ASKEY, HERON, AND COMPUTER ALGEBRA

PETER PAULE
RISC, JOHANNES KEPLER UNIVERSITY LINZ, AND
CENTER FOR APPLIED MATHEMATICS, TIANJIN UNIVERSITY

In memory of, and gratitude to, Dick Askey

1. Introduction

It was always a pleasure to exchange with Dick, on mathematical and also non-math topics. Usually there was some message to take away; at various occasions he succeeded to surprise me.

Years ago, when working in my office in the RISC castle, my secretary forwarded a phone call to me, "an American wants to speak to you urgently." It was Dick. He just finished reading an article dealing with didactical aspects of math education. Some statements made by the author, an Austrian, upset him so much that he decided to call me instantly. "What is going on with math education in Austria?" was his introductory sentence, and some more excitement followed. I was not following any kind of such discussions in Austria, so it took me a while to understand the background of his anger. It also took me a while to calm him down.

The reasons of Dick's worry, which I could perfectly well understand, were related to "strange" proposals the author made on the usage of computer algebra systems in school. Dick was interested in computer algebra and was following developments there. So he could very well understand how problematic the arguments in this article were.

Of special interest to Dick were applications of computer algebra in mathematical research. For example, he appreciated Pillwein's work in special functions combining traditional with symbolic methods; an example is her proof of a conjecture of Schöberl concerning the positivity of sums over ultraspherical Jacobi polynomials [7]. Discussions with Dick and members of my group at the occasion of his RISC visit in August 2011 are unforgettable.

Another occasion where Dick surprised me was related to elementary geometry. I was sitting with him on a bus; this was at the occasion of some conference years ago. Actually, I forgot which one; I can only guess that this happened before his 2010 article [1] was published. Until then, I connected Dick mainly with the areas

Date: December 31, 2024, revised January 10, 2025.

of classical analysis and special functions. So it came very much as a surprise to me when he started to tell me about his interests in elementary geometry. And later, the same or the next day, when Dick gave his conference talk (which was announced without title in the program) my surprise was complete: it was devoted to aspects of Ptolemy's theorem!

Apart from being surprised, I was very pleased by Dick's geometry interest: a topic I learned to like from my early student days on. I told him that from time to time I do "geometry proving" just for fun. (I.e., trying to prove elementary geometry theorems from first principles.) I also told him about recent developments of using computer algebra for automated geometric theorem proving. And, I promised to show him a concrete example, something I came up with when preparing a talk illustrating the virtues of symbolic computation: a computer-assisted derivation of Heron's theorem using the Mathematica system. But as life goes, this never happened! It is the objective of this note to finally present this computer demonstration—announced to Dick many years ago.

Section 2 presents such an automated computer algebra derivation of Heron's formula (1); i.e., a derivation "without thinking." Some remarks on history and Gröbner bases background are added. In addition, with help of computer algebra the law of cosines (2) is derived, again "without thinking." Section 3 presents an automated derivation of Ptolemy's formula (3); it can be viewed as a computer algebra version of the proof presented in Askey's [1]. Moreover, connections between Ptolemy's formula and Heron's are drawn, including a famous formula by Brahmagupta (7) and its extension (6) by Coolidge. Open problems for interested readers are stated.

The note concludes with expressing thanks to Dick Askey.

2. Heron

2.1. **Automatic derivation of Heron's formula.** To derive Heron's formula for the area of a triangle by using the Mathematica system, consider a general triangle as in Figure 1.

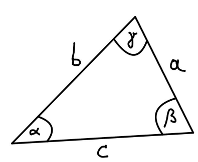

Figure 1.

By h_a, h_b, and h_c we denote the altitudes on the sides a, b, and c, respectively. As input we only use basic knowledge such as the geometric definition of the sine function,

$$\sin(\alpha) = \frac{h_c}{b},$$

the angle addition formula for sine,

$$\sin(\gamma) = \sin(\pi - \alpha - \beta) = \sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

and the Pythagorean identity involving sin and cos:

In[1]:= $\mathbf{Eliminate[\{F == \frac{c \cdot h_c}{2}, F == \frac{b \cdot h_b}{2}, F == \frac{a \cdot h_a}{2},}$
$\mathbf{h_c == b\,sin[\alpha], h_b == a\,sin[\gamma], h_a == c\,sin[\beta],}$
$\mathbf{sin[\gamma] == sin[\alpha]cos[\beta] + cos[\alpha]sin[\beta],}$
$\mathbf{sin[\alpha]^2 + cos[\alpha]^2 == 1, sin[\beta]^2 + cos[\beta]^2 == 1\},}$
$\mathbf{\{h_a, h_b, h_c, sin[\alpha], sin[\beta], sin[\gamma], cos[\alpha], cos[\beta]\}]}$

Out[1]= $(a^4 - 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4)F + 16F^3 = 0$

The Mathematica command `Eliminate` does the following: It takes the given list

$$\{F = \frac{c \cdot h_c}{2}, \ldots, \texttt{sin[}\beta\texttt{]}^2 + \texttt{cos[}\beta\texttt{]}^2 = 1\}$$

as an input system of equations. Note that Mathematica interprets the expressions `sin[`α`]`, etc., as variables; in contrast, the built-in Mathematica sine function would be written as `Sin[`α`]`. From this system of equations, the procedure `Eliminate` tries to derive algebraic relations being free of all the variables from the second input list as specified in `In[1]`,

$$\{h_a, \ldots, \texttt{cos[}\beta\texttt{]}\}.$$

In our case `Eliminate` finds one such equation, `Out[1]`. Solving it for F and factoring the non-trivial positive solution gives Heron's formula:

In[2]:= $\mathbf{Solve[(a^4 - 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4)F + 16F^3 == 0, F]}$

Out[2]= $\{\{F \to 0\}, \{F \to -(1/4)\mathrm{Sqrt}[-a^4 + 2a^2b^2 - b^4 + 2a^2c^2 + 2b^2c^2 - c^4]\},$
$\{F \to 1/4\,\mathrm{Sqrt}[-a^4 + 2a^2b^2 - b^4 + 2a^2c^2 + 2b^2c^2 - c^4]\}\}$

In[3]:= $\mathbf{Factor[-a^4 + 2a^2b^2 - b^4 + 2a^2c^2 + 2b^2c^2 - c^4]}$

Out[3]= $(-a + b + c)(a - b + c)(a + b - c)(a + b + c)$

Summarizing, this way one can derive Heron's formula, often written in the form,

$$F = \sqrt{(s-a)(s-b)(s-c)s} \text{ where } s = \frac{a+b+c}{2}. \tag{1}$$

without any thought! We stress the fact that to this end, only basic trigonometry was used.

Remark 1. *To our knowledge there are (still) gaps in the history of Heron's formula. For example, Coxeter and Greitzer* [5, p. 59]*: "Although this is named after Heron of Alexandria (about 60 A.D.), van der Waerden [...] supports Bell [...] in attributing it to Archimedes (third century B.C.)."*

Remark 2. *The algorithm underlying Mathematica's* ***Eliminate*** *is Buchberger's Gröbner basis algorithm* [3]. *Informally, for the ideal generated by the polynomials as given in input* ***In[1]****, this algorithm computes a very special basis. This basis, the Gröbner basis, is computed with respect to a certain term order induced by the list of variables specified in* ***In[1]****. In Mathematica these basis elements are obtained as follows:*

In[4]:= $\mathbf{GroebnerBasis[\{F == \frac{c \cdot h_c}{2}, F == \frac{b \cdot h_b}{2}, F == \frac{a \cdot h_a}{2},}$
$\mathbf{h_c == b\sin[\alpha], h_b == a\sin[\gamma], h_a == c\sin[\beta],}$
$\mathbf{\sin[\gamma] == \sin[\alpha]\cos[\beta] + \cos[\alpha]\sin[\beta],}$
$\mathbf{\sin[\alpha]^2 + \cos[\alpha]^2 == 1, \sin[\beta]^2 + \cos[\beta]^2 == 1\},}$
$\mathbf{\{h_a, h_b, h_c, \sin[\alpha], \sin[\beta], \sin[\gamma], \cos[\alpha], \cos[\beta]\}]}$

Out[4]= $\{a^4F - 2a^2b^2F + b^4F - 2a^2c^2F - 2b^2c^2F + c^4F + 16F^3, -a^2F + b^2F - c^2F + 2ac\,F\cos[\beta],$
$a^4c - a^2b^2c + 4c\,F^2 - 8aF^2\cos[\beta] - a^4c\cos[\beta]^2 + a^2b^2c\cos[\beta]^2, \cdots,$
$-1 + \cos[\alpha]^2 + \sin[\alpha]^2, -b\sin[\alpha] + h_c, -a\sin[\gamma] + h_b, -c\sin[\beta] + h_a\}$

In full length the output consists of 44 *polynomials; we only display the first and the last three of those. Note that the first entry of the list is the desired Heron polynomial* ***Out[1]*** *in which none of the elimination variables occurs. Already the second entry contains* ***cos[β]****, in addition to F. Each of the following polynomials may contain one or more elimination variables as the preceding one. It is this kind of "stair-case" feature of Gröbner bases which is used in procedures such as* ***Eliminate****.*

2.2. **The cosine formula by a simple variation.** We want to note that the law of cosines, in the notation of Figure 1,

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha), \tag{2}$$

can be obtained by a simple variation. Namely, one uses the same input as before but with one exception: instead of asking to eliminate the variable cos[α], standing for the cosine function in α, we ask to eliminate the area variable F:

In[5]:= $\mathbf{Eliminate[\{F == \frac{c \cdot h_c}{2}, F == \frac{b \cdot h_b}{2}, F == \frac{a \cdot h_a}{2},}$
$\mathbf{h_c == b\sin[\alpha], h_b == a\sin[\gamma], h_a == c\sin[\beta],}$
$\mathbf{\sin[\gamma] == \sin[\alpha]\cos[\beta] + \cos[\alpha]\sin[\beta],}$
$\mathbf{\sin[\alpha]^2 + \cos[\alpha]^2 == 1, \sin[\beta]^2 + \cos[\beta]^2 == 1\},}$
$\mathbf{\{h_a, h_b, h_c, \sin[\alpha], \sin[\beta], \sin[\gamma], F, \cos[\beta]\}]}$

Out[5]= $b^2c^2\cos[\alpha](2 - 2\cos[\alpha]^2) + bc^3(-1 + \cos[\alpha]^2) + bc(a^2 - b^2 - a^2\cos[\alpha]^2 + b^2\cos[\alpha]^2) = 0$

Again the result of the elimination is only one equation. Solving this polynomial expression with respect to cos[α] gives:

In[6]:= $\mathbf{Solve[b^2c^2\cos[\alpha](2 - 2\cos[\alpha]^2) + bc^3(-1 + \cos[\alpha]^2) +}$
$\mathbf{bc(a^2 - b^2 - a^2\cos[\alpha]^2 + b^2\cos[\alpha]^2) == 0, \cos[\alpha]]}$

Out[6]= $\{\{\cos[\alpha] \to -1\}, \{\cos[\alpha] \to 1\}, \{\cos[\alpha] \to \frac{-a^2 + b^2 + c^2}{2bc}\}\}$

The non-trivial solution is nothing but the cosine formula (2).

We want to stress that the computation timings of all these Mathematica commands are in the scale of fractions of a second on a standard laptop. For elimination problems involving larger systems with significant more variables, computation time can be an issue owing to the complexity of Buchberger's algorithm.

3. Ptolemy

3.1. **Automatic derivation of Ptolemy's formula.** Denote by a, b, c, d the sides and by x, y the diagonals of a quadrilateral inscribed into a circle, in short: a cyclic quadrilateral; see Figure 2.

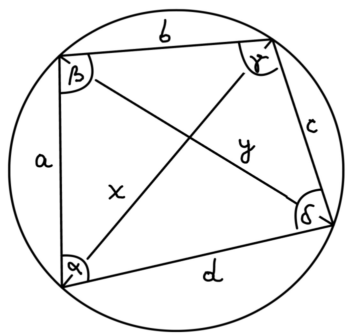

Figure 2.

To such cyclic quadrilaterals the formula of Ptolemy applies,

$$xy = a\,c + b\,d. \tag{3}$$

We will derive Ptolemy's formula by using the Mathematica system. To this end, as input we again use only basic knowledge such as the law of cosines (2) which we implement into Mathematica,

In[7]:= **CosLaw[a,b,c,γ] := c² − (a² + b² − 2ab cos[γ])**

and the fact that opposite angles of cyclic quadrilaterals sum to 180 degrees; for example,

$$\cos(\delta) = \cos(\pi - \beta) = -\cos(\beta).$$

First, by elimination of the variables `cos[β]` and `cos[δ]` we derive a formula for the diagonal x:

In[8]:= **Eliminate[{CosLaw[a, b, x, β] == 0, CosLaw[c, d, x, δ] == 0,
cos[δ] == −cos[β]}, {cos[β], cos[δ]}]**

Out[8]= $abc^2 + cd(a^2 + b^2 - x^2) = ab(-d^2 + x^2)$

In[9]:= **Solve**$[abc^2 + cd(a^2 + b^2 - x^2) == ab(-d^2 + x^2), x]$

Out[9]= $\left\{\left\{x \to -\frac{\sqrt{a^2cd + abc^2 + abd^2 + b^2cd}}{\sqrt{ab + cd}}\right\}, \left\{x \to \frac{\sqrt{a^2cd + abc^2 + abd^2 + b^2cd}}{\sqrt{ab + cd}}\right\}\right\}$

In[10]:= $\frac{a^2cd + abc^2 + abd^2 + b^2cd}{ab + cd}$ //**Factor**

Out[10]= $\frac{(ad + bc)(ac + bd)}{ab + cd}$

This means, we obtained for the first diagonal,

$$x^2 = \frac{(ad + bc)(ac + bd)}{ab + cd}. \tag{4}$$

By symmetry, or by analogous computation, we have for the second diagonal,

$$y^2 = \frac{(ac + bd)(ab + cd)}{ad + bc}. \tag{5}$$

Finally, combining (4) and (5) results in Ptolemy's formula (3).

Remark 3.1. *This derivation can be considered as a computerized version of the proof given in Askey's article* [1, Sec. 2]*. A beautiful proof "without words" is given in* [6]*.*

3.2. **Connecting Ptolemy's formula with Heron's.** Let a, b, c, d denote the sides of a general quadrilateral (i.e., not necessarily inscribed into a circle) with diagonals x and y. Under this assumption Coolidge [4] proved for its area F,

$$F = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{4}\left((a\,c + b\,d)^2 - x^2y^2\right)} \tag{6}$$

where

$$s = \frac{a + b + c + d}{2}.$$

In case this quadrilateral is a cyclic one, Ptolemy's formula (3) applies and one has

$$F = \sqrt{(s-a)(s-b)(s-c)(s-d)} \quad \text{where} \quad s = \frac{a + b + c + d}{2}. \tag{7}$$

This relation was discovered in the seventh century A.D. by the Hindu mathematician Brahmagupta; for a classical proof see, e.g., [5, Thm. 3.22]. For $d = 0$ the cyclic quadrilateral degenerates to a triangle and we have Heron's formula (1).

In 1939 Coolidge wrote [4, p. 345–346]: "The amount of information available in the literature of mathematics bearing on the quadrilateral [. . .] is discouragingly large. Yet there seems to be no part of the science so far from exhaustion as elementary geometry. It has seemed to me that there must be connecting links between different known formulas which were worth investigating, not only for their own sake but also for historical reasons."

Dick Askey's note [1] is a remarkable confirmation of this statement. Besides its mathematical content, concerning the history of the formulas (4) and (5), Bogomolny on his marvelous web site wrote [2]: "Most of the sources attribute this result to the great 9th century Indian mathematician Mahavira (or Mahaviracharya, meaning Mahavira the Teacher). However, according to Richard Askey [1] with a reference to Henry Thomas Colebrooke the formulas have been known to another great Indian mathematician Brahmagupta already in the 7th century."

Concerning the mathematical aspect of Coolidge's statement, computer algebra developments such as Buchberger's Gröbner bases brought another new wind into the area. We want to conclude this section by challenging the interested reader with two problems: Find a computer algebra derivation of (7) "without thinking"; i.e., in the spirit of our derivation of Heron's formula (1). Even more challenging, find such a proof for Coolidge's extension (6) of Brahmagupta's formula.

4. Conclusion

I want to conclude this note by expressing my sincerest thanks to Dick Askey for inspiration and his strong support at various occasions. For example, to arrange with manifold other obligations, he took the pain of going transatlantic from Madison to Austria for two days only: he wanted to be personally present at an important decision meeting of the Austrian Science Foundation, deciding about RISC funding! Thanks go also to Howard Cohl for his careful reading of the manuscript.

References

[1] R. Askey. *Completing Brahmagupta's extension of Ptolemy's theorem*, pages 191–197 in: *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*, K. Alladi et al. (eds.), Springer, 2010.

[2] A. Bogomolny. *Brahmagupta-Mahavira Identities*, `https://www.cut-the-knot.org/proofs/PtolemyDiagonals.shtml`.

[3] B. Buchberger. *Theoretical basis for the reduction of polynomials to canonical forms*, SIGSAM Bull. 39 (1976), 19–24.

[4] J.L. Coolidge. *A historically interesting formula for the area of a quadrilateral*, Amer. Math. Monthly 46:6 (1939), 345–347.

[5] H.S.M. Coxeter and S.L. Greitzer. *Geometry Revisted*, Math. Assoc. of Amer., Washington D.C., 1967.

[6] W. Derrick and J. Herstein. *Proof without words: Ptolemy's theorem*, The College Mathematics Journal 43:5 (2012), 386.

[7] V. Pillwein. *Positivity of certain sums over Jacobi kernel polynomials*, Adv. in Appl. Math. 41:3 (2008), 365–377.