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Recurrence relations for the moments of discrete semiclassical functionals of class $s \leq 2$.

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Abstract

We study recurrence relations satisfied by the moments $\lambda_n(z)$ of discrete linear functionals whose first moment satisfies a holonomic differential equation. We consider all cases when the order of the ODE is less or equal than 3.

1 Introduction

Let $\mathbb{K}$ be a field (we mostly think of $\mathbb{K}$ as $\mathbb{R}$ or $\mathbb{C}$) and $\mathbb{N}_0$ be the set of *nonnegative integers*

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

Let $\mathbb{F} = \mathbb{K}[[z]]$ denote the ring of *formal power series* in the variable z

$$\mathbb{F} = \left\{ \sum_{n=0}^{\infty} c_n z^n : \quad c_n \in \mathbb{K} \right\},$$

and $\vartheta : \mathbb{F} \to \mathbb{F}$ be the differential operator [49, 16.8.2]

$$\vartheta = z\partial_z, \tag{1}$$

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where ∂_z is the *derivative operator* $\partial_z = \frac{\partial}{\partial z}$. We use the notation

$$(x+\mathbf{c}) = (x+c_1)\cdots(x+c_m), \quad \mathbf{c} \in \mathbb{K}^m, \quad m \in \mathbb{N},$$

and for $m = 0$ we understand that $\mathbb{K}^0 = \emptyset$ and

$$(x+\emptyset) = x.$$

We denote by $\delta_{k,n}$ the *Kronecker delta*

$$\delta_{k,n} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}, \quad k,n \in \mathbb{N}_0,$$

and say that $\{\Lambda_n\}_{n\geq 0} \subset \mathbb{K}[x]$ is a *monic basis* of $\mathbb{K}[x]$ if Λ_n is monic and $\deg(\Lambda_n) = n$ for all $n \in \mathbb{N}_0$. The *Pochhammer symbol* $(c)_n$ is defined by [49, 5.2.4]

$$(c)_0 = 1, \quad (c)_n = \prod_{j=0}^{n-1}(c+j), \quad n \in \mathbb{N}, \quad c \in \mathbb{K},$$

and for $\mathbf{c} \in \mathbb{K}^m$ we will use the notation [49, 16.1]

$$(\mathbf{c})_n = (c_1)_n \cdots (c_m)_n, \quad (\emptyset)_n = 1, \quad n \in \mathbb{N}_0.$$

In this article, we continue the work started in [21], where we studied the moments $\lambda_n = L_{p,q}[\Lambda_n]$ of linear functionals $L_{p,q} : \mathbb{K}[x] \to \mathbb{F}$ (acting on the variable x) defined by

$$L_{p,q}[u] = \sum_{x=0}^{\infty} u(x) \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad u \in \mathbb{K}[x], \tag{2}$$

and we **always** take $\mathbf{a} \in \mathbb{K}^p$, $\mathbf{b} \in \mathbb{K}^q$. It follows from (2) that the first moment $\lambda_0(z) = L[1]$ satisfies a differential equation (in the variable z) with polynomial coefficients $\Theta_{p,q}[y] = 0$, where the differential operator $\Theta_{p,q}$ is defined by

$$\Theta_{p,q} = \vartheta(\vartheta+\mathbf{b}) - z(\vartheta+\mathbf{a}),$$

and we **always** assume that x and z are **independent variables**.

The ODE $\Theta_{p,q}[y] = 0$ is the (generalized) *hypergeometric differential equation* [49, 16.8.3] of order $\mathrm{o} = \max\{p, q+1\}$, and the first moment $\lambda_0(z)$ can be represented as

$$\lambda_0(z) = {}_pF_q\left(\begin{matrix}\mathbf{a} \\ \mathbf{b}+1\end{matrix}; z\right),$$

where the (generalized) *hypergeometric function* ${}_pF_q$ is defined by [49, 16.2.1], [54],

$$ {}_pF_q \left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} ; z \right) = \sum_{x=0}^{\infty} \frac{(\mathbf{a})_x}{(\mathbf{b})_x} \frac{z^x}{x!}. $$

Functionals of the form (2) are called *discrete semiclassical* [24], [43], [47]. If we define the polynomials

$$ \sigma(x) = x(x+\mathbf{b}), \quad \tau(x) = (x+\mathbf{a}), \tag{3} $$

then the *class* s of the functional $L_{p,q}$ is given by

$$ s = \max\{\deg(\sigma) - 1, \deg(\tau) - 1\} = \mathrm{o} - 1, \tag{4} $$

and semiclassical functionals of class $s = 0$ are called *classical*. In [22], we classified all discrete semiclassical linear functionals of class $s \leq 1$ (see also [45], [52], [53]). We extended our results in [23] to the class $s = 2$.

In this paper, we will find recurrence relations for the moments of all discrete semiclassical linear functionals of class $s \leq 2$.

2 Previous results

In this section, we give a brief description of the results from [21] that will be needed in this paper. For more details and proofs, we refer the reader to [21].

Since the operator ϑ defined in (1) satisfies

$$ \vartheta[z^x f] = z^x (\vartheta + x)[f], \quad f \in \mathbb{F}, $$

it follows using linearity that for all $u \in \mathbb{K}[x]$,

$$ u(\vartheta)[z^x f] = z^x u(\vartheta + x)[f], \quad f \in \mathbb{F}. \tag{5} $$

Let $L : \mathbb{K}[x] \to \mathbb{F}$ be a *discrete functional*

$$ L[u] = \sum_{x=0}^{\infty} u(x)\rho(x) z^x, \quad u \in \mathbb{K}[x], \tag{6} $$

where $\rho : \mathbb{N}_0 \to \mathbb{K}$ is a given function. If we set $f = 1$ in (5), we get

$$ u(\vartheta)[z^x] = u(x) z^x, \quad u \in \mathbb{K}[x] $$

and therefore

$$L[u] = u(\vartheta) L[1], \quad u \in \mathbb{K}[x]. \tag{7}$$

We conclude from (7) that the *moments* $\lambda_n(z)$ of L on any monic basis $\{\Lambda_n\}_{n\geq 0}$ are **completely determined** by the first moment

$$\lambda_n = L[\Lambda_n] = \Lambda_n(\vartheta) L[1] = \Lambda_n(\vartheta)[\lambda_0], \quad n \in \mathbb{N}_0. \tag{8}$$

2.1 Newtonian bases

A convenient choice for $\{\Lambda_n\}_{n\geq 0}$ is the basis of *Newton polynomials [61], [62]*

$$\Lambda_n(x) = \prod_{k=0}^{n-1} (x - \xi_k), \quad n \in \mathbb{N}, \quad \Lambda_0(x) = 1,$$

where $\{\xi_k\}_{k\geq 0} \subset \mathbb{K}$ is a **fixed** sequence. These polynomials satisfy the $2-$term recurrence relation

$$x\Lambda_n(x) = \Lambda_{n+1}(x) + \xi_n \Lambda_n(x). \tag{9}$$

Setting $x = \vartheta$ in (9), multiplying by $\lambda_0(z)$ and using (8), we obtain.

$$\vartheta[\lambda_n] = (\mathcal{S} + \xi_n)[\lambda_n], \tag{10}$$

where $\mathcal{S}$ denotes the *shift operator* in n

$$\mathcal{S}[c_n] = c_{n+1}. \tag{11}$$

Remark 1 *From (10) and (11) we see that*

$$\vartheta[\lambda_{n+1}] = \lambda_{n+2} + \xi_{n+1}\lambda_{n+1} = \mathcal{S}[\lambda_{n+1} + \xi_n \lambda_n] = \mathcal{S}(\mathcal{S} + \xi_n)[\lambda_n], \tag{12}$$

which is ***different*** *from*

$$(\mathcal{S} + \xi_n)\mathcal{S}[\lambda_n] = (\mathcal{S} + \xi_n)[\lambda_{n+1}] = \lambda_{n+2} + \xi_n \lambda_{n+1},$$

so caution must be exercised when ξ_n depends on n.

The Newton polynomials $\Lambda_n(x)$ satisfy the *change of bases formula*

$$x^n = \sum_{i=0}^{n} \left\{ {n \atop i} \right\} \Lambda_i(x), \tag{13}$$

where the coefficients $\left\{ {n \atop i} \right\}$ satisfy the recurrence

$$\left\{ {n+1 \atop i} \right\} = \left\{ {n \atop i-1} \right\} + \xi_i \left\{ {n \atop i} \right\}, \quad \left\{ {n \atop n} \right\} = 1, \tag{14}$$

with boundary conditions

$$\left\{ {n \atop i} \right\} = 0, \quad i \notin [0, n].$$

Among all Newtonian bases, we will consider the *monomial basis* $(\xi_k = 0)$ $\Lambda_n(x) = x^n$ and the basis of *falling factorial polynomials* $(\xi_k = k)$ $\Lambda_n(x) = \phi_n(x)$. The main reason for choosing the polynomials $\phi_n(x)$ is that from their definition

$$\phi_n(x) = \prod_{k=0}^{n-1} (x-k), \quad n \in \mathbb{N}, \tag{15}$$

it follows that

$$\phi_{n+1}(x) = (x-n)\,\phi_n(x) = x\phi_n(x-1), \quad n \geq 0, \tag{16}$$

and therefore $\phi_n(x)$ is well suited for dealing with shifts in x. We will call the moments of $L_{p,q}$ on the monomial basis *standard moments*

$$\mu_n(z) = L_{p,q}[x^n], \quad n \in \mathbb{N}_0,$$

and the moments of $L_{p,q}$ on the falling factorial polynomials basis *modified moments*

$$\nu_n(z) = L_{p,q}[\phi_n], \quad n \in \mathbb{N}_0.$$

Since

$$z^n \partial_z^n [z^x] = z^n \phi_n(x)\, z^{x-n} = \phi_n(x)\, z^x,$$

we have $\nu_n(z) = z^n \partial_z^n [\nu_0]$ and using the formula [49, 16.3.1]

$$\partial_z^n \left[{}_pF_q \left(\begin{matrix} \mathbf{a} \\ \mathbf{b}+1 \end{matrix} ; z \right) \right] = \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} \, {}_pF_q \left(\begin{matrix} \mathbf{a}+n \\ \mathbf{b}+n+1 \end{matrix} ; z \right),$$

we obtain the hypergeometric representation

$$\nu_n(z) = z^n \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} \, {}_pF_q\left(\begin{matrix} \mathbf{a}+n \\ \mathbf{b}+n+1 \end{matrix} ; z \right), \quad n \in \mathbb{N}_0.$$

Multiplying the ODE $\Theta_{p,q}[y] = 0$ by ϑ^n and using (5), we get

$$\left[\vartheta^{n+1}(\vartheta+\mathbf{b}) - z(\vartheta+1)^n(\vartheta+\mathbf{a})\right][\mu_0] = 0. \tag{17}$$

Using (10) with $\xi_n = 0$ in (17) we have $\Phi_{p,q}[\mu_0] = 0$, where the *standard moments recurrence operator* $\Phi_{p,q}$ is defined by

$$\Phi_{p,q} = \mathcal{S}^{n+1}(\mathcal{S}+\mathbf{b}) - z(\mathcal{S}+1)^n(\mathcal{S}+\mathbf{a}). \tag{18}$$

The polynomials $(x+\mathbf{c})$ can be written in the monomial basis as

$$(x+\mathbf{c}) = \sum_{k=0}^{m} e_{m-k}(\mathbf{c})\, x^k, \quad \mathbf{c} \in \mathbb{K}^m, \tag{19}$$

where the *elementary symmetric polynomials* $e_n(\mathbf{c})$ are defined by the generating function [41]

$$\sum_{n=0}^{\infty} e_n(\mathbf{c})\, t^n = \prod_{i=1}^{m}(1+tc_i), \quad \mathbf{c} \in \mathbb{K}^m.$$

Using (19), we can rewrite $\Phi_{p,q}$ in extended form

$$\Phi_{p,q} = \sum_{k=0}^{q} e_{q-k}(\mathbf{b})\, \mathcal{S}^{n+k+1} - z\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{p} e_{p-j}(\mathbf{a})\, \mathcal{S}^{k+j},$$

and the equation $\Phi_{p,q}[\mu_0] = 0$ gives a recurrence for the standard moments

$$\sum_{k=0}^{q} e_{q-k}(\mathbf{b})\, \mu_{n+k+1} - z\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{p} e_{p-j}(\mathbf{a})\, \mu_{k+j} = 0$$

of order $n+s+1$, where s is the class of the functional $L_{p,q}$ defined in (4).

Similarly, multiplying the ODE $\Theta_{p,q}[y] = 0$ by $\phi_n(\vartheta-1)$ and using (5) and (16), we get

$$[(\vartheta+\mathbf{b})\,\phi_{n+1}(\vartheta) - z(\vartheta+\mathbf{a})\,\phi_n(\vartheta)][\nu_0] = 0.$$

Using (8) and (12), we conclude that $\Psi_{p,q}[\nu_n] = 0$, where the *modified moments recurrence operator* $\Psi_{p,q}$ is defined by

$$\Psi_{p,q} = \mathcal{S}(\mathcal{S}+n+\mathbf{b}) - z(\mathcal{S}+n+\mathbf{a}). \tag{20}$$

When $\xi_k = k$, the coefficients $S_{n,i} = \left\{ {n \atop i} \right\}$ defined by (14) are called *Stirling numbers of the second kind [49, 26.8].* Using (13) in (19), we obtain

$$(x+\mathbf{c}) = \sum_{k=0}^{m} e_{m-k}(\mathbf{c}) \sum_{i=0}^{k} S_{k,i}\phi_i(x),$$

and from (12) we conclude that

$$(\mathcal{S}+n+\mathbf{c}) = \sum_{k=0}^{m} e_{m-k}(\mathbf{c}+n) \sum_{i=0}^{k} S_{k,i}\mathcal{S}^i. \tag{21}$$

Using (21), we can rewrite $\Psi_{p,q}$ in extended form

$$\Psi_{p,q} = \sum_{k=0}^{q} e_{m-k}(\mathbf{b}+n+1) \sum_{i=0}^{k} S_{k,i}\mathcal{S}^{i+1} - z\sum_{k=0}^{p} e_{p-k}(\mathbf{a}+n) \sum_{i=0}^{k} S_{k,i}\mathcal{S}^i,$$

and the equation $\Psi_{p,q}[\nu_n] = 0$ gives a recurrence for the modified moments

$$\sum_{k=0}^{q} e_{m-k}(\mathbf{b}+n+1) \sum_{i=0}^{k} S_{k,i}\nu_{n+i+1} - z\sum_{k=0}^{p} e_{p-k}(\mathbf{a}+n) \sum_{i=0}^{k} S_{k,i}\nu_{n+i} = 0,$$

of **minimal** order $s+1$, where s is the class of the functional $L_{p,q}$ defined in (4).

2.2 Transformations

There are 4 canonical transformations of the functional $L_{p,q}$:

1) The *Christoffel transformation* at ω, which we define by

$$L_{p,q}^{C}[u] = \sum_{x=0}^{\infty} (x-\omega)\, u(x) \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad \omega \in \mathbb{K}, \quad u \in \mathbb{K}[x]. \tag{22}$$

Using (9) and (22) we obtain

$$\lambda_n^C = L_{p,q}^C[\Lambda_n] = L_{p,q}[(x-\omega)\Lambda_n] = \lambda_{n+1} + (\xi_n - \omega)\lambda_n,$$

and in particular

$$\nu_n^C = \nu_{n+1} + (n-\omega)\,\nu_n. \tag{23}$$

Since from (15) it follows that

$$x-\omega = -\omega \frac{(-\omega+1)_x}{(-\omega)_x}, \tag{24}$$

the first moment has the hypergeometric representation

$$\lambda_0^C(z;\omega) = -\omega \;{}_{p+1}F_{q+1}\left(\begin{array}{c} \mathbf{a}, -\omega+1 \\ \mathbf{b}+1, -\omega \end{array} ;z\right).$$

Thus, λ_0^C is a solution of the ODE $\Theta_{p,q}^C[y]=0$, where the differential operator $\Theta_{p,q}^C$ is defined by

$$\Theta_{p,q}^C = \sigma(\vartheta)(\vartheta-\omega-1) - z\tau(\vartheta)(\vartheta-\omega+1), \tag{25}$$

and the polynomials σ, τ where defined in (3).

From (18) and (25) we see that $\Phi_{p,q}^C\left[\mu_0^C\right] = 0$, where the recurrence operator $\Phi_{p,q}^C$ is defined by

$$\Phi_{p,q}^C = \mathcal{S}^{n+1}(\mathcal{S}+\mathbf{b})(\mathcal{S}-\omega-1) - z(\mathcal{S}+1)^n(\mathcal{S}+\mathbf{a})(\mathcal{S}-\omega+1).$$

Similarly, (20) and (25) give $\Psi_{p,q}^C\left[\nu_n^C\right]=0$, with

$$\Psi_{p,q}^C = \mathcal{S}(\mathcal{S}+n+\mathbf{b})(\mathcal{S}+n-\omega-1) - z(\mathcal{S}+n+\mathbf{a})(\mathcal{S}+n-\omega+1).$$

2) The *Geronimus transformation* at ω, which we define by

$$L_{p,q}^G[u] = \sum_{x=0}^{\infty} \frac{u(x)}{x-\omega} \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad \omega \in \mathbb{K}\setminus\mathbb{N}_0, \quad u \in \mathbb{K}[x]. \tag{26}$$

Using (24), we see that the first moment has the hypergeometric representation

$$\lambda_0^G(z;\omega) = -\omega^{-1} \;{}_{p+1}F_{q+1}\left(\begin{array}{c} \mathbf{a}, -\omega \\ \mathbf{b}+1, -\omega+1 \end{array} ;z\right),$$

and therefore is a solution of the ODE $\Theta_{p,q}^G[y]=0$, where the differential operator $\Theta_{p,q}^G$ is defined by

$$\Theta_{p,q}^G = \sigma(\vartheta)(\vartheta-\omega) - z\tau(\vartheta)(\vartheta-\omega) = \Theta_{p,q}(\vartheta-\omega). \tag{27}$$

From (9) and (26), we see that

$$\lambda_{n+1}^{G} + (\xi_n - \omega)\,\lambda_n^{G} = L_{p,q}^{G}\left[(x-\omega)\,\Lambda_n\right] = L_{p,q}\left[\Lambda_n\right] = \lambda_n,$$

and in particular

$$\nu_{n+1}^{G} + (n-\omega)\,\nu_n^{G} = \nu_n. \tag{28}$$

From (18) and (27) we see that $\Phi_{p,q}^{G}\left[\mu_0^{G}\right] = 0$, where the recurrence operator $\Phi_{p,q}^{G}$ is defined by

$$\Phi_{p,q}^{G} = \Phi_{p,q}\left(\mathcal{S}-\omega\right).$$

Similarly, (20) and (27) give $\Psi_{p,q}^{G}\left[\nu_n^{G}\right] = 0$, with

$$\Psi_{p,q}^{G} = \Psi_{p,q}\left(\mathcal{S}+n-\omega\right).$$

3) The *Uvarov transformation* at ω, which we define by

$$L_{p,q}^{U}[u] = L_{p,q}[u] + \eta u(\omega)\, z^{\omega}, \quad \eta \in \mathbb{K}, \quad u \in \mathbb{K}[x].$$

The differential operator $\Theta_{p,q}^{U}$ is defined by

$$\Theta_{p,q}^{U} = (\vartheta-\omega)(\vartheta-\omega-1)\,\Theta_{p,q}, \tag{29}$$

since

$$\begin{aligned}\Theta_{p,q}^{U} &= \sigma(\vartheta)(\vartheta-\omega)(\vartheta-\omega-1) - z\tau(\vartheta)(\vartheta-\omega+1)(\vartheta-\omega)\\ &= \left[\sigma(\vartheta)(\vartheta-\omega-1) - z\tau(\vartheta)(\vartheta-\omega+1)\right](\vartheta-\omega),\end{aligned}$$

and therefore

$$\Theta_{p,q}^{U}[\lambda_0] = 0, \quad \Theta_{p,q}^{U}[z^{\omega}] = 0.$$

Comparing (29) with (25) and (27), we see that

$$\Theta_{p,q}^{U} = \Theta_{p,q}^{G,C} = \Theta_{p,q}^{C,G}$$

in the sense of applying a double transformation to the operator $\Theta_{p,q}$.

From (18) and (29) we see that $\Phi_{p,q}^{U}\left[\mu_0^{U}\right] = 0$, where the recurrence operator $\Phi_{p,q}^{U}$ is defined by

$$\Phi_{p,q}^{U} = (\mathcal{S}-\omega)(\mathcal{S}-\omega-1)\,\Phi_{p,q}.$$

Similarly, (20) and (29) give $\Psi_{p,q}^{U}\left[\nu_n^{U}\right] = 0$, with

$$\Psi_{p,q}^{U} = (\mathcal{S}+n-\omega)(\mathcal{S}+n-\omega-1)\,\Psi_{p,q}.$$

Remark 2 *Because $\vartheta - \omega$ annihilates the function z^{ω}, we have $\Phi_{p,q}^{G}\left[z^{\omega}\right] = 0$ and could have defined*

$$L_{p,q}^{G}\left[u\right] = \sum_{x=0}^{\infty} \frac{u\left(x\right)}{x-\omega} \frac{\left(\mathbf{a}\right)_{x}}{\left(\mathbf{b}+1\right)_{x}} \frac{z^{x}}{x!} + \eta u\left(\omega\right) z^{\omega}, \quad \eta \in \mathbb{K},$$

as some authors do. Thus, we will not consider the double transformation $\Phi_{p,q}^{U,G}$, since we have the reduction

$$\Phi_{p,q}^{U,G} = \Phi_{p,q}^{G}.$$

If $\sigma\left(\zeta\right) = 0$, then the differential operator

$$\Theta_{p,q}^{U(\zeta)} = \left(\vartheta - \zeta - 1\right)\Theta_{p,q}$$

is called a *reduced-Uvarov transformation*, since

$$\Theta_{p,q}^{U(\zeta)} = \sigma\left(\vartheta\right)\left(\vartheta - \zeta - 1\right) - z\tau\left(\vartheta\right)\left(\vartheta - \zeta\right) = \left[\widetilde{\sigma}\left(\vartheta\right)\left(\vartheta - \zeta - 1\right) - z\tau\left(\vartheta\right)\right]\left(\vartheta - \zeta\right),$$

with $\sigma\left(\vartheta\right) = \left(\vartheta - \zeta\right)\widetilde{\sigma}\left(\vartheta\right)$, and therefore $\Theta_{p,q}^{U(\zeta)}\left[z^{\zeta}\right] = 0$. In this case, we have

$$\Phi_{p,q}^{U(\zeta)} = \left(\mathcal{S} - \zeta - 1\right)\Phi_{p,q},$$

and

$$\Psi_{p,q}^{U(\zeta)} = \left(\mathcal{S} + n - \zeta\right)\Psi_{p,q}.$$

The second possibility for a reduced-Uvarov transformation happens when $\tau\left(\zeta\right) = 0$. We now have

$$\Theta_{p,q}^{U(\zeta)} = \left(\vartheta - \zeta\right)\Theta_{p,q},$$

$$\Phi_{p,q}^{U(\zeta)} = \left(\mathcal{S} - \zeta\right)\Phi_{p,q},$$

and

$$\Psi_{p,q}^{U(\zeta)} = \left(\mathcal{S} + n - \zeta - 1\right)\Psi_{p,q}.$$

4) The *truncation transformation* at N, which we define by

$$L_{p,q}^{T}\left[u\right] = \sum_{x=0}^{N} u\left(x\right) \frac{\left(\mathbf{a}\right)_{x}}{\left(\mathbf{b}+1\right)_{x}} \frac{z^{x}}{x!}, \quad N \in \mathbb{N}_{0}, \quad u \in \mathbb{K}[x].$$

The first moment admits the hypergeometric representation

$$\lambda_0^T(z) = {}_{p+1}F_{q+1}\left(\begin{array}{c} \mathbf{a}, -N \\ \mathbf{b}+1, -N \end{array}; z\right),$$

and therefore is a solution of the ODE $\Theta_{p,q}^T[y] = 0$, where the differential operator $\Theta_{p,q}^T$ is defined by

$$\Theta_{p,q}^T = \sigma(\vartheta)(\vartheta - N - 1) - z\tau(\vartheta)(\vartheta - N) = (\vartheta - N - 1)\,\Theta_{p,q}. \tag{30}$$

From (18) and (30) we see that $\Phi_{p,q}^T\left[\mu_0^T\right] = 0$, where the recurrence operator $\Phi_{p,q}^T$ is defined by

$$\Phi_{p,q}^T = (\mathcal{S} - N - 1)\,\Phi_{p,q}.$$

Similarly, (20) and (30) give $\Psi_{p,q}^T\left[\nu_n^T\right] = 0$, with

$$\Psi_{p,q}^T = (\mathcal{S} + n - N)\,\Psi_{p,q}.$$

3 Examples

We now illustrate the application of the formulas that we have derived. We will consider all discrete semiclassical functionals of class $s \leq 2$, and also look at the subclasses obtained by applying one or more of the transformations from the previous section.

3.1 Functionals of class 0 (discrete classical functionals)

The discrete classical orthogonal polynomials (Charlier, Meixner, Krawtchouk) first appeared in the literature in the years 1905–1934, and were considered at the time as a generalization of the continuous classical polynomials (Hermite, Laguerre, Jacobi).

The last member of this class (Hahn polynomials) were introduced by Chebyshev (1875) and Hahn (1949), but we don't consider them by themselves since they are a special case ($z = 1$) of the Generalized Hahn polynomials (see Section 3.2.4).

We will use the notation $(p, q; N)$ to indicate that one of the upper parameters in the hypergeometric representation of the first moment is a negative integer $-N$, $N \in \mathbb{N}$.

For additional references, see [13], [18], [19], [32], [48], [50], [1], [60].

3.1.1 Functional of type $(0,0)$ (Charlier)

The Charlier polynomials were introduced by Carl Vilhelm Ludwig Charlier (1862–1934) in his paper [14].

Linear functional

$$L_{0,0}[u] = \sum_{x=0}^{\infty} u(x) \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{0,0} = \vartheta - z.$$

Standard moments recurrence operator

$$\Theta_{0,0} = \mathcal{S}^{n+1} - z(\mathcal{S}+1)^n .$$

Modified moments hypergeometric representation

$$\nu_n(z) = z^n \,{}_0F_0\left(\begin{matrix} - \\ - \end{matrix}; z\right) = z^n e^z. \tag{31}$$

Modified moments recurrence operator

$$\Psi_{0,0} = \mathcal{S} - z.$$

Remark 3 *The Charlier polynomials have the hypergeometric representation [49, 18.20.8]*

$$P_n^{(0,0)}(x;z) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix}; -z^{-1}\right). \tag{32}$$

3.1.2 Functional of type $(1,0)$ (Meixner)

The Meixner polynomials were introduced by Josef Meixner (1908 – 1994) in his paper [46], although Ladislav Truksa (1891–?) already considered them in his 1931 papers [56], [57], [58], [12].

Linear functional

$$L_{1,0}[u] = \sum_{x=0}^{\infty} u(x)\ (a)_x \frac{z^x}{x!}, \quad z \neq 1.$$

First moment differential operator

$$\Theta_{1,0} = \vartheta - z(\vartheta + a).$$

Standard moments recurrence operator

$$\Phi_{1,0} = \mathcal{S}^{n+1} - z\left(\mathcal{S}+1\right)^{n}\left(\mathcal{S}+a\right).$$

Modified moments hypergeometric representation

$$\nu_{n}\left(z\right) = z^{n}\left(a\right)_{n}\ {}_{1}F_{0}\left[\begin{array}{c} a+n \\ - \end{array};z\right] = z^{n}\left(a\right)_{n}\left(1-z\right)^{-a-n}. \tag{33}$$

Modified moments recurrence operator

$$\Psi_{1,0} = \mathcal{S} - z\left(\mathcal{S}+n+a\right).$$

3.1.2.1 Functional of type $(1,0;N)$ (Krawtchouk)

The Krawtchouk polynomials were introduced by Mykhailo Pylypovych Kravchuk (1892 –1942) in his paper [37]. These polynomials are a particular case of the Meixner polynomials, with $-a = N \in \mathbb{N}$.

Linear functional

$$L_{1,0;N}\left[u\right] = \sum_{x=0}^{N} u\left(x\right)\ \left(-N\right)_{x}\frac{z^{x}}{x!},\quad z\neq 1.$$

First moment differential operator

$$\Theta_{1,0;N} = \vartheta - z\left(\vartheta - N\right).$$

Standard moments recurrence

$$\Phi_{1,0;N} = \mathcal{S}^{n+1} - z\left(\mathcal{S}+1\right)^{n}\left(\mathcal{S}-N\right).$$

Modified moments

$$\nu_{n}\left(z\right) = z^{n}\left(-N\right)_{n}\left(1-z\right)^{N-n}.$$

Modified moments recurrence operator

$$\Psi_{1,0;N} = \mathcal{S} - z\left(\mathcal{S}+n-N\right).$$

3.2 Functionals of class 1

In [22] and [23], we classified the discrete semiclassical functionals of class $s = 1$. There are 4 main families and 9 subfamilies, obtained by applying transformations to the Charlier and Meixner functionals.

For additional references, see [9], [31], [3], [2], [39], [11], [51], [40], [44], [27], [34], [6], [15].

3.2.1 Functional of type $(0,1)$ (Generalized Charlier)

Linear functional

$$L_{0,1}[u] = \sum_{x=0}^{\infty} u(x) \; \frac{1}{(b+1)_x} \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{0,1} = \vartheta(\vartheta + b) - z.$$

Standard moments recurrence operator

$$\Phi_{0,1} = \mathcal{S}^{n+1}(\mathcal{S} + b) - z(\mathcal{S} + 1)^n .$$

Modified moments hypergeometric representation

$$\begin{aligned} \nu_n(z) &= \frac{z^n}{(b+1)_n} \, {}_0F_1 \left[\begin{array}{c} - \\ b+1+n \end{array} ; z \right] \\ &= z^{\frac{n-b}{2}} \Gamma(b+1) \, I_{b+n}\left(2\sqrt{z}\right), \end{aligned}$$

where $I_v(z)$ denotes the *modified Bessel function of the first kind* [49, 10.25.2].

Modified moments recurrence operator

$$\Psi_{0,1} = \mathcal{S}(\mathcal{S} + n + b) - z = \mathcal{S}^2 + (n+1+b)\mathcal{S} - z. \tag{34}$$

Remark 4 *If we write*

$$\nu_n = A^n p_n,$$

then the recurrence (34) becomes

$$p_{n+1} + \frac{(n+b)}{A} p_n - \frac{z}{A^2} p_{n-1} = 0.$$

Choosing

$$\frac{1}{A} = -2x, \quad -\frac{z}{A^2} = 1,$$

we get

$$p_{n+1} - 2(n+b)\, x p_n + p_{n-1} = 0. \tag{35}$$

The orthogonal polynomials satisfying the 3-term recurrence relation (35) with initial conditions

$$p_0 = 1, \quad p_1 = 2bx$$

are the modified Lommel polynomials *having the hypergeometric representation*

$$p_n(x) = (b)_n (2x)^n \; {}_2F_3\left(\begin{array}{c} -\frac{n}{2}, -\frac{n-1}{2} \\ b, -n, 1-b-n \end{array} ; -x^{-2} \right).$$

See [17], [28], [38], [42] .

Another possibility is to define

$$\nu_n = (-1)^n q_n,$$

where the monic polynomials $q_n(b)$ *satisfy the* 3*-term recurrence relation*

$$b q_n = q_{n+1} - n q_n + z q_{n-1}, \quad q_{-1} = 0, \quad q_0 = 1.$$

For additional references on the generalized Charlier polynomials, see [16], [35], [55], [59].

3.2.2 Functional of type $(1,1)$ (Generalized Meixner)

Linear functional

$$L_{1,1}[u] = \sum_{x=0}^{\infty} u(x) \; \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{1,1} = \vartheta(\vartheta + b) - z(\vartheta + a). \tag{36}$$

Standard moments recurrence

$$\Phi_{1,1} = \mathcal{S}^{n+1}(\mathcal{S} + b) - z(\mathcal{S} + 1)^n (\mathcal{S} + a).$$

Modified moments hypergeometric representation

$$\nu_n(z) = z^n \frac{(a)_n}{(b+1)_n} \, {}_1F_1 \left[\begin{array}{c} a+n \\ b+1+n \end{array} ; z \right].$$

Modified moments recurrence operator

$$\Psi_{1,1} = \mathcal{S}(\mathcal{S}+n+b) - z(\mathcal{S}+n+a) = \mathcal{S}^2 + (n+1+b-z)\mathcal{S} - z(n+a).$$

Remark 5 *If we define*

$$\nu_n = (-1)^n p_n,$$

then the monic polynomials $p_n(b)$ satisfy the 3*-term recurrence relation*

$$bp_n = p_{n+1} - (n-z)p_n + z(n+a-1)p_{n-1}.$$

For additional references on the generalized Meixner polynomials, see [10], [16], [29].

3.2.2.1 Christoffel Charlier functional

Linear functional

$$L_{0,0}^C[u] = \sum_{x=0}^{\infty} (x-\omega)\, u(x) \, \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{0,0}^C = \vartheta(\vartheta-\omega-1) - z(\vartheta-\omega+1),$$

which is a special case of (36) with

$$a = -\omega+1, \quad b = -\omega-1.$$

Standard moments recurrence operator

$$\Phi_{0,0}^C = \mathcal{S}^{n+1}(\mathcal{S}-\omega-1) - z(\mathcal{S}+1)^n(\mathcal{S}-\omega+1).$$

Modified moments hypergeometric representation

$$\nu_n^C(z) = (n-\omega)\, z^n \, {}_1F_1 \left(\begin{array}{c} n-\omega+1 \\ n-\omega \end{array} ; z \right).$$

Using Kummer's transformation [49, 13.2.39]

$$ {}_1F_1\left(\begin{matrix} a \\ b \end{matrix} ; z\right) = e^z \ {}_1F_1\left(\begin{matrix} b-a \\ b \end{matrix} ; -z\right), $$

we get

$$ \begin{aligned} \nu_n^C(z;\omega) &= (n-\omega)\, z^n e^z \ {}_1F_1\left(\begin{matrix} -1 \\ n-\omega \end{matrix} ; -z\right) \\ &= (n-\omega)\, z^n e^z \left(1 + \frac{z}{n-\omega}\right), \end{aligned} $$

in agreement with (23), since

$$ \nu_n^C = \nu_{n+1} + (n-\omega)\,\nu_n = (z-\omega+n)\, z^n e^z. \tag{37} $$

Modified moments recurrence operator

$$ \begin{aligned} &\Psi_{0,0}^C = \mathcal{S}\,(\mathcal{S}+n-\omega-1) - z\,(\mathcal{S}+n-\omega+1) \\ &= \mathcal{S}^2 + (n-\omega-z)\,\mathcal{S} - z\,(n-\omega+1)\,. \end{aligned} \tag{38} $$

Remark 6 *Using (37), we see that the modified moments satisfy the* first order *recurrence* $\psi_{0,0}^C[\nu] = 0$, *with*

$$ \psi_{0,0}^C = (n-\omega+z)\,\mathcal{S} - z\,(n-\omega+1+z)\,. $$

This agrees with (38), since

$$ (\mathcal{S}+n+1-\omega)\,\psi_{0,0}^C = (z+n+1-\omega)\,\Psi_{0,0}^C. $$

3.2.2.2 Geronimus Charlier functional

Linear functional

$$ L_{0,0}^G[u] = \sum_{x=0}^{\infty} \frac{u(x)}{x-\omega}\,\frac{z^x}{x!}, \quad \omega \notin \mathbb{N}_0. $$

First moment differential operator

$$ \Theta_{0,0}^G = \Theta_{0,0}^G(\vartheta-\omega) = \vartheta\,(\vartheta-\omega) - z\,(\vartheta-\omega)\,, $$

which is a special case of (36) with

$$a = -\omega, \quad b = -\omega.$$

Standard moments recurrence operator

$$\Phi_{0,0}^{G} = \Phi_{0,0}\left(\mathcal{S} - \omega\right) = \left[\mathcal{S}^{n+1} - z\left(\mathcal{S} + 1\right)^{n}\right]\left(\mathcal{S} - \omega\right).$$

Modified moments hypergeometric representation

$$\nu_n^{G}\left(z;\omega\right) = \frac{z^n}{n-\omega}\ {}_1F_1\left(\begin{array}{c} n-\omega \\ n-\omega+1 \end{array};z\right).$$

Using the identity [49, 13.6.5]

$${}_1F_1\left(\begin{array}{c} a \\ a+1 \end{array};-z\right) = az^{-a}\gamma\left(a,z\right),$$

where $\gamma\left(a,z\right)$ is the *incomplete gamma function* defined by [49, 8.2.1]

$$\gamma\left(a,z\right) = \int_0^z t^{a-1}e^{-t}dt, \quad \operatorname{Re}\left(a\right) > 0,$$

we obtain

$$\nu_n^{G}\left(z;\omega\right) = \left(-1\right)^n\left(-z\right)^{\omega}\gamma\left(n-\omega,-z\right).$$

Since the function $\gamma\left(a,z\right)$ satisfies the recurrence [49, 8.8.1]

$$\gamma\left(a+1,z\right) = a\gamma\left(a,z\right) - z^a e^{-z},$$

we see that

$$\nu_{n+1}^{G} + \left(n-\omega\right)\nu_n^{G} = z^n e^z = \nu_n,$$

in agreement with (28).

Modified moments recurrence operator

$$\Psi_{0,0}^{G} = \Psi_{0,0}\left(\mathcal{S} + n - \omega\right) = \mathcal{S}^2 + \left(n+1-\omega-z\right)\mathcal{S} - z\left(n-\omega\right).$$

3.2.2.3 Reduced-Uvarov Charlier functional

Since for the Charlier functional

$$\sigma(\vartheta) = \vartheta,$$

we will have a reduced Uvarov transformation $U(\zeta)$ for it if $\zeta = 0$.

Linear functional

$$L_{0,0}^{U(0)}[u] = \sum_{x=0}^{\infty} u(x) \ \frac{z^x}{x!} + \eta u(0).$$

First moment differential operator

$$\Theta_{0,0}^{U(0)} = (\vartheta - 1)\,\Theta_{0,0} = \vartheta(\vartheta - 1) - z\vartheta,$$

which is a special case of (36) with

$$a = 0, \quad b = -1.$$

Standard moments recurrence operator

$$\Phi_{0,0}^{U(0)} = (\mathcal{S} - 1)\,\Phi_{0,0} = \mathcal{S}^{n+1}(\mathcal{S} - 1) - z(\mathcal{S} + 1)^n \mathcal{S}.$$

Modified moments recurrence operator

$$\Psi_{0,0}^{U(0)} = (\mathcal{S} + n)\,\Psi_{0,0} = \mathcal{S}^2 + (n - z)\,\mathcal{S} - nz.$$

For additional references, see [7], [26].

3.2.2.4 Truncated Charlier functional

Linear functional

$$L_{0,0}^{T}[u] = \sum_{x=0}^{N} u(x) \ \frac{z^x}{x!}, \quad N \in \mathbb{N}_0.$$

First moment

$$\lambda_0^T(z) = \sum_{x=0}^{N} \frac{z^x}{x!} = \epsilon_N(z),$$

where $\epsilon_N(z)$ denotes the *truncated exponential series* [49, 8.4.11].

First moment differential operator

$$\Theta_{0,0}^{T} = (\vartheta - N - 1)\,\Theta_{0,0} = \vartheta\,(\vartheta - N - 1) - z\,(\vartheta - N)\,,$$

which is a special case of (36) with

$$a = -N, \quad b = -N - 1.$$

Standard moments recurrence operator

$$\Phi_{0,0}^{T} = (\mathcal{S} - N - 1)\,\Phi_{0,0} = \mathcal{S}^{n+1}\,(\mathcal{S} - N - 1) - z\,(\mathcal{S} + 1)^{n}\,(\mathcal{S} - N)\,.$$

Modified moments hypergeometric representation

$$\nu_n^{T}(z) = z^{n}\ {}_1F_1\left(\begin{array}{c} n - N \\ n - N \end{array}; z\right) = \frac{z^{n}}{(N-n)!}U\,(n - N, n - N, z)\,,$$

where $U\,(a, b, z)$ is *Tricomi's function [49, 13.2.6]*. Using the identity [49, 13.6.6]

$$U\,(a, a, z) = e^{z}\Gamma\,(1 - a, z)\,,$$

where $\Gamma\,(a, z)$ is the incomplete gamma function defined by [49, 8.6.5]

$$\Gamma\,(a, z) = \int_{z}^{\infty} t^{a-1} e^{t} dt,$$

we conclude that

$$\nu_n^{T}(z) = \frac{z^{n} e^{z}}{(N-n)!}\Gamma\,(N - n + 1, z)\,.$$

Comparing with (31), we see that

$$\nu_n^{T}(z) = \frac{\Gamma\,(N - n + 1, z)}{(N-n)!}\nu_n(z)\,, \quad 0 \leq n \leq N.$$

Modified moments recurrence operator

$$\Psi_{0,0}^{T} = (\mathcal{S} + n - N)\,\Psi_{0,0} = \mathcal{S}^{2} + (n - N - z)\,\mathcal{S} - (n - N)\,z.$$

For additional references, see [33].

3.2.3 Functional of type $(2,0;N)$ (Generalized Krawtchouk)

Linear functional

$$L_{2,0;N}[u] = \sum_{x=0}^{N} u(x)\ (-N)_x (a)_x \frac{z^x}{x!}, \quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\Theta_{2,0;N} = \vartheta - z(\vartheta - N)(\vartheta + a).$$

Standard moments recurrence operator

$$\Phi_{2,0;N} = \mathcal{S}^{n+1} - z(\mathcal{S}+1)^n (\mathcal{S} - N)(\mathcal{S} + a).$$

Modified moments hypergeometric representation

$$\nu_n(z) = z^n (-N)_n (a)_n \ {}_2F_0 \left[\begin{matrix} n-N, a+n \\ - \end{matrix} ; z \right].$$

Using the hypergeometric representation (32) of the Charlier polynomials $P_n^{(0,0)}(x;z)$, we can write

$$\nu_n(z) = z^n (-N)_n (a)_n \ P_{N-n}^{(0,0)}\left(-a-n; -z^{-1}\right).$$

Modified moments recurrence operator

$$\begin{aligned} \Psi_{2,0;N} &= \mathcal{S} - z(\mathcal{S} + n - N)(\mathcal{S} + n + a) \\ &= -z\mathcal{S}^2 + [1 - z(2n + 1 - N + a)]\mathcal{S} + z(N-n)(n+a). \end{aligned}$$

Remark 7 *If we set $z^{-1} = x$, we see that the modified moments are a family of monic orthogonal polynomials $p_n(x)$, satisfying the 3-term recurrence relation*

$$xp_n = p_{n+1} + (2n - 1 - N + a)p_n + (n + a - 1)(n - N - 1)p_{n-1}.$$

3.2.4 Functional of type $(2,1)$ (Generalized Hahn of type I)

Linear functional

$$L_{2,1}[u] = \sum_{x=0}^{\infty} u(x) \; \frac{(a_1)_x (a_2)_x}{(b+1)_x} \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{2,1} = \vartheta (\vartheta + b) - z (\vartheta + a_1)(\vartheta + a_2). \tag{39}$$

Standard moments recurrence operator

$$\Phi_{2,1} = \mathcal{S}^{n+1} (\mathcal{S} + b) - z (\mathcal{S} + 1)^n (\mathcal{S} + a_1)(\mathcal{S} + a_2).$$

Modified moments hypergeometric representation

$$\nu_n(z) = z^n \frac{(a_1)_n (a_2)_n}{(b+1)_n} \, {}_2F_1 \left[\begin{array}{c} a_1 + n, a_2 + n \\ b + 1 + n \end{array} ; z \right]. \tag{40}$$

Modified moments recurrence operator

$$\begin{aligned} \Psi_{2,1} &= \mathcal{S} (\mathcal{S} + n + b) - z (\mathcal{S} + n + a_1)(\mathcal{S} + n + a_2) \\ &= (1-z) \mathcal{S}^2 + [b + n + 1 - z (2n + 1 + a_1 + a_2)] \mathcal{S} - z (n + a_1)(n + a_2). \end{aligned} \tag{41}$$

Remark 8 *If we set $b = -x$, we see that the modified moments are a family of orthogonal polynomials $p_n(x)$, satisfying the 3-term recurrence relation*

$$\begin{aligned} x p_n &= (1-z) p_{n+1} + [n - z (2n - 1 + a_1 + a_2)] p_n \\ &\quad + z (n - 1 + a_1)(n - 1 + a_2) p_{n-1}. \end{aligned}$$

3.2.4.1 Hahn functional

When $z = 1$, the generalized Hahn functional of type I becomes the *Hahn functional* [36]. Note that in this case (40) can be reduced using the identity [49, 15.4.20]

$${}_2F_1 \left[\begin{array}{c} a_1, a_2 \\ b \end{array} ; 1 \right] = \frac{(b - a_2)_{-a_1}}{(b)_{-a_1}}, \quad \mathrm{Re}(b - a_2) > \mathrm{Re}(a_1).$$

Choosing $a_1 = -N$, $N \in \mathbb{N}$, $a_2 = a$, we get

$$\nu_n(1) = \frac{(-N)_n (a)_n}{(b+1)_n} \frac{(b+1-a)_{N-n}}{(b+1+n)_{N-n}},$$

or

$$\nu_n(1) = (-1)^n \frac{(b+1-a)_N}{(b+1)_N} \frac{(-N)_n (a)_n}{(a-b-N)_n},$$

which is a solution of (41) when $z = 1$, $a_1 = -N$, and $a_2 = a$

$$[(b-a+N-n)\mathcal{S} - (n-N)(n+a)][\nu_n(1)] = 0.$$

For additional references, see [20], [30].

3.2.4.2 Christoffel Meixner functional

Linear functional

$$L_{1,0}^C[u] = \sum_{x=0}^{\infty} (x-\omega)\, u(x)\ (a)_x \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{1,0}^C = \vartheta(\vartheta - \omega - 1) - z(\vartheta + a)(\vartheta - \omega + 1),$$

which is a special case of (39) with

$$a_1 = a, \quad a_2 = -\omega + 1, \quad b = -\omega - 1.$$

Standard moments recurrence operator

$$\Phi_{1,0}^C = \mathcal{S}^{n+1}(\mathcal{S} - \omega - 1) - z(\mathcal{S}+1)^n (\mathcal{S}+a)(\mathcal{S} - \omega + 1).$$

Modified moments hypergeometric representation

$$\nu_n^C(z) = (n-\omega)(a)_n\ z^n\ {}_2F_1\left(\begin{array}{c} n+a, n-\omega+1 \\ n-\omega \end{array}; z\right).$$

Using the identity

$${}_2F_1\left(\begin{array}{c} a, b \\ c \end{array}; z\right) = (1-z)^{c-a-b}\ {}_2F_1\left(\begin{array}{c} c-a, c-b \\ c \end{array}; z\right),$$

we get

$$\nu_n^C(z;\omega) = (n-\omega)(a)_n \ z^n (1-z)^{-a-n-1} \ {}_2F_1\left(\begin{matrix} -\omega-a,-1 \\ n-\omega \end{matrix} ;z \right)$$
$$= (n-\omega)(a)_n \ z^n (1-z)^{-a-n-1} \left[1 + \frac{(a+\omega)z}{n-\omega} \right],$$

in agreement with (23), since

$$\nu_n^C = \nu_{n+1} + (n-\omega)\nu_n = (z\omega + az + n - \omega)(1-z)^{-a-n-1}(a)_n \ z^n. \quad (42)$$

Modified moments recurrence operator

$$\Psi_{1,0}^C = \mathcal{S}(\mathcal{S}+n-\omega-1) - z(\mathcal{S}+n+a)(\mathcal{S}+n-\omega+1)$$
$$= (1-z)\mathcal{S}^2 + [n-\omega-z(2n+2+a-\omega)]\mathcal{S} - z(n+1-\omega)(n+a). \quad (43)$$

Remark 9 *From (42), we see that the modified moments satisfy the* first order *recurrence* $\psi_{1,0}^C[\nu] = 0$, *where*

$$\psi_{1,0}^C = (1-z)(n-\omega+z\omega+az)\mathcal{S} - z(n+a)(n+1-\omega+z\omega+az).$$

This agrees with the second order recurrence (43), since

$$(\mathcal{S}+n+1-\omega)\psi_{1,0}^C = (n+1-\omega+z\omega+az)\Psi_{1,0}^C.$$

3.2.4.3 Geronimus Meixner functional

Linear functional

$$L_{1,0}^G[u] = \sum_{x=0}^{\infty} \frac{u(x)}{x-\omega} \ (a)_x \ \frac{z^x}{x!}, \quad \omega \notin \mathbb{N}_0.$$

First moment differential operator

$$\Theta_{1,0}^G = \Theta_{1,0}(\vartheta-\omega) = \vartheta(\vartheta-\omega) - z(\vartheta+a)(\vartheta-\omega),$$

which is a special case of (39) with

$$a_1 = a, \quad a_2 = -\omega, \quad b = -\omega.$$

Standard moments recurrence operator

$$\Phi_{1,0}^{G} = \Phi_{1,0}\left(\mathcal{S} - \omega\right) = \left[\mathcal{S}^{n+1} - z\left(\mathcal{S} + 1\right)^{n}\left(\mathcal{S} + a\right)\right]\left(\mathcal{S} - \omega\right).$$

Modified moments hypergeometric representation

$$\nu_{n}^{G}\left(z;\omega\right) = \frac{z^{n}}{n-\omega}\left(a\right)_{n}\ {}_{2}F_{1}\left[\begin{array}{c} a+n, n-\omega \\ n-\omega+1 \end{array};z\right].$$

Using the identity [49, 8.17.7]

$${}_{2}F_{1}\left[\begin{array}{c} a, 1-b \\ a+1 \end{array};z\right] = az^{-a}B_{z}\left(a,b\right),$$

we conclude that

$$\nu_{n}^{G}\left(z;\omega\right) = \left(a\right)_{n}\ z^{\omega}B_{z}\left(n-\omega, 1-a-n\right),$$

where $B_{z}\left(a,b\right)$ is the *incomplete beta function* defined by [49, 8.17.1]

$$B_{z}\left(a,b\right) = z^{a}\int_{0}^{1} t^{a-1}\left(1-zt\right)^{b-1} dt. \tag{44}$$

Since the function $B_{z}\left(a,b\right)$ satisfies the recurrence [49, 8.17(iv)]

$$aB_{z}\left(a,b+1\right) - bB_{z}\left(a+1,b\right) = z^{a}\left(1-z\right)^{b},$$

we see that

$$\nu_{n+1}^{G} + \left(n-\omega\right)\nu_{n}^{G} = z^{n}\left(a\right)_{n}\left(1-z\right)^{-a-n} = \nu_{n},$$

in agreement with (28).

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{G} &= \Psi_{1,0}\left(\mathcal{S} + n - \omega\right) \\ &= \left(1-z\right)\mathcal{S}^{2} + \left[n-\omega+1-z\left(1+a-\omega+2n\right)\right]\mathcal{S} - z\left(n-\omega\right)\left(n+a\right).\end{aligned}$$

3.2.4.4 Reduced-Uvarov Meixner functional

Since for the Meixner functional we have

$$\sigma\left(\vartheta\right) = \vartheta, \quad \tau\left(\vartheta\right) = \vartheta + a,$$

we will have reduced cases for its Uvarov transformation $U\left(\zeta\right)$ if

$$\zeta = 0, -a.$$

i) $\zeta = 0$ Linear functional

$$L_{1,0}^{U(0)}[u] = \sum_{x=0}^{\infty} u(x)\ (a)_x\ \frac{z^x}{x!} + \eta u(0).$$

First moment differential operator

$$\Theta_{1,0}^{U(0)} = (\vartheta - 1)\,\Theta_{1,0} = \vartheta(\vartheta - 1) - z(\vartheta + a)\,\vartheta,$$

which is a special case of (39) with

$$a_1 = a, \quad a_2 = 0, \quad b = -1.$$

Standard moments recurrence operator

$$\Theta_{1,0}^{U(0)} = (\mathcal{S} - 1)\,\Theta_{1,0} = \mathcal{S}^{n+1}(\mathcal{S} - 1) - z(\mathcal{S} + 1)^n(\mathcal{S} + a)\,\mathcal{S}.$$

Modified moments recurrence operator

$$\Psi_{1,0}^{U(0)} = (\mathcal{S} + n)\,\Psi_{1,0} = (1 - z)\,\mathcal{S}^2 + [n - z(2n + 1 + a)]\,\mathcal{S} - zn(n + a).$$

For additional references, see [4], [8], [25].

ii) $\zeta = -a$ Linear functional

$$L_{1,0}^{U(-a)}[u] = \sum_{x=0}^{\infty} u(x)\ (a)_x\ \frac{z^x}{x!} + \eta u(-a)\, z^{-a}.$$

First moment differential operator

$$\Theta_{1,0}^{U(-a)} = (\vartheta + a)\,\Theta_{1,0} = \vartheta(\vartheta + a) - z(\vartheta + a)(\vartheta + a + 1),$$

which is a special case of (39) with

$$a_1 = a, \quad a_2 = a + 1, \quad b = a.$$

Standard moments recurrence operator

$$\Phi_{1,0}^{U(-a)} = (\mathcal{S} + a)\,\Phi_{1,0} = \left[\mathcal{S}^{n+1} - z(\mathcal{S} + 1)^n(\mathcal{S} + a + 1)\right](\mathcal{S} + a).$$

Modified moments' recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(-a)} &= (\mathcal{S} + n + a + 1)\,\Psi_{1,0} \\ &= (1 - z)\,\mathcal{S}^2 - (n + a + 1)\left[(2z - 1)\,\mathcal{S} + z(n + a)\right].\end{aligned}$$

3.2.4.5 Truncated Meixner functional

Linear functional

$$L_{1,0}^{T}\left[u\right]=\sum_{x=0}^{N}u\left(x\right)\ \left(a\right)_{x}\ \frac{z^{x}}{x!},\quad N\in\mathbb{N}_{0}.$$

First moment differential operator

$$\Theta_{1,0}^{T}=\left(\vartheta-N-1\right)\Theta_{1,0}=\vartheta\left(\vartheta-N-1\right)-z\left(\vartheta+a\right)\left(\vartheta-N\right),$$

which is a special case of (39) with

$$a_{1}=a,\quad a_{2}=-N,\quad b=-N-1.$$

Standard moments recurrence operator

$$\Phi_{1,0}^{T}=\left(\mathcal{S}-N-1\right)\Phi_{1,0}=\mathcal{S}^{n+1}\left(\mathcal{S}-N-1\right)-z\left(\mathcal{S}+1\right)^{n}\left(\mathcal{S}+a\right)\left(\mathcal{S}-N\right).$$

Modified moments hypergeometric representation

$$\nu_{n}^{T}\left(z\right)=\left(a\right)_{n}\ z^{n}\ {}_{2}F_{1}\left(\begin{array}{c}n-N,n+a\\ n-N\end{array};z\right).$$

Using the transformation [49, 15.8.7]

$${}_{2}F_{1}\left(\begin{array}{c}-N,b\\ c\end{array};z\right)=\frac{\left(c-b\right)_{N}}{\left(c\right)_{N}}\ {}_{2}F_{1}\left(\begin{array}{c}-N,b\\ b-c-N+1\end{array};1-z\right),$$

we obtain

$$\nu_{n}^{T}\left(z\right)=\left(a\right)_{n}\ z^{n}\frac{\left(-N-a\right)_{N-n}}{\left(n-N\right)_{N-n}}\ {}_{2}F_{1}\left(\begin{array}{c}n+a,n-N\\ n+a+1\end{array};1-z\right).$$

Since the incomplete beta function (44) has the hypergeometric representation [49, 8.17.7]

$$B_{z}\left(a,b\right)=\frac{z^{a}}{a}\ {}_{2}F_{1}\left(\begin{array}{c}1-b,a\\ a+1\end{array};z\right),$$

we conclude that

$$\nu_{n}^{T}\left(z\right)=\frac{\left(a\right)_{N+1}}{\left(N-n\right)!}z^{n}\left(1-z\right)^{-a-n}B_{1-z}\left(a+n,N-n+1\right),$$

and comparing with (33), we see that

$$\nu_n^T(z) = \frac{(a+n)_{N-n+1}}{(N-n)!} B_{1-z}(a+n, N-n+1)\, \nu_n(z) \ .$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^T &= (\mathcal{S}+n-N)\, \Psi_{1,0} \\ &= (1-z)\, \mathcal{S}^2 + [n-N-z(2n+1-N+a)]\, \mathcal{S} - z(n-N)(n+a)\,.\end{aligned}$$

3.3 Functionals of class 2

In [23], we classified the discrete semiclassical functionals of class $s=2$. There are 6 main families and 58 subfamilies, obtained by applying transformations to the functionals of class $s=1$, or double transformations to the functionals of class $s=0$.

3.3.1 Functional of type (0,2)

Linear functional

$$L_{0,2}[u] = \sum_{x=0}^{\infty} u(x) \ \frac{1}{(b_1+1)_x (b_2+1)_x} \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{0,2} = \vartheta(\vartheta+b_1)(\vartheta+b_2) - z.$$

Standard moments recurrence operator

$$\Phi_{0,2} = \mathcal{S}^{n+1}(\mathcal{S}+b_1)(\mathcal{S}+b_2) - z(\mathcal{S}+1)^n = 0.$$

Modified moments hypergeometric representation

$$\nu_n(z) = \frac{z^n}{(b_1+1)_n (b_2+1)_n} \ {}_0F_2\left[\begin{array}{c} - \\ b_1+1+n, b_2+1+n \end{array}; z\right].$$

Modified moments recurrence operator

$$\Psi_{0,2} = \mathcal{S}(\mathcal{S}+n+b_1)(\mathcal{S}+n+b_2) - z.$$

3.3.2 Functional of type (1,2)

Linear functional

$$L_{1,2}[u] = \sum_{x=0}^{\infty} u(x) \; \frac{(a)_x}{(b_1+1)_x (b_2+1)_x} \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{1,2} = \vartheta (\vartheta + b_1)(\vartheta + b_2) - z(\vartheta + a). \tag{45}$$

Standard moments recurrence operator

$$\Phi_{1,2} = \mathcal{S}^{n+1} (\mathcal{S} + b_1)(\mathcal{S} + b_2) - z(\mathcal{S}+1)^n (\mathcal{S} + a).$$

Modified moments hypergeometric representation

$$\nu_n(z) = \frac{z^n (a)_n}{(b_1+1)_n (b_2+1)_n} \; {}_1F_2 \left[\begin{array}{c} a+n \\ b_1+1+n, b_2+1+n \end{array} ; z \right].$$

Modified moments recurrence operator

$$\Psi_{1,2} = \mathcal{S}(\mathcal{S}+n+1+b_1)(\mathcal{S}+n+1+b_2) - z(\mathcal{S}+n+a).$$

3.3.2.1 Christoffel Generalized Charlier functional

Linear functional

$$L_{0,1}^{C}[u] = \sum_{x=0}^{\infty} (x-\omega) u(x) \; \frac{1}{(b+1)_x} \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{0,1}^{C} = \vartheta(\vartheta+b)(\vartheta-\omega-1) - z(\vartheta-\omega+1),$$

which is a special case of (45) with

$$a = -\omega+1, \quad b_1 = b, \quad b_2 = -\omega-1.$$

Standard moments recurrence

$$\Phi_{0,1}^{C} = \mathcal{S}^{n+1}(\mathcal{S}+b)(\mathcal{S}-\omega-1) - z(\mathcal{S}+1)^n(\mathcal{S}-\omega+1).$$

Modified moments hypergeometric representation

$$\nu_n^{C}(z) = \frac{n-\omega}{(b+1)_n} \; z^n \; {}_1F_2 \left(\begin{array}{c} n-\omega+1 \\ b+1+n, n-\omega \end{array} ; z \right).$$

Modified moments recurrence operator

$$\Psi_{0,1}^{C} = \mathcal{S}(\mathcal{S}+n+b)(\mathcal{S}+n-\omega-1) - z(\mathcal{S}+n-\omega+1).$$

3.3.2.2 Geronimus Generalized Charlier functional

Linear functional

$$L_{0,1}^{G}[u]=\sum_{x=0}^{\infty}\frac{u(x)}{x-\omega}\,\frac{1}{(b+1)_x}\,\frac{z^x}{x!},\quad \omega\notin\mathbb{N}_0.$$

First moment differential operator

$$\Theta_{0,1}^{G}=\Theta_{0,1}(\vartheta-\omega)=\vartheta(\vartheta+b)(\vartheta-\omega)-z(\vartheta-\omega),$$

which is a special case of (45) with

$$a=-\omega,\quad b_1=b,\quad b_2=-\omega.$$

Standard moments recurrence operator

$$\Phi_{0,1}^{G}=\Phi_{0,1}(\mathcal{S}-\omega)=\left[\mathcal{S}^{n+1}(\mathcal{S}+b)-z(\mathcal{S}+1)^{n}\right](\mathcal{S}-\omega).$$

Modified moments hypergeometric representation

$$\nu_n^{G}(z)=\frac{1}{n-\omega}\frac{z^n}{(b+1)_n}\;{}_1F_2\left(\begin{array}{c} n-\omega \\ b+1+n,n-\omega+1 \end{array};z\right).$$

Modified moments recurrence operator

$$\Psi_{0,1}^{G}=\Psi_{0,1}(\mathcal{S}+n-\omega)=\left[\mathcal{S}(\mathcal{S}+n+b)-z\right](\mathcal{S}+n-\omega).$$

3.3.2.3 Reduced-Uvarov Generalized Charlier functional

Since for the Generalized Charlier functional we have

$$\sigma(\vartheta)=\vartheta(\vartheta+b),$$

we will have reduced cases for its Uvarov transformation $U(\zeta)$ if

$$\zeta=0,-b.$$

i) $\zeta = 0$ Linear functional

$$L_{0,1}^{U(0)}[u] = \sum_{x=0}^{\infty} u(x) \; \frac{1}{(b+1)_x} \; \frac{z^x}{x!} + \eta u(0).$$

First moment differential operator

$$\Theta_{0,1}^{U(0)} = (\vartheta - 1)\,\Theta_{0,1} = \vartheta(\vartheta + b)(\vartheta - 1) - z\vartheta,$$

which is a special case of (45) with

$$a = 0, \quad b_1 = b, \quad b_2 = -1.$$

Standard moments recurrence operator

$$\Phi_{0,1}^{U(0)} = (\mathcal{S} - 1)\,\Phi_{0,1} = \mathcal{S}^{n+1}(\mathcal{S} + b)(\mathcal{S} - 1) - z(\mathcal{S} + 1)^n \mathcal{S}.$$

Modified moments recurrence operator

$$\Psi_{0,1}^{U(0)} = (\mathcal{S} + n)\,\Psi_{0,1} = \mathcal{S}(\mathcal{S} + n + b)(\mathcal{S} + n - 1) - z(\mathcal{S} + n).$$

ii) $\zeta = -b$ Linear functional

$$L_{0,1}^{U(-b)}[u] = \sum_{x=0}^{\infty} u(x) \; \frac{1}{(b+1)_x} \; \frac{z^x}{x!} + \eta u(-b).$$

First moment differential operator

$$\Theta_{0,1}^{U(-b)} = (\vartheta + b - 1)\,\Theta_{0,1} = \vartheta(\vartheta + b)(\vartheta + b - 1) - z(\vartheta + b),$$

which is a special case of (45) with

$$a = b, \quad b_1 = b, \quad b_2 = b - 1.$$

Standard moments recurrence operator

$$\Phi_{0,1}^{U(-b)} = (\mathcal{S} + b - 1)\,\Phi_{0,1} = \left[\mathcal{S}^{n+1}(\mathcal{S} + b - 1) - z(\mathcal{S} + 1)^n\right](\mathcal{S} + b).$$

Modified moments recurrence operator

$$\Psi_{0,1}^{U(-b)} = (\mathcal{S} + n + b)\,\Psi_{0,1} = \left[\mathcal{S}(\mathcal{S} + n + b - 1) - z\right](\mathcal{S} + n + b).$$

3.3.2.4 Truncated Generalized Charlier functional

Linear functional

$$L_{0,1}^{T}[u]=\sum_{x=0}^{N}u(x)\ \frac{1}{(b+1)_x}\frac{z^x}{x!},\quad N\in\mathbb{N}_0.$$

First moment differential operator

$$\Theta_{0,1}^{T}=(\vartheta-N-1)\,\Theta_{0,1}=\vartheta(\vartheta+b)(\vartheta-N-1)-z(\vartheta-N),$$

which is a special case of (45) with

$$a=-N,\quad b_1=b,\quad b_2=-N-1.$$

Standard moments recurrence operator

$$\Phi_{0,1}^{T}=(\mathcal{S}-N-1)\,\Phi_{0,1}=\mathcal{S}^{n+1}(\mathcal{S}+b)(\mathcal{S}-N-1)-z(\mathcal{S}+1)^{n}(\mathcal{S}-N).$$

Modified moments recurrence operator

$$\Psi_{0,1}^{T}=(\mathcal{S}+n-N)\,\Psi_{0,1}=\mathcal{S}(\mathcal{S}+n+b)(\mathcal{S}+n-N-1)-z(\mathcal{S}+n-N).$$

3.3.3 Functional of type (2,2)

Linear functional

$$L_{2,2}[u]=\sum_{x=0}^{\infty}u(x)\ \frac{(a_1)_x(a_2)_x}{(b_1+1)_x(b_2+1)_x}\frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{2,2}=\vartheta(\vartheta+b_1)(\vartheta+b_2)-z(\vartheta+a_1)(\vartheta+a_2). \tag{46}$$

Standard moments recurrence operator

$$\Phi_{2,2}=\mathcal{S}^{n+1}(\mathcal{S}+b_1)(\mathcal{S}+b_2)-z(\mathcal{S}+1)^{n}(\mathcal{S}+a_1)(\mathcal{S}+a_2).$$

Modified moments hypergeometric representation

$$\nu_n(z)=\frac{z^n(a_1)_n(a_2)_n}{(b_1+1)_n(b_2+1)_n}\ {}_2F_2\left[\begin{matrix}a_1+n,a_2+n\\ b_1+1+n,b_2+1+n\end{matrix};z\right].$$

Modified moments recurrence operator

$$\Psi_{2,2}=\mathcal{S}(\mathcal{S}+n+b_1)(\mathcal{S}+n+b_2)-z(\mathcal{S}+n+a_1)(\mathcal{S}+n+a_2).$$

3.3.3.1 Uvarov Charlier functional

Linear functional

$$L_{0,0}^{U}[u]=\sum_{x=0}^{\infty}u(x)\ \frac{z^{x}}{x!}+\eta u(\omega)z^{\omega},\quad \omega\neq 0.$$

First moment differential operator

$$\begin{aligned}\Theta_{0,0}^{U}&=(\vartheta-\omega)(\vartheta-\omega-1)\Theta_{0,0}\\&=\vartheta(\vartheta-\omega)(\vartheta-\omega-1)-z(\vartheta-\omega+1)(\vartheta-\omega),\end{aligned}$$

which is a special case of (46) with

$$a_1=-\omega+1,\quad a_2=-\omega,\quad b_1=-\omega,\quad b_2=-\omega-1.$$

Standard moments recurrence operator

$$\begin{aligned}&\Phi_{0,0}^{U}=(\mathcal{S}-\omega)(\mathcal{S}-\omega-1)\Phi_{0,0}\\&=\left[\mathcal{S}^{n+1}(\mathcal{S}-\omega-1)-z(\mathcal{S}+1)^{n}(\mathcal{S}-\omega+1)\right](\mathcal{S}-\omega).\end{aligned}\tag{47}$$

Modified moments recurrence operator

$$\begin{aligned}&\Psi_{0,0}^{U}=(\mathcal{S}+n-\omega)(\mathcal{S}+n-\omega+1)\Psi_{0,0}\\&=\left[\mathcal{S}(\mathcal{S}+n-\omega-1)-z(\mathcal{S}+n-\omega+1)\right](\mathcal{S}+n-\omega).\end{aligned}\tag{48}$$

3.3.3.2 Double Christoffel Charlier functional

Linear functional

$$L_{0,0}^{C^2}[u]=\sum_{x=0}^{\infty}(x-\omega_1)(x-\omega_2)u(x)\ \frac{z^{x}}{x!}.$$

First moment differential operator

$$\Theta_{0,0}^{C^2}=\vartheta(\vartheta-\omega_1-1)(\vartheta-\omega_2-1)-z(\vartheta-\omega_1+1)(\vartheta-\omega_2+1),$$

which is a special case of (46) with

$$a_1=-\omega_1+1,\quad a_2=-\omega_2+1,\quad b_1=-\omega_1-1,\quad b_2=-\omega_2-1.$$

Standard moments recurrence operator

$$\Phi_{0,0}^{C^2}=\mathcal{S}^{n+1}(\mathcal{S}-\omega_1-1)(\mathcal{S}-\omega_2-1)-z(\mathcal{S}+1)^{n}(\mathcal{S}-\omega_1+1)(\mathcal{S}-\omega_2+1).$$

Modified moments

$$\begin{aligned}\nu_n^{C^2} &= \nu_{n+1}^C + (n-\omega_2)\,\nu_n^C \\ &= \nu_{n+2} + (2n+1-\omega_1-\omega_2)\,\nu_{n+1} + (n-\omega_1)(n-\omega_2)\,\nu_n \\ &= \left[z^2 + (2n+1-\omega_1-\omega_2)\,z + (n-\omega_2)(n-\omega_1)\right] z^n e^z.\end{aligned}$$

Modified moments hypergeometric representation

$$\nu_n^{C^2}(z) = (n-\omega_1)(n-\omega_2)\ z^n\ {}_2F_2\left[\begin{array}{c} n-\omega_1+1, n-\omega_2+1 \\ n-\omega_1, n-\omega_2 \end{array}; z\right].$$

Modified moments recurrence operator

$$\Psi_{0,0}^{C^2} = \mathcal{S}(\mathcal{S}+n-\omega_1-1)(\mathcal{S}+n-\omega_2-1) - z(\mathcal{S}+n-\omega_1+1)(\mathcal{S}+n-\omega_2+1).$$

3.3.3.3 Geronimus Christoffel Charlier functional

Linear functional

$$L_{0,0}^{G,C}[u] = \sum_{x=0}^{\infty} \frac{x-\omega_1}{x-\omega_2} u(x)\ \frac{z^x}{x!}, \quad \omega_1 \neq \omega_2, \quad \omega_2 \notin \mathbb{N}_0.$$

First moment differential operator

$$\Theta_{0,0}^{G,C} = \Theta_{0,0}^{C}(\vartheta-\omega_2) = \vartheta(\vartheta-\omega_1-1)(\vartheta-\omega_2) - z(\vartheta-\omega_1+1)(\vartheta-\omega_2),$$

which is a special case of (46) with

$$a_1 = -\omega_1+1, \quad a_2 = -\omega_2, \quad b_1 = -\omega_1-1, \quad b_2 = -\omega_2.$$

Standard moments recurrence operator

$$\Phi_{0,0}^{G,C} = \Phi_{0,0}^{C}(\mathcal{S}-\omega_2) = \left[\mathcal{S}^{n+1}(\mathcal{S}-\omega_1-1) - z(\mathcal{S}+1)^n(\mathcal{S}-\omega_1+1)\right](\mathcal{S}-\omega_2).$$

Modified moments hypergeometric representation

$$\nu_n^{G,C}(z) = \frac{n-\omega_1}{n-\omega_2}\ z^n\ {}_2F_2\left[\begin{array}{c} n-\omega_1+1, n-\omega_2 \\ n-\omega_1, n-\omega_2+1 \end{array}; z\right].$$

Modified moments recurrence operator

$$\Psi_{0,0}^{G,C} = \Psi_{0,0}^{C}(\mathcal{S}+n-\omega_2) = \left[\mathcal{S}(\mathcal{S}+n-\omega_1-1) - z(\mathcal{S}+n-\omega_1+1)\right](\mathcal{S}+n-\omega_2).$$

3.3.3.4 Reduced-Uvarov Christoffel Charlier functional

Since for the Christoffel Charlier functional we have

$$\sigma(\vartheta) = \vartheta(\vartheta - \omega - 1), \quad \tau(\vartheta) = \vartheta - \omega + 1,$$

we will have reduced cases for its Uvarov transformation $U(\zeta)$ if

$$\zeta = 0, \omega + 1, \omega - 1.$$

i) $\zeta = 0$

Linear functional

$$L_{0,0}^{U(0),C}[u] = \sum_{x=0}^{\infty} (x - \omega) u(x) \ \frac{z^x}{x!} + \eta u(0).$$

First moment differential operator

$$\Theta_{0,0}^{U(0),C} = (\vartheta - 1)\Theta_{0,0}^{C} = \vartheta(\vartheta - \omega - 1)(\vartheta - 1) - z(\vartheta - \omega + 1)\vartheta,$$

which is a special case of (46) with

$$a_1 = -\omega + 1, \quad a_2 = 0, \quad b_1 = -\omega - 1, \quad b_2 = -1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{0,0}^{U(0),C} &= (\mathcal{S} - 1)\Phi_{0,0}^{C} = \mathcal{S}^{n+1}(\mathcal{S} - \omega - 1)(\mathcal{S} - 1) \\ &- z(\mathcal{S} + 1)^n (\mathcal{S} - \omega + 1)\mathcal{S}.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{0,0}^{U(0),C} &= (\mathcal{S} + n)\Psi_{0,0}^{C} = \mathcal{S}(\mathcal{S} + n - \omega - 1)(\mathcal{S} + n - 1) \\ &- z(\mathcal{S} + n - \omega + 1)(\mathcal{S} + n).\end{aligned}$$

ii) $\zeta = \omega + 1$

Linear functional

$$L_{0,0}^{U(\omega+1),C}[u] = \sum_{x=0}^{\infty} (x - \omega) u(x) \ \frac{z^x}{x!} + \eta u(\omega + 1) z^{\omega+1}.$$

First moment differential operator

$$\begin{aligned}\Theta_{0,0}^{U(\omega+1),C} &= (\vartheta - \omega - 2)\Theta_{0,0}^{C} = \vartheta(\vartheta - \omega - 1)(\vartheta - \omega - 2) \\ &- z(\vartheta - \omega + 1)(\vartheta - \omega - 1),\end{aligned}$$

which is a special case of (46) with

$$a_1 = -\omega + 1, \quad a_2 = -\omega - 1, \quad b_1 = -\omega - 1, \quad b_2 = -\omega - 2.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{0,0}^{U(\omega+1),C} &= (\mathcal{S} - \omega - 2)\, \Phi_{0,0}^{C} \\ &= \left[\mathcal{S}^{n+1} (\mathcal{S} - \omega - 2) - z\, (\mathcal{S} + 1)^{n} (\mathcal{S} - \omega + 1)\right] (\mathcal{S} - \omega - 1)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{0,0}^{U(\omega+1),C} &= (\mathcal{S} + n - \omega - 1)\, \Psi_{0,0}^{C} \\ &= \left[\mathcal{S}\, (\mathcal{S} + n - \omega - 2) - z\, (\mathcal{S} + n - \omega + 1)\right] (\mathcal{S} + n - \omega - 1)\,.\end{aligned}$$

iii) $\zeta = \omega - 1$
Linear functional

$$L_{0,0}^{U(\omega-1),C}[u] = \sum_{x=0}^{\infty} (x - \omega)\, u(x)\, \frac{z^x}{x!} + \eta u(\omega - 1)\, z^{\omega - 1}.$$

First moment differential operator

$$\Theta_{0,0}^{U(\omega-1),C} = (\vartheta - \omega + 1)\, \Theta_{0,0}^{C} = \vartheta\, (\vartheta - \omega - 1)(\vartheta - \omega + 1) - z\, (\vartheta - \omega + 1)(\vartheta - \omega + 2)\,,$$

which is a special case of (46) with

$$a_1 = -\omega + 1, \quad a_2 = -\omega + 2, \quad b_1 = -\omega - 1, \quad b_2 = -\omega + 1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{0,0}^{U(\omega-1),C} &= (\mathcal{S} - \omega + 1)\, \Phi_{0,0}^{C} \\ &= \left[\mathcal{S}^{n+1} (\mathcal{S} - \omega - 1) - z\, (\mathcal{S} + 1)^{n} (\mathcal{S} - \omega + 2)\right] (\mathcal{S} - \omega + 1)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{0,0}^{U(\omega-1),C} &= (\mathcal{S} + n - \omega + 2)\, \Psi_{0,0}^{C} \\ &= \left[\mathcal{S}\, (\mathcal{S} + n - \omega - 1) - z\, (\mathcal{S} + n - \omega + 2)\right] (\mathcal{S} + n - \omega + 1)\,.\end{aligned}$$

3.3.3.5 Truncated Christoffel Charlier functional

Linear functional

$$L_{0,0}^{T,C}[u] = \sum_{x=0}^{N} (x-\omega)\, u(x)\ \frac{z^x}{x!}, \quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{0,0}^{T,C} &= (\vartheta - N - 1)\,\Theta_{0,0}^{C} = \vartheta\,(\vartheta - \omega - 1)\,(\vartheta - N - 1)\\ &\quad - z\,(\vartheta - \omega + 1)\,(\vartheta - N),\end{aligned}$$

which is a special case of (46) with

$$a_1 = -\omega + 1, \quad a_2 = -N, \quad b_1 = -\omega - 1, \quad b_2 = -N - 1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{0,0}^{T,C} &= (\mathcal{S} - N - 1)\,\Phi_{0,0}^{C} = \mathcal{S}^{n+1}\,(\mathcal{S} - \omega - 1)\,(\mathcal{S} - N - 1)\\ &\quad - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} - \omega + 1)\,(\mathcal{S} - N).\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{0,0}^{T,C} &= (\mathcal{S} + n - N)\,\Psi_{0,0}^{C} = \mathcal{S}\,(\mathcal{S} + n - \omega - 1)\,(\mathcal{S} + n - N - 1)\\ &\quad - z\,(\mathcal{S} + n - \omega + 1)\,(\mathcal{S} + n - N).\end{aligned}$$

3.3.3.6 Double Geronimus Charlier functional

Linear functional

$$L_{0,0}^{G^2}[u] = \sum_{x=0}^{\infty} \frac{1}{(x-\omega_1)\,(x-\omega_2)} u(x)\ \frac{z^x}{x!}, \quad \omega_1, \omega_2 \notin \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{0,0}^{G^2} &= \Theta_{0,0}^{G}\,(\vartheta - \omega_2) = \Theta_{0,0}\,(\vartheta - \omega_1)\,(\vartheta - \omega_2)\\ &= \vartheta\,(\vartheta - \omega_1)\,(\vartheta - \omega_2) - z\,(\vartheta - \omega_1)\,(\vartheta - \omega_2),\end{aligned}$$

which is a special case of (46) with

$$a_1 = -\omega_1, \quad a_2 = -\omega_2, \quad b_1 = -\omega_1, \quad b_2 = -\omega_2.$$

Standard moments recurrence operator

$$\Phi_{0,0}^{G^2} = \Phi_{0,0}^{G}\left(\mathcal{S}-\omega_2\right) = \Phi_{0,0}\left(\mathcal{S}-\omega_1\right)\left(\mathcal{S}-\omega_2\right)$$
$$= \left[\mathcal{S}^{n+1} - z\left(\mathcal{S}+1\right)^n\right]\left(\mathcal{S}-\omega_1\right)\left(\mathcal{S}-\omega_2\right).$$

Modified moments hypergeometric representation

$$\nu_n^{G^2}(z) = \frac{z^n}{\left(n-\omega_1\right)\left(n-\omega_2\right)}\ {}_2F_2\left[\begin{array}{c} n-\omega_1, n-\omega_2 \\ n-\omega_1+1, n-\omega_2+1 \end{array}; z\right].$$

Modified moments recurrence operator

$$\Psi_{0,0}^{G^2} = \Psi_{0,0}^{G}\left(\mathcal{S}+n-\omega_2\right) = \Psi_{0,0}\left(\mathcal{S}+n-\omega_1\right)\left(\mathcal{S}+n-\omega_2\right)$$
$$= \left(\mathcal{S}-z\right)\left(\mathcal{S}+n-\omega_1\right)\left(\mathcal{S}+n-\omega_2\right).$$

3.3.3.7 Reduced-Uvarov Geronimus Charlier functional

Since for the Geronimus Charlier functional we have

$$\sigma(\vartheta) = \vartheta\left(\vartheta-\omega\right), \quad \tau(\vartheta) = \vartheta-\omega, \quad \omega \notin \mathbb{N}_0,$$

we will have a reduced case for its Uvarov transformation $U(\zeta)$ if $\zeta = 0$.

Linear functional

$$L_{0,0}^{U(0),G}[u] = \sum_{x=0}^{\infty} \frac{u(x)}{x-\omega}\ \frac{z^x}{x!} + \eta u(0), \quad \omega \notin \mathbb{N}_0.$$

First moment differential operator

$$\Theta_{0,0}^{U(0),G} = (\vartheta-1)\,\Theta_{0,0}^{G} = (\vartheta-1)\,\Theta_{0,0}(\vartheta-\omega)$$
$$= \vartheta(\vartheta-1)(\vartheta-\omega) - z\vartheta(\vartheta-\omega),$$

which is a special case of (46) with

$$a_1 = 0, \quad a_2 = -\omega, \quad b_1 = -1, \quad b_2 = -\omega.$$

Standard moments recurrence operator

$$\Phi_{0,0}^{U(0),G} = (\mathcal{S}-1)\,\Phi_{0,0}^{G} = (\mathcal{S}-1)\,\Phi_{0,0}(\mathcal{S}-\omega)$$
$$= \left[\mathcal{S}^{n+1}(\mathcal{S}-1) - z(\mathcal{S}+1)^n\,\mathcal{S}\right](\mathcal{S}-\omega).$$

Modified moments recurrence operator

$$\Psi_{0,0}^{U(0),G} = (\mathcal{S}+n)\,\Psi_{0,0}^{G} = (\mathcal{S}+n)\,\Psi_{0,0}(\mathcal{S}+n-\omega)$$
$$= \left[\mathcal{S}(\mathcal{S}+n-1) - z(\mathcal{S}+n)\right](\mathcal{S}+n-\omega).$$

3.3.3.8 Truncated Geronimus Charlier functional

Linear functional

$$L_{0,0}^{T,G}[u] = \sum_{x=0}^{N} \frac{u(x)}{x-\omega} \frac{z^x}{x!}, \quad N \in \mathbb{N}_0, \quad \omega \notin \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{0,0}^{T,G} &= (\vartheta - N - 1)\,\Theta_{0,0}^{G} = (\vartheta - N - 1)\,\Theta_{0,0}(\vartheta - \omega) \\ &= \vartheta(\vartheta - N - 1)(\vartheta - \omega) - z(\vartheta - N)(\vartheta - \omega),\end{aligned}$$

which is a special case of (46) with

$$a_1 = -N, \quad a_2 = -\omega, \quad b_1 = -N-1, \quad b_2 = -\omega.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{0,0}^{T,G} &= (\mathcal{S} - N - 1)\,\Phi_{0,0}^{G} = (\mathcal{S} - N - 1)\,\Phi_{0,0}(\mathcal{S} - \omega) \\ &= \left[\mathcal{S}^{n+1}(\mathcal{S} - N - 1) - z(\mathcal{S}+1)^n(\mathcal{S} - N)\right](\mathcal{S} - \omega).\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{0,0}^{T,G} &= (\mathcal{S} + n - N)\,\Psi_{0,0}^{G} = (\mathcal{S} + n - N)\,\Psi_{0,0}(\mathcal{S} + n - \omega) \\ &= \left[\mathcal{S}(\mathcal{S} + n - N - 1) - z(\mathcal{S} + n - N)\right](\mathcal{S} + n - \omega).\end{aligned}$$

3.3.3.9 Double Uvarov Charlier functional

Since for the Reduced Uvarov Charlier functional we have

$$\sigma_{0,0}^{U(0)}(\vartheta) = \vartheta(\vartheta - 1),$$

we will have a reduced case for its Uvarov transformation $U(\zeta)$ if $\zeta = 1$.

Linear functional

$$L_{0,0}^{U^2(1,0)}[u] = \sum_{x=0}^{\infty} u(x) \frac{z^x}{x!} + \eta_1 u(0) + \eta_2 u(1)\,z.$$

First moment differential operator

$$\begin{aligned}\Theta_{0,0}^{U^2(1,0)} &= (\vartheta - 2)\,\Theta_{0,0}^{U(0)} = (\vartheta - 2)(\vartheta - 1)\,\Theta_{0,0} \\ &= \vartheta(\vartheta - 1)(\vartheta - 2) - z\vartheta(\vartheta - 1),\end{aligned}$$

which is a special case of (47) with $\omega = 1$.

Standard moments recurrence operator

$$\begin{aligned}\Phi_{0,0}^{U^2(1,0)} &= (\mathcal{S}-2)\,\Phi_{0,0}^{U(0)} = (\mathcal{S}-2)\,(\mathcal{S}-1)\,\Phi_{0,0}\\ &= \left[\mathcal{S}^{n+1}\,(\mathcal{S}-2) - z\,(\mathcal{S}+1)^n\,\mathcal{S}\right](\mathcal{S}-1)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}&\Psi_{0,0}^{U^2(1,0)} = (\mathcal{S}+n-1)\,\Psi_{0,0}^{U(0)} = (\mathcal{S}+n-1)\,(\mathcal{S}+n)\,\Psi_{0,0}\\ &= \left[\mathcal{S}\,(\mathcal{S}+n-2) - z\,(\mathcal{S}+n)\right](\mathcal{S}+n-1)\,.\end{aligned} \tag{49}$$

3.3.3.10 Reduced-Uvarov Truncated Charlier functional

Since for the Truncated Charlier functional we have

$$\sigma(\vartheta) = \vartheta\,(\vartheta - N - 1)\,,\quad \tau(\vartheta) = \vartheta - N,\quad N \in \mathbb{N}_0,$$

we will have reduced cases for its Uvarov transformation $U(\zeta)$ if

$$\zeta = 0, N+1, N.$$

i) $\zeta = 0$

Linear functional

$$L_{0,0}^{U(0),T}[u] = \sum_{x=0}^{N} u(x)\,\frac{z^x}{x!} + \eta u(0)\,,\quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{0,0}^{U(0),T} &= (\vartheta-1)\,\Theta_{0,0}^{T} = (\vartheta-1)\,(\vartheta-N-1)\,\Theta_{0,0}\\ &= \vartheta\,(\vartheta-N-1)\,(\vartheta-1) - z\,(\vartheta-N)\,\vartheta,\end{aligned}$$

which is a special case of (46) with

$$a_1 = -N,\quad a_2 = 0,\quad b_1 = -N-1,\quad b_2 = -1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{0,0}^{U(0),T} &= (\mathcal{S}-1)\,\Phi_{0,0}^{T} = (\mathcal{S}-1)\,(\mathcal{S}-N-1)\,\Phi_{0,0}\\ &= \mathcal{S}^{n+1}\,(\mathcal{S}-1)\,(\mathcal{S}-N-1) - z\,(\mathcal{S}+1)^n\,\mathcal{S}\,(\mathcal{S}-N)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{0,0}^{U(0),T} &= (\mathcal{S}+n)\,\Psi_{0,0}^{T} = (\mathcal{S}+n)\,(\mathcal{S}+n-N)\,\Psi_{0,0} \\ &= \mathcal{S}\,(\mathcal{S}+n-1)\,(\mathcal{S}+n-N-1) - z\,(\mathcal{S}+n)\,(\mathcal{S}+n-N)\,.\end{aligned}$$

ii) $\zeta = N+1$
Linear functional

$$L_{0,0}^{U(N+1),T}[u] = \sum_{x=0}^{N} u(x)\ \frac{z^x}{x!} + \eta u(N+1)\,z^{N+1}, \quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{0,0}^{U(N+1),T} &= (\vartheta-N-2)\,\Theta_{0,0}^{T} = \vartheta\,(\vartheta-N-1)\,(\vartheta-N-2) \\ &\quad - z\,(\vartheta-N)\,(\vartheta-N-1)\,,\end{aligned}$$

which is a special case of (47) with $\omega = N+1$.

Standard moments recurrence operator

$$\begin{aligned}\Phi_{0,0}^{U(N+1),T} &= (\mathcal{S}-N-2)\,\Phi_{0,0}^{T} = (\mathcal{S}-N-2)\,(\mathcal{S}-N-1)\,\Phi_{0,0} \\ &= \left[\mathcal{S}^{n+1}\,(\mathcal{S}-N-2) - z\,(\mathcal{S}+1)^{n}\,(\mathcal{S}-N)\right](\mathcal{S}-N-1)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{0,0}^{U(N+1),T} &= (\mathcal{S}+n-N-1)\,\Psi_{0,0}^{T} = (\mathcal{S}+n-N-1)\,(\mathcal{S}+n-N)\,\Psi_{0,0} \\ &= \left[\mathcal{S}\,(\mathcal{S}+n-N-2) - z\,(\mathcal{S}+n-N)\right](\mathcal{S}+n-N-1)\,.\end{aligned}$$

iii) $\zeta = N$
Linear functional

$$L_{0,0}^{U(N),T}[u] = \sum_{x=0}^{N} u(x)\ \frac{z^x}{x!} + \eta u(N)\,, \quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{0,0}^{U(N),T} &= (\vartheta-N)\,\Theta_{0,0}^{T} = (\vartheta-N)\,(\vartheta-N-1)\,\Theta_{0,0} \\ &= \vartheta\,(\vartheta-N-1)\,(\vartheta-N) - z\,(\vartheta-N)\,(\vartheta-N+1)\,,\end{aligned}$$

which is a special case of (47) with $\omega = N$.

Standard moments recurrence operator

$$\begin{aligned}\Phi_{0,0}^{U(N),T} &= (\mathcal{S}-N)\,\Phi_{0,0}^{T} = (\mathcal{S}-N)\,(\mathcal{S}-N-1)\,\Phi_{0,0} \\ &= \left[\mathcal{S}^{n+1}\,(\mathcal{S}-N-1) - z\,(\mathcal{S}+1)^{n}\,(\mathcal{S}-N+1)\right](\mathcal{S}-N)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{0,0}^{U(N),T} &= (\mathcal{S}+n-N+1)\,\Psi_{0,0}^{T} = (\mathcal{S}+n-N+1)\,(\mathcal{S}+n-N)\,\Psi_{0,0} \\ &= \left[\mathcal{S}\,(\mathcal{S}+n-N-1) - z\,(\mathcal{S}+n-N+1)\right](\mathcal{S}+n-N)\,.\end{aligned}$$

3.3.3.11 Christoffel Generalized Meixner functional

Linear functional

$$L_{1,1}^{C}\,[u] = \sum_{x=0}^{\infty} (x-\omega)\,u\,(x)\;\frac{(a)_x}{(b+1)_x}\frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{0,1}^{C} = \vartheta\,(\vartheta+b)\,(\vartheta-\omega-1) - z\,(\vartheta+a)\,(\vartheta-\omega+1)\,,$$

which is a special case of (46) with

$$a_1 = a, \quad a_2 = -\omega+1, \quad b_1 = b, \quad b_2 = -\omega-1.$$

Standard moments recurrence

$$\Phi_{1,1}^{C} = \mathcal{S}^{n+1}\,(\mathcal{S}+b)\,(\mathcal{S}-\omega-1) - z\,(\mathcal{S}+1)^{n}\,(\mathcal{S}+a)\,(\mathcal{S}-\omega+1)\,.$$

Modified moments hypergeometric representation

$$\nu_n^{C}\,(z) = (n-\omega)\,\frac{(a)_n}{(b+1)_n}\;z^n\;{}_2F_2\left(\begin{array}{c} a+n, n-\omega+1 \\ b+1+n, n-\omega \end{array};z\right).$$

Modified moments recurrence operator

$$\Psi_{1,1}^{C} = \mathcal{S}\,(\mathcal{S}+n+b)\,(\mathcal{S}+n-\omega-1) - z\,(\mathcal{S}+n+a)\,(\mathcal{S}+n-\omega-1)\,.$$

3.3.3.12 Geronimus Generalized Meixner functional

Linear functional

$$L_{1,1}^{G}\left[u\right]=\sum_{x=0}^{\infty}\frac{u\left(x\right)}{x-\omega}\ \frac{\left(a\right)_{x}}{\left(b+1\right)_{x}}\ \frac{z^{x}}{x!},\quad \omega\notin\mathbb{N}_{0}.$$

First moment differential operator

$$\Theta_{1,1}^{G}=\Theta_{1,1}\left(\vartheta-\omega\right)=\vartheta\left(\vartheta+b\right)\left(\vartheta-\omega\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega\right),$$

which is a special case of (45) with

$$a_{1}=a,\quad a_{2}=-\omega,\quad b_{1}=b,\quad b_{2}=-\omega.$$

Standard moments recurrence operator

$$\Phi_{1,1}^{G}=\Phi_{1,1}\left(\mathcal{S}-\omega\right)=\left[\mathcal{S}^{n+1}\left(\mathcal{S}+b\right)-z\left(\mathcal{S}+1\right)^{n}\left(\mathcal{S}+a\right)\right]\left(\mathcal{S}-\omega\right)$$

Modified moments hypergeometric representation

$$\nu_{n}^{G}\left(z\right)=\frac{1}{n-\omega}\frac{\left(a\right)_{n}}{\left(b+1\right)_{n}}z^{n}\ {}_{2}F_{2}\left(\begin{array}{c}a+n,n-\omega\\ b+1+n,n-\omega+1\end{array};z\right).$$

Modified moments recurrence operator

$$\Psi_{1,1}^{G}=\Psi_{1,1}\left(\mathcal{S}+n-\omega\right)=\left[\mathcal{S}\left(\mathcal{S}+n+b\right)-z\left(\mathcal{S}+n+a\right)\right]\left(\mathcal{S}+n-\omega\right).$$

3.3.3.13 Reduced-Uvarov Generalized Meixner functional

Since for the generalized Meixner functional we have

$$\sigma\left(\vartheta\right)=\vartheta\left(\vartheta+b\right),\quad \tau\left(\vartheta\right)=\vartheta+a,$$

we will have reduced cases for its Uvarov transformation $U\left(\zeta\right)$ if

$$\zeta=0,-b,-a.$$

i) $\zeta=0$

Linear functional

$$L_{1,1}^{U(0)}\left[u\right]=\sum_{x=0}^{\infty}u\left(x\right)\ \frac{\left(a\right)_{x}}{\left(b+1\right)_{x}}\frac{z^{x}}{x!}+\eta u\left(0\right).$$

First moment differential operator

$$\Theta_{1,1}^{U(0)} = (\vartheta - 1)\,\Theta_{1,1} = \vartheta\,(\vartheta + b)\,(\vartheta - 1) - z\,(\vartheta + a)\,\vartheta,$$

which is a special case of (46) with

$$a_1 = a, \quad a_2 = 0, \quad b_1 = b, \quad b_2 = -1.$$

Standard moments recurrence operator

$$\Phi_{1,1}^{U(0)} = (\mathcal{S} - 1)\,\Phi_{1,1} = \mathcal{S}^{n+1}\,(\mathcal{S} + b)\,(\mathcal{S} - 1) - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} + a)\,\mathcal{S}.$$

Modified moments recurrence operator

$$\Psi_{1,1}^{U(0)} = (\mathcal{S} + n)\,\Psi_{1,1} = \mathcal{S}\,(\mathcal{S} + n + b)\,(\mathcal{S} + n - 1) - z\,(\mathcal{S} + n + a)\,(\mathcal{S} + n)\,.$$

ii) $\zeta = -a$
Linear functional

$$L_{1,1}^{U(-a)}[u] = \sum_{x=0}^{\infty} u\,(x)\,\,\frac{(a)_x}{(b+1)_x}\frac{z^x}{x!} + \eta u\,(-a)\,z^{-a}.$$

First moment differential operator

$$\Theta_{1,1}^{U(-a)} = (\vartheta + a)\,\Theta_{1,1} = \vartheta\,(\vartheta + b)\,(\vartheta + a) - z\,(\vartheta + a)\,(\vartheta + a + 1)\,,$$

which is a special case of (46) with

$$a_1 = a, \quad a_2 = a + 1, \quad b_1 = b, \quad b_2 = a.$$

Standard moments recurrence operator

$$\Phi_{1,1}^{U(-a)} = (\mathcal{S} + a)\,\Phi_{1,1} = \left[\mathcal{S}^{n+1}\,(\mathcal{S} + b) - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} + a + 1)\right](\mathcal{S} + a)\,.$$

Modified moments recurrence operator

$$\Psi_{1,1}^{U(-a)} = (\mathcal{S} + n + a + 1)\,\Psi_{1,1} = [\mathcal{S}\,(\mathcal{S} + n + b) - z\,(\mathcal{S} + n + a + 1)]\,(\mathcal{S} + n + a)\,.$$

iii) $\zeta = -b$
Linear functional

$$L_{1,1}^{U(-b)}[u] = \sum_{x=0}^{\infty} u\,(x)\,\,\frac{(a)_x}{(b+1)_x}\frac{z^x}{x!} + \eta u\,(-b)\,z^{-b}.$$

First moment differential operator

$$\Theta_{1,1}^{U(-b)} = (\vartheta + b - 1)\,\Theta_{1,1} = \vartheta\,(\vartheta + b)\,(\vartheta + b - 1) - z\,(\vartheta + a)\,(\vartheta + b)\,,$$

which is a special case of (46) with

$$a_1 = a, \quad a_2 = b, \quad b_1 = b, \quad b_2 = b - 1.$$

Standard moments recurrence operator

$$\Phi_{1,1}^{U(-b)} = (\mathcal{S} + b - 1)\,\Phi_{1,1} = \left[\mathcal{S}^{n+1}\,(\mathcal{S} + b - 1) - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} + a)\right](\mathcal{S} + b)\,.$$

Modified moments recurrence operator

$$\Psi_{1,1}^{U(-b)} = (\mathcal{S} + n + b)\,\Psi_{1,1} = \left[\mathcal{S}\,(\mathcal{S} + n + b - 1) - z\,(\mathcal{S} + n + a)\right](\mathcal{S} + n + b)\,.$$

3.3.3.14 Truncated Generalized Meixner functional

Linear functional

$$L_{1,1}^{T}[u] = \sum_{x=0}^{N} u(x)\,\frac{(a)_x}{(b+1)_x}\,\frac{z^x}{x!}, \quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\Theta_{1,1}^{T} = (\vartheta - N - 1)\,\Theta_{1,1} = \vartheta\,(\vartheta + b)\,(\vartheta - N - 1) - z\,(\vartheta + a)\,(\vartheta - N)\,,$$

which is a special case of (46) with

$$a_1 = a, \quad a_2 = -N, \quad b_1 = b, \quad b_2 = -N - 1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,1}^{T} = (\mathcal{S} - N - 1)\,\Phi_{1,1} = \mathcal{S}^{n+1}\,&(\mathcal{S} + b)\,(\mathcal{S} - N - 1)\\ &- z\,(\mathcal{S} + 1)^n\,(\mathcal{S} + a)\,(\mathcal{S} - N)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,1}^{T} = (\mathcal{S} + n - N)\,\Psi_{1,1} = \mathcal{S}\,&(\mathcal{S} + n + b)\,(\mathcal{S} + n - N - 1)\\ &- z\,(\mathcal{S} + n + a)\,(\mathcal{S} + n - N)\,.\end{aligned}$$

3.3.4 Functional of type (3,0;N)

Linear functional

$$L_{3,0;N}[u] = \sum_{x=0}^{N} u(x) \ (-N)_x (a_1)_x (a_2)_x \frac{z^x}{x!}, \quad N \in \mathbb{N}_0,$$

First moment differential operator

$$\Theta_{3,0;N} = \vartheta - z(\vartheta - N)(\vartheta + a_1)(\vartheta + a_2).$$

Standard moments recurrence operator

$$\Phi_{3,0;N} = \mathcal{S}^{n+1} - z(\mathcal{S}+1)^n (\mathcal{S} - N)(\mathcal{S} + a_1)(\mathcal{S} + a_2).$$

Modified moments hypergeometric representation

$$\nu_n(z) = (-N)_n (a_1)_n (a_2)_n \ z^n \ {}_3F_0 \left[\begin{matrix} n-N, a_1+n, a_2+n \\ - \end{matrix} ; z \right].$$

Modified moments recurrence operator

$$\Psi_{3,0;N} = \mathcal{S} - z(\mathcal{S} + n - N)(\mathcal{S} + n + a_1)(\mathcal{S} + n + a_2).$$

3.3.5 Functional of type (3,1;N)

Linear functional

$$L_{3,1;N}[u] = \sum_{x=0}^{N} u(x) \frac{(-N)_x (a_1)_x (a_2)_x}{(b+1)_x} \frac{z^x}{x!}, \quad N \in \mathbb{N}_0.$$

First moment

$$\lambda_0(z) = \ {}_3F_1 \left[\begin{matrix} -N, a_1, a_2 \\ b+1 \end{matrix} ; z \right].$$

First moment differential operator

$$\Theta_{3,1;N} = \vartheta(\vartheta + b) - z(\vartheta - N)(\vartheta + a_1)(\vartheta + a_2). \tag{50}$$

Standard moments recurrence operator

$$\Phi_{3,1;N} = \mathcal{S}^{n+1}(\mathcal{S} + b) - z(\mathcal{S}+1)^n (\mathcal{S} - N)(\mathcal{S} + a_1)(\mathcal{S} + a_2).$$

Modified moments hypergeometric representation

$$\nu_n(z) = \frac{(-N)_n (a_1)_n (a_2)_n}{(b+1)_n} \; z^n \; {}_3F_1 \left[\begin{array}{c} n-N, a_1+n, a_2+n \\ b+1+n \end{array} ; z \right].$$

Modified moments recurrence operator

$$\Psi_{3,1;N} = \mathcal{S}(\mathcal{S}+n+b) - z(\mathcal{S}+n-N)(\mathcal{S}+n+a_1)(\mathcal{S}+n+a_2).$$

3.3.5.1 Christoffel Generalized Krawtchouk functional

Linear functional

$$L^C_{2,0;N}[u] = \sum_{x=0}^{N} (x-\omega)\, u(x) \; (-N)_x (a)_x \frac{z^x}{x!}, \quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\Theta^C_{2,0;N} = \vartheta(\vartheta-\omega-1) - z(\vartheta-N)(\vartheta+a)(\vartheta-\omega+1)$$

which is a special case of (50) with

$$a_1 = -N, \quad a_2 = a, \quad a_3 = -\omega+1, \quad b = -\omega-1.$$

Standard moments recurrence

$$\Phi^C_{2,0;N} = \mathcal{S}^{n+1}(\mathcal{S}-\omega-1) - z(\mathcal{S}+1)^n(\mathcal{S}-N)(\mathcal{S}+a)(\mathcal{S}-\omega+1).$$

Modified moments hypergeometric representation

$$\nu^C_n(z) = (n-\omega)\frac{(-N)_n (a)_n}{(b+1)_n} \; z^n \; {}_3F_1 \left(\begin{array}{c} n-N, a+n, n-\omega+1 \\ n-\omega \end{array} ; z \right).$$

Modified moments recurrence operator

$$\Psi^C_{2,0;N} = \mathcal{S}(\mathcal{S}+n-\omega-1) - z(\mathcal{S}+n-N)(\mathcal{S}+n+a)(\mathcal{S}+n-\omega-1).$$

3.3.5.2 Geronimus Generalized Krawtchouk functional

Linear functional

$$L_{2,0;N}^{G}[u]=\sum_{x=0}^{N}\frac{u(x)}{x-\omega}\ (-N)_x (a)_x\ \frac{z^x}{x!},\quad N\in\mathbb{N}_0,\quad \omega\notin\mathbb{N}_0.$$

First moment differential operator

$$\Theta_{2,0;N}^{G}=\Theta_{2,0;N}(\vartheta-\omega)=\vartheta(\vartheta-\omega)-z(\vartheta-N)(\vartheta+a)(\vartheta-\omega)$$

which is a special case of (50) with

$$a_1=-N,\quad a_2=a,\quad a_3=-\omega,\quad b=-\omega.$$

Standard moments recurrence operator

$$\Phi_{2,0;N}^{G}=\Phi_{2,0;N}(\mathcal{S}-\omega)=\left[\mathcal{S}^{n+1}-z(\mathcal{S}+1)^n(\mathcal{S}-N)(\mathcal{S}+a)\right](\mathcal{S}-\omega)$$

Modified moments hypergeometric representation

$$\nu_n^{G}(z)=\frac{1}{n-\omega}\frac{(-N)_n (a)_n}{(b+1)_n}z^n\ {}_3F_1\left(\begin{array}{c}n-N,a+n,n-\omega\\ n-\omega+1\end{array};z\right).$$

Modified moments recurrence operator

$$\Psi_{2,0;N}^{G}=\Psi_{2,0;N}(\mathcal{S}+n-\omega)=[\mathcal{S}-z(\mathcal{S}+n-N)(\mathcal{S}+n+a)](\mathcal{S}+n-\omega).$$

3.3.5.3 Reduced-Uvarov Generalized Krawtchouk functional

Since for the Generalized Krawtchouk functional we have

$$\sigma(\vartheta)=\vartheta,\quad \tau(\vartheta)=(\vartheta-N)(\vartheta+a),\quad N\in\mathbb{N}_0,$$

we will have reduced cases for its Uvarov transformation $U(\zeta)$ if

$$\zeta=0,N,-a.$$

i) $\zeta=0$

Linear functional

$$L_{2,0;N}^{U(0)}[u]=\sum_{x=0}^{N}u(x)\ (-N)_x (a)_x\ \frac{z^x}{x!}+\eta u(0),\quad N\in\mathbb{N}_0.$$

First moment differential operator

$$\Theta_{2,0;N}^{U(0)} = (\vartheta - 1)\,\Theta_{2,0;N} = \vartheta\,(\vartheta - 1) - z\,(\vartheta - N)\,(\vartheta + a)\,\vartheta,$$

which is a special case of (50) with

$$a_1 = -N, \quad a_2 = a, \quad a_3 = 0, \quad b = -1.$$

Standard moments recurrence operator

$$\Phi_{2,0;N}^{U(0)} = (\mathcal{S} - 1)\,\Phi_{2,0;N} = \mathcal{S}^{n+1}\,(\mathcal{S} - 1) - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} - N)\,(\mathcal{S} + a)\,\mathcal{S}.$$

Modified moments recurrence operator

$$\Psi_{2,0;N}^{U(0)} = (\mathcal{S} + n)\,\Psi_{2,0;N} = \mathcal{S}\,(\mathcal{S} + n - 1) - z\,(\mathcal{S} + n - N)\,(\mathcal{S} + n + a)\,(\mathcal{S} + n)\,.$$

ii) $\zeta = N$

Linear functional

$$L_{2,0;N}^{U(N)}[u] = \sum_{x=0}^{N} u(x)\ (-N)_x\,(a)_x\,\frac{z^x}{x!} + \eta u(N)\,z^N.$$

First moment differential operator

$$\Theta_{2,0;N}^{U(N)} = (\vartheta - N)\,\Theta_{2,0;N} = \vartheta\,(\vartheta - N) - z\,(\vartheta - N)\,(\vartheta + a)\,(\vartheta - N + 1)\,,$$

which is a special case of (50) with

$$a_1 = -N, \quad a_2 = a, \quad a_3 = -N + 1, \quad b = -N.$$

Standard moments recurrence operator

$$\begin{aligned} \Phi_{2,0;N}^{U(N)} &= (\mathcal{S} - N)\,\Phi_{2,0;N} \\ &= \left[\mathcal{S}^{n+1} - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} + a)\,(\mathcal{S} - N + 1)\right](\mathcal{S} - N)\,. \end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned} \Psi_{2,0;N}^{U(N)} &= (\mathcal{S} + n - N + 1)\,\Psi_{2,0;N} \\ &= \left[\mathcal{S} - z\,(\mathcal{S} + n + a)\,(\mathcal{S} + n - N + 1)\right](\mathcal{S} + n - N)\,. \end{aligned}$$

iii) $\zeta = -a$

Linear functional

$$L_{2,0;N}^{U(-a)}[u]=\sum_{x=0}^{N}u(x)\ \ (-N)_x(a)_x\frac{z^x}{x!}+\eta u(-a)z^{-a}.$$

First moment differential operator

$$\begin{aligned}\Theta_{2,0;N}^{U(-a)}&=(\vartheta+a)\,\Theta_{2,0;N}\\&=\vartheta(\vartheta+a)-z(\vartheta-N)(\vartheta+a)(\vartheta+a+1),\end{aligned}$$

which is a special case of (50) with

$$a_1=-N,\quad a_2=a,\quad a_3=a+1,\quad b=a.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{2,0;N}^{U(-a)}&=(\mathcal{S}+a)\,\Phi_{2,0;N}\\&=\left[\mathcal{S}^{n+1}-z(\mathcal{S}+1)^n(\mathcal{S}-N)(\mathcal{S}+a+1)\right](\mathcal{S}+a).\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{2,0;N}^{U(-a)}&=(\mathcal{S}+n+a+1)\,\Psi_{2,0;N}\\&=\left[\mathcal{S}-z(\mathcal{S}+n-N)(\mathcal{S}+n+a+1)\right](\mathcal{S}+n+a).\end{aligned}$$

3.3.6 Functional of type (3,2)

Linear functional

$$L_{3,2}[u]=\sum_{x=0}^{\infty}u(x)\frac{(a_1)_x(a_2)_x(a_3)_x}{(b_1+1)_x(b_2+1)_x}\frac{z^x}{x!}.$$

First moment

$$\lambda_0(z)={}_3F_2\left[\begin{matrix}a_1,a_2,a_3\\b_1+1,b_2+1\end{matrix};z\right].$$

First moment differential operator

$$\Theta_{3,2}=\vartheta(\vartheta+b_1)(\vartheta+b_2)-z(\vartheta+a_1)(\vartheta+a_2)(\vartheta+a_3). \tag{51}$$

Standard moments recurrence operator

$$\Phi_{3,2} = \mathcal{S}^{n+1} (\mathcal{S}+b_1)(\mathcal{S}+b_2) \\ - z(\mathcal{S}+1)^n (\mathcal{S}+a_1)(\mathcal{S}+a_2)(\mathcal{S}+a_3).$$

Modified moments hypergeometric representation

$$\nu_n(z) = \frac{(-N)_n (a_1)_n (a_2)_n}{(b_1+1)_n (b_2+1)_n} z^n \, {}_3F_1 \left[\begin{matrix} n-N, a_1+n, a_2+n \\ b+1+n, b_2+1+n \end{matrix} ; z \right].$$

Modified moments recurrence operator

$$\Psi_{3,2} = \mathcal{S}(\mathcal{S}+n+b_1)(\mathcal{S}+n+b_2) \\ - z(\mathcal{S}+n+a_1)(\mathcal{S}+n+a_2)(\mathcal{S}+n+a_3).$$

3.3.6.1 Uvarov Meixner functional

Linear functional

$$L_{1,0}^U[u] = \sum_{x=0}^{\infty} u(x)\ (a)_x\ \frac{z^x}{x!} + \eta u(\omega) z^\omega, \quad \omega \neq 0, -a.$$

First moment differential operator

$$\Theta_{1,0}^U = (\vartheta-\omega)(\vartheta-\omega-1)\Theta_{1,0} \\ = \vartheta(\vartheta-\omega)(\vartheta-\omega-1) - z(\vartheta+a)(\vartheta-\omega+1)(\vartheta-\omega),$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -\omega+1, \quad a_3 = -\omega, \quad b_1 = -\omega, \quad b_2 = -\omega-1.$$

Standard moments recurrence operator

$$\Phi_{1,0}^U = (\mathcal{S}-\omega)(\mathcal{S}-\omega-1)\Phi_{1,0} \\ = \left[\mathcal{S}^{n+1}(\mathcal{S}-\omega-1) - z(\mathcal{S}+1)^n(\mathcal{S}+a)(\mathcal{S}-\omega+1)\right](\mathcal{S}-\omega).$$

Modified moments recurrence operator

$$\Psi_{1,0}^U = (\mathcal{S}+n-\omega)(\mathcal{S}+n-\omega+1)\Psi_{1,0} \\ = [\mathcal{S}(\mathcal{S}+n-\omega-1) - z(\mathcal{S}+n+a)(\mathcal{S}+n-\omega+1)](\mathcal{S}+n-\omega).$$

3.3.6.2 Geronimus Christoffel Meixner functional

Linear functional

$$L_{1,0}^{G,C}[u] = \sum_{x=0}^{\infty} \frac{x-\omega_1}{x-\omega_2} u(x) \ (a)_x \ \frac{z^x}{x!}, \quad \omega_1 \neq \omega_2, \quad \omega_2 \notin \mathbb{N}_0.$$

First moment differential operator

$$\Theta_{1,0}^{G,C} = \Theta_{1,0}^{C}(\vartheta-\omega_2) = \vartheta(\vartheta-\omega_1-1)(\vartheta-\omega_2) - z(\vartheta+a)(\vartheta-\omega_1+1)(\vartheta-\omega_2),$$

which is a special case of (46) with

$$a_1 = a, \quad a_2 = -\omega_1+1, \quad a_3 = -\omega_2, \quad b_1 = -\omega_1-1, \quad b_2 = -\omega_2.$$

Standard moments recurrence operator

$$\begin{aligned} \Phi_{1,0}^{G,C} &= \Phi_{1,0}^{C}(\mathcal{S}-\omega_2) \\ &= \left[\mathcal{S}^{n+1}(\mathcal{S}-\omega_1-1) - z(\mathcal{S}+1)^n(\mathcal{S}+a)(\mathcal{S}-\omega_1+1)\right](\mathcal{S}-\omega_2). \end{aligned}$$

Modified moments hypergeometric representation

$$\nu_n^{G,C}(z) = \frac{n-\omega_1}{n-\omega_2}(a)_n \ z^n \ {}_3F_2\left[\begin{matrix} a+n, n-\omega_1+1, n-\omega_2 \\ n-\omega_1, n-\omega_2+1 \end{matrix}; z\right].$$

Modified moments recurrence operator

$$\begin{aligned} \Psi_{1,0}^{G,C} &= \Psi_{1,0}^{C}(\mathcal{S}+n-\omega_2) \\ &= \left[(\mathcal{S}+n-\omega_1-1)\mathcal{S} - z(\mathcal{S}+n+a)(\mathcal{S}+n-\omega_1+1)\right](\mathcal{S}+n-\omega_2). \end{aligned}$$

3.3.6.3 Double Christoffel Meixner functional

Linear functional

$$L_{1,0}^{C^2}[u] = \sum_{x=0}^{\infty}(x-\omega_1)(x-\omega_2)u(x) \ (a)_x \ \frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{1,0}^{C^2} = \vartheta(\vartheta-\omega_1-1)(\vartheta-\omega_2-1) - z(\vartheta+a)(\vartheta-\omega_1+1)(\vartheta-\omega_2+1),$$

which is a special case of (46) with

$$a_1 = a, \quad a_2 = -\omega_1+1, \quad a_3 = -\omega_2+1, \quad b_1 = -\omega_1-1, \quad b_2 = -\omega_2-1.$$

Standard moments recurrence operator

$$\Phi_{1,0}^{C^2} = \mathcal{S}^{n+1}\left(\mathcal{S}-\omega_1-1\right)\left(\mathcal{S}-\omega_2-1\right) - z\left(\mathcal{S}+1\right)^n\left(\mathcal{S}+a\right)\left(\mathcal{S}-\omega_1+1\right)\left(\mathcal{S}-\omega_2+1\right).$$

Modified moments

$$\begin{aligned}\nu_n^{C^2} &= \nu_{n+1}^{C} + \left(n-\omega_2\right)\nu_n^{C} \\ &= \nu_{n+2} + \left(2n+1-\omega_1-\omega_2\right)\nu_{n+1} + \left(n-\omega_1\right)\left(n-\omega_2\right)\nu_n \\ &= \left(\upsilon_2 z^2 + \upsilon_1 z + \upsilon_0\right) z^n \left(a\right)_n \left(1-z\right)^{-a-n-2},\end{aligned}$$

where

$$\begin{aligned}\upsilon_2 &= \left(a+\omega_1\right)\left(a+\omega_2\right), \\ \upsilon_1 &= a-(a+\omega_2)\omega_1-(a+\omega_1)\omega_2+n(1+2a+\omega_1+\omega_2) \\ \upsilon_0 &= \left(n-\omega_1\right)\left(n-\omega_2\right).\end{aligned}$$

Modified moments hypergeometric representation

$$\nu_n^{C^2}\left(z\right) = \left(n-\omega_1\right)\left(n-\omega_2\right)\left(a\right)_n \ z^n \ {}_3F_2\left[\begin{array}{c} a+n, n-\omega_1+1, n-\omega_2+1 \\ n-\omega_1, n-\omega_2 \end{array}; z\right].$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{C^2} &= \mathcal{S}\left(\mathcal{S}+n-\omega_1-1\right)\left(\mathcal{S}+n-\omega_2-1\right) \\ &\quad - z\left(\mathcal{S}+n+a\right)\left(\mathcal{S}+n-\omega_1+1\right)\left(\mathcal{S}+n-\omega_2+1\right).\end{aligned}$$

3.3.6.4 Reduced-Uvarov Christoffel Meixner functional

Since for the Christoffel Meixner functional we have

$$\sigma\left(\vartheta\right) = \vartheta\left(\vartheta-\omega-1\right), \quad \tau\left(\vartheta\right) = \left(\vartheta+a\right)\left(\vartheta-\omega+1\right),$$

we will have reduced cases for its Uvarov transformation $U\left(\zeta\right)$ if

$$\zeta = 0, \omega+1, -a, \omega-1.$$

i) $\zeta = 0$

Linear functional

$$L_{1,0}^{U(0),C}\left[u\right] = \sum_{x=0}^{\infty}\left(x-\omega\right)u\left(x\right)\ \left(a\right)_x \frac{z^x}{x!} + \eta u\left(0\right).$$

First moment differential operator

$$\Theta_{1,0}^{U(0),C} = (\vartheta - 1)\,\Theta_{1,0}^{C} = \vartheta\,(\vartheta - \omega - 1)\,(\vartheta - 1) - z\,(\vartheta + a)\,(\vartheta - \omega + 1)\,\vartheta,$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -\omega + 1, \quad a_3 = 0, \quad b_1 = -\omega - 1, \quad b_2 = -1.$$

Standard moments recurrence operator

$$\Phi_{1,0}^{U(0),C} = (\mathcal{S} - 1)\,\Phi_{1,0}^{C} = \mathcal{S}^{n+1}\,(\mathcal{S} - \omega - 1)\,(\mathcal{S} - 1) - z\,(\mathcal{S} + 1)^{n}\,(\mathcal{S} + a)\,(\mathcal{S} - \omega + 1)\,\mathcal{S}.$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(0),C} &= (\mathcal{S} + n)\,\Psi_{1,0}^{C} = \mathcal{S}\,(\mathcal{S} + n - \omega - 1)\,(\mathcal{S} + n - 1)\\ &\quad - z\,(\mathcal{S} + a)\,(\mathcal{S} + n - \omega + 1)\,(\mathcal{S} + n)\,.\end{aligned}$$

ii) $\zeta = \omega + 1$
Linear functional

$$L_{1,0}^{U(\omega+1),C}[u] = \sum_{x=0}^{\infty} (x - \omega)\,u(x)\ (a)_x\,\frac{z^x}{x!} + \eta u(\omega + 1)\,z^{\omega+1}.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{U(\omega+1),C} &= (\vartheta - \omega - 2)\,\Theta_{1,0}^{C}\\ &= \vartheta\,(\vartheta - \omega - 1)\,(\vartheta - \omega - 2) - z\,(\vartheta + a)\,(\vartheta - \omega + 1)\,(\vartheta - \omega - 1)\,,\end{aligned}$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -\omega + 1, \quad a_3 = -\omega - 1, \quad b_1 = -\omega - 1, \quad b_2 = -\omega - 2.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{U(\omega+1),C} &= (\mathcal{S} - \omega - 2)\,\Phi_{1,0}^{C}\\ &= \left[\mathcal{S}^{n+1}\,(\mathcal{S} - \omega - 2) - z\,(\mathcal{S} + 1)^{n}\,(\mathcal{S} + a)\,(\mathcal{S} - \omega + 1)\right](\mathcal{S} - \omega - 1)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(\omega+1),C} &= (\mathcal{S} + n - \omega - 1)\,\Psi_{1,0}^{C}\\ &= \left[\mathcal{S}\,(\mathcal{S} + n - \omega - 2) - z\,(\mathcal{S} + n + a)\,(\mathcal{S} + n - \omega + 1)\right](\mathcal{S} + n - \omega - 1)\,.\end{aligned}$$

iii) $\zeta = -a$
Linear functional

$$L_{0,0}^{U(-a),C}[u] = \sum_{x=0}^{\infty} (x-\omega)\, u(x)\ (a)_x \frac{z^x}{x!} + \eta u(-a)\, z^{-a}.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{U(-a),C} &= (\vartheta + a)\, \Theta_{1,0}^{C} \\ &= \vartheta(\vartheta - \omega - 1)(\vartheta + a) - z(\vartheta + a)(\vartheta - \omega + 1)(\vartheta + a + 1),\end{aligned}$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -\omega + 1, \quad a_3 = a + 1, \quad b_1 = -\omega - 1, \quad b_2 = a.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{U(-a),C} &= (\mathcal{S} + a)\, \Phi_{1,0}^{C} \\ &= \left[\mathcal{S}^{n+1}(\mathcal{S} - \omega - 1) - z(\mathcal{S} + 1)^n (\mathcal{S} - \omega + 1)(\mathcal{S} + a + 1)\right](\mathcal{S} + a).\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(-a),C} &= (\mathcal{S} + n + a + 1)\, \Psi_{1,0}^{C} \\ &= \left[\mathcal{S}(\mathcal{S} + n - \omega - 1) - z(\mathcal{S} + n - \omega + 1)(\mathcal{S} + n + a + 1)\right](\mathcal{S} + n + a).\end{aligned}$$

iv) $\zeta = \omega - 1$
Linear functional

$$L_{1,0}^{U(\omega-1),C}[u] = \sum_{x=0}^{\infty} (x-\omega)\, u(x)\ (a)_x \frac{z^x}{x!} + \eta u(\omega - 1)\, z^{\omega-1}.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{U(\omega-1),C} &= (\vartheta - \omega + 1)\, \Theta_{1,0}^{C} \\ &= \vartheta(\vartheta - \omega - 1)(\vartheta - \omega + 1) - z(\vartheta + a)(\vartheta - \omega + 1)(\vartheta - \omega + 2),\end{aligned}$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -\omega + 1, \quad a_3 = -\omega + 2, \quad b_1 = -\omega - 1, \quad b_2 = -\omega + 1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{U(\omega-1),C} &= (\mathcal{S}-\omega+1)\,\Phi_{1,0}^{C}\\ &= \left[\mathcal{S}^{n+1}(\mathcal{S}-\omega-1) - z(\mathcal{S}+1)^{n}(\mathcal{S}+a)(\mathcal{S}-\omega+2)\right](\mathcal{S}-\omega+1)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(\omega-1),C} &= (\mathcal{S}+n-\omega+2)\,\Psi_{1,0}^{C}\\ &= \left[\mathcal{S}(\mathcal{S}+n-\omega-1) - z(\mathcal{S}+n+a)(\mathcal{S}+n-\omega+2)\right](\mathcal{S}+n-\omega+1)\,.\end{aligned}$$

3.3.6.5 Truncated Christoffel Meixner functional

Linear functional

$$L_{1,0}^{T,C}[u] = \sum_{x=0}^{N}(x-\omega)\,u(x)\ \ (a)_x\,\frac{z^x}{x!},\quad N\in\mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{T,C} &= (\vartheta-N-1)\,\Theta_{1,0}^{C}\\ &= \vartheta(\vartheta-\omega-1)(\vartheta-N-1) - z(\vartheta+a)(\vartheta-\omega+1)(\vartheta-N)\,,\end{aligned}$$

which is a special case of (51) with

$$a_1=a,\quad a_2=-\omega+1,\quad a_3=-N,\quad b_1=-\omega-1,\quad b_2=-N-1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{T,C} = (\mathcal{S}-N-1)\,\Phi_{1,0}^{C} &= \mathcal{S}^{n+1}(\mathcal{S}-\omega-1)(\mathcal{S}-N-1)\\ &- z(\mathcal{S}+1)^{n}(\mathcal{S}+a)(\mathcal{S}-\omega+1)(\mathcal{S}-N)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{T,C} = (\mathcal{S}+n-N)\,\Psi_{1,0}^{C} &= \mathcal{S}(\mathcal{S}+n-\omega-1)(\mathcal{S}+n-N-1)\\ &- z(\mathcal{S}+n+a)(\mathcal{S}+n-\omega+1)(\mathcal{S}+n-N)\,.\end{aligned}$$

3.3.6.6 Double Geronimus Meixner functional

Linear functional

$$L_{1,0}^{G^2}[u] = \sum_{x=0}^{\infty} \frac{1}{(x-\omega_1)(x-\omega_2)} u(x) \ (a)_x \frac{z^x}{x!}, \quad \omega_1, \omega_2 \notin \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned} \Theta_{1,0}^{G^2} &= \Theta_{1,0}^{G} (\vartheta - \omega_2) = \Theta_{1,0} (\vartheta - \omega_1)(\vartheta - \omega_2) \\ &= \vartheta (\vartheta - \omega_1)(\vartheta - \omega_2) - z(\vartheta + a)(\vartheta - \omega_1)(\vartheta - \omega_2), \end{aligned}$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -\omega_1, \quad a_3 = -\omega_2, \quad b_1 = -\omega_1, \quad b_2 = -\omega_2.$$

Standard moments recurrence operator

$$\begin{aligned} \Phi_{1,0}^{G^2} &= \Phi_{1,0}^{G} (\mathcal{S} - \omega_2) = \Phi_{1,0} (\mathcal{S} - \omega_1)(\mathcal{S} - \omega_2) \\ &= \left[\mathcal{S}^{n+1} - z(\mathcal{S}+1)^n (\mathcal{S}+a)\right] (\mathcal{S} - \omega_1)(\mathcal{S} - \omega_2). \end{aligned}$$

Modified moments hypergeometric representation

$$\nu_n^{G^2}(z) = \frac{(a)_n \ z^n}{(n-\omega_1)(n-\omega_2)} \ {}_3F_2 \left[\begin{matrix} a+n, n-\omega_1, n-\omega_2 \\ n-\omega_1+1, n-\omega_2+1 \end{matrix} ; z \right].$$

Modified moments recurrence operator

$$\begin{aligned} \Psi_{1,0}^{G^2} &= \Psi_{1,0}^{G} (\mathcal{S} + n - \omega_2) = \Psi_{1,0} (\mathcal{S} + n - \omega_1)(\mathcal{S} + n - \omega_2) \\ &= \left[\mathcal{S} - z(\mathcal{S} + n + a)\right] (\mathcal{S} + n - \omega_1)(\mathcal{S} + n - \omega_2). \end{aligned}$$

3.3.6.7 Reduced-Uvarov Geronimus Meixner functional

Since for the Geronimus Meixner functional we have

$$\sigma(\vartheta) = \vartheta(\vartheta - \omega), \quad \tau(\vartheta) = (\vartheta + a)(\vartheta - \omega), \quad \omega \notin \mathbb{N}_0,$$

we will have a reduced case for its Uvarov transformation $U(\zeta)$ if

$$\zeta = 0, -a.$$

i) $\zeta = 0$

Linear functional

$$L_{1,0}^{U(0),G}[u] = \sum_{x=0}^{\infty} \frac{u(x)}{x-\omega} \, (a)_x \, \frac{z^x}{x!} + \eta u(0), \quad \omega \notin \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned} \Theta_{1,0}^{U(0),G} &= (\vartheta - 1) \Theta_{1,0}^{G} = (\vartheta - 1) \Theta_{1,0} (\vartheta - \omega) \\ &= \vartheta (\vartheta - \omega)(\vartheta - 1) - z (\vartheta + a)(\vartheta - \omega) \vartheta, \end{aligned}$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -\omega, \quad a_3 = 0, \quad b_1 = -\omega, \quad b_2 = -1.$$

Standard moments recurrence operator

$$\begin{aligned} \Phi_{1,0}^{U(0),G} &= (\mathcal{S} - 1) \Phi_{1,0}^{G} = (\mathcal{S} - 1) \Phi_{1,0} (\mathcal{S} - \omega) \\ &= \left[\mathcal{S}^{n+1} (\mathcal{S} - 1) - z (\mathcal{S} + 1)^n (\mathcal{S} + a) \mathcal{S}\right] (\mathcal{S} - \omega). \end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned} \Psi_{1,0}^{U(0),G} &= (\mathcal{S} + n) \Psi_{1,0}^{G} = (\mathcal{S} + n) \Psi_{1,0} (\mathcal{S} + n - \omega) \\ &= [\mathcal{S} (\mathcal{S} + n - 1) - z (\mathcal{S} + n + a)(\mathcal{S} + n)] (\mathcal{S} + n - \omega). \end{aligned}$$

ii) $\zeta = -a$

Linear functional

$$L_{1,0}^{U(-a),G}[u] = \sum_{x=0}^{\infty} \frac{u(x)}{x-\omega} \, (a)_x \, \frac{z^x}{x!} + \eta u(-a) z^{-a}, \quad \omega \notin \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned} \Theta_{1,0}^{U(-a),G} &= (\vartheta + a) \Theta_{1,0}^{G} = (\vartheta + a) \Theta_{1,0} (\vartheta - \omega) \\ &= \vartheta (\vartheta - \omega)(\vartheta + a) - z (\vartheta + a)(\vartheta - \omega)(\vartheta + a + 1), \end{aligned}$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -\omega, \quad a_3 = a + 1, \quad b_1 = -\omega, \quad b_2 = a.$$

Standard moments recurrence operator

$$\begin{aligned} \Phi_{1,0}^{U(-a),G} &= (\mathcal{S} + a) \Phi_{1,0}^{G} = (\mathcal{S} + a) \Phi_{1,0} (\mathcal{S} - \omega) \\ &= \left[\mathcal{S}^{n+1} - z (\mathcal{S} + 1)^n (\mathcal{S} + a + 1)\right] (\mathcal{S} + a)(\mathcal{S} - \omega). \end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned} \Psi_{1,0}^{U(-a),G} &= (\mathcal{S} + a + n + 1) \Psi_{1,0}^{G} = (\mathcal{S} + a + n + 1) \Psi_{1,0} (\mathcal{S} + n - \omega) \\ &= [\mathcal{S} - z (\mathcal{S} + a + n + 1)] (\mathcal{S} + a + n)(\mathcal{S} + n - \omega). \end{aligned}$$

3.3.6.8 Truncated Geronimus Meixner functional

Linear functional

$$L_{1,0}^{T,G}\left[u\right]=\sum_{x=0}^{N}\frac{u\left(x\right)}{x-\omega}\ \left(a\right)_{x}\frac{z^{x}}{x!},\quad N\in\mathbb{N}_{0},\quad \omega\notin\mathbb{N}_{0}.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{T,G}&=\left(\vartheta-N-1\right)\Theta_{1,0}^{G}=\left(\vartheta-N-1\right)\Theta_{1,0}\left(\vartheta-\omega\right)\\&=\vartheta\left(\vartheta-\omega\right)\left(\vartheta-N-1\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega\right)\left(\vartheta-N\right),\end{aligned}$$

which is a special case of (51) with

$$a_{1}=a,\quad a_{2}=-\omega,\quad a_{3}=-N\quad b_{1}=-\omega,\quad b_{2}=-N-1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{T,G}&=\left(\mathcal{S}-N-1\right)\Phi_{1,0}^{G}=\left(\mathcal{S}-N-1\right)\Phi_{1,0}\left(\mathcal{S}-\omega\right)\\&=\left[\mathcal{S}^{n+1}\left(\mathcal{S}-N-1\right)-z\left(\mathcal{S}+1\right)^{n}\left(\mathcal{S}+a\right)\left(\mathcal{S}-N\right)\right]\left(\mathcal{S}-\omega\right).\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{T,G}&=\left(\mathcal{S}+n-N\right)\Psi_{1,0}^{G}=\left(\mathcal{S}+n-N\right)\Psi_{1,0}\left(\mathcal{S}+n-\omega\right)\\&=\left[\mathcal{S}\left(\mathcal{S}+n-N-1\right)-z\left(\mathcal{S}+n+a\right)\left(\mathcal{S}+n-N\right)\right]\left(\mathcal{S}+n-\omega\right).\end{aligned}$$

3.3.6.9 Double Uvarov Meixner functional

Since for the Reduced-Uvarov Meixner functional we have

$$\begin{aligned}\sigma_{1,0}^{U(0)}\left(\vartheta\right)&=\vartheta\left(\vartheta-1\right),\quad \tau_{1,0}^{U(0)}\left(\vartheta\right)=\vartheta\left(\vartheta+a\right)\\\sigma_{1,0}^{U(-a)}\left(\vartheta\right)&=\vartheta\left(\vartheta+a\right),\quad \tau_{1,0}^{U(-a)}\left(\vartheta\right)=\left(\vartheta+a\right)\left(\vartheta+a+1\right),\end{aligned}$$

we will have a reduced case for their Uvarov transformations $U\left(\zeta\right)$ if

$$\zeta=1,-a,\ \text{ or }\ \zeta=0,-a-1.$$

i) $\zeta=1$

Linear functional

$$L_{1,0}^{U(1,0)}\left[u\right]=\sum_{x=0}^{\infty}u\left(x\right)\ \left(a\right)_{x}\frac{z^{x}}{x!}+\eta_{1}u\left(0\right)+\eta_{2}u\left(1\right)z.$$

First moment differential operator

$$\Theta_{1,0}^{U(1,0)} = (\vartheta - 2)\,\Theta_{1,0}^{U(0)} = (\vartheta - 2)\,(\vartheta - 1)\,\Theta_{1,0}$$
$$= \vartheta\,(\vartheta - 1)\,(\vartheta - 2) - z\,(\vartheta + a)\,\vartheta\,(\vartheta - 1)\,,$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = 0, \quad a_3 = -1, \quad b_1 = -1, \quad b_2 = -2.$$

Standard moments recurrence operator

$$\Phi_{1,0}^{U(1,0)} = (\mathcal{S} - 2)\,\Phi_{1,0}^{U(0)} = (\mathcal{S} - 2)\,(\mathcal{S} - 1)\,\Phi_{1,0}$$
$$= \left[\mathcal{S}^{n+1}\,(\mathcal{S} - 2) - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} + a)\,\mathcal{S}\right](\mathcal{S} - 1)\,.$$

Modified moments recurrence operator

$$\Psi_{1,0}^{U(1,0)} = (\mathcal{S} + n - 1)\,\Psi_{1,0}^{U(0)} = (\mathcal{S} + n - 1)\,(\mathcal{S} + n)\,\Psi_{1,0}$$
$$= \left[\mathcal{S}\,(\mathcal{S} + n - 2) - z\,(\mathcal{S} + n + a)\,(\mathcal{S} + n)\right](\mathcal{S} + n - 1)\,.$$

ii) $\zeta = -a$
Linear functional

$$L_{1,0}^{U(-a,0)}[u] = \sum_{x=0}^{\infty} u\,(x)\ \ (a)_x\ \ \frac{z^x}{x!} + \eta_1 u\,(0) + \eta_2 u\,(-a)\,z^{-a}.$$

First moment differential operator

$$\Theta_{1,0}^{U(-a,0)} = (\vartheta + a)\,\Theta_{1,0}^{U(0)} = (\vartheta + a)\,(\vartheta - 1)\,\Theta_{1,0}$$
$$= \vartheta\,(\vartheta - 1)\,(\vartheta + a) - z\,(\vartheta + a)\,\vartheta\,(\vartheta + a + 1)\,,$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = 0, \quad a_3 = a + 1, \quad b_1 = -1, \quad b_2 = a.$$

Standard moments recurrence operator

$$\Phi_{1,0}^{U(-a,0)} = (\mathcal{S} + a)\,\Phi_{1,0}^{U(1)} = (\mathcal{S} + a)\,(\mathcal{S} - 1)\,\Phi_{1,0}$$
$$= \left[\mathcal{S}^{n+1}\,(\mathcal{S} - 1) - z\,(\mathcal{S} + 1)^n\,\mathcal{S}\,(\mathcal{S} + a + 1)\right](\mathcal{S} + a)\,.$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(-a,0)} &= (\mathcal{S}+n+a+1)\,\Psi_{1,0}^{U(0)} = (\mathcal{S}+n+a+1)\,(\mathcal{S}+n)\,\Psi_{1,0}\\ &= [\mathcal{S}\,(\mathcal{S}+n-1) - z\,(\mathcal{S}+n)\,(\mathcal{S}+n+a+1)]\,(\mathcal{S}+n+a)\,.\end{aligned}$$

iii) $\zeta = -a-1$

Linear functional

$$L_{1,0}^{U(-a-1,-a)}[u] = \sum_{x=0}^{\infty} u(x)\ (a)_x\ \frac{z^x}{x!} + \eta_1 u(-a)\,z^{-a} + \eta_2 u(-a-1)\,z^{-a-1}.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{U(-a-1,-a)} &= (\vartheta+a+1)\,\Theta_{1,0}^{U(-a)} = (\vartheta+a+1)\,(\vartheta+a)\,\Theta_{1,0}\\ &= \vartheta\,(\vartheta+a)\,(\vartheta+a+1) - z\,(\vartheta+a)\,(\vartheta+a+1)\,(\vartheta+a+2)\,,\end{aligned}$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = a+1, \quad a_3 = a+2, \quad b_1 = a, \quad b_2 = a+1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{U(-a-1,-a)} &= (\mathcal{S}+a+1)\,\Phi_{1,0}^{U(-a)} = (\mathcal{S}+a+1)\,(\mathcal{S}+a)\,\Phi_{1,0}\\ &= \left[\mathcal{S}^{n+1} - z\,(\mathcal{S}+1)^n\,(\mathcal{S}+a+2)\right](\mathcal{S}+a)\,(\mathcal{S}+a+1)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(-a-1,-a)} &= (\mathcal{S}+n+a+2)\,\Psi_{1,0}^{U(-a)} = (\mathcal{S}+n+a+2)\,(\mathcal{S}+n+a+1)\,\Psi_{1,0}\\ &= [\mathcal{S} - z\,(\mathcal{S}+n+a+2)]\,(\mathcal{S}+n+a)\,(\mathcal{S}+n+a+1)\,.\end{aligned}$$

3.3.6.10 Reduced-Uvarov Truncated Meixner functional

Since for the Truncated Meixner functional we have

$$\sigma(\vartheta) = \vartheta\,(\vartheta-N-1)\,, \quad \tau(\vartheta) = (\vartheta+a)\,(\vartheta-N)\,, \quad N \in \mathbb{N}_0,$$

we will have reduced cases for its Uvarov transformation $U(\zeta)$ if

$$\zeta = 0, N+1, -a, N.$$

i) $\zeta = 0$

Linear functional

$$L_{1,0}^{U(0),T}[u]=\sum_{x=0}^{N}u(x)\ (a)_x\frac{z^x}{x!}+\eta u(0),\quad N\in\mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{U(0),T}&=(\vartheta-1)\,\Theta_{1,0}^{T}=(\vartheta-1)(\vartheta-N-1)\,\Theta_{1,0}\\&=\vartheta(\vartheta-N-1)(\vartheta-1)-z(\vartheta+a)(\vartheta-N)\,\vartheta,\end{aligned}$$

which is a special case of (51) with

$$a_1=a,\quad a_2=-N,\quad a_3=0,\quad b_1=-N-1,\quad b_2=-1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{U(0),T}&=(\mathcal{S}-1)\,\Phi_{1,0}^{T}=(\mathcal{S}-1)(\mathcal{S}-N-1)\,\Phi_{1,0}\\&=\mathcal{S}^{n+1}(\mathcal{S}-N-1)(\mathcal{S}-1)-z(\mathcal{S}+1)^n(\mathcal{S}+a)(\mathcal{S}-N)\,\mathcal{S}.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(0),T}&=(\mathcal{S}+n)\,\Psi_{1,0}^{T}=(\mathcal{S}+n)(\mathcal{S}+n-N)\,\Psi_{1,0}\\&=\mathcal{S}(\mathcal{S}+n-N-1)(\mathcal{S}+n-1)-z(\mathcal{S}+a)(\mathcal{S}+n-N)(\mathcal{S}+n).\end{aligned}$$

ii) $\zeta=N+1$
Linear functional

$$L_{1,0}^{U(N+1),T}[u]=\sum_{x=0}^{N}u(x)\ (a)_x\frac{z^x}{x!}+\eta u(N+1)\,z^{N+1},\quad N\in\mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{U(N+1),T}&=(\vartheta-N-2)\,\Theta_{1,0}^{T}=(\vartheta-N-2)(\vartheta-N-1)\,\Theta_{1,0}\\&=\vartheta(\vartheta-N-1)(\vartheta-N-2)-z(\vartheta+a)(\vartheta-N)(\vartheta-N-1),\end{aligned}$$

which is a special case of (51) with

$$a_1=a,\quad a_2=-N,\quad a_3=-N-1,\quad b_1=-N-1,\quad b_2=-N-2.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{U(N+1),T} &= (\mathcal{S}-N-2)\,\Phi_{1,0}^{T} = (\mathcal{S}-N-2)\,(\mathcal{S}-N-1)\,\Phi_{1,0} \\ &= \left[\mathcal{S}^{n+1}\,(\mathcal{S}-N-2) - z\,(\mathcal{S}+1)^{n}\,(\mathcal{S}+a)\,(\mathcal{S}-N)\right](\mathcal{S}-N-1)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(N+1),T} &= (\mathcal{S}+n-N-1)\,\Psi_{1,0}^{T} = (\mathcal{S}+n-N-1)\,(\mathcal{S}+n-N)\,\Psi_{1,0} \\ &= \left[\mathcal{S}\,(\mathcal{S}+n-N-2) - z\,(\mathcal{S}+n+a)\,(\mathcal{S}+n-N)\right](\mathcal{S}+n-N-1)\,.\end{aligned}$$

iii) $\zeta = -a$
Linear functional

$$L_{1,0}^{U(-a),T}[u] = \sum_{x=0}^{N} u(x)\ (a)_x\,\frac{z^x}{x!} + \eta u(-a)\,z^{-a}, \quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{U(-a),T} &= (\vartheta+a)\,\Theta_{1,0}^{T} = (\vartheta+a)\,(\vartheta-N-1)\,\Theta_{1,0} \\ &= \vartheta\,(\vartheta-N-1)\,(\vartheta+a) - z\,(\vartheta+a)\,(\vartheta-N)\,(\vartheta+a+1)\,,\end{aligned}$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -N, \quad a_3 = a+1, \quad b_1 = -N-1, \quad b_2 = a.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{U(-a),T} &= (\mathcal{S}+a)\,\Phi_{1,0}^{T} = (\mathcal{S}+a)\,(\mathcal{S}-N-1)\,\Phi_{1,0} \\ &= \left[\mathcal{S}^{n+1}\,(\mathcal{S}-N-1) - z\,(\mathcal{S}+1)^{n}\,(\mathcal{S}-N)\,(\mathcal{S}+a+1)\right](\mathcal{S}+a)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(-a),T} &= (\mathcal{S}+n+a+1)\,\Psi_{1,0}^{T} = (\mathcal{S}+n+a+1)\,(\mathcal{S}+n-N)\,\Psi_{1,0} \\ &= \left[\mathcal{S}\,(\mathcal{S}+n-N-1) - z\,(\mathcal{S}+n-N)\,(\mathcal{S}+n+a+1)\right](\mathcal{S}+n+a)\,.\end{aligned}$$

iv) $\zeta = N$
Linear functional

$$L_{1,0}^{U(N),T}[u] = \sum_{x=0}^{N} u(x)\ (a)_x\,\frac{z^x}{x!} + \eta u(N)\,z^{N}, \quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{1,0}^{U(N),T} &= (\vartheta - N)\,\Theta_{1,0}^{T} = (\vartheta - N)\,(\vartheta - N - 1)\,\Theta_{1,0} \\ &= \vartheta\,(\vartheta - N - 1)\,(\vartheta - N) - z\,(\vartheta + a)\,(\vartheta - N)\,(\vartheta - N + 1)\,,\end{aligned}$$

which is a special case of (51) with

$$a_1 = a, \quad a_2 = -N, \quad a_3 = -N + 1, \quad b_1 = -N - 1, \quad b_2 = -N.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{1,0}^{U(N),T} &= (\mathcal{S} - N)\,\Phi_{1,0}^{T} = (\mathcal{S} - N)\,(\mathcal{S} - N - 1)\,\Phi_{1,0} \\ &= \left[\mathcal{S}^{n+1}\,(\mathcal{S} - N - 1) - z\,(\mathcal{S} + 1)^{n}\,(\mathcal{S} + a)\,(\mathcal{S} - N + 1)\right](\mathcal{S} - N)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{1,0}^{U(N),T} &= (\mathcal{S} + n - N + 1)\,\Psi_{1,0}^{T} = (\mathcal{S} + n - N + 1)\,(\mathcal{S} + n - N)\,\Psi_{1,0} \\ &= [\mathcal{S}\,(\mathcal{S} + n - N - 1) - z\,(\mathcal{S} + n + a)\,(\mathcal{S} + n - N + 1)]\,(\mathcal{S} + n - N)\,.\end{aligned}$$

3.3.6.11 Christoffel Generalized Hahn functional

Linear functional

$$L_{2,1}^{C}[u] = \sum_{x=0}^{\infty} (x - \omega)\,u(x)\;\frac{(a_1)_x\,(a_2)_x}{(b+1)_x}\frac{z^x}{x!}.$$

First moment differential operator

$$\Theta_{2,1}^{C} = \vartheta\,(\vartheta + b)\,(\vartheta - \omega - 1) - z\,(\vartheta + a_1)\,(\vartheta + a_2)\,(\vartheta - \omega + 1)\,,$$

which is a special case of (51) with

$$a_3 = -\omega + 1, \quad b_1 = b, \quad b_2 = -\omega - 1.$$

Standard moments recurrence

$$\Phi_{2,1}^{C} = \mathcal{S}^{n+1}\,(\mathcal{S} + b)\,(\mathcal{S} - \omega - 1) - z\,(\mathcal{S} + 1)^{n}\,(\mathcal{S} + a_1)\,(\mathcal{S} + a_2)\,(\mathcal{S} - \omega + 1)\,.$$

Modified moments hypergeometric representation

$$\nu_n^{C}(z) = (n - \omega)\,\frac{(a_1)_n\,(a_2)_n}{(b+1)_n}\;z^n\;{}_3F_2\left[\begin{matrix} a_1 + n, a_2 + n, n - \omega + 1 \\ b + 1 + n, n - \omega \end{matrix};z\right].$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{2,1}^{C} &= \mathcal{S}\,(\mathcal{S} + n + b)\,(\mathcal{S} + n - \omega - 1) \\ &\quad - z\,(\mathcal{S} + n + a_1)\,(\mathcal{S} + n + a_2)\,(\mathcal{S} + n - \omega + 1)\,.\end{aligned}$$

3.3.6.12 Geronimus Generalized Hahn functional

Linear functional

$$L_{2,1}^{G}[u]=\sum_{x=0}^{\infty}\frac{u(x)}{x-\omega}\ \frac{(a_1)_x(a_2)_x}{(b+1)_x}\ \frac{z^x}{x!},\quad \omega\notin\mathbb{N}_0.$$

First moment differential operator

$$\Theta_{2,1}^{G}=\Theta_{2,1}(\vartheta-\omega)=\vartheta(\vartheta+b)(\vartheta-\omega)-z(\vartheta+a_1)(\vartheta+a_2)(\vartheta-\omega),$$

which is a special case of (51) with

$$a_3=-\omega,\quad b_1=b,\quad b_2=-\omega.$$

Standard moments recurrence operator

$$\Phi_{2,1}^{G}=\Phi_{2,1}(\mathcal{S}-\omega)=\left[\mathcal{S}^{n+1}(\mathcal{S}+b)-z(\mathcal{S}+1)^n(\mathcal{S}+a_1)(\mathcal{S}+a_2)\right](\mathcal{S}-\omega)$$

Modified moments hypergeometric representation

$$\nu_n^{G}(z)=\frac{1}{n-\omega}\frac{(a_1)_n(a_2)_n}{(b+1)_n}z^n\ {}_3F_2\left(\begin{matrix}a_1+n,a_2+n,n-\omega\\ b+1+n,n-\omega+1\end{matrix};z\right).$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{2,1}^{G}&=\Psi_{2,1}(\mathcal{S}+n-\omega)\\&=\left[\mathcal{S}(\mathcal{S}+n+b)-z(\mathcal{S}+n+a_1)(\mathcal{S}+n+a_2)\right](\mathcal{S}+n-\omega).\end{aligned}$$

3.3.6.13 Reduced-Uvarov Generalized Hahn functional

Since for the generalized Hahn functional we have

$$\sigma(\vartheta)=\vartheta(\vartheta+b),\quad \tau(\vartheta)=(\vartheta+a_1)(\vartheta+a_2),$$

we will have reduced cases for its Uvarov transformation $U(\zeta)$ if

$$\zeta=0,-b,-a_1,-a_2.$$

i) $\zeta=0$

Linear functional

$$L_{2,1}^{U(0)}[u]=\sum_{x=0}^{\infty}u(x)\ \frac{(a_1)_x(a_2)_x}{(b+1)_x}\frac{z^x}{x!}+\eta u(0).$$

First moment differential operator

$$\Theta_{2,1}^{U(0)} = (\vartheta - 1)\,\Theta_{2,1} = \vartheta\,(\vartheta + b)\,(\vartheta - 1) - z\,(\vartheta + a_1)\,(\vartheta + a_2)\,\vartheta,$$

which is a special case of (51) with

$$a_3 = 0, \quad b_1 = b, \quad b_2 = -1.$$

Standard moments recurrence operator

$$\Phi_{2,1}^{U(0)} = (\mathcal{S} - 1)\,\Phi_{2,1} = \mathcal{S}^{n+1}\,(\mathcal{S} + b)\,(\mathcal{S} - 1) \\ - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} + a_1)\,(\mathcal{S} + a_2)\,\mathcal{S}.$$

Modified moments recurrence operator

$$\Psi_{2,1}^{U(0)} = (\mathcal{S} + n)\,\Psi_{2,1} = \mathcal{S}\,(\mathcal{S} + n + b)\,(\mathcal{S} + n - 1) \\ - z\,(\mathcal{S} + n + a_1)\,(\mathcal{S} + n + a_2)\,(\mathcal{S} + n)\,.$$

For the special case $z = 1$, see [5].
ii) $\zeta = -b$
Linear functional

$$L_{2,1}^{U(-b)}\,[u] = \sum_{x=0}^{\infty} u\,(x)\;\frac{(a_1)_x\,(a_2)_x}{(b+1)_x}\frac{z^x}{x!} + \eta u\,(-b)\,z^{-b}.$$

First moment differential operator

$$\Theta_{2,1}^{U(-b)} = (\vartheta + b - 1)\,\Theta_{2,1} = \vartheta\,(\vartheta + b)\,(\vartheta + b - 1) \\ - z\,(\vartheta + a_1)\,(\vartheta + a_2)\,(\vartheta + b)\,,$$

which is a special case of (51) with

$$a_3 = b, \quad b_1 = b, \quad b_2 = b - 1.$$

Standard moments recurrence operator

$$\Phi_{2,1}^{U(-b)} = (\mathcal{S} + b - 1)\,\Phi_{2,1} \\ = \left[\mathcal{S}^{n+1}\,(\mathcal{S} + b - 1) - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} + a_1)\,(\mathcal{S} + a_2)\right](\mathcal{S} + b)\,.$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{2,1}^{U(0)} &= (\mathcal{S}+n+b)\,\Psi_{2,1}\\ &= \left[\mathcal{S}\,(\mathcal{S}+n+b-1) - z\,(\mathcal{S}+n+a_1)\,(\mathcal{S}+n+a_2)\right](\mathcal{S}+n+b)\,.\end{aligned}$$

iii) $\zeta = -a_1$
Linear functional

$$L_{2,1}^{U(-a_1)}[u] = \sum_{x=0}^{\infty} u(x)\ \frac{(a_1)_x\,(a_2)_x}{(b+1)_x}\frac{z^x}{x!} + \eta u(-a_1)\,z^{-a_1}.$$

First moment differential operator

$$\begin{aligned}\Theta_{2,1}^{U(-a_1)} = (\vartheta+a_1)\,\Theta_{2,1} &= \vartheta\,(\vartheta+b)\,(\vartheta+a_1)\\ &- z\,(\vartheta+a_1)\,(\vartheta+a_2)\,(\vartheta+a_1+1)\,,\end{aligned}$$

which is a special case of (51) with

$$a_3 = a_1+1, \quad b_1 = b, \quad b_2 = a_1.$$

Standard moments recurrence operator

$$\begin{aligned}\Phi_{2,1}^{U(-a_1)} &= (\mathcal{S}+a_1)\,\Phi_{2,1}\\ &= \left[\mathcal{S}^{n+1}\,(\mathcal{S}+b) - z\,(\mathcal{S}+1)^n\,(\mathcal{S}+a_2)\,(\mathcal{S}+a_1+1)\right](\mathcal{S}+a_1)\,.\end{aligned}$$

Modified moments recurrence operator

$$\begin{aligned}\Psi_{2,1}^{U(-a_1)} &= (\mathcal{S}+n+a_1+1)\,\Psi_{2,1}\\ &= \left[\mathcal{S}\,(\mathcal{S}+n+b) - z\,(\mathcal{S}+n+a_2)\,(\mathcal{S}+n+a_1+1)\right](\mathcal{S}+n+a_1)\,.\end{aligned}$$

3.3.6.14 Truncated Generalized Hahn functional

Linear functional

$$L_{2,1}^{T}[u] = \sum_{x=0}^{N} u(x)\ \frac{(a_1)_x\,(a_2)_x}{(b+1)_x}\frac{z^x}{x!}, \quad N \in \mathbb{N}_0.$$

First moment differential operator

$$\begin{aligned}\Theta_{2,1}^{T} = (\vartheta-N-1)\,\Theta_{2,1} &= \vartheta\,(\vartheta+b)\,(\vartheta-N-1)\\ &- z\,(\vartheta+a_1)\,(\vartheta+a_2)\,(\vartheta-N)\,,\end{aligned}$$

which is a special case of (51) with

$$a_3 = -N, \quad b_1 = b, \quad b_2 = -N - 1.$$

Standard moments recurrence operator

$$\Phi_{2,1}^T = (\mathcal{S} - N - 1)\,\Phi_{2,1} = \mathcal{S}^{n+1}\,(\mathcal{S} + b)\,(\mathcal{S} - N - 1) \\ - z\,(\mathcal{S} + 1)^n\,(\mathcal{S} + a_1)\,(\mathcal{S} + a_2)\,(\mathcal{S} - N)\,.$$

Modified moments recurrence operator

$$\Psi_{2,1}^T = (\mathcal{S} + n - N)\,\Psi_{2,1} = \mathcal{S}\,(\mathcal{S} + n + b)\,(\mathcal{S} + n - N - 1) \\ - z\,(\mathcal{S} + n + a_1)\,(\mathcal{S} + n + a_2)\,(\mathcal{S} + n - N)\,.$$

4 Summary

In this section, we list all the functionals and their transformations. Note that we have

$s = 0$	$s = 1$	$s = 2$
Charlier $(0,0)$	4 cases $(1,1)$	15 cases $(2,2)$
Meixner $(1,0)$	5 cases $(2,1)$	20 cases $(3,2)$
	Generalized Charlier $(0,1)$	5 cases $(1,2)$
	Generalized Meixner $(1,1)$	6 cases $(2,2)$
	Generalized Krawtchouk $(2,0;N)$	5 cases $(3,1;N)$
	Generalized Hahn $(2,1)$	7 cases $(3,2)$

Charlier

Differential operator:

$$\Theta_{00} = \vartheta - z.$$

(i) Christoffel transformation (of type $1,1$):

$$\Theta_{0,0}^C = \vartheta\,(\vartheta - \omega - 1) - z\,(\vartheta - \omega + 1)\,.$$

Double transformations (of type $2,2$):

$$\Theta_{0,0}^{C^2} = \vartheta\,(\vartheta - \omega_1 - 1)\,(\vartheta - \omega_2 - 1) - z\,(\vartheta - \omega_1 + 1)\,(\vartheta - \omega_2 + 1)\,,$$

$$\Theta_{0,0}^{G,C}=\vartheta\left(\vartheta-\omega_1-1\right)\left(\vartheta-\omega_2\right)-z\left(\vartheta-\omega_1+1\right)\left(\vartheta-\omega_2\right),$$

$$\begin{aligned}\Theta_{0,0}^{U(0),C}&=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-1\right)-z\left(\vartheta-\omega+1\right)\vartheta,\\ \Theta_{0,0}^{U(\omega+1),C}&=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-\omega-2\right)-z\left(\vartheta-\omega+1\right)\left(\vartheta-\omega-1\right),\\ \Theta_{0,0}^{U(\omega-1),C}&=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-\omega+1\right)-z\left(\vartheta-\omega+1\right)\left(\vartheta-\omega+2\right),\end{aligned}$$

$$\Theta_{0,0}^{T,C}=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-N-1\right)-z\left(\vartheta-\omega+1\right)\left(\vartheta-N\right).$$

(ii) Geronimus transformation (of type $1,1$):

$$\Theta_{0,0}^{G}=\vartheta\left(\vartheta-\omega\right)-z\left(\vartheta-\omega\right).$$

Double transformations (of type $2,2$):

$$\Theta_{0,0}^{C,G}=\Theta_{0,0}^{G,C},$$

$$\Theta_{0,0}^{G^2}=\vartheta\left(\vartheta-\omega_1\right)\left(\vartheta-\omega_2\right)-z\left(\vartheta-\omega_1\right)\left(\vartheta-\omega_2\right),$$

$$\begin{aligned}\Theta_{0,0}^{U(0),G}&=\vartheta\left(\vartheta-\omega\right)\left(\vartheta-1\right)-z\left(\vartheta-\omega\right)\vartheta,\\ \Theta_{0,0}^{U(\omega),G}&=\Theta_{0,0}^{G},\end{aligned}$$

$$\Theta_{0,0}^{T,G}=\vartheta\left(\vartheta-\omega\right)\left(\vartheta-N-1\right)-z\left(\vartheta-\omega\right)\left(\vartheta-N\right).$$

(iii) Reduced-Uvarov transformation $U\left(0\right)$ (of type $1,1$):

$$\Theta_{0,0}^{U(0)}=\vartheta\left(\vartheta-1\right)-z\vartheta.$$

Double transformations (of type $2,2$):

$$\Theta_{0,0}^{C,U(0)}=\Theta_{0,0}^{U(0),C},\quad \Theta_{0,0}^{G,U(0)}=\Theta_{0,0}^{U(0),G},$$

$$\begin{aligned}\Theta_{0,0}^{U^2(1,0)}&=\vartheta\left(\vartheta-1\right)\left(\vartheta-2\right)-z\vartheta\left(\vartheta-1\right),\\ \Theta_{0,0}^{T,U(0)}&=\vartheta\left(\vartheta-1\right)\left(\vartheta-N-1\right)-z\vartheta\left(\vartheta-N\right).\end{aligned}$$

(iv) Truncation transformation (of type $1,1$):

$$\Theta_{0,0}^{T}=\vartheta\left(\vartheta-N-1\right)-z\left(\vartheta-N\right).$$

Double transformations (of type $3,2$):

$$\Theta_{0,0}^{C,T}=\Theta_{0,0}^{T,C},\quad \Theta_{0,0}^{G,T}=\Theta_{0,0}^{T,G},\quad \Theta_{0,0}^{U(0),T}=\Theta_{0,0}^{T,U(0)},$$

$$\begin{aligned}\Theta_{0,0}^{U(N+1),T}&=\vartheta\left(\vartheta-N-1\right)\left(\vartheta-N-2\right)-z\left(\vartheta-N\right)\left(\vartheta-N-1\right),\\ \Theta_{0,0}^{U(N),T}&=\vartheta\left(\vartheta-N-1\right)\left(\vartheta-N\right)-z\left(\vartheta-N\right)\left(\vartheta-N+1\right).\end{aligned}$$

(v) Uvarov transformation (of type $2,2$):

$$\Theta_{0,0}^{U}=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-\omega\right)-z\left(\vartheta-\omega\right)\left(\vartheta-\omega+1\right),\quad \omega\neq 0.$$

Meixner

Differential operator:

$$\Theta_{1,0}=\vartheta-z\left(\vartheta+a\right).$$

(i) Christoffel transformation (of type $2,1$):

$$\Theta_{1,0}^{C}=\vartheta\left(\vartheta-\omega-1\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega+1\right).$$

Double transformations (of type $3,2$):

$$\Theta_{1,0}^{C^2}=\vartheta\left(\vartheta-\omega_1-1\right)\left(\vartheta-\omega_2-1\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega_1+1\right)\left(\vartheta-\omega_2+1\right),$$

$$\Theta_{1,0}^{G,C}=\vartheta\left(\vartheta-\omega_1-1\right)\left(\vartheta-\omega_2\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega_1+1\right)\left(\vartheta-\omega_2\right),$$

$$\begin{aligned}\Theta_{1,0}^{U(0),C}&=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-1\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega+1\right)\vartheta,\\ \Theta_{1,0}^{U(\omega+1),C}&=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-\omega-2\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega+1\right)\left(\vartheta-\omega-1\right),\\ \Theta_{1,0}^{U(-a),C}&=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta+a\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega+1\right)\left(\vartheta+a+1\right),\\ \Theta_{1,0}^{U(\omega-1),C}&=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-\omega+1\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega+1\right)\left(\vartheta-\omega+2\right),\end{aligned}$$

$$\Theta_{1,0}^{T,C}=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-N-1\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega+1\right)\left(\vartheta-N\right).$$

(ii) Geronimus transformation (of type $2,1$):

$$\Theta_{1,0}^{G} = \vartheta\left(\vartheta-\omega\right) - z\left(\vartheta+a\right)\left(\vartheta-\omega\right).$$

Double transformations (of type $3,2$):

$$\Theta_{1,0}^{C,G} = \Theta_{1,0}^{G,C},$$

$$\Theta_{1,0}^{G^2} = \vartheta\left(\vartheta-\omega_1\right)\left(\vartheta-\omega_2\right) - z\left(\vartheta+a\right)\left(\vartheta-\omega_1\right)\left(\vartheta-\omega_2\right),$$

$$\begin{aligned}
\Theta_{1,0}^{U(0),G} &= \vartheta\left(\vartheta-\omega\right)\left(\vartheta-1\right) - z\left(\vartheta+a\right)\left(\vartheta-\omega\right)\vartheta,\\
\Theta_{1,0}^{U(-a),G} &= \vartheta\left(\vartheta-\omega\right)\left(\vartheta+a\right) - z\left(\vartheta+a\right)\left(\vartheta-\omega\right)\left(\vartheta+a+1\right),\\
\Theta_{1,0}^{U(\omega),G} &= \Theta_{1,0}^{G},
\end{aligned}$$

$$\Theta_{1,0}^{G,T} = \vartheta\left(\vartheta-\omega\right)\left(\vartheta-N-1\right) - z\left(\vartheta+a\right)\left(\vartheta-\omega\right)\left(\vartheta-N\right).$$

(iii) Reduced-Uvarov transformation $U\left(0\right)$ (of type $2,1$):

$$\Theta_{1,0}^{U(0)} = \vartheta\left(\vartheta-1\right) - z\left(\vartheta+a\right)\vartheta.$$

Double transformations (of type $3,2$):

$$\Theta_{1,0}^{C,U(0)} = \Theta_{1,0}^{U(0),C}, \quad \Theta_{1,0}^{G,U(0)} = \Theta_{1,0}^{U(0),G},$$

$$\begin{aligned}
\Theta_{1,0}^{U^2(1,0)} &= \vartheta\left(\vartheta-1\right)\left(\vartheta-2\right) - z\left(\vartheta+a\right)\vartheta\left(\vartheta-1\right),\\
\Theta_{1,0}^{U^2(-a,0)} &= \vartheta\left(\vartheta-1\right)\left(\vartheta+a\right) - z\left(\vartheta+a\right)\vartheta\left(\vartheta+a+1\right),
\end{aligned}$$

$$\Theta_{1,0}^{T,U(0)} = \vartheta\left(\vartheta-1\right)\left(\vartheta-N-1\right) - z\left(\vartheta+a\right)\vartheta\left(\vartheta-N\right).$$

(iv) Reduced-Uvarov transformation $U\left(-a\right)$ (of type $2,1$):

$$\Theta_{1,0}^{U(-a)} = \vartheta\left(\vartheta+a\right) - z\left(\vartheta+a\right)\left(\vartheta+a+1\right).$$

Double transformations (of type $3,2$):

$$\Theta_{1,0}^{C,U(-a)} = \Theta_{1,0}^{U(-a),C}, \quad \Theta_{1,0}^{G,U(-a)} = \Theta_{1,0}^{U(-a),G}, \quad \Theta_{1,0}^{U^2(0,-a)} = \Theta_{1,0}^{U^2(-a,0)},$$

$$\begin{aligned}
\Theta_{1,0}^{U^2(-a-1,-a)} &= \vartheta\left(\vartheta+a\right)\left(\vartheta+a+1\right) - z\left(\vartheta+a\right)\left(\vartheta+a+1\right)\left(\vartheta+a+2\right),\\
\Theta_{1,0}^{T,U(-a)} &= \vartheta\left(\vartheta+a\right)\left(\vartheta-N-1\right) - z\left(\vartheta+a\right)\left(\vartheta+a+1\right)\left(\vartheta-N\right).
\end{aligned}$$

(v) Truncation transformation (of type $2,1$):

$$\Theta_{1,0}^{T}=\vartheta\left(\vartheta-N-1\right)-z\left(\vartheta+a\right)\left(\vartheta-N\right).$$

Double transformations (of type $3,2$):

$$\Theta_{1,0}^{C,T}=\Theta_{1,0}^{T,C},\quad \Theta_{1,0}^{G,T}=\Theta_{1,0}^{T,G},\quad \Theta_{1,0}^{U(0),T}=\Theta_{1,0}^{T,U(0)},\quad \Theta_{1,0}^{U(-a),T}=\Theta_{1,0}^{T,U(-a)},$$

$$\begin{aligned}\Theta_{1,0}^{U(N+1),T}&=\vartheta\left(\vartheta-N-1\right)\left(\vartheta-N-2\right)-z\left(\vartheta+a\right)\left(\vartheta-N\right)\left(\vartheta-N-1\right),\\ \Theta_{1,0}^{U(N),T}&=\vartheta\left(\vartheta-N-1\right)\left(\vartheta-N\right)-z\left(\vartheta+a\right)\left(\vartheta-N\right)\left(\vartheta-N+1\right).\end{aligned}$$

(vi) Uvarov transformation (of type $3,2$):

$$\Theta_{1,0}^{U}=\vartheta\left(\vartheta-\omega-1\right)\left(\vartheta-\omega\right)-z\left(\vartheta+a\right)\left(\vartheta-\omega\right)\left(\vartheta-\omega+1\right),\quad \omega\neq 0,-a.$$

Generalized Charlier

Differential operator:

$$\Theta_{0,1}=\vartheta\left(\vartheta+b\right)-z.$$

Transformations (of type $1,2$):

(i) Christoffel

$$\Theta_{0,1}^{C}=\vartheta\left(\vartheta+b\right)\left(\vartheta-\omega-1\right)-z\left(\vartheta-\omega+1\right),$$

(ii) Geronimus

$$\Theta_{0,1}^{G}=\vartheta\left(\vartheta+b\right)\left(\vartheta-\omega\right)-z\left(\vartheta-\omega\right),$$

(iii) Reduced-Uvarov

$$\begin{aligned}\Theta_{0,1}^{U(0)}&=\vartheta\left(\vartheta+b\right)\left(\vartheta-1\right)-z\vartheta,\\ \Theta_{0,1}^{U(-b)}&=\vartheta\left(\vartheta+b\right)\left(\vartheta+b-1\right)-z\left(\vartheta+b\right),\end{aligned}$$

(iv) Truncation

$$\Theta_{0,1}^{T}=\vartheta\left(\vartheta+b\right)\left(\vartheta-N-1\right)-z\left(\vartheta-N\right).$$

Generalized Meixner

Differential operator:

$$\Theta_{1,1} = \vartheta\left(\vartheta + b\right) - z\left(\vartheta + a\right).$$

Transformations (of type $2, 2$):

(i) Christoffel

$$\Theta_{1,1}^{C} = \vartheta\left(\vartheta + b\right)\left(\vartheta - \omega - 1\right) - z\left(\vartheta + a\right)\left(\vartheta - \omega + 1\right),$$

(ii) Geronimus

$$\Theta_{1,1}^{G} = \vartheta\left(\vartheta + b\right)\left(\vartheta - \omega\right) - z\left(\vartheta + a\right)\left(\vartheta - \omega\right),$$

(iii) Reduced-Uvarov

$$\begin{aligned}\Theta_{1,1}^{U(0)} &= \vartheta\left(\vartheta + b\right)\left(\vartheta - 1\right) - z\left(\vartheta + a\right)\vartheta,\\ \Theta_{1,1}^{U(-b)} &= \vartheta\left(\vartheta + b\right)\left(\vartheta + b - 1\right) - z\left(\vartheta + a\right)\left(\vartheta + b\right),\\ \Theta_{1,1}^{U(-a)} &= \vartheta\left(\vartheta + b\right)\left(\vartheta + a\right) - z\left(\vartheta + a\right)\left(\vartheta + a + 1\right),\end{aligned}$$

(iv) Truncation

$$\Theta_{1,1}^{T} = \vartheta\left(\vartheta + b\right)\left(\vartheta - N - 1\right) - z\left(\vartheta + a\right)\left(\vartheta - N\right).$$

Generalized Krawtchouk

Differential operator:

$$\Theta_{2,0;N} = \vartheta - z\left(\vartheta - N\right)\left(\vartheta + a\right)$$

Transformations (of type $3, 1; N$):

(i) Christoffel

$$\Theta_{2,0;N}^{C} = \vartheta\left(\vartheta - \omega - 1\right) - z\left(\vartheta - N\right)\left(\vartheta + a\right)\left(\vartheta - \omega + 1\right),$$

(ii) Geronimus

$$\Theta_{2,0;N}^{G} = \vartheta\left(\vartheta - \omega\right) - z\left(\vartheta - N\right)\left(\vartheta + a\right)\left(\vartheta - \omega\right),$$

(iii) Reduced-Uvarov

$$\begin{aligned}\Theta_{2,0;N}^{U(0)} &= \vartheta\,(\vartheta-1) - z\,(\vartheta-N)\,(\vartheta+a)\,\vartheta,\\ \Theta_{2,0;N}^{U(N)} &= \vartheta\,(\vartheta-N) - z\,(\vartheta-N)\,(\vartheta+a)\,(\vartheta-N+1)\,,\\ \Theta_{2,0;N}^{U(-a)} &= \vartheta\,(\vartheta+a) - z\,(\vartheta-N)\,(\vartheta+a)\,(\vartheta+a+1)\,.\end{aligned}$$

Generalized Hahn functional of type I

Differential operator:

$$\Theta_{2,1} = \vartheta\,(\vartheta+b) - z\,(\vartheta+a_1)\,(\vartheta+a_2)\,.$$

Transformations (of type $3,2$):

(i) Christoffel

$$\Theta_{2,1}^{C} = \vartheta\,(\vartheta+b)\,(\vartheta-\omega-1) - z\,(\vartheta+a_1)\,(\vartheta+a_2)\,(\vartheta-\omega+1)\,,$$

(ii) Geronimus

$$\Theta_{2,1}^{G} = \vartheta\,(\vartheta+b)\,(\vartheta-\omega) - z\,(\vartheta+a_1)\,(\vartheta+a_2)\,(\vartheta-\omega)\,,$$

(iii) Reduced-Uvarov

$$\begin{aligned}\Theta_{2,1}^{U(0)} &= \vartheta\,(\vartheta+b)\,(\vartheta-1) - z\,(\vartheta+a_1)\,(\vartheta+a_2)\,\vartheta,\\ \Theta_{2,1}^{U(-b)} &= \vartheta\,(\vartheta+b)\,(\vartheta+b-1) - z\,(\vartheta+a_1)\,(\vartheta+a_2)\,(\vartheta+b)\,,\\ \Theta_{2,1}^{U(-a_1)} &= \vartheta\,(\vartheta+b)\,(\vartheta+a_1) - z\,(\vartheta+a_1)\,(\vartheta+a_2)\,(\vartheta+a_1+1)\,,\\ \Theta_{2,1}^{U(-a_2)} &= \vartheta\,(\vartheta+b)\,(\vartheta+a_2) - z\,(\vartheta+a_1)\,(\vartheta+a_2)\,(\vartheta+a_2+1)\,,\end{aligned}$$

(iv) Truncation

$$\Theta_{2,1}^{T} = \vartheta\,(\vartheta+b)\,(\vartheta-N-1) - z\,(\vartheta+a_1)\,(\vartheta+a_2)\,(\vartheta-N)\,.$$

5 Conclusion

We have studied the discrete functionals (6) characterized by the hypergeometric differential equation satisfied by their first moment $\lambda_0(z) = L[1]$,

$$[\vartheta(\vartheta + \mathbf{b}) - z(\vartheta + \mathbf{a})][\lambda_0] = 0, \quad \mathbf{a} \in \mathbb{K}^p, \quad \mathbf{b} \in \mathbb{K}^q.$$

We obtained recurrence relations for the moments on the monomial and falling factorial polynomial bases, and gave examples for all functionals of class $s \leq 2$, where $s = \max\{p-1, q\}$.

We are currently working on further applications of our results to study some properties of the orthogonal polynomials themselves (representations, recurrence-relation coefficients, generating functions, etc.)

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