RESEARCH INSTITUTE FOR
SYMBOLIC COMPUTATION

JOHANNES KEPLER
UNIVERSITY LINZ

Asymptotics for the reciprocal and shifted quotient of the partition function

Koustav Banerjee, Peter Paule, Cristian-Silviu Radu, Carsten Schneider

December 2024

RISC Report Series No. 24-06

ISSN: 2791-4267 (online)

Available at https://doi.org/10.35011/risc.24-06

JOHANNES KEPLER UNIVERSITY LINZ
Altenberger Str. 69
4040 Linz, Austria
www.jku.at
DVR 0093696

arXiv:2412.02257 [math.NT]
RISC Report Series 24-06

ASYMPTOTICS FOR THE RECIPROCAL AND SHIFTED QUOTIENT OF THE PARTITION FUNCTION

KOUSTAV BANERJEE, PETER PAULE, CRISTIAN-SILVIU RADU, AND CARSTEN SCHNEIDER

Abstract. Let $p(n)$ denote the partition function. In this paper our main goal is to derive an asymptotic expansion up to order N (for any fixed positive integer N) along with estimates for error bounds for the shifted quotient of the partition function, namely $p(n+k)/p(n)$ with $k \in \mathbb{N}$, which generalizes a result of Gomez, Males, and Rolen. In order to do so, we derive asymptotic expansions with error bounds for the shifted version $p(n+k)$ and the multiplicative inverse $1/p(n)$, which is of independent interest.

Contents

1. Introduction and summary of results

A partition λ of a positive integer n is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\sum_{j=1}^{r} \lambda_j = n$. The total number of partitions of n is denoted by $p(n)$ with $p(0) := 1$. A rigorous study on $p(n)$ from the analytic point of view began with the seminal work of Hardy and Ramanujan [18]. In [18], developing a widely celebrated tool known as Circle Method, they established the asymptotic growth of $p(n)$ which states that

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}} \quad \text{as } n \to \infty.$$

Later Rademacher [33] refined the Circle Method to give an exact formula for $p(n)$. Lehmer [22, 23] estimated the error term after truncating Rademacher's convergent series for $p(n)$. Among many other applications, Rademacher's formula along with Lehmer's estimate have been used to prove inequalities for $p(n)$. The first such instance has been documented in Nicolas' work [26] on the *log-concavity* of $p(n)$, where a sequence (of positive real numbers) $(a_n)_{n\geq 0}$ is called *log-concave* if

2020 *Mathematics Subject Classification.* 05A16, 05A20, 11P82.
Key words and phrases. integer partitions, Hardy-Ramanujan Rademacher formula, asymptotic expansion, shifted quotient of partitions.

$a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 1$. We refer to [10, 38] for a more detailed study on *log-concavity* and its higher order analogues known as *higher order Turán inequalities*. After a standstill for more than thirty years, DeSalvo and Pak's work [12] on the *log-concavity* of $p(n)$ and its associated inequalities not only confirmed the conjectures of Chen [9] but also resurrected this particular area of research on $p(n)$. Since then an extensive amount of work on inequalities for $p(n)$ and its variations have been recorded. Among many others, we refer the readers to the work of Bringmann, Kane, Rolen, and Tripp [8] on inequalities for the *fractional partition function* $p_\alpha(n)$ arising from the famous *Nekrasov–Okounkov formula* [25] and to Ono, Pujahari, and Rolen's work [30] on Turán inequalities for the *plane partition function* $\mathrm{PL}(n)$. For a detailed study on $\mathrm{PL}(n)$, see [24].

In the following we shall restrict ourselves to inequalities for $p(n)$. The motivation for doing so is as follows. We call a polynomial with real coefficients *hyperbolic* if all roots are real. The sequence of so-called *Jensen polynomials* associated with $p(n)$ is defined as

$$J_p^{d,n}(x) := \sum_{j=0}^{d} \binom{d}{j} p(n+j)x^j. \tag{1.1}$$

From work of Nicolas [26] and DeSalvo and Pak [12] we already know that $J_p^{2,n}(x)$ is hyperbolic for $n \geq 26$. The case for $d = 3$ has been settled by Chen, Jia, and Wang [11]. Soon after, in their fundamental work on a problem entangled with variants of the *Riemann hypothesis*, Griffin, Ono, Rolen, and Zagier [16] confirmed that for any $d \in \mathbb{Z}_{\geq 0}$, $J_p^{d,n}(x)$ is eventually hyperbolic which settled the conjecture of Chen, Jia, and Wang [11]. In [16, Theorem 3], Griffin, Ono, Rolen, and Zagier developed a generic framework to decide the *hyperbolicity* of Jensen polynomials associated with a family of sequences satisfying *certain* growth conditions. In particular, for $p(n)$ they proved the hyperbolicity of $J_p^{d,n}(x)$ for sufficiently large n by showing the following uniform convergence property:

$$\lim_{n\to\infty} \left(\frac{\delta(n)^d}{p(n)} J_p^{d,n} \left(\frac{\delta(n)x - 1}{\exp\left(A(n)\right)} \right) \right) = H_d(x), \tag{1.2}$$

with $H_d(x)$ the d-th Hermite polynomial and where the sequences $(\delta(n))_{n\geq 0}$, $(A(n))_{n\geq 0}$ are determined by

$$\log\left(\frac{p(n+j)}{p(n)}\right) = A(n)j - \delta(n)^2 j^2 + o\left(\delta(n)^d\right) \quad \text{as } n \to \infty.$$

Larson and Wagner [21, Theorem 1.3] derived an effective cutoff $N(d)$ for n such that $J_p^{d,n}(x)$ is hyperbolic for all $n \geq N(d)$. In order to arrive at their main result, one of the quintessential steps was to get an asymptotic expansion of the quotient $p(n+j)/p(n)$; see [21, Lemmas 2.2 and 2.3]. Recently, Gomez, Males, and Rolen [14, Theorem 1.1] obtained a more effective error bound for the asymptotic expansion of the ratio $p(n-j)/p(n)$ truncating after the first three terms of the asymptotic series, which, in turn, has many applications including a shifted convexity property of $p(n)$ (see [4, 15, 17, 20, 29]) and new estimates of the k-rank partition function $N_k(m,n)$ defined by Garvan [13]. In this article we generalize the result [14, Theorem 1.1] of Gomez, Males, and Rolen as stated below in Theorem 1.1 but with a slight variant; namely, for quotients with positive shifts $p(n+k)/p(n)$.

Throughout, we write $f(x) = O_{\leq c}\left(g(x)\right)$ if $|f(x)| \leq cg(x)$ for a positive function g and for a domain of x which we will specify in each given context. With this convention we can state the main result of this article. Moreover, we use $\mathbb{N}$ to denote the set of positive integers.

Theorem 1.1. *Let $(k, N) \in \mathbb{N}^2$. Then for all $n \geq n_N(k)$*

$$\frac{p(n+k)}{p(n)} = \sum_{m=0}^{N} \frac{c_k(m)}{n^{\frac{m}{2}}} + O_{\leq E_N(k)}\left(n^{-\frac{N+1}{2}}\right),$$

where $c_k(0) = 1$, *the constants* $(c_k(m))_{m\geq 1}$, $n_N(k)$, *and* $E_N(k)$ *are determined effectively for any fixed* N.

Remark 1.2. *The* $(c_k(m))_{m\geq 1}$, $n_N(k)$, *and* $E_N(k)$ *are stated explicitly in* (5.3), (5.1), *and* (5.28) *respectively.*

Remark 1.3. (1) *In their study on asymptotic* r*-log-concavity for* $p(n)$, *Hou and Zhang* [19, Theorem 1.3] *determined an asymptotic expansion of* $p(n+1)/p(n)$. *Theorem 1.1 is a straightforward generalization of their result. In particular, now we have estimated effectively the cutoff* $n_N(k)$, *the error bound* $E_N(k)$, *and also provided a concrete description of the coefficient sequence* $(c_k(m))_{m\geq 0}$.

(2) *In light of the remark* [14, Remark 1.3] *made by Gomez, Males, and Rolen on positivity of* $\Delta_j^r(p(n))$, *where* Δ_j^r *is the* r*-fold application of the difference operator* Δ_j *defined by* $(\Delta_j p)(n) = p(n) - p(n-j)$, *it seems that by making the shift* $n \mapsto n + j$, *one might use Theorem 1.1 by choosing the truncation point* $N(r)$[1] *(depending on* $r \in \mathbb{N}$*) in the asymptotic expansion of* $p(n+k)/p(n)$ *appropriately so as to obtain the asymptotic growth of* $\Delta_j^r(p(n))$. *In addition to that, we can estimate a cutoff* $n_{N(r),r}(j)$ *such that* $\Delta_j^r(p(n)) \geq 0$ *for all* $n \geq n_{N(r),r}(j)$. *The case* $r = 2$ *was settled in* [14, Theorem 1.2].

We summarize in brief the main tools which will be used in proving Theorem 1.1. First, we will start with the infinite family of inequalities for $p(n)$ given in [7, Theorem 4.4] that states: for $m \in \mathbb{Z}_{\geq 2}$ and $n > g(m)$ (see Theorem 2.1 below),

$$p(n) = \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} + O_{\leq 1}\left(\mu(n)^{-m}\right)\right) \quad \text{with } \mu(n) := \frac{\pi}{6}\sqrt{24n-1}.$$

This result has twofold applications in the present context: (i) by considering the shift $n \mapsto n+k$, after Taylor series expansion of the major term $\frac{\sqrt{12}e^{\mu(n+k)}}{24(n+k)-1}\left(1 - \frac{1}{\mu(n+k)}\right)$, we extract the coefficient sequence in the asymptotic expansion of $p(n+k)$ and similarly, (ii) for exhibiting the coefficient sequence in the asymptotic expansion of $1/p(n)$, we use Taylor series expansion for the inverse of the major term; i.e., $\frac{24n-1}{\sqrt{12}e^{\mu(n)}}\left(1 - \frac{1}{\mu(n)}\right)^{-1}$.

In addition, we shall use the *symbolic summation* package `Sigma`, developed by the third author [35], to simplify the coefficients appearing in the asymptotic expansion for $1/p(n)$ so as to obtain precise error bound estimates for the asymptotic expansion under consideration.

The rest of this paper is organized as follows: in Section 2, we will recall a list of preliminary results from [6, 7] which will be helpful in carrying out estimates in the subsequent sections. In Sections 3 and 4, we will provide asymptotic expansions for the shifted partition function $p(n+k)$ (see Theorem 3.1) and the inverse $1/p(n)$ (see Theorem 4.18), respectively. The proof of Theorem 1.1 is given in Section 5. Finally in Section 6, we lay out a detailed exposition on the usage of the symbolic summation package `Sigma`.

2. Preliminaries

The following results from [6] and [7] will be needed. Throughout, for $n \in \mathbb{N}$,

$$\mu(n) := \frac{\pi}{6}\sqrt{24n-1}.$$

Theorem 2.1. [7, Theorem 4.4] *For* $m \in \mathbb{Z}_{\geq 2}$, *define*

$$\widehat{g}(m) := \frac{1}{24}\left(\frac{36}{\pi^2} \cdot \nu(m)^2 + 1\right),$$

[1] For instance, one can choose $N(r) = r + 1$, see [5, Chap. 6, Sec. 6.7.2].

where
$$\nu(m) := 2\log 6 + (2\log 2)m + 2m\log m + 2m\log\log m + \frac{5m\log\log m}{\log m}.$$
Then for all $m \in \mathbb{Z}_{\geq 2}$ and $n > \widehat{g}(m)$ such that $(n,m) \neq (6,2)$, we have
$$p(n) = \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} + O_{\leq 1}\left(\mu(n)^{-m}\right)\right). \tag{2.1}$$
Lemma 2.2. [6, Lemma 4.1] *Let $x_1, x_2, \ldots, x_n$ and $y_1, \ldots, y_n$ be non-negative real numbers such that $x_j \leq 1$ for $1 \leq j \leq n$. Then*
$$\frac{(1-x_1)(1-x_2)\cdots(1-x_n)}{(1+y_1)(1+y_2)\cdots(1+y_n)} \geq 1 - \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} y_j.$$

From [34, eqs. (1) and (2)], we obtain the following inequality for the central binomial coefficients.

Lemma 2.3. *For $n \in \mathbb{N}$, we have*
$$\frac{4^n}{\sqrt{\pi n}}\left(1 - \frac{1}{8n}\right) < \binom{2n}{n} < \frac{4^n}{\sqrt{\pi n}}.$$
Lemma 2.4. [6, Lemma 4.6] *For $t \geq 1$ and $k \in \{0,1,2,3\}$, with $\alpha = \frac{\pi}{6}$, we have*
$$\sum_{u=t+1}^{\infty} \frac{u^k \alpha^{2u}}{(2u)!} \leq \frac{C_k}{t^2} \quad \text{with} \quad C_k = \frac{\alpha^4 2^k}{18}.$$

We generalize Lemma 2.4 only for the case $k = 0$ stated below.

Lemma 2.5. *For $t \geq 1$ and $0 < c < 2.2$, we have*
$$\sum_{u=t+1}^{\infty} \frac{c^{2u}}{(2u)!} \leq \frac{c^4}{18 \cdot t^2}.$$

Proof. The statement is obtained by a routine calculation following the proof of Lemma 2.4 (cf. [6, Section 8, p. 38]). ☐

3. Asymptotics of $p(n+k)$

In this section we shall prove the following theorem on the asymptotic expansion of $p(n+k)$.

Theorem 3.1. *Let $(k,N) \in \mathbb{N}^2$ and $\widehat{g}(m)$ be as in Theorem 2.1. For $n > n_N^{[1]}(k)$ we have*
$$p(n+k) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{N} \frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}} + O_{\leq E_N^{[1]}(k)}\left(n^{-\frac{N+1}{2}}\right)\right), \tag{3.1}$$
where
$$\omega_k^{[1]}(t) = \left(\frac{24k-1}{4\sqrt{6}}\right)^t \sum_{\ell=0}^{\frac{t+1}{2}} \binom{t+1}{\ell} \frac{(t+1-\ell)}{(t+1-2\ell)!}(-1)^\ell \left(\frac{\pi}{6}\right)^{t-2\ell}\left(\frac{1}{24k-1}\right)^\ell, \tag{3.2}$$
and where $n_N^{[1]}(k), E_N^{[1]}(k)$ are determined effectively in (3.33) and (3.36) respectively.

In order to prove Theorem 3.1 we need some preparatory estimates documented below. First, applying Theorem 2.1 with $m \mapsto N+1$, and taking the shift $n \mapsto n+k$, we have the following.

Lemma 3.2. *Let $(k,N) \in \mathbb{N}^2$ and $\widehat{g}(m)$ be as in (2.1). For $n > \widehat{g}(N+1)$ we have*
$$p(n+k) = \frac{\sqrt{12}e^{\mu(n+k)}}{24(n+k)-1}\left(1 - \frac{1}{\mu(n+k)} + O_{\leq 1}\left(\mu(n+k)^{-N-1}\right)\right).$$

Next, extracting the factor $\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$ from the major term $\frac{\sqrt{12}e^{\mu(n+k)}}{24(n+k)-1}\left(1-\frac{1}{\mu(n+k)}\right)$ and by Taylor series expansion of the residual part, we get the following.

Lemma 3.3. *Let $(\omega_k^{[1]}(t))_{t\geq 0}$ be as in (3.2). For $k \in \mathbb{N}$,*

$$\frac{\sqrt{12}e^{\mu(n+k)}}{24(n+k)-1}\left(1-\frac{1}{\mu(n+k)}\right) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\sum_{t=0}^{\infty}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}.$$

Proof. Analogous to the proof of [28, Proposition 4.4]. □

Next, we truncate the Taylor series $\sum_{t=0}^{\infty}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}$ at N and estimate the absolute value of the remainder part by estimating $\left|\omega_k^{[1]}(t)\right|$ independence of t even or odd.

For $k \in \mathbb{N}$, we define

$$\alpha_k := \mu(k) = \frac{\pi}{6}\sqrt{24k-1} \tag{3.3}$$

and

$$C_1(k) := \frac{3k\left(3\lceil\sqrt{k}\rceil+1\right)^2\alpha_k^{6\lceil\sqrt{k}\rceil+4}}{2\lceil\sqrt{k}\rceil\left(6\lceil\sqrt{k}\rceil+4\right)!}. \tag{3.4}$$

Lemma 3.4. *Let α_k and $C_1(k)$ be as in (3.3) and (3.4), respectively. Then for all $(k,t) \in \mathbb{N}^2$,*

$$\left|\omega_k^{[1]}(2t+1)\right| \leq \left(\frac{24k-1}{24}\right)^t\left(\frac{6}{\pi^3}\right)^{\frac{1}{2}}\sqrt{t+1}\,|\cos(\alpha_k)|\left(1+\frac{C_1^*(k)}{t}\right),$$

with

$$C_1^*(k) := \frac{\alpha_k^2(\cosh(\alpha_k)-1)+4\cdot C_1(k)}{4\cdot|\cos(\alpha_k)|}. \tag{3.5}$$

Proof. By (3.2) and (3.3),

$$\begin{aligned}
\omega_k^{[1]}(2t+1) &= \left(\frac{24k-1}{4\sqrt{6}}\right)^{2t+1}\sum_{\ell=0}^{t+1}\binom{2t+2}{\ell}\frac{(2t+2-\ell)}{(2t+2-2\ell)!}(-1)^{\ell}\left(\frac{\pi}{6}\right)^{2t+1-2\ell}\left(\frac{1}{24k-1}\right)^{\ell}\\
&\overset{(\ell\mapsto t+1-\ell)}{=} (-1)^{t+1}\frac{(24k-1)^t}{(4\sqrt{6})^{2t+1}}\sum_{\ell=0}^{t+1}\binom{2t+2}{t+1-\ell}(t+1+\ell)\frac{(-1)^{\ell}}{(2\ell)!}\left(\frac{\pi}{6}\right)^{2\ell-1}(24k-1)^{\ell}\\
&= \frac{(24k-1)^t}{(4\sqrt{6})^{2t+1}}\frac{6}{\pi}\binom{2t+2}{t+1}(t+1)(-1)^{t+1}\sum_{\ell=0}^{t+1}\frac{\binom{2t+2}{t+1-\ell}(t+1+\ell)}{\binom{2t+2}{t+1}(t+1)}\frac{(-1)^{\ell}}{(2\ell)!}\alpha_k^{2\ell}\\
&=: \frac{(24k-1)^t}{(4\sqrt{6})^{2t+1}}\frac{6}{\pi}\binom{2t+2}{t+1}(t+1)(-1)^{t+1}S_k^{[1]}(t).
\end{aligned} \tag{3.6}$$

Now, we bound $|S_k^{[1]}|(t)$ as

$$\begin{aligned}
\left|S_k^{[1]}(t)\right| &\leq |\cos(\alpha_k)| + \left|S_k^{[1]}(t)-\cos(\alpha_k)\right|\\
&= |\cos(\alpha_k)| + \left|\sum_{\ell=0}^{t+1}\frac{\binom{2t+2}{t+1-\ell}(t+1+\ell)}{\binom{2t+2}{t+1}(t+1)}\frac{(-1)^{\ell}}{(2\ell)!}\alpha_k^{2\ell} - \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{(2\ell)!}\alpha_k^{2\ell}\right|
\end{aligned}$$

$$
\begin{aligned}
&\le |\cos(\alpha_k)| + \sum_{\ell=0}^{t+1} \left| \frac{\binom{2t+2}{t+1-\ell}(t+1+\ell)}{\binom{2t+2}{t+1}(t+1)} - 1 \right| \frac{\alpha_k^{2\ell}}{(2\ell)!} + \sum_{\ell=t+2}^{\infty} \frac{\alpha_k^{2\ell}}{(2\ell)!} \\
&= |\cos(\alpha_k)| + \sum_{\ell=2}^{t+1} \left| \frac{\binom{2t+2}{t+1-\ell}(t+1+\ell)}{\binom{2t+2}{t+1}(t+1)} - 1 \right| \frac{\alpha_k^{2\ell}}{(2\ell)!} + \sum_{\ell=t+2}^{\infty} \frac{\alpha_k^{2\ell}}{(2\ell)!} \\
&\qquad \left(\text{as } \frac{\binom{2t+2}{t+1-\ell}(t+1+\ell)}{\binom{2t+2}{t+1}(t+1)} = 1 \text{ for } \ell \in \{0,1\} \right) \\
&=: |\cos(\alpha_k)| + S_{k,1}^{[1]}(t) + S_{k,2}^{[1]}(t). \qquad (3.7)
\end{aligned}
$$

Analogous to the proof of [5, Lemma 6.2.7], one can derive that for $t \ge 1$,

$$
S_{k,2}^{[1]}(t) \le \frac{C_1(k)}{t^2}. \qquad (3.8)
$$

To obtain an upper bound of $S_{k,1}^{[1]}(t)$, we first note that

$$
\frac{\binom{2t+2}{t+1-\ell}(t+1+\ell)}{\binom{2t+2}{t+1}(t+1)} = \prod_{j=2}^{\ell} \frac{t+2-j}{t+j},
$$

which implies

$$
\frac{\binom{2t+2}{t+1-\ell}(t+1+\ell)}{\binom{2t+2}{t+1}(t+1)} - 1 \le 0 \quad \text{as} \quad \frac{t+2-j}{t+j} \le 1 \quad \text{for all } j \ge 1. \qquad (3.9)
$$

Moreover, by using Lemma 2.2,

$$
\prod_{j=2}^{\ell} \frac{t+2-j}{t+j} = \prod_{j=2}^{\ell} \frac{1-\frac{j-2}{t}}{1+\frac{j}{t}} \ge 1 - \sum_{j=2}^{\ell} \frac{j-2}{t} - \sum_{j=2}^{\ell} \frac{j}{t} = 1 - \frac{\ell^2-\ell}{t},
$$

we have

$$
\frac{\binom{2t+2}{t+1-\ell}(t+1+\ell)}{\binom{2t+2}{t+1}(t+1)} - 1 \ge -\frac{\ell^2-\ell}{t}. \qquad (3.10)
$$

Combining (3.9) and (3.10), it follows that

$$
\left| \frac{\binom{2t+2}{t+1-\ell}(t+1+\ell)}{\binom{2t+2}{t+1}(t+1)} - 1 \right| \le \frac{\ell^2-\ell}{t}. \qquad (3.11)
$$

Now, using (3.11) and $\ell \ge 2$, we bound $S_{k,1}^{[1]}(t)$ against

$$
S_{k,1}^{[1]}(t) \le \sum_{\ell=2}^{t+1} \frac{\ell^2-\ell}{t} \frac{\alpha_k^{2\ell}}{(2\ell)!} \le \frac{\alpha_k^2}{4t} \sum_{\ell=2}^{\infty} \frac{\alpha_k^{2\ell-2}}{(2\ell-2)!} = \frac{\alpha_k^2(\cosh(\alpha_k)-1)}{4t}. \qquad (3.12)
$$

Applying (3.8) and (3.12) to (3.7) gives

$$
\begin{aligned}
\left| S_k^{[1]}(t) \right| &\le |\cos(\alpha_k)| + \frac{\alpha_k^2(\cosh(\alpha_k)-1)}{4t} + \frac{C_1(k)}{t^2} \\
&\le |\cos(\alpha_k)| + \frac{\alpha_k^2(\cosh(\alpha_k)-1) + 4 \cdot C_1(k)}{4t} \quad (\text{as } t \ge 1)
\end{aligned}
$$

$$= |\cos(\alpha_k)| \left(1 + \frac{\alpha_k^2 (\cosh(\alpha_k) - 1) + 4 \cdot C_1(k)}{4 |\cos(\alpha_k)| \cdot t}\right) = |\cos(\alpha_k)| \left(1 + \frac{C_1^*(k)}{t}\right), \tag{3.13}$$

where the last equality is by (3.5). Summarizing,

$$\begin{aligned}
\left|\omega_k^{[1]}(2t+1)\right| &\le \frac{(24k-1)^t}{(4\sqrt{6})^{2t+1}} \frac{6}{\pi} \binom{2t+2}{t+1} (t+1) \left|S_k^{[1]}(t)\right| \quad \text{(by (3.6))} \\
&\le \frac{(24k-1)^t}{(4\sqrt{6})^{2t+1}} \frac{6}{\pi} \binom{2t+2}{t+1} (t+1) |\cos(\alpha_k)| \left(1 + \frac{C_1^*(k)}{t}\right) \quad \text{(by (3.13))} \\
&\le \frac{(24k-1)^t}{(4\sqrt{6})^{2t+1}} \frac{6}{\pi} \frac{4^{t+1}}{\sqrt{\pi}} \sqrt{t+1}\, |\cos(\alpha_k)| \left(1 + \frac{C_1^*(k)}{t}\right) \quad \text{(by Lemma 2.3)} \\
&= \left(\frac{24k-1}{24}\right)^t \left(\frac{6}{\pi^3}\right)^{\frac{1}{2}} \sqrt{t+1}\, |\cos(\alpha_k)| \left(1 + \frac{C_1^*(k)}{t}\right),
\end{aligned}$$

which conclude the proof of Lemma 3.4. ☐

For $k \in \mathbb{N}$ define

$$C_2(k) := \frac{9k \left(3\lceil\sqrt{k}\rceil + 1\right)^2 \alpha_k^{6\lceil\sqrt{k}\rceil+3}}{\left(6\lceil\sqrt{k}\rceil + 1\right)\left(6\lceil\sqrt{k}\rceil + 3\right)!}. \tag{3.14}$$

Lemma 3.5. *Let* α_k *be as in* (3.3) *and* $C_2(k)$ *be as in and* (3.14)*. Then for all* $(k,t) \in \mathbb{N}^2$*,*

$$\left|\omega_k^{[1]}(2t)\right| \le \left(\frac{24k-1}{24}\right)^t \frac{2\sqrt{t}}{\sqrt{\pi}\alpha_k} |\sin(\alpha_k)| \left(1 + \frac{C_2^*(k)}{t}\right)$$

with

$$C_2^*(k) := \frac{1}{2}\left(1 + 3\frac{\alpha_k^2 \cdot \sinh(\alpha_k) + 4 \cdot C_2(k)}{4|\sin(\alpha_k)|}\right). \tag{3.15}$$

Proof. By (3.2) and (3.3),

$$\begin{aligned}
\omega_k^{[1]}(2t) &= \left(\frac{24k-1}{4\sqrt{6}}\right)^{2t} \sum_{\ell=0}^{t} \binom{2t+1}{\ell} \frac{(2t+1-\ell)}{(2t+1-2\ell)!} (-1)^\ell \left(\frac{\pi}{6}\right)^{2t-2\ell} \left(\frac{1}{24k-1}\right)^\ell \\
&\overset{(\ell\mapsto t-\ell)}{=} \frac{(24k-1)^{t+\frac{1}{2}}}{(4\sqrt{6})^{2t}} \frac{6}{\pi} \binom{2t+1}{t} (t+1)(-1)^t \sum_{\ell=0}^{t} \frac{\binom{2t+1}{t-\ell}(t+1+\ell)}{\binom{2t+1}{t}(t+1)} \frac{(-1)^\ell}{(2\ell+1)!} \alpha_k^{2\ell+1} \\
&= \frac{(24k-1)^{t+\frac{1}{2}}}{(4\sqrt{6})^{2t}} \frac{6}{\pi} \binom{2t}{t} (2t+1)(-1)^t \sum_{\ell=0}^{t} \frac{\binom{2t+1}{t-\ell}(t+1+\ell)}{\binom{2t+1}{t}(t+1)} \frac{(-1)^\ell}{(2\ell+1)!} \alpha_k^{2\ell+1} \\
&=: \frac{(24k-1)^{t+\frac{1}{2}}}{(4\sqrt{6})^{2t}} \frac{6}{\pi} \binom{2t}{t} (2t+1)(-1)^t S_k^{[2]}(t).
\end{aligned} \tag{3.16}$$

Now, we bound $|S_k^{[2]}(t)|$ as

$$\begin{aligned}
\left|S_k^{[2]}(t)\right| &\le |\sin(\alpha_k)| + \left|S_k^{[2]}(t) - \sin(\alpha_k)\right| \\
&= |\sin(\alpha_k)| + \left|\sum_{\ell=0}^{t} \frac{\binom{2t+1}{t-\ell}(t+1+\ell)}{\binom{2t+1}{t}(t+1)} \frac{(-1)^\ell}{(2\ell+1)!} \alpha_k^{2\ell+1} - \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)!} \alpha_k^{2\ell+1}\right|
\end{aligned}$$

$$\leq |\sin(\alpha_k)| + \sum_{\ell=1}^{t} \left| \frac{\binom{2t+1}{t-\ell}(t+1+\ell)}{\binom{2t+1}{t}(t+1)} - 1 \right| \frac{\alpha_k^{2\ell+1}}{(2\ell+1)!} + \sum_{\ell=t+1}^{\infty} \frac{\alpha_k^{2\ell+1}}{(2\ell+1)!}$$

$$=: |\cos(\alpha_k)| + S_{k,1}^{[2]}(t) + S_{k,2}^{[2]}(t). \tag{3.17}$$

Similar to (3.8), following the proof of [5, Lemma 6.2.7], it follows that for $t \geq 1$,

$$S_{k,2}^{[2]}(t) \leq \frac{C_2(k)}{t^2}. \tag{3.18}$$

To obtain an upper bound of $S_{k,1}^{[2]}(t)$, observe that

$$\frac{\binom{2t+1}{t-\ell}(t+1+\ell)}{\binom{2t+1}{t}(t+1)} = \prod_{j=1}^{\ell} \frac{t+1-j}{t+j},$$

which implies

$$\frac{\binom{2t+1}{t-\ell}(t+1+\ell)}{\binom{2t+1}{t}(t+1)} - 1 \leq 0 \quad \text{as} \quad \frac{t+1-j}{t+j} \leq 1 \quad \text{for all } j \geq 1. \tag{3.19}$$

Moreover, by using Lemma 2.2,

$$\prod_{j=1}^{\ell} \frac{t+1-j}{t+j} = \prod_{j=1}^{\ell} \frac{1-\frac{j-1}{t}}{1+\frac{j}{t}} \geq 1 - \sum_{j=1}^{\ell} \frac{j-1}{t} - \sum_{j=1}^{\ell} \frac{j}{t} = 1 - \frac{\ell^2}{t},$$

we have

$$\frac{\binom{2t+1}{t-\ell}(t+1+\ell)}{\binom{2t+1}{t}(t+1)} - 1 \geq -\frac{\ell^2}{t}. \tag{3.20}$$

Combining (3.19) and (3.20), it follows that

$$\left| \frac{\binom{2t+1}{t-\ell}(t+1+\ell)}{\binom{2t+1}{t}(t+1)} - 1 \right| \leq \frac{\ell^2}{t}. \tag{3.21}$$

Now, using (3.21) and $\ell \geq 1$, we bound $S_{k,1}^{[2]}(t)$ against

$$S_{k,1}^{[2]}(t) \leq \sum_{\ell=1}^{t} \frac{\ell^2}{t} \frac{\alpha_k^{2\ell+1}}{(2\ell+1)!} \leq \frac{\alpha_k^2}{4t} \sum_{\ell=1}^{\infty} \frac{\alpha_k^{2\ell-1}}{(2\ell-1)!} = \frac{\alpha_k^2 \cdot \sinh(\alpha_k)}{4t}. \tag{3.22}$$

Applying (3.18) and (3.22) to (3.17) gives

$$\left| S_k^{[2]}(t) \right| \leq |\sin(\alpha_k)| + \frac{\alpha_k^2 \cdot \sinh(\alpha_k)}{4t} + \frac{C_2(k)}{t^2} \leq |\sin(\alpha_k)| + \frac{\alpha_k^2 \cdot \sinh(\alpha_k) + 4C_2(k)}{4t} \quad (\text{as } t \geq 1)$$

$$= |\sin(\alpha_k)| \left(1 + \frac{\alpha_k^2 \cdot \sinh(\alpha_k) + 4C_2(k)}{|\sin(\alpha_k)| \cdot 4t} \right). \tag{3.23}$$

Summarizing,

$$\left| \omega_k^{[1]}(2t) \right| \leq \frac{(24k-1)^{t+\frac{1}{2}}}{(4\sqrt{6})^{2t}} \frac{6}{\pi} \binom{2t}{t} (2t+1) \left| S_k^{[2]}(t) \right| \quad (\text{by (3.16)})$$

$$\leq \frac{(24k-1)^{t+\frac{1}{2}}}{(4\sqrt{6})^{2t}} \frac{6}{\pi} \binom{2t}{t} (2t+1) |\sin(\alpha_k)| \left(1 + \frac{\alpha_k^2 \cdot \sinh(\alpha_k) + 4C_2(k)}{|\sin(\alpha_k)| \cdot 4t} \right) \quad (\text{by (3.23)})$$

$$\leq \frac{(24k-1)^{t+\frac{1}{2}}}{(4\sqrt{6})^{2t}}\frac{6}{\pi}\frac{4^t}{\sqrt{\pi t}}(2t+1)\left|\sin(\alpha_k)\right|\left(1+\frac{\alpha_k^2\cdot\sinh(\alpha_k)+4C_2(k)}{|\sin(\alpha_k)|\cdot 4t}\right) \quad \text{(by Lemma 2.3)}$$
$$= \left(\frac{24k-1}{24}\right)^t\frac{2}{\sqrt{\pi}\alpha_k}\sqrt{t}\left|\sin(\alpha_k)\right|\left(1+\frac{\alpha_k^2\cdot\sinh(\alpha_k)+4C_2(k)}{|\sin(\alpha_k)|\cdot 4t}\right)\left(1+\frac{1}{2t}\right)$$
$$\leq \left(\frac{24k-1}{24}\right)^t\frac{2\sqrt{t}}{\sqrt{\pi}\alpha_k}\left|\sin(\alpha_k)\right|\left(1+\frac{C_2^*(k)}{t}\right) \quad (\text{as } t\geq 1),$$

which concludes the proof of Lemma 3.5. □

Remark 3.6. *In Lemmas 3.4 and 3.5 we have seen that the factors* $|\cos(\alpha_k)|$ *and* $|\sin(\alpha_k)|$ *appear in the estimates of upper bounds. Following the definition of* α_k *in* (3.3)*, it is a routine check to verify that* $\sin(\alpha_k)\neq 0$ *and* $\cos(\alpha_k)\neq 0$ *for all* $k\in\mathbb{N}$ *which, in turn, justifies that both* $C_1^*(k)$ *and* $C_2^*(k)$ *(cf.* (3.5) *and* (3.15)*) are well-defined.*

Now, applying the bounds from Lemmas 3.4 and 3.5, we now estimate an upper bound for the absolute value of $\sum_{t=N+1}^{\infty}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}$.

Lemma 3.7. *Let* $(k,N)\in\mathbb{N}^2$ *and* $\omega_k^{[1]}(t)$ *be as in* (3.2)*. Let* $\widehat{g}(m)$ *be as in Theorem 2.1. Then for all* $n\geq \max\limits_{k,N\geq 1}\{24k-1,\widehat{g}(N+1)\}$,

$$\left|\sum_{t=N+1}^{\infty}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}\right|\leq E_{N,1}^{[1]}(k)\cdot n^{-\frac{N+1}{2}},$$

where $E_{N,1}^{[1]}(k)$ *is defined below in* (3.31).

Proof. We split the series depending on t odd or even:

$$\sum_{t=N+1}^{\infty}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}=\sum_{\substack{t=N+1\\ t\equiv\ 0\ (\mathrm{mod}\ 2)}}^{\infty}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}+\sum_{\substack{t=N+1\\ t\equiv\ 1\ (\mathrm{mod}\ 2)}}^{\infty}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}=\sum_{t\geq\frac{N+1}{2}}\frac{\omega_k^{[1]}(2t)}{n^t}+\sum_{t\geq\frac{N}{2}}\frac{\omega_k^{[1]}(2t+1)}{n^{t+\frac{1}{2}}},$$

and so,

$$\left|\sum_{t=N+1}^{\infty}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}\right|\leq\sum_{t\geq\frac{N+1}{2}}\frac{\left|\omega_k^{[1]}(2t)\right|}{n^t}+n^{-\frac{1}{2}}\sum_{t\geq\frac{N}{2}}\frac{\left|\omega_k^{[1]}(2t+1)\right|}{n^t}. \tag{3.24}$$

Recalling (3.15), we set

$$C_e(k):=2C_2^*(k)+\frac{(24k-1)(1+C_2^*(k))}{6}. \tag{3.25}$$

First, we estimate

$$\sum_{t\geq\frac{N+1}{2}}\frac{\left|\omega_k^{[1]}(2t)\right|}{n^t}$$
$$\leq\frac{2\left|\sin(\alpha_k)\right|}{\sqrt{\pi}\alpha_k}\left(1+\frac{2C_2^*(k)}{N+1}\right)\sum_{t\geq\frac{N+1}{2}}\left(\frac{24k-1}{24n}\right)^t\sqrt{t} \quad \text{(by Lemma 3.5)}$$
$$=\frac{\sqrt{2}\left|\sin(\alpha_k)\right|}{\sqrt{\pi}\alpha_k}\left(1+\frac{2C_2^*(k)}{N+1}\right)\left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+1}\,n^{-\frac{N+1}{2}}\sum_{t\geq 0}\left(\frac{24k-1}{24n}\right)^t\sqrt{1+\frac{2t}{N+1}}$$

$$\leq \frac{\sqrt{2}\,|\sin(\alpha_k)|}{\sqrt{\pi}\alpha_k}\left(1+\frac{2C_2^*(k)}{N+1}\right)\left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+1}n^{-\frac{N+1}{2}}\sum_{t\geq 0}\left(\frac{24k-1}{24n}\right)^t\sqrt{1+t}$$

$$\left(\text{since for } N\geq 1, \frac{2t}{N+1}\leq t\right)$$

$$\leq \frac{\sqrt{2}\,|\sin(\alpha_k)|}{\sqrt{\pi}\alpha_k}\left(1+\frac{2C_2^*(k)}{N+1}\right)\left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+1}n^{-\frac{N+1}{2}}\sum_{t\geq 0}\left(\frac{24k-1}{12n}\right)^t$$

$$\left(\text{as } \sqrt{1+t}\leq 2^t \text{ for } t\geq 0\right)$$

$$\leq \frac{\sqrt{2}\,|\sin(\alpha_k)|}{\sqrt{\pi}\alpha_k}\left(1+\frac{2C_2^*(k)}{N+1}\right)\left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+1}\left(1+\frac{24k-1}{6n}\right)n^{-\frac{N+1}{2}}$$

$$\left(\text{since for } 0<x\leq\frac{1}{2},\ (1-x)^{-1}\leq 1+2x \text{ and choose } x\mapsto\frac{24k-1}{12n} \text{ as } n\geq 24k-1\right)$$

$$\leq \frac{\sqrt{2}\,|\sin(\alpha_k)|}{\sqrt{\pi}\alpha_k}\left(1+\frac{2C_2^*(k)}{N+1}\right)\left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+1}\left(1+\frac{24k-1}{6N}\right)n^{-\frac{N+1}{2}}$$

(as for all $N\in\mathbb{N}, n>\widehat{g}(N+1)\geq N$ by definition of $\widehat{g}(m)$ in Theorem 2.1)

$$\leq \frac{\sqrt{2}\,|\sin(\alpha_k)|}{\sqrt{\pi}\alpha_k}\left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+1}\left(1+\frac{C_e(k)}{N}\right)n^{-\frac{N+1}{2}} = E^{[1]}_{N,1,e}(k)\cdot n^{-\frac{N+1}{2}} \tag{3.26}$$

with

$$E^{[1]}_{N,1,e}(k) := \frac{\sqrt{2}\,|\sin(\alpha_k)|}{\sqrt{\pi}\alpha_k}\left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+1}\left(1+\frac{C_e(k)}{N}\right). \tag{3.27}$$

Recalling (3.5), we set

$$C_o(k) := 2C_1^*(k)+\frac{(24k-1)(1+2C_1^*(k))}{6}. \tag{3.28}$$

Next, we have

$$n^{-\frac{1}{2}}\sum_{t\geq\frac{N+1}{2}}\frac{\left|\omega_k^{[1]}(2t+1)\right|}{n^t}$$

$$\leq \left(\frac{6}{\pi^3}\right)^{\frac{1}{2}}|\cos(\alpha_k)|\left(1+\frac{2C_1^*(k)}{N}\right)n^{-\frac{1}{2}}\sum_{t\geq\frac{N}{2}}\left(\frac{24k-1}{24n}\right)^t\sqrt{t+1}\quad\text{(by Lemma 3.4)}$$

$$= \left(\frac{3}{\pi^3}\right)^{\frac{1}{2}}|\cos(\alpha_k)|\left(1+\frac{2C_1^*(k)}{N}\right)\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N+2}n^{-\frac{N+1}{2}}\sum_{t\geq 0}\left(\frac{24k-1}{24n}\right)^t\sqrt{1+\frac{2t}{N+2}}$$

$$\leq \left(\frac{3}{\pi^3}\right)^{\frac{1}{2}}|\cos(\alpha_k)|\left(1+\frac{2C_1^*(k)}{N}\right)\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N+2}n^{-\frac{N+1}{2}}\sum_{t\geq 0}\left(\frac{24k-1}{24n}\right)^t\sqrt{1+t}$$

$$\left(\text{since for } N\geq 1, \frac{2t}{N+2}\leq t\right)$$

$$\leq \left(\frac{3}{\pi^3}\right)^{\frac{1}{2}}|\cos(\alpha_k)|\left(1+\frac{2C_1^*(k)}{N}\right)\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N+2}n^{-\frac{N+1}{2}}\sum_{t\geq 0}\left(\frac{24k-1}{12n}\right)^t$$

$$\left(\text{as } \sqrt{1+t}\leq 2^t \text{ for } t\geq 0\right)$$

$$\le \left(\frac{3}{\pi^3}\right)^{\frac{1}{2}} |\cos(\alpha_k)| \left(1+\frac{2C_1^*(k)}{N}\right)\left(\frac{24k-1}{24}\right)^{\frac{N}{2}} \sqrt{N+2}\left(1+\frac{24k-1}{6n}\right) n^{-\frac{N+1}{2}}$$

$$\le \left(\frac{3}{\pi^3}\right)^{\frac{1}{2}} |\cos(\alpha_k)| \left(1+\frac{2C_1^*(k)}{N}\right)\left(\frac{24k-1}{24}\right)^{\frac{N}{2}} \sqrt{N+2}\left(1+\frac{24k-1}{6N}\right) n^{-\frac{N+1}{2}}$$

(as for all $N \in \mathbb{N}, n > \widehat{g}(N+1) \ge N$ by definition of $\widehat{g}(m)$ in Theorem 2.1)

$$\le \left(\frac{3}{\pi^3}\right)^{\frac{1}{2}} |\cos(\alpha_k)| \left(\frac{24k-1}{24}\right)^{\frac{N}{2}} \sqrt{N+2}\left(1+\frac{C_o(k)}{N}\right) n^{-\frac{N+1}{2}} = E^{[1]}_{N,1,o}(k)\cdot n^{-\frac{N+1}{2}} \tag{3.29}$$

with

$$E^{[1]}_{N,1,o}(k) := \left(\frac{3}{\pi^3}\right)^{\frac{1}{2}} |\cos(\alpha_k)| \left(\frac{24k-1}{24}\right)^{\frac{N}{2}} \sqrt{N+2}\left(1+\frac{C_o(k)}{N}\right). \tag{3.30}$$

Recalling (3.27) and (3.30), set

$$E^{[1]}_{N,1}(k) := E^{[1]}_{N,1,e}(k) + E^{[1]}_{N,1,o}(k), \tag{3.31}$$

and then applying (3.26) and (3.29) to (3.24), we finally get for $n \ge \max_{k,N\ge 1}\{24k-1, \widehat{g}(N+1)\}$,

$$\left|\sum_{t=N+1}^{\infty} \frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}\right| \le E^{[1]}_{N,1}(k)\cdot n^{-\frac{N+1}{2}},$$

which finishes the proof of Lemma 3.7. □

Therefore, applying Lemma 3.7 to Lemma 3.3, we get for $(k,N)\in\mathbb{N}^2$ and $n \ge \max_{k,N\ge 1}\{24k-1, \widehat{g}(N+1)\}$,

$$\frac{\sqrt{12}e^{\mu(n+k)}}{24(n+k)-1}\left(1-\frac{1}{\mu(n+k)}\right) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{N}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}} + O_{\le E^{[1]}_{N,1}(k)}\left(n^{-\frac{N+1}{2}}\right)\right). \tag{3.32}$$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: Recalling the definition of $\widehat{g}(m)$ in Theorem 2.1, for $(k,N)\in\mathbb{N}^2$ we define

$$n^{[1]}_N(k) := \max_{(N,k)\in\mathbb{N}^2}\left\{\widehat{g}(N+1), (24k-1)^2\right\}. \tag{3.33}$$

Applying (3.32) to Lemma 3.2, we have for $(k,N)\in\mathbb{N}^2$ and for all $n \ge n^{[1]}_N(k)$,

$$p(n+k) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{N}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}} + O_{\le E^{[1]}_{N,1}(k)}\left(n^{-\frac{N+1}{2}}\right)\right) + \frac{\sqrt{12}e^{\mu(n+k)}}{24(n+k)-1}\cdot O_{\le 1}\left(\mu(n+k)^{-N-1}\right). \tag{3.34}$$

Finally, it remains to estimate the second factor on the right hand side of (3.34).

$$\frac{\sqrt{12}e^{\mu(n+k)}}{24(n+k)-1}\mu(n+k)^{-N-1}$$

$$= \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\exp\left(\mu(n+k)-\pi\sqrt{2n/3}\right)\left(1+\frac{24k-1}{24n}\right)^{-1}\mu(n+k)^{-N-1}$$

$$\le \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\exp\left(\mu(n+k)-\pi\sqrt{2n/3}\right)\mu(n+k)^{-N-1} \quad \left(\text{as } 1+\frac{24k-1}{24n}\ge 1 \text{ for } (k,n)\in\mathbb{N}^2\right)$$

$$\le \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\exp\left(\mu(n+k)-\pi\sqrt{2n/3}\right)\left(\frac{6}{\pi\sqrt{24}}\right)^{N+1} n^{-\frac{N+1}{2}}$$

$$\left(\text{as } \mu(n+k) = \frac{\pi}{6}\sqrt{24n+24k-1} \geq \frac{\pi}{6}\sqrt{24n} \text{ for } (k,n) \in \mathbb{N}^2\right)$$

$$= \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\left(\sqrt{1+\frac{24k-1}{24n}}-1\right)\right)\left(\frac{6}{\pi\sqrt{24}}\right)^{N+1} n^{-\frac{N+1}{2}}$$

$$\leq \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\cdot\frac{24k-1}{48n}\right)\left(\frac{6}{\pi\sqrt{24}}\right)^{N+1} n^{-\frac{N+1}{2}}$$

$$\left(\text{applying } (1+x)^r \leq 1+rx \text{ for } 0 \leq r \leq 1 \text{ and } x \geq -1 \;\text{ with } (x,r) = \left(\frac{24k-1}{24n}, \frac{1}{2}\right)\right)$$

$$\leq \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \left(1+\frac{\pi}{24}\sqrt{\frac{2}{3n}}(24k-1)\right)\left(\frac{6}{\pi\sqrt{24}}\right)^{N+1} n^{-\frac{N+1}{2}}$$

$$\left(\text{applying } e^x < 1+2x \text{ for } 0<x<1 \text{ with } x = \frac{\pi\sqrt{2}(24k-1)}{48\sqrt{3n}} \text{ and } 0<x<1 \text{ as } n \geq (24k-1)^2\right)$$

$$\leq \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \left(1+\frac{3.7\cdot k}{N}\right)\left(\frac{6}{\pi\sqrt{24}}\right)^{N+1} n^{-\frac{N+1}{2}} \quad \left(\text{as } n \geq \widehat{g}(N+1) \geq \frac{1}{2}N^2\right). \tag{3.35}$$

Recalling (3.31), we set

$$E_N^{[1]}(k) := E_{N,1}^{[1]}(k) + \left(1+\frac{3.7\cdot k}{N}\right)\left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}. \tag{3.36}$$

Finally, combining (3.34) and (3.35) concludes the proof of Theorem 3.1. □

4. Asymptotics of $\frac{1}{p(n)}$

This section is devoted to provide an asymptotic expansion for the inverse of the partition function $1/p(n)$ along with computations for error bounds to obtain a result similar to Theorem 3.1. We shall state the main result later (cf. Theorem 4.18) as it involves certain technical parameters which we will define later.

Lemma 4.1. *Let $\widehat{g}(m)$ be as in Theorem 2.1 and $N \in \mathbb{N}$. Then for all $n \geq \widehat{g}(N+1)$,*

$$\frac{1}{p(n)} = \frac{24n-1}{\sqrt{12}} e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1} + 4n\sqrt{3}\; e^{-\pi\sqrt{2n/3}} \cdot O_{\leq E_{N,2}^{[2]}}\left(n^{-\frac{N+1}{2}}\right),$$

where

$$E_{N,2}^{[2]} := \left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}\left(1+\frac{4}{N}\right). \tag{4.1}$$

Proof. Applying Theorem 2.1 with $m = N+1$ and $N \in \mathbb{N}$, for all $n > \widehat{g}(N+1)$ it follows that

$$\frac{24n-1}{\sqrt{12}} e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1}\left(1+\mu(n)^{-N-1}\left(1-\frac{1}{\mu(n)}\right)^{-1}\right)^{-1} < \frac{1}{p(n)}$$

$$< \frac{24n-1}{\sqrt{12}} e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1}\left(1-\mu(n)^{-N-1}\left(1-\frac{1}{\mu(n)}\right)^{-1}\right)^{-1}. \tag{4.2}$$

Next, for all $N \in \mathbb{N}$ this gives

$$\left(1+\mu(n)^{-N-1}\left(1-\frac{1}{\mu(n)}\right)^{-1}\right)^{-1} \geq 1-\mu(n)^{-N-1}\left(1-\frac{1}{\mu(n)}\right)^{-1}$$

$$\left(\text{applying } (1+x)^{-1} > 1-x \text{ for } x>0 \text{ with } x=\mu(n)^{-N-1}\left(1-\frac{1}{\mu(n)}\right)^{-1}\right)$$

$$\geq 1-\left(\frac{6}{\pi\sqrt{24}}\right)^{\frac{N+1}{2}} n^{-\frac{N+1}{2}}\left(1-\frac{1}{24n}\right)^{-\frac{N+1}{2}}\left(1+\frac{1}{\sqrt{2n}}\right)$$

$$\left(\text{as } \left(1-\frac{1}{\mu(n)}\right)^{-1} \leq 1+\frac{1}{\sqrt{2n}} \text{ for } n\in\mathbb{N}\right)$$

$$\geq 1-\left(\frac{6}{\pi\sqrt{24}}\right)^{\frac{N+1}{2}} n^{-\frac{N+1}{2}}\left(1-\frac{N+1}{48n}\right)^{-1}\left(1+\frac{1}{\sqrt{2n}}\right)$$

$$\left(\text{applying } (1+x)^r \geq 1+rx \text{ for } (r,x)\in\mathbb{R}_{\geq 1}\times\mathbb{R}_{>-1} \text{ with } (r,x)=\left(\frac{N+1}{2},-\frac{1}{24n}\right)\right)$$

$$\geq 1-\left(\frac{6}{\pi\sqrt{24}}\right)^{\frac{N+1}{2}} n^{-\frac{N+1}{2}}\left(1-\frac{1}{12N}\right)^{-1}\left(1+\frac{1}{N}\right)$$

$$\left(\text{as for } n>\widehat{g}(N+1)\geq\frac{N^2}{2}, \sqrt{2n}\geq N \text{ and } \frac{N+1}{48n}\leq\frac{1}{12N}\right)$$

$$> 1-\left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}\left(1+\frac{2}{N}\right) n^{-\frac{N+1}{2}}. \tag{4.3}$$

Similarly, we obtain for $n\in\mathbb{Z}_{\geq 4}$ and $N\in\mathbb{N}$,

$$\left(1-\mu(n)^{-N-1}\left(1-\frac{1}{\mu(n)}\right)^{-1}\right)^{-1} < 1+\left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}\left(1+\frac{3}{N}\right) n^{-\frac{N+3}{2}}. \tag{4.4}$$

Define

$$E^{[2]}_{N,1} := \left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}\left(1+\frac{3}{N}\right). \tag{4.5}$$

Applying (4.3) and (4.4) to (4.2) gives

$$\begin{aligned}
\frac{1}{p(n)} &= \frac{24n-1}{\sqrt{12}} e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1}\left(1+O_{\leq E^{[2]}_{N,1}}\left(n^{-\frac{N+1}{2}}\right)\right)\\
&= \frac{24n-1}{\sqrt{12}} e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1} + \frac{24n-1}{\sqrt{12}} e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1}\cdot O_{\leq E^{[2]}_{N,1}}\left(n^{-\frac{N+1}{2}}\right)\\
&= \frac{24n-1}{\sqrt{12}} e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1}\\
&\quad + 4n\sqrt{3}e^{-\pi\sqrt{2n/3}}\left(1-\frac{1}{24n}\right)\exp\left(-\pi\sqrt{2n/3}\left(\sqrt{1-\frac{1}{24n}}-1\right)\right)\cdot O_{\leq E^{[2]}_{N,1}}\left(n^{-\frac{N+1}{2}}\right).
\end{aligned}$$

Consequently, using the fact that for all $N\in\mathbb{N}$, $n\geq\widehat{g}(N+1)>\frac{1}{2}N^2$, we can bound the term involving the factor $n^{-\frac{N+1}{2}}$ as

$$\begin{aligned}
&E^{[2]}_{N,1}\left(1-\frac{1}{24n}\right)\exp\left(-\pi\sqrt{2n/3}\left(\sqrt{1-\frac{1}{24n}}-1\right)\right) n^{-\frac{N+1}{2}}\\
&= \left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}\left(1+\frac{3}{N}\right)\left(1-\frac{1}{24n}\right)\exp\left(-\pi\sqrt{2n/3}\left(\sqrt{1-\frac{1}{24n}}-1\right)\right) n^{-\frac{N+1}{2}} \quad \text{(by (4.5))}
\end{aligned}$$

$$\leq \left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}\left(1+\frac{3}{N}\right)\exp\left(-\pi\sqrt{2n/3}\left(\sqrt{1-\frac{1}{24n}}-1\right)\right)n^{-\frac{N+1}{2}}$$
$$\leq \left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}\left(1+\frac{3}{N}\right)\left(1+\frac{1}{4\sqrt{2n}}\right)n^{-\frac{N+1}{2}} \leq \left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}\left(1+\frac{3}{N}\right)\left(1+\frac{1}{4N}\right)n^{-\frac{N+1}{2}}$$
$$\left(\text{as } n \geq \widehat{g}(N+1) > \frac{1}{2}N^2\right)$$
$$\leq \left(\frac{6}{\pi\sqrt{24}}\right)^{N+1}\left(1+\frac{4}{N}\right)n^{-\frac{N+1}{2}} = E^{[2]}_{N,2}n^{-\frac{N+1}{2}}.$$

Thus we obtain

$$\frac{1}{p(n)} = \frac{24n-1}{\sqrt{12}}e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1} + 4n\sqrt{3}\ e^{-\pi\sqrt{2n/3}}\cdot O_{\leq E^{[2]}_{N,2}}\left(n^{-\frac{N+1}{2}}\right),$$

which finishes the proof of Lemma 4.1. □

Our next task is to derive the Taylor series expansion of the dominant term $\frac{24n-1}{\sqrt{12}}e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1}$ after extracting the factor $4n\sqrt{3}e^{-\pi\sqrt{2n/3}}$. To do so, we first observe that

$$\frac{24n-1}{\sqrt{12}}e^{-\mu(n)}\left(1-\frac{1}{\mu(n)}\right)^{-1} = 4n\sqrt{3}e^{-\pi\sqrt{2n/3}}\exp\left(-\mu(n)+\pi\sqrt{\frac{2n}{3}}\right)\left(1-\frac{1}{24n}\right)\left(1-\frac{1}{\mu(n)}\right)^{-1}$$
$$= 4n\sqrt{3}e^{-\pi\sqrt{2n/3}}\cdot T_1(n)\cdot T_2(n),$$

where

$$T_1(n) := \exp\left(-\mu(n)+\pi\sqrt{\frac{2n}{3}}\right) \quad \text{and} \quad T_2(n) := \left(1-\frac{1}{24n}\right)\left(1-\frac{1}{\mu(n)}\right)^{-1}.$$

Throughout the rest, set

$$\alpha := \frac{\pi}{6}, \tag{4.6}$$

and define

$$(a)_k := \begin{cases} a(a+1)\cdots(a+k-1), & \text{if } k \geq 1 \\ 1, & \text{if } k = 0 \end{cases}.$$

Define for all $t \in \mathbb{Z}_{\geq 0}$,

$$E_1(t) := \begin{cases} 1, & \text{if } t = 0 \\ \left(\frac{-1}{24}\right)^t \frac{\left(\frac{1}{2}-t\right)_{t+1}}{t}\sum_{u=1}^{t}\frac{(-1)^u(-t)_u}{(t+u)!(2u-1)!}\alpha^{2u}, & \text{if } t \geq 1 \end{cases}, \tag{4.7}$$

and

$$O_1(t) := \frac{\pi}{12\sqrt{6}}\frac{(-1)^t\left(\frac{1}{2}-t\right)_{t+1}}{24^t}\sum_{u=0}^{t}\frac{(-1)^u(-t)_u}{(t+u+1)!(2u)!}\alpha^{2u}. \tag{4.8}$$

Lemma 4.2. *Let $E_1(t)$ and $O_1(t)$ be as in (4.7) and (4.8), respectively. Then we have*

$$T_1(n) = \sum_{t=0}^{\infty}\frac{E_1(t)}{\sqrt{n}^{2t}} + \sum_{t=0}^{\infty}\frac{O_1(t)}{\sqrt{n}^{2t+1}}. \tag{4.9}$$

Proof. From [6, Lemma 3.4], we already know the coefficients from the Taylor series expansion of $T_1(n)^{-1}$ which take the following shape,

$$\exp\left(\mu(n)-\pi\sqrt{\frac{2n}{3}}\right)=\sum_{t=0}^{\infty}\frac{E_1(t)}{\sqrt{n}^{2t}}-\sum_{t=0}^{\infty}\frac{O_1(t)}{\sqrt{n}^{2t+1}}.$$

Using the fact that for the coefficient functional $[t^n]\sum_{k\geq 0}a_kt^k:=a_n$,

$$[t^{2k}]\left(e^t\right)=[t^{2k}]\left(e^{-t}\right)\quad\text{and}\quad[t^{2k+1}]\left(e^t\right)=-[t^{2k+1}]\left(e^{-t}\right),\ k\geq 0,$$

we conclude (4.9). ∎

For $t\in\mathbb{Z}_{\geq 0}$, define

$$e_2(t):=\begin{cases}1, & \text{if } t=0\\ \frac{36}{\pi^2+36}\left(\frac{(1+\alpha^{-2})}{24}\right)^t, & \text{if } t\geq 1\end{cases},\quad\text{and}\quad o_2(t):=\frac{6}{\pi\sqrt{24}}\left(\frac{-1}{24}\right)^t\sum_{m=0}^{t}\left(-\alpha^{-2}\right)^m\binom{-\frac{2m+1}{2}}{t-m}. \tag{4.10}$$

Following (4.10), we set

$$E_2(t):=\begin{cases}1, & \text{if } t=0\\ \frac{36-\pi^2}{24\pi^2}, & \text{if } t=1\\ \frac{54}{\pi^4(1+\alpha^{-2})}\left(\frac{(1+\alpha^{-2})}{24}\right)^{t-1}, & \text{if } t\geq 2\end{cases},\quad\text{and}\quad O_2(t):=\begin{cases}o_2(t), & \text{if } t=0\\ o_2(t)-\frac{o_2(t-1)}{24}, & \text{if } t\geq 1.\end{cases} \tag{4.11}$$

Lemma 4.3. *Let $E_2(t)$ and $O_2(t)$ be as in (4.10). Then*

$$T_2(n)=\sum_{t=0}^{\infty}\frac{E_2(t)}{\sqrt{n}^{2t}}+\sum_{t=0}^{\infty}\frac{O_2(t)}{\sqrt{n}^{2t+1}}. \tag{4.12}$$

Proof. Taylor series expansion gives

$$\begin{aligned}\left(1-\frac{1}{\mu(n)}\right)^{-1}&=\sum_{m=0}^{\infty}\left(\frac{6}{\pi\sqrt{24}}\right)^m n^{-\frac{m}{2}}\left(1-\frac{1}{24n}\right)^{-\frac{m}{2}}\\&=\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\left(\frac{6}{\pi\sqrt{24}}\right)^m\binom{-\frac{m}{2}}{\ell}\left(\frac{-1}{24}\right)^{\ell}\left(\frac{1}{\sqrt{n}}\right)^{m+2\ell}\\&=\sum_{t=0}^{\infty}\sum_{m=0}^{t}\left(\frac{6}{\pi\sqrt{24}}\right)^{2m}\binom{-m}{t-m}\left(\frac{-1}{24}\right)^{t-m}\left(\frac{1}{\sqrt{n}}\right)^{2t}\\&\qquad+\sum_{t=0}^{\infty}\sum_{m=0}^{t}\left(\frac{6}{\pi\sqrt{24}}\right)^{2m+1}\binom{-\frac{2m+1}{2}}{t-m}\left(\frac{-1}{24}\right)^{t-m}\left(\frac{1}{\sqrt{n}}\right)^{2t+1}\\&=\sum_{t=0}^{\infty}\frac{e_2(t)}{\sqrt{n}^{2t}}+\sum_{t=0}^{\infty}\frac{o_2(t)}{\sqrt{n}^{2t+1}}.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}T_2(n)=\left(1-\frac{1}{24n}\right)\left(1-\frac{1}{\mu(n)}\right)^{-1}&=\left(1-\frac{1}{24n}\right)\left(\sum_{t=0}^{\infty}\frac{e_2(t)}{\sqrt{n}^{2t}}+\sum_{t=0}^{\infty}\frac{o_2(t)}{\sqrt{n}^{2t+1}}\right)\\&=\sum_{t=0}^{\infty}\frac{e_2(t)}{\sqrt{n}^{2t}}+\sum_{t=0}^{\infty}\frac{o_2(t)}{\sqrt{n}^{2t+1}}-\frac{1}{24}\sum_{t=0}^{\infty}\frac{e_2(k)}{\sqrt{n}^{2t+2}}-\frac{1}{24}\sum_{t=0}^{\infty}\frac{o_2(t)}{\sqrt{n}^{2t+3}}\end{aligned}$$

$$= \sum_{t=0}^{\infty} \frac{E_2(t)}{\sqrt{n}^{2t}} + \sum_{t=0}^{\infty} \frac{O_2(t)}{\sqrt{n}^{2t+1}}.$$

In the last step we used the fact that $E_2(k) = e_2(k) - \dfrac{e_2(k-1)}{24}$ for all $k \geq 2$. This finishes the proof of Lemma 4.3. □

Definition 4.4. *For $t \in \mathbb{Z}_{\geq 0}$, define*

$$g_{e,1}(t) := \begin{cases} 1, & \text{if } t = 0 \\ \dfrac{\pi^4 - 288\pi^2 + 10368}{6912\pi^2}, & \text{if } t = 1 \\ S_1(t) + \frac{3(1-\alpha^2)}{2\pi^2} S_1(t-1) + \frac{3}{2\pi^2(1+\alpha^2)} \left(\frac{(1+\alpha^{-2})}{24} \right)^{t-1} (1 + S_2(t)), & \text{if } t \geq 2 \end{cases} \tag{4.13}$$

with

$$S_1(t) := \left(\frac{-1}{24} \right)^t \frac{\left(\frac{1}{2} - t\right)_{t+1}}{t} \sum_{u=1}^{t} \frac{(-1)^u (-t)_u}{(t+u)!(2u-1)!} \alpha^{2u} \tag{4.14}$$

and

$$S_2(t) := \sum_{s=1}^{t-2} \frac{\left(-\left(1 + \alpha^{-2}\right)\right)^{-s} \left(\frac{1}{2} - s\right)_{s+1}}{s} \sum_{u=1}^{s} \frac{(-1)^u (-s)_u}{(s+u)!(2u-1)!} \alpha^{2u}. \tag{4.15}$$

Definition 4.5. *For $t \in \mathbb{Z}_{\geq 0}$, define*

$$g_{e,2}(t) := \frac{1}{(24)^t} \left(\frac{1}{1+\alpha^2} S_3(t) - \frac{8}{(1+\alpha^{-2})} S_4(t) + S_5(t) \right) \tag{4.16}$$

with

$$S_3(t) := (-1)^{t-1} \sum_{s=0}^{t-2} \left(\frac{1}{2} - s \right)_{s+1} \sum_{u=0}^{s} \frac{(-1)^u (-s)_u}{(s+u+1)!(2u)!} \alpha^{2u} \sum_{r=0}^{t-s-1} \left(\frac{-1}{\alpha^2} \right)^r \binom{-\frac{2r+1}{2}}{t-s-r-1}, \tag{4.17}$$

$$S_4(t) := \sum_{s=0}^{t-2} \frac{(-1)^s \left(\frac{1}{2} - s\right)_{s+1}}{4^{t-s}(2t-2s-3)} \binom{2t-2s-3}{t-s-1} \sum_{u=0}^{s} \frac{(-1)^u (-s)_u}{(s+u+1)!(2u)!} \alpha^{2u}, \tag{4.18}$$

and

$$S_5(t) := (-1)^{t-1} \left(\frac{3}{2} - t \right)_t \sum_{u=0}^{t-1} \frac{(-1)^u (-t+1)_u}{(t+u)!(2u)!} \alpha^{2u}. \tag{4.19}$$

Definition 4.6. *Let $S_1(t)$ be as in (4.14). For $t \in \mathbb{Z}_{\geq 0}$, define*

$$g_{o,1}(t) := \begin{cases} \frac{\sqrt{6}}{2\pi}, & \text{if } t = 0 \\ \frac{\pi^4 - 144\pi^2 + 10368}{2306\sqrt{6}\pi^3}, & \text{if } t = 1 \\ \frac{\sqrt{6}}{2\pi} \left(S_1(t) + \frac{\left(\frac{1}{24}\right)^t}{1+\alpha^2} \left(S_6(t) + S_7(t)\right) - \frac{2}{(1+\alpha^{-2}) \cdot 96^t} \frac{\binom{2t-1}{t}}{2t-1} - \frac{2\alpha^2}{(1+\alpha^2) \cdot 24^t} S_8(t) \right), & \text{if } t \geq 2 \end{cases}, \tag{4.20}$$

with

$$S_6(t) := (-1)^t \sum_{s=0}^{t} \left(\frac{-1}{\alpha^2} \right)^s \binom{-\frac{2s+1}{2}}{t-s}, \tag{4.21}$$

$$S_7(t) := (-1)^t \sum_{s=1}^{t-1} \frac{\left(\frac{1}{2} - s\right)_{s+1}}{s} \sum_{u=1}^{s} \frac{(-1)^u (-s)_u}{(s+u)!(2u-1)!} \alpha^{2u} \sum_{r=0}^{t-s} \left(\frac{-1}{\alpha^2} \right)^r \binom{-\frac{2r+1}{2}}{t-s-r}, \tag{4.22}$$

and

$$S_8(t) := \sum_{s=1}^{t-1} \frac{(-1)^s \left(\frac{1}{2}-s\right)_{s+1} \binom{2t-2s-1}{t-s}}{s \cdot 4^{t-s}(2t-2s-1)} \sum_{u=1}^{s} \frac{(-1)^u(-s)_u}{(s+u)!(2u-1)!}\alpha^{2u}. \tag{4.23}$$

Definition 4.7. *Let $S_5(t)$ be as in (4.19). For $t \in \mathbb{Z}_{\geq 0}$, define*

$$g_{o,2}(t) := \begin{cases} \frac{\pi}{24\sqrt{6}}, & \text{if } t = 0 \\ \frac{\pi}{12\sqrt{6}} \left(\frac{1}{24}\right)^t \left(\frac{(1+\alpha^{-2})^t}{(1+\alpha^2)^2} S_9(t) + \frac{1-\alpha^2}{\alpha^2} S_5(t) + S_5(t+1)\right), & \text{if } t \geq 1 \end{cases} \tag{4.24}$$

with

$$S_9(t) := \sum_{s=0}^{t-2} \left(\frac{-1}{(1+\alpha^{-2})}\right)^s \left(\frac{1}{2}-s\right)_{s+1} \sum_{u=0}^{s} \frac{(-1)^u(-s)_u}{(s+u+1)!(2u)!}\alpha^{2u}. \tag{4.25}$$

Definition 4.8. *For $j \in \{1,2\}$, let $g_{e,j}(t)$ and $g_{o,j}(t)$ be as in Definitions 4.4 to 4.7. For $t \in \mathbb{Z}_{\geq 0}$, define*

$$g_e(t) := g_{e,1}(t) + g_{e,2}(t) \quad \text{and} \quad g_o(t) := g_{o,1}(t) + g_{o,2}(t).$$

Lemma 4.9. *For $t \in \mathbb{Z}_{\geq 0}$, let $g_e(t)$ and $g_o(t)$ be as in Definition 4.8. Then*

$$\frac{24n-1}{\sqrt{12}} e^{-\mu(n)} \left(1 - \frac{1}{\mu(n)}\right)^{-1} = 4n\sqrt{3}\ e^{-\pi\sqrt{2n/3}} \sum_{t=0}^{\infty} \frac{g(t)}{n^{\frac{t}{2}}}, \tag{4.26}$$

where

$$g(2t) = g_e(t) \quad \text{and} \quad g(2t+1) = g_o(t). \tag{4.27}$$

Proof. Applying the Cauchy product for the power series representations of $T_1(n)$ and $T_2(n)$ from Lemmas 4.2 and 4.3, we obtain the convoluted coefficients $(g(2t))_{t\geq 0}$ and $(g(2t+1))_{t\geq 0}$. □

Next, we estimate an error bound for the absolute value of remainder terms $\sum_{t\geq N+1} g(t)/n^{\frac{t}{2}}$ for $N \geq 1$. In order to do so, we need to estimate bounds for $|g_e(t)|$ and $|g_o(t)|$ for $t \geq 2$ and therefore, following Definitions 4.4 to 4.7, we see that deriving estimates for the sums $(S_j(t))_{1\leq j\leq 9}$ are prerequisites.

Lemma 4.10. *For all $t \in \mathbb{Z}_{\geq 2}$,*

$$S_1(t) = \frac{\alpha\sinh(\alpha)}{2\sqrt{\pi}} \frac{1}{24^t \cdot t^{\frac{3}{2}}} \left(1 + O_{\leq 2.6}\left(\frac{1}{t}\right)\right), \tag{4.28}$$

$$S_2(t) = \left(\cosh\left(\sqrt{1+\alpha^2}-1\right) - 1\right)\left(1 + O_{\leq 54.9}\left(\frac{1}{t}\right)\right), \tag{4.29}$$

$$S_3(t) = \frac{\sinh\left(\sqrt{1+\alpha^2}-1\right)}{1+\alpha^2} \left(\frac{1+\alpha^2}{\alpha^2}\right)^t \left(1 + O_{\leq 15.3}\left(\frac{1}{t}\right)\right), \tag{4.30}$$

$$S_4(t) = \frac{\alpha\cosh(\alpha) + \sinh(\alpha)}{16\sqrt{\pi}\cdot\alpha} \frac{1}{t^{\frac{3}{2}}} \left(1 + O_{\leq 6.7}\left(\frac{1}{t}\right)\right), \tag{4.31}$$

$$S_5(t) = \frac{\cosh(\alpha)}{2\sqrt{\pi}\cdot t^{\frac{3}{2}}} \left(1 + O_{\leq 1.2}\left(\frac{1}{t}\right)\right), \tag{4.32}$$

$$S_6(t) = \frac{1}{(1+\alpha^2)^{\frac{1}{2}}} \left(\frac{1+\alpha^2}{\alpha^2}\right)^t \left(1 + O_{\leq 0.2}\left(\frac{1}{t}\right)\right), \tag{4.33}$$

$$S_7(t) = \frac{\cosh\left(\sqrt{1+\alpha^2}-1\right) - 1}{\sqrt{1+\alpha^2}} \left(\frac{1+\alpha^2}{\alpha^2}\right)^t \left(1 + O_{\leq 14}\left(\frac{1}{t}\right)\right), \tag{4.34}$$

$$S_8(t) = \frac{\alpha\sinh(\alpha)+\cosh(\alpha)-1}{4\sqrt{\pi}\cdot t^{\frac{3}{2}}}\left(1+O_{\leq 9}\left(\frac{1}{t}\right)\right), \tag{4.35}$$

$$S_9(t) = \frac{\sqrt{1+\alpha^2}\sinh\left(\sqrt{1+\alpha^2}-1\right)}{\alpha^2}\left(1+O_{\leq 7.7}\left(\frac{1}{t}\right)\right). \tag{4.36}$$

Proof. See Section 6.2. □

Now, we apply Lemma 4.10 to derive bounds for $g_{e,j}(t)$ and $g_{o,j}(t)$ with $j \in \{1,2\}$ and consequently for $(g(2t))_{t\geq 0}$ and $(g(2t+1))_{t\geq 0}$ in accordance with Definition 4.8 and (4.27).

Lemma 4.11. *For* $t \geq 1$,

$$g_{e,1}(t) = \frac{3}{2\pi^2(1+\alpha^2)}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1}\cosh\left(\sqrt{1+\alpha^2}-1\right)\left(1+O_{\leq 1.2}\left(\frac{1}{t}\right)\right).$$

Proof. By (4.28) with $t \mapsto t-1$, for all $t \geq 3$, we have

$$\begin{aligned} S_1(t-1) &= \frac{\alpha\sinh(\alpha)}{2\sqrt{\pi}}\frac{24}{24^t\cdot t^{\frac{3}{2}}}\left(1-\frac{1}{t}\right)^{-\frac{3}{2}}\left(1+O_{\leq 2.6}\left(\frac{1}{t-1}\right)\right) \\ &= \frac{\alpha\sinh(\alpha)}{2\sqrt{\pi}}\frac{24}{24^t\cdot t^{\frac{3}{2}}}\left(1+O_{\leq 3.7}\left(\frac{1}{t}\right)\right)\left(1+O_{\leq 5.2}\left(\frac{1}{t}\right)\right) \\ &= \frac{\alpha\sinh(\alpha)}{2\sqrt{\pi}}\frac{24}{24^t\cdot t^{\frac{3}{2}}}\left(1+O_{\leq 19}\left(\frac{1}{t}\right)\right), \end{aligned}$$

which implies that

$$\frac{3(1-\alpha^2)}{2\pi^2}S_1(t-1) = \frac{1-\alpha^2}{\alpha^2}\frac{\alpha\sinh(\alpha)}{2\sqrt{\pi}}\frac{1}{24^t\cdot t^{\frac{3}{2}}}\left(1+O_{\leq 19}\left(\frac{1}{t}\right)\right). \tag{4.37}$$

We note the fact that for

$$f(t) = A_1\cdot g(t)\left(1+O_{\leq E_1}\left(\frac{1}{t}\right)\right) \quad \text{and} \quad h(t) = A_2\cdot g(t)\left(1+O_{\leq E_2}\left(\frac{1}{t}\right)\right)$$

with $(A_1, A_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, then

$$f(t)+h(t) = (A_1+A_2)\cdot g(t)\left(1+O_{\leq\left|\frac{A_1\cdot E_1+A_2\cdot E_2}{A_1+A_2}\right|}\left(\frac{1}{t}\right)\right). \tag{4.38}$$

Combining (4.28) and (4.37) and then applying (4.38) with

$$\begin{aligned} &(f(t), g(t), h(t), A_1, A_2, E_1, E_2) \\ &\quad\mapsto \left(S_1(t), \frac{1}{24^t\cdot t^{\frac{3}{2}}}, \frac{3(1-\alpha^2)}{2\pi^2}S_1(t-1), \frac{\alpha\sinh(\alpha)}{2\sqrt{\pi}}, \frac{(1-\alpha^2)\sinh(\alpha)}{\alpha\cdot 2\sqrt{\pi}}, 2.6, 19\right), \end{aligned}$$

we obtain

$$\begin{aligned} S_1(t)+\frac{3(1-\alpha^2)}{2\pi^2}S_1(t-1) &= \frac{\sinh(\alpha)}{\alpha\cdot 2\sqrt{\pi}\cdot 24^t\cdot t^{\frac{3}{2}}}\left(1+O_{\leq 2.6\alpha^2+\frac{19(1-\alpha^2)}{\alpha}}\left(\frac{1}{t}\right)\right) \\ &= \frac{\sinh(\alpha)}{\alpha\cdot 2\sqrt{\pi}\cdot 24^t\cdot t^{\frac{3}{2}}}\left(1+O_{\leq 27.1}\left(\frac{1}{t}\right)\right). \end{aligned} \tag{4.39}$$

Next, noting the implication

$$f(t) = C_1\left(1+O_{\leq E_1}\left(\frac{1}{t}\right)\right) \implies 1+f(t) = (1+C_1)\left(1+O_{\leq\frac{E_1\cdot C_1}{C_1+1}}\left(\frac{1}{t}\right)\right),$$

and applying it with

$$(f(t), C_1, E_1) \mapsto \left(S_2(t), \cosh\left(\sqrt{1+\alpha^2}-1\right)-1, 54.9\right),$$

one obtains from (4.29) that

$$\begin{aligned}1+S_2(t) &= \cosh\left(\sqrt{1+\alpha^2}-1\right)\left(1+O_{\le \frac{54.9\left(\cosh\left(\sqrt{1+\alpha^2}-1\right)-1\right)}{\cosh\left(\sqrt{1+\alpha^2}-1\right)}}\left(\frac{1}{t}\right)\right)\\ &= \cosh\left(\sqrt{1+\alpha^2}-1\right)\left(1+O_{\le 0.5}\left(\frac{1}{t}\right)\right),\end{aligned}$$

and hence,

$$\frac{3}{2\pi^2(1+\alpha^2)}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1}(1+S_2(t)) = \frac{3\cosh\left(\sqrt{1+\alpha^2}-1\right)}{2\pi^2(1+\alpha^2)}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1}\left(1+O_{\le 0.5}\left(\frac{1}{t}\right)\right). \tag{4.40}$$

Finally, applying (4.39) and (4.40) to (4.13), it follows that for $t \ge 3$,

$$\begin{aligned}&g_{e,1}(t)\\ &= \frac{\sinh(\alpha)}{\alpha\cdot 2\sqrt{\pi}\cdot 24^t\cdot t^{\frac{3}{2}}}\left(1+O_{\le 27.1}\left(\frac{1}{t}\right)\right) + \frac{3\cosh\left(\sqrt{1+\alpha^2}-1\right)}{2\pi^2(1+\alpha^2)}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1}\left(1+O_{\le 0.5}\left(\frac{1}{t}\right)\right)\\ &= \frac{3\cosh\left(\sqrt{1+\alpha^2}-1\right)}{2\pi^2(1+\alpha^2)}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1}\\ &\qquad\cdot\left(1+O_{\le 0.5}\left(\frac{1}{t}\right)+\frac{\pi^{\frac{3}{2}}\sinh(\alpha)(1+\alpha^2)}{72\alpha\cosh\left(\sqrt{1+\alpha^2}-1\right)}\frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^{t-1}}{t^{\frac{3}{2}}}\left(1+O_{\le 27.1}\left(\frac{1}{t}\right)\right)\right)\\ &= \frac{3\cosh\left(\sqrt{1+\alpha^2}-1\right)}{2\pi^2(1+\alpha^2)}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1}\left(1+O_{\le 1.2}\left(\frac{1}{t}\right)\right).\end{aligned}$$

We verify the remaining cases $1 \le t \le 2$ by numerical checks with computer algebra which altogether concludes the proof. □

Lemma 4.12. *For $t \in \mathbb{Z}_{\ge 1}$,*

$$g_{e,2}(t) = \frac{\sinh\left(\sqrt{1+\alpha^2}-1\right)}{(1+\alpha^2)^2}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^t\left(1+O_{\le 21}\left(\frac{1}{t}\right)\right).$$

Proof. Applying (4.30)-(4.32) to (4.16), we obtain the result for $t \ge 2$; we verify the case $t = 1$ by numerical checks with computer algebra which altogether concludes the proof. We omit details of the proof due to its similarity with that of Lemma 4.11. □

Lemma 4.13. *For $t \in \mathbb{Z}_{\ge 1}$, we have*

$$|g(2t)| < C_1(\alpha)\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1}\left(1+\frac{3.5}{t}\right),$$

where

$$C_1(\alpha) := \frac{3\left(\cosh\left(\sqrt{1+\alpha^2}-1\right)+\sinh\left(\sqrt{1+\alpha^2}-1\right)\right)}{2\pi^2(1+\alpha^2)}. \tag{4.41}$$

Proof. Applying Lemmas 4.11 and 4.12 to Definitions 4.8 and (4.27), proves the statement. □

Lemma 4.14. *For $t \in \mathbb{Z}_{\geq 1}$,*

$$g_{o,1}(t) = \frac{\sqrt{6}}{2\pi(1+\alpha^2)^{\frac{3}{2}}} \left(\frac{1+\alpha^2}{24\alpha^2}\right)^t \cosh\left(\sqrt{1+\alpha^2}-1\right)\left(1+O_{\leq 0.7}\left(\frac{1}{t}\right)\right).$$

Proof. Applying (4.28), (4.33)-(4.35) and using Lemma 2.3 to (4.20), we obtain the result for $t \geq 2$ and we verify the case $t = 1$ by numerical checks with computer algebra which concludes the proof. □

Lemma 4.15. *For $t \in \mathbb{Z}_{\geq 1}$,*

$$g_{o,2}(t) = \frac{\pi}{12\sqrt{6}\alpha^2(1+\alpha^2)^{\frac{3}{2}}} \left(\frac{1+\alpha^2}{24\alpha^2}\right)^t \sinh\left(\sqrt{1+\alpha^2}-1\right)\left(1+O_{\leq 13}\left(\frac{1}{t}\right)\right).$$

Proof. Applying (4.32) and (4.36) to (4.24), we obtain the result for $t \geq 2$; we verify the case $t = 1$ by numerical checks with computer algebra which altogether concludes the proof. □

Lemma 4.16. *For $t \in \mathbb{Z}_{\geq 1}$, we have*

$$|g(2t+1)| < C_2(\alpha)\left(\frac{1+\alpha^2}{24\alpha^2}\right)^t\left(1+\frac{0.5}{t}\right),$$

where

$$C_2(\alpha) := \frac{1}{\pi}\sqrt{\frac{3}{2}}\frac{\left(\cosh\left(\sqrt{1+\alpha^2}-1\right)+\sinh\left(\sqrt{1+\alpha^2}-1\right)\right)}{(1+\alpha^2)^{\frac{3}{2}}}. \tag{4.42}$$

Proof. Applying Lemmas 4.14 and 4.15 to Definitions 4.8 and (4.27) gives (4.42). □

Now we are ready to present both the statement and the proof of the main theorem of this section.

Definition 4.17. *Let $E^{[2]}_{N,2}$ be as in Lemma 4.1. Let $C_1(\alpha)$ and $C_2(\alpha)$ be as in Lemmas 4.13 and 4.16, respectively. Then for $N \in \mathbb{N}$, define*

$$E^{[2]}_{N,e} := C_1(\alpha)\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N-1}{2}}\left(1+\frac{8}{N}\right), \tag{4.43}$$

$$E^{[2]}_{N,o} := C_2(\alpha)\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}}\left(1+\frac{3}{N}\right), \tag{4.44}$$

$$E^{[2]}_{N} := E^{[2]}_{N,e} + E^{[2]}_{N,o} + E^{[2]}_{N,2}. \tag{4.45}$$

Theorem 4.18. *Let $N \in \mathbb{N}$. Let $\widehat{g}(m)$ be as in Theorem 2.1, $(g(t))_{t\geq 0}$ as in (4.26), and $E^{[2]}_N$ as in (4.45). Then for $n > \widehat{g}(N+1)$, we have*

$$\frac{1}{p(n)} = 4n\sqrt{3}\ e^{-\pi\sqrt{2n/3}}\left(\sum_{t=0}^{N}\frac{g(t)}{n^{\frac{t}{2}}} + O_{\leq E^{[2]}_N}\left(n^{-\frac{N+1}{2}}\right)\right). \tag{4.46}$$

Proof. From Lemmas 4.1 and 4.9, for all $n > \widehat{g}(N+1)$ we obtain

$$\frac{1}{p(n)} = 4n\sqrt{3}\ e^{-\pi\sqrt{2n/3}}\left(\sum_{t=0}^{N}\frac{g(t)}{n^{\frac{t}{2}}} + \sum_{t\geq N+1}\frac{g(t)}{n^{\frac{t}{2}}} + O_{\leq E^{[2]}_{N,2}}\left(n^{-\frac{N+1}{2}}\right)\right)$$

$$= 4n\sqrt{3}\, e^{-\pi\sqrt{2n/3}} \left(\sum_{t=0}^{N} \frac{g(t)}{n^{\frac{t}{2}}} + \underbrace{\sum_{t\geq \frac{N+1}{2}} \frac{g(2t)}{n^t}}_{=:S_e(n)} + \underbrace{\sum_{t\geq \frac{N}{2}} \frac{g(2t+1)}{n^{t+\frac{1}{2}}}}_{=:S_o(n)} + O_{\leq E^{[2]}_{N,2}} \left(n^{-\frac{N+1}{2}}\right) \right). \quad (4.47)$$

First, we bound $S_e(n)$,

$$|S_e(n)| \leq \sum_{t\geq \frac{N+1}{2}} \frac{|g(2t)|}{n^t} \leq C_1(\alpha) \sum_{t\geq \frac{N+1}{2}} \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1} \frac{1}{n^t} \left(1+\frac{3.5}{t}\right) \quad \text{(by Lemma 4.13)}$$

$$\leq C_1(\alpha)\left(1+\frac{7}{N+1}\right) \sum_{t\geq \frac{N+1}{2}} \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1} \frac{1}{n^t} \quad \left(\text{as } t \geq \frac{N+1}{2}\right)$$

$$= C_1(\alpha)\left(1+\frac{7}{N+1}\right) \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N-1}{2}} n^{-\frac{N+1}{2}} \sum_{t\geq 0} \left(\frac{1+\alpha^2}{24\alpha^2\cdot n}\right)^{t}$$

$$\leq C_1(\alpha)\left(1+\frac{7}{N+1}\right) \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N-1}{2}} n^{-\frac{N+1}{2}} \sum_{t\geq 0} \left(\frac{1+\alpha^2}{12\alpha^2\cdot N^2}\right)^{t} \quad \left(\text{as } n \geq \widehat{g}(N+1) \geq \frac{N^2}{2}\right)$$

$$\leq C_1(\alpha)\left(1+\frac{7}{N+1}\right) \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N-1}{2}} n^{-\frac{N+1}{2}} \sum_{t\geq 0} \left(\frac{1+\alpha^2}{12\alpha^2\cdot N}\right)^{t} \quad (\text{as } N \geq 1)$$

$$\leq C_1(\alpha)\left(1+\frac{7}{N+1}\right) \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N-1}{2}} n^{-\frac{N+1}{2}} \left(1+\frac{2}{3N}\right)$$

$$\leq C_1(\alpha)\left(1+\frac{8}{N}\right) \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N-1}{2}} n^{-\frac{N+1}{2}} = E^{[2]}_{N,e} \cdot n^{-\frac{N+1}{2}} \quad (\text{by (4.43)}). \quad (4.48)$$

Next, we bound $S_o(n)$,

$$|S_o(n)| \leq \sum_{t\geq \frac{N}{2}} \frac{|g(2t+1)|}{n^{t+\frac{1}{2}}} \leq C_2(\alpha) \sum_{t\geq \frac{N}{2}} \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t} \frac{1}{n^t} \left(1+\frac{0.5}{t}\right) n^{-\frac{1}{2}} \quad \text{(by Lemma 4.16)}$$

$$\leq C_2(\alpha)\left(1+\frac{1}{N}\right) n^{-\frac{1}{2}} \sum_{t\geq \frac{N}{2}} \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t} \frac{1}{n^t} \quad \left(\text{as } t \geq \frac{N+1}{2}\right)$$

$$= C_2(\alpha)\left(1+\frac{1}{N}\right) \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}} n^{-\frac{N+1}{2}} \sum_{t\geq 0} \left(\frac{1+\alpha^2}{24\alpha^2\cdot n}\right)^{t}$$

$$\leq C_2(\alpha)\left(1+\frac{1}{N}\right) \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}} n^{-\frac{N+1}{2}} \left(1+\frac{2}{3N}\right) \quad \left(\text{as } n > \widehat{g}(N+1) \geq \frac{N^2}{2}\right)$$

$$\leq C_2(\alpha)\left(1+\frac{3}{N}\right) \left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}} n^{-\frac{N+1}{2}} = E^{[2]}_{N,o} \cdot n^{-\frac{N+1}{2}} \quad (\text{by (4.44)}). \quad (4.49)$$

Applying (4.48) and (4.49) to (4.47) the statement of the theorem is proven. □

5. Asymptotics of $p(n+k)/p(n)$ (Proof of Theorem 1.1)

For all $(k,N)\in\mathbb{N}^2$, define

$$n_N(k) := \max_{(N,k)\in\mathbb{N}^2}\left\{\widehat{g}(N+1),(24k-1)^2\right\} \tag{5.1}$$

Combining Theorems 3.1 and 4.18, for all $n\geq n_N(k)$ we have

$$\begin{aligned}
\frac{p(n+k)}{p(n)} &= \left(\sum_{t=0}^{N}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}+O_{\leq E_N^{[1]}(k)}\left(n^{-\frac{N+1}{2}}\right)\right)\left(\sum_{t=0}^{N}\frac{g(t)}{n^{\frac{t}{2}}}+O_{\leq E_N^{[2]}}\left(n^{-\frac{N+1}{2}}\right)\right)\\
&= \sum_{t=0}^{N}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}\cdot\sum_{t=0}^{N}\frac{g(t)}{n^{\frac{t}{2}}}+\sum_{t=0}^{N}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}\cdot O_{\leq E_N^{[2]}}\left(n^{-\frac{N+1}{2}}\right)+\sum_{t=0}^{N}\frac{g(t)}{n^{\frac{t}{2}}}\cdot O_{\leq E_N^{[1]}(k)}\left(n^{-\frac{N+1}{2}}\right)\\
&\qquad + O_{\leq E_N^{[1]}(k)}\left(n^{-\frac{N+1}{2}}\right)\cdot O_{\leq E_N^{[2]}}\left(n^{-\frac{N+1}{2}}\right)\\
&= \sum_{t=0}^{N}\frac{1}{n^{\frac{t}{2}}}\sum_{s=0}^{t}\omega_k^{[1]}(s)\cdot g(t-s)+n^{-\frac{N+1}{2}}\sum_{t=0}^{N-1}\frac{1}{n^{\frac{t}{2}}}\sum_{s=t}^{N-1}\omega_k^{[1]}(s+1)g(N+t-s)\\
&\qquad +\sum_{t=0}^{N}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}\cdot O_{\leq E_N^{[2]}}\left(n^{-\frac{N+1}{2}}\right)+\sum_{t=0}^{N}\frac{g(t)}{n^{\frac{t}{2}}}\cdot O_{\leq E_N^{[1]}(k)}\left(n^{-\frac{N+1}{2}}\right)\\
&\qquad + O_{\leq E_N^{[1]}(k)}\left(n^{-\frac{N+1}{2}}\right)\cdot O_{\leq E_N^{[2]}}\left(n^{-\frac{N+1}{2}}\right)\\
&=: \sum_{t=0}^{N}\frac{c_k(t)}{n^{\frac{t}{2}}}+S_k^{[1]}(n;N)+S_k^{[2]}(n;N)+S_k^{[3]}(n;N)+S_k^{[4]}(n;N),
\end{aligned} \tag{5.2}$$

where for $t\in\mathbb{Z}_{\geq 0}$,

$$c_k(t)=\sum_{s=0}^{t}\omega_k^{[1]}(s)\cdot g(t-s). \tag{5.3}$$

Next, to determine the error bound, we estimate an upper bound for each of the sums $S_k^{[j]}(n;N), 1\leq j\leq 4$, individually.

To begin with, we first estimate $S_k^{[4]}(n;N)$. To estimate the error bound we need to refine the error bounds $E_N^{[1]}(k)$ and $E_N^{[2]}$. From (3.27) it follows that

$$\begin{aligned}
E_{N,1,e}^{[1]}(k) &= \frac{\sqrt{2}\,|\sin(\alpha_k)|}{\sqrt{\pi}\alpha_k}\left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+1}\left(1+\frac{C_e(k)}{N}\right)\\
&\leq \left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+1}\,(1+C_e(k))\ \ (\text{as } \alpha_k\geq 1 \text{ for } k\in\mathbb{N} \text{ and } N\geq 1)\\
&\leq \left(\frac{24k-1}{12}\right)^{\frac{N+1}{2}}(1+C_e(k))\quad \left(\text{as } \sqrt{N+1}\leq 2^{N+1} \text{ for } N\geq 1\right).
\end{aligned} \tag{5.4}$$

Similarly from (3.30), we obtain

$$\begin{aligned}
E_{N,1,o}^{[1]}(k) &= \left(\frac{3}{\pi^3}\right)^{\frac{1}{2}}|\cos(\alpha_k)|\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N+2}\left(1+\frac{C_o(k)}{N}\right)\\
&\leq \sqrt{\frac{24}{24k-1}}\left(\frac{24k-1}{24}\right)^{\frac{N+1}{2}}\sqrt{N+2}\,(1+C_o(k))\ \ (\text{as } N\geq 1)
\end{aligned}$$

$$\leq \left(\frac{24k-1}{12}\right)^{\frac{N+1}{2}} (1.1 + 1.1 \cdot C_o(k)) \quad \left(\text{as } \sqrt{N+2} \leq 2^{N+1} \text{ for } N \geq 1\right). \tag{5.5}$$

Applying (5.4) and (5.5) to (3.31) we obtain

$$E^{[1]}_{N,1}(k) \leq \left(\frac{24k-1}{12}\right)^{\frac{N+1}{2}} (2.1 + C_e(k) + 1.1 \cdot C_o(k)),$$

which by (3.36) implies

$$\begin{aligned}
E^{[1]}_{N}(k) &\leq \left(\frac{24k-1}{12}\right)^{\frac{N+1}{2}} (2.1 + C_e(k) + 1.1 \cdot C_o(k)) + \left(1 + \frac{3.7 \cdot k}{N}\right)\left(\frac{6}{\pi\sqrt{24}}\right)^{N+1} \\
&= \left(\frac{24k-1}{12}\right)^{\frac{N+1}{2}} \left(2.1 + C_e(k) + 1.1 \cdot C_o(k) + \left(1 + \frac{3.7 \cdot k}{N}\right)\left(\frac{6}{\pi\sqrt{2(24k-1)}}\right)^{N+1}\right) \\
&\leq \left(\frac{24k-1}{12}\right)^{\frac{N+1}{2}} \left(2.1 + C_e(k) + 1.1 \cdot C_o(k) + \left(1 + \frac{3.7 \cdot k}{N}\right)\left(\frac{6}{\pi\sqrt{46}}\right)^{2}\right) \quad \left(\text{as } (k,N) \in \mathbb{N}^2\right) \\
&\leq \left(\frac{24k-1}{12}\right)^{\frac{N+1}{2}} \left(2.2 + C_e(k) + 1.1 \cdot C_o(k) + \frac{0.3 \cdot k}{N}\right).
\end{aligned} \tag{5.6}$$

Analogously, from (4.45), it follows that

$$E^{[2]}_{N} \leq 4.1. \tag{5.7}$$

Following (5.2), we estimate the sum $S^{[4]}_k(n;N)$ as

$$\begin{aligned}
\left|S^{[4]}_k(n;N)\right| &\leq 4.1 \left(\frac{24k-1}{12}\right)^{\frac{N+1}{2}} \left(2.2 + C_e(k) + 1.1 \cdot C_o(k) + \frac{0.3 \cdot k}{N}\right) n^{-N-1} \quad (\text{by (5.6) and (5.7)}) \\
&= 4.1 \left(\frac{24k-1}{12n}\right)^{\frac{N+1}{2}} \left(2.8 + C_e(k) + 1.1 \cdot C_o(k) + \frac{0.3 \cdot k}{N}\right) n^{-\frac{N+1}{2}} \\
&\leq 4.1 \frac{24k-1}{12n} (2.2 + C_e(k) + 1.1 \cdot C_o(k) + 0.3 \cdot k)\, n^{-\frac{N+1}{2}} \\
&\qquad\qquad \left(\text{as for } n \geq (24k-1)^2 \text{ and } k \geq 1,\ \frac{24k-1}{12n} < 1\right) \\
&\leq 4.1 \frac{1}{12\sqrt{n}} (2.2 + C_e(k) + 1.1 \cdot C_o(k) + 0.3 \cdot k)\, n^{-\frac{N+1}{2}} \quad \left(\text{as } n \geq (24k-1)^2\right) \\
&\leq 4.1 \frac{\sqrt{2}}{12} (2.2 + C_e(k) + 1.1 \cdot C_o(k) + 0.3 \cdot k)\, \frac{1}{N} n^{-\frac{N+1}{2}} \\
&\qquad\qquad \left(\text{as } n \geq \widehat{g}(N+1) \geq \frac{1}{2}N^2 \text{ for all } N \geq 1\right) \\
&\leq E_{N,1}(k) \cdot n^{-\frac{N+1}{2}}
\end{aligned} \tag{5.8}$$

with

$$E_{N,1}(k) := \frac{1.1 + 0.5 \cdot C_e(k) + 0.6 \cdot C_o(k) + 0.2 \cdot k}{N}. \tag{5.9}$$

Moving on, following (5.2) we bound $S_k^{[3]}(n;N)$,

$$\begin{aligned}
\left|S_k^{[3]}(n;N)\right| &\leq E_N^{[1]}(k)\cdot n^{-\frac{N+1}{2}}\left|\sum_{t=0}^{N}\frac{g(t)}{n^{\frac{t}{2}}}\right| \leq E_N^{[1]}(k)\cdot n^{-\frac{N+1}{2}}\sum_{t=0}^{N}\frac{|g(t)|}{n^{\frac{t}{2}}}\\
&= E_N^{[1]}(k)\cdot n^{-\frac{N+1}{2}}\left(1+\frac{|g(1)|}{\sqrt{n}}+\sum_{t=1}^{\frac{N}{2}}\frac{|g(2t)|}{n^t}+\frac{1}{\sqrt{n}}\sum_{t=1}^{\frac{N-1}{2}}\frac{|g(2t+1)|}{n^t}\right)\\
&= E_N^{[1]}(k)\cdot n^{-\frac{N+1}{2}}\left(1+\frac{\frac{2239488-432\pi^4+\pi^6}{497664\sqrt{6}\pi^3}}{\sqrt{n}}+\sum_{t=1}^{\frac{N}{2}}\frac{|g(2t)|}{n^t}+\frac{1}{\sqrt{n}}\sum_{t=1}^{\frac{N-1}{2}}\frac{|g(2t+1)|}{n^t}\right)\\
&\qquad\text{(by Definitions 4.6 and 4.7)}\\
&\leq E_N^{[1]}(k)\cdot n^{-\frac{N+1}{2}}\left(1+\frac{0.1}{N}+\sum_{t=1}^{\frac{N}{2}}\frac{|g(2t)|}{n^t}+\frac{1}{\sqrt{n}}\sum_{t=1}^{\frac{N-1}{2}}\frac{|g(2t+1)|}{n^t}\right)\quad\left(\text{as } n>\widehat{g}(N+1)\geq\frac{N^2}{2}\right)\\
&\leq E_N^{[1]}(k)\cdot n^{-\frac{N+1}{2}}\left(1+\frac{0.1}{N}+\frac{1}{N}\sum_{t\geq 1}|g(2t)|+\frac{1}{N}\sum_{t\geq 1}|g(2t+1)|\right)\\
&\leq E_N^{[1]}(k)\cdot n^{-\frac{N+1}{2}}\left(1+\frac{0.1}{N}+\frac{4.5\cdot C_1(\alpha)}{N}\sum_{t\geq 1}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t-1}+\frac{1.5\cdot C_2(\alpha)}{N}\sum_{t\geq 1}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{t}\right)\\
&\qquad\text{(by Lemmas 4.13 and 4.16)}\\
&\leq E_{N,2}(k)\cdot n^{-\frac{N+1}{2}}\quad\text{(using (4.41) and (4.42))}
\end{aligned}\tag{5.10}$$

with

$$E_{N,2}(k) := E_N^{[1]}(k)\left(1+\frac{1}{N}\right). \tag{5.11}$$

Again, following (5.2) we bound $S_k^{[2]}(n;N)$,

$$\begin{aligned}
\left|S_k^{[2]}(n;N)\right| &\leq E_N^{[2]}\cdot n^{-\frac{N+1}{2}}\left|\sum_{t=0}^{N}\frac{\omega_k^{[1]}(t)}{n^{\frac{t}{2}}}\right| \leq E_N^{[2]}\cdot n^{-\frac{N+1}{2}}\sum_{t=0}^{N}\frac{\left|\omega_k^{[1]}(t)\right|}{n^{\frac{t}{2}}}\\
&= E_N^{[2]}\cdot n^{-\frac{N+1}{2}}\left(1+\frac{\left|\omega_k^{[1]}(1)\right|}{\sqrt{n}}+\sum_{t=1}^{\frac{N}{2}}\frac{\left|\omega_k^{[1]}(2t)\right|}{n^t}+\frac{1}{\sqrt{n}}\sum_{t=1}^{\frac{N-1}{2}}\frac{\left|\omega_k^{[1]}(2t+1)\right|}{n^t}\right)\\
&= E_N^{[2]}\cdot n^{-\frac{N+1}{2}}\left(1+\frac{\frac{\pi^2(24k-1)-72}{\pi\cdot 24\sqrt{6}}}{\sqrt{n}}+\sum_{t=1}^{\frac{N}{2}}\frac{|\omega_k^{[1]}(2t)|}{n^t}+\frac{1}{\sqrt{n}}\sum_{t=1}^{\frac{N-1}{2}}\frac{|\omega_k^{[1]}(2t+1)|}{n^t}\right)\\
&\qquad\text{(applying (3.2) for } t=1)\\
&\leq E_N^{[2]}\cdot n^{-\frac{N+1}{2}}\left(1+\frac{1.9\cdot k}{N}+\sum_{t=1}^{\frac{N}{2}}\frac{|\omega_k^{[1]}(2t)|}{n^t}+\frac{\sqrt{2}}{N}\sum_{t=1}^{\frac{N-1}{2}}\frac{|\omega_k^{[1]}(2t+1)|}{n^t}\right)\ \left(\text{as } n\geq\widehat{g}(N+1)\geq\frac{N^2}{2}\right).
\end{aligned}\tag{5.12}$$

Using Lemma 3.5 it follows that

$$\sum_{t=1}^{\frac{N}{2}} \frac{|\omega_k^{[1]}(2t)|}{n^t} \leq \frac{2\,|\sin(\alpha_k)|}{\sqrt{\pi}\alpha_k} \sum_{t=1}^{\frac{N}{2}} \left(\frac{24k-1}{24n}\right)^t \sqrt{t}\left(1+\frac{C_2^*(k)}{t}\right)$$
$$\leq \frac{12}{\pi^{\frac{3}{2}}\sqrt{23}}\left(1+C_2^*(k)\right)\sum_{t=1}^{\frac{N}{2}}\left(\frac{24k-1}{24n}\right)^t\sqrt{t}$$
$$\leq \frac{12}{\pi^{\frac{3}{2}}\sqrt{23}}\left(1+C_2^*(k)\right)\sum_{t=1}^{\frac{N}{2}}\left(\frac{24k-1}{12n}\right)^t \quad \left(\text{as } \sqrt{t}\leq 2^t \text{ for } t\geq 1\right)$$
$$= \frac{12}{\pi^{\frac{3}{2}}\sqrt{23}}\left(1+C_2^*(k)\right)\sum_{t=1}^{\frac{N}{2}}\left(\frac{24k-1}{12\sqrt{n}}\right)^t\frac{1}{n^{\frac{t}{2}}} \leq \frac{12}{\pi^{\frac{3}{2}}\sqrt{23}}\frac{\left(1+C_2^*(k)\right)}{\sqrt{n}}\sum_{t=1}^{\frac{N}{2}}\left(\frac{24k-1}{12\sqrt{n}}\right)^t$$
$$\leq \frac{12\sqrt{2}}{\pi^{\frac{3}{2}}\sqrt{23}}\frac{\left(1+C_2^*(k)\right)}{N}\sum_{t=1}^{\frac{N}{2}}\left(\frac{24k-1}{12\sqrt{n}}\right)^t \quad \left(\text{as } n\geq \widehat{g}(N+1)\geq \frac{N^2}{2}\right)$$
$$\leq \frac{12\sqrt{2}}{\pi^{\frac{3}{2}}\sqrt{23}}\frac{\left(1+C_2^*(k)\right)}{N}\sum_{t\geq 1}\left(\frac{1}{12}\right)^t \quad \left(\text{as } n\geq (24k-1)^2\right) = \frac{12\sqrt{2}}{11\pi^{\frac{3}{2}}\sqrt{23}}\frac{\left(1+C_2^*(k)\right)}{N}$$
$$\leq \frac{6\cdot 10^{-2}\left(1+C_2^*(k)\right)}{N}. \tag{5.13}$$

Similarly, using Lemma 3.4 we obtain

$$\sum_{t=1}^{\frac{N-1}{2}} \frac{|\omega_k^{[1]}(2t+1)|}{n^t} \leq \frac{3\cdot 10^{-2}\left(1+C_1^*(k)\right)}{N}. \tag{5.14}$$

Applying (5.13) and (5.13) to (5.12) we obtain

$$\left|S_k^{[2]}(n;N)\right| \leq E_{N,3}(k)\cdot n^{-\frac{N+1}{2}}, \tag{5.15}$$

with

$$E_{N,3}(k) = E_N^{[2]}\left(1+\frac{1.9\cdot k + 6\cdot 10^{-2}\left(1+C_2^*(k)\right)+5\cdot 10^{-2}\left(1+C_1^*(k)\right)}{N}\right). \tag{5.16}$$

We split the remaining sum $S_k^{[1]}(n;N)$ from (5.2) as follows,

$$S_k^{[1]}(n;N) = \left(\sum_{t=0}^{N-1}\frac{1}{n^{\frac{t}{2}}}\sum_{s=t}^{N-1}\omega_k^{[1]}(s+1)g(N+t-s)\right)n^{-\frac{N+1}{2}}$$
$$= \left(\sum_{s=0}^{N-1}\omega_k^{[1]}(s+1)g(N-s) + \sum_{t=1}^{N-1}\frac{1}{n^{\frac{t}{2}}}\sum_{s=t}^{N-1}\omega_k^{[1]}(s+1)g(N+t-s)\right)n^{-\frac{N+1}{2}}$$
$$=: S_{k,1}^{[1]}(n;N) + S_{k,2}^{[1]}(n;N). \tag{5.17}$$

Observe that for $N=1$,

$$S_{k,2}^{[1]}(n;N) = 0, \tag{5.18}$$

and so in this case, we get

$$\left|S_k^{[1]}(n;1)\right| = \left|\omega_k^{[1]}(1)\cdot g(1)\right|\cdot n^{-1} \leq \frac{0.1\cdot k}{n}. \tag{5.19}$$

By plugging in $t = 1$ into Definitions 4.6 and 4.7, along with (3.2) we estimate an upper bound for $\left|S_k^{[1]}(n;N)\right|$ with $t \geq 1$. First, using Lemmas (3.4) and (3.5) we obtain for all $t \geq 2$,

$$\left|\omega_k^{[1]}(t)\right| \leq \frac{1}{2}\left(\frac{24k-1}{24}\right)^{\frac{t}{2}} \sqrt{t}\,(1 + C^*(k)) \text{ with } C^*(k) = \max_{k\geq 1}\{C_1^*(k), C_2^*(k)\}. \tag{5.20}$$

In a similar way, for $t \geq 2$ it follows that

$$|g(t)| \leq 3.2\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{t}{2}}. \tag{5.21}$$

For all $N \geq 2$ one obtains

$$\begin{aligned}
&\left|S_k^{[1]}(n;N)\right| \\
&\leq \left(\sum_{s=0}^{N-1}\left|\omega_k^{[1]}(s+1)\right||g(N-s)|\right) n^{-\frac{N+1}{2}} \\
&\leq \left(\left|\omega_k^{[1]}(1)\right||g(N)| + \sum_{s=1}^{N-2}\left|\omega_k^{[1]}(s+1)\right||g(N-s)| + \left|\omega_k^{[1]}(N)\right||g(1)|\right) n^{-\frac{N+1}{2}} \\
&\leq \left(1.3\cdot k\,|g(N)| + \sum_{s=1}^{N-2}\left|\omega_k^{[1]}(s+1)\right||g(N-s)| + 6\cdot 10^{-2}\left|\omega_k^{[1]}(N)\right|\right) n^{-\frac{N+1}{2}} \\
&\qquad\qquad \text{(by Definitions 4.6 and (4.7), and using (3.2) for } t = 1) \\
&\leq \left(4.2\cdot k\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}} + \sum_{s=1}^{N-2}\left|\omega_k^{[1]}(s+1)\right|\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N-s}{2}} + 6\cdot 10^{-2}\left|\omega_k^{[1]}(N)\right|\right) n^{-\frac{N+1}{2}} \\
&\qquad\qquad \text{(by (5.21))} \\
&\leq \left(4.2\cdot k\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}} + \frac{(1+C^*(k))(24k-1)}{48}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}}\sum_{s=1}^{N-2}\left(\frac{\alpha^2(24k-1)}{1+\alpha^2}\right)^{\frac{s}{2}}\sqrt{s+1}\right. \\
&\qquad \left. + 3\cdot 10^{-2}\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\,(1+C^*(k))\right) n^{-\frac{N+1}{2}} \quad \text{(by (5.20))} \\
&\leq \left(4.2\cdot k\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}} + \frac{(1+C^*(k))(24k-1)}{48}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}}\sqrt{N}\sum_{s=1}^{N-2}\left(\frac{\alpha^2(24k-1)}{1+\alpha^2}\right)^{\frac{s}{2}}\right. \\
&\qquad \left. + 3\cdot 10^{-2}\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\,(1+C^*(k))\right) n^{-\frac{N+1}{2}} \quad \text{(as } s \leq N-2) \\
&= \left(4.2\cdot k\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}} + \frac{(1+C^*(k))(24k-1)}{48}\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\sum_{s=2}^{N-1}\left(\frac{1+\alpha^2}{\alpha^2(24k-1)}\right)^{\frac{s}{2}}\right. \\
&\qquad \left. + 3\cdot 10^{-2}\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\,(1+C^*(k))\right) n^{-\frac{N+1}{2}} \\
&\leq \left(4.2\cdot k\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}} + \frac{(1+C^*(k))(24k-1)}{48}\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\sum_{s\geq 2}\left(\frac{1+\alpha^2}{\alpha^2(24k-1)}\right)^{\frac{s}{2}}\right.
\end{aligned}$$

$$+ 3\cdot 10^{-2}\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\,(1+C^*(k))\Bigg)n^{-\frac{N+1}{2}}$$

$$= \left(4.2\cdot k\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}} + \left(\frac{(1+C^*(k))(24k-1)}{96} + 3\cdot 10^{-2}\right)\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\right)n^{-\frac{N+1}{2}}$$

$$= \left(3\cdot 10^{-2} + \frac{(1+C^*(k))(24k-1)}{96} + \frac{4.2\cdot k\left(\frac{1+\alpha^2}{\alpha^2(24k-1)}\right)^{\frac{N}{2}}}{\sqrt{N}}\right)\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\cdot n^{-\frac{N+1}{2}}$$

$$\leq \left(3\cdot 10^{-2} + \frac{(1+C^*(k))(24k-1)}{96} + \frac{0.9\cdot k}{\sqrt{N}}\right)\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\cdot n^{-\frac{N+1}{2}} \quad \left(\text{as } (k,N)\in\mathbb{N}^2\right)$$

$$= E^{[1]}_{N,4}(k)\cdot n^{-\frac{N+1}{2}} \tag{5.22}$$

with

$$E^{[1]}_{N,4}(k) := \left(3\cdot 10^{-2} + \frac{(1+C^*(k))(24k-1)}{96} + \frac{0.9\cdot k}{\sqrt{N}}\right)\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}. \tag{5.23}$$

Finally, we proceed to estimate $S^{[1]}_{k,2}(n;N)$ for $N\geq 2$.

$$\left|S^{[1]}_{k,2}(n;N)\right|$$

$$\leq \sum_{t=1}^{N-1}\frac{1}{n^{\frac{t}{2}}}\sum_{s=t}^{N-1}\left|\omega^{[1]}_k(s+1)\right|\left|g(N+t-s)\right|n^{-\frac{N+1}{2}} \quad (\text{by } (5.17))$$

$$\leq 1.6\,(1+C^*(k))\sqrt{\frac{24k-1}{24}}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}}\sum_{t=1}^{N-1}\left(\frac{1+\alpha^2}{24\alpha^2 n}\right)^{\frac{t}{2}}\sum_{s=t}^{N-1}\left(\frac{\alpha^2(24k-1)}{1+\alpha^2}\right)^{\frac{s}{2}}\sqrt{s+1}$$

(by (5.20) and (5.21))

$$\leq 1.6\,(1+C^*(k))\sqrt{\frac{24k-1}{24}}\left(\frac{1+\alpha^2}{24\alpha^2}\right)^{\frac{N}{2}}\sqrt{N}\sum_{t=1}^{N-1}\left(\frac{1+\alpha^2}{24\alpha^2 n}\right)^{\frac{t}{2}}\sum_{s=t}^{N-1}\left(\frac{\alpha^2(24k-1)}{1+\alpha^2}\right)^{\frac{s}{2}} \quad (\text{as } s\leq N-1)$$

$$= 1.6\sqrt{\frac{1+\alpha^2}{24\alpha^2}}\,(1+C^*(k))\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\sum_{t=1}^{N-1}\left(\frac{1+\alpha^2}{24\alpha^2 n}\right)^{\frac{t}{2}}\sum_{s=0}^{N-t-1}\left(\frac{1+\alpha^2}{\alpha^2(24k-1)}\right)^{\frac{s}{2}}$$

$$\leq 0.8\,(1+C^*(k))\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\sum_{t=1}^{N-1}\left(\frac{1+\alpha^2}{24\alpha^2 n}\right)^{\frac{t}{2}}\sum_{s=0}^{N-t-1}\left(\frac{1}{2}\right)^{s} \quad (\text{as } k\geq 1)$$

$$\leq 0.8\,(1+C^*(k))\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\sum_{t=1}^{N-1}\left(\frac{1+\alpha^2}{24\alpha^2 n}\right)^{\frac{t}{2}}\sum_{s\geq 0}\left(\frac{1}{2}\right)^{s}$$

$$= 1.6\,(1+C^*(k))\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\sum_{t=1}^{N-1}\left(\frac{1+\alpha^2}{24\alpha^2 n}\right)^{\frac{t}{2}}$$

$$\leq 1.6\,(1+C^*(k))\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\sqrt{N}\left(\frac{1+\alpha^2}{24\alpha^2\cdot n}\right)^{\frac{1}{2}}\sum_{t\geq 0}\left(\frac{1+\alpha^2}{24\alpha^2 n}\right)^{\frac{t}{2}}$$

$$\leq 1.6\,(1+C^*(k))\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\frac{1}{\sqrt{N}}\left(\frac{1+\alpha^2}{12\alpha^2}\right)^{\frac{1}{2}}\sum_{t\geq 0}\left(\frac{1+\alpha^2}{24\alpha^2 n}\right)^{\frac{t}{2}} \quad \left(\text{as } n>\widehat{g}(N+1)\geq\frac{N^2}{2}\right)$$

$$\leq 1.6\,(1+C^*(k))\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\frac{1}{\sqrt{N}}\left(\frac{1+\alpha^2}{12\alpha^2}\right)^{\frac{1}{2}}\sum_{t\geq 0}\left(\frac{1+\alpha^2}{48\alpha^2}\right)^{\frac{t}{2}}\quad\left(\text{as } n>\frac{N^2}{2}\geq 2 \text{ for } N\geq 2\right)$$

$$\leq 0.8\,(1+C^*(k))\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}\frac{1}{\sqrt{N}}. \tag{5.24}$$

Combining (5.18) and (5.24), it follows that for $N \geq 1$,

$$\left|S_{k,2}^{[1]}(n;N)\right| \leq E_{N,4}^{[2]}(k)\cdot n^{-\frac{N+1}{2}} \tag{5.25}$$

with

$$E_{N,4}^{[2]}(k) := 0.8\frac{(1+C^*(k))}{\sqrt{N}}\left(\frac{24k-1}{24}\right)^{\frac{N}{2}}. \tag{5.26}$$

Combining (5.22) and (5.25),

$$\left|S_{k}^{[1]}(n;N)\right| \leq \left(E_{N,4}^{[1]}(k)+E_{N,4}^{[2]}(k)\right)n^{-\frac{N+1}{2}} =: E_{N,4}(k)n^{-\frac{N+1}{2}}. \tag{5.27}$$

Finally, setting

$$E_N(k) := \sum_{m=1}^{4} E_{N,m}(k), \tag{5.28}$$

and combining (5.8), (5.10), (5.15), and (5.27) concludes the proof of the theorem. □

6. Appendix

This section is divided into two Subsections 6.1 and 6.2 where we utilize Schneider's Symbolic Summation machinery, implemented in Mathematica in the form of the `Sigma` package, to simplify the sums $(S_j(t))_{1\leq j\leq 9}$ defined in Lemma 4.10. These simplifications are crucial to obtain the fundamental estimates as specified in Lemma 4.10. We observed that there are two classes: (i) the sums $S_j(t)$ for $j \in \{1,4,5,6,8\}$ which are (or reduce to) single sums, and (ii) the sums $S_j(t)$ with $j \in \{2,3,7,9\}$ which can be simplified to combinations of certain double sums. We restrict to show details of the summation machinery of `Sigma` by choosing the double sum representations of $S_2(t)$ and $S_3(t)$ from class (ii) mentioned before. To illustrate the full power of `Sigma` some further aspects are given for the more involved triple sum $S_3(t)$. The `Sigma` simplification of the remaining sums works similarly.

6.1. **Simplification of the sums $S_2(t)$ and $S_3(t)$ using `Sigma`.** Before we apply our summation tools to $S_2(t)$ we observe that the sum representation in (4.15) can be rewritten in the form

$$S_2(t) = \sum_{u=1}^{t-2}\frac{(-1)^u\big(\alpha^2\big)^u}{(2u-1)!}\sum_{s=0}^{t-u-2}\frac{\big(-(1+\alpha^{-2})\big)^{-s-u}(-s-u)_u\big(\frac{1}{2}-s-u\big)_{u+s+1}}{(s+u)(s+2u)!};$$

recall that $\alpha = \frac{\pi}{6}$. In the following we will focus on the inner sum which reads as

$$T(t,u) = \sum_{s=0}^{t-u-2}\frac{\pi^{2(s+u)}\big(\frac{-1}{36+\pi^2}\big)^{s+u}(-s-u)_u\big(\frac{1}{2}-s-u\big)_{1+s+u}}{(s+u)(s+2u)!}.$$

Using the summation package `Sigma` (or alternatively, any other package, like [31], that can deal with the creative telescoping paradigm [39] for hypergeometric products) one obtains the linear recurrence

$$\pi^2 uT(t,u) + 2\big(72+\pi^2\big)(1+u)T(t,u+1) + \pi^2(2+u)T(t,u+2)$$

$$= \frac{(-1)^t \pi^{-2+2t}(-3+t)(-2+t)(-7+2t)(-5+2t)(-3+2t)(1+u)(4-t)_u \left(\frac{9}{2}-t\right)_{-3+t}}{2(-3+t-u)(-2+t-u)(t+u)(-3+t+u)(-2+t+u)(-1+t+u)(36+\pi^2)^{t-2}(-4+t+u)!}. \quad (6.1)$$

Next, we are interested in d'Alembertian solutions of the found recurrence, i.e., in the set of all solutions that can be given in terms of indefinite nested sums defined over hypergeometric products. Using `Sigma` we find two linearly independent solutions

$$\frac{1}{u}\Big(\frac{-72-\pi^2+12\sqrt{36+\pi^2}}{\pi^2}\Big)^u \quad \text{and} \quad \frac{(-1)^u}{u}\Big(\frac{-72-\pi^2+12\sqrt{36+\pi^2}}{\pi^2}\Big)^u \qquad (6.2)$$

of the homogeneous version of the recurrence (6.1). To this end, we apply a generalized version [2] of Petkovšek's Hyper algorithm [32]; we remark that the bases $\frac{-72-\pi^2+12\sqrt{36+\pi^2}}{\pi^2}$ and $-\frac{72+\pi^2+12\sqrt{36+\pi^2}}{\pi^2}$ of the powers in (6.2) are the roots of the characteristic polynomial

$$q(x) = \pi^2 x^2 + 2\left(72+\pi^2\right)x + \pi^2 \qquad (6.3)$$

that appears in the underlying algorithm [32].

In the next step, we can apply the algorithm given in [3] to find the particular solution

$$\frac{(36+\pi^2)^2(-3+t)(-2+t)(-7+2t)(-5+2t)(-3+2t)(-1)^t\pi^{2t}\left(-72-\pi^2+12\sqrt{36+\pi^2}\right)^u\left(\frac{9}{2}-t\right)_{-3+t}}{2u\pi^{4+2u}(36+\pi^2)^t}\times$$
$$\times\sum_{i=1}^{u}\frac{(-1)^i\left(72+\pi^2+12\sqrt{36+\pi^2}\right)^i}{\left(-72-\pi^2+12\sqrt{36+\pi^2}\right)^i}\sum_{j=1}^{i}\frac{(-1)^j(-1+j)\pi^{2j}(4-t)_j}{(j-t)(1+j-t)(2+j-t)(3+j-t)(-3+j+t)(-2+j+t)\left(72+\pi^2+12\sqrt{36+\pi^2}\right)^j(-4+j+t)!}$$

of the recurrence (6.1) in terms of an indefinite nested double sum over the powers given in (6.2). Then a crucial step for further processing, i.e., for carrying out the estimates in Section 6.2, is the simplification of the found particular solution. Here again the summation package `Sigma` can assist; for a survey of the used telescoping algorithms we refer, e.g., to [37]. However, with the given limitations of the computer algebra system Mathematica, the underlying implementation does not work properly when dealing with algebraic elements such as $\sqrt{36+\pi^2}$. To overcome this situation, we perform the substitution

$$\pi \mapsto \frac{6\sqrt{1-b^2}}{b} \qquad (6.4)$$

for a new variable b; note that the inverse substitution is

$$b \mapsto \frac{6}{\sqrt{36+\pi^2}}. \qquad (6.5)$$

In this way, the technically challenging element $\sqrt{36+\pi^2}$ vanishes; it simplifies to

$$\sqrt{36+\pi^2} = \frac{6}{b}.$$

We remark that such rewritings (also called rationalizing transformations) appear frequently when dealing with nested integrals in the research area of elementary particle physics; for a survey and applications we refer to [27].

After this preprocessing step we continue with the sum representation

$$\tilde{S}_2(t) = \sum_{u=1}^{t-2}\frac{(-1)^u\left(\frac{1-b^2}{b^2}\right)^u}{(2u-1)!}\tilde{T}(t,u)$$

where

$$\tilde{T}(t,u) = \sum_{s=0}^{t-u-2}\frac{\left(b^2-1\right)^{s+u}(-s-u)_u\left(\frac{1}{2}-s-u\right)_{1+s+u}}{(s+u)(s+2u)!}.$$

By construction, if we apply the substitution (6.5) to $\tilde{S}_2(t)$ one gets $S_2(t)$ back. Repeating the above calculations in this new representation we obtain the improved recurrence

$$(-1+b)(1+b)u\tilde{T}(t,u) - 2\big(1+b^2\big)(1+u)\tilde{T}(t,u+1) + (-1+b)(1+b)(2+u)\tilde{T}(t,u+2)$$
$$= \frac{(-1+b)^{-1+t}(1+b)^{-1+t}(-3+t)(-2+t)(-7+2t)(-5+2t)(-3+2t)(1+u)(4-t)_u\left(\frac{9}{2}-t\right)_{-3+t}}{2(-3+t-u)(-2+t-u)(t+u)(-3+t+u)(-2+t+u)(-1+t+u)(-4+t+u)!}.$$

We remark that the correctness of the recurrence can be verified by the summand recurrence (not printed here) that is provided by the creative telescoping method. Solving this recurrence we get now the two linearly independent solutions

$$\tilde{H}_1(t,u) = \frac{(b-1)^u}{u(1+b)^u} \quad \text{and} \quad \tilde{H}_2(t,u) = \frac{(1+b)^u}{u(b-1)^u}$$

of the homogeneous version; note that these solutions agree with (6.2) with the substitutions in (6.5), resp. (6.4). Most relevant, using `Sigma` the found particular solution can be simplified to the form

$$\begin{aligned}\tilde{P}(t,u) = {}& \frac{(-1+b)^{t+u}(1+b)^{-3+t-u}\left(\frac{1}{2}-t\right)_t}{t!}R_1(t,u) + \frac{(-1+b)^{-3+t-u}(1+b)^{t+u}\left(\frac{1}{2}-t\right)_t}{t!}R_2(t,u)\\ &+ \frac{(-1+b)^{-1+t}(1+b)^{-1+t}\left(\frac{1}{2}-t\right)_t(-t)_u}{(t+u)!}R_3(t,u) - \frac{(-1+b)^{t-u}(1+b)^{t+u}\left(\frac{1}{2}-t\right)_t}{2u}\sum_{i=1}^{u}\frac{(-1+b)^i(1+b)^{-i}(-t)_i}{\big(t+i\big)!}\\ &- \frac{(-1+b)^{t+u}(1+b)^{t-u}\left(\frac{1}{2}-t\right)_t}{2u}\sum_{i=1}^{u}\frac{(-1+b)^{-i}(1+b)^{i}(-t)_i}{\big(t+i\big)!}\end{aligned}$$

for some rational functions $R_1(t,u)$, $R_2(t,u)$ and $R_3(t,u)$ in the variables b,u,t. We remark that correctness of the computed solutions can be verified easily by plugging them into the recurrence, carrying out the shift operator on the indefinite nested sums and products, and by finally simplification by performing simple rational function arithmetic.

In summary, using `Sigma` we found two linearly independent solutions of the homogeneous version together with the a particular solution $\tilde{P}(t,u)$ of the recurrence itself. Due to the linearly independence of the homogeneous solutions, there exist unique constants $c_1(t)$ and $c_2(t)$ (free of u) such that

$$\tilde{T}(t,u) = c_1(t)\,\tilde{H}_1(t,u) + c_2(t)\,\tilde{H}_2(t,u) + \tilde{P}(t,u) \tag{6.6}$$

holds for all nonnegative integers t and u with $t \geq 2$ and $1 \leq u \leq t-2$. To determine these $c_j(t)$, we consider the two initial values $\tilde{T}(t,1)$ and $\tilde{T}(t,2)$ which can be simplified with the same technology as above (where t has taken over the role of u) to the form

$$\tilde{T}(t,1) = \tfrac{1}{2} - \frac{(-1+b)^{-1+t}(1+b)^{-1+t}\left(\frac{1}{2}-t\right)_t}{(-1+2t)t!} + \frac{1}{(-1+b)(1+b)}\sum_{i=1}^{t}\frac{(-1+b)^i(1+b)^i\big(\frac{1}{2}-i\big)_i}{(-1+2i)i!},$$

$$\tilde{T}(t,2) = \frac{3+b^2}{4(-1+b)(1+b)} - \frac{(-1+b)^{-1+t}(1+b)^{-1+t}\left(\frac{1}{2}-t\right)_t}{(-1+2t)t!} + \frac{(1+b^2)}{(-1+b)^2(1+b)^2}\sum_{i=1}^{t}\frac{(-1+b)^i(1+b)^i\big(\frac{1}{2}-i\big)_i}{(-1+2i)i!}.$$

By setting $u=1,2$ in (6.6) and solving the underlying system we compute the $c_1(t)$ and $c_2(t)$ (which we do not print here) and obtain the desired simplification

$$
\begin{aligned}
\tilde{T}(t,u) =& \frac{\big((1+b)(-1+b)^{2u}+(-1+b)(1+b)^{2u}\big)}{4bu(-1+b)^{u}(1+b)^{u}}\\
&-\frac{\big((1+b)(-1+b)^{2u}+(-1+b)(1+b)^{2u}\big)(-1+b)^{t-u}(1+b)^{t-u}\big(\frac{1}{2}-t\big)_t}{4bu\,t!}\\
&+\frac{\big((-1+b)(1+b)(-1+t)(-1+2t)+2tu+2u^2\big)(t-u)(-1+b)^{-1+t}(1+b)^{-1+t}\big(\frac{1}{2}-t\big)_t(-t)_u}{2(-1+t)t(-1+2t)u(t+u)!}\\
&-\frac{(-1+b)^{t-u}(1+b)^{t+u}\big(\frac{1}{2}-t\big)_t}{2u}\sum_{i=1}^{u}\frac{(-1+b)^i(-t)_i}{(i+t)!(1+b)^i}\\
&+\frac{\big((1+b)^{2u}-(-1+b)^{2u}\big)(-1+b)^{-u}(1+b)^{-u}}{4bu}\sum_{i=1}^{t}\frac{(-1+b)^i(1+b)^i\big(\frac{1}{2}-i\big)_i}{(-1+2i)i!}\\
&-\frac{(-1+b)^{t+u}(1+b)^{t-u}\big(\frac{1}{2}-t\big)_t}{2u}\sum_{i=1}^{u}\frac{(1+b)^i(-t)_i}{(-1+b)^i(i+t)!}.
\end{aligned}
\tag{6.7}
$$

Finally, we replace in

$$\tilde{S}_2(t,u)=\sum_{u=1}^{t-2}\frac{(-1)^u\big(\frac{1-b^2}{b^2}\big)^u}{(-1+2u)!}\tilde{T}(t,u)$$

the sum $\tilde{T}(t,u)$ by the right-hand side of (6.7) and apply the substitution (6.4). This yields an expression that agrees with $S_2(t)$ for all $t\geq 2$ from which one can read of the splittings (6.15)–(6.16) of $S_2(t)$ which in Section 6.2 are used to obtain the estimate (4.29) in Lemma 4.10.

As indicated above, all calculations can be verified independently from the steps of their algorithmic derivations and thus lead to a rigorous proof for the correctness of the simplification. However, all the steps are tedious to handle manually. To assist such calculations (that are similarly to those that we frequently tackle in particle physics [1]), we provide besides `Sigma` the package `EvaluateMultiSums` [36] which contains all the experience of the `Sigma`–developer to carry out these calculations fully automatically. More precisely, after loading the packages into the computer algebra system Mathematica

In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-JKU

In[2]:= **<< EvaluateMultiSums.m**

EvaluateMultiSum by Carsten Schneider © RISC-JKU

we can insert the sum

In[3]:= $\mathbf{\tilde{T}=\sum_{s=0}^{t-u-2}\frac{(b^2-1)^{s+u}(-s-u)_u\big(\frac{1}{2}-s-u\big)_{1+s+u}}{(s+u)(s+2u)!};}$

and get with the command

In[4]:= **EvaluateMultiSum[$\tilde{\mathbf{T}}$, {}, {u, t}, {1, 2}, {t − 2, ∞}]**

Out[4]= ...

the output "..." that is equivalent to (6.7). Actually, one obtains a slight variant of it that one can transform to the version (6.7) using the command `SigmaReduce`; for a concrete application of such a transformation we refer to In[7] and In[8] below.

Using this technology, we focus now on the more involved triple sum $S_3(t)$ defined in (4.17). Similarly to $S_2(t)$ we rewrite $S_3(t)$ and work with the alternative sum representation

$$S_3(t) = \sum_{u=0}^{t-2} \frac{(-1)^u \left(\frac{\pi}{6}\right)^{2u}}{(2u)!} \sum_{s=0}^{t-u-2} \frac{\left(\frac{1}{2}-s-u\right)_{1+s+u}(-s-u)_u}{(1+s+2u)!} \sum_{r=0}^{t-u-s-1} (-1)^r \left(\frac{6}{\pi}\right)^{2r} \binom{-\frac{1}{2}-r}{-1-r-s+t-u}.$$

When trying to apply the summation tools on these summations, it is reasonable to start with the inner most sum, say $U(t,s,u)$. One finds the representation

$$\begin{aligned} U(t,s,u) := &\sum_{r=0}^{t-u-s-1} (-1)^r \left(\frac{6}{\pi}\right)^{2r} \binom{-\frac{1}{2}-r}{-1-r-s+t-u} \\ = &-(-1)^{s+t+u}\pi^{2+2s-2t+2u}\left(36+\pi^2\right)^{-1-s+t-u} - \frac{(-1)^{s+u}\pi^{2+2s+2u}\left(36+\pi^2\right)^{-1-s-u}\binom{-\frac{1}{2}}{t}}{-1+2t} \\ &+(-1)^{s+t+u}\pi^{2+2s-2t+2u}\left(36+\pi^2\right)^{-1-s+t-u}\sum_{i=1}^{t}\frac{\pi^{2i}\left(-\frac{1}{36+\pi^2}\right)^i\binom{-\frac{1}{2}}{i}}{-1+2i} \\ &+\frac{1}{2}(-1)^{s+u}\pi^{2+2s+2u}\left(36+\pi^2\right)^{-1-s-u}\sum_{i=1}^{u}\frac{\pi^{-2i}\left(-36-\pi^2\right)^i\binom{-\frac{1}{2}}{-1-i+t}}{-i+t} \\ &+\frac{1}{2}(-1)^{s}\pi^{2+2s}\left(36+\pi^2\right)^{-1-s}\sum_{i=1}^{s}\frac{\pi^{-2i}\left(-36-\pi^2\right)^i\binom{-\frac{1}{2}}{-1-i+t-u}}{-i+t-u}. \end{aligned}$$

In the next step, one may deal with the second summation

$$T(t,u) = \sum_{s=0}^{t-u-2} \frac{\left(\frac{1}{2}-s-u\right)_{1+s+u}(-s-u)_u}{(1+s+2u)!} U(t,u,s)$$

using the simplification of $U(t,u,s)$. Using `Sigma` one obtains the recurrence

$$\pi^2(1+2u)T(t,u) + 2\left(72+\pi^2\right)(3+2u)T(t,u+1) + \pi^2(5+2u)T(t,u+2) = R(t,u)$$

with a right-hand side $R(t,u)$ represented in terms of indefinite nested sums too large for being printed here. Solving the homogeneous version of the recurrence gives the two solutions

$$\frac{1}{2u+1}\left(\frac{-72-\pi^2+12\sqrt{36+\pi^2}}{\pi^2}\right)^u \quad \text{and} \quad \frac{(-1)^u}{2u+1}\left(\frac{-72-\pi^2+12\sqrt{36+\pi^2}}{\pi^2}\right)^u$$

which are closely related to the solutions (6.2) when dealing with $S_2(t)$. Indeed, the algorithm [32] (and the generalized version [2] implemented in `Sigma`) asks again for the roots of the same polynomial (6.3) that establishes the powers involving the algebraic element $\sqrt{36+\pi^2}$. Thus using the same rationalizing transformation (6.4) that we used above for the $S_2(t)$ one obtains homogeneous solutions that are free of any algebraic elements. This suggests to reconsider the sum $S_3(t)$ by applying first the substitution (6.4) leading to

$$\tilde{S}_3(t) = \sum_{u=0}^{t-2} \frac{(-1)^u\left(\frac{1-b^2}{b^2}\right)^u}{(2u)!} \sum_{s=0}^{t-i-2} \frac{(-s-u)_u\left(\frac{1}{2}-s-u\right)_{1+s+u}}{(1+s+2u)!} \sum_{r=0}^{t-u-s-1} (-1)^r \left(\frac{b^2}{1-b^2}\right)^r \binom{-\frac{1}{2}-r}{-1-r-s+t-u};$$

i.e., $S_3(t)$ equals to $\tilde{S}_3(t)$ after applying the inverse transformation (6.5). To this end, we insert the double sum

In[5]:= $$\tilde{\mathbf{T}} = \sum_{\mathbf{s}=0}^{\mathbf{t-i-2}} \frac{(\mathbf{-s-u})_{\mathbf{u}}\left(\frac{1}{2}-\mathbf{s}-\mathbf{u}\right)_{\mathbf{1+s+u}}}{(\mathbf{1+s+2u})!} \sum_{\mathbf{r}=0}^{\mathbf{t-u-s-1}} (-1)^{\mathbf{r}}\left(\frac{\mathbf{b}^2}{1-\mathbf{b}^2}\right)^{\mathbf{r}} \binom{-\frac{1}{2}-r}{-1-r-s+t-u};$$

and apply our summation toolbox by executing

In[6]:= $\mathbf{closedForm\tilde{T} = EvaluateMultiSum[\tilde{T}, \{\}, \{u, t\}, \{0, 2\}, \{t-2, \infty\}, SplitSums \to False]}$

$$\text{Out[6]=}\ \frac{\big((1-b+b^2)(-1+b)^{2u}+(1+b+b^2)(1+b)^{2u}\big)(-1+b)^{1-t-u}(1+b)^{1-t-u}}{2b^2(1+2u)}$$
$$-\frac{(-3+2t)\big(-1+2b^2t\big)\big((-1+b)^{1+2u}-(1+b)^{1+2u}\big)(-1+b)^{-u}(1+b)^{-u}\big(\tfrac{5}{2}-t\big)_{-1+t}}{4b^2(1+2u)t!}$$
$$+\frac{b(-3+2t)(-1+2t)(-1+b)^{-1-u}(1+b)^{1+u}\big(\tfrac{5}{2}-t\big)_{-1+t}}{2t(1+2u)}\sum_{i=1}^{u}\frac{(-1+b)^{i}(1+b)^{-i}(-t)_{i}}{(-1+i+t)!}$$
$$+\frac{3\big((-1+b)^{1+2u}-(1+b)^{1+2u}\big)(-1+b)^{-t-u}(1+b)^{-t-u}}{4b^2(1+2u)}\sum_{i=1}^{t}\frac{(-1+b)^{i}(1+b)^{i}\big(\tfrac{5}{2}-i\big)_{-1+i}}{i!}$$
$$-\frac{b(-3+2t)(-1+2t)(-1+b)^{1+u}(1+b)^{-1-u}\big(\tfrac{5}{2}-t\big)_{-1+t}}{2t(1+2u)}\sum_{i=1}^{u}\frac{(-1+b)^{-i}(1+b)^{i}(-t)_{i}}{(-1+i+t)!}$$

In order to represent the expression in terms of the products $u!, (-t)_u$ and $t!, (t+u)!, (1/2-t)_t$, one may use the following commands:

In[7]:= $\mathbf{closedForm\tilde{T} = SigmaReduce[closedForm\tilde{T}, u, Tower \to \{u!, (-t)_u\}]}$

In[8]:= $\mathbf{closedForm\tilde{T} = SigmaReduce[closedForm\tilde{T}, t, Tower \to \{t!, (t+u)!, (1/2-t)_t\}]}$

$$\text{Out[8]=}\ \frac{\big((-1+b)^{2u}+(1+b)^{2u}\big)}{2(1+2u)(-1+b)^{t+u-1}(1+b)^{t+u-1}}-\frac{\big((-1+b)^{2u}+(1+b)^{2u}\big)(-1+b)^{1-u}(1+b)^{1-u}\big(\tfrac{1}{2}-t\big)_{t}}{2(1+2u)t!}$$
$$+\frac{(t-u)\left(2b^2t-b^2-2t+2u+2\right)\left(\tfrac{1}{2}-t\right)_t(-t)_u}{t(2t-1)(2u+1)(t+u)!}$$
$$+\frac{\big((-1+b)^{2u}+(1+b)^{2u}-b(-1+b)^{2u}+b(1+b)^{2u}\big)}{2(1+2u)(-1+b)^{t+u}(1+b)^{t+u}}\sum_{i=1}^{t}\frac{(-1+b)^{i}(1+b)^{i}\big(\tfrac{1}{2}-i\big)_{i}}{(-1+2i)i!}$$
$$-\frac{b(1+b)^{1+u}\big(\tfrac{1}{2}-t\big)_{t}}{(1+2u)(-1+b)^{u}}\sum_{i=1}^{u}\frac{(-1+b)^{i}(1+b)^{-i}(-t)_{i}}{(i+t)!}-\frac{b(-1+b)^{1+u}\big(\tfrac{1}{2}-t\big)_{t}}{(1+2u)(1+b)^{u}}\sum_{i=1}^{u}\frac{(-1+b)^{-i}(1+b)^{i}(-t)_{i}}{(i+t)!}$$

Denoting the result in Out[9] by $T'(t,u)$ we obtain

$$\tilde{S}_3(t)=\sum_{u=0}^{t-2}\frac{(-1)^u\left(\frac{1-b^2}{b^2}\right)^u}{(2u)!}T'(t,u). \tag{6.8}$$

Finally, applying the substitution (6.5) to the right-hand side of (6.8) yields the simplification of $S_3(t)$ that has been used to perform the estimate (4.30) stated in Lemma 4.10; for further details we refer to Section 6.2.

To our knowledge this is the first time that rationalizing transformations, usually applied to simplify integrals, have been applied non-trivially to symbolic summation. As pointed out, in our concrete calculations such transformations can be discovered automatically when one looks for "nice" hypergeometric solutions and sum solutions over such products of a given linear recurrence. It would be interesting to see where this extra feature could be also exploited in other applications.

6.2. **Estimates for the sums** $(S_j(t))_{1\le j\le 9}$**.** We restrict to give detailed proofs for the bound estimates of $S_1(t)$ and $S_2(t)$ in Lemma 4.10. The derivations of the cases $3\le j\le 9$ work analogously.

Proof of Lemma 4.10: We start with the definition (4.14),

$$S_1(t)=\left(\frac{-1}{24}\right)^t\frac{\left(\frac{1}{2}-t\right)_{t+1}}{t}\sum_{u=1}^{t}\frac{(-1)^u(-t)_u}{(t+u)!(2u-1)!}\alpha^{2u}$$
$$=\frac{\binom{2t}{t}}{2t\cdot 96^t}\sum_{u=1}^{t}\left(\prod_{j=1}^{u}\frac{1-\frac{j-1}{t}}{1+\frac{j}{t}}\right)\frac{\alpha^{2u}}{(2u-1)!}$$

$$= \frac{1}{2\sqrt{\pi}} \frac{1}{24^t \cdot t^{\frac{3}{2}}} \left(1 + O_{\le \frac{1}{8}}\left(\frac{1}{t}\right)\right) \sum_{u=1}^{t} \left(\prod_{j=1}^{u} \frac{1 - \frac{j-1}{t}}{1 + \frac{j}{t}} \right) \frac{\alpha^{2u}}{(2u-1)!} \quad \text{(by Lemma 2.3)}$$

$$= \frac{1}{2\sqrt{\pi}} \frac{1}{24^t \cdot t^{\frac{3}{2}}} \left(1 + O_{\le \frac{1}{8}}\left(\frac{1}{t}\right)\right) \sum_{u=1}^{t} \left(1 + O_{\le u^2}\left(\frac{1}{t}\right)\right) \frac{\alpha^{2u}}{(2u-1)!}$$

$$\left(\text{by Lemma 2.2 with } (x_j, y_j, n) = \left(\frac{j-1}{t}, \frac{j}{t}, u\right)\right)$$

$$= \frac{1}{2\sqrt{\pi}} \frac{1}{24^t \cdot t^{\frac{3}{2}}} \left(1 + O_{\le \frac{1}{8}}\left(\frac{1}{t}\right)\right) \left(\underbrace{\sum_{u=1}^{t} \frac{\alpha^{2u}}{(2u-1)!}}_{=:S_1^{[1]}(t)} + \underbrace{\sum_{u=1}^{t} \frac{\alpha^{2u}}{(2u-1)!} \cdot O_{\le u^2}\left(\frac{1}{t}\right)}_{=:S_1^{[2]}(t)} \right). \tag{6.9}$$

First we bound $S_1^{[2]}(t)$,

$$\left|S_1^{[2]}(t)\right| \le \frac{1}{t} \sum_{u=1}^{t} \frac{u^2 \alpha^{2u}}{(2u-1)!} \le \frac{1}{t} \sum_{u \ge 1} \frac{u^2 \alpha^{2u}}{(2u-1)!} = \frac{\frac{\alpha}{4}\left(3\alpha \cosh(\alpha) + (\alpha^2+1)\sinh(\alpha)\right)}{t}. \tag{6.10}$$

Now it remains to estimate $S_1^{[1]}(t)$. Note that

$$S_1^{[1]}(t) = \sum_{u=1}^{\infty} \frac{\alpha^{2u}}{(2u-1)!} - \sum_{u=t+1}^{\infty} \frac{\alpha^{2u}}{(2u-1)!} = \alpha \sinh(\alpha) - \sum_{u=t+1}^{\infty} \frac{\alpha^{2u}}{(2u-1)!}.$$

By Lemma 2.4 with $k = 1$,

$$0 < \sum_{u=t+1}^{\infty} \frac{\alpha^{2u}}{(2u-1)!} = 2 \sum_{u=t+1}^{\infty} \frac{u \cdot \alpha^{2u}}{(2u)!} \le \frac{2\alpha^4}{9 \cdot t^2} \le \frac{2\alpha^3}{t},$$

which implies that

$$S_1^{[1]}(t) = \alpha \sinh(\alpha) + O_{\le 2\alpha^3}\left(\frac{1}{t}\right). \tag{6.11}$$

Combining (6.10) and (6.11) gives

$$\begin{aligned} &S_1(t) \\ &= \frac{1}{2\sqrt{\pi}} \frac{1}{24^t \cdot t^{\frac{3}{2}}} \left(1 + O_{\le \frac{1}{8}}\left(\frac{1}{t}\right)\right) \left(\alpha \sinh(\alpha) + O_{\le 2\alpha^3}\left(\frac{1}{t}\right) + O_{\le \frac{\alpha}{4}(3\alpha\cosh(\alpha)+(\alpha^2+1)\sinh(\alpha))}\left(\frac{1}{t}\right)\right) \\ &= \frac{\alpha \sinh(\alpha)}{2\sqrt{\pi}} \frac{1}{24^t \cdot t^{\frac{3}{2}}} \left(1 + O_{\le \frac{1}{8}}\left(\frac{1}{t}\right)\right) \left(1 + O_{\le \frac{2\alpha^3}{\alpha \sinh(\alpha)}}\left(\frac{1}{t}\right) + O_{\le \frac{\frac{\alpha}{4}\left(3\alpha\cosh(\alpha)+(\alpha^2+1)\sinh(\alpha)\right)}{\alpha\sinh(\alpha)}}\left(\frac{1}{t}\right)\right) \\ &= \frac{\alpha \sinh(\alpha)}{2\sqrt{\pi}} \frac{1}{24^t \cdot t^{\frac{3}{2}}} \left(1 + O_{\le 2.6}\left(\frac{1}{t}\right)\right), \end{aligned}$$

which finishes the proof of (4.28).

To estimate the sum $S_2(t)$ defined in (4.15) we list three basic facts which be used for the estimates of $S_2(t)$. For $t \ge 2$,

$$\frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^{t-1}}{\sqrt{t}} \le \frac{1}{2t}, \tag{6.12}$$

for $t\in\mathbb{Z}_{\geq 0}$,

$$(-1)^t\left(\frac{1}{2}-t\right)_t=\frac{\binom{2t}{t}\cdot t!}{4^t}, \tag{6.13}$$

and for $|x|<1$,

$$1-\sqrt{1-x}=\sum_{m=1}^{\infty}\frac{\binom{2m}{m}}{2m-1}\left(\frac{x}{4}\right)^m. \tag{6.14}$$

Next, using Sigma (with the summand representation (6.7)), we split the sum $S_2(t)$ as

$$S_2(t)=\sum_{k=1}^{5}S_2^{[k]}(t),$$

with

$$S_2^{[1]}(t):=\left(\frac{1}{2}-t\right)_t(-1)^t\left(\frac{\alpha^2}{1+\alpha^2}\right)^{t-1}\sum_{u=1}^{t-2}\frac{\alpha^{2u}}{(2u)!}\frac{(-1)^u(-t)_u}{(t+u)!}\left(-2\frac{(t-u)u(t+u)}{(t-1)t(2t-1)}+\frac{t-u}{t}\frac{\alpha^2}{1+\alpha^2}\right), \tag{6.15}$$

$$S_2^{[2]}(t):=\frac{1}{2}\sum_{u=1}^{t-2}\left(\left(\sqrt{1+\alpha^2}+1\right)\frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!}-\left(\sqrt{1+\alpha^2}-1\right)\frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}\right), \tag{6.16}$$

$$\begin{aligned}S_2^{[3]}(t):=&\frac{\left(\frac{1}{2}-t\right)_t(-1)^{t+1}}{2\cdot t!}\left(\frac{\alpha^2}{1+\alpha^2}\right)^{t-1}\\&\times\sum_{u=1}^{t-2}\frac{\left(\sqrt{1+\alpha^2}+1\right)\left(\sqrt{1+\alpha^2}-1\right)^{2u}-\left(\sqrt{1+\alpha^2}-1\right)\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!},\end{aligned} \tag{6.17}$$

$$\begin{aligned}S_2^{[4]}(t):=&\left(\frac{1}{2}-t\right)_t(-1)^{t+1}\left(\frac{\alpha^2}{1+\alpha^2}\right)^t\sum_{u=1}^{t-2}\frac{1}{(2u)!}\\&\times\left(\left(\sqrt{1+\alpha^2}+1\right)^{2u}\sum_{s=1}^{u}\frac{(-t)_s(-1)^s}{(t+s)!}\left(\frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1}\right)^s\right.\\&\left.+\left(\sqrt{1+\alpha^2}-1\right)^{2u}\sum_{s=1}^{u}\frac{(-t)_s(-1)^s}{(t+s)!}\left(\frac{\sqrt{1+\alpha^2}+1}{\sqrt{1+\alpha^2}-1}\right)^s\right),\end{aligned} \tag{6.18}$$

$$S_2^{[5]}(t):=\frac{\sqrt{1+\alpha^2}}{2}\sum_{u=1}^{t}\frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}-\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!}\sum_{s=1}^{t}\frac{\left(\frac{1}{2}-s\right)_s(-1)^s}{(2s-1)s!}\left(\frac{\alpha^2}{1+\alpha^2}\right)^s. \tag{6.19}$$

We treat the $S_2^{[k]}(t)$ successively. First for $S_2^{[1]}(t)$, after simplifying the right hand side of (6.15), we obtain

$$\begin{aligned}&\left|S_2^{[1]}(t)\right|\\&\overset{(6.13)}{=}\frac{\binom{2t}{t}t!}{4^t}\left(\frac{\alpha^2}{1+\alpha^2}\right)^{t-1}\left|\sum_{u=1}^{t-2}\frac{\alpha^{2u}}{(2u)!}\frac{(-1)^u(-t)_u}{(t+u)!}\left(-2\frac{(t-u)u(t+u)}{(t-1)t(2t-1)}+\frac{t-u}{t}\frac{\alpha^2}{1+\alpha^2}\right)\right|\end{aligned}$$

$$\leq \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^{t-1}}{\sqrt{\pi t}} \sum_{u=1}^{t-2} \frac{\alpha^{2u}}{(2u)!} \left(\prod_{j=1}^{u} \frac{t+1-j}{t+j}\right) \left(2\frac{(t-u)u(t+u)}{(t-1)t(2t-1)} + \frac{t-u}{t}\frac{\alpha^2}{1+\alpha^2}\right) \quad \text{(by Lemma 2.3)}$$

$$\leq \left(2+\frac{\alpha^2}{1+\alpha^2}\right) \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^{t-1}}{\sqrt{\pi t}} \sum_{u=1}^{t-2} \frac{\alpha^{2u}}{(2u)!}$$

$$\left(\text{as } \frac{t+1-j}{t+j} \leq 1 \text{ for } j \geq 1, \frac{(t-u)t(t+u)}{(t-1)t(2t-1)} \leq 1, \text{ and } \frac{t-u}{t} \leq 1 \text{ for } 1 \leq u \leq t-2\right)$$

$$\leq \left(2+\frac{\alpha^2}{1+\alpha^2}\right) \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^{t-1}}{\sqrt{\pi t}} \sum_{u\geq 1} \frac{\alpha^{2u}}{(2u)!} = \left(2+\frac{\alpha^2}{1+\alpha^2}\right) (\cosh(\alpha)-1) \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^{t-1}}{\sqrt{\pi t}}$$

$$\overset{(6.12)}{\leq} \left(2+\frac{\alpha^2}{1+\alpha^2}\right) \frac{(\cosh(\alpha)-1)}{\sqrt{\pi}} \frac{1}{2t} \leq \frac{0.1}{t},$$

which implies that

$$S_2^{[1]}(t) = O_{\leq 0.1}\left(\frac{1}{t}\right). \tag{6.20}$$

According to (6.16) we write $S_2^{[2]}(t)$ as the difference of two sums

$$S_2^{[2]}(t) := \frac{1}{2}\left(S_{2,1}^{[2]}(t) - S_{2,2}^{[2]}(t)\right), \tag{6.21}$$

with

$$S_{2,1}^{[2]}(t) = \left(\sqrt{1+\alpha^2}+1\right) \sum_{u=1}^{t-2} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!} \text{ and } S_{2,2}^{[2]}(t) = \left(\sqrt{1+\alpha^2}-1\right) \sum_{u=1}^{t-2} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}.$$

Next, observe that

$$\begin{aligned} S_{2,1}^{[2]}(t) &= \left(\sqrt{1+\alpha^2}+1\right) \left(\sum_{u=1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!} - \sum_{u=t-1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!}\right) \\ &= \left(\sqrt{1+\alpha^2}+1\right) \left(\cosh\left(\sqrt{1+\alpha^2}-1\right) - 1 - \sum_{u=t-1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!}\right). \end{aligned}$$

By Lemma 2.5 with $c = \sqrt{1+\alpha^2}-1$ and $t \mapsto t-1$, we obtain for $t \geq 2$,

$$0 < \sum_{u=t-1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!} \leq \frac{\left(\sqrt{1+\alpha^2}-1\right)^4}{18\cdot(t-1)^2} \leq \frac{\left(\sqrt{1+\alpha^2}-1\right)^4}{9\cdot t} \leq \frac{4\cdot 10^{-5}}{t},$$

which in turn implies that

$$\begin{aligned} S_{2,1}^{[2]}(t) &= \left(\sqrt{1+\alpha^2}+1\right) \left(\cosh\left(\sqrt{1+\alpha^2}-1\right) - 1 + O_{\leq 4\cdot 10^{-5}}\left(\frac{1}{t}\right)\right) \\ &= \left(\sqrt{1+\alpha^2}+1\right) \left(\cosh\left(\sqrt{1+\alpha^2}-1\right) - 1\right) \left(1 + O_{\leq \frac{4\cdot 10^{-5}}{\cosh\left(\sqrt{1+\alpha^2}-1\right)-1}}\left(\frac{1}{t}\right)\right) \\ &= \left(\sqrt{1+\alpha^2}+1\right) \left(\cosh\left(\sqrt{1+\alpha^2}-1\right) - 1\right) \left(1 + O_{\leq 5\cdot 10^{-3}}\left(\frac{1}{t}\right)\right). \end{aligned} \tag{6.22}$$

Similarly,

$$
\begin{aligned}
S_{2,2}^{[2]}(t) &= \left(\sqrt{1+\alpha^2}-1\right)\left(\sum_{u=1}^{\infty}\frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}-\sum_{u=t-1}^{\infty}\frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}\right)\\
&= \left(\sqrt{1+\alpha^2}-1\right)\left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-1-\sum_{u=t-1}^{\infty}\frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}\right).
\end{aligned}
$$

By Lemma 2.5 with $c=\sqrt{1+\alpha^2}+1$ and $t\mapsto t-1$, we obtain for $t\geq 2$,

$$
0<\sum_{u=t-1}^{\infty}\frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}\leq\frac{\left(\sqrt{1+\alpha^2}+1\right)^{4}}{18\cdot(t-1)^2}\leq\frac{\left(\sqrt{1+\alpha^2}+1\right)^{4}}{9\cdot t}\leq\frac{2.3}{t},
$$

which implies that

$$
\begin{aligned}
S_{2,2}^{[2]}(t) &= \left(\sqrt{1+\alpha^2}-1\right)\left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-1+O_{\leq 2.3}\left(\frac{1}{t}\right)\right)\\
&= \left(\sqrt{1+\alpha^2}-1\right)\left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-1\right)\left(1+O_{\leq\frac{2.3}{\cosh\left(\sqrt{1+\alpha^2}+1\right)-1}}\left(\frac{1}{t}\right)\right)\\
&= \left(\sqrt{1+\alpha^2}-1\right)\left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-1\right)\left(1+O_{\leq 0.8}\left(\frac{1}{t}\right)\right). \qquad (6.23)
\end{aligned}
$$

Consequently, setting

$$
\begin{aligned}
f(t) &\mapsto S_{2,1}^{[2]}(t),\\
g(t) &= 1,\\
h(t) &\mapsto S_{2,2}^{[2]}(t),\\
A_1 &= \left(\sqrt{1+\alpha^2}+1\right)\left(\cosh\left(\sqrt{1+\alpha^2}-1\right)-1\right),\\
A_2 &= -\left(\sqrt{1+\alpha^2}-1\right)\left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-1\right),\\
(E_1,E_2) &= \left(5\cdot 10^{-3},0.8\right),
\end{aligned}
$$

in (4.38) and applying (6.22) and (6.23) to (6.21), it follows that

$$
\begin{aligned}
S_2^{[2]}(t) = \frac{1}{2}\Big(&\left(\sqrt{1+\alpha^2}+1\right)\cosh\left(\sqrt{1+\alpha^2}-1\right)-\left(\sqrt{1+\alpha^2}-1\right)\cosh\left(\sqrt{1+\alpha^2}+1\right)-2\Big)\\
&\times\left(1+O_{\leq 0.9}\left(\frac{1}{t}\right)\right). \qquad (6.24)
\end{aligned}
$$

Recalling (6.17), we obtain

$$
\begin{aligned}
&\left|S_2^{[3]}(t)\right|\\
&\overset{(6.13)}{=}\frac{\binom{2t}{t}}{2\cdot 4^t}\left(\frac{\alpha^2}{1+\alpha^2}\right)^t\sum_{u=1}^{t-2}\frac{\left(\sqrt{1+\alpha^2}+1\right)\left(\sqrt{1+\alpha^2}-1\right)^{2u}-\left(\sqrt{1+\alpha^2}-1\right)\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}\\
&\overset{\text{Lemma 2.3}}{\leq}\frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{2\sqrt{\pi}\sqrt{t}}\sum_{u=1}^{t-2}\frac{\left(\sqrt{1+\alpha^2}+1\right)\left(\sqrt{1+\alpha^2}-1\right)^{2u}-\left(\sqrt{1+\alpha^2}-1\right)\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}
\end{aligned}
$$

$$\le \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{2\sqrt{\pi}\sqrt{t}} \sum_{u=1}^{t-2} \frac{\left(\sqrt{1+\alpha^2}+1\right)\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!} \le \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{2\sqrt{\pi}\sqrt{t}} \sum_{u\ge 1} \frac{\left(\sqrt{1+\alpha^2}+1\right)\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!}$$

$$= \frac{\left(\sqrt{1+\alpha^2}+1\right)}{2\sqrt{\pi}} \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{\sqrt{t}} \left(\cosh\left(\sqrt{1+\alpha^2}-1\right)-1\right) \overset{(6.12)}{\le} \frac{3\cdot 10^{-3}}{t},$$

and therefore,

$$S_2^{[3]}(t) = O_{\le 3\cdot 10^{-3}}\left(\frac{1}{t}\right). \tag{6.25}$$

Starting with (6.18), we obtain

$$\left|S_2^{[4]}(t)\right|$$

$$\overset{(6.13)}{=} \frac{\binom{2t}{t}\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{4^t} \sum_{u=1}^{t-2} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!} \left(\sum_{s=1}^{u} \frac{(-t)_s(-1)^s}{(t+1)_s}\left(\frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1}\right)^s \right.$$

$$\left. + \sum_{s=1}^{u} \frac{(-t)_s(-1)^s}{(t+1)_s}\left(\frac{\sqrt{1+\alpha^2}+1}{\sqrt{1+\alpha^2}-1}\right)^{s-2u}\right)$$

$$\le \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{\sqrt{\pi\cdot t}} \sum_{u=1}^{t-2} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!} \left(\sum_{s=1}^{u} \left(\frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1}\right)^s + \sum_{s=1}^{u} \left(\frac{\sqrt{1+\alpha^2}+1}{\sqrt{1+\alpha^2}-1}\right)^{s-2u}\right)$$

$$\left(\text{by Lemma 2.3 and by } \frac{(-t)_s(-1)^s}{(t+1)_s} = \prod_{j=1}^{s} \frac{t+1-j}{t+j} \le 1 \text{ as } t+1-j \le t+j \text{ for } j \ge 1\right)$$

$$= \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{\sqrt{\pi\cdot t}} \sum_{u=1}^{t-2} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!} \left(\sum_{s=1}^{u} \left(\frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1}\right)^s + \sum_{s=u}^{2u-1} \left(\frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1}\right)^{s}\right)$$

$$\le \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{\sqrt{\pi\cdot t}} \sum_{u=1}^{t-2} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!} \left(\sum_{s\ge 1} \left(\frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1}\right)^s + \sum_{s\ge 1} \left(\frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1}\right)^{s}\right) \quad (\text{as } s \ge u \ge 1)$$

$$\le \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{\sqrt{\pi\cdot t}} \sum_{u=1}^{t-2} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!} \left(2\sum_{s\ge 1}\left(\frac{1}{16}\right)^s\right) \quad \left(\text{as } \frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1} \le \frac{1}{16}\right)$$

$$= \frac{2}{15\sqrt{\pi}} \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{\sqrt{t}} \sum_{u=1}^{t-2} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!} \le \frac{2}{15\sqrt{\pi}} \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{\sqrt{t}} \sum_{u\ge 1} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}$$

$$= \frac{2}{15\sqrt{\pi}} \frac{\left(\frac{\alpha^2}{1+\alpha^2}\right)^t}{\sqrt{t}} \left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-1\right) \overset{(6.12)}{\le} \frac{0.2}{t},$$

which implies that

$$S_2^{[4]}(t) = O_{\le 0.2}\left(\frac{1}{t}\right). \tag{6.26}$$

Finally we rewrite $S_2^{[5]}(t)$ as

$$S_2^{[5]}(t) \overset{(6.13)}{=} \frac{\sqrt{1+\alpha^2}}{2} \sum_{u=1}^{t} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u} - \left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!} \sum_{s=1}^{t} \frac{\binom{2s}{s}}{(2s-1)} \left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s$$
$$=: \frac{\sqrt{1+\alpha^2}}{2} \left(S_{2,1}^{[5]}(t) - S_{2,2}^{[5]}(t)\right) S_{2,3}^{[5]}(t), \tag{6.27}$$

with

$$S_{2,1}^{[5]}(t) = \sum_{u=1}^{t} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}, \; S_{2,2}^{[5]}(t) = \sum_{u=1}^{t} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!},$$
$$S_{2,3}^{[5]}(t) = \sum_{s=1}^{t} \frac{\binom{2s}{s}}{(2s-1)} \left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s.$$

Observe that

$$S_{2,1}^{[5]}(t) = \sum_{u=1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!} - \sum_{u=t+1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}$$
$$= \cosh\left(\sqrt{1+\alpha^2}+1\right) - 1 - \sum_{u=t+1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!}.$$

Hence by Lemma 2.5 with $c = \sqrt{1+\alpha^2}+1$, we get

$$0 < \sum_{u=t+1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}+1\right)^{2u}}{(2u)!} \le \frac{\left(\sqrt{1+\alpha^2}+1\right)^{4}}{18 \cdot t^2} \le \frac{1.2}{t},$$

which implies that

$$S_{2,1}^{[5]}(t) = \cosh\left(\sqrt{1+\alpha^2}+1\right) - 1 + O_{\le 1.2}\left(\frac{1}{t}\right)$$
$$= \left(\cosh\left(\sqrt{1+\alpha^2}+1\right) - 1\right) \left(1 + O_{\le \frac{1.2}{\cosh\left(\sqrt{1+\alpha^2}+1\right)-1}}\left(\frac{1}{t}\right)\right)$$
$$= \left(\cosh\left(\sqrt{1+\alpha^2}+1\right) - 1\right) \left(1 + O_{\le 0.4}\left(\frac{1}{t}\right)\right). \tag{6.28}$$

Similarly, for $S_{2,2}^{[5]}(t)$ we obtain

$$S_{2,2}^{[5]}(t) = \sum_{u=1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!} - \sum_{u=t+1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!}$$
$$= \cosh\left(\sqrt{1+\alpha^2}-1\right) - 1 - \sum_{u=t+1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!},$$

and by Lemma 2.5 with $c = \sqrt{1+\alpha^2} - 1$, we obtain

$$0 < \sum_{u=t+1}^{\infty} \frac{\left(\sqrt{1+\alpha^2}-1\right)^{2u}}{(2u)!} \leq \frac{\left(\sqrt{1+\alpha^2}-1\right)^{4}}{18 \cdot t^2} \leq \frac{2 \cdot 10^{-5}}{t},$$

which implies that

$$\begin{aligned} S_{2,2}^{[5]}(t) &= \cosh\left(\sqrt{1+\alpha^2}-1\right) - 1 + O_{\leq 2\cdot 10^{-5}}\left(\frac{1}{t}\right) \\ &= \left(\cosh\left(\sqrt{1+\alpha^2}-1\right)-1\right)\left(1 + O_{\leq \frac{2\cdot 10^{-5}}{\cosh\left(\sqrt{1+\alpha^2}-1\right)-1}}\left(\frac{1}{t}\right)\right) \\ &= \left(\cosh\left(\sqrt{1+\alpha^2}-1\right)-1\right)\left(1 + O_{\leq 3\cdot 10^{-3}}\left(\frac{1}{t}\right)\right). \end{aligned} \tag{6.29}$$

Applying (4.38) with

$$\begin{aligned} f(t) &\mapsto S_{5,1}^{[2]}(t), \\ g(t) &= 1, \\ h(t) &\mapsto S_{5,2}^{[2]}(t), \\ A_1 &= \left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-1\right), \\ A_2 &= -\left(\cosh\left(\sqrt{1+\alpha^2}-1\right)-1\right), \\ (E_1, E_2) &= \left(0.4, 3\cdot 10^{-3}\right), \end{aligned}$$

from (6.28) and (6.29) we obtain

$$\begin{aligned} &S_{2,1}^{[5]}(t) - S_{2,2}^{[5]}(t) \\ &= \left(\cosh\left(\sqrt{1+\alpha^2}+1\right) - \cosh\left(\sqrt{1+\alpha^2}-1\right)\right)\left(1 + O_{\leq 0.5}\left(\frac{1}{t}\right)\right). \end{aligned} \tag{6.30}$$

Finally, for $S_{2,3}^{[5]}(t)$ it follows that

$$\begin{aligned} S_{2,3}^{[5]}(t) &= \sum_{s=1}^{t} \frac{\binom{2s}{s}}{(2s-1)}\left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s = \sum_{s=1}^{\infty} \frac{\binom{2s}{s}}{(2s-1)}\left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s - \sum_{s=t+1}^{\infty} \frac{\binom{2s}{s}}{(2s-1)}\left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s \\ &= 1 - \sqrt{1 - \frac{\alpha^2}{1+\alpha^2}} - \sum_{s=t+1}^{\infty} \frac{\binom{2s}{s}}{(2s-1)}\left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s \quad \left(\text{by (6.14) with } x \mapsto \frac{\alpha^2}{1+\alpha^2}\right) \\ &= \frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}} - \sum_{s=t+1}^{\infty} \frac{\binom{2s}{s}}{(2s-1)}\left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s. \end{aligned} \tag{6.31}$$

Concerning the last series, for $t \geq 2$,

$$\begin{aligned} 0 < \sum_{s=t+1}^{\infty} \frac{\binom{2s}{s}}{(2s-1)}\left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s &\leq \frac{1}{2\sqrt{\pi}\cdot t^{\frac{3}{2}}} \sum_{s=t+1}^{\infty} \left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s \\ &\qquad \text{(by Lemma 2.3 and } 2s-1 \geq 2t \text{ for } s \geq t+1) \\ &\leq \frac{1}{2\sqrt{\pi}\cdot t^{\frac{3}{2}}} \sum_{s\geq 3} \left(\frac{\alpha^2}{4(1+\alpha^2)}\right)^s \quad (\text{as } t \geq 2) \end{aligned}$$

$$\leq \frac{1}{2\sqrt{\pi}\cdot t^{\frac{3}{2}}} \sum_{s\geq 3} \left(\frac{1}{18}\right)^s \leq \frac{6\cdot 10^{-5}}{t}. \tag{6.32}$$

Applying (6.32) to (6.31), we obtain

$$S_{2,3}^{[5]}(t) = \frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}} + O_{\leq 6\cdot 10^{-5}}\left(\frac{1}{t}\right) = \frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}}\left(1+O_{\leq 6\cdot 10^{-4}}\left(\frac{1}{t}\right)\right). \tag{6.33}$$

Applying (6.30) and (6.33) to (6.27), it follows that

$$\begin{aligned}
&S_2^{[5]}(t)\\
&= \frac{\left(\sqrt{1+\alpha^2}-1\right)\left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-\cosh\left(\sqrt{1+\alpha^2}-1\right)\right)}{2}\\
&\qquad \cdot \left(1+O_{\leq 0.5}\left(\frac{1}{t}\right)\right)\left(1+O_{\leq 6\cdot 10^{-4}}\left(\frac{1}{t}\right)\right)\\
&= \frac{\left(\sqrt{1+\alpha^2}-1\right)\left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-\cosh\left(\sqrt{1+\alpha^2}-1\right)\right)}{2}\left(1+O_{\leq 0.6}\left(\frac{1}{t}\right)\right).
\end{aligned} \tag{6.34}$$

Applying

$$\begin{aligned}
f(t) &\mapsto S_2^{[2]}(t),\\
h(t) &\mapsto S_2^{[5]}(t),\\
g(t) &= 1,\\
A_1 &= \frac{1}{2}\left(\left(\sqrt{1+\alpha^2}+1\right)\cosh\left(\sqrt{1+\alpha^2}-1\right)-\left(\sqrt{1+\alpha^2}-1\right)\cosh\left(\sqrt{1+\alpha^2}+1\right)-2\right),\\
A_2 &= \frac{1}{2}\left(\left(\sqrt{1+\alpha^2}-1\right)\left(\cosh\left(\sqrt{1+\alpha^2}+1\right)-\cosh\left(\sqrt{1+\alpha^2}-1\right)\right)\right),\\
(E_1,E_2) &= (0.9, 0.6).
\end{aligned}$$

in (4.38), from (6.24) and (6.34) we obtain

$$S_2^{[2]}(t)+S_2^{[5]}(t) = \left(\cosh\left(\sqrt{1+\alpha^2}-1\right)-1\right)\left(1+O_{\leq 6.7}\left(\frac{1}{t}\right)\right). \tag{6.35}$$

From (6.20), (6.25), (6.26), and (6.35) it follows that

$$\begin{aligned}
S_2(t) &= \left(\cosh\left(\sqrt{1+\alpha^2}-1\right)-1\right)\left(1+O_{\leq 6.7}\left(\frac{1}{t}\right)\right)+O_{\leq 0.4}\left(\frac{1}{t}\right)\\
&= \left(\cosh\left(\sqrt{1+\alpha^2}-1\right)-1\right)\left(1+O_{\leq 54.9}\left(\frac{1}{t}\right)\right),
\end{aligned}$$

which concludes the proof of (4.29).

To prove the remaining sums $(S_j(t))_{3\leq j\leq 9}$ one applies arguments very similar to those presented above. □

Acknowledgements

Banerjee was funded by the Austrian Science Fund (FWF): W1214-N15, project DK6. Paule and Radu were supported by grant SFB F50-06 of the FWF. This research was funded in whole or in part by the Austrian Science Fund (FWF) grants 10.55776/P20347 and 10.55776/PAT1332123. For open access purposes, the authors have applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission.

Data Availability Statement No data were generated or used in the preparation of this paper.

Disclosure of potential conflicts of interest The authors confirm that there is no conflict of interest in connection with this paper.

References

[1] J. Ablinger, J. Blümlein, A. De Freitas, M. Saragnese, C. Schneider, K. Schönwald. *The three-loop polarized pure singlet operator matrix element with two different masses*. Nuclear Physics B 952(114916), pp. 1-18. 2020.

[2] S.A. Abramov, M. Bronstein, M. Petkovšek, and C. Schneider. On rational and hypergeometric solutions of linear ordinary difference equations in $\Pi\Sigma^*$-field extensions. *J. Symb. Comput.*, 107:23–66, 2021. arXiv:2005.04944 [cs.SC].

[3] S.A. Abramov and M. Petkovšek, *D'Alembertian solutions of linear differential and difference equations*, Proc. ISSAC'94 (J. von zur Gathen, ed.), ACM Press, 1994, pp. 169–174.

[4] G. Almkvist, *On the differences of the partition function*, Acta Arith., **61** (1992), 173–181.

[5] K. Banerjee, *New Asymptotics and Inequalities Related to the Partition Function*, Ph.D. Thesis, Johannes Kepler University Linz, 2022. Available at `http://www3.risc.jku.at/publications/download/risc_6712/Ph.D.%20thesis%20(revised%20version).pdf`

[6] K. Banerjee, P. Paule, C. Radu, and C. Schneider, *Error bounds for the asymptotic expansion of the partition function*, Rocky Mountain J. Math. (accepted for publication) (2023).

[7] K. Banerjee, P. Paule, C. Radu, and W. Zeng, *New inequalities for* $p(n)$ *and* $\log p(n)$, Ramanujan J., **61** (2023), 1295–1338.

[8] K. Bringmann, B. Kane, L. Rolen, and Z. Tripp, *Fractional partitions and conjectures of Chern–Fu–Tang and Heim–Neuhauser*, Trans. Amer. Math. Soc. Ser. B, **8** (2021), 615–634.

[9] W. Y. C. Chen, *Recent developments on log-concavity and q-log-concavity of combinatorial polynomials*, In: FPSAC 2010 Conference Talk Slides, `http://www.billchen.org/talks/2010-FPSAC.pdf` (2010).

[10] W. Y. C. Chen, *The spt-function of Andrews*, Surveys in combinatorics, Cambridge Univ. Press, Cambridge, (2017), 141–203.

[11] W. Y. C. Chen, D. Jia, and L. Wang, *Higher order Turán inequalities for the partition function*, Trans. Amer. Math. Soc., **372** (2019), 2143–2165.

[12] S. DeSalvo and I. Pak, *Log-concavity of the partition function*, Ramanujan J., **38** (2015), 61–73.

[13] F. Garvan, *Generalizations of Dyson's rank and non-Rogers-Ramanujan partitions*, Manuscripta Math., **84** (1994), 343–359.

[14] K. Gomez, J. Males, and L. Rolen, *The second shifted difference of partitions and its applications*, Bull. Aust. Math. Soc., **107** (2023), 66–78.

[15] I. J. Good, *The differences of the partition function*, Problem 6137, Amer. Math. Monthly, **84** (1977), p. 141.

[16] M. Griffin, K. Ono, L. Rolen, and D. Zagier, *Jensen polynomials for the Riemann zeta function and other sequences*, Proc. Natl. Acad. Sci. USA, **116** (2019), 11103–11110.

[17] H. Gupta, *Finite differences of the partition function*, Math. Comp., **32** (1978), 1241–1243.

[18] G. H. Hardy and S. Ramanujan. *Asymptotic formulae in combinatory analysis*, Proc. Lond. Math. Soc. (3), **2** (1918), 75–115.

[19] Q. H. Hou and Z. R. Zhang, *r-log-concavity of partition functions*, Ramanujan J., **48** (2019), 117–129.

[20] C. Knessl and J. B. Keller, *Asymptotic behavior of high-order differences of the partition function*, Comm. Pure Appl. Math., **44** (1991), 1033–1045.

[21] H. Larson and I. Wagner, *Hyperbolicity of the partition Jensen polynomials*, Res. Number Theory, **5** (2019), 1–12.

[22] D. H. Lehmer, *On the remainders and convergence of the series for the partition function*, Trans. Amer. Math. Soc., **46** (1939), 362–373.

[23] D. H. Lehmer, *On the series for the partition function*, Trans. Amer. Math. Soc., **43** (1938), 271–295.

[24] P. A. MacMahon, *Combinatory analysis, Vol. I, II*, Dover Publ., Mineola, New York, 2004.

[25] N. Nekrasov and A. Okounkov, *Seiberg–Witten theory and random partitions*, The unity of mathematics, Birkhäuser Boston (2006), 525–596.

[26] J. L. Nicolas, *Sur les entiers N pour lesquels il y a beaucoup de groupes abéliens d'ordre N*, Ann. Inst. Fourier, **28** (1978), 1–16.

[27] C.G. Raab, *Nested Integrals and Rationalizing Transformations*, In: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (ed.), Texts and Monographs in Symbolic Computation , pp. 395-422. 2021. Springer.

[28] C. O'Sullivan, *Detailed asymptotic expansions for partitions into powers*, Int. J. Number Theory **19** (2023), 2163–2196.

[29] A. M. Odlyzko, *Differences of the partition function*, Acta Arith., **49** (1988), 237–254.
[30] K. Ono, S. Pujahari, and L. Rolen, *Turán inequalities for the plane partition function*, Adv. Math. **409** (2022), p. 108692.
[31] P. Paule and M. Schorn. A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities. J. Symbolic Comput., 20:673–698, 1995.
[32] M. Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. *J. Symbolic Comput.*, 14(2-3):243–264, 1992.
[33] H. Rademacher, *A convergent series for the partition function* $p(n)$, Proc. Natl. Acad. Sci. USA, **23** (1937), 78–84.
[34] H. Robbins, *A remark on Stirling's formula*, The American mathematical monthly **61:1** (1955), 26–29.
[35] C. Schneider, *Symbolic summation assists combinatorics*, Sémin. Lothar. Comb., **56** (2007), 1–36.
[36] C. Schneider. *Simplifying multiple sums in difference fields.* In: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, C. Schneider, J. Blümlein (ed.), Texts and Monographs in Symbolic Computation , pp. 325-360. 2013. Springer.
[37] C. Schneider, *Term algebras, canonical representations and difference ring theory for symbolic summation.* In: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (ed.), Texts and Monographs in Symbolic Computation , pp. 423-485. 2021. Springer.
[38] R. P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Ann. New York Acad. Sci., **576** (1989), 500–535.
[39] D. Zeilberger, *The method of creative telescoping*, J. Symbolic Comput. **11** (1991), 195–204.

Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Altenberger Strasse 69, A-4040 Linz, Austria. In addition to RISC, Peter Paule is affiliated as guest professor to the Center for Applied Mathematics (TCAM), Tianjin University, Tianjin 300072, P. R. China. The first author is affiliated at Department of Mathematics and Computer Science, Division of Mathematics, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany

Email address: kbanerj1@uni-koeln.de, Koustav.Banerjee@risc.jku.at

Email address: Peter.Paule@risc.jku.at

Email address: sradu@risc.jku.at

Email address: Carsten.Schneider@risc.jku.at