INVARIANTS OF THE QUARTIC BINARY FORM AND PROOFS OF CHEN'S CONJECTURES ON INEQUALITIES FOR THE PARTITION FUNCTION AND THE ANDREWS' SPT FUNCTION

KOUSTAV BANERJEE

Abstract. An extensive amount of study has been done on inequalities for the partition function, emerged primarily through works of Chen. In particular, the Turán inequality and the higher order Turán inequalities for $p(n)$ has been one of the most predominant theme. Among many others, one of the most notable one is Griffin, Ono, Rolen, and Zagier's result in which they proved that for every integer $d \geq 1$, there exists an integer $N(d)$ such that the Jensen polynomial of degree d and shift n associated with the partition function, denoted by $J_p^{d,n}(x)$, has only distinct real roots for all $n \geq N(d)$, earlier conjectured by Chen, Jia, and Wang and Ono independently. Later, Larson and Wagner have provided an estimate of upper bound for $N(d)$. This phenomena in turn implies that the discriminant of $J_p^{d,n}(x)$ is positive; i.e., $\mathrm{Disc}_x(J_p^{d,n}) > 0$. For $d = 2$, $\mathrm{Disc}_x(J_p^{2,n}) > 0$ when $n \geq N(2) = 26$ is equivalent to the fact that $(p(n))_{n\geq 26}$ is log-concave. In 2017, Chen undertook a comprehensive investigation on inequalities for $p(n)$ through the lens of invariant theory of binary forms of degree n. Positivity of the invariant of a quadratic binary form (resp. cubic binary form) associated with $p(n)$ reflects that the sequence $(p(n))_{n\geq 26}$ satisfies the Turán inequality (resp. $(p(n))_{n\geq 95}$ satisfies the higher order Turán inequality). Chen further studied on the two invariants for a quartic binary form where its coefficients are shifted values of integer partitions and conjectured four inequalities for $p(n)$. In this paper, we give explicit error bounds for the asymptotic expansion of the shifted partition function $p(n-\ell)$ for any non-negative integer ℓ. As an application of these infinite family of inequalities, we confirm the conjectures of Chen. Moreover, three family of inequalities related to the partition function have been studied in this paper, namely, higher order Laguerre inequalities, higher order shifted differences, and higher order log-concavity. In context of higher order Laguerre inequalities for $p(n)$, we settle a conjecture of Wagner. For higher order shifted difference of $p(n)$, we extend a result of Gomez, Males, and Rolen. In context of higher order log-concavity for $p(n)$, we prove discuss on the asymptotic growth for the r-fold applications (with $r \in \{1, 2, 3\}$) of the operator $\mathcal{L}$ on $p(n)$ defined by $\mathcal{L}(p(n)) = p(n)^2 - p(n-1)p(n+1)$ and propose a conjecture on infinite log-concavity in this regard. Furthermore, we will show how to construct a unified framework to prove partition function inequalities of the above types and discuss a few possible applications of such construction. Finally, we prove all the Chen's conjectures related to the inequalities for the Andrews' spt function, denoted by $\mathrm{spt}(n)$, arising from invariants of quartic binary form using inequalities for the shifted partition function.

Contents

1. Introduction 2
2. Plan of paper 6
3. Preliminaries 6
4. Set Up 8
4.1. Coefficients in the asymptotic expansion of $p(n-\ell)$ 12
4.2. Estimation of $(S_i(t,\ell))$ 14
4.3. Error bounds 15

2020 *Mathematics Subject Classification.* 05A16, 05A20, 11P82, 13A50.

Key words and phrases. the partition function, Andrews' spt function, Hardy-Ramanujan-Rademacher formula, invariants of binary forms, log-concavity and its higher analogs, higher order Turán inequalities, higher order Laguerre inequalities, higher order shifted differences.

5. Inequalities for $p(n-\ell)$ 20
6. Proof of Theorems 1.5, 1.7, and 1.9 22
7. Applications 24
7.1. Higher order Laguerre inequalities 24
7.2. Higher order shifted difference 25
7.3. Higher order log-concavity 30
8. A unified framework to prove inequalities for $p(n)$ 31
9. Chen's conjectures on the Andrews' spt function 35
9.1. Inequalities for $\mathrm{spt}(n)$ 37
10. Concluding Remarks 43
11. Appendix 47
References 52

1. Introduction

Throughout this paper, we consider only sequences of real numbers. A sequence $(a_n)_{n\geq 0}$ is said to satisfy the Turán inequlaities or to be log-concave, if

$$a_n^2 - a_{n-1}a_{n+1} \geq 0 \quad \text{for all} \quad n \geq 1, \tag{1.1}$$

see [62]. We say that a sequence $(a_n)_{n\geq 0}$ is said to satisfy the higher order Turán inequlaities if for all $n \geq 1$,

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2}) - (a_n a_{n+1} - a_{n-1}a_{n+2})^2 \geq 0. \tag{1.2}$$

The Turán inequalities and the higher order Turán inequalities are related to the Laguerre-Pólya class of real entire functions [24, 65]. A real entire function

$$\psi(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!} \tag{1.3}$$

is said to be in Laguerre-Pólya class, denoted by $\psi(x) \in \mathcal{LP}$, if it is of the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \Big(1 + \frac{x}{x_k}\Big) e^{-\frac{x}{x_k}},$$

where c, β, x_k are real numbers, $\alpha \geq 0$, $m \in \mathbb{Z}_{\geq 0}$, and $\displaystyle\sum_{k=1}^{\infty} x_k^{-2}$ converges. Any sequence of polynomials with only real zeroes, say $(P_n(x))_{n\geq 0}$, converges uniformly to a function $P(x) \in \mathcal{LP}$. For a more detailed study on the theory of the $\mathcal{LP}$ class, we refer to [59]. Jensen [37] proved that a real entire function $\psi(x)$ is in $\mathcal{LP}$ class if and only if for any $d \in \mathbb{Z}_{\geq 1}$, the Jensen polynomial of degree d associated with a sequence $(a_n)_{n\geq 0}$:

$$J_a^d(x) = \sum_{k=0}^{d} \binom{d}{k} a_k x^k$$

has only real zeroes. Pólya and Schur [61] proved that for a real entire function $\psi(x) \in \mathcal{LP}$ and for any $n \geq \mathbb{Z}_{\geq 0}$, the n-th derivative $\psi^{(n)}(x)$ of $\psi(x)$ also belongs to the $\mathcal{LP}$ class, that is, the Jensen polynomial associated with $\psi^{(n)}(x)$

$$J_a^{d,n}(x) = \sum_{k=0}^{d} \binom{d}{k} a_{n+k} x^k$$

has only real zeroes. Observe that for $d = 2$ and for all nonnegative integer n, the real-rootedness of $J_a^{d,n}(x)$ implies that the discriminant $4(a_{n+1}^2 - a_n a_{n+2})$ is nonnegative. Pólya's work [54] on $\mathcal{LP}$ class is closely connected with the Riemann hypothesis. He showed that the Riemann hypothesis is equivalent to real rootedness of the Jensen polynomial $J_a^{d,n}(x)$ for all nonnegative integers d and n, where the coefficient sequence $\{a_n\}_{n\geq 0}$ is defined by

$$(-1+4z^2)\ \Lambda\Big(\frac{1}{2}+z\Big) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^{2n},$$

with $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s)$, where ζ denotes the Riemann zeta function and Γ denotes the Gamma function. In 2019, Griffin, Ono, Rolen, and Zagier [32, Theorem 1] proved that for all $d \geq 1$, $J_a^{d,n}(x)$ has only real roots for all sufficiently large n.

Now we discuss in brief the inequalities of the partition function. A partition of a positive integer n is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Let $p(n)$ denote the number of partitions of n. Estimates on the partition function systematically began with the work of Hardy and Ramanujan [34] in 1918 and independently by Uspensky [66] in 1920:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty. \tag{1.4}$$

Hardy and Ramanujan's proof involved an important tool called the Circle Method which has manifold applications in analytic number theory. For a well documented exposition on this collaboration, see [45]. During 1937-1943, Rademacher [55, 56, 57] improved the work of Hardy and Ramanujan and found a convergent series for $p(n)$ and Lehmer's [43, 44] considerations were on the estimation for the remainder term of the series for $p(n)$. The Hardy-Ramanujan-Rademacher formula reads

$$p(n) = \frac{\sqrt{12}}{24n-1} \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left[\left(1 - \frac{k}{\mu(n)}\right) e^{\mu(n)/k} + \left(1 + \frac{k}{\mu(n)}\right) e^{-\mu(n)/k} \right] + R_2(n, N), \tag{1.5}$$

where

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}, \quad A_k(n) = \sum_{\substack{h \mod k \\ (h,k)=1}} e^{-2\pi i n h/k + \pi i s(h,k)}$$

with

$$s(h,k) = \sum_{\mu=1}^{k-1} \left(\frac{\mu}{k} - \Big\lfloor \frac{\mu}{k} \Big\rfloor - \frac{1}{2} \right) \left(\frac{h\mu}{k} - \Big\lfloor \frac{h\mu}{k} \Big\rfloor - \frac{1}{2} \right),$$

and

$$|R_2(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left[\left(\frac{N}{\mu(n)} \right)^3 \sinh \frac{\mu(n)}{N} + \frac{1}{6} - \left(\frac{N}{\mu(n)} \right)^2 \right]. \tag{1.6}$$

Independently Nicolas [50] and DeSalvo and Pak [23, Theorem 1.1] proved that the partition function $(p(n))_{n\geq 26}$ is log-concave, conjectured by Chen [15]. DeSalvo and Pak [23, Theorem 4.1] also proved that for all $n \geq 2$,

$$\frac{p(n-1)}{p(n)} \left(1 + \frac{1}{n} \right) > \frac{p(n)}{p(n+1)}, \tag{1.7}$$

conjectured by Chen [15]. Further, they improved the term $(1 + \frac{1}{n})$ in (1.7) and proved that for all $n \geq 7$,

$$\frac{p(n-1)}{p(n)} \left(1 + \frac{240}{(24n)^{3/2}} \right) > \frac{p(n)}{p(n+1)}, \tag{1.8}$$

see [23, p. 4.2]. DeSalvo and Pak [23] finally came up with the conjecture that the coefficient of $1/n^{3/2}$ in (1.8) can be improved to $\pi/\sqrt{24}$; i.e., for all $n \geq 45$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}, \tag{1.9}$$

which was proved by Chen, Wang and Xie [18, Sec. 2]. Paule, Radu, Zeng, and the author [8, Theorem 7.6] confirmed that the coefficient of $1/n^{3/2}$ is indeed $\pi/\sqrt{24}$, which is the optimal; i.e., they proved that for all $n \geq 120$,

$$p(n)^2 > \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{1}{n^2}\right)p(n-1)p(n+1). \tag{1.10}$$

Chen [16] conjectured that $p(n)$ satisfies the higher order Turán inequalities for all $n \geq 95$ which was proved by Chen, Jia, and Wang [17, Theorem 1.3] and analogous to the inequality (1.9), they conjectured that for all $n \geq 2$,

$$4(1-u_n)(1-u_{n+1}) < \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right)(1-u_nu_{n+1})^2 \ \text{ with } \ u_n := \frac{p(n+1)p(n-1)}{p(n)^2}, \tag{1.11}$$

settled by Larson and Wagner [42, Theorem 1.2]. In [17], Chen, Jia, and Wang conjectured[1] that for any integer $d \geq 1$ there exists an integer $N(d)$ such that the Jensen polynomial of degree d and shift n associated with $p(n)$ has only real roots which was settled by Griffin, Ono, Rolen, and Zagier [32, Theorem 5] and inspired by their work, Larson and Wagner [42, Theorem 1.3] proved that $N(d) \leq (3d)^{24d}(50d)^{3d^2}$. Proofs of the inequalities, stated before, primarily relies on the Hardy-Ramanujan-Rademacher formula (1.5) and Lehmer's error bound (1.6) but with different methodology.

While studying on the higher order Turán inequality for $p(n)$, Chen [16] undertook a comprehensive study on inequalities pertaining to invariants of a binary form. A binary form $P(x,y)$ of degree d is a homogeneous polynomial of degree d in two variables x and y is defined by

$$P_d(x,y) := \sum_{i=0}^{d}\binom{n}{i}a_ix^iy^{n-i},$$

where $(a_i)_{1\leq i\leq n} \in \mathbb{C}^n$. But we restrict a_i to be real numbers. The binary form $P_d(x,y)$ is transformed into a new binary form, say $Q(\overline{x},\overline{y})$ with

$$Q_d(\overline{x},\overline{y}) = \sum_{i=0}^{d}\binom{n}{i}c_i\overline{x}^i\overline{y}^{n-i}$$

under the action of $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix} \in GL_2(\mathbb{R})$ as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}.$$

The transformed coefficients $(c_i)_{0\leq i\leq d}$ are polynomials in $(a_i)_{0\leq i\leq d}$ and entries of the matrix M. For $k \in \mathbb{Z}_{\geq 0}$, a polynomial $I(a_0,a_1,\ldots,a_d)$ in the coefficients $(a_i)_{0\leq i\leq d}$ is called an invariant of index of k of the binary form $P_d(x,y)$ if for any $M \in GL_2(\mathbb{R})$,

$$I(\overline{a}_0,\overline{a}_1,\ldots,\overline{a}_d) = (\det M)^k I(a_0,a_1,\ldots,a_n).$$

[1] Independently conjectured by K. Ono

For a more detailed study on the theory of invariants, see, for example, Hilbert [35], Kung and Rota [40], and Sturmfels [63]. We observe that $I(a_0,a_1,a_2)=a_1^2-a_0a_2$ is an invariant of the quadratic binary form

$$P_2(x,y)=a_2x^2+2a_1xy+a_0y^2$$

and the discriminant is $4I(a_0,a_1,a_2)$. For a sequence $(a_n)_{n\geq 0}$, define

$$I_{n-1}(a_0,a_1,a_2):=I(a_{n-1},a_n,a_{n+1})=a_n^2-a_{n-1}a_{n+1}.$$

Therefore, if we choose $a_n=p(n)$, then $I_{n-1}(p(0),p(1),p(2))>0$ for all $n\geq 26$ is the same thing as saying $(p(n))_{n\geq 26}$ is log-concave. For degree 3,

$$I(a_0,a_1,a_2,a_3)=4(a_1^2-a_0a_2)(a_2^2-a_1a_3)-(a_1a_2-a_0a_3)^2$$

is an invariant of the cubic binary form $P_3(x,y)=a_3x^3+3a_2x^2y+3a_1xy^2+a_0y^3$ and the discriminant is $27I(a_0,a_1,a_2,a_3)$. Similarly, setting $a_n=p(n)$, the positivity of $I_{n-1}(a_0,a_1,a_2,a_3)$ for all $n\geq 95$ is equivalent to state that $(p(n))_{n\geq 95}$ satisfies the higher order Turán inequality. Two invariants of the quartic binary form

$$P_4(x,y)=a_4x^4+4a_3x^3y+6a_2x^2y^2+4a_1xy^3+a_0y^4$$

are of the following form

$$\begin{aligned}A(a_0,a_1,a_2,a_3,a_4)&=a_0a_4-4a_1a_3+3a_2^2,\\ B(a_0,a_1,a_2,a_3,a_4)&=-a_0a_2a_4+a_2^3+a_0a_3^2+a_1^2a_4-2a_1a_2a_3.\end{aligned}$$

Setting $a_n=p(n)$, Chen [16] conjectured that

$$A(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3})>0 \ \text{ and } \ B(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3})>0,$$

along with the associated companion inequalities in the spirit of (1.9) and (1.11). Here we list all the four conjectures with $a_n=p(n)$.

Conjecture 1.1 (Eqn. (6.17), [16])**.**

$$a_{n-1}a_{n+3}+3a_{n+1}^2>4a_na_{n+2} \quad \text{for all} \quad n\geq 185. \tag{1.12}$$

Conjecture 1.2 (Conjecture 6.15, [16])**.** *We have*

$$4\Big(1+\frac{\pi^2}{16n^3}\Big)a_na_{n+2}>a_{n-1}a_{n+3}+3a_{n+1}^2 \quad \text{for all} \quad n\geq 218. \tag{1.13}$$

Conjecture 1.3 (Eqn. (6.18), [16])**.**

$$a_{n+1}^3+a_{n-1}a_{n+2}^2+a_n^2a_{n+3}>2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3} \quad \text{for all} \quad n\geq 221. \tag{1.14}$$

Conjecture 1.4 (Conjecture 6.16, [16])**.** *We have*

$$\Big(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}\Big)(2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3})>a_{n+1}^3+a_{n-1}a_{n+2}^2+a_n^2a_{n+3} \quad \text{for all} \quad n\geq 244. \tag{1.15}$$

We prove all the four conjectures along with the confirmation that the rate of decay $\pi^2/16n^3$ (resp. $\pi^3/72\sqrt{6}n^{9/2}$) in (1.2) (resp. in (1.4)) is the optimal one, as stated in Theorem 1.5 (resp. Theorem 1.7). We also ensure that the rate of decay is $\pi/\sqrt{24}n^{3/2}$ in context of (1.11) can not be improved further by proving Theorem 1.9. Let $a_n:=p(n)$.

Theorem 1.5. *For all $n\geq 218$,*

$$4\Big(1+\frac{\pi^2}{16n^3}\Big)a_na_{n+2}>a_{n-1}a_{n+3}+3a_{n+1}^2>4\Big(1+\frac{\pi^2}{16n^3}-\frac{6}{n^{7/2}}\Big)a_na_{n+2}. \tag{1.16}$$

Corollary 1.6. *Conjecture 1.1 and 1.2 is true.*

Theorem 1.7. *For all $n \geq 244$,*

$$\begin{aligned}\Big(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}\Big)(2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3}) &> a_{n+1}^3+a_{n-1}a_{n+2}^2+a_n^2a_{n+3}\\ &> \Big(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}-\frac{8}{n^5}\Big)(2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3}).\end{aligned} \tag{1.17}$$

Corollary 1.8. *Conjecture 1.3 and 1.4 is true.*

Theorem 1.9. *For all $n \geq 115$,*

$$\begin{aligned}\Big(1+\frac{\pi}{\sqrt{24}n^{3/2}}\Big)(a_na_{n+1}-a_{n-1}a_{n+2})^2 &> 4(a_n^2-a_{n-1}a_{n+1})(a_{n+1}^2-a_na_{n+2})\\ &> \Big(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{3}{n^2}\Big)(a_na_{n+1}-a_{n-1}a_{n+2})^2.\end{aligned} \tag{1.18}$$

Remark 1.10. *We observe that Theorem 1.9 immediately implies the following three statements:*

(1) $(p(n))_{n\geq 95}$ *satisfies the higher order Turán inequalities [17, Theorem 1.3].*
(2) *For all $n \geq 2$, (1.11) holds [42, Theorem 1.2].*
(3) $\frac{\pi}{\sqrt{24}n^{3/2}}$ *is the optimal rate of decay of the quotient*

$$4(a_n^2-a_{n-1}a_{n+1})(a_{n+1}^2-a_na_{n+2})/(a_na_{n+1}-a_{n-1}a_{n+2})^2.$$

2. Plan of paper

The rest of this paper is organized as follows. In Section 3, we shall present a couple of lemmas from [8, 7] that will be helpful in later sections. Section 4 prepares the set up by determining the coefficients in the asymptotic expansion of $p(n-\ell)$ with $\ell \in \mathbb{N}$ along with its estimates. An infinite family of inequalities for $p(n-\ell)$ is presented in Section 5. Section 6 presents proofs of the Theorems 1.5, 1.7, and 1.9. In Section 7, we presents further applications including higher order Laguerre inequalities, higher order shifted difference, and higher order log-concavity for $p(n)$. Section 8 presents a unified framework on proving inequalities for $p(n)$ of types discussed in this paper. Finally, we conclude this paper by proving all the Chen's conjectures stated before in context of the Andrews' spt function in Section 9.

A note for the reader: We intentionally did not define (and consequently, did not make any reference to the works done in the respective contexts) the notions of higher order Laguerre inequalities, higher order shifted difference, and higher order log-concavity for a sequence. We refer the reader to see Subsections 7.1-7.3 for the details. Similarly, the reader will find a comprehensive detail on Andrews's spt function in Section 9.

3. Preliminaries

This section presents all the preliminary lemmas required for the proofs of the lemmas presented in subsequent sections.

Lemma 3.1. *[7, Lemma 3.3] For j, $k \in \mathbb{Z}_{\geq 0}$ with $k < 2j$*[1],

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{i/2}{j} = \begin{cases} 1, & j=k=0\\ (-1)^j 2^{k-2j}\frac{k}{j}\binom{2j-k-1}{j-k}, & \textit{otherwise}\end{cases}. \tag{3.1}$$

Lemma 3.2. *[7, Lemma 4.1] Let $x_1, x_2, \dots, x_n \leq 1$ and $y_1, \dots, y_1$ be non-negative real numbers. Then*

$$\frac{(1-x_1)(1-x_2)\cdots(1-x_n)}{(1+y_1)(1+y_2)\cdots(1+y_n)} \geq 1-\sum_{j=1}^{n}x_j-\sum_{j=1}^{n}y_j.$$

[1] This condition has been tacitly assumed in the proof of [7, Lemma 3.3] but not written explicitly.

Lemma 3.3. *[7, Lemma 4.2] For $t \geq 1$ and non-negative integer $u \leq t$, we have*

$$\frac{1}{2t} \geq \frac{t(-t)_u(-1)^u}{(1+2t)(t+u)(t)_u} \geq \frac{1}{2t}\left(1 - \frac{u^2+\frac{1}{2}}{t}\right).$$

Lemma 3.4. *[7, Lemma 4.3] For $t \geq 1$ and non-negative integer $u \leq t$, we have*

$$\frac{2u+1}{2t} \geq \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i} \geq \frac{2u+1}{2t} - \frac{4u^3+6u^2+8u+3}{12t^2}.$$

Throughout the rest of this paper,

$$\alpha_\ell := \frac{\pi}{6}\sqrt{1+24\ell}.$$

Lemma 3.5. *[7, Lemma 4.4] We have*

$$\sum_{u=0}^{\infty}\frac{\alpha^{2u}}{(2u)!} = \cosh(\alpha),\ \sum_{u=0}^{\infty}\frac{u\alpha^{2u}}{(2u)!} = \frac{1}{2}\alpha\sinh(\alpha),\ \sum_{u=0}^{\infty}\frac{u^2\alpha^{2u}}{(2u)!} = \frac{\alpha^2}{4}\cosh(\alpha) + \frac{\alpha}{4}\sinh(\alpha),$$
$$\sum_{u=0}^{\infty}\frac{u^3\alpha^{2u}}{(2u)!} = \frac{3\alpha^2}{8}\cosh(\alpha) + \frac{\alpha(\alpha^2+1)}{8}\sinh(\alpha),$$
$$\text{and } \sum_{u=0}^{\infty}\frac{u^4\alpha^{2u}}{(2u)!} = \frac{1}{16}\alpha^2(\alpha^2+7)\cosh(\alpha) + \frac{1}{16}(6\alpha^3+\alpha)\sinh(\alpha).$$

Lemma 3.6. *[7, Lemma 4.5] Let $u \in \mathbb{Z}_{\geq 0}$. Assume that $a_{n+1} - a_n \geq b_{n+1} - b_n$ for all $n \geq u$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Then*

$$b_n \geq a_n \text{ for all } n \geq u.$$

Lemma 3.7. *For $t \geq 1$ and $k \in \{0, 1, 2, 3\}$ we have*

$$\sum_{u=t+1}^{\infty}\frac{u^k\alpha_\ell^{2u}}{(2u)!} \leq \frac{C_k(\ell)}{t^2},$$

where

$$C_k(\ell) = \begin{cases} C_k = \dfrac{\alpha_\ell^4 \cdot 2^k}{18}, & \ell = 0 \\ \dfrac{\lceil\sqrt{\ell}\rceil^2\left(1+\lceil\sqrt{\ell}\rceil\right)^{k+2}\alpha_\ell^{2(1+\lceil\sqrt{\ell}\rceil)}}{(1+2\lceil\sqrt{\ell}\rceil)(2+2\lceil\sqrt{\ell}\rceil)!}, & \ell \geq 1 \end{cases}.$$

Proof. Applying Lemma 3.6 with $a_n = \sum_{u=n+1}^{\infty}\frac{u^k\alpha_\ell^{2u}}{(2u)!}$ and $b_n = \frac{C_k(\ell)}{n^2}$, $b_{n+1} - b_n \leq a_{n+1} - a_n$ is equivalent to show that $f(n) := \frac{n^2(n+1)^{k+2}\alpha_\ell^{2n+2}}{(2n+1)(2n+2)!} \leq C_k(\ell)$. To prove $f(n) \leq C_k(\ell)$, it is sufficient to show that $f(m) \leq C_k(\ell)$ for a minimal m such that $f(m)$ is maximal. In order to find such m, it is enough to that $\frac{f(n+1)}{f(n)} \leq 1$ for all $n \geq \max\{\lceil\sqrt{\ell}\rceil, 1\}$, and therefore, $\max\limits_{n\in\mathbb{Z}_{\geq 0}} f(n) = f(\lceil\sqrt{\ell}\rceil) = C_k(\ell)$ for all $\ell \geq 1$ and for $\ell = 0$, $\max\limits_{n\in\mathbb{Z}_{\geq 0}} f(n) = f(1) = C_k(0)$. Now, $\frac{f(n+1)}{f(n)} = \frac{\alpha_\ell^2(n+2)^{k+2}(2n+1)}{(2n+4)(2n+3)^2(n+1)^k n^2} \leq 1$ holds for all all $n \geq \max\{\lceil\sqrt{\ell}\rceil, 1\}$. □

Lemma 3.8. *[8, Equation 7.5, Lemma 7.3] For $n, k, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$ let*

$$b_{k,n}(s) := \frac{4\sqrt{s}}{\sqrt{s+k-1}}\binom{s+k-1}{s-1}\frac{1}{n^k},$$

then

$$0 < \sum_{t=k}^{\infty} \binom{-\frac{2s-1}{2}}{t} \frac{(-1)^k}{n^k} < b_{k,n}(s). \tag{3.2}$$

Lemma 3.9. *[8, Equation 7.9, Lemma 7.5] For $m, n, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$ let*

$$c_{m,n}(s) := \frac{2}{m} \frac{s^m}{n^m},$$

then

$$-\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty} \binom{1/2}{k} \frac{(-1)^k s^k}{n^k} < 0. \tag{3.3}$$

Lemma 3.10. *[8, Equation 7.7, Lemma 7.4] For $n, s \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{N}$ and $n > 2s$ let*

$$\beta_{m,n}(s) := \frac{2}{n^m} \binom{s+m-1}{s-1},$$

then

$$0 < \sum_{k=m}^{\infty} \binom{-s}{k} \frac{(-1)^k}{n^k} < \beta_{m,n}(s). \tag{3.4}$$

4. Set Up

Using the Hardy-Ramanujan-Rademacher formula for $p(n)$ and Lehmer's error bound, we have the following inequality for $p(n)$ due to Chen, Jia, and Wang.

Lemma 4.1. *[17, Lemma 2.2] For all $n \geq 1206$,*

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^{10}}\right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^{10}}\right), \tag{4.1}$$

where for $n \geq 1$, $\mu(n) := \frac{\pi}{6}\sqrt{24n-1}$.

The definition of $\mu(n)$ is kept throughout this paper. Paule, Radu, Zeng, and the author extended Lemma 4.1 as follows.

Theorem 4.2. *[8, Theorem 4.4] For $k \in \mathbb{Z}_{\geq 2}$, define*

$$\widehat{g}(k) := \frac{1}{24}\left(\frac{36}{\pi^2} \cdot \nu(k)^2 + 1\right),$$

where $\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \dfrac{5k\log\log k}{\log k}$. Then for all $k \in \mathbb{Z}_{\geq 2}$ and $n > \widehat{g}(k)$ such that $(n,k) \neq (6,2)$, we have

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^{k}}\right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^{k}}\right). \tag{4.2}$$

By making the shift $n - \ell$ in $p(n)$ for any $\ell \geq 0$, we obtain the following result.

Lemma 4.3. *Let $\ell \in \mathbb{Z}_{\geq 0}$. For $k \in \mathbb{Z}_{\geq 2}$, let $\widehat{g}(k)$ be as in Theorem 4.2. Then for all $k \in \mathbb{Z}_{\geq 2}$ and $n > \widehat{g}(k) + \ell$ such that $(n,k) \neq (6,2)$, we have*

$$\frac{\sqrt{12}e^{\mu(n-\ell)}}{24(n-\ell)-1}\left(1 - \frac{1}{\mu(n-\ell)} - \frac{1}{\mu(n-\ell)^{k}}\right) < p(n-\ell) < \frac{\sqrt{12}e^{\mu(n-\ell)}}{24(n-\ell)-1}\left(1 - \frac{1}{\mu(n-\ell)} + \frac{1}{\mu(n-\ell)^{k}}\right). \tag{4.3}$$

Rewrite the term $\dfrac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\dfrac{1}{\mu(n-\ell)}\Big)$ in the following way:

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big)=\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}\underbrace{e^{\pi\sqrt{2n/3}\ \left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)}}_{=:A_1(n,\ell)}\underbrace{\Big(1-\frac{1+24\ell}{24n}\Big)^{-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big)}_{=:A_2(n,\ell)}. \tag{4.4}$$

Now we compute the Taylor expansion of the residue parts of $A_1(n,\ell)$ and $A_2(n,\ell)$, defined in (4.4).

Definition 4.4. *For $t,\ell\in\mathbb{Z}_{\geq 0}$, define*

$$e_1(t,\ell):=\begin{cases}1, & \text{if } t=0\\ \dfrac{(-1)^t(1+24\ell)^t}{(24)^t}\dfrac{(1/2-t)_{t+1}}{t}\displaystyle\sum_{u=1}^{t}\frac{(-1)^u(-t)_u}{(t+u)!(2u-1)!}\alpha_\ell^{2u}, & \text{otherwise}\end{cases}, \tag{4.5}$$

and

$$E_1\Big(\frac{1}{\sqrt{n}},\ell\Big):=\sum_{t=0}^{\infty}e_1(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t},\ n\geq 1. \tag{4.6}$$

Definition 4.5. *For $t,\ell\in\mathbb{Z}_{\geq 0}$, define*

$$o_1(t,\ell):=-\frac{\pi}{12\sqrt{6}}(1+24\ell)\left(\frac{(-1)^t(1/2-t)_{t+1}(1+24\ell)^t}{(24)^t}\sum_{u=0}^{t}\frac{(-1)^u(-t)_u}{(t+u+1)!(2u)!}\alpha_\ell^{2u}\right) \tag{4.7}$$

and

$$O_1\Big(\frac{1}{\sqrt{n}},\ell\Big):=\sum_{t=0}^{\infty}o_1(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1},\ n\geq 1. \tag{4.8}$$

Lemma 4.6. *Let $A_1(n,\ell)$ be defined as in* (4.4)*. Let $E_1(n,\ell)$ be as in Definition 4.4 and $O_1(n,\ell)$ as in Definition 4.5. Then*

$$A_1(n,\ell)=E_1\Big(\frac{1}{\sqrt{n}},\ell\Big)+O_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{4.9}$$

Proof. From (4.4), we get

$$\begin{aligned}A_1(n,\ell)&=e^{\pi\sqrt{2n/3}\ \left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)}\\&=\sum_{k=0}^{\infty}\frac{(\pi\sqrt{2n/3})^k}{k!}\left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)^k\\&=\sum_{k=0}^{\infty}\frac{(\pi\sqrt{2/3})^k}{k!}(\sqrt{n})^k\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(\sqrt{1-\frac{1+24\ell}{24n}}\right)^i\\&=\sum_{k=0}^{\infty}\frac{(\pi\sqrt{2/3})^k}{k!}(\sqrt{n})^k\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=0}^{\infty}\binom{i/2}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j}\\&=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{\infty}\frac{(\pi\sqrt{2/3})^k}{k!}\frac{(-1)^{k-i+j}(1+24\ell)^j}{(24)^j}\binom{k}{i}\binom{i/2}{j}(\sqrt{n})^{k-2j}.\end{aligned} \tag{4.10}$$

Setting $z:=\frac{1}{\sqrt{n}}$ and $c:=\pi\sqrt{\frac{2}{3}}$, $A_1\left(\frac{1}{z^2},\ell\right)=e^{\frac{c}{z}\left(\sqrt{1-\frac{1+24\ell}{24}z^2}-1\right)}$ is an analytic function in z, and therefore its Taylor expansion in a neighborhood of 0 is of the form $\sum_{t=0}^{\infty}a_t(\ell)z^t$; i.e., $\sum_{t=0}^{\infty}\frac{a_t(\ell)}{\sqrt{n}^t}$

for some constants $(a_t(\ell))_{t\geq 0}$. Hence, due to the uniqueness of the Taylor expansion of $A_1(n,\ell)$, we have for $k>2j$ in (4.10),

$$\sum_{i=0}^{k}\binom{k}{i}\binom{i/2}{j}=0.$$

Consequently, we need only to consider the range $0\leq k\leq 2j$ in (4.10). Split $S:=\big\{(k,i,j)\in\mathbb{Z}^3_{\geq 0}:0\leq i\leq k\leq 2j\big\}=:\bigcup_{t\in\mathbb{Z}_{\geq 0}}V(t)$, where for each $t\in\mathbb{Z}_{\geq 0}$,

$$V(2t)=\big\{(2u,i,u+t)\in\mathbb{Z}^3_{\geq 0}:0\leq i\leq 2u\big\}$$

and

$$V(2t+1)=\big\{(2u+1,i,u+t+1)\in\mathbb{Z}^3_{\geq 0}:0\leq i\leq 2u+1\big\}.$$

For $r=(k,i,j)\in S$, we define

$$S(r):=\frac{(\pi\sqrt{2/3})^k}{k!}\frac{(-1)^{k-i+j}(1+24\ell)^j}{(24)^j}\binom{k}{i}\binom{i/2}{j}\quad\text{and}\quad f(r):=k-2j.$$

Rewrite (4.10) as

$$A_1(n,\ell)\;=\;\sum_{t=0}^{\infty}\sum_{r\in V(2t)}S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}+\sum_{t=0}^{\infty}\sum_{r\in V(2t+1)}S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}.\tag{4.11}$$

Now

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t)}S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}\;=\;\sum_{t=0}^{\infty}\frac{(-1)^t(1+24\ell)^t}{(24)^t}\Bigg(\sum_{u=0}^{\infty}\frac{(-1)^u}{(2u)!}\alpha_\ell^{2u}\mathcal{E}_1(u,t)\Bigg)\Big(\frac{1}{\sqrt{n}}\Big)^{2t},\tag{4.12}$$

where by Lemma 3.1,

$$\mathcal{E}_1(u,t):=\sum_{i=0}^{2u}(-1)^i\binom{2u}{i}\binom{i/2}{u+t}=\begin{cases}1, & \text{if } u=t=0\\ 0, & \text{if } u>t\\ \frac{2u(1/2-t)_{t+1}(-t)_u}{t(t+u)!}, & \text{otherwise}\end{cases}.$$

Consequently, we have

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t)}S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}=E_1\Big(\frac{1}{\sqrt{n}},\ell\Big).\tag{4.13}$$

After simplifying, it follows that

$$\begin{aligned}&\sum_{t=0}^{\infty}\sum_{r\in V(2t+1)}S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\\ &=-\frac{\pi(1+24\ell)}{12\sqrt{6}}\sum_{t=0}^{\infty}\frac{(-1)^t(1+24\ell)^t}{(24)^t}\Bigg(\sum_{u=0}^{\infty}\frac{(-1)^u}{(2u+1)!}\alpha_\ell^{2u}\mathcal{O}_1(u,t)\Bigg)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1},\end{aligned}\tag{4.14}$$

where by Lemma 3.1,

$$\mathcal{O}_1(u,t):=\sum_{i=0}^{2u+1}(-1)^i\binom{2u+1}{i}\binom{i/2}{u+t+1}=\begin{cases}0, & \text{if } u>t\\ -\frac{(2u+1)(1/2-t)_{t+1}(-t)_u}{(t+u+1)!}, & \text{otherwise}\end{cases}.$$

Therefore, we have

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t+1)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} = O_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{4.15}$$

From (4.11), (4.13), and (4.15), we get (4.9). □

Definition 4.7. *For $t\in\mathbb{Z}_{\geq 0}$, define*

$$E_2\Big(\frac{1}{\sqrt{n}},\ell\Big) := \sum_{t=0}^{\infty} e_2(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \ \text{with}\ \ e_2(t,\ell) := \frac{(1+24\ell)^t}{(24)^t}. \tag{4.16}$$

Definition 4.8. *For $t\in\mathbb{Z}_{\geq 0}$, define*

$$O_2\Big(\frac{1}{\sqrt{n}},\ell\Big) := \sum_{t=0}^{\infty} o_2(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \ \text{with}\ \ o_2(t,\ell) := -\frac{6}{\pi\sqrt{24}}\binom{-3/2}{t}\frac{(-1)^t(1+24\ell)^t}{(24)^t}. \tag{4.17}$$

Lemma 4.9. *Let $A_2(n,\ell)$ be defined as in (4.4). Let $E_2(n,\ell)$ be as in Definition 4.7 and $O_2(n,\ell)$ as in Definition 4.8. Then*

$$A_2(n,\ell) = E_2\Big(\frac{1}{\sqrt{n}},\ell\Big) + O_2\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{4.18}$$

Proof. Following the definition of $A_2(n,\ell)$ from (4.4) and expand it as follows:

$$\begin{aligned} A_2(n,\ell) &= \Big(1-\frac{1+24\ell}{24n}\Big)^{-1} - \frac{6}{\pi\sqrt{24}}\frac{1}{\sqrt{n}}\Big(1-\frac{1+24\ell}{24n}\Big)^{-3/2} \\ &= E_2\Big(\frac{1}{\sqrt{n}},\ell\Big) + O_2\Big(\frac{1}{\sqrt{n}},\ell\Big). \end{aligned} \tag{4.19}$$

This completes the proof of (4.18). □

Definition 4.10. *Following the Definitions 4.4-4.8, we define*

$$S_{e,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) := E_1\Big(\frac{1}{\sqrt{n}},\ell\Big)E_2\Big(\frac{1}{\sqrt{n}},\ell\Big), \tag{4.20}$$

$$S_{e,2}\Big(\frac{1}{\sqrt{n}},\ell\Big) := O_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big), \tag{4.21}$$

$$S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) := E_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big), \tag{4.22}$$

and

$$S_{o,2}\Big(\frac{1}{\sqrt{n}},\ell\Big) := E_2\Big(\frac{1}{\sqrt{n}},\ell\Big)O_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{4.23}$$

Lemma 4.11. *For each $i\in\{1,2\}$, let $S_{e,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ and $S_{o,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ be as in Definition 4.10. Then*

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big) = \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}\sum_{i=1}^{2}\Bigg(S_{e,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)+S_{o,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)\Bigg). \tag{4.24}$$

Proof. The proof follows immediately by applying Lemmas 4.6 and 4.9 to (4.4). □

4.1. **Coefficients in the asymptotic expansion of $p(n-\ell)$.**

Definition 4.12. *For $t, \ell \in \mathbb{Z}_{\geq 0}$, define*

$$S_1(t,\ell) := \sum_{s=1}^{t} \frac{(-1)^s (1/2-s)_{s+1}}{s} \sum_{u=1}^{s} \frac{(-1)^u (-s)_u}{(s+u)!(2u-1)!} \alpha_\ell^{2u}, \tag{4.25}$$

and

$$g_{e,1}(t,\ell) := \frac{(1+24\ell)^t}{(24)^t} \Big(1 + S_1(t,\ell)\Big). \tag{4.26}$$

Lemma 4.13. *Let $S_{e,1}\Big(\frac{1}{\sqrt{n}}, \ell\Big)$ be as in (4.20). Let $g_{e,1}(t,\ell)$ be as in Definition 4.12. Then*

$$S_{e,1}\Big(\frac{1}{\sqrt{n}}, \ell\Big) = \sum_{t=0}^{\infty} g_{e,1}(t,\ell) \Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{4.27}$$

Proof. From (4.6), (4.16), and (4.20), we have

$$S_{e,1}\Big(\frac{1}{\sqrt{n}}, \ell\Big) = 1 + \sum_{t=1}^{\infty} \left(e_1(t,\ell) + e_2(t,\ell) + \sum_{s=1}^{t-1} e_1(s,\ell) e_2(t-s,\ell) \right) \Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{4.28}$$

Combining (4.5) and (4.16), we obtain

$$e_1(t,\ell) + e_2(t,\ell) + \sum_{s=1}^{t-1} e_1(s,\ell) e_2(t-s,\ell) = \frac{(1+24\ell)^t}{(24)^t} \Big(1 + S_1(t,\ell)\Big) = g_{e,1}(t,\ell), \tag{4.29}$$

which concludes the proof of (4.27). □

Definition 4.14. *For $t \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$, define*

$$S_2(t,\ell) := \sum_{s=0}^{t-1} (1/2-s)_{s+1} \binom{-3/2}{t-s-1} \sum_{u=0}^{s} \frac{(-1)^u (-s)_u}{(s+u+1)!(2u)!} \alpha_\ell^{2u}, \tag{4.30}$$

and

$$g_{e,2}(t,\ell) := \frac{(-1)^{t-1}(1+24\ell)^t}{(24)^t} S_2(t,\ell). \tag{4.31}$$

Lemma 4.15. *Let $S_{e,2}\Big(\frac{1}{\sqrt{n}}, \ell\Big)$ as in (4.21) and $g_{e,2}(t,\ell)$ as in Definition 4.14. Then*

$$S_{e,2}\Big(\frac{1}{\sqrt{n}}, \ell\Big) = \sum_{t=1}^{\infty} g_{e,2}(t,\ell) \Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{4.32}$$

Proof. From (4.8), (4.18) and (4.21), we have

$$\begin{aligned} S_{e,2}\Big(\frac{1}{\sqrt{n}}, \ell\Big) &= O_1\Big(\frac{1}{\sqrt{n}}, \ell\Big) O_2\Big(\frac{1}{\sqrt{n}}, \ell\Big) \\ &= \sum_{t=1}^{\infty} \left(\sum_{s=0}^{t-1} o_1(s,\ell) o_2(t-s-1,\ell) \right) \Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\ &= \sum_{t=1}^{\infty} g_{e,2}(t,\ell) \Big(\frac{1}{\sqrt{n}}\Big)^{2t} \text{ (by (4.7) and (4.17)).} \end{aligned} \tag{4.33}$$

□

Definition 4.16. *For $t\in\mathbb{Z}_{\geq 2}$ and $\ell\in\mathbb{Z}_{\geq 0}$, define*

$$S_3(t,\ell):=\sum_{s=1}^{t}\frac{(1/2-s)_{s+1}\binom{-3/2}{t-s}}{s}\sum_{u=1}^{s}\frac{(-1)^u(-s)_u}{(s+u)!(2u-1)!}\alpha_\ell^{2u}, \tag{4.34}$$

and

$$g_{o,1}(t,\ell):=\begin{cases}-\dfrac{6}{\pi\sqrt{24}}\dfrac{(-1)^t(1+24\ell)^t}{(24)^t}\left(\binom{-3/2}{t}+S_3(t,\ell)\right), & \text{if } t\geq 2\\ -\dfrac{432+(1+24\ell)\pi^2}{2304\sqrt{6}\pi}, & \text{if } t=1\ .\\ -\dfrac{6}{\pi\sqrt{24}}, & \text{if } t=0\end{cases} \tag{4.35}$$

Lemma 4.17. *Let $S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ as in (4.22) and $g_{o,1}(t,\ell)$ be as in Definition 4.16. Then*

$$S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big)=\sum_{t=0}^{\infty}g_{o,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \tag{4.36}$$

Proof. From (4.6), (4.17) and (4.22), it follows that

$$\begin{aligned}S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) &= E_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big)\\ &= g_{o,1}(0,\ell)\frac{1}{\sqrt{n}}+g_{o,1}(1,\ell)\frac{1}{\sqrt{n}^3}+\sum_{t=2}^{\infty}\left(o_2(t)+\sum_{s=1}^{t}e_1(s,\ell)o_2(t-s,\ell)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\\ &= g_{o,1}(0,\ell)\frac{1}{\sqrt{n}}+g_{o,1}(1,\ell)\frac{1}{\sqrt{n}^3}+\sum_{t=2}^{\infty}g_{o,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\text{(by (4.5) and (4.17)).}\end{aligned} \tag{4.37}$$

□

Definition 4.18. *For $t\in\mathbb{Z}_{\geq 1}$ and $\ell\in\mathbb{Z}_{\geq 0}$, define*

$$S_4(t,\ell):=\sum_{s=0}^{t}(-1)^s(1/2-s)_{s+1}\sum_{u=0}^{s}\frac{(-1)^u(-s)_u}{(s+u+1)!(2u)!}\alpha_\ell^{2u}, \tag{4.38}$$

and

$$g_{o,2}(t,\ell):=-\frac{\pi(1+24\ell)}{12\sqrt{6}}\frac{(1+24\ell)^t}{(24)^t}S_4(t,\ell). \tag{4.39}$$

Lemma 4.19. *Let $S_{o,2}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ be as in (4.23) and $g_{o,2}(t,\ell)$ be as in Definition 4.18. Then*

$$S_{o,2}\Big(\frac{1}{\sqrt{n}},\ell\Big)=\sum_{t=0}^{\infty}g_{o,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \tag{4.40}$$

Proof. From (4.8), (4.16) and (4.23), it follows that

$$\begin{aligned}S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) &= O_1\Big(\frac{1}{\sqrt{n}},\ell\Big)E_2\Big(\frac{1}{\sqrt{n}},\ell\Big)\\ &= \sum_{t=0}^{\infty}\left(\sum_{s=0}^{t}o_1(s,\ell)e_2(t-s,\ell)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\\ &= \sum_{t=0}^{\infty}g_{o,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\ \text{(by (4.8) and (4.16)).}\end{aligned} \tag{4.41}$$

□

Definition 4.20. *For each $i \in \{1,2\}$, let $g_{e,i}(t,\ell)$ and $g_{o,i}(t,\ell)$ be as in Definitions 4.12-4.18. We define a power series*

$$G(n,\ell) := \sum_{t=0}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t = \sum_{t=0}^{\infty} g(2t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \sum_{t=0}^{\infty} g(2t+1,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1},$$

where

$$g(2t,\ell) := g_{e,1}(t,\ell) + g_{e,2}(t,\ell) \quad and \quad g(2t+1,\ell) := g_{o,1}(t,\ell) + g_{o,2}(t,\ell). \tag{4.42}$$

Lemma 4.21. *Let $G(n,\ell)$ be as in Definition 4.20. Then*

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big) = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \cdot G(n,\ell). \tag{4.43}$$

Proof. Applying Lemmas 4.13-4.19 to Lemma 4.9, we have (4.43). □

Remark 4.22. *Using* `Sigma` *due to Schneider [60] and* `GeneratingFunctions` *due to* Mallinger *[46], we observe that for all $t \geq 0$,*

$$g(2t,\ell) = g_{e,1}(t,\ell) + g_{e,2}(t,\ell) = \omega_{2t,\ell} \quad and \quad g(2t+1,\ell) = g_{o,1}(t,\ell) + g_{o,2}(t,\ell) = \omega_{2t+1,\ell}, \tag{4.44}$$

where

$$g(t,\ell) = \omega_{t,\ell} = \frac{(1+24\ell)^t}{(-4\sqrt{6})^t} \sum_{k=0}^{\frac{t+1}{2}} \binom{t+1}{k} \frac{t+1-k}{(t+1-2k)!} \Big(\frac{\pi}{6}\Big)^{t-2k} \frac{1}{(1+24\ell)^k}. \tag{4.45}$$

Note that for $\ell = 0$, we retrieve ω_t as in O'Sullivan's [51, Proposition 4.4] work. Adapting the proof methodology in [51, Proposition 4.4] for $p(n)$, can do similarly for $p(n-\ell)$ and conclude the identity (4.45) *by uniqueness of Taylor expansion of an analytic function (without using the computer algebra packages mentioned above).*

4.2. **Estimation of $(S_i(t,\ell))$.** We present the Lemmas 4.24-4.30 which will be needed in the Subsection 4.3. A brief sketch of proofs of these lemmas are presented in the Section 11.

Definition 4.23. *Let $C_k(\ell)$ be as in Lemma 3.7. Define*

$$C_1^{\mathcal{L}}(\ell) := \frac{\cosh(\alpha_\ell)-1}{4} + C_0(\ell) + \frac{\alpha_\ell^2\cosh(\alpha_\ell)+\alpha_\ell\sinh(\alpha_\ell)}{8},$$
$$C_1^{\mathcal{U}}(\ell) := C_1(\ell) + \frac{\alpha_\ell^2+1}{4}\cosh(\alpha_\ell) + \frac{\alpha_\ell(\alpha_\ell^2+12)}{24}\sinh(\alpha_\ell).$$

Lemma 4.24. *Let $S_1(t,\ell)$ be as in Definition 4.12 and $C_1^{\mathcal{L}}(\ell), C_1^{\mathcal{U}}(\ell)$ as in Definition 4.23. Then for all $t \geq 1$,*

$$-\frac{C_1^{\mathcal{L}}(\ell)}{t^2} < \frac{S_1(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\big(\cosh(\alpha_\ell)-1\big) + \frac{1}{2t}\alpha_\ell\sinh(\alpha_\ell) < \frac{C_1^{\mathcal{U}}(\ell)}{t^2}. \tag{4.46}$$

Definition 4.25. *Let $C_k(\ell)$ be as in Lemma 3.7. Define*

$$\begin{aligned}
C_{2,1}^{\mathcal{L}}(\ell) &:= \frac{\cosh(\alpha_\ell)}{4} + \frac{\sinh(\alpha_\ell)}{4\alpha_\ell} + \frac{\alpha_\ell\sinh(\alpha_\ell)}{4} + \frac{2C_1(\ell)}{\alpha_\ell^2},\\
C_{2,1}^{\mathcal{U}}(\ell) &:= -\frac{\cosh(\alpha_\ell)}{2} + \frac{\sinh(\alpha_\ell)}{2\alpha_\ell} + \frac{2C_2(\ell)}{\alpha_\ell^2},\\
\mathrm{csh}(\ell) &:= \cosh(\alpha_\ell) + \alpha_\ell\sinh(\alpha_\ell),\\
C_{2,2}(\ell) &:= \frac{8C_3(\ell)}{\alpha_\ell^2} + \frac{(\alpha_\ell^2+1)\cosh(\alpha_\ell)}{4} + \frac{(\alpha_\ell^3+12\alpha_\ell)\sinh(\alpha_\ell)}{24},
\end{aligned}$$

$$\begin{aligned}
C_2^{\mathcal{L}}(\ell) &:= C_{2,1}^{\mathcal{U}}(\ell) + \frac{\mathrm{csh}(\ell)}{2} + \frac{4C_2(\ell)}{\alpha_\ell^2},\\
C_2^{\mathcal{U}}(\ell) &:= C_{2,1}^{\mathcal{L}}(\ell) - \frac{\mathrm{csh}(\ell)}{2} + C_{2,2}(\ell).
\end{aligned}$$

Lemma 4.26. *Let $S_2(t,\ell)$ be as in Definition 4.14 and $C_2^{\mathcal{L}}(\ell), C_2^{\mathcal{U}}(\ell)$ as in Definition 4.25. Then for all $t \geq 1$,*

$$-\frac{C_2^{\mathcal{L}}(\ell)}{t} < \frac{S_2(t,\ell)}{\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha_\ell) + \frac{\sinh(\alpha_\ell)}{\alpha_\ell} < \frac{C_2^{\mathcal{U}}(\ell)}{t}. \tag{4.47}$$

Definition 4.27. *Let $C_k(\ell)$ be as in Lemma 3.7. Define*

$$\begin{aligned}
C_{3,1}(\ell) &:= \frac{3\alpha_\ell^2\cosh(\alpha_\ell) + 7\alpha_\ell\sinh(\alpha_\ell) + 2\cosh(\alpha_\ell) - 2}{8} + C_0(\ell),\\
C_{3,2}(\ell) &:= \frac{9\alpha_\ell^3\sinh(\alpha_\ell) + (\alpha_\ell^4 + 24\alpha_\ell^2)\cosh(\alpha_\ell) + 18\alpha_\ell\sinh(\alpha_\ell)}{24} + 2C_2(\ell) + C_1(\ell),\\
\mathrm{sch}(\ell) &:= \alpha_\ell^2\cosh(\alpha_\ell) + 2\alpha_\ell\sinh(\alpha_\ell),\\
C_3^{\mathcal{L}}(\ell) &:= C_{3,1}(\ell) + C_{3,2}(\ell) - \frac{\mathrm{sch}(\ell)}{2},\\
C_3^{\mathcal{U}}(\ell) &:= 3C_1(\ell) + \frac{\mathrm{sch}(\ell)}{2}.
\end{aligned}$$

Lemma 4.28. *Let $S_3(t,\ell)$ be as in Definition 4.16 and $C_3^{\mathcal{L}}(\ell), C_3^{\mathcal{U}}(\ell)$ as in Definition 4.27. Then for all $t \geq 2$,*

$$-\frac{C_3^{\mathcal{L}}(\ell)}{t} < \frac{S_3(t,\ell)}{\binom{-\frac{3}{2}}{t}} + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\alpha_\ell\sinh(\alpha_\ell) + 1 - \cosh(\alpha_\ell) < \frac{C_3^{\mathcal{U}}(\ell)}{t}. \tag{4.48}$$

Definition 4.29. *Let $C_k(\ell)$ be as in Lemma 3.7. Define*

$$\begin{aligned}
C_{4,1}(\ell) &:= \frac{\alpha_\ell^4}{72} + \frac{(\alpha_\ell^2+6)\cosh(\alpha_\ell) + 3\alpha_\ell\sinh(\alpha_\ell)}{16},\\
C_4^{\mathcal{L}}(\ell) &:= C_{4,1}(\ell) - \frac{\cosh(\alpha_\ell)}{4} + \frac{2C_0(\ell)}{3},\\
C_4^{\mathcal{U}}(\ell) &:= \frac{(\alpha_\ell^2+12)\cosh(\alpha_\ell) + 3\alpha_\ell\sinh(\alpha_\ell) + 12C_0(\ell)}{24}.
\end{aligned}$$

Lemma 4.30. *Let $S_4(t,\ell)$ be as in Definition 4.18 and $C_4^{\mathcal{L}}(\ell), C_4^{\mathcal{U}}(\ell)$ as in Definition 4.29. Then for $t \geq 1$,*

$$-\frac{C_4^{\mathcal{L}}(\ell)}{t^2} < \frac{S_4(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{1}{2t}\cosh(\alpha_\ell) < \frac{C_4^{\mathcal{U}}(\ell)}{t^2}. \tag{4.49}$$

4.3. **Error bounds.**

Lemma 4.31. *For all $k \in \mathbb{Z}_{\geq 1}$, $\ell \in \mathbb{Z}_{\geq 0}$, and $n \geq \ell + 1$,*

$$\frac{(1+24\ell)^k}{(24n)^k} < \sum_{t=k}^{\infty}\frac{(1+24\ell)^t}{(24n)^t} \leq \frac{24(\ell+1)}{23}\frac{(1+24\ell)^k}{(24n)^k}. \tag{4.50}$$

Proof. Equation (4.50) follows from

$$\sum_{t=k}^{\infty}\frac{(1+24\ell)^t}{(24n)^t} = \frac{(1+24\ell)^k}{(24n)^k}\frac{24n}{24n-24\ell-1} \quad \text{and} \quad 1 < \frac{24n}{24n-24\ell-1} \leq \frac{24(\ell+1)}{23} \quad \text{for all } n \geq \ell+1.$$

□

Lemma 4.32. *For all $n, k, s \in \mathbb{Z}_{\geq 1}$, $\ell \in \mathbb{Z}_{\geq 0}$, and $n \geq \ell + 1$,*

$$\frac{1}{(k+1)^{s-\frac{1}{2}}}\frac{(1+24\ell)^k}{(24n)^k} < \sum_{t=k}^{\infty}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^s}\frac{(1+24\ell)^t}{(24n)^t} < \frac{12(\ell+1)}{5(k+1)^{s-\frac{1}{2}}}\frac{(1+24\ell)^t}{(24n)^k}. \tag{4.51}$$

Proof. We observe that

$$\sum_{t=k}^{\infty}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^s}\frac{1}{(24n)^t} = \sum_{t=k}^{\infty}\frac{\binom{2t+2}{t+1}}{4^t}\frac{t+1}{2t^s}\frac{(1+24\ell)^t}{(24n)^t}. \tag{4.52}$$

For all $t \geq 1$,

$$\frac{4^t}{2\sqrt{t}} \leq \binom{2t}{t} \leq \frac{4^t}{\sqrt{\pi t}}.$$

From (4.52) we obtain

$$\sum_{t=k}^{\infty}\frac{\sqrt{t+1}}{t^s}\frac{(1+24\ell)^t}{(24n)^t} \leq \sum_{t=k}^{\infty}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^s}\frac{1}{(24n)^t} \leq \frac{4}{\sqrt{\pi}}\sum_{t=k}^{\infty}\frac{\sqrt{t+1}}{2t^s}\frac{(1+24\ell)^t}{(24n)^t}. \tag{4.53}$$

For all $k \geq 1$,

$$\sum_{t=k}^{\infty}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^s}\frac{(1+24\ell)^t}{(24n)^t} \geq \sum_{t=k}^{\infty}\frac{\sqrt{t+1}}{t^s}\frac{(1+24\ell)^t}{(24n)^t} > \frac{1}{(k+1)^{s-\frac{1}{2}}}\frac{(1+24\ell)^k}{(24n)^k} \tag{4.54}$$

and

$$\begin{aligned}
\sum_{t=k}^{\infty}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^s}\frac{(1+24\ell)^t}{(24n)^t} &< \frac{4}{\sqrt{\pi}}\sum_{t=k}^{\infty}\frac{1}{(t+1)^{s-\frac{1}{2}}}\frac{(1+24\ell)^t}{(24n)^t}\\
&\leq \frac{4}{\sqrt{\pi}(k+1)^{s-\frac{1}{2}}}\sum_{t=k}^{\infty}\frac{(1+24\ell)^t}{(24n)^t}\\
&< \frac{4\cdot 24(\ell+1)}{23\cdot\sqrt{\pi}}\frac{1}{(k+1)^{s-\frac{1}{2}}}\frac{(1+24\ell)^k}{(24n)^k}\ \text{(by (4.50)).}\\
&< \frac{12}{5}\frac{(\ell+1)}{(k+1)^{s-\frac{1}{2}}}\frac{1}{(24n)^k}.
\end{aligned} \tag{4.55}$$

Equations (4.54) and (4.55) imply (4.51). □

Lemma 4.33. *For $n \in \mathbb{Z}_{\geq 1}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, and $n \geq 4\ell + 1$,*

$$0 < \sum_{t=k}^{\infty}\binom{-\frac{3}{2}}{t}\frac{(-1)^t(1+24\ell)^t}{(24n)^t} < 4\sqrt{2}\frac{\sqrt{k+1}(1+24\ell)^k}{(24n)^k}. \tag{4.56}$$

Proof. Setting $(n, s) \mapsto \left(\frac{24n}{24\ell+1}, 2\right)$ in (3.2), it follows that for all $n \geq 4\ell + 1$,

$$0 < \sum_{t=k}^{\infty}\binom{-\frac{3}{2}}{t}\frac{(-1)^t}{(24n)^t} < 4\sqrt{2}\frac{\sqrt{k+1}(1+24\ell)^k}{(24n)^k}.$$

□

Definition 4.34. *Let $C_1^{\mathcal{L}}(\ell)$ and $C_1^{\mathcal{U}}(\ell)$ be as in Definition 4.23. Then for all $k \geq 1$ and $\ell \geq 0$, define*

$$L_1(k,\ell) := \left(\cosh(\alpha_\ell) - \frac{6\alpha_\ell \sinh(\alpha_\ell)(\ell+1)}{5\sqrt{k+1}} - \frac{12(\ell+1)}{5(k+1)^{3/2}} C_1^{\mathcal{L}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k}$$

and

$$U_1(k,\ell) := \left(\frac{24(\ell+1)\cosh(\alpha_\ell)}{23} - \frac{\alpha_\ell \sinh(\alpha_\ell)}{2\sqrt{k+1}} + \frac{12(\ell+1)}{5(k+1)^{3/2}} C_1^{\mathcal{U}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k}.$$

Lemma 4.35. *Let $L_1(k,\ell)$ and $U_1(k,\ell)$ be as in Definition 4.34. Let $g_{e,1}(t,\ell)$ be as in Definition 4.12. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell+1$,*

$$L_1(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k} < \sum_{t=k}^{\infty} g_{e,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} < U_1(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k}. \tag{4.57}$$

Proof. From (4.26) and (4.46), it follows that for $t \geq 1$,

$$\begin{aligned}\cosh(\alpha_\ell) - \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{2t}\alpha_\ell \sinh(\alpha_\ell) - \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2} C_1^{\mathcal{L}}(\ell) &< \Big(\frac{24}{1+24\ell}\Big)^t g_{e,1}(t) = 1 + S_1(t,\ell)\\ &< \cosh(\alpha_\ell) - \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{2t}\alpha_\ell \sinh(\alpha_\ell) + \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2} C_1^{\mathcal{U}}(\ell).\end{aligned} \tag{4.58}$$

Applying (4.50) and (4.51) with $s = 1$ and 2, respectively, to (4.58), we obtain (4.57). ☐

Definition 4.36. *Let $C_2^{\mathcal{L}}(\ell)$ and $C_2^{\mathcal{U}}(\ell)$ be as in Definition 4.25. For all $k \geq 1$ and $\ell \geq 0$, define*

$$L_2(k,\ell) := \left(-\frac{24(\ell+1)\cosh(\alpha_\ell)}{23} - \frac{12(\ell+1)}{5\sqrt{k+1}} C_2^{\mathcal{U}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k}$$

and

$$U_2(k,\ell) := \left(-\cosh(\alpha_\ell) + \frac{4\sqrt{2}\sinh(\alpha_\ell)}{\alpha_\ell}\sqrt{k+1} + \frac{12(\ell+1)}{5\sqrt{k+1}} C_2^{\mathcal{L}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k}.$$

Lemma 4.37. *Let $L_2(k,\ell)$ and $U_2(k,\ell)$ be as in Definition 4.36. Let $g_{e,2}(t,\ell)$ be as in Definition 4.14. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell+1$,*

$$L_2(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k} < \sum_{t=k}^{\infty} g_{e,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} < U_2(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k}. \tag{4.59}$$

Proof. From (4.31) and (4.47), it follows that for $t \geq 1$,

$$\begin{aligned}-\cosh(\alpha_\ell) + (-1)^t\binom{-\frac{3}{2}}{t}\frac{\sinh(\alpha_\ell)}{\alpha_\ell} - \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t} C_2^{\mathcal{U}}(\ell) &< \Big(\frac{24}{1+24\ell}\Big)^t g_{e,2}(t,\ell) = (-1)^{t-1} S_2(t,\ell)\\ &< -\cosh(\alpha_\ell) + (-1)^t\binom{-\frac{3}{2}}{t}\frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t} C_2^{\mathcal{L}}(\ell).\end{aligned} \tag{4.60}$$

Applying (4.50), (4.51) with $s = 1$ and (4.56) to (4.60), we get (4.59). ☐

Definition 4.38. *Let $C_3^{\mathcal{L}}(\ell)$ and $C_3^{\mathcal{U}}(\ell)$ be as in Definition 4.27. For all $k \geq 1$ and $\ell \geq 0$, define*

$$L_3(k,\ell) := -\left(-\frac{6\alpha_\ell \sinh(\alpha_\ell)}{\pi\sqrt{1+24\ell}} + \frac{24\sqrt{2}\cosh(\alpha_\ell)\sqrt{k+1}}{\pi\sqrt{1+24\ell}} + \frac{72(\ell+1)}{5\pi\sqrt{1+24\ell}}\frac{C_3^{\mathcal{U}}(\ell)}{\sqrt{k+1}}\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k+1}$$

and

$$U_3(k,\ell) := \left(\frac{6\cdot 24(\ell+1)}{23\pi\sqrt{1+24\ell}}\alpha_\ell \sinh(\alpha_\ell) + \frac{72(\ell+1)}{5\pi\sqrt{1+24\ell}}\frac{C_3^{\mathcal{L}}(\ell)}{\sqrt{k+1}}\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k+1}.$$

Lemma 4.39. *Let $L_3(k,\ell)$ and $U_3(k,\ell)$ be as in Definition 4.38. Let $g_{o,1}(t,\ell)$ be as in Definition 4.16. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell+1$,*

$$L_3(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1} < \sum_{t=k}^{\infty} g_{o,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} < U_3(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1}. \tag{4.61}$$

Proof. Define $c_1(t,\ell) := -\dfrac{6}{\pi\sqrt{1+24\ell}}(-1)^t\binom{-\frac{3}{2}}{t}$. From (4.35) and (4.48), it follows that for $t \geq 2$,

$$\begin{aligned}
&\frac{6\alpha_\ell \sinh(\alpha_\ell)}{\pi\sqrt{1+24\ell}} - \frac{6\cosh(\alpha_\ell)}{\pi\sqrt{1+24\ell}}(-1)^t\binom{-\frac{3}{2}}{t} - \frac{6C_3^{\mathcal{U}}(\ell)}{\pi\sqrt{1+24\ell}}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}\\
&< \left(\sqrt{\frac{24}{24\ell+1}}\right)^{2t+1} g_{o,1}(t,\ell) = c_1(t,\ell)\left(1+\frac{S_3(t,\ell)}{\binom{-\frac{3}{2}}{t}}\right)\\
&< \frac{6\alpha_\ell \sinh(\alpha_\ell)}{\pi\sqrt{1+24\ell}} - \frac{6\cosh(\alpha_\ell)}{\pi\sqrt{1+24\ell}}(-1)^t\binom{-\frac{3}{2}}{t} + \frac{6C_3^{\mathcal{L}}(\ell)}{\pi\sqrt{1+24\ell}}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}.
\end{aligned} \tag{4.62}$$

We observe that (4.62) also holds for $t \in \{0,1\}$; see (4.35). Now, applying (4.50), (4.51) with $s=1$, and (4.56) to (4.62), we conclude the proof. ☐

Definition 4.40. *Let $C_4^{\mathcal{L}}(\ell)$ and $C_4^{\mathcal{U}}(\ell)$ be as in Definition 4.29. For all $k \geq 1$ and $\ell \geq 0$, define*

$$L_4(k,\ell) := -\frac{\pi\sqrt{1+24\ell}}{6}\left(-\frac{\cosh(\alpha_\ell)}{2\sqrt{k+1}} + \frac{24(\ell+1)\sinh(\alpha_\ell)}{23\alpha_\ell} + \frac{12(\ell+1)C_4^{\mathcal{U}}(\ell)}{5(k+1)^{3/2}}\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k+1}$$

and

$$U_4(k,\ell) := \frac{\pi\sqrt{1+24\ell}}{6}\left(\frac{6(\ell+1)\cosh(\alpha_\ell)}{5\sqrt{k+1}} - \frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{12(\ell+1)C_4^{\mathcal{L}}(\ell)}{5(k+1)^{3/2}}\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k+1}.$$

Lemma 4.41. *Let $L_4(k,\ell)$ and $U_4(k,\ell)$ be as in Definition 4.40. Let $g_{o,2}(t,\ell)$ be as in Definition 4.18. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell+1$,*

$$L_4(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1} < \sum_{t=k}^{\infty} g_{o,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} < U_4(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1}. \tag{4.63}$$

Proof. Define $c_2(t,\ell) := -\dfrac{\pi\sqrt{1+24\ell}}{6}(-1)^t\binom{-\frac{3}{2}}{t}$. From (4.39) and (4.49), it follows that for $t \geq 1$,

$$\begin{aligned}&\frac{\pi\sqrt{1+24\ell}\cosh(\alpha_\ell)}{12}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}-\frac{\pi\sqrt{1+24\ell}\sinh(\alpha_\ell)}{6\alpha_\ell}-\frac{\pi\sqrt{1+24\ell}\ C_4^{\mathcal{U}}(\ell)}{6}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}\\ &<\left(\sqrt{\frac{24}{24\ell+1}}\right)^{2t+1}g_{o,2}(t,\ell)=c_2(t,\ell)\frac{S_4(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}}\\ &<\frac{\pi\sqrt{1+24\ell}\cosh(\alpha_\ell)}{12}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}-\frac{\pi\sqrt{1+24\ell}\sinh(\alpha_\ell)}{6\alpha_\ell}+\frac{\pi\sqrt{1+24\ell}\ C_4^{\mathcal{L}}(\ell)}{6}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}.\end{aligned}\tag{4.64}$$

Now, applying (4.50) and (4.51) with $s = 1$ and 2, respectively, to (4.64), we have (4.63). ☐

Definition 4.42. *For $k \geq 1$ and $\ell \geq 0$, define*

$$n_0(k,\ell) := \max_{k\geq 1,\ell\geq 0}\left\{\frac{(24\ell+1)^2}{16}, \frac{(k+3)(24\ell+1)}{24}\right\}.$$

Definition 4.43. *Let $n_0(k,\ell)$ be as in Definition 4.42. For $k \geq 1$ and $\ell \geq 0$, define*

$$\widehat{L}_2(k,\ell) := \frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{4\sqrt{n_0(k,l)}}\right) \text{ and } \widehat{U}_2(k,\ell) := \frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1+\frac{k(1+24\ell)}{3\cdot n_0(k,l)}\right).$$

Lemma 4.44. *Let $\widehat{L}_2(k,\ell)$, and $\widehat{U}_2(k,\ell)$ be as in Definition 4.43. Let $n_0(k,\ell)$ be as in Definition 4.42. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n > n_0(k,\ell)$,*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{L}_2(k,\ell)}{\sqrt{n}^k} < \frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\frac{1}{\mu(n-\ell)^k} < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{U}_2(k,\ell)}{\sqrt{n}^k}.\tag{4.65}$$

Proof. For all $k \geq 1$ and $\ell \geq 0$, define

$$\mathcal{E}(n,k,\ell) := \frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\frac{1}{\mu(n-\ell)^k},\ \mathcal{U}(n,k,\ell) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{1}{\sqrt{n}^k}$$

and

$$\mathcal{Q}(n,k,\ell) := \frac{\mathcal{E}(n,k,\ell)}{\mathcal{U}(n,k,\ell)} = \frac{e^{\pi\sqrt{\frac{2n}{3}}\left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)}}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}.$$

Using (3.3) with $(m,n,s) \mapsto (1, 24n, 24\ell+1)$, we obtain for all $n \geq 2\ell+1$,

$$-\frac{1+24\ell}{12n} < \sqrt{1-\frac{1+24\ell}{24n}}-1 = \sum_{m=1}^{\infty}\binom{1/2}{m}\frac{(-1)^m}{(24n)^m} < 0,$$

and consequently for $n \geq n_0(k,\ell)$,

$$\left(1-\frac{1+24\ell}{4\sqrt{n_0(k,\ell)}}\right) < e^{-\frac{\pi(1+24\ell)}{6\sqrt{6n}}} < e^{\pi\sqrt{\frac{2n}{3}}\left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)} < 1.\tag{4.66}$$

Therefore

$$\frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}\left(1-\frac{1}{4\sqrt{n_0(k,\ell)}}\right) < \mathcal{Q}(n,k,\ell) < \frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}.\tag{4.67}$$

We estimate $\left(1-\dfrac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}$ by splitting it into two cases depending on whether k is even or odd.
For $k=2r$ with $r\in\mathbb{Z}_{\geq 0}$:

$$\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}=\left(1-\frac{1+24\ell}{24n}\right)^{-(r+1)}=1+\sum_{j=1}^{\infty}\binom{-(r+1)}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j}.$$

From (3.4) with $(m,s,n)\mapsto(1,r+1,\frac{24n}{24\ell+1})$, for all $n>\frac{(r+1)(1+24\ell)}{12}$, we get

$$0<\sum_{j=1}^{\infty}\binom{-(r+1)}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j}<\frac{(r+1)(24\ell+1)}{12n},$$

which is equivalent to

$$1<\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}<1+\frac{(k+2)(24\ell+1)}{24n}\quad\text{for all}\quad n>n_0(k,\ell). \tag{4.68}$$

For $k=2r+1$ with $r\in\mathbb{Z}_{\geq 0}$:

$$\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}=\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{2r+3}{2}}=1+\sum_{j=1}^{\infty}\binom{-\frac{2r+3}{2}}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j}.$$

Using (3.2) with $(m,s,n)\mapsto(1,r+2,\dfrac{24n}{24\ell+1})$, for all $n>\frac{(r+2)(1+24\ell)}{12}$, we get

$$0<\sum_{j=1}^{\infty}\binom{-\frac{2\ell+3}{2}}{j}\frac{(-1)^j}{(24n)^j}<\frac{(r+2)(1+24\ell)}{6n}$$

which is equivalent to

$$1<\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}<1+\frac{k(1+24\ell)}{3n}\quad\text{for all}\quad n>n_0(k,\ell). \tag{4.69}$$

From (4.68) and (4.69), for all $n>n_0(k,\ell)$ it follows that

$$1<\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}<1+\frac{k(1+24\ell)}{3\cdot n_0(k,\ell)}. \tag{4.70}$$

From (4.67) and (4.70), we conclude the proof.

□

5. Inequalities for $p(n-\ell)$

Definition 5.1. *Let $(L_i(k,\ell))_{1\leq i\leq 4}$ and $(U_i(k,\ell))_{1\leq i\leq 4}$ be as in Definitions 4.34-4.40. Let $\widehat{U}_2(k,\ell)$ be as in Definition 4.43. Then for all $w\in\mathbb{Z}_{\geq 1}$ with $\lceil w/2\rceil\geq 1$, define*

$$L(w,\ell):=L_1\Big(\Big\lceil\frac{w}{2}\Big\rceil,\ell\Big)+L_2\Big(\Big\lceil\frac{w}{2}\Big\rceil,\ell\Big)+L_3\Big(\Big\lfloor\frac{w}{2}\Big\rfloor,\ell\Big)+L_4\Big(\Big\lfloor\frac{w}{2}\Big\rfloor,\ell\Big)-\widehat{U}_2(w,\ell)$$

and

$$U(w,\ell):=U_1\Big(\Big\lceil\frac{w}{2}\Big\rceil,\ell\Big)+U_2\Big(\Big\lceil\frac{w}{2}\Big\rceil,\ell\Big)+U_3\Big(\Big\lfloor\frac{w}{2}\Big\rfloor,\ell\Big)+U_4\Big(\Big\lfloor\frac{w}{2}\Big\rfloor,\ell\Big)+\widehat{U}_2(w,\ell).$$

Lemma 5.2. *Let $\widehat{g}(k)$ be as in Theorem 4.2 and $n_0(k,\ell)$ as in Definition 4.42. Let $g(t,\ell)$ be as in (4.45). Let $L(w,\ell)$ and $U(w,\ell)$ be as in Definition 5.1. If $m\in\mathbb{Z}_{\geq 1}$ and $n>\max\{1,n_0(2m,\ell),\widehat{g}(2m)+\ell\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1}\frac{g(t,\ell)}{\sqrt{n}^t}+\frac{L(2m,\ell)}{\sqrt{n}^{2m}}\right)<p(n-\ell)<\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1}\frac{g(t,\ell)}{\sqrt{n}^t}+\frac{U(2m,\ell)}{\sqrt{n}^{2m}}\right).$$

Proof. Following Definition 4.20 and from Lemma 4.21, we have

$$\begin{aligned}\sum_{t=0}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t &= \sum_{t=0}^{2m-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=2m}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t \\ &= \sum_{t=0}^{2m-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=m}^{\infty} g(2t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \sum_{t=m}^{\infty} g(2t+1,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \\ &= \sum_{t=0}^{2m-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=m}^{\infty}(g_{e,1}(t,\ell)+g_{e,2}(t,\ell))\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\ &\qquad + \sum_{t=m}^{\infty}(g_{o,1}(t,\ell)+g_{o,2}(t,\ell))\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}.\end{aligned} \tag{5.1}$$

Using Lemmas 4.35-4.41 by making the substitution $k \mapsto m$, it follows that

$$\begin{aligned}\frac{L_1(m,\ell)+L_2(m,\ell)}{\sqrt{n}^{2m}} + \frac{L_3(m,\ell)+L_4(m,\ell)}{\sqrt{n}^{2m+1}} &< \sum_{t=2m}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t \\ &< \frac{U_1(m,\ell)+U_2(m,\ell)}{\sqrt{n}^{2m}} + \frac{U_3(m,\ell)+U_4(m,\ell)}{\sqrt{n}^{2m+1}}.\end{aligned} \tag{5.2}$$

Moreover, by Lemma 4.44 with $k = 2m$, it follows that

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\frac{1}{\mu(n-\ell)^{2m}} < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{U}_2(2m,\ell)}{\sqrt{n}^{2m}}. \tag{5.3}$$

Combining (5.2) and (5.3), and applying to Lemma 4.3, we conclude the proof. ☐

Lemma 5.3. *Let $\widehat{g}(k)$ be as in Theorem 4.2 and $n_0(k,\ell)$ as in Definition 4.42. Let $g(t,\ell)$ be as in Equation (4.45). Let $L(w,\ell)$ and $U(w,\ell)$ be as in Definition 5.1. If $m \in \mathbb{Z}_{\geq 0}$ and $n > \max\{1, n_0(2m+1,\ell), \widehat{g}(2m+1)+\ell\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m}\frac{g(t,\ell)}{\sqrt{n}^t} + \frac{L(2m+1,\ell)}{\sqrt{n}^{2m+1}}\right) < p(n-\ell) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m}\frac{g(t,\ell)}{\sqrt{n}^t} + \frac{U(2m+1,\ell)}{\sqrt{n}^{2m+1}}\right).$$

Proof. The proof is analogous to the proof of Lemma 5.2. ☐

Definition 5.4. *Let $g(t,\ell)$ be as in (4.45), $L(w,\ell), U(w,\ell)$ as in Definition 5.1. If $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$, define*

$$\mathcal{L}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{L(w,\ell)}{\sqrt{n}^w} \quad \text{and} \quad \mathcal{U}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{U(w,\ell)}{\sqrt{n}^w}.$$

Theorem 5.5. *Let $\widehat{g}(k)$ be as in Theorem 4.2 and $n_0(k,\ell)$ as in Definition 4.42. Let $\mathcal{L}_n(w,\ell)$ and $\mathcal{U}_n(w,\ell)$ be as in Definition 5.4. If $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$ and $n > \max\{\widehat{g}(w)+\ell, n_0(w,\ell)\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{L}_n(w,\ell) < p(n-\ell) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{U}_n(w,\ell). \tag{5.4}$$

Proof. Putting Lemmas 5.2 and 5.3 together, we obtain (5.4). ☐

6. Proof of Theorems 1.5, 1.7, and 1.9

Proof of Theorem 1.5: To prove the lower bound of (1.16), it is equivalent to show that

$$p(n-4)p(n)+3p(n-2)^2>4\left(1+\frac{\pi^2}{16(n-3)^3}-\frac{6}{(n-3)^{7/2}}\right)p(n-3)p(n-1). \tag{6.1}$$

Since $1+\frac{\pi^2}{16n^3}-\frac{5}{n^{7/2}}>1+\frac{\pi^2}{16(n-3)^3}-\frac{6}{(n-3)^{7/2}}$ for all $n\geq 5$, it is enough to show that

$$p(n-4)p(n)+3p(n-2)^2>4\left(1+\frac{\pi^2}{16n^3}-\frac{5}{n^{7/2}}\right)p(n-3)p(n-1). \tag{6.2}$$

Choosing $w=12$ and applying Theorem 5.5, for all $n\geq 2329$, we have

$$p(n-4)p(n)+3p(n-2)^2>\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{L}_n(12,4)\cdot\mathcal{L}_n(12,0)+3\,\mathcal{L}_n^2(12,2)\right), \tag{6.3}$$

and

$$p(n-3)p(n-1)<\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{U}_n(12,3)\cdot\mathcal{U}_n(12,1)\right). \tag{6.4}$$

Therefore, it suffices to show that

$$\mathcal{L}_n(12,4)\cdot\mathcal{L}_n(12,0)+3\,\mathcal{L}_n^2(12,2)>4\left(1+\frac{\pi^2}{16n^3}-\frac{5}{n^{7/2}}\right)\mathcal{U}_n(12,3)\cdot\mathcal{U}_n(12,1). \tag{6.5}$$

Using the `Reduce`[1] command within Mathematica, it can be easily checked that for all $n\geq 625$, (6.5) holds.

Similarly, to prove the upper bound of (1.16), it is equivalent to prove that

$$p(n-4)p(n)+3p(n-2)^2<4\left(1+\frac{\pi^2}{16(n-3)^3}\right)p(n-3)p(n-1). \tag{6.6}$$

Since $1+\frac{\pi^2}{16n^3}<1+\frac{\pi^2}{16(n-3)^3}$ for all $n\geq 4$, it is enough to show that

$$p(n-4)p(n)+3p(n-2)^2<4\left(1+\frac{\pi^2}{16n^3}\right)p(n-3)p(n-1). \tag{6.7}$$

Choosing $w=12$ and applying Theorem 5.5, for all $n\geq 2329$, we have

$$p(n-4)p(n)+3p(n-2)^2<\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{U}_n(12,4)\cdot\mathcal{U}_n(12,0)+3\,\mathcal{U}_n^2(12,2)\right), \tag{6.8}$$

and

$$p(n-3)p(n-1)>\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{L}_n(12,3)\cdot\mathcal{L}_n(12,1)\right). \tag{6.9}$$

Therefore, it suffices to show that

$$\mathcal{U}_n(12,4)\cdot\mathcal{U}_n(12,0)+3\,\mathcal{U}_n^2(12,2)<4\left(1+\frac{\pi^2}{16n^3}\right)\mathcal{L}_n(12,3)\cdot\mathcal{L}_n(12,1). \tag{6.10}$$

[1]`Reduce` uses cylindrical algebraic decomposition for polynomials over real domains which is based on Collin's algorithm [20]. Cylindrical Algebraic Decomposition (CAD) is an algorithm which proves that a given polynomial in several variables is positive (non-negative).

In a similar way as stated before, it can be easily checked that for all $n \geq 784$, (6.5) holds. We conclude the proof of Theorem 1.5 by verifying the inequality (1.16) for all $218 \leq n \leq 2328$ with Mathematica. $\square$

Proof of Theorem 1.7: To prove the lower bound of (1.17), it is equivalent to show that

$$p(n-2)^3 + p(n-4)p(n-1)^2 + p(n-3)^2p(n) > \left(1 + \frac{\pi^3}{72\sqrt{6}(n-3)^{9/2}} - \frac{8}{(n-3)^5}\right)\big(2p(n-3)p(n-2)p(n-1) + p(n-4)p(n-2)p(n)\big). \tag{6.11}$$

As $1 + \frac{\pi^3}{72\sqrt{6}n^{9/2}} - \frac{7}{n^5} > 1 + \frac{\pi^3}{72\sqrt{6}(n-3)^{9/2}} - \frac{8}{(n-3)^5}$ for all $n \geq 4$, it suffices to show that

$$p(n-2)^3 + p(n-4)p(n-1)^2 + p(n-3)^2p(n) > \left(1 + \frac{\pi^3}{72\sqrt{6}n^{9/2}} - \frac{7}{n^5}\right)\big(2p(n-3)p(n-2)p(n-1) + p(n-4)p(n-2)p(n)\big). \tag{6.12}$$

Choosing $w = 15$ and applying Theorem 5.5, for all $n \geq 4047$, we have

$$p(n-2)^3 + p(n-4)p(n-1)^2 + p(n-3)^2p(n) > \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^3\left(\mathcal{L}_n^3(15,2) + \mathcal{L}_n(15,4)\cdot\mathcal{L}_n^2(15,1) + \mathcal{L}_n^2(15,3)\cdot\mathcal{L}_n(15,0)\right), \tag{6.13}$$

and

$$2p(n-3)p(n-2)p(n-1) + p(n-4)p(n-2)p(n) < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^3\left(2\cdot\mathcal{U}_n(15,3)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,1) + \mathcal{U}_n(15,4)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,0)\right). \tag{6.14}$$

Similar to the proof of (6.5), it can be easily checked that for all $n \geq 1444$,

$$\mathcal{L}_n^3(15,2) + \mathcal{L}_n(15,4)\cdot\mathcal{L}_n^2(15,1) + \mathcal{L}_n^2(15,3)\cdot\mathcal{L}_n(15,0) > \left(1 + \frac{\pi^3}{72\sqrt{6}n^{9/2}} - \frac{7}{n^5}\right)\left(2\cdot\mathcal{U}_n(15,3)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,1) + \mathcal{U}_n(15,4)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,0)\right) \tag{6.15}$$

Analogously, one can prove that for all $n \geq 2916$,

$$\mathcal{U}_n^3(15,2) + \mathcal{U}_n(15,4)\cdot\mathcal{U}_n^2(15,1) + \mathcal{U}_n^2(15,3)\cdot\mathcal{U}_n(15,0) < \left(1 + \frac{\pi^3}{72\sqrt{6}n^{9/2}}\right)\left(2\cdot\mathcal{L}_n(15,3)\cdot\mathcal{L}_n(15,2)\cdot\mathcal{L}_n(15,1) + \mathcal{L}_n(15,4)\cdot\mathcal{L}_n(15,2)\cdot\mathcal{L}_n(15,0)\right), \tag{6.16}$$

which is sufficient to prove the upper bound of (1.17). We conclude the proof of Theorem 1.7 by verifying the inequality (1.17) for all $244 \leq n \leq 4047$ with Mathematica. $\square$

Proof of Theorem 1.9: Corresponding to (1.18), we show

$$\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right)\Big(p(n-2)p(n-1) - p(n-3)p(n)\Big)^2 > 4\Big(p(n-2)^2 - p(n-3)p(n-1)\Big)\Big(p(n-1)^2 - p(n-2)p(n)\Big), \tag{6.17}$$

and

$$4\Big(p(n-2)^2-p(n-3)p(n-1)\Big)\Big(p(n-1)^2-p(n-2)p(n)\Big) > \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{2}{n^2}\right)\Big(p(n-2)p(n-1)-p(n-3)p(n)\Big)^2. \tag{6.18}$$

Applying Theorem 5.5 with $w = 13$, and following the similar method worked out in the proof of Theorem 1.5, we obtain (1.18) for all $n \geq 2842$. For $115 \leq n \leq 2841$, we verified (1.18) numerically with Mathematica. □

7. Applications

7.1. **Higher order Laguerre inequalities.** If a polynomial $p(x)$ satisfies

$$p'(x)^2 - p(x)\cdot p''(x) \geq 0, \tag{7.1}$$

then we say that $p(x)$ satisfies Laguerre inequality. Laguerre [41] proved that if $p(x)$ is a polynomial having only roots, then $p(x)$ satisfies (7.1). Later in 1913, Jensen [37] obtained a generalization of (7.1)

$$L_n(p(x)) := \frac{1}{2}\sum_{k=0}^{2n}(-1)^{n+k}\binom{2n}{k}p^{(k)}(x)p^{(2n-k)}(x) \geq 0, \tag{7.2}$$

where $p^{(k)}(x)$ denotes the kth derivative of $p(x)$. The case $n = 1$ gives the classical Laguerre inequality (7.1). For a detailed study on the Laguerre inequalities of order m, we refer to [21, 67]. Considering the discrete version of (7.2), define that a sequence $(a_n)_{n\geq 0}$ satisfies Laguerre inequalities of order m if

$$L_m(a_n) := \frac{1}{2}\sum_{k=0}^{2m}(-1)^{m+k}\binom{2n}{k}a(n+k)a(2m-k+n) \geq 0. \tag{7.3}$$

Wagner [67] proved that $p(n)$ satisfies (7.3) with $m \geq 1$ and for sufficiently large n. Moreover, he proposed the following conjecture.

Conjecture 7.1. *[67] For $1 \leq m \leq 10$, $p(n)$ satisfies the Laguerre inequality of order m for $n \geq N(m)$, where*

m	1	2	3	4	5	6	7	8	9	10
$N(m)$	25	184	531	1102	1923	3014	4391	6070	8063	10382

The case $m = 1$ was settled by DeSalvo and Pak [23]. Wang and Yang [68, Theorem 2.1] settled the case $m = 2$. Recently, Dou and Wang [25, Sections 2 and 3] resolved the cases $3 \leq m \leq 9$. Dou and Wang [25, page 8] also proved that $p(n)$ satisfies the Laguerre inequality of order 10 for all $n > 218573927203706866261$ but in order to conclude for $m = 10$, they had to verify the remaining *quintillion* gap which was impossible to check with computer and therefore, the case $m = 10$ remains open.

For $2 \leq m \leq 15$, let $N(m)$ denotes the actual cut off for n such that $(p(n))_{n\geq N(m)}$ satisfies the Laguerre inequality of order m, $w(m)$ denotes the truncation point as given in Theorem 5.5, and $N_B(m)$ denotes the cut-off from which point on we are able to show (using Theorem 5.5) that $(p(n))_{n\geq N_B(m)}$ satisfies Laguerre inequalities of order m. $T(m)$ denotes the time (in seconds) taken in computation with 'Reduce' command in Mathematica.

Enumeration of cut-off				
m	$N(m)$	$w(m)$	$N_B(m)$	$T(m)$
2	184	11	1873	0.76
3	531	15	4049	1.53
4	1102	20	8164	4.61
5	1923	23	11436	7.51
6	3014	30	21577	11.46

Enumeration of cut-off				
m	$N(m)$	$w(m)$	$N_B(m)$	$T(m)$
7	4391	34	29034	25.34
8	6070	39	40138	40.88
9	8063	45	56180	126.91
10	10382	50	71893	177.34
11	13037	55	89803	366.15

Enumeration of cut-off				
m	$N(m)$	$w(m)$	$N_B(m)$	$T(m)$
12	16038	63	109966	419.18
13	19393	68	132433	659.63
14	23110	72	157254	673.56
15	27199	78	184471	754.29

From the above tables, we have the following theorem.

Theorem 7.2. *For $2 \leq m \leq 15$,*

$$L_m(p(n-2m)) > 0 \quad \text{for all} \quad n \geq N(m). \tag{7.4}$$

Remark 7.3. *We observe that Theorem 7.2 settles the Conjecture 7.1. Now, in spite of having Wagner's proof on positivity of $L_m(p(n))$ for all but finitely many n, a natural question arises: what is the growth of $L_m(p(n))$ as $n \to \infty$? More explicitly, how to get an effective estimate of $N(m)$ such that for all $n > N(m)$, $L_m(p(n)) > 0$?*

Based on numerical evidences (checked $1 \leq m \leq 200$), we propose the following conjecture.

Conjecture 7.4. *For all $m \in \mathbb{Z}_{\geq 1}$,*

$$\frac{(-1)^m}{2}\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\sum_{s=0}^{3m} g(s,2m-k)g(3m-s,k) = \left(\frac{\pi}{2\sqrt{6}}\right)^m \frac{(2m-1)!}{(m-1)!},$$

and for all $m \in \mathbb{Z}_{\geq 1}$ and $0 \leq \nu \leq 3m-1$,

$$\frac{(-1)^m}{2}\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\sum_{s=0}^{\nu} g(s,2m-k)g(3m-s,k) = 0.$$

If the above conjecture is true for $m \geq 1$, then by choosing $w(m) = 3m+1$ in Theorem 5.5, can derive an effective estimate for $N(w(m))$ and prove that $L_m(p(n-2m)) > 0$ for all $n > N(m)$. As a consequence, it will follow that as $n \to \infty$,

$$L_m(p(n)) \sim \left(\frac{\pi}{2\sqrt{6}}\right)^m \frac{(2m-1)!}{(m-1)!}\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2.$$

7.2. **Higher order shifted difference.** Let Δ be the difference operator defined on a sequence $(a(n))_{n\geq 0}$ by $\Delta(a(n)) := a(n+1) - a(n)$. A r-fold applications of Δ is denoted by Δ^r. Recently, Gomez, Males, and Rolen [27] generalized the Δ operator by introducing a shift parameter j, defined as $\Delta_j^2(a(n)) := a(n) - 2a(n-j) + a(n-2j)$, and studied the positivity of $\Delta_j^2(p(n))$. Consequently, they also proved that $N_k(m,n) - N_k(m+1,n) > 0$, where the k-rank function $N_k(m,n)$ which counts the number of partitions of n into at least $(k-1)$ successive Durfee squares with k-rank equal to m (see [26]). Following Theorem 5.5, we obtain the asymptotic expansion of $\Delta_j^r(p(n)) := \sum_{m=0}^{r}(-1)^m\binom{m}{r}p(n-m\cdot j)$ for any positive integer r, which finally leads to a completion the work of Odlyzko [52] on $\Delta^r p(n)$ (setting $j=1$) by proving its asymptotic growth. Works related to the positivity of $\Delta^r p(n)$ can be found in [30, 33, 2, 39].

Following the notation from [31], here $\left\{ {n \atop m} \right\}$ denotes the Stirling number of second kind which counts the number of ways to partition a set of n elements into m nonempty subsets. Here we state three facts about $\left\{ {n \atop m} \right\}$ which will be referenced later.

Fact 7.5. *[31, Table 264]* $\left\{ {n \atop m} \right\} = \binom{n}{m} = 0,$

Fact 7.6. *[31, Table 264]* $\left\{ {n \atop n} \right\} = \binom{n}{n} = 1,$

Fact 7.7. *[31, Table 265, Eq. (6.19)]* $m!\left\{ {n \atop m} \right\} = \displaystyle\sum_{k=0}^{m} \binom{m}{k} k^n (-1)^{m-k}.$

Lemma 7.8. *Let $g(t,\ell)$ be as in Equation 4.45. Then for all $r \geq 1$,*

$$\begin{aligned}\sum_{m=0}^{r}(-1)^m\binom{r}{m}\sum_{t=0}^{r+1}\frac{g(t,m\cdot j)}{\sqrt{n}^t} =&\Big(\frac{\pi\cdot j}{\sqrt{6}}\Big)^r\frac{1}{\sqrt{n}^r}-\\ &\Big(\frac{\pi\cdot j}{\sqrt{6}}\Big)^{r-1}\frac{j}{4}\left[\frac{\pi^2}{36}(1+12jr)+(r^2+3r+2)\right]\frac{1}{\sqrt{n}^{r+1}}.\end{aligned} \tag{7.5}$$

Proof. Following (4.45), we have

$$\begin{aligned}&\sum_{m=0}^{r}(-1)^m\binom{r}{m}\sum_{t=0}^{r+1}\frac{g(t,m\cdot j)}{\sqrt{n}^t}\\
&=\sum_{m=0}^{r}(-1)^m\binom{r}{m}\sum_{t=0}^{r+1}\Big(\frac{1+24m\cdot j}{-4\sqrt{6n}}\Big)^t\\
&\qquad\cdot\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\Big(\frac{\pi}{6}\Big)^{t-2k}\frac{1}{(1+24m\cdot j)^k}\\
&=\sum_{t=0}^{r+1}\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\Big(\frac{\pi}{6}\Big)^{t-2k}\Big(\frac{1}{-4\sqrt{6n}}\Big)^t\\
&\qquad\cdot\sum_{m=0}^{r}(-1)^m\binom{r}{m}(1+24m\cdot j)^{t-k}\\
&=\sum_{t=0}^{r+1}\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\Big(\frac{\pi}{6}\Big)^{t-2k}\Big(\frac{1}{-4\sqrt{6n}}\Big)^t\\
&\qquad\cdot\sum_{\ell=0}^{t-k}\binom{t-k}{\ell}(24j)^\ell\sum_{m=0}^{r}(-1)^m\binom{r}{m}m^\ell\\
&=\sum_{t=0}^{r+1}\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\Big(\frac{\pi}{6}\Big)^{t-2k}\Big(\frac{1}{-4\sqrt{6n}}\Big)^t\\
&\qquad\cdot\sum_{\ell=0}^{t-k}\binom{t-k}{\ell}(24j)^\ell(-1)^r r!\left\{ {\ell \atop r} \right\}\ \text{(by Fact 7.7)}.\end{aligned} \tag{7.6}$$

From Fact 7.5, we have $\left\{ {\ell \atop r} \right\} = 0$ for all $\ell < r$ and by Fact 7.6, $\left\{ {r \atop r} \right\} = 1$. Therefore, the minimal choice for $(t,k,\ell) = (r,0,r)$ so that the sum on the right hand side of (7.6) to be non-zero. For $t = r+1$, we have two choices; i.e., $(k,\ell) = (1,r)$ and for $k = 0$, $\ell \in \{r, r+1\}$. Therefore, we have

$$\sum_{m=0}^{r}(-1)^m\binom{r}{m}\sum_{t=0}^{r+1}\frac{g(t,m\cdot j)}{\sqrt{n}^t} \tag{7.7}$$

$$=\sum_{t=r}^{r+1}\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\left(\frac{\pi}{6}\right)^{t-2k}\left(\frac{1}{-4\sqrt{6n}}\right)^t \cdot\sum_{\ell=0}^{t-k}\binom{t-k}{\ell}(24j)^\ell(-1)^r r!\left\{\begin{matrix}\ell\\ r\end{matrix}\right\}$$

$$=\left(\frac{\pi\cdot j}{\sqrt{6}}\right)^r\frac{1}{\sqrt{n}^r} +\left[\sum_{k=0}^{1}\binom{r+2}{k}\frac{r+2-k}{(r+2-2k)!}\left(\frac{\pi}{6}\right)^{r+1-2k}\sum_{\ell=r}^{r+1-k}(24j)^\ell(-1)^r r!\left\{\begin{matrix}\ell\\ r\end{matrix}\right\}\right]\frac{1}{(-4\sqrt{6n})^{r+1}}$$

$$=\left(\frac{\pi\cdot j}{\sqrt{6}}\right)^r\frac{1}{\sqrt{n}^r}-\left(\frac{\pi\cdot j}{\sqrt{6}}\right)^{r-1}\frac{j}{4}\left[\frac{\pi^2}{36}(1+12jr)+(r^2+3r+2)\right]\frac{1}{\sqrt{n}^{r+1}}. \tag{7.8}$$

$\square$

Definition 7.9. *For all $r\geq 1$, define*

$$C_r(j):=\left(\frac{\pi\cdot j}{\sqrt{6}}\right)^r,$$

$$C_{r+1}(j):=\left(\frac{\pi\cdot j}{\sqrt{6}}\right)^{r-1}\frac{j}{4}\left[\frac{\pi^2}{36}(1+12jr)+(r^2+3r+2)\right],$$

$$\widetilde{U}_r(j):=\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r}{2m}U(r+2,2m\cdot j)-\sum_{m=0}^{\lfloor (r-1)/2\rfloor}\binom{r}{2m+1}L(r+2,(2m+1)j)$$

and

$$\widetilde{L}_r(j):=\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r}{2m}L(r+2,2m\cdot j)-\sum_{m=0}^{\lfloor (r-1)/2\rfloor}\binom{r}{2m+1}U(r+2,(2m+1)j).$$

Lemma 7.10. *For all $n>\max\left\{\widehat{g}(r+2)+r\cdot j, n_0(r+2,r\cdot j)\right\}$, we have*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\widetilde{M}_r(n,j)+\frac{\widetilde{L}_r(j)}{\sqrt{n}^{r+2}}\right)<\Delta_j^r(p(n))<\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\widetilde{M}_r(n,j)+\frac{\widetilde{U}_r(j)}{\sqrt{n}^{r+2}}\right), \tag{7.9}$$

where

$$\widetilde{M}_r(n,j)=\frac{C_r(j)}{\sqrt{n}^r}-\frac{C_{r+1}(j)}{\sqrt{n}^{r+1}}.$$

Proof. We split $\Delta_j^r(p(n))$ as follows:

$$\begin{aligned}\Delta_j^r(p(n)) &= \sum_{m=0}^{r}(-1)^m\binom{m}{r}p(n-m\cdot j)\\ &= \sum_{m=0}^{\lfloor r/2\rfloor}\binom{r}{2m}p(n-2m\cdot j)-\sum_{m=0}^{\lfloor (r-1)/2\rfloor}\binom{r}{2m+1}p(n-(2m+1)\cdot j).\end{aligned} \tag{7.10}$$

Applying Theorem 5.5 for each of the above two factors, we obtain (7.9). □

Consequently, from Lemma 7.10, we obtain the following result.

Corollary 7.11. *For all* $r, j \in \mathbb{Z}_{\geq 1}$*,*

$$\Delta_j^r(p(n)) \sim \left(\frac{\pi \cdot j}{\sqrt{6n}}\right)^r \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \text{ as } n \to \infty. \tag{7.11}$$

Corollary 7.12. *For* $j = 1$ *and* $r \in \mathbb{Z}_{\geq 1}$*, we have*

$$\Delta^r(p(n)) \sim \left(\frac{\pi}{\sqrt{6n}}\right)^r \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \text{ as } n \to \infty.$$

More generally, we have the asymptotic expansion of $\Delta_j^r(p(n))$ of the following form stated below.

Theorem 7.13. *For all* $r, j \in \mathbb{Z}_{\geq 1}$*,*

$$\Delta_j^r(p(n)) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \left(\frac{\pi \cdot j}{\sqrt{6n}}\right)^r \sum_{t \geq 0} \frac{\widetilde{g}_{r,j}(t)}{\sqrt{n}^t} \text{ as } n \to \infty, \tag{7.12}$$

where

$$\widetilde{g}_{r,j}(t) = \frac{(t+r+1)r!}{(-4\sqrt{6})^t} \sum_{k=0}^{t} \sum_{\ell=0}^{t-k} \binom{t+r}{\ell+r} \binom{t-\ell}{k} \frac{1}{(t+r+1-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} (24j)^\ell \left\{ \begin{matrix} \ell+r \\ r \end{matrix} \right\}.$$

Proof. Letting $w \to \infty$, from (5.4) it follows that

$$\begin{aligned} \Delta_j^r(p(n)) &= \sum_{m=0}^{r} (-1)^m \binom{m}{r} p(n - m \cdot j) \\ &\underset{n\to\infty}{\sim} \left(\sum_{m=0}^{r} (-1)^m \binom{r}{m} \sum_{t \geq 0} \frac{g(t, m \cdot j)}{\sqrt{n}^t} \right) \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}. \end{aligned}$$

From Lemma 7.8, for $0 \leq t \leq r-1$ we have

$$\sum_{m=0}^{r} (-1)^m \binom{r}{m} \sum_{t=0}^{r-1} \frac{g(t, m \cdot j)}{\sqrt{n}^t} = 0, \tag{7.13}$$

and therefore from (7.13) and (7.13), as $n \to \infty$, it follows that

$$\Delta_j^r(p(n)) \sim \sum_{m=0}^{r} (-1)^m \binom{r}{m} \sum_{t \geq r} \frac{g(t, m \cdot j)}{\sqrt{n}^t}. \tag{7.14}$$

Now,

$$\begin{aligned} &\sum_{t \geq r} \sum_{k=0}^{(t+1)/2} \frac{1}{(-4\sqrt{6n})^t} \binom{t+1}{k} \frac{t+1-k}{(t+1-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} \sum_{m=0}^{r} (-1)^m \binom{r}{m} (1 + 24m \cdot j)^{t-k} \\ &= \sum_{t \geq r} \sum_{k=0}^{(t+1)/2} \sum_{\ell=0}^{t-k} \frac{1}{(-4\sqrt{6n})^t} \binom{t+1}{k} \binom{t-k}{\ell} \frac{t+1-k}{(t+1-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} (24j)^\ell \sum_{m=0}^{r} (-1)^m \binom{r}{m} m^\ell \\ &= (-1)^r r! \sum_{t \geq r} \sum_{k=0}^{(t+1)/2} \sum_{\ell=0}^{t-k} \frac{1}{(-4\sqrt{6n})^t} \binom{t+1}{k} \binom{t-k}{\ell} \frac{t+1-k}{(t+1-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} (24j)^\ell \left\{ \begin{matrix} \ell \\ r \end{matrix} \right\} \\ &=: (-1)^r r! \sum_{t \geq r} A(t, r) = (-1)^r r! \sum_{t \geq 0} A(t+r, r) \end{aligned}$$

$$= \frac{(4\pi\cdot j)^r}{(-4\sqrt{6n})^{r+t}} \sum_{k=0}^{t} \sum_{\ell=0}^{t-k} \binom{t+r}{\ell+r} \binom{t-\ell}{k} \frac{t+r+1}{(t+r+1-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} (24j)^\ell \left\{ \begin{matrix} \ell+r \\ r \end{matrix} \right\}. \tag{7.15}$$

Applying (7.15) to (7.14), we finally obtain (7.12). □

Corollary 7.14. *For $j \in \mathbb{Z}_{\geq 1}$,*

$$\Delta_j^1(p(n)) \sim \frac{e^{\pi\sqrt{2n/3}}}{12\sqrt{2}n^{3/2}} \pi j \sum_{t\geq 0} \frac{\widetilde{g}_{1,j}(t)}{\sqrt{n}^t} \text{ as } n \to \infty, \tag{7.16}$$

where

$$\widetilde{g}_{1,j}(t) = \frac{(t+2)}{(-4\sqrt{6})^t} \sum_{k=0}^{t} \sum_{\ell=0}^{t-k} \binom{t+1}{\ell+1} \binom{t-\ell}{k} \frac{1}{(t+2-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} (24j)^\ell (\ell+1).$$

Remark 7.15. *Replacing $n \mapsto n-k-m+1 := n_k$ and plugging $j=1$ in Corollary 7.14, for all $m > n/2$, we have the full asymptotic expansion of $N_k(m,n)$ with respect to the base $\frac{1}{\sqrt{n_k}^t}$. But in order to get the asymptotic expansion with respect to the base $\frac{1}{\sqrt{n}^t}$, we directly employ Theorem 5.5 and obtain for $m > n/2$,*

$$N_k(m,n) \underset{n\to\infty}{\sim} \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \sum_{t\geq 0} \frac{\overline{g}_k(t)}{\sqrt{n}^t}, \tag{7.17}$$

where

$$\overline{g}_k(t) := g(t, k+m-1) - g(t, k+m).$$

For $k = 1, 2$ we get the asymptotic expansion of $M(m,n)$ and $N(m,n)$ respectively.

Corollary 7.16. *For $j \in \mathbb{Z}_{\geq 1}$,*

$$\Delta_j^2(p(n)) \sim \frac{e^{\pi\sqrt{2n/3}}}{24\sqrt{3}n^2} \pi^2 j^2 \sum_{t\geq 0} \frac{\widetilde{g}_{2,j}(t)}{\sqrt{n}^t} \text{ as } n \to \infty, \tag{7.18}$$

where

$$\widetilde{g}_{2,j}(t) = \frac{(2t+6)}{(-4\sqrt{6})^t} \sum_{k=0}^{t} \sum_{\ell=0}^{t-k} \binom{t+2}{\ell+2} \binom{t-\ell}{k} \frac{1}{(t+3-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} (24j)^\ell (2^{\ell+1}-1).$$

Remark 7.17. *By making the substitution $n \mapsto n-k-m+1 := n_k$ and plugging $j=1$ in Corollary 7.16, for all $m > n/2$, we have the full asymptotic expansion of $N_k(m,n) - N_k(m+1,n)$ with respect to the base $\frac{1}{\sqrt{n_k}^t}$. But in order to get the asymptotic expansion with respect to the base $\frac{1}{\sqrt{n}^t}$, we directly employ Theorem 5.5 and obtain for $m > n/2$,*

$$N_k(m,n) - N_k(m+1,n) \underset{n\to\infty}{\sim} \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \sum_{t\geq 0} \frac{\widetilde{g}_k(t)}{\sqrt{n}^t}, \tag{7.19}$$

where

$$\widetilde{g}_k(t) := g(t, k+m-1) - 2g(t, k+m) + g(t, k+m+1).$$

For $k = 1, 2$ we get the asymptotic expansion of $M(m,n) - M(m+1)$ and $N(m,n) - N(m+1,n)$ respectively.

7.3. **Higher order log-concavity.** The notion of log-concavity can be further generalized in the following way. Consider the operator $\mathcal{L}$ defined on a sequence $\mathcal{A} := (a_n)_{n\geq 0} \subset \mathbb{R}_{>0}$ by $\mathcal{L}(\mathcal{A}) := \mathcal{A}_1 := (b_n)_{n\geq 0}$ with

$$b_0 = a_0^2 \text{ and } b_n = a_n^2 - a_{n-1}a_{n+1}, \text{ for } n \geq 1.$$

Hence a sequence $\mathcal{A}$ is log-concave if and only if $\mathcal{L}(\mathcal{A})$ is a non-negative sequence. A sequence is k-log-concave if j-fold applications of $\mathcal{L}$ on $\mathcal{A}$, denoted by $\mathcal{L}^j(\mathcal{A})$, is a non-negative sequence for all $0 \leq j \leq k$. A sequence is called infinitely log-concave if it is k-log-concave for all $k \geq 1$. For example, Brändén [10] proved that the sequence of binomial coefficients $\left(\binom{n}{k}\right)_{0\leq k\leq n}$ is infinitely log-concave for all $n \geq 0$ which was conjectured by Boros and Moll [9]. For a more detailed study on infinite log-concavity of sequences, we refer the reader to [49]. In context of the partition function, DeSalvo and Pak proved that $(p(n))_{n\geq 26}$ is log-concave. Hou and Zhang [36, Page 128], Jia and Wang [38, Theorem 1.6] proved that $(p(n))_{n\geq 222}$ is 2-log-concave.

Theorem 7.18. *For $r \in \{1, 2, 3\}$ and $n > \max\Big\{\widehat{g}(3\cdot 2^r - 2) + 2r, n_0(3\cdot 2^r - r, 2r)\Big\} =: N(r)$,*

$$\mathcal{L}^r(p(n-r)) = \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^{2^r}\left(\frac{\pi^{2^r-1}}{\sqrt{2}^{r^2+r+1}\sqrt{3}^{r^2-r+1}\sqrt{n}^{3(2^r-1)}} + O\left(\frac{1}{\sqrt{n}^{3\cdot 2^r-2}}\right)\right) \tag{7.20}$$

Proof. For $r = 1$, $\mathcal{L}(p(n-1)) = p(n-1)^2 - p(n)p(n-2)$. Applying Theorem 5.5 with $w = 4$, for all $n > N(4) = 151$ we have

$$\begin{aligned}\mathcal{L}(p(n-1)) &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\sum_{t=0}^{3}\frac{g(t,1)^2 - g(t,0)g(t,2)}{\sqrt{n}^t} + O\Big(\frac{1}{n^2}\Big)\right)\\ &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\frac{\pi}{2\sqrt{6}n^{3/2}} + O\Big(\frac{1}{n^2}\Big)\right).\end{aligned} \tag{7.21}$$

Define

$$g_2(t,\ell) := g(t,\ell)^2 - g(t,\ell-1)g(t,\ell+1).$$

Now for $r = 2$, applying Theorem 5.5 with $w = 10$, for all $n > N(10) = 1473$ it follows that

$$\begin{aligned}\mathcal{L}^2(p(n-2)) &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^4\left(\sum_{t=0}^{9}\frac{g_2(t,2)^2 - g_2(t,3)g_2(t,1)}{\sqrt{n}^t} + O\Big(\frac{1}{n^5}\Big)\right)\\ &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^4\left(\frac{\pi^3}{24\sqrt{6}n^{9/2}} + O\Big(\frac{1}{n^5}\Big)\right).\end{aligned} \tag{7.22}$$

Define

$$g_3(t,\ell) := g_2(t,\ell)^2 - g_2(t,\ell-1)g_2(t,\ell+1).$$

Finally for $r = 3$, from Theorem 5.5 with $w = 22$, for all $n > N(22) = 10273$ we get

$$\begin{aligned}\mathcal{L}^3(p(n-3)) &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^8\left(\sum_{t=0}^{21}\frac{g_3(t,3)^2 - g_3(t,4)g_3(t,2)}{\sqrt{n}^t} + O\Big(\frac{1}{n^{11}}\Big)\right)\\ &= \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^8\left(\frac{\pi^7}{1728\sqrt{6}n^{21/2}} + O\Big(\frac{1}{n^{11}}\Big)\right).\end{aligned} \tag{7.23}$$

∎

For $r \in \{1,2\}$, we obtain the following two inequalities for $\mathcal{L}^r(p(n-r))$ using (5.4). For all $n > 676$,

$$\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\frac{\pi}{2\sqrt{6}n^{3/2}}-\frac{4}{n^2}\right) < \mathcal{L}^1(p(n-1)) < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\frac{\pi}{2\sqrt{6}n^{3/2}}+\frac{4}{n^2}\right), \tag{7.24}$$

and for all $n > 5499$,

$$\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^4\left(\frac{\pi^3}{24\sqrt{6}n^{9/2}}-\frac{10}{n^5}\right) < \mathcal{L}^2(p(n-2)) < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^4\left(\frac{\pi^3}{24\sqrt{6}n^{9/2}}+\frac{10}{n^5}\right). \tag{7.25}$$

Equations (7.24) and (7.25) retrieve that $(p(n))_{n\geq 26}$ is log-concave and $(p(n))_{n\geq 222}$ is 2-log-concave respectively along with respective asymptotic growths.
Following the proof of Theorem 7.18, it suggests that for all $n > N(r)$,

$$\mathcal{L}^r(p(n-r)) = \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^{2^r}\left(\sum_{t=0}^{3(2^r-1)}\frac{g_r(t,r)^2-g_r(t,r+1)g_r(t,r-1)}{\sqrt{n}^t}+O\left(\frac{1}{\sqrt{n}^{3\cdot 2^r-2}}\right)\right),$$

where $g_r(t,r) = g_{r-1}(t,r)^2 - g_{r-1}(t,r-1)g_{r-1}(t,r+1)$ for all $r \geq 2$ and $g_1(t,r) = g(t,r)$. Moreover, following (7.21)-(7.23), it further suggests that

$$\sum_{t=0}^{3(2^r-1)}\frac{g_r(t,r)^2-g_r(t,r+1)g_r(t,r-1)}{\sqrt{n}^t} = \frac{G_r}{\sqrt{n}^{3(2^r-1)}},$$

where $G_r = g_r(3(2^r-1),r)^2 - g_r(3(2^r-1),r+1)g_r(3(2^r-1),r-1)$. This finally leads us to make the following conjecture.

Conjecture 7.19. *Let $N(r)$ be as in Theorem 7.18 for $r \in \mathbb{Z}_{\geq 1}$. Then for all $n > N(r)$,*

$$\mathcal{L}^r p(n-r) \sim \frac{\pi^{2^r-1}}{\sqrt{2}^{r^2+r+1}\sqrt{3}^{r^2-r+1}\sqrt{n}^{3(2^r-1)}}\left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^{2^r} \quad \text{as } n \to \infty. \tag{7.26}$$

In other words, $p(n)$ is infinitely log-concave.

8. A unified framework to prove inequalities for $p(n)$

In [5, Section 5], the author provided a unified framework to prove multiplicative inequalities for $p(n)$ given in the following form: for $T \in \mathbb{N}$ and $(s_i, r_i) \in \mathbb{Z}^2_{\geq 0}$ for all $1 \leq i \leq T$,

$$\prod_{i=1}^{T} p(n+s_i) \geq \prod_{i=1}^{T} p(n+r_i),$$

using an infinite family of inequalities for $\log p(n+j)$ with $j \in \mathbb{Z}_{\geq 0}$, see [5, Theorem 3.9]. But in the literature, we found several examples for inequalities of $p(n)$ which do not fit into the multiplicative set up, for example the higher order Turán inequalities. In this context, the author also discussed explicitly in [5, Section 7] the limitation of using [5, Theorem 3.9] to construct a framework to prove inequalities for $p(n)$ which is not multiplicative type and indicated what might be a possible way to prove such inequalities in a systematic way. First, we note that all the inequalities for the partition function stated in previous sections can be written in the following form: for $(c_j, d_j) \in \mathbb{N}^2$, $(s_{i,j}, r_{i,j}) \in \mathbb{Z}^2_{\geq 0}$ with $1 \leq i \leq T$ and $1 \leq j \leq \max\{M_1, M_2\}$,

$$\sum_{j=1}^{M_1} c_j \prod_{i=1}^{T} p(n-s_{i,j}) \geq \sum_{j=1}^{M_2} d_j \prod_{i=1}^{T} p(n-r_{i,j}).$$

We call the above inequality is an *additive-multiplicative* inequality for $p(n)$.
Define

$$P_{M_1,T}(\mathbf{s},n) := \sum_{j=1}^{M_1} c_j \prod_{i=1}^{T} p(n-s_{i,j}) \quad \text{and} \quad P_{M_2,T}(\mathbf{r},n) := \sum_{j=1}^{M_2} d_j \prod_{i=1}^{T} p(n-r_{i,j}).$$

In this subsection, we will construct a unified framework to determine

(1) sign of $(P_{M_1,T}(\mathbf{s},n) - P_{M_2,T}(\mathbf{r},n))$,
(2) and the cut off $N_{M_1,M_2,T}(\mathbf{s},\mathbf{r})$ such that for all $n \geq N_{M_1,M_2,T}(\mathbf{s},\mathbf{r})$,

$$\text{sign of } (P_{M_1,T}(\mathbf{s},n) - P_{M_2,T}(\mathbf{r},n)) \text{ is consistent.}$$

Here we list down the steps so as to determine two facts stated above which finally verify whether a given *additive-multiplicative* inequality for $p(n)$ holds or not. To begin with, let us define

$$D_{M_1,M_2,T}(\mathbf{s},\mathbf{r},n) := P_{M_1,T}(\mathbf{s},n) - P_{M_2,T}(\mathbf{r},n).$$

- (Step 0): Associate

$$p(n-\ell) \mapsto \sum_{t=0}^{\infty} \frac{g(t,\ell)}{\sqrt{n}^t}.$$

Note that here we excluded the exponential term $\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}$ because both factors $P_{M_1,T}(\mathbf{s},n)$ and $P_{M_2,T}(\mathbf{s},n)$ has same number of products T, and therefore, we can extract out $\left(\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\right)^T$ from $D_{M_1,M_2,T}(\mathbf{s},\mathbf{r},n)$, which is always positive. So, in order to determine the sign of $D_{M_1,M_2,T}(\mathbf{s},\mathbf{r},n)$, it is sufficient to consider the power series $\sum_{t=0}^{\infty} \frac{g(t,\ell)}{\sqrt{n}^t}$ associated with $p(n-\ell)$.

- (Step 1): Map

$$D_{M_1,M_2,T}(\mathbf{s},\mathbf{r},n) \mapsto \underbrace{\sum_{j=1}^{M_1} c_j \prod_{i=1}^{T} \sum_{t=0}^{\infty} \frac{g(t,s_{i,j})}{\sqrt{n}^t} - \sum_{j=1}^{M_2} d_j \prod_{i=1}^{T} \sum_{t=0}^{\infty} \frac{g(t,r_{i,j})}{\sqrt{n}^t}}_{=: \sum_{t=0}^{\infty} \frac{G_{\mathbf{s},\mathbf{r}}(t)}{\sqrt{n}^t} =: P_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right)}.$$

- (Step 2): Now we can decide the sign of $D_{M_1,M_2,T}(\mathbf{s},\mathbf{r},n)$ if $P_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right) \neq 0$. Consequently, for $P_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right) \neq 0$, let

$$\text{ord}\left(P_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right)\right) =: m \quad \text{with } m \in \mathbb{N}.$$

- (Step 3): So,

$$\text{sign}\left(D_{M_1,M_2,T}(\mathbf{s},\mathbf{r},n)\right) = \text{sign}\left(G_{\mathbf{s},\mathbf{r}}(m)\right), \text{where } G_{\mathbf{s},\mathbf{r}}(m) = \left[\frac{1}{\sqrt{n}^m}\right]\left(P_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right)\right).$$

Now, there are two cases: (i) $G_{\mathbf{s},\mathbf{r}}(m) > 0$ and (ii) $G_{\mathbf{s},\mathbf{r}}(m) < 0$.

- (Step 4): In order to verify the consistency of sign of $D_{M_1,M_2,T}(\mathbf{s},\mathbf{r},n)$, choose $w = m+1$, where w is the truncation point stated in Theorem 5.5.

- (Step 5): Consider the case $G_{\mathbf{s},\mathbf{r}}(m)>0$. Applying Theorem 5.5 with $w\mapsto m+1$, we get
$$D_{M_1,M_2,T}(\mathbf{s},\mathbf{r},n)>\sum_{j=1}^{M_1}c_j\prod_{i=1}^{T}\mathcal{L}_n(m+1,s_{i,j})-\sum_{j=1}^{M_2}d_j\prod_{i=1}^{T}\mathcal{U}_n(m+1,r_{i,j})=:p^{[1]}_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right),$$
where
$$p^{[1]}_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right)=\sum_{k=m}^{d}\frac{A_{\mathbf{s},\mathbf{r}}(k)}{\sqrt{n}^k}\text{ with }A_{\mathbf{s},\mathbf{r}}(m)=G_{\mathbf{s},\mathbf{r}}(m)\text{ and }d\le T\cdot(m+1).$$
- (Step 6): Determine the cutoff $N_{M_1,M_2,T}(\mathbf{s},\mathbf{r})$, such that for all $n\ge N_{M_1,M_2,T}(\mathbf{s},\mathbf{r})$, $p^{[1]}_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right)>0$.
- (Step 7): Consider the case $G_{\mathbf{s},\mathbf{r}}(m)<0$. Applying Theorem 5.5 with $w\mapsto m+1$, we get
$$D_{M_1,M_2,T}(\mathbf{s},\mathbf{r},n)<\sum_{j=1}^{M_1}c_j\prod_{i=1}^{T}\mathcal{U}_n(m+1,s_{i,j})-\sum_{j=1}^{M_2}d_j\prod_{i=1}^{T}\mathcal{L}_n(m+1,r_{i,j})=:p^{[2]}_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right),$$
where
$$p^{[2]}_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right)=\sum_{k=m}^{d}\frac{B_{\mathbf{s},\mathbf{r}}(k)}{\sqrt{n}^k}\text{ with }B_{\mathbf{s},\mathbf{r}}(m)=G_{\mathbf{s},\mathbf{r}}(m)\text{ and }d\le T\cdot(m+1).$$
In this case, estimate $N_{M_1,M_2,T}(\mathbf{s},\mathbf{r})$, such that for all $n\ge N_{M_1,M_2,T}(\mathbf{s},\mathbf{r})$, $p^{[2]}_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right)<0$.

Now we will discuss in brief about motivation behind considering the framework given above in context of works done on inequalities for $p(n)$.

We mainly focus on two type of inequalities for the partition function: the higher order Turán inequality and determinantal inequalities associated with the Hankel matrix.

(1) Recall that $(p(n))_{n\ge 95}$ satisfies the higher order Turán inequality if for all $n\ge 95$,
$$4\left(p(n)^2-p(n-1)p(n+1)\right)\left(p(n+1)^2-p(n)p(n+2)\right)\\ -\left(p(n)p(n+1)-p(n-1)p(n+2)\right)^2\ge 0.$$
Making the shift $n\mapsto n-2$, can reformulate the left hand side of the above inequality into the following form:
$$D_{2,3,4}(\mathbf{s},\mathbf{r},n)=\sum_{j=1}^{2}c_j\prod_{i=1}^{4}p(n-s_{i,j})-\sum_{j=1}^{3}d_j\prod_{i=1}^{4}p(n-r_{i,j}),$$
with $(M_1,M_2,T)=(2,3,4)$, $(c_j)_{1\le j\le 2}$,$(d_j)_{1\le j\le 3}$, $(s_{i,j})_{\substack{1\le i\le 4\\ 1\le j\le 2}}$, and $(r_{i,j})_{\substack{1\le i\le 4\\ 1\le j\le 3}}$ are explicitly determined. Next, following Step 1, we see that
$$D_{2,3,4}(\mathbf{s},\mathbf{r},n)\mapsto\sum_{t=1}^{\infty}\frac{G_{\mathbf{s},\mathbf{r}}(t)}{\sqrt{n}^t}=:P_{\mathbf{s},\mathbf{r}}\Big(\frac{1}{\sqrt{n}}\Big)=\frac{\pi^3}{12\sqrt{6}}\frac{1}{\sqrt{n}^9}+\dots.$$
Hence, $\mathrm{ord}\left(P_{\mathbf{s},\mathbf{r}}\Big(\frac{1}{\sqrt{n}}\Big)\right)=9$ and sign of $D_{2,3,4}(\mathbf{s},\mathbf{r},n)$ is positive which proves that $p(n)$ satisfies the higher order Turán inequality for sufficiently large n. So, in order to determine a finite cutoff $N(w)$ (so as to verify the remaining cases using any computer algebra system) such that for $n\ge N(w)$, the inequality holds for $p(n)$, we need to choose a w so that we can apply Theorem 5.5 and follows the Steps 4, 5, and 6. According to Step

4, it is immediate that the minimal choice for w is $\operatorname{ord}\left(P_{\mathbf{s},\mathbf{r}}\left(\frac{1}{\sqrt{n}}\right)\right)+1=9+1=10$.

Therefore, we see that the unified framework explains why Chen, Jia, and Wang [17, Lemma 2.2] started with the following inequality (cf. Lemma 4.1) for $p(n)$:

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{10}}\right)<p(n)<\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{10}}\right)$$

to prove the higher order Turán inequality for $p(n)$.

(2) Jia and Wang [38, Theorem 1.5] proved that $\det\left(M_3(p(n))\right)>0$ for all $n\geq 222$, where $M_k(p(n))=(p(n-i+j))_{1\leq i,j\leq k}$ with $k\in\mathbb{Z}_{\geq 2}$. This in turn proves that the Hankel matrix with entries in $p(n)$ and its shifted values is totally positive. Note that for the case $k=2$, $\det\left(M_2(p(n))\right)>0$ for all $n\geq 26$ due to DeSalvo and Pak [23, Theorem 1.1]. Jia and Wang [38, Conjecture 1.7] conjectured that for any $k\geq 2$, there exists $N(k)$ such that $\det\left(M_k(p(n))\right)>0$ for all $n\geq N(k)$. Recently, Wang and Yang [69, Theorem 4.1] settled the conjecture of Jia and Wang for the case $k=4$ and proved the inequality with $N(4)=656$. Moreover in [69, Section 6], they draw an outline of extending their iterative construction to prove $\det\left(M_k(p(n))\right)>0$. But what precludes to determine an effective cutoff $N(k)$ so that the inequality $\det\left(M_k(p(n))\right)>0$ holds is to determine the truncation point $w(k)$ depending on $k\in\mathbb{Z}_{\geq 2}$ because the key tool they used in proving these inequalities was

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^{w(k)}}\right)<p(n)<\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^{w(k)}}\right).$$

Wang and Yang concludes the paper by saying that for large k, one only needs to find $w(k)$ in order to follow their set up to settle the conjecture of Jia and Wang.
Following our framework, first make a shift $n\mapsto n-k$ and consider the matrix $\widehat{M}_k(p(n))=(p(n-k-i+j))_{1\leq i,j\leq k}$. Using the formula of determinant, we have

$$\det\left(\widehat{M}_k(p(n))\right)=\sum_{\sigma\in\mathfrak{S}_k}\left(\operatorname{sgn}(\sigma)\prod_{i=1}^{k}p(n-k-i+\sigma(i))\right).$$

Next, associating $p(n-k-i+\sigma(i))$ to $\sum\limits_{t\geq 0}\dfrac{g(t,k+i-\sigma(i))}{\sqrt{n}^t}$, we finally needs to compute the order of the power series $P_k\left(\frac{1}{\sqrt{n}}\right)$, where

$$P_k\left(\frac{1}{\sqrt{n}}\right)=\sum_{t=0}^{\infty}\left(\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)\prod_{i=1}^{k}g(t,k+i-\sigma(i))\right)\frac{1}{\sqrt{n}^t}.$$

So, the minimal choice for $w(k)$ is $\operatorname{ord}\left(P_k\left(\frac{1}{\sqrt{n}}\right)\right)+1$. Still there is a subtle problem to conclude the proof (following the setup of Wang and Yang) of the conjecture because we have to ensure that for each $k\in\mathbb{Z}_{\geq 2}$ and for some $t\geq 0$,

$$\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)\prod_{i=1}^{k}g\left(\nu,k+i-\sigma(i)\right)=0.$$

For $k\in\{2,3,4,5\}$, we have $(w(k))_{2\leq k\leq 5}=(4,10,19,31)$ which seems to suggest that $w(k)=\frac{3k(k-1)}{2}+1$. Therefore to settle the conjecture [38, Conjecture 1.7], it remains to prove that for $k \in \mathbb{Z}_{\geq 2}$,

$$\sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) \prod_{i=1}^{k} g\left(\frac{3k(k-1)}{2}, k+i-\sigma(i)\right) \neq 0,$$

and for all $0 \leq \nu < \frac{3k(k-1)}{2}$,

$$\sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) \prod_{i=1}^{k} g\left(\nu, k+i-\sigma(i)\right) = 0.$$

We leave this as an open problem.

9. Chen's conjectures on the Andrews' spt function

Andrews [4] defined the smallest parts function $\mathrm{spt}(n)$ for any integer $n \geq 1$ as

$$\mathrm{spt}(n) = \sum_{\lambda \vdash n} \mathrm{mult}\left(s(\lambda)\right),$$

where $\mathrm{mult}\left(s(\lambda)\right)$ denotes the multiplicity of the smallest part $s(\lambda)$ in a partition λ of n. For $n = 4$, $\mathrm{spt}(n) = 10$. The generating function of $\mathrm{spt}(n)$ is given by

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_\infty},$$

where $(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$. Ahlgren and Andersen [1, Theorem 1] obtained a Hardy-Ramanujan-Rademacher type exact formula for $\mathrm{spt}(n)$ as the following conditionally convergent infinite series

$$\mathrm{spt}(n) = \frac{\pi}{6}(24n-1)^{\frac{1}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \left(I_{1/2} - I_{3/2}\right)\left(\frac{\pi\sqrt{24n-1}}{6k}\right),$$

where $A_K(n)$ is the Kloosterman sum defined after (1.5). To obtain an estimate for the error term of this conditionally convergent series in spirit of the work done by Lehmer for $p(n)$, Ahlgren and Andersen used the spectral theory of automorphic forms. Using the algebraic formula (see [1, Theorem 2]) and traces of singular moduli, Dawsey and Masri [22, Theorem 1.1] gave an asymptotic formula for $\mathrm{spt}(n)$. Recently, González [29, Theorem 1.2] substantially improved the result of Dawsey and Masri by proving that for $n \geq 1$,

$$\mathrm{spt}(n) = \frac{\sqrt{3}}{\pi\sqrt{24n-1}} e^{\mu(n)} + E_s(n) \text{ with } |E_s(n)| < 4.1 e^{\frac{\mu(n)}{2}},$$

where $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$. Moreover due to González, we have the following estimation of $\mathrm{spt}(n)$ in terms of $p(n)$ which we will use to prove inequalities for $\mathrm{spt}(n)$.

Lemma 9.1. *[29, Corollary 1.4] Let $\mu(n)$ be as stated before. Then for all $n \geq 1$,*

$$\mathrm{spt}(n) = \frac{\sqrt{24n-1}}{2\pi} p(n) + \frac{6\sqrt{3}}{\pi^2(24n-1)} e^{\mu(n)} + E(n),$$

where $|E(n)| < 4.11 e^{\frac{\mu(n)}{2}}$.

Dawsey and Masri [22, Theorem 1.2] settled the conjectures of Chen (see [16, Conjectures 6.7-6.11]) on inequalities for $\mathrm{spt}(n)$. But conjectures on postivity of invariants of cubic form and quartic binary forms with coefficients in $\mathrm{spt}(n)$ made by Chen still remains open. Here we list all the remaining conjectures of Chen.

Let $b_n := \mathrm{spt}(n)$.

Conjecture 9.2. *[16, Conjectures 6.12 and 6.14]*

(1) $(b_n)_{n\geq 108}$ *satisfies the higher order Turán inequality.*

(2) *Let*

$$v_n = \frac{b_{n+1}b_{n-1}}{b_n^2}.$$

Then for all $n \geq 2$,

$$\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right)(1-v_nv_{n+1})^2 > 4(1-v_n)(1-v_{n+1}).$$

Conjecture 9.3. *[16, Eqn. (6.20) and Conjecture 6.15]*

(1) *For all* $n \geq 205$,

$$b_{n-1}b_{n+3} + 3b_{n+1}^2 > 4b_nb_{n+2}.$$

(2) *For all* $n \geq 260$,

$$4\left(1+\frac{\pi^2}{16n^3}\right)b_nb_{n+2} > b_{n-1}b_{n+3} + 3b_{n+1}^2.$$

Conjecture 9.4. *[16, Eqn. (6.21) and Conjecture 6.16]*

(1) *For all* $n \geq 241$,

$$b_{n+1}^3 + b_{n-1}b_{n+2}^2 + b_n^2b_{n+3} > 2b_nb_{n+1}b_{n+2} + b_{n-1}b_{n+1}b_{n+3}.$$

(2) *For all* $n \geq 290$,

$$\left(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}\right)(2b_nb_{n+1}b_{n+2} + b_{n-1}b_{n+1}b_{n+3}) > b_{n+1}^3 + b_{n-1}b_{n+2}^2 + b_n^2b_{n+3}.$$

Theorem 9.5. (1) *For all* $n \geq 74$,

$$\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{b_n^2}{b_{n-1}b_{n+1}} > \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{1}{n^2}\right). \tag{9.1}$$

(2) *For all* $n \geq 143$,

$$\begin{aligned}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right)(b_nb_{n+1}-b_{n-1}b_{n+2})^2 &> 4(b_n^2-b_{n-1}b_{n+1})(b_{n+1}^2-b_nb_{n+2})\\ &> \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{2}{n^2}\right)(b_nb_{n+1}-b_{n-1}b_{n+2})^2.\end{aligned} \tag{9.2}$$

(3) *For all* $n \geq 265$,

$$4\left(1+\frac{\pi^2}{16n^3}\right)b_nb_{n+2} > b_{n-1}b_{n+3}+3b_{n+1}^2 > 4\left(1+\frac{\pi^2}{16n^3}-\frac{3}{n^{7/2}}\right)b_nb_{n+2}. \tag{9.3}$$

(4) *For all* $n \geq 290$,

$$\begin{aligned}\left(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}\right)(2b_nb_{n+1}b_{n+2}+b_{n-1}b_{n+1}b_{n+3}) &> b_{n+1}^3+b_{n-1}b_{n+2}^2+b_n^2b_{n+3}\\ &> \left(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}-\frac{5}{n^5}\right)(2b_nb_{n+1}b_{n+2}+b_{n-1}b_{n+1}b_{n+3}).\end{aligned} \tag{9.4}$$

Corollary 9.6. *Conjectures 9.2-9.4 are true.*

Before we move on to the proof of Theorem 9.5, in the following subsection, we shall derive an infinite family of inequalities for $\mathrm{spt}(n-\ell)$ in Theorem 9.9 using Lemmas 9.1 and 4.3.

9.1. **Inequalities for $\mathrm{spt}(n)$.** Let $\widehat{g}(k)$ be as in Theorem 4.2 and $\mu(n)$ be as in Lemma 4.1. For $k \in \mathbb{Z}_{\geq 1}$, define

$$\begin{aligned} g^*(k) &:= \frac{1}{24}\left(1+\left(\left(\frac{24k+48}{\pi}\right)\log(k+4)\right)^2\right), \\ n(k,\ell) &:= \max\{\widehat{g}(k), g^*(k)\}+\ell. \end{aligned} \tag{9.5}$$

Lemma 9.7. *For $\ell \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 2}$, and $n > n(k,\ell)$,*

$$\frac{\sqrt{3}}{\pi}\frac{e^{\mu(n-\ell)}}{\sqrt{24(n-\ell)-1}}\left(1-\frac{2}{\mu(n-\ell)^k}\right) < \mathrm{spt}(n-\ell) < \frac{\sqrt{3}}{\pi}\frac{e^{\mu(n-\ell)}}{\sqrt{24(n-\ell)-1}}\left(1+\frac{2}{\mu(n-\ell)^k}\right).$$

Proof. From Lemmas 9.1 and 4.2, we have for $n > \widehat{g}(k)$

$$\begin{aligned} \mathrm{spt}(n) &< \frac{\sqrt{24n-1}}{2\pi}p(n) + \frac{6\sqrt{3}}{\pi^2(24n-1)}e^{\mu(n)} + 4.11e^{\frac{\mu(n)}{2}} \\ &< \frac{\sqrt{24n-1}}{2\pi}\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^k}\right) + \frac{6\sqrt{3}}{\pi^2(24n-1)}e^{\mu(n)} + 4.11e^{\frac{\mu(n)}{2}} \\ &= \frac{\sqrt{3}}{\pi}\frac{e^{\mu(n)}}{\sqrt{24n-1}}\left(1+\frac{1}{\mu(n)^k}+\frac{4.11\cdot 12}{\sqrt{3}}e^{-\frac{\mu(n)}{2}}\frac{\mu(n)}{2}\right), \end{aligned} \tag{9.6}$$

and

$$\begin{aligned} \mathrm{spt}(n) &> \frac{\sqrt{24n-1}}{2\pi}p(n) + \frac{6\sqrt{3}}{\pi^2(24n-1)}e^{\mu(n)} - 4.11e^{\frac{\mu(n)}{2}} \\ &> \frac{\sqrt{24n-1}}{2\pi}\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^k}\right) + \frac{6\sqrt{3}}{\pi^2(24n-1)}e^{\mu(n)} - 4.11e^{\frac{\mu(n)}{2}} \\ &= \frac{\sqrt{3}}{\pi}\frac{e^{\mu(n)}}{\sqrt{24n-1}}\left(1-\frac{1}{\mu(n)^k}-\frac{4.11\cdot 12}{\sqrt{3}}e^{-\frac{\mu(n)}{2}}\frac{\mu(n)}{2}\right). \end{aligned} \tag{9.7}$$

Next, we show that for all $n > g^*(k)$,

$$\frac{4.11\cdot 12}{\sqrt{3}}e^{-\frac{\mu(n)}{2}}\frac{\mu(n)}{2} < \frac{1}{\mu(n)^k}. \tag{9.8}$$

Since $\dfrac{4.11\cdot 12}{\sqrt{3}} < 28.5$, it suffices to prove $28.5\ e^{-\frac{\mu(n)}{2}}\frac{\mu(n)}{2} < \frac{1}{\mu(n)^k}$. Setting $x = \frac{\mu(n)}{2}$, we aim to show that $28.5\ e^{-x}x < \frac{1}{2^k x^k}$ which is equivalent to prove the following inequality:

$$x-(k+1)\log x > k\log 2+\log 28.5.$$

Define $f_k(x) := x-(k+1)\log x$ and observe that $f_k(x)$ is increasing for all $x \geq k+1$. So it is enough to prove that $f_k(x_0) > k\log 2+\log 28.5$ with $x_0 > k+1$. Choose $x_0 = (2k+4)\log(k+4)$ and observe that for all $k \geq 1$, $x_0 > k+1$. Now, it can be easily derived that for $k \geq 1$,

$$f_k(x_0) = (2k+4)\log(k+4)-(k+1)\log\left((2k+4)\log(k+4)\right) > k+4 > k\log 2+\log 28.5.$$

Therefore, for all $x > x_0$, we see that $f_k(x_0) > 0$, which in turn, concludes the proof of (9.8). Applying (9.8) to (9.6) and (9.7), we obtain for $n > \max\{\widehat{g}(k), g^*(k)\}$

$$\frac{\sqrt{3}}{\pi}\frac{e^{\mu(n)}}{\sqrt{24n-1}}\left(1-\frac{2}{\mu(n)^k}\right) < \mathrm{spt}(n) < \frac{\sqrt{3}}{\pi}\frac{e^{\mu(n)}}{\sqrt{24n-1}}\left(1+\frac{2}{\mu(n)^k}\right). \tag{9.9}$$

Consequently, applying the shift $n \mapsto n-\ell$ in (9.9), we conclude the proof. □

Next, we proceed to determine explicitly the coefficient sequence in the Taylor expansion of $\frac{\sqrt{3}}{\pi}\frac{e^{\mu(n-\ell)}}{\sqrt{24(n-\ell)-1}}$ after extracting out the factor $\frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}}$.

Lemma 9.8. *For $t \in \mathbb{Z}_{\geq 0}$, define*

$$\omega^*(t,\ell) = \frac{(1+24\ell)^t}{(-4\sqrt{6})^t} \sum_{k=0}^{\frac{t}{2}} \binom{t}{k} \frac{1}{(t-2k)!} \left(\frac{\pi}{6}\right)^{t-2k} \frac{1}{(1+24\ell)^k}.$$

Then

$$\frac{\sqrt{3}}{\pi} \frac{e^{\mu(n-\ell)}}{\sqrt{24(n-\ell)-1}} = \frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}} \cdot \sum_{t\geq 0} \frac{\omega^*(t,\ell)}{\sqrt{n}^t}.$$

Proof. Applying Lemma 4.6, we have

$$\frac{\sqrt{3}}{\pi} \frac{e^{\mu(n-\ell)}}{\sqrt{24(n-\ell)-1}} = \frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}} \left(E_1\left(\frac{1}{\sqrt{n}},\ell\right) + O_1\left(\frac{1}{\sqrt{n}},\ell\right)\right) E_2^*\left(\frac{1}{\sqrt{n}},\ell\right), \tag{9.10}$$

where

$$E_2^*\left(\frac{1}{\sqrt{n}},\ell\right) = \left(1-\frac{24\ell+1}{24n}\right)^{-\frac{1}{2}} = \sum_{t\geq 0}(-1)^t\binom{-\frac{1}{2}}{t}\left(\frac{24\ell+1}{24}\right)^t\left(\frac{1}{\sqrt{n}}\right)^{2t} =: \sum_{t\geq 0}\frac{e_2^*(t,\ell)}{\sqrt{n}^{2t}}. \tag{9.11}$$

Now,

$$\begin{aligned} E_1\left(\frac{1}{\sqrt{n}},\ell\right) \cdot E_2^*\left(\frac{1}{\sqrt{n}},\ell\right) &= \left(\sum_{t\geq 0}\frac{e_1(t,\ell)}{\sqrt{n}^{2t}}\right)\left(\sum_{t\geq 0}\frac{e_2^*(t,\ell)}{\sqrt{n}^{2t}}\right) \quad \text{(by (4.6) and (9.11))} \\ &= \sum_{t\geq 0}\left(\sum_{k=0}^{t} e_1(k,\ell)e_2^*(t-k,\ell)\right)\frac{1}{\sqrt{n}^{2t}} \\ &= \sum_{t\geq 0}\left(e_2^*(t,\ell)+\sum_{k=1}^{t} e_1(k,\ell)e_2^*(t-k,\ell)\right)\frac{1}{\sqrt{n}^{2t}} =: \sum_{t\geq 0}\frac{g^*(2t,\ell)}{\sqrt{n}^{2t}}, \end{aligned}$$

with

$$g^*(2t,\ell) = (-1)^t\left(\frac{24\ell+1}{24}\right)^t\left(\binom{-\frac{1}{2}}{t}+\sum_{k=1}^{t}\binom{-\frac{1}{2}}{t-k}\frac{\left(\frac{1}{2}-k\right)_{k+1}}{k}\sum_{u=1}^{k}\frac{(-1)^u(-k)_u}{(k+u)!(2u-1)!}\alpha_\ell^{2u}\right) \quad \text{(by (4.5) and (9.11))}. \tag{9.12}$$

Similarly, applying (4.8) and (9.11), we get

$$O_1\left(\frac{1}{\sqrt{n}},\ell\right) \cdot E_2^*\left(\frac{1}{\sqrt{n}},\ell\right) = \sum_{t\geq 0}\left(\sum_{k=0}^{t} o_1(k,\ell)e_2^*(t-k,\ell)\right)\frac{1}{\sqrt{n}^{2t+1}} =: \sum_{t\geq 0}\frac{g^*(2t+1,\ell)}{\sqrt{n}^{2t+1}},$$

and subsequently, by (4.7) and (9.11), it follows that

$$\begin{aligned} g^*(2t+1,\ell) = (-1)^{t+1}&\frac{\pi}{12\sqrt{6}}(24\ell+1)\left(\frac{24\ell+1}{24}\right)^t \\ &\cdot\sum_{k=0}^{t}\binom{-\frac{1}{2}}{t-k}\left(\frac{1}{2}-k\right)_{k+1}\sum_{u=0}^{k}\frac{(-1)^u(-k)_u}{(k+u+1)!(2u)!}\alpha_\ell^{2u}. \end{aligned} \tag{9.13}$$

Combining (9.12) and (9.13), we rewrite (9.10) as

$$\frac{\sqrt{3}}{\pi}\frac{e^{\mu(n-\ell)}}{\sqrt{24(n-\ell)-1}}=\frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}}\cdot\sum_{t\geq 0}\frac{g^*(t,\ell)}{\sqrt{n}^t}. \tag{9.14}$$

Now, it remains to show that $g^*(t,\ell)=\omega^*(t,\ell)$ for $t\geq 0$. Set $\gamma:=\gamma_n=\pi\sqrt{\frac{2n}{3}}$ and $w:=w_n=-\frac{24\ell+1}{24n}$. Then

$$\frac{\sqrt{3}}{\pi}\frac{e^{\mu(n-\ell)}}{\sqrt{24(n-\ell)-1}}=\frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}}\cdot\frac{e^{\gamma\left(\sqrt{1+w}-1\right)}}{\sqrt{1+w}}=:\frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}}\sum_{m\geq 0}c^*_m(\gamma)w^m.$$

Integrating once $\dfrac{e^{\gamma\left(\sqrt{1+w}-1\right)}}{\sqrt{1+w}}$ with respect to w, we see that

$$\begin{aligned}
c^*_m(\gamma)&=(m+1)\,\mathrm{coeff}_{[w^{m+1}]}\left(\frac{2}{\gamma}e^{\gamma\left(\sqrt{1+w}-1\right)}\right)\\
&=(m+1)\,\mathrm{coeff}_{[w^{m+1}]}\left(\frac{2}{\gamma}\sum_{k=0}^{m+1}\frac{\gamma^k}{k!}\left(\sqrt{1+w}-1\right)^k\right)\\
&=(m+1)\sum_{k=0}^{m+1}\left(\frac{\gamma}{2}\right)^{k-1}\frac{1}{k!}\,\mathrm{coeff}_{[w^{m+1-k}]}\left(\frac{2}{w}\left(\sqrt{1+w}-1\right)\right)^k\\
&=(m+1)\sum_{k=0}^{m+1}\left(\frac{\gamma}{2}\right)^{k-1}\frac{1}{k!}\,\mathrm{coeff}_{[w^{m+1-k}]}\left(\frac{1+z}{2}\right)^{-k}\quad\left(\text{by setting } z:=z(w)=\sqrt{1+w}\right)\\
&=(m+1)\sum_{k=0}^{m+1}\left(\frac{\gamma}{2}\right)^{m-k}\frac{1}{(m+1-k)!}\,\mathrm{coeff}_{[w^m]}\left(\frac{1+z}{2}\right)^{-m-1+k}.
\end{aligned}\tag{9.15}$$

From [31, p. 203], we know that for $y=\sqrt{1+x}$ and $m\in\mathbb{C}\setminus\mathbb{Z}_{\geq 0}$,

$$\left(\frac{1+y}{2}\right)^m=\sum_{j=0}^{\infty}\frac{m}{m-j}\binom{m-j}{m}\frac{x^j}{4^j}. \tag{9.16}$$

Applying (9.16) to (9.15), after simplifying, we have

$$c^*_m(\gamma)=\sum_{k=0}^{m}\frac{\gamma^{m-k}}{2^{m+k}}\frac{(-1)^k}{(m-k)!}\binom{m+k}{k},$$

and consequently, it follows that

$$\begin{aligned}
\sum_{m\geq 0}c^*_m(\gamma)w^m&=\sum_{m\geq 0}\sum_{k=0}^{m}\frac{1}{2^{m+k}}\frac{(-1)^k}{(m-k)!}\binom{m+k}{k}\left(\pi\sqrt{\frac{2n}{3}}\right)^{m-k}\left(-\frac{24\ell+1}{24n}\right)^m\\
&=\sum_{m\geq 0}\sum_{k=0}^{m}\frac{1}{2^{m+k}}\frac{(-1)^k}{(m-k)!}\binom{m+k}{k}\left(\pi\sqrt{\frac{2}{3}}\right)^{m-k}\left(-\frac{24\ell+1}{24}\right)^m\frac{1}{\sqrt{n}^{m+k}}\\
&=\sum_{r\geq 0}\left(\sum_{\substack{k+m=r\\ k\leq m}}\frac{1}{2^{m+k}}\frac{(-1)^k}{(m-k)!}\binom{m+k}{k}\left(\pi\sqrt{\frac{2}{3}}\right)^{m-k}\left(-\frac{24\ell+1}{24}\right)^m\right)\frac{1}{\sqrt{n}^r}
\end{aligned}$$

$$= \sum_{r\geq 0}\left(\left(-\frac{24\ell+1}{4\sqrt{6}}\right)^r \sum_{k=0}^{\frac{r}{2}}\binom{r}{k}\frac{1}{(r-2k)!}\left(\frac{\pi}{6}\right)^{r-2k}\frac{1}{(1+24\ell)^k}\right)\frac{1}{\sqrt{n}^r}. \qquad (9.17)$$

Due to uniqueness property of the Taylor expansion of an analytic function, we conclude that $g^*(t,\ell)=\omega^*(t,\ell)$ for all $t\geq 0$. ☐

Next, we move on to estimate an error bound for the absolute value of the remainder part of the series $\sum_{t\geq 0}\frac{\omega^*(t,\ell)}{\sqrt{n}^t}$ truncated at $k-1$ for $k\in\mathbb{Z}_{\geq 2}$ using estimations for the coefficient sequences $(g^*(2t,\ell))_{t\geq 0}$ and $(g^*(2t+1,\ell))_{t\geq 0}$ in the following lemma.

Let $n_0(k,\ell)$ be as in Definition 4.42. For $k\in\mathbb{Z}_{\geq 2}$, define

$$\begin{aligned}
E^{[1]}(k,\ell) &:= \left(\frac{2}{\sqrt{\pi}}+\frac{\alpha_\ell\zeta\left(\frac{3}{2}\right)}{\sqrt{\pi}}\sinh(\alpha_\ell)\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2\lceil\frac{k}{2}\rceil},\\
E^{[2]}(k,\ell) &:= \sqrt{24\ell+1}\left(\frac{\pi}{6\sqrt{2}}+\frac{\sqrt{\pi}\zeta\left(\frac{3}{2}\right)}{6\sqrt{2}}\cosh(\alpha_\ell)\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2\lceil\frac{k-1}{2}\rceil+1},\\
E^{[3]}(k,\ell) &:= \frac{1}{(\alpha_0\sqrt{24})^k}\left(1+\frac{k(1+24\ell)}{3n_0(k,\ell)}\right)\\
E^*(k,\ell) &:= E^{[1]}(k,\ell)+E^{[2]}(k,\ell)+2E^{[3]}(k,\ell). \qquad (9.18)
\end{aligned}$$

Theorem 9.9. *For $w\in\mathbb{Z}_{\geq 2}$, $\ell\in\mathbb{Z}_{\geq 0}$, and $n>\max\{n(w,\ell),n_0(w,\ell)\}$,*

$$\frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}}\left(\sum_{t=0}^{w-1}\frac{\omega^*(t,\ell)}{\sqrt{n}^t}-\frac{E^*(w,\ell)}{\sqrt{n}^w}\right)<\operatorname{spt}(n-\ell)<\frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}}\left(\sum_{t=0}^{w-1}\frac{\omega^*(t,\ell)}{\sqrt{n}^t}+\frac{E^*(w,\ell)}{\sqrt{n}^w}\right).$$

Proof. Split the series in Lemma 9.8 as

$$\begin{aligned}
\sum_{t\geq 0}\frac{\omega^*(t,\ell)}{\sqrt{n}^t} &= \sum_{t=0}^{w-1}\frac{\omega^*(t,\ell)}{\sqrt{n}^t}+\sum_{t\geq w}\frac{\omega^*(t,\ell)}{\sqrt{n}^t} = \sum_{t=0}^{w-1}\frac{\omega^*(t,\ell)}{\sqrt{n}^t}+\sum_{t\geq\lceil\frac{w}{2}\rceil}\frac{\omega^*(2t,\ell)}{\sqrt{n}^{2t}}+\sum_{t\geq\lceil\frac{w-1}{2}\rceil}\frac{\omega^*(2t+1,\ell)}{\sqrt{n}^{2t+1}}\\
&= \sum_{t=0}^{w-1}\frac{\omega^*(t,\ell)}{\sqrt{n}^t}+\sum_{t\geq\lceil\frac{w}{2}\rceil}\frac{g^*(2t,\ell)}{\sqrt{n}^{2t}}+\sum_{t\geq\lceil\frac{w-1}{2}\rceil}\frac{g^*(2t+1,\ell)}{\sqrt{n}^{2t+1}}. \qquad (9.19)
\end{aligned}$$

For all $t\geq 1$ and using (9.12), we have

$$\begin{aligned}
|g^*(2t,\ell)| &\leq \left(\frac{1+24\ell}{24}\right)^t\left(\left|\binom{-\frac{1}{2}}{t}\right|+\sum_{k=1}^{t}\left|\binom{-\frac{1}{2}}{t-k}\right|\frac{\left|\left(\frac{1}{2}-k\right)_{k+1}\right|}{k}\sum_{u=1}^{k}\frac{|(-1)^u(-k)_u|}{(k+u)!(2u-1)!}\alpha_\ell^{2u}\right)\\
&= \left(\frac{1+24\ell}{24}\right)^t\left(\frac{1}{4^t}\binom{2t}{t}+\sum_{k=1}^{t}\left|\binom{-\frac{1}{2}}{t-k}\right|\frac{\binom{2k}{k}}{2k\cdot 4^k}\sum_{u=1}^{k}\frac{\prod_{j=0}^{u-1}(k-j)}{\prod_{j=0}^{u-1}(k+j+1)}\frac{\alpha_\ell^{2u}}{(2u-1)!}\right)\\
&\leq \left(\frac{1+24\ell}{24}\right)^t\left(\frac{1}{4^t}\binom{2t}{t}+\sum_{k=1}^{t}\frac{\binom{2k}{k}}{2k\cdot 4^k}\sum_{u=1}^{k}\frac{\alpha_\ell^{2u}}{(2u-1)!}\right)\\
&= \left(\frac{1+24\ell}{24}\right)^t\left(\frac{1}{4^t}\binom{2t}{t}+\sum_{k=1}^{t}\frac{\alpha_\ell\binom{2k}{k}}{2k\cdot 4^k}\sum_{u=0}^{k-1}\frac{\alpha_\ell^{2u+1}}{(2u+1)!}\right)
\end{aligned}$$

$$\le \left(\frac{1+24\ell}{24}\right)^t \left(\frac{1}{4^t}\binom{2t}{t} + \alpha_\ell \sinh(\alpha_\ell) \sum_{k=1}^{t} \frac{\binom{2k}{k}}{2k \cdot 4^k}\right)$$

$$\le \left(\frac{1+24\ell}{24}\right)^t \left(\frac{1}{\sqrt{\pi \cdot t}} + \frac{\alpha_\ell \sinh(\alpha_\ell)}{2\sqrt{\pi}} \sum_{k=1}^{t} \frac{1}{k^{3/2}}\right) \quad \left(\text{since } \binom{2m}{m} \le \frac{4^m}{\sqrt{\pi m}} \text{ for } m \ge 1\right)$$

$$\le \left(\frac{1+24\ell}{24}\right)^t \left(\frac{1}{\sqrt{\pi}} + \frac{\alpha_\ell \sinh(\alpha_\ell)}{2\sqrt{\pi}} \zeta\left(\frac{3}{2}\right)\right),$$

and consequently, it follows that

$$\begin{aligned}
\left|\sum_{t \ge \lceil \frac{w}{2} \rceil} \frac{g^*(2t,\ell)}{\sqrt{n}^{2t}}\right| &\le \sum_{t \ge \lceil \frac{w}{2} \rceil} \frac{|g^*(2t,\ell)|}{\sqrt{n}^{2t}} \\
&\le \left(\frac{1}{\sqrt{\pi}} + \frac{\alpha_\ell \sinh(\alpha_\ell)}{2\sqrt{\pi}} \zeta\left(\frac{3}{2}\right)\right) \sum_{t \ge \lceil \frac{w}{2} \rceil} \left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2t} \\
&= \left(\frac{1}{\sqrt{\pi}} + \frac{\alpha_\ell \sinh(\alpha_\ell)}{2\sqrt{\pi}} \zeta\left(\frac{3}{2}\right)\right) \left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2\lceil \frac{w}{2} \rceil} \sum_{t \ge 0} \left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2t} \\
&\le \left(\frac{1}{\sqrt{\pi}} + \frac{\alpha_\ell \sinh(\alpha_\ell)}{2\sqrt{\pi}} \zeta\left(\frac{3}{2}\right)\right) \left(\sqrt{\frac{1+24\ell}{24}}\right)^{2\lceil \frac{w}{2} \rceil} \frac{1}{\sqrt{n}^w} \sum_{t \ge 0} \left(\frac{1+24\ell}{24n}\right)^{t} \\
&\le \left(\frac{1}{\sqrt{\pi}} + \frac{\alpha_\ell \sinh(\alpha_\ell)}{2\sqrt{\pi}} \zeta\left(\frac{3}{2}\right)\right) \left(\sqrt{\frac{1+24\ell}{24}}\right)^{2\lceil \frac{w}{2} \rceil} \frac{1}{\sqrt{n}^w} \sum_{t \ge 0} \frac{1}{2^t} \\
&\qquad\qquad (\text{as } n > n_0(k,\ell) > 2\ell+1) \\
&= \frac{E^{[1]}(w,\ell)}{\sqrt{n}^w}.
\end{aligned} \tag{9.20}$$

Next, for all $t \ge 0$ and applying (9.13), we get

$$\begin{aligned}
|g^*(2t+1,\ell)| &\le \frac{\pi}{12\sqrt{6}}(24\ell+1)\left(\frac{1+24\ell}{24}\right)^t \sum_{k=0}^{t} \left|\binom{-\frac{1}{2}}{t-k}\right| \left|\left(\frac{1}{2}-k\right)_{k+1}\right| \sum_{u=0}^{k} \frac{|(-1)^u(-k)_u|\,\alpha_\ell^{2u}}{(k+u+1)!(2u)!} \\
&\le \frac{\pi}{12\sqrt{6}}(24\ell+1)\left(\frac{1+24\ell}{24}\right)^t \sum_{k=0}^{t} \left|\left(\frac{1}{2}-k\right)_{k+1}\right| \sum_{u=0}^{k} \frac{|(-1)^u(-k)_u|\,\alpha_\ell^{2u}}{(k+u+1)!(2u)!} \\
&= \frac{\pi}{12\sqrt{6}}(24\ell+1)\left(\frac{1+24\ell}{24}\right)^t \left(\frac{1}{2} + \sum_{k=1}^{t} \frac{\binom{2k}{k}}{2 \cdot 4^k} \sum_{u=0}^{k} \frac{\prod_{j=0}^{u-1}(k-j)}{\prod_{j=0}^{u}(k+j+1)} \frac{\alpha_\ell^{2u}}{(2u)!}\right) \\
&\le \frac{\pi}{24\sqrt{6}}(24\ell+1)\left(\frac{1+24\ell}{24}\right)^t \left(1 + \sum_{k=1}^{t} \frac{1}{\sqrt{\pi k}} \sum_{u=0}^{k} \frac{1}{k+u+1} \frac{\alpha_\ell^{2u}}{(2u)!}\right) \\
&\le \frac{\pi}{24\sqrt{6}}(24\ell+1)\left(\frac{1+24\ell}{24}\right)^t \left(1 + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{t} \frac{1}{k^{3/2}} \sum_{u=0}^{k} \frac{\alpha_\ell^{2u}}{(2u)!}\right) \\
&\le \frac{\pi}{24\sqrt{6}}(24\ell+1)\left(\frac{1+24\ell}{24}\right)^t \left(1 + \frac{\cosh(\alpha_\ell)}{\sqrt{\pi}} \sum_{k=1}^{t} \frac{1}{k^{3/2}}\right)
\end{aligned}$$

$$\le \frac{\pi}{24\sqrt{6}}(24\ell+1)\left(\frac{1+24\ell}{24}\right)^t\left(1+\frac{\cosh(\alpha_\ell)\zeta(\frac{3}{2})}{\sqrt{\pi}}\right),$$

and therefore,

$$\begin{aligned}
&\left|\sum_{t\ge\lceil\frac{w-1}{2}\rceil}\frac{g^*(2t+1,\ell)}{\sqrt{n}^{2t+1}}\right| \le \sum_{t\ge\lceil\frac{w-1}{2}\rceil}\frac{|g^*(2t+1,\ell)|}{\sqrt{n}^{2t+1}}\\
&\le \frac{\pi}{24\sqrt{6}}(24\ell+1)\left(1+\frac{\cosh(\alpha_\ell)\zeta(\frac{3}{2})}{\sqrt{\pi}}\right)\sum_{t\ge\lceil\frac{w-1}{2}\rceil}\left(\frac{1+24\ell}{24}\right)^t\frac{1}{\sqrt{n}^{2t+1}}\\
&= \sqrt{24\ell+1}\left(\frac{\pi}{12}+\frac{\sqrt{\pi}\cosh(\alpha_\ell)\zeta(\frac{3}{2})}{12}\right)\sum_{t\ge\lceil\frac{w-1}{2}\rceil}\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2t+1}\\
&\le \sqrt{24\ell+1}\left(\frac{\pi}{12}+\frac{\sqrt{\pi}\cosh(\alpha_\ell)\zeta(\frac{3}{2})}{12}\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2\lceil\frac{w-1}{2}\rceil+1}\frac{1}{\sqrt{n}^w}\sum_{t\ge 0}\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2t+1}\\
&\le \sqrt{24\ell+1}\left(\frac{\pi}{12}+\frac{\sqrt{\pi}\cosh(\alpha_\ell)\zeta(\frac{3}{2})}{12}\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2\lceil\frac{w-1}{2}\rceil+1}\frac{1}{\sqrt{n}^w}\sum_{t\ge 0}\left(\frac{1}{\sqrt{2}}\right)^{2t+1}\\
&\qquad\qquad (\text{as } n>n_0(k,\ell)>2\ell+1)\\
&= \frac{E^{[2]}(w,\ell)}{\sqrt{n}^w}. \qquad (9.21)
\end{aligned}$$

Applying (9.20) and (9.21) to Lemma 9.8, we obtain for $n>\max\{\widehat{g}(w),g^*(w),n_0(w,\ell)\}$,

$$\begin{aligned}
\frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}}\left(\sum_{t=0}^{w-1}\frac{\omega^*(t,\ell)}{\sqrt{n}^t}-\frac{E^{[1]}(w,\ell)+E^{[2]}(w,\ell)}{\sqrt{n}^w}\right) &< \frac{\sqrt{3}}{\pi}\frac{e^{\mu(n-\ell)}}{\sqrt{24(n-\ell)-1}}\\
&< \frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}}\left(\sum_{t=0}^{w-1}\frac{\omega^*(t,\ell)}{\sqrt{n}^t}+\frac{E^{[1]}(w,\ell)+E^{[2]}(w,\ell)}{\sqrt{n}^w}\right).
\end{aligned} \tag{9.22}$$

Finally, it can be deduce[2] that for $n>n_0(w,\ell)$,

$$\frac{\sqrt{3}}{\pi}\frac{e^{\mu(n-\ell)}}{\sqrt{24(n-\ell)-1}}\cdot\frac{2}{\mu(n-\ell)^k} < \frac{e^{\pi\sqrt{2n/3}}}{\pi\sqrt{8n}}\frac{2E^{[3]}(k,\ell)}{\sqrt{n}^w}. \tag{9.23}$$

We finish the proof by applying (9.22) and (9.23) to Lemma 9.7. ☐

Here we only provide brief sketch of proof of Theorem 9.5 as it is analogous to the proofs given in Section 6.

Proof of Theorem 9.5: Applying Theorem 9.9 with $w=7,12,11,15$, we conclude that (9.1), (9.2), (9.3), (9.4) holds for $n\ge 1135,3667,3018,6097$ respectively. For the remaining cases, we verified numerically with Mathematica. ☐

Remark 9.10. *Using Theorem 9.9, one can also prove the higher order Laguerre inequalities, higher order* log*-concavity (for example,* 2*-*log*-concavity), determinantal inequalities, and higher order shifted differences for* spt(n)*, to name a few. Moreover, adapting the set up devised by Griffin, Ono, Rolen, and Zagier [32, Theorem 3], one can appropriately choose the sequences*

[2]We omit the detail due to similarity with the proof of Lemma 4.44

$(A(n))_{n\geq 0}$ and $(\delta(n))_{n\geq 0}$ so as to conclude that for all but finitely many n, the Jensen polynomial $J^{d,n}_{\mathrm{spt}}(x)$ associated with the sequence $(\mathrm{spt}(n))_{n\geq 0}$ has all real roots.

10. Concluding Remarks

In this section, we discuss on two aspects of the work done on inequalities for $p(n)$: (1) regarding the proof methodology adapted for Theorem 5.5 and (2) advantages and disadvantages of the unified framework presented in Section 8.

(1) First, we show schematically how one can derive a similar family of inequalities for $p(n-\ell)$ from [7, Theorem 7.5], and then will discuss about why might not be classified a standard proof methodology.

From [7, Theorem 7.5], we have for $w \in \mathbb{N}$ and for $n > N(w)$,

$$p(n) = \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1}\frac{g(t)}{\sqrt{n}^t} + O_{\leq C_w}\left(\frac{1}{\sqrt{n}^w}\right)\right),$$

with $N(w)$ and C_w are explicitly determined. Therefore making the shift $n \mapsto n-\ell$ with $\ell \in \mathbb{Z}_{\geq 0}$, for $n > N(w)+\ell$, we immediately get

$$p(n-\ell) = \frac{e^{\pi\sqrt{\frac{2n-2\ell}{3}}}}{4(n-\ell)\sqrt{3}}\left(\sum_{t=0}^{w-1}\frac{g(t)}{(n-\ell)^{\frac{t}{2}}} + O_{\leq C^{[1]}_{w,\ell}}\left(\frac{1}{\sqrt{n}^w}\right)\right).$$

Now in order to derive analogous result stated in Theorem 5.5, first we need to extract the factor $\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}$ and it takes the following form:

$$\begin{aligned}
p(n-\ell) &= \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\frac{e^{\pi\sqrt{\frac{2n-2\ell}{3}}-\pi\sqrt{\frac{2n}{3}}}}{1-\frac{\ell}{n}}\left(\sum_{t=0}^{w-1}\frac{g(t)}{\sqrt{n}^t}\sum_{m=0}^{\infty}\binom{-\frac{t}{2}}{m}\left(-\frac{\ell}{n}\right)^m + O_{\leq C^{[1]}_{w,\ell}}\left(\frac{1}{\sqrt{n}^w}\right)\right)\\
&= \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\frac{e^{\pi\sqrt{\frac{2n-2\ell}{3}}-\pi\sqrt{\frac{2n}{3}}}}{1-\frac{\ell}{n}}\left(\sum_{t=0}^{w-1}g(t)\sum_{m=0}^{\infty}\binom{-\frac{t}{2}}{m}(-\ell)^m\left(\frac{1}{\sqrt{n}}\right)^{2m+t}\right.\\
&\qquad\left. + O_{\leq C^{[1]}_{w,\ell}}\left(\frac{1}{\sqrt{n}^w}\right)\right).
\end{aligned}$$

After splitting the inner infinite sum as

$$\begin{aligned}
\sum_{t=0}^{w-1}g(t)\sum_{m=0}^{\infty}\binom{-\frac{t}{2}}{m}(-\ell)^m\left(\frac{1}{\sqrt{n}}\right)^{2m+t} &= \sum_{t=0}^{w-1}g(t)\sum_{m=0}^{\left\lfloor\frac{w-1-t}{2}\right\rfloor}\binom{-\frac{t}{2}}{m}(-\ell)^m\left(\frac{1}{\sqrt{n}}\right)^{2m+t}\\
&\quad+\sum_{t=0}^{w-1}g(t)\sum_{m=\left\lfloor\frac{w-1-t}{2}\right\rfloor+1}^{\infty}\binom{-\frac{t}{2}}{m}(-\ell)^m\left(\frac{1}{\sqrt{n}}\right)^{2m+t}\\
&=: \sum_{t=0}^{w-1}\frac{g^{[1]}(t,\ell)}{\sqrt{n}^t} + E^{[1]}(w,\ell),
\end{aligned}$$

can compute an error bound for $E^{[1]}(w,\ell)$ of the following form

$$E^{[1]}(w,\ell) = O_{\leq C^{[2]}_{w,\ell}}\left(\frac{1}{\sqrt{n}^w}\right).$$

Similarly, the coefficients can be obtained explicitly from the Taylor expansion of $\frac{e^{\pi\sqrt{\frac{2n-2\ell}{3}}-\pi\sqrt{\frac{2n}{3}}}}{1-\frac{\ell}{n}}$ and after truncating the series at $w-1$, it follows that

$$\begin{aligned}\frac{e^{\pi\sqrt{\frac{2n-2\ell}{3}}-\pi\sqrt{\frac{2n}{3}}}}{1-\dfrac{\ell}{n}} &=: \sum_{t=0}^{\infty}\frac{g^{[2]}(t,\ell)}{\sqrt{n}^{t}} = \sum_{t=0}^{w-1}\frac{g^{[2]}(t,\ell)}{\sqrt{n}^{t}} + \sum_{t=w}^{\infty}\frac{g^{[2]}(t,\ell)}{\sqrt{n}^{t}}\\ &=: \sum_{t=0}^{w-1}\frac{g^{[2]}(t,\ell)}{\sqrt{n}^{t}} + E^{[2]}(w,\ell),\end{aligned}$$

and consequently, estimate the error term as

$$E^{[2]}(w,\ell) = O_{\leq C^{[3]}_{w,\ell}}\left(\frac{1}{\sqrt{n}^{w}}\right).$$

Altogether these estimations lead to the following asymptotic estimate:

$$\begin{aligned}p(n-\ell) = \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}&\left(\sum_{t=0}^{w-1}\frac{g^{[2]}(t,\ell)}{\sqrt{n}^{t}} + O_{\leq C^{[3]}_{w,\ell}}\left(\frac{1}{\sqrt{n}^{w}}\right)\right)\\ &\cdot\left(\sum_{t=0}^{w-1}\frac{g^{[1]}(t,\ell)}{\sqrt{n}^{t}} + O_{\leq C^{[2]}_{w,\ell}+C^{[1]}_{w,\ell}}\left(\frac{1}{\sqrt{n}^{w}}\right)\right).\end{aligned}$$

Writing the product of two polynomials each of degree $w-1$ as

$$\sum_{t=0}^{w-1}a_t x^t\cdot\sum_{t=0}^{w-1}b_t x^t = \sum_{t=0}^{w-1}\sum_{k=0}^{t}a_k b_{t-k}x^t + x^w\sum_{t=0}^{w-2}\sum_{k=t}^{w-2}a_{k+1}b_{w+m-1-k}x^t,$$

we have

$$\begin{aligned}p(n-\ell) &= \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1}\sum_{k=0}^{t}g^{[2]}(k,\ell)g^{[1]}(t-k,\ell)\left(\frac{1}{\sqrt{n}}\right)^{t}\right.\\ &\quad+\left(\frac{1}{\sqrt{n}}\right)^{w}\sum_{t=0}^{w-2}\sum_{k=t}^{w-2}g^{[2]}(k+1,\ell)g^{[1]}(w+m-1-k,\ell)\left(\frac{1}{\sqrt{n}}\right)^{t}\\ &\quad\left.+\sum_{t=0}^{w-1}\frac{g^{[2]}(t,\ell)}{\sqrt{n}^{t}}O_{\leq C^{[2]}_{w,\ell}+C^{[1]}_{w,\ell}}\left(\frac{1}{\sqrt{n}}\right)^{w}+\sum_{t=0}^{w-1}\frac{g^{[1]}(t,\ell)}{\sqrt{n}^{t}}O_{\leq C^{[3]}_{w,\ell}}\left(\frac{1}{\sqrt{n}}\right)^{w}\right)\\ &= \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1}\frac{g(t,\ell)}{\sqrt{n}^{t}}\right.\\ &\quad+\left(\frac{1}{\sqrt{n}}\right)^{w}\sum_{t=0}^{w-2}\sum_{k=t}^{w-2}g^{[2]}(k+1,\ell)g^{[1]}(w+m-1-k,\ell)\left(\frac{1}{\sqrt{n}}\right)^{t}\\ &\quad\left.+\sum_{t=0}^{w-1}\frac{g^{[2]}(t,\ell)}{\sqrt{n}^{t}}O_{\leq C^{[2]}_{w,\ell}+C^{[1]}_{w,\ell}}\left(\frac{1}{\sqrt{n}}\right)^{w}+\sum_{t=0}^{w-1}\frac{g^{[1]}(t,\ell)}{\sqrt{n}^{t}}O_{\leq C^{[3]}_{w,\ell}}\left(\frac{1}{\sqrt{n}}\right)^{w}\right),\end{aligned}$$

where $g(t,\ell)$ be as in (4.45). Finally, one needs to bound the three residual sums in the above equation so as to obtain a result of the form

$$p(n-\ell) = \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\left(\sum_{t=0}^{w-1}\frac{g(t,\ell)}{\sqrt{n}^{t}} + O_{\leq C_{w,\ell}}\left(\frac{1}{\sqrt{n}^{w}}\right)\right).$$

We abandoned this approach because, our goal was to determine an asymptotic estimate of the coefficients $(g(t,\ell))_{t\geq 0}$ appeared in the asymptotic expansion of $p(n-\ell)$ in the following form:

$$f(t,\ell)-l(t,\ell)\leq g(t,\ell)\leq f(t,\ell)+u(t,\ell), \tag{10.1}$$

with $\lim_{t\to\infty}\frac{g(t,\ell)}{f(t,\ell)}=1$, $\lim_{t\to\infty}\frac{l(t,\ell)}{f(t,\ell)}=0$, and $\lim_{t\to\infty}\frac{u(t,\ell)}{f(t,\ell)}=0$. Precise descriptions for such an estimations are given in Section 4 along with the inequalities of the form (10.1). In order to derive a similar asymptotic estimation in the present context (the alternative approach we have discussed in the previous paragraph), we need to derive inequalities for $(g^{[1]}(k,\ell))_{k\geq 0}$, $(g^{[2]}(k,\ell))_{k\geq 0}$, and the convolution of these two sequences. Even if we can derive such an estimate, this alternative approach to prove inequalities related to the partition function (for instance, log-concavity, higher order Turán inequality etc.) is somewhat a pretentious one. The reason is that the methodology we have depicted reflects that in order to prove the inequalities (mentioned above) for a sequence, say $(c(n))_{n\geq 0}$, arising from Fourier expansion of a modular form of negative weight which admits a Rademacher type exact formula, one first needs to estimate error bounds for the asymptotic expansion $c(n)$ and then follow this framework to get a similar result for the shifted version $c(n-\ell)$. Whereas, what we have shown in this paper is that, we can estimate error bounds for asymptotic expansion of $c(n-\ell)$ explicitly in a more straight forward way. In brevity, we showed that how one can generalize systematically the framework constructed by Paule, Radu, Schneider, and author in [7] earlier so as to prove inequalities for $p(n)$, and inevitably it shows that to prove inequalities for $p(n)$, we need not to follow the result [7, Theorem 7.5], which has already been subsumed in the present work.

(2) Now, we discuss about advantages and disadvantages of the framework presented in Section 8.

In Section 8, we have shown that how one can prove inequalities for the partition function of the form:

$$\sum_{j=1}^{M_1}c_j\prod_{i=1}^{T}p(n-s_{i,j})\geq\sum_{j=1}^{M_2}d_j\prod_{i=1}^{T}p(n-r_{i,j}),$$

along with determination of cutoff $N(M_1,M_2,T)$ for n. One can construct a similar structure to prove inequalities (of above type) for a broader class of sequences associated with *modular* objects. In the following points, we shall consider sequences (and provide example of such a sequence) arising from Fourier expansion of modular forms (of certain types), mock modular forms, and mixed mock modular forms. Here we omit the definitions of modular/mock modular/mixed mock modular forms, for a detailed exposition on these class of functions, we refer the reader to [11, 53].

(a) We know that the Fourier coefficients, say $(a_f(n))_{n\geq 0}$, of a modular form f of negative weight (half-integral or integral) over a subgroup of finite index of the full modular group admits a convergent series representation (analogous to $p(n)$) due to work of Rademacher and Zuckerman [58, 70]. Having such an expression for $a_f(n)$ (cf. [58, 70, Theorem 1.1]) in hand and after estimating the Kloosterman sums following the footsteps of Lehmer's work [43], it remains to estimate the error bound for the $I_{\nu(f)}$-Bessel function to get an infinite family of inequalities for $a_f(n-\ell)$. Now estimations for the error bound of the asymptotic expansion of I-Bessel function has been done in [6]. Therefore, one can immediately adapt the framework done in Section 8 to prove log-concavity, higher order Turán inequality, etc. for $a_f(n)$. For instance, can choose $a_f(n)$ to be the cubic partition function, following the Rademacher type exact formula due to Mauth [47] and construct a similar framework as demonstrated before. In a more general setting, one can adapt our set up to prove inequalities for sequences arising from Fourier expansion of a certain class of eta-quotients which admit Rademacher type series expansion (cf. [19, 64]).

(b) In case of mock modular form, one of the most notable example is the coefficient sequence $(\alpha_f(n))_{n\geq 0}$ arising from Ramanujan's third order mock theta function $f(q)$ (which is mock modular form of weight $\frac{1}{2}$). Using the theory of Maass-Poincaré series, Bringmann and Ono [13, Theorem 1.1] obtained a Rademacher type conditional convergent series expression for $\alpha_f(n)$ which settled a conjecture of Andrews and Dragonette. Now, in order to get an asymptotic expansion for $\alpha_f(n)$ along with estimations of error bound, first we need to estimate an error bound after truncating the condinitional convergent series for $\alpha_f(n)$ at a point, say N. Following the work done in [29] for the Andrews' spt function, it seems that one can derive a similar error bound for $\alpha_f(n)$, which will presumably improve the bound obtained by Gomez and Zhu [28]. After getting such an asymptotic estimation for $\alpha_f(n)$, the rest is straight forward.

(c) Last, but not the least, consider the function counting partitions without sequences, denoted by $p_2(n)$, whose generating function is a mixed mock modular form of weight 0. For a more detailed study on the function, we refer the reader to [3]. Extending the Hardy-Ramanujan circle method, Bringmann and Mahlburg [12] obtained a Rademacher type formula for $p_2(n)$. Very recently, Mauth [48] obtained an asymptotic estimate for $p_2(n)$ to prove that $(p_2(n))_{n\geq 482}$ is log-concave. One can make use of [48, Lemma 2.11] to derive an asymptotic expansion for the shifted version; i.e., $p_2(n-\ell)$ so as to adapt our set up to prove the class of inequalities discussed before.

Summarizing the points above, it can be fairly state that whenever we have a Rademacher type exact for a sequence arising from modular or mock modular or mixed mock modular form, one can adapt our framework and prove certain class of inequalities discussed so far.

In order to demonstrate the disadvantages of the framework built up in Section 8, let us prepare the premise by recalling a result due to Griffin, Ono, Rolen, and Zagier [32, Theorem 5] which states that $J_p^{d,n}(x)$ has all real roots for sufficiently large n and consequently, conjectured the cutoff $N(d) \approx 10d^2 \log d$ such that for all $n \geq N(d)$, $J_p^{d,n}(x)$ has all real roots. Larson and Wagner [42] estimated an upper bound for $N(d) \leq (3d)^{24d}(50d)^{3d^2}$ which is of super exponential growth, far away from $O(d^2 \log d)$. One of the main reason for deviation in a large scale is that the interpretation of real rootedness property of a polynomial was done in the realm of algebra; that is, by using the Hermite-Sylvester criteria, see [42, Section 2]. Now, reformulating the Hermite-Sylvester criteria in terms of the generic inequalities for the shifted partition function:

$$\sum_{j=1}^{M_1} c_j \prod_{i=1}^{T} p(n-s_{i,j}) \geq \sum_{j=1}^{M_2} d_j \prod_{i=1}^{T} p(n-r_{i,j}),$$

we see that as the degree d of $J_p^{d,n}(x)$ increases, the number of product T also increases and here the difficulty lies in to get a *good* approximation for $N(d)$. Without going into much detail, it would be safe to say that the reason behind it is in the asymptotic estimation of $\prod_{i=1}^{T} p(n-s_{i,j})$ (using Theorem 5.5 for T times), multinomial expressions turns up in the coefficients of the asymptotic expansion and estimating such coefficients in general with an asymptotic precision seems to be a difficult task. So, instead adapting the algebraic set up, it would be nice to develop an analytic one by studying the polynomials $J_p^{d,n}(x)$ and the Hermite polynomial $H_d(x)$.

11. Appendix

In the proofs of Lemmas 4.24-4.30, we follow the same notations and the proof strategy as in [7, Subsection 5.2].

Proof of Lemma 4.24: Following Definition 4.12, write $S_1(t,\ell)$ as follows:

$$\begin{aligned} S_1(t,\ell) &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\sum_{s=u}^{t}\frac{(-1)^s}{s}\Big(\frac{1}{2}-s\Big)_{s+1}\frac{(-s)_u}{(s+u)!} \\ &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\underbrace{\sum_{s=0}^{t-u}\frac{(-1)^{s+u}}{s+u}\Big(\frac{1}{2}-s-u\Big)_{s+u+1}\frac{(-s-u)_u}{(s+2u)!}}_{=:S_1(t,u)}. \end{aligned} \tag{11.1}$$

From [7, Eqn. (5.6)], we have

$$S_1(t,u)=(-1)^t\binom{-\frac{3}{2}}{t}\frac{(-1)^u}{2u}A_1(t,u), \tag{11.2}$$

where

$$A_1(t,u)=\frac{t(-t)_u(-1)^u}{(1+2t)(t+u)(t)_u}-\left(\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}+\frac{1}{(1+2t)}+\frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i}\right).$$

Now by Lemmas 3.3 and 3.4,

$$\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}-\frac{u}{t}-\frac{u^2}{2t^2}\le A_1(t,u)\le\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}+\frac{1}{4t^2}+u\Big(\frac{2}{3t^2}-\frac{1}{t}\Big)+\frac{u^2}{2t^2}+\frac{u^3}{3t^2}. \tag{11.3}$$

Equations (11.1) and (11.2), it follows that

$$S_1(t,\ell)=(-1)^t\binom{-\frac{3}{2}}{t}\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}A_1(t,u)}{(2u)!}. \tag{11.4}$$

Applying (11.3) to (11.4), we get the following lower bound of $S_1(t,\ell)$,

$$\begin{aligned} \frac{S_1(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}} &\ge \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}\right)\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u)!}-\frac{1}{t}\sum_{u=1}^{t}\frac{u\alpha_\ell^{2u}}{(2u)!}-\frac{1}{2t^2}\sum_{u=1}^{t}\frac{u^2\alpha_\ell^{2u}}{(2u)!} \\ &\ge \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}\right)\left(\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}-1-\sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}\right)-\frac{1}{t}\sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u)!}-\frac{1}{2t^2}\sum_{u=0}^{\infty}\frac{u^2\alpha_\ell^{2u}}{(2u)!}. \\ &> \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}\right)\left(\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}-1-\frac{C_0(\ell)}{t^2}\right)-\frac{1}{t}\sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u)!}-\frac{1}{2t^2}\sum_{u=0}^{\infty}\frac{u^2\alpha_\ell^{2u}}{(2u)!} \\ &\qquad \left(\text{by Lemma 3.7 and } \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}>\frac{1}{4t^2} \text{ for all } t\ge 1\right) \\ &> \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}\right)\Big(\cosh(\alpha_\ell)-1\Big)-\frac{C_0(\ell)}{t^2}-\frac{\alpha_\ell\sinh(\alpha_\ell)}{2t} \\ &\qquad -\frac{1}{2t^2}\Big(\frac{\alpha_\ell^2}{4}\cosh(\alpha_\ell)+\frac{\alpha_\ell}{4}\sinh(\alpha_\ell)\Big) \\ &\qquad \left(\text{by Lemma 3.5 and } \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}<1 \text{ for all } t\ge 1\right) \end{aligned}$$

$$= \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha_\ell)-1) - \frac{\alpha_\ell \sinh(\alpha_\ell)}{2t} - \frac{C_1^{\mathcal{L}}(\ell)}{2t^2} \quad \Big(\text{by Definition 4.23}\Big). \tag{11.5}$$

For the upper bound estimation, we have for all $t \geq 1$,

$$\frac{S_1(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}}$$

$$\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{t}\sum_{u=1}^{t}\frac{u\alpha_\ell^{2u}}{(2u)!} + \frac{1}{4t^2}\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u)!} + \frac{2}{3t^2}\sum_{u=1}^{t}\frac{u\alpha_\ell^{2u}}{(2u)!} + \frac{1}{2t^2}\sum_{u=1}^{t}\frac{u^2\alpha_\ell^{2u}}{(2u)!} + \frac{1}{3t^2}\sum_{u=1}^{t}\frac{u^3\alpha_\ell^{2u}}{(2u)!}$$

$$\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha_\ell)-1) - \frac{1}{2t}\alpha_\ell\sinh(\alpha_\ell) + \frac{C_1(\ell)}{t^3} + \frac{1}{4t^2}\cosh(\alpha_\ell) + \frac{1}{3t^2}\alpha_\ell\sinh(\alpha_\ell)$$

$$+\frac{1}{2t^2}\left(\frac{\alpha_\ell^2}{4}\cosh(\alpha_\ell) + \frac{\alpha_\ell}{4}\sinh(\alpha_\ell)\right) + \frac{1}{3t^2}\left(\frac{3\alpha_\ell^2}{8}\cosh(\alpha_\ell) + \frac{\alpha_\ell(\alpha_\ell^2+1)}{8}\sinh(\alpha_\ell)\right)$$

$$\Big(\text{by Lemmas 3.5 and} \quad 3.7\Big)$$

$$\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha_\ell)-1) - \frac{1}{2t}\alpha_\ell\sinh(\alpha_\ell) + \frac{C_1^{\mathcal{U}}(\ell)}{t^2} \quad \Big(\text{by Definition 4.23}\Big). \tag{11.6}$$

Combining (11.5) and (11.6), we arrive at (4.46) which concludes the proof. ☐

Proof of Lemma 4.26: Following Definition 4.14, write $S_2(t,\ell)$ as follows:

$$\begin{aligned} S_2(t,\ell) &= \sum_{u=0}^{t-1}\frac{(-1)^u\alpha_\ell^{2u}}{(2u)!}\sum_{s=u}^{t-1}\Big(\frac{1}{2}-s\Big)_{s+1}\binom{-\frac{3}{2}}{t-s-1}\frac{(-s)_u}{(s+u+1)!} \\ &= \sum_{u=0}^{t-1}\frac{(-1)^u\alpha_\ell^{2u}}{(2u)!}\underbrace{\sum_{s=0}^{t-u-1}\Big(\frac{1}{2}-s-u\Big)_{s+u+1}\binom{-\frac{3}{2}}{t-s-u-1}\frac{(-s-u)_u}{(s+2u+1)!}}_{=:S_2(t,u)}. \end{aligned} \tag{11.7}$$

From [7, Eqn. (5.13)], we have

$$S_2(t,u) = \binom{-\frac{3}{2}}{t}(-1)^{u+1}\Big(A_{2,1}(t,u) + A_{2,2}(t,u)\Big), \tag{11.8}$$

where

$$A_{2,1}(t,u) = \frac{2t(t-u)(-t)_u(-1)^u}{(1+2t)(1+2u)(t+u)(t)_u}$$

and

$$A_{2,2}(t,u) = \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} + \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-1)^i(-t)_i}{(t+i)(t)_i}.$$

Combining (11.7) and (11.8), we get

$$S_2(t,\ell) = -\binom{-\frac{3}{2}}{t}\Big(s_{2,1}(t,\ell) + s_{2,2}(t,\ell)\Big), \tag{11.9}$$

where

$$s_{2,1}(t,\ell) = \sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u)!}A_{2,1}(t,u) \quad \text{and} \quad s_{2,2}(t,\ell) = \sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u)!}A_{2,2}(t,u). \tag{11.10}$$

By Lemma 3.3, we have

$$\frac{1}{1+2u} - \frac{u^2+u+\frac{1}{2}}{t(1+2u)} \leq A_{2,1}(t,u) \leq \frac{t-u}{t(1+2u)}. \tag{11.11}$$

Applying (11.11) into (11.10) we obtain

$$\sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u+1)!}-\frac{1}{t}\sum_{u=0}^{t-1}\frac{u^2+u+\frac{1}{2}}{(2u+1)!}\alpha_\ell^{2u}\le s_{2,1}(t)\le\sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u+1)!}-\frac{1}{t}\sum_{u=0}^{t-1}\frac{u\alpha_\ell^{2u}}{(2u+1)!},$$

and consequently,

$$\begin{aligned}\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!}-\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!}-\frac{1}{t}\sum_{u=0}^{\infty}\frac{u^2+u+\frac{1}{2}}{(2u+1)!}\alpha_\ell^{2u}\le s_{2,1}(t,\ell)\le\\ \sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!}-\frac{1}{t}\left(\sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u+1)!}-\sum_{u=t}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u+1)!}\right).\end{aligned}\tag{11.12}$$

By Lemma 3.7, it follows that

$$\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!}\le\frac{2C_1(\ell)}{\alpha_\ell^2\ t^2}\quad\text{and}\quad\sum_{u=t}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u+1)!}\le\frac{2C_2(\ell)}{\alpha_\ell^2t^2}.\tag{11.13}$$

Applying (11.13) into (11.12) and by Lemma 3.5, we obtain

$$\frac{\sinh(\alpha_\ell)}{\alpha_\ell}-\frac{C_{2,1}^{\mathcal{L}}(\ell)}{t}\le s_{2,1}(t,\ell)\le\frac{\sinh(\alpha_\ell)}{\alpha_\ell}+\frac{C_{2,1}^{\mathcal{U}}(\ell)}{t}.\tag{11.14}$$

Next we apply Lemma 3.4 and get

$$\frac{2u+1}{2t}-\frac{4u^3+6u^2+8u+3}{12t^2}+\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\le A_{2,2}(t,u)\le\frac{2u+1}{2t}+\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}.\tag{11.15}$$

Plugging (11.15) into (11.10), we obtain

$$\begin{aligned}\frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!}-\frac{1}{2t}\sum_{u=t}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!}+\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}-\frac{1}{12t^2}\sum_{u=0}^{\infty}\frac{p_3(u)\alpha_\ell^{2u}}{(2u)!}\\ \le s_{2,2}(t,\ell)\le\frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!}+\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}-\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!},\end{aligned}\tag{11.16}$$

where $p_3(u)=4u^3+6u^2+8u+3$. By Lemma 3.7 we obtain

$$\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}\le\frac{4C_2(\ell)}{\alpha_\ell^2\ t^2}\quad\text{and}\quad\sum_{u=t}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!}\le\frac{8C_3(\ell)}{\alpha_\ell^2\ t^2}.\tag{11.17}$$

Note that for all $t\ge 1$,

$$\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}=\frac{2^{2t+1}}{t+1}\frac{1}{\binom{2t+2}{t+1}}<1.\tag{11.18}$$

Combining (11.17) with (11.18) and applying Lemma 3.7 to (11.16), we obtain

$$\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha_\ell)+\frac{\operatorname{csh}(\alpha_\ell)}{2t}-\frac{C_{2,2}(\alpha_\ell)}{t^2}\le s_{2,2}(t,\ell)\le\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha_\ell)+\frac{\operatorname{csh}(\alpha_\ell)}{2t}+\frac{4C_2(\ell)}{\alpha_\ell^2\ t^2}.\tag{11.19}$$

Applying (11.14) and (11.19) to (11.9), we obtain (4.47). $\square$

Proof of Lemma 4.28: Recalling Definition 4.16, rewrite $S_3(t,\ell)$ as follows:

$$\begin{aligned} S_3(t,\ell) &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\sum_{s=u}^{t}\frac{1}{s}\Big(\frac{1}{2}-s\Big)_{s+1}\binom{-\frac{3}{2}}{t-s}\frac{(-s)_u}{(s+u)!} \\ &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\underbrace{\sum_{s=0}^{t-u}\frac{1}{s+u}\Big(\frac{1}{2}-s-u\Big)_{s+u+1}\binom{-\frac{3}{2}}{t-s-u}\frac{(-s-u)_u}{(s+2u)!}}_{=:S_3(t,u)}. \end{aligned} \tag{11.20}$$

From [7, Eqn. (5.34)], we have

$$S_3(t,u) = \binom{-\frac{3}{2}}{t}(-1)^u\Big(A_{3,1}(t,u)+A_{3,2}(t,u)\Big), \tag{11.21}$$

where

$$A_{3,1}(t,u) = \frac{t(1+2t-2u)(-t)_u(-1)^u}{2(1+2t)u(t+u)(t)_u}$$

and

$$A_{3,2}(t,u) = \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} + \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i}.$$

From (11.20) and (11.21), it follows that

$$S_3(t,\ell) = \binom{-\frac{3}{2}}{t}\Big(s_{3,1}(t)+s_{3,2}(t)\Big), \tag{11.22}$$

with

$$s_{3,1}(t,\ell) = \sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u-1)!}A_{3,1}(t,u) \quad\text{and}\quad s_{3,2}(t,\ell) = \sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u-1)!}A_{3,2}(t,u). \tag{11.23}$$

By Lemma 3.3, we have

$$-\frac{3u^2+2u+\frac{1}{2}}{4ut} \leq A_{3,1}(t,u) - \frac{1}{2u} \leq 0. \tag{11.24}$$

Applying (11.24) into (11.23) and by Lemmas 3.7 and 3.5, we obtain

$$-\frac{C_{3,1}(\ell)}{t} \leq s_{3,1}(t,\ell) + 1 - \cosh(\alpha_\ell) \leq 0. \tag{11.25}$$

Now, by Lemma 3.4, we obtain

$$-\frac{4u^3+6u^2+8u+3}{12t^2} \leq A_{3,2}(t,u) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t} \leq 0. \tag{11.26}$$

Applying (11.26) to (11.23), it follows that

$$s_{3,2}(t,\ell) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u-1)!} - \frac{1}{2t}\sum_{u=1}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!} \leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u-1)!}, \tag{11.27}$$

and

$$\begin{aligned} s_{3,2}(t,\ell) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u-1)!} - \frac{1}{2t}\sum_{u=1}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!} &\geq \\ -\frac{1}{12t^2}\sum_{u=1}^{\infty}\frac{p_3(u)\alpha_\ell^{2u}}{(2u-1)!} - \frac{1}{2t}\sum_{u=t+1}^{\infty}&\frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!}, \end{aligned} \tag{11.28}$$

where $p_3(u) = 4u^3 + 6u^2 + 8u + 3$ is as in (11.16). By Lemma 3.7 we obtain

$$\sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u-1)!} \le \frac{2C_1(\ell)}{t^2} \quad \text{and} \quad \sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!} \le \frac{4C_2(\ell)+2C_1(\ell)}{t^2}. \tag{11.29}$$

Applying (11.29) and Lemma Lemma 3.5 into (11.27) and (11.28), we have

$$-\frac{C_{3,2}(\ell)}{t^2} \le s_{3,2}(t,\ell) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \alpha_\ell \sinh(\alpha_\ell) - \frac{1}{2t}\mathrm{sch}(\alpha_\ell) \le \frac{3C_1(\ell)}{t^2}. \tag{11.30}$$

Applying (11.25) and (11.30) into (11.22) we arrive at (4.48). □

Proof of Lemma 4.30: Following Definition 4.18, write $S_4(t,\ell)$ as follows:

$$\begin{aligned} S_4(t,\ell) &= \sum_{u=0}^{t} \frac{(-1)^u \alpha_\ell^{2u}}{(2u)!} \sum_{s=u}^{t} (-1)^s \left(\frac{1}{2} - s\right)_{s+1} \frac{(-s)_u}{(s+u+1)!} \\ &= \sum_{u=0}^{t} \frac{(-1)^u \alpha_\ell^{2u}}{(2u)!} \underbrace{\sum_{s=0}^{t-u} (-1)^{s+u} \left(\frac{1}{2} - s - u\right)_{s+u+1} \frac{(-s-u)_u}{(s+2u+1)!}}_{=:S_4(t,u)}. \end{aligned} \tag{11.31}$$

From [7, Eqn. (5.53)], we have

$$S_4(t,u) = \binom{-\frac{3}{2}}{t} (-1)^{u+t} \Big(A_{4,1}(t,u) + A_{4,2}(t,u)\Big), \tag{11.32}$$

where

$$A_{4,1}(t,u) = \frac{t(-t)_u(-1)^u}{2(1+2t)(t+u)(t+u+1)(t)_u}$$

and

$$A_{4,2}(t,u) = \frac{1}{1+2u}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{1+2t} - \frac{2t}{1+2t}\sum_{i=1}^{u} \frac{(-1)^i(-t)_i}{(t+i)(t)_i}\right).$$

From (11.31) and (11.32) it follows that

$$S_4(t,\ell) = (-1)^t \binom{-\frac{3}{2}}{t} \Big(s_{4,1}(t,\ell) + s_{4,2}(t,\ell)\Big), \tag{11.33}$$

where

$$s_{4,1}(t,\ell) = \sum_{u=0}^{t} \frac{\alpha_\ell^{2u}}{(2u)!} A_{4,1}(t,u) \quad \text{and} \quad s_{4,2}(t) := \sum_{u=0}^{t} \frac{\alpha_\ell^{2u}}{(2u)!} A_{4,2}(t). \tag{11.34}$$

Lemmas 3.2 and 3.3 imply that

$$\frac{1}{4t^2}\left(1 - \frac{u^2+u+\frac{3}{2}}{t}\right) \le A_{4,1}(t,u) \le \frac{1}{4t^2}. \tag{11.35}$$

From (11.35) and (11.34), we obtain

$$\frac{1}{4t^2}\sum_{u=0}^{\infty} \frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{4t^2}\sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{4t^3}\sum_{u=0}^{\infty} \frac{(u^2+u+\frac{3}{2})\alpha_\ell^{2u}}{(2u)!} \le s_{4,1}(t,\ell) \le \frac{1}{4t^2}\sum_{u=0}^{\infty} \frac{\alpha_\ell^{2u}}{(2u)!}. \tag{11.36}$$

Applying Lemmas 3.7 and 3.5 to (11.36), it follows that

$$\frac{1}{4t^2}\cosh(\alpha_\ell) - \frac{C_{4,1}(\ell)}{t^3} \le s_{4,1}(t,\ell) \le \frac{1}{4t^2}\cosh(\alpha_\ell). \tag{11.37}$$

Now, by Lemma 3.4, we obtain

$$0 \le A_{4,2}(t,u) - \frac{1}{1+2u}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t}\right) \le \frac{1}{1+2u}\frac{p_3(u)}{12t^2}, \tag{11.38}$$

where $p_3(u)$ is as in (11.16). Plugging (11.38) into (11.34), it follows that

$$\begin{aligned} -\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u+1)!} \le & s_{4,2}(t,\ell) - \sum_{u=0}^{\infty} \frac{\alpha_\ell^{2u}}{(2u+1)!}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{2u+1}{2t}\right) \le \\ & \frac{1}{12t^2}\sum_{u=0}^{\infty}\frac{p_3(u)\alpha_\ell^{2u}}{(2u+1)!} + \frac{1}{2t}\sum_{u=t+1}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u+1)!}. \end{aligned} \tag{11.39}$$

Using Lemma 3.7, we get

$$\sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u+1)!} \le \frac{C_0(\ell)}{t^2} \quad \text{and} \quad \sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u+1)!} = \sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u)!} \le \frac{C_0(\ell)}{t^2}. \tag{11.40}$$

Plugging (11.40) to (11.39) and using Lemma 3.5, we finally obtain

$$-\frac{2C_0(\ell)}{3t^2} \le s_{4,2}(t,\ell) - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{\cosh(\alpha_\ell)}{2t} \le \frac{(\alpha^2+6)\cosh(\alpha_\ell) + 3\alpha_\ell\sinh(\alpha_\ell) + 12C_0(\ell)}{24t^2}. \tag{11.41}$$

We conclude the proof by combining (11.37), (11.41), and (11.33). □

Acknowledgements

The author is extremely grateful to Prof. William Y. C. Chen for his extensive study leading to the conjectures, discussed in this paper and also for explaining his thoughts and remarks on the subject in email correspondence [14]. The author would like to express his sincere gratitude to Prof. Paule his kind initiation of the correspondence with Prof. Chen. The research was funded by the Austrian Science Fund (FWF): W1214-N15, project DK6.

References

[1] S. Ahlgren and N. Andersen. Algebraic and transcendental formulas for the smallest parts function. *Advances in Mathematics*, **289** (2016), pp. 411–437.

[2] G. Almkvist. On the differences of the partition function. *Acta Arithmetica*, **61** (1992), pp. 173–181.

[3] G. E. Andrews. Partitions with short sequences and mock theta functions. *Proceedings of the National Academy of Sciences*, **102** (2005), pp. 4666–4671.

[4] G. E. Andrews. The number of smallest parts in the partitions of n. *Journal für die Reine und Angewandte Mathematik*, **624** (2008), pp. 133–142.

[5] K. Banerjee. A unified framework to prove multiplicative inequalities for the partition function. *Advances in Applied Mathematics*, **152** (2024), p. 102590.

[6] K. Banerjee. Inequalities for the modified Bessel function of first kind of non-negative order. *Journal of Mathematical Analysis and Applications*, **524** (2023), p. 127082.

[7] K. Banerjee, P. Paule, C. S. Radu, and C.. Schneider. Error bounds for the asymptotic expansion of the partition function. *Rocky Mountain Journal of Mathematics* (accepted for publication), Available at `https://projecteuclid.org/journals/rmjm/rocky-mountain-journal-of-mathematics/DownloadAcceptedPapers/230220-Radu.pdf`, (2023).

[8] K. Banerjee, P. Paule, C. S. Radu, and W. H. Zeng. New inequalities for $p(n)$ and $\log p(n)$. *Ramanujan Journal*, **61** (2023), pp. 1295–1338.

[9] G. Boros and V. H. Moll. *Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals*. Cambridge University Press, (2004).

[10] P. Brändén. Iterated sequences and the geometry of zeros. *Journal für die Reine und Angewandte Mathematik*, **658** (2011), pp. 115–131.

[11] K. Bringmann, A. Folsom, K. Ono, and L. Rolen. Harmonic Maass forms and mock modular forms: theory and applications. American Mathematical Society Colloquium Publications (American Mathematical Society, Providence, RI), Vol. 64, 2017.

[12] K. Bringmann and K. Mahlburg. An extension of the Hardy-Ramanujan circle method and applications to partitions without sequences. *American Journal of Mathematics*, **133** (2011), pp. 1151–1178.

[13] K. Bringmann and K. Ono. The $f(q)$ mock theta function conjecture and partition ranks. *Inventiones mathematicae*, **165** (2006), pp. 243–266.

[14] W. Y. C. Chen. Private Communication. 2020.

[15] W. Y. C. Chen. Recent developments on log-concavity and q-log-concavity of combinatorial polynomials. *DMTCS Proceeding of 22nd International Conference on Formal Power Series and Algebraic Combinatorics*, 2010.

[16] W. Y. C. Chen. The spt-function of Andrews. *Surveys in combinatorics, London Math. Soc. Lecture Note Ser. 440*, Cambridge Univ. Press, Cambridge, 2017, pp. 141–203.

[17] W. Y. C. Chen, D. Jia, and L. Wang. Higher order Turán inequalities for the partition function. *Transactions of the American Mathematical Society*, **372** (2019), pp. 2143–2165.

[18] W. Y. C. Chen, L. Wang, and G. Xie. Finite differences of the logarithm of the partition function. *Mathematics of Computation*, **85** (2016), pp. 825–847.

[19] S. Chern. Asymptotics for the Fourier coefficients of eta-quotients. *Journal of Number Theory*, **199** (2019), pp. 168–191.

[20] G. E. Collins. Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition. *Lecture Notes in Computer Science*, **33** (1975), 134–183, pp. 134–183.

[21] G. Csordas and R. S. Varga. Necessary and sufficient conditions and the Riemann hypothesis. *Advances in Applied Mathematics*, **11** (1990), pp. 328–357.

[22] M. Dawsey and R. Masri. Effective bounds for the Andrews spt-function. *Forum Mathematicum*, **31** (2019), pp. 743–767.

[23] S. DeSalvo and I. Pak. Log-concavity of the partition function. *Ramanujan Journal*, **38** (2015), pp. 61–73.

[24] D. Dimitrov. Higher order Turán inequalities. *Proceedings of the American Mathematical Society*, **126** (1998), pp. 2033–2037.

[25] L. M. Dou and L. X. W. Wang. Higher order Laguerre inequalities for the partition function. *Discrete Mathematics*, **346** (2023), p. 113366.

[26] F. G. Garvan. Generalizations of Dyson's rank and non-Rogers-Ramanujan partitions. *Manuscripta Mathematica*, **84** (1994), pp. 343–359.

[27] K. Gomez, J. Males, and L. Rolen. The second shifted difference of partitions and its applications. *Bulletin of the Australian Mathematical Society*, **107** (2023), pp. 66–78.

[28] K. Gomez and E. Zhu. Bounds for coefficients of the $f(q)$ mock theta function and applications to partition ranks. *Journal of Number Theory*, **226** (2021), pp. 1–23.

[29] O. E. González. Effective estimates for the smallest parts function. *Journal of Number Theory*, **241** (2022), pp. 364–386.

[30] I. J. Good. The differences of the partition function, Problem 6137. *American Mathematical Monthly*, **84** (1977), p. 141.

[31] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison Wesley, Reading MA, second edition, 1994.

[32] M. Griffin, Ken Ono, Larry Rolen, and Don Zagier. Jensen polynomials for the Riemann zeta function and other sequences. Proceedings of the National Academy of Sciences, **116** (2019), pp. 11103–11110.

[33] H. Gupta. Finite differences of the partition function. *Mathematics of Computation*, **32** (1978), pp. 1241–1243.

[34] G. H. Hardy and S. Ramanujan. Asymptotic formulæ in combinatory analysis. *Proceedings of the London Mathematical Society*, **2** (1918), pp. 75–115.

[35] D. Hilbert. *Theory of algebraic invariants*. Cambridge university press, 1993.

[36] Q. H. Hou and Z. R. Zhang. r-log-concavity of partition functions. *Ramanujan Journal*, **48** (2019), pp. 117–129.

[37] J. L. W. V. Jensen. Recherches sur la théorie des équations. *Acta Mathematica*, **36** (1913), pp. 181–195.

[38] D. X. Q. Jia and L. X. W. Wang. Determinantal inequalities for the partition function. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, **150** (2020), pp. 1451–1466.

[39] C. Knessl and J. B. Keller. Asymptotic behavior of high-order differences of the partition function. *Communications on pure and applied mathematics*, **44** (1991), pp. 1033–1045.

[40] J. P. S. Kung and G. C. Rota. The invariant theory of binary forms. *Bulletin of the American Mathematical Society*, **10** (1984), pp. 27–85.

[41] E. Laguerre. *Oeuvres, vol. 1.* Gaauthier-Villars, Paris, 1989.

[42] H. Larson and I. Wagner. Hyperbolicity of the partition Jensen polynomials. *Research in Number Theory*, **5** (2019), pp. 1–12.

[43] D. H. Lehmer. On the remainders and convergence of the series for the partition function. *Transactions of the American Mathematical Society*, **46** (1939), pp. 362–373.

[44] D. H. Lehmer. On the series for the partition function. *Transactions of the American Mathematical Society*, **43** (1938), pp. 271–295.

[45] J. E. Littlewood. *Littlewood's miscellany*. Cambridge University Press, 1986.

[46] C. Mallinger. Algorithmic manipulations and transformations of univariate holonomic functions and sequences. *Master's thesis, RISC, J. Kepler University, Linz*, August 1996.

[47] L. Mauth. Exact formula for cubic partitions. *arXiv preprint arXiv:2305.03396*, (2023).

[48] L. Mauth. Log-concavity for partitions without sequences. *arXiv preprint arXiv:2306.07459*, (2023).

[49] P. R. W. McNamara and B. E. Sagan. Infinite log-concavity: developments and conjectures. *Advances in Applied Mathematics*, **44** (2010), pp. 1–15.

[50] J. L. Nicolas. Sur les entiers N pour lesquels il y a beaucoup de groupes abéliens d'ordre N. *Annales de l'institut Fourier*, **28** (1978), pp. 1–16.

[51] C. O'Sullivan. Detailed asymptotic expansions for partitions into powers. *International Journal of Number Theory*. DOI:`https://doi.org/10.1142/S1793042123501063`., (2023).

[52] A. M. Odlyzko. Differences of the partition function. *Acta Arithmetica*, **49** (1988), pp. 237–254.

[53] K. Ono. The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series. American Mathematical Society, Providence, 2004.

[54] G. Pólya. Über die algebraisch-funktionentheoretische Untersuchungen von J. L. W. V. Jensen. Kgl Danske Vid Sel Math-Fys Medd, **7** (1927), pp. 3–33.

[55] H. Rademacher. A convergent series for the partition function $p(n)$. *Proceedings of the National Academy of Sciences of the United States of America*, **23** (1937), p. 78.

[56] H. Rademacher. On the expansion of the partition function in a series. *Annals of Mathematics*, **44** (1943), pp. 416–422.

[57] H. Rademacher. On the partition function $p(n)$. *Proceedings of the London Mathematical Society*, **2** (1938), pp. 241–254.
[58] H. Rademacher and H. S. Zuckerman. On the Fourier coefficients of certain modular forms of positive dimension. *Annals of Mathematics*, 39(1938), pp. 433–462.
[59] Q. I. Rahman and G. Schmeisser. *Analytic theory of polynomials.* Oxford University Press, 2002.
[60] C. Schneider. Symbolic summation assists combinatorics. *Séminaire Lotharingien de Combinatoire*, **56** (2007), 1–36.
[61] J. Schur and G. Pólya. Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen. *J. Reine Angew. Math.*, **144** (1914), pp. 89–113.
[62] R. P. Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. *Ann. New York Acad. Sci.*, **576** (1989), pp. 500–535.
[63] B. Sturmfels. *Algorithms in invariant theory.* Springer Science and Business Media, 2008.
[64] E. Sussman. Rademacher Series for η-quotients. *arXiv preprint arXiv:1710.03415*, (2017).
[65] G Szegö. On an inequality of P. Turán concerning Legendre polynomials. *Bulletin of the American Mathematical Society*, **54** (1948), pp. 401–405.
[66] J. V. Uspensky. Asymptotic formulae for numerical functions which occur in the theory of partitions. *Bulletin of the Russian Academy of Sciences*, **14** (1920), pp. 199–218.
[67] I. Wagner. On a new class of Laguerre-Pólya type functions with applications in number theory. *Pacific Journal of Mathematics*, **320** (2022), pp. 177–192.
[68] L. X. W. Wang and E. Y. Y. Yang. Laguerre inequalities for discrete sequences. *Advances in Applied Mathematics*, **139** (2022), p. 102357.
[69] L. X. W. Wang and N. N. Y. Yang. Positivity of the determinants of the partition function and the overpartition function. *Mathematics of Computation*, **92** (2023), pp. 1383–1402.
[70] H. S. Zuckerman. On the coefficients of certain modular forms belonging to subgroups of the modular group. *Transactions of the American Mathematical Society*, **45** (1939), pp. 298–321.

Research Institute for Symbolic Computation, Johannes Kepler University, Altenberger Strasse 69, A-4040 Linz, Austria

Email address: Koustav.Banerjee@risc.uni-linz.ac.at, koustavbanerjee58@gmail.com