exist a step $P; \delta; \sigma \Longrightarrow_{\mathsf{AM}} P'; \delta'; \sigma'$, and a degree ζ such that $\zeta \otimes \delta' = \varepsilon \otimes \delta$ and (τ, ζ) solves $P'; \delta'; \sigma'$.*

Proof. Write $P = \{\!\{l_1 \ll r_1, \ldots, l_n \ll r_n\}\!\} \uplus \{\!\{t \ll r\}\!\}$, where t is App-pure and s is ground, as in the definition of AM. As (τ, ε) solves $P; \delta; \sigma$, there are degrees $\zeta, \eta_1, \ldots, \eta_n$ with $\zeta \otimes \eta_1 \otimes \cdots \otimes \eta_n \succsim \varepsilon$ such that $\zeta \Vdash t\tau =_{E^\epsilon} r\tau$ and $\eta_i \Vdash l_i\tau =_{E^\epsilon} r_i\tau$ for $i = 1, \ldots, n$. Since r is ground, we have $\zeta \Vdash t\tau =_{E^\epsilon} r$. By Lemma 36 (and using that Ex is crisp), there exists a substitution τ' such that $\zeta \Vdash t\tau' =_{App} r$ and $\tau =_{Ex} \tau'$. Thus, τ' is an App-matcher of t to r. Since r is ground and App is regular, any such matcher must be a ground substitution; consequently, we have $(\tau', \vartheta) \in \mathrm{CSM}_{App}(t, s)$ for some $\vartheta \succsim \zeta$. Thus, we obtain the step

$$P; \delta; \sigma \Longrightarrow_{\mathsf{AM}} \{\!\{l_1\tau' \ll r_1\tau', \ldots, l_n\tau' \ll r_n\tau'\}\!\}; \delta \otimes \vartheta; \sigma\tau'.$$

We claim that $(\tau, \eta_1 \otimes \cdots \otimes \eta_n)$ solves this configuration. First, we need to show that $\sigma\tau' \leqq_{Ex} \tau$. As σ is idempotent, it suffices to show that $\sigma\tau'\tau =_{Ex} \tau$ by Remark 24. Since τ' is also idempotent, $\tau' \leqslant_{Ex} \tau$ means $\tau'\tau =_{Ex} \tau$ (again by Remark 24). Hence, $\sigma\tau'\tau =_{Ex} \sigma\tau =_{Ex} \tau$ (as $\sigma \leqq_{Ex} \tau$). Moreover, $\tau'\tau =_{Ex} \tau$ also means that $\eta_i \Vdash l_i\tau'\tau =_{E^\epsilon} r_i\tau'\tau$ holds for $i = 1, \ldots, n$. Thus, $(\tau, \eta_1 \otimes \cdots \otimes \eta_n)$ solves the configuration $\{\!\{l_1\tau' \ll r_1\tau', \ldots, l_n\tau' \ll r_n\tau'\}\!\}; \delta \otimes \vartheta; \sigma\tau'$, and $\eta_1 \otimes \cdots \otimes \eta_n \otimes \delta \otimes \vartheta \succsim \varepsilon \otimes \delta$, as desired. □

Lemma 39. *If (τ, ε) solves a configuration $\mathfrak{C}$ and $\mathfrak{C} \Longrightarrow_{\mathsf{ASA}} \mathfrak{C}'$, then τ can be extended to a substitution $\tau' = \tau \cup \rho$ such that (τ', ε) solves $\mathfrak{C}'$.*

Proof. We can write $\mathfrak{C}$ as $\{\!\{t \ll r\}\!\} \uplus P; \delta; \sigma$, and $\mathfrak{C}'$ as $\{\!\{t[x]_p \ll r,\ \ t|_p \ll x\}\!\} \cup P; \delta; \sigma$, as in the definition of ASA. Setting $\tau' := \tau \cup \{\!\{x \mapsto (t|_p)\tau\}\!\}$, it is straight-forward to verify that τ' solves $\mathfrak{C}'$. □

The result below summarises Lemmas 37 to 39.

Corollary 40. *Suppose that Ex and App commute, and let τ be a substitution, $\varepsilon \succ \bot$ some degree. If (τ, ε) solves the non-terminal configuration $\mathfrak{C}$ given by $P; \delta; \sigma$, then there exist a step $\mathfrak{C} \Longrightarrow \mathfrak{C}' = P'; \delta; \sigma$, a degree ε' and an extension $\tau' = \tau \cup \rho$ such that (τ', ε') solves $\mathfrak{C}'$ and $\varepsilon \otimes \delta \precsim \varepsilon' \otimes \delta'$.*

The completeness result below is obtained by combining Corollary 40 and Lemma 30.

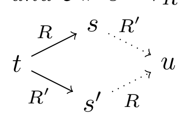
Theorem 41 (Completeness). *Let $E^{\epsilon} = App \cup Ex$ be regular, suppose that $=_{App}$ and $=_{Ex}$ commute, and let P be a left-linear, well-moded E^{ϵ}-matching problem. If (τ, ε) is a solution for P, then there exists a derivation $P; \kappa; Id \Longrightarrow^* \emptyset; \delta; \sigma$ in* QCombine *such that $\delta \succsim \varepsilon$ and $\sigma|_{\mathcal{V}(P)} \leqq_{Ex} \tau|_{\mathcal{V}(P)}$.*

This concludes the completeness proof under the assumption that *Ex* and *App* are disjoint and commute. We now proceed to investigate syntactic conditions that grant their commutativity.

Definition 42. *Two terms t and s are said to* overlap *if there exist non-variable positions $p \in NPos(t), q \in NPos(s)$ and substitutions σ, τ such that $(t|_p)\sigma = (s|_q)\tau$.*

The next two results, which are inspired by a similar result from [44], gives a syntactic condition for the commutativity of two quantitative equational relations.

Lemma 43 (Analogous to Prop. 10 in [44]). *If R and R' are two linear, regular, and collapse-free relations such that no overlap occurs between a left-hand side of a rule from R and a left-hand side of a rule from R', then the following property is satisfied: If $\varepsilon \Vdash t \to_R s$ and $\delta \Vdash t \to_{R'} s'$, then there exists a term u such that $\delta \Vdash s \to_{R'} u$ and $\varepsilon \Vdash s' \to_R u$.*



Proof. By definition, there are positions $p, q \in NPos(t)$, substitutions σ, τ, and rules $\varepsilon' \Vdash l \to r \in R$, $\delta' \Vdash l' \to r' \in R'$ with $\varepsilon' \succsim \varepsilon$ and $\delta' \succsim \delta$ such that $t|_p = l\sigma$ and $s = t[r\sigma]_p$ as well as $t|_q = l'\tau$ and $s' = t[r'\tau]_q$.

If p and q are parallel positions (i.e. none is a subposition of the other), then we have $s|_q = l'\tau$ and $s'|_p = l\sigma$, whence we obtain $\delta' \Vdash s \to_R u$ and $\varepsilon' \Vdash s' \to_{R'} u$ for $u = t[r\sigma]_p[r'\tau]_q$.

If p and q are not orthogonal, we first consider the case where p is a subposition of q and write $p = q.p'$. We have $t|_q = l'\tau$ and $(l'|_{p'})\tau = (l'\tau)|_{p'} = t|_p = l\sigma$. Since l and l' do not overlap, this means that either l or $l'|_{p'}$ is a variable; therefore, $l'|_{p'} = x$ is a variable (as $l \notin \mathcal{V}$ by collapse-freeness). Thus, $x\tau = l\sigma$. Since $s|_q = (t[r\sigma]_p)|_q = (t|_q)[r\sigma]_{p'} = (l'\tau)[r\sigma]_{p'}$. Since the only occurrence of x in l' is in the position p', we can write $(l'\tau)[r\sigma]_{p'} = l'\tau'$, where τ' is the substitution defined by $x\tau' := r\sigma$ and $y\tau' := y\tau$ for any $y \in \mathcal{V} \setminus \{x\}$. Thus, we obtain the step $\delta' \Vdash s \to_{R'} t[r'\tau']_q$. It remains to show that there is also a step

$\varepsilon' \Vdash s' \to_R t[r'\tau']_q$. By regularity and linearity, there is a unique occurrence of x in r'; say in the position q'. Then $r'|_{q'} = x$, so $(r'\tau')|_{q'} = l\sigma$; thus, we obtain a step $\varepsilon' \Vdash t[r'\tau]_q \to_R t[r'\tau[r\sigma]'_q]_q$. Now note that we have $r'\tau[r\sigma]q' = r'\tau'$ (again as x only occurs in position q' of r'), whence $t[r'\tau[r\sigma]'_q]_q = t[r'\tau']_q$, as desired.

The case where q is a subposition of p is treated similarly. □

Corollary 44 (Analogous to Cor. 3 in [44]). *Suppose that R_1 and R_2 are regular and linear, and that no right hand side of a rule in R_1 overlaps the left hand side of a rule in R_2. Then $\to_{R_1}$ commutes over $\to_{R_2}$.*

Proof. Apply Lemma 43 with $R = R_1^{-1}$ and $R' = R_2$. □

Proposition 45. *If E_1 and E_2 are regular, linear, collapse-free, and disjoint theories over a totally ordered quantale, then $=_{E_1}$ and $=_{E_2}$ commute.*

Proof. Since E_1 and E_2 are disjoint and collapse-free, there is no overlap between a side of an equation in E_1 and a side of an equation in E_2. Thus, $\leftrightarrow_{E_1}$ and $\leftrightarrow_{E_2}$ commute by Corollary 44. As a consequence, also $\leftrightarrow^*_{E_1}$ and $\leftrightarrow^*_{E_2}$ commute by Lemma 34.

Now to show that $=_{E_1}$ and $=_{E_2}$ commute, assume that $\varepsilon \Vdash t =_{E_1} s$ and $\delta \Vdash s =_{E_2} u$; then by Corollary 3 we have $\varepsilon' \Vdash t \leftrightarrow^*_{E_1} s$ and $\delta' \Vdash s \leftrightarrow^*_{E_2} u$ for some degrees $\varepsilon' \gtrsim \varepsilon$ and $\delta' \gtrsim \delta$. As $\leftrightarrow^*_{E_1}$ and $\leftrightarrow^*_{E_2}$ commute, there exist degrees ζ and η as well as a term r such that $\zeta \Vdash t \leftrightarrow^*_{E_2} r$ and $\eta \Vdash r \leftrightarrow^*_{E_1} s$ with $\zeta \otimes \eta \gtrsim \varepsilon' \gtrsim \delta' \gtrsim \varepsilon \otimes \delta$. Thus, $\zeta \Vdash t =_{E_2} r$ and $\eta \Vdash r =_{E_1} s$, concluding the proof. □

Corollary 46. *If Ex and App are signature-disjoint, linear, regular, collapse-free theories over a totally ordered quantale, each admitting a complete matching algorithm, then* QCombine *is a complete E^ϵ-matching algorithm (in the sense of Theorem 41).*

7.2 Quantitative Matching over Primitive Theories

We now present an *App*-matching algorithm for a specific class of theories, using a restricted version of the calculus for quantitative unification from [20]. We assume that all axioms of *App* have the form

$$\varepsilon \Vdash f(x_1, \ldots, x_n) \approx g(x_1, \ldots, x_n),$$

where f and g are n-ary function symbols and $x_1, \ldots, x_n$ are variables. We refer to such theories as *primitive.*

In a primitive theory, it makes sense to speak about the *approximation degree* of two function symbols, which is defined as

$$\mathfrak{d}_{App}(f, g) := \bigvee \{\varepsilon \mid \varepsilon \Vdash f(x_1, \ldots, x_n) =_{App} g(x_1, \ldots, x_m)\}$$

for an n-ary f and m-ary g. (Obviously, $\mathfrak{d}_{App}(f, g) = \bot$ if $n \neq m$.)

Remark 47. Since the equational theories we consider here are finitely presented, the degree of two function symbols of the same arity can be effectively computed as

$$\mathfrak{d}_{App}(f,g) = \bigvee\{\gamma \mid \gamma \Vdash f(x_1,\ldots,x_n) = g(x_1,\ldots,x_n) \in C\},$$

where C is the closure of the presentation of App under the (Trans) and (Symm) rules.

A calculus QUnif for unification over primitive theories has been presented in [20]. We present an adapted version for matching, which we denote by QMatch.

The configurations we work with have the same form as the configurations of QCombine. The matching rules are presented in Figure 6. We refer to this

Dec: **Decompose**

$$\{\!\{f(t_1,\ldots,t_n) \ll g(s_1,\ldots,s_n)\}\!\} \uplus P; \delta; \sigma \Longrightarrow$$
$$\{\!\{t_1 \ll s_1,\ldots,t_n \ll s_n\}\!\} \cup P; \delta \otimes \mathfrak{d}_{App}(f,g); \sigma$$

where $\delta \otimes \mathfrak{d}_{App}(f,g) \succsim \varepsilon$.

Cla: **Clash**

$$\{\!\{f(t_1,\ldots,t_n) \ll g(s_1,\ldots,s_m)\}\!\} \uplus P; \delta; \sigma \Longrightarrow \mathbf{F},$$

if $\delta \otimes \mathfrak{d}_{App}(f,g) \not\succsim \varepsilon$.

L-Sub: **Substitute (lazy)**

$$\{\!\{x \ll f(s_1,\ldots,s_n)\}\!\} \uplus P; \delta; \sigma \Longrightarrow$$
$$\{\!\{x_1 \ll s_1,\ldots,x_n \ll s_n\}\!\} \cup P\rho; \delta \otimes \mathfrak{d}_{App}(f,g); \sigma\rho,$$

where $\rho = \{x \mapsto g(x_1,\ldots,x_n)\}$ with $x_1,\ldots,x_n$ being fresh variables and $\delta \otimes \mathfrak{d}_{App}(f,g) \succsim \varepsilon$.

Fig. 6: QMatch rules

calculus by QMatch.

Theorem 48. *Let App be a primitive quantitative equational theory. The calculus* QMatch *constitutes a terminating, sound and complete App-matching procedure.*

Here, by "complete", we mean the following property:

Completeness: *If s is a ground term and $\varepsilon \Vdash t\tau =_{App} s$ holds for some $\varepsilon \succ \bot$, then* QMatch *admits a derivation*

$$\{\!\{t \ll s\}\!\}; \kappa; Id \Longrightarrow^* \emptyset; \delta; \sigma$$

such that $\sigma|_{\mathcal{V}(t)} = \tau|_{\mathcal{V}(t)}$ and $\delta \succsim \varepsilon$.

Proof. Soundness and termination follow from the corresponding properties of QUnif [20].

For completeness, using the results from [20], we obtain that QUnif admits a derivation

$$\{t \ll s\}; \kappa; Id \Longrightarrow^* \emptyset; \sigma; \delta$$

such that $\sigma|_{\mathcal{V}(t)} \ll \tau|_{\mathcal{V}(t)}$ and $\delta \succsim \varepsilon$. Since App is simple and $\varepsilon \Vdash t\tau =_{App} s$, we must have $\mathcal{V}(t\sigma) = \mathcal{V}(s)$, so $t\tau$ must be ground. By soundness, we also have $\delta \vdash t\sigma =_{App} s$, so $t\sigma$ is ground as well. Thus, $\sigma|_{\mathcal{V}(t)} \ll \tau|_{\mathcal{V}(t)}$ implies that $\sigma|_{\mathcal{V}(t)} = \tau|_{\mathcal{V}(t)}$. So QUnif is complete in the above sense; it remains to be shown that the same holds for the restricted calculus QMatch.

To do so, it suffices to show that for any derivation $\{t \ll s\}; \kappa; Id \Longrightarrow^* \mathfrak{C}$ in QUnif, there is a derivation $\{t \ll s\}; \kappa; Id \Longrightarrow^* \mathfrak{C}$ using only Dec, Cla, and L-Sub. To this end, suppose that $P; \delta; \sigma \Longrightarrow \mathfrak{C}'$ is a Tri, CCh, or Ori step in the aforementioned derivation in QUnif.

Since s is ground and right-groundness is preserved under Dec and Cla, P is a set of right-ground equations. Thus, the step in question cannot be an application of CCh or Ori. If it is a Tri step, then we may write $P = P_0 \cup \{t \ll t\}$, where t is a ground term. Then there is a derivation $P; \delta; \sigma \Longrightarrow^*_{\mathsf{Dec}} \mathfrak{C}$ using only Dec steps. Thus, we obtain a derivation $\{t \ll s\}; \kappa; Id \Longrightarrow^* \mathfrak{C}$ in QMatch, as desired. □

In this setting, we also have good control over the length of derivations.

Proposition 49. *If s is a ground term, then any derivation in* QMatch *starting with the initial configuration $\{\!\{t \ll s\}\!\}; Id; \kappa$ terminates after at most $|s|$ steps.*

Proof. For a given configuration $\{\!\{l_1 \ll r_1, \dots, l_n \ll r_n\}\!\}; \delta; \sigma$, consider the combined size $|r_1| + \cdots + |r_n|$ of the right hand sides, and observe that this quantity decreases under every Dec and L-Sub application. □

8 Related Work

Since the notion of quantitative equational reasoning (resp. quantitative rewriting) includes as a particular case the standard, non-quantitative version (see Remark 8), it is clear that some of the relations shown in Table 2 correspond to well-known relations. For example, if we consider an empty set of quantitative rewrite rules with a trivial notion of distance, i.e., an equational theory generated by an empty set of equations, then the quantitative relation $\rightarrow_{R^\epsilon/E^\epsilon}$ is simply the standard notion of syntactic equality of terms. More interestingly, if R^ϵ is empty and E^ϵ consists only of *crisp* equations, we obtain the standard notion of equational reasoning whereas if R^ϵ contains only *crisp* rules and E^ϵ is empty, we obtain the standard notion of term rewriting. If R^ϵ is a set of quantitative rewrite rules and E^ϵ is empty we obtain the notion of quantitative term rewriting from [25]; fuzzy term rewriting [33] can be seen as a particular case.

The notion of weighted term rewriting [3] is also a ternary relation but the degrees are not taken from a quantale, instead they are elements of a set with a monoidal structure and a partial order. The weighted rewriting relation is non-expansive by definition. A notion of weighted rewriting can be obtained as a particular case of quantitative rewriting by relaxing the conditions on the quantale. To our knowledge, no notion of *equational* weighted rewriting has been defined in prior work.

Starting with Peterson & Stickel's seminal paper [43] there has been extensive work on rewriting modulo equational theories (also called equational rewriting, or rewriting modulo), see for example [8,27,30,38,39]. If both R and E are crisp, $\to_{R^\epsilon/E^\epsilon}$ corresponds to the standard notion of equational rewriting, which is the foundation of programming languages such as Maude [39]. More recently, Real-Time Maude [41] and model checkers such as Prism [32] have been developed, which are based on notions of quantitative rewriting modulo crisp theories, i.e., the relation denoted $\to_{R^\epsilon/E}$ in Table 2.

Fuzzy rewriting [18], which has application in systems working on partial information, is a notion of rewriting where the matching algorithm uses a fuzzy (or approximate) notion equality. In other words, fuzzy rewriting is standard rewriting using fuzzy matching. Since approximate equality is defined via a notion of proximity degree (i.e., a term matches a pattern if it is in the same proximity class as an instance of the pattern [18]), which is a particular case of quantitative equality, fuzzy rewriting (when restricted to similarity relations) is a particular case of $\to_{R/E^\epsilon}$. More generally, the notion of rewriting modulo quantitative equational theories $\to_{R/E^\epsilon}$ obtained as a particular case of $\to_{R^\epsilon/E^\epsilon}$ could be used to provide a semantics in settings where the notion of approximation involves equational reasoning.

To the best of our knowledge, no definitions of quantitative rewriting with fuzzy matching exist in prior work. Such a notion is a particular case of $\to_{R^\epsilon/E^\epsilon}$ that could be used as a basis for the definition of probabilistic or real time versions of fuzzy programming languages.

The notion of quantitative rewriting modulo quantitative equational theories $\to_{R^\epsilon/E^\epsilon}$, which is the subject of this paper, subsumes all the other relations mentioned in Table 2. It could be used to provide a semantics to computations where both the notion of equality and the notion of rewriting require quantification, e.g., to specify computations in neural networks or to specify notions of protocol analysis that take resources into account.

9 Concluding Remarks

In this work, we introduced a quantitative extension of rewriting logic, where both computation and deduction explicitly account for quantitative information. We defined a system of inference rules for quantitative rewrite sentences and proved its equivalence to the reflexive-transitive closure of quantitative rewriting modulo a quantitative equational theory (the $\to_{R^\epsilon/E^\epsilon}$ relation). We also presented an operational counterpart (the $\to_{R^\epsilon,E^\epsilon}$ relation), based on quantitative matching, and showed that it accurately models provability under the property of strict coherence. This establishes quantitative analogues of classical results while providing a framework for reasoning about quantitative system behaviour. We further investigated the underlying matching problem in the context of a combination of exact and strictly quantitative components.

There are several promising directions for future work. A natural next step is to consider fully general rewrite theories, as in the classical setting allowing sorts [37], conditional rules, and parallelism. Integrating our framework with real-time rewrite frameworks such as Real-Time Maude [42] could enable modelling of probabilistic timed systems, cost-aware scheduling, and other time-sensitive quantitative features. Finally, relaxing the non-expansiveness requirement and adopting a graded quantitative rewriting approach [25], in which the effect of a rewrite may depend on its context, would further enhance expressivity, making the framework suitable for systems whose behaviour is both quantitative and context-dependent.

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