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MACMAHON'S PARTITION ANALYSIS XIV: PARTITIONS WITH n COPIES OF n

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ABSTRACT. We apply the methods of partition analysis to partitions with n copies of n. This allows us to obtain multivariable generating functions related to classical Rogers-Ramanujan type identities. Also, partitions with n copies of n are extended to partition diamonds yielding numerous new results including a natural connection to overpartitions and a variety of partition congruences.

AMS Mathematics Subject Classification. Primary: 05A17, 11P83, 11F33; secondary: 05A15, 05A19, 11Y99.

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1. INTRODUCTION

This paper is devoted to the study of partitions with n copies of n by means of partition analysis. Our basic objects are subscripted positive integers wherein the subscript does not exceed the integer. We order these objects lexicographically

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < \dots$$

We now define the weighted difference between m_i and n_j to be

$$((m_i - n_j)) := m - n - i - j.$$

Additionally, we shall sometimes consider partitions with $(n+1)$ copies of n. Here the subscript 0 will be allowed, and lexicographic order is maintained; i.e.,

$$1_0 < 1_1 < 2_0 < 2_1 < 2_2 < 3_0 < 3_1 < 3_2 < 3_3 < \dots$$

The study of partitions with n copies of n had its origins in Regime III of the hard hexagon model [3]. It was made explicit in [4] where two theorems were presented linking partitions with n copies of n to ordinary partitions.

Theorem 1 ([4], p. 41)**.** *The number of partitions of N using n copies of n in which the weighted difference between parts is non-negative equals the number of ordinary partitions of N into parts $\not\equiv 0, \pm 6 \pmod{14}$.*

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Theorem 2 ([4], p. 41)**.** *The number of partitions of N using n copies of n in which the weighted difference between parts is positive equals the number of ordinary partitions of N into parts* $\not\equiv 0, \pm 4 \pmod{10}$.

In [1], A.K. Agarwal proved that the generating function for partitions with n copies of n and weighted difference between parts $\geq r$ with $r \geq -2$ is given by

$$\sum_{n\geq 0} \frac{q^{n^2+r\binom{n}{2}}}{(q;q)_n(q;q^2)_n}, \tag{1.1}$$

where

$$(A;q)_m := (1-A)(1-Aq)\dots(1-Aq^{m-1}).$$

Setting $r=0$ in the generating function (1.1), from [17, p. 158, eq. (61)] Agarwal deduced Theorem 1, and setting $r=1$ in (1.1), from equation (46) in [17] he deduced Theorem 2. In addition, he noted that $r=-1$ matches equation (88) in [17].

In [7] the theory of separable integer partition classes was applied to deduce all of the above results results.

The object of this paper is to apply partition analysis to the study of partitions with n copies of n, and also partitions with $(n+1)$ copies of n. Partition analysis easily provides us with multivariable generating functions which, in turn, reveal theorems not readily discovered by other methods.

In Section 2, we derive a multivariable version of (1.1) in Theorem 5; moreover,we present new proofs of Theorems 1 and 2, and draw a connection to one of Ramanujan's fifth order mock theta functions.

Section 3 is devoted to partitions with $(n+1)$ copies of n. As a corollary of this study, we prove

Theorem 3. *The number of overpartitions of N equals the number of partitions of N with $(n+1)$ copies of n wherein the weighted difference between parts is non-negative.*

Remark. Overpartitions (first defined by Corteel and Lovejoy [9]) are ordinary partitions with the addition of one part of each part size possibly being overlined.

As an example of Theorem 3, take $N=4$: The 14 overpartitions of 4 are

$$4, \bar{4}, 31, \bar{3}1, 3\bar{1}, \bar{3}\bar{1}, 22, \bar{2}2, 211, \bar{2}11, 2\bar{1}1, \bar{2}\bar{1}1, 1111, \bar{1}111.$$

The 14 partitions of 4 into $(n+1)$ copies of n with non-negative weighted difference between parts are

$$4_0, 4_1, 4_2, 4_3, 4_4, 3_0 1_0, 3_0 1_1, 3_1 1_0, 3_1 1_1, 3_2 1_0, 2_0 2_0, 2_0 1_0 1_0, 2_1 1_0 1_0, 1_0 1_0 1_0 1_0.$$

In Section 4 we extend the concept of partition diamonds to partitions with n (and $(n+1)$) copies of n. Partition diamonds were first studied in [5]. In this setting summands a_i are placed at the vertices of a directed graph with the inequalities between parts as indicated in Figure 1; an arrow pointing from a_i to a_j means that $a_i \geq a_j$.

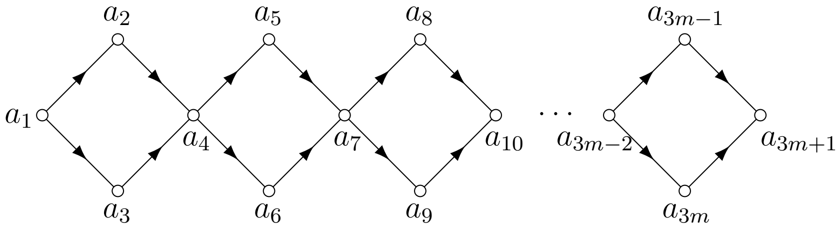

Figure 1. A plane partition diamond of length m.

To extend the n copies of n (or $(n+1)$ copies of n) concept to this setting, we introduce the following conditions:

(A) The summands at the links (i.e., $a_1, a_4, a_7, a_{10}, \dots$) may have a full set of subscripts, namely $1, 2, \dots, n$ for n copies of n, or $0, 1, \dots, n$ for $(n+1)$ copies of n.

(B) All other summands may only have the subscript 0.

(C) The "$\geq$" sign means that the weighted difference between parts is ≥ 0.

Here as before we shall obtain multivariable generating functions. We shall also consider Schmidt-type theorems as we did in [6]. Perhaps the most striking result here is the following, proven in Section 3:

Theorem 4. *The number of partitions of N into parts of three colors equals the number of partitions with $(n+1)$ copies of n wherein the weighted difference between parts is non-negative and the first, third, fifth, $\dots$ summands add up to N.*

The rest of our article is structured as follows. In Section 5, with Theorem 9 we begin to present congruences for the partition numbers PDN1(N); i.e., the number of partition diamonds with $(n+1)$ copies of n where summing the parts at the links gives N. In Section 6 we describe the algorithmic tool we use to derive further congruences: Radu's Ramanujan-Kolberg algorithm [13] and Smoot's Mathematica package RaduRK [18] which implements it. In Section 7 we consider diamond partition congruences on arithmetic subsequences $5n+j$, $0 \leq j \leq 4$. Section 8 presents some more congruences, for instance, $\mathrm{PDN1}(7m+5) \equiv 0 \pmod{7}$ in Corollary 7. Other divisibility properties, mod 25 and mod 49, are proven in Theorem 17 and Theorem 15. The conclusion is made by the curious congruences (8.23) and (8.25).

2. Partitions with n copies of n and partition analysis

Theorem 5. *The generating function for partitions with n copies of n with m parts in which the weighted difference between parts $\geq r$ with $r \geq -2$ is given by*

$$\frac{x_1^{(m-1)r+(2m-1)} x_2^{(m-2)r+(2m-3)} \dots x_m^{0\cdot r+1}}{\prod_{i=1}^{m}(1-x_1x_2\dots x_i)\cdot(1-x_1)\prod_{i=2}^{m}(1-x_1^2x_2^2\dots x_{i-1}^2x_i)}, \tag{2.1}$$

where the exponent of x_i accounts for the ith part of the partition in question.

Before we begin the proof of Theorem 5, we must recall the basic ideas of partition analysis [11].

We shall be concerned with the operator $\Omega_{\geq}$. It operates on multivariate Laurent series as follows:

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r}. \tag{2.2}$$

Thus $\Omega_{\geq}$ annihilates all terms in which any λ_i exponent is negative, and it then sets all $\lambda_i = 1$ in the remaining terms. In the analytic setting, the domain of the $A_{s_1,\ldots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. In addition, the $A_{s_1,\ldots,s_r}$ are required to be such that any of the series involved is absolutely convergent within the domain of the definition of $A_{s_1,\ldots,s_r}$.

To prove Theorem 5, we require the following identities from partition analysis, [6, p. 98, eq. (2.2)] and [11, p. 102, §348, 2nd eq.]:

$$\underset{\geq}{\Omega} \frac{\lambda^{-s}}{(1-A\lambda)(1-B\lambda^{-r})\ldots(1-C\lambda^{-t})} = \frac{A^s}{(1-A)(1-A^rB)\ldots(1-A^tC)}, \tag{2.3}$$

where $s, r, \ldots, t \in \mathbb{Z}_{\geq 0}$, and

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)(1-\frac{z}{\lambda})} = \frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)}. \tag{2.4}$$

Proof of Theorem 5. Partition analysis allows us to embed the various difference conditions in the exponents of the λ's. Also we note that the role played by the subscripts of the n copies of n only appears in the exponents of the λ's where the requirements for the weighted differences between parts is effectuated.

Hence, in terms of partition analysis our generating function is given by

$$\begin{aligned}
&\underset{\geq}{\Omega} \sum_{\substack{n_1,\ldots,n_m \geq 0 \\ 1 \leq i_j \leq n_j, 1 \leq j \leq m}} x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m} \lambda_1^{n_1-n_2-i_1-i_2-r} \lambda_2^{n_2-n_3-i_2-i_3-r} \ldots \lambda_{m-1}^{n_{m-1}-n_m-i_{m-1}-i_m-r} \lambda_m^{n_m-1} \\
&= \underset{\geq}{\Omega} \sum_{\substack{i_1,\ldots,i_m \geq 1 \\ n_1,\ldots,n_m \geq 0}} x_1^{n_1+i_1} x_2^{n_2+i_2} \ldots x_m^{n_m+i_m} \lambda_1^{n_1-n_2-2i_2-r} \lambda_2^{n_2-n_3-2i_3-r} \ldots \lambda_{m-1}^{n_{m-1}-n_m-2i_m-r} \lambda_m^{n_m+i_m-1} \\
&= \frac{x_1 x_2 \ldots x_m}{1-x_1} \underset{\geq}{\Omega} \frac{\lambda_1^{-r-2} \lambda_2^{-r-2} \ldots \lambda_{m-1}^{-r-2} \lambda_m^{-1}}{(1-x_1\lambda_1)(1-x_2\frac{\lambda_2}{\lambda_1}) \ldots (1-x_{m-1}\frac{\lambda_{m-1}}{\lambda_{m-2}})(1-x_m\frac{\lambda_m}{\lambda_{m-1}})} \\
&\qquad \times \frac{1}{(1-\frac{x_2}{\lambda_1^2})(1-\frac{x_3}{\lambda_2^2}) \ldots (1-\frac{x_{m-1}}{\lambda_{m-2}^2})(1-\lambda_m \frac{x_m}{\lambda_{m-1}^2})}.
\end{aligned} \tag{2.5}$$

Now if $m = 1$, the above is merely

$$\frac{x_1}{1-x_1}.$$

If $m = 2$, then

$$\frac{x_1x_2}{1-x_1}\underset{\geq}{\Omega}\frac{\lambda_1^{-r-2}}{(1-x_1\lambda_1)(1-x_2\frac{\lambda_2}{\lambda_1})(1-x_2\frac{\lambda_2}{\lambda_1^2})}=\frac{x_1^{r+3}x_2}{(1-x_1)(1-x_1x_2)\cdot(1-x_1)(1-x_1^2x_2)}$$

by (2.3). In general, if we eliminate λ_1 from (2.5) by means of (2.3), we obtain

$$\frac{x_1^{r+3}x_2\dots x_m}{(1-x_1)^2(1-x_1x_2)}\underset{\geq}{\Omega}\frac{\lambda_2^{-r-2}\dots\lambda_{m-1}^{-r-2}}{(1-x_1x_2\lambda_2)(1-x_3\frac{\lambda_3}{\lambda_2})\dots(1-x_{m-1}\frac{\lambda_{m-1}}{\lambda_{m-2}})(1-x_m\frac{\lambda_m}{\lambda_{m-1}})}$$
$$\times\frac{1}{(1-\frac{x_3}{\lambda_2^2})(1-\frac{x_4}{\lambda_3^2})\dots(1-\frac{x_{m-1}}{\lambda_{m-2}^2})(1-\lambda_m\frac{x_m}{\lambda_{m-1}^2})}.$$

Now the above $\Omega_{\geq}$ expression is just the original expression with m replaced by $m-1$, each $x_i \to x_{i+1}$ for $i = 2, \dots, m-1$, and x_1 replaced by x_1x_2. Thus iteration yields that all λ's may be eliminated in exactly the same way yielding (2.1) as desired. ☐

Corollary 1 ([1])**.** *The generating function for partitions with n copies of n in which the weighted difference between parts is $\geq r$ with $r \geq -2$ is given by*

$$\sum_{m\geq 0}\frac{q^{m^2+r\binom{m}{2}}}{(q;q)_m(q;q^2)_m}. \tag{2.6}$$

Proof. Set all $x_i = q$ in Theorem 5, and then sum over all $m \geq 0$. ☐

Note: Corollary 1 was originally proved by A.K. Agarwal in [1] using q-difference equations. A second proof appeared in [7] where the partitions with n copies of n are analyzed using the method of separable integer partition classes.

Proof of Theorem 1. Take $r = 0$ in (2.6) and recall [17, p. 158, eq. (61)],

$$\sum_{n\geq 0}\frac{q^{m^2}}{(q;q)_m(q;q^2)_m}=\prod_{\substack{n=1\\ n\not\equiv 0,\pm 6\ (\mathrm{mod}\ 14)}}\frac{1}{1-q^n}.$$

☐

Proof of Theorem 2. Take $r = 1$ in (2.6) and recall [17, p. 156, eq. (46)],

$$\sum_{n\geq 0}\frac{q^{m(3m-1)/2}}{(q;q)_m(q;q^2)_m}=\prod_{\substack{n=1\\ n\not\equiv 0,\pm 4\ (\mathrm{mod}\ 10)}}\frac{1}{1-q^n}.$$

☐

2.1. **On the $r = -2$ case for n copies of n.** The special case $r = -2$ of Corollary 1 leads to a connection to one of the fifth order mock theta functions of Ramanujan. Namely, setting $r = -2$ in (2.6) gives,

$$\begin{aligned}
\sum_{n\geq 0} \frac{q^n}{(q;q)_n(q;q^2)_n} &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} \frac{q^n}{(q;q)_n}(q^{2n+1};q^2)_\infty \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} \frac{q^n}{(q;q)_n} \sum_{m\geq 0} \frac{(-1)^m q^{m^2+2mn}}{(q^2;q^2)_m} \quad \text{(by [2, eq. (2.2.6)])} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m\geq 0} \frac{(-1)^m q^{m^2}}{(q^2;q^2)_m} \frac{1}{(q^{2m+1};q)_\infty} \quad \text{(by [2, eq. (2.2.5)])} \\
&= \frac{1}{(q^2;q^2)_\infty (q;q)_\infty} \sum_{m\geq 0} \frac{(-1)^m q^{m^2}(q;q)_{2m}}{(q^2;q^2)_m} \\
&= \frac{1}{(q^2;q^2)_\infty (q;q)_\infty} \cdot \sum_{m\geq 0} (-1)^m q^{m^2} (q;q^2)_m \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \cdot \phi_0(-q) \quad \text{(by [2, eq. (1.2.5)])}, \qquad (2.7)
\end{aligned}$$

where ϕ_0 is one of Ramanujan's fifth order mock theta functions. This leads us to include the following "Ramanujanesque" theorem.

Theorem 6. *The generating function for partitions with n copies of n in which the weighted difference between parts is ≥ -2 is given by*

$$\frac{\phi_0(-q)}{\phi(-q)}, \qquad (2.8)$$

where ϕ_0 is one of Ramanujan's fifth order mock theta functions,

$$\phi_0(q) = \sum_{m\geq 0} q^{m^2}(-q;q^2)_m,$$

and, in Ramanujan's notation,

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Proof. In view of (2.7), the proof is completed by noting that

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2},$$

a fact already known to Gauß [2, eq. (2.2.12)]). ☐

For example,

$$\frac{\phi_0(-q)}{\phi(-q)} = 1 + q + 3q^2 + 6q^3 + 11q^4 + 19q^5 + 32q^6 + \dots;$$

the 11 relevant partitions of 4 are

$$4_4, 4_3, 4_2, 4_1, 3_3 + 1_1, 3_2 + 1_1, 3_1 + 1_1, 2_1 + 2_1, 2_2 + 1_1 + 1_1, 2_1 + 1_1 + 1_1, 1_1 + 1_1 + 1_1 + 1_1.$$

3. Partitions with $(n+1)$ copies of n and partition analysis

The treatment of partitions with $(n+1)$ copies of n is simpler than that of n copies of n and leads to some striking results. We restrict our considerations to the simplest case.

Theorem 7. *The generating function for partitions with $(n+1)$ copies of n in which the weighted difference between parts is ≥ 0 is given by*

$$\frac{1}{\prod_{i=1}^{\infty}(1 - x_1x_2\dots x_i)\prod_{i=1}^{\infty}(1 - x_1^2x_2^2\dots x_{i-1}^2x_i)}, \tag{3.1}$$

where the exponent of x_i accounts for the ith part of the partition in question.

Proof. We begin with the $\Omega_{\geq}$ version of the generating function,

$$\begin{aligned}
&\lim_{m\to\infty} \underset{\geq}{\Omega} \sum_{\substack{n_1,\dots,n_m\geq 0\\ 0\leq i_j\leq n_j, 1\leq j\leq m}} x_1^{n_1}x_2^{n_2}\dots x_m^{n_m}\lambda_1^{n_1-n_2-i_1-i_2}\lambda_2^{n_2-n_3-i_2-i_3}\dots\lambda_{m-1}^{n_{m-1}-n_m-i_{m-1}-i_m}\lambda_m^{n_m}\\
&= \frac{1}{1-x_1}\lim_{m\to\infty} \underset{\geq}{\Omega} \frac{1}{(1-x_1\lambda_1)(1-x_2\frac{\lambda_2}{\lambda_1})\dots(1-x_{m-1}\frac{\lambda_{m-1}}{\lambda_{m-2}})(1-x_m\frac{\lambda_m}{\lambda_{m-1}})}\\
&\qquad\times\frac{1}{(1-\frac{x_2}{\lambda_1^2})(1-\frac{x_3}{\lambda_2^2})\dots(1-\frac{x_{m-1}}{\lambda_{m-2}^2})(1-\lambda_m\frac{x_m}{\lambda_{m-1}^2})}\\
&= \lim_{m\to\infty}\frac{1}{\prod_{i=1}^{m}(1-x_1x_2\dots x_i)\prod_{i=1}^{m}(1-x_1^2x_2^2\dots x_{i-1}^2x_i)}.
\end{aligned}$$

We have telescoped the proof because it is exactly the same as the proof of Theorem 5 except there are no numerator entries. □

Proof of Theorem 3. In Theorem 7, set all $x_i = q$. Thus the generating function for partitions with $(n+1)$ copies of n wherein the weighted difference between parts is non-negative is given by

$$\prod_{n=1}^{\infty}\frac{1}{1-q^n}\prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}} = \prod_{n=1}^{\infty}\frac{1+q^n}{1-q^n} \quad \text{(by [2, p. 5, eq. (1.2.5)])},$$

and the last product is the generating function for overpartitions [9, eq. (1.1)]. □

Proof of Theorem 4. In Theorem 7, set odd subscripted x's equal to q, and set even subscripted x's equal to 1. This yields the generating function for the Schmidt-type partitions described in Theorem 4, and this generating function is

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^3},$$

which is the generating function for partitions using three colors. ☐

4. Partition diamonds

Partition diamonds were first discussed by us in [5]. The one variable generating function for partition diamonds (or plane partition diamonds as we called them in[5]) is given by [5, p. 237, Cor. 2.1],

$$\prod_{n=1}^{\infty} \frac{1+q^{3n+1}}{1-q^n} = 1 + q + 3q^2 + 4q^3 + 7q^4 + 11q^5 + \dots \tag{4.1}$$

For example, Figure 2 shows the eleven partition diamonds of 5.

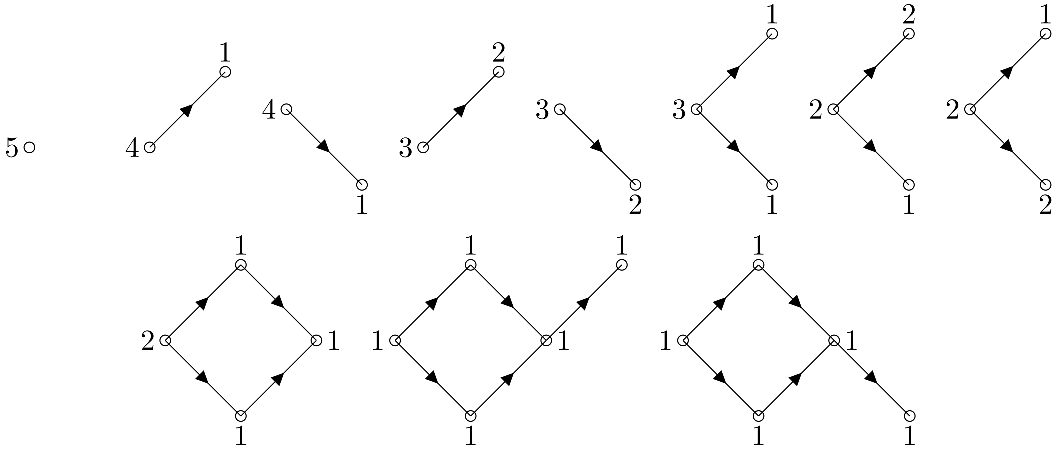

Figure 2. Eleven partition diamonds of 5.

Additionally in [6], we looked at extending Schmidt-type partitions to partition diamonds. Here we only added up the summands at the links. We found the generating function to be [6, p. 102, eq. (4.1)],

$$\frac{(-q;q)_\infty}{(q;q)_\infty^3} = 1 + 4q + 13q^2 + 36q^3 + 90q^4 + \dots,$$

where

$$(A;q)_\infty = \lim_{n\to\infty} (A;q)_n.$$

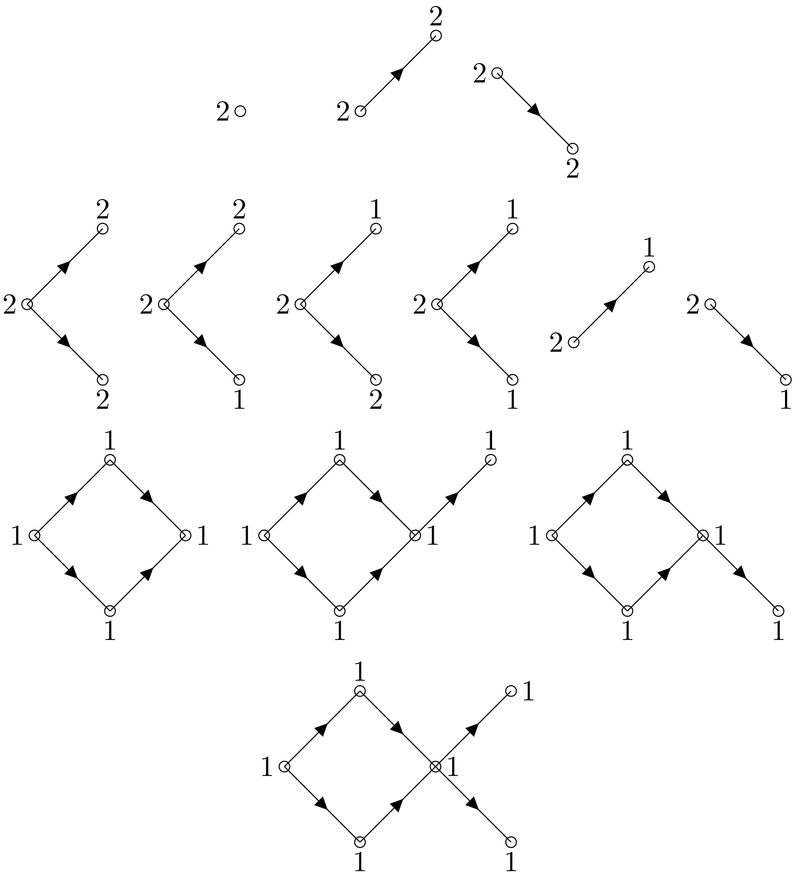

FIGURE 3. Thirteen partitions with link sum 2.

For example, Figure 3 shows the thirteen partitions where the summands at the links sum up to 2.

We shall restrict ourselves here to considering partition diamonds with $(n+1)$ copies of n modified as follows. As in the original case, we place summands at the vertices of the directed graph in Figure 4.

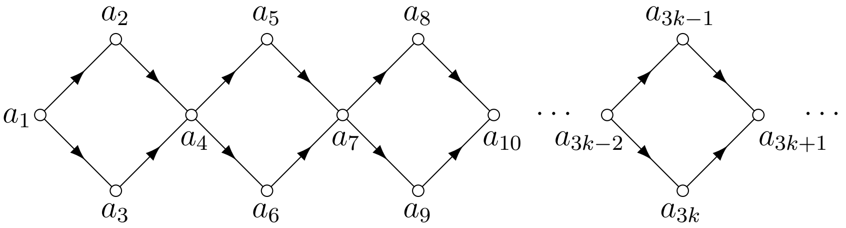

FIGURE 4. A plane partition diamond.

(A) The summands at the links (i.e., $a_1, a_4, a_7, a_{10}, \ldots$) are allowed to be $(n+1)$ copies of n.

(B) All other a_i ($i \not\equiv 1 \pmod 3$) can only have the subscript 0.

(C) An arrow pointing from a_i to a_j means that $a_i \geq a_j$; the "$\geq$" sign means that the weighted difference between parts is ≥ 0.

The following theorem is on partition diamonds of length m as depicted in Figure 1.

Theorem 8. *The generating function for partitions with* $(n+1)$ *copies of* n *on diamonds of length* m *as modified in (A), (B), and (C) above is given by*

$$\frac{\prod_{j=1}^{m}(1 - x_1^2 x_2^2 \dots x_{3j-2}^2 x_{3j-1} x_{3j})}{\prod_{j=1}^{3m+1}(1 - x_1 x_2 \dots x_j) \prod_{j=0}^{m}\left(1 - \frac{x_1^2 x_2^2 \dots x_{3j+1}^2}{x_{3j+1}}\right) \prod_{j=1}^{m}\left(1 - \frac{x_1 x_2 \dots x_{3j}}{x_{3j-1}}\right)}, \tag{4.2}$$

where the exponent of x_i *accounts for the* i*th part of the partition in question.*

To prove Theorem 8 it is convenient to introduce another identity from partition analysis.

Lemma 4.1.

$$\begin{aligned} &\underset{\geq}{\Omega} \frac{1}{(1 - z_1\lambda_1\lambda_2)\left(1 - z_2\frac{\lambda_3}{\lambda_1}\right)\left(1 - z_3\frac{\lambda_4}{\lambda_2}\right)\left(1 - z_4\frac{1}{\lambda_3\lambda_4}\right)\left(1 - z_5\frac{1}{\lambda_3^2\lambda_4^2}\right)} \\ &= \frac{1 - z_1^2 z_2 z_3}{(1 - z_1)(1 - z_1 z_2)(1 - z_1 z_3)(1 - z_1 z_2 z_3)(1 - z_1 z_2 z_3 z_4)(1 - z_1^2 z_2^2 z_3^2 z_5)}. \end{aligned} \tag{4.3}$$

Proof. In the first step we use (2.3) to eliminate λ_4 which reduces the left side of (4.3) to

$$\underset{\geq}{\Omega} \frac{1}{(1 - z_1\lambda_1\lambda_2)\left(1 - z_2\frac{\lambda_3}{\lambda_1}\right)\left(1 - \frac{z_3}{\lambda_2}\right)\left(1 - \frac{z_3 z_4}{\lambda_2\lambda_3}\right)\left(1 - \frac{z_3^2 z_5}{\lambda_2^2\lambda_3^2}\right)}.$$

Using (2.3) again, we eliminate λ_3 and λ_2, in this order, to obtain

$$\frac{1}{(1 - z_1 z_2 z_3 z_4)(1 - z_1^2 z_2^2 z_3^2 z_5)} \underset{\geq}{\Omega} \frac{1}{(1 - z_1\lambda_1)(1 - z_1 z_3\lambda_1)\left(1 - \frac{z_2}{\lambda_1}\right)}.$$

Finally, elimination of λ_1 using (2.4) yields the desired result. ☐

Proof of Theorem 8. The $\Omega_{\geq}$ form of the generating function is the following,

$$\begin{aligned}
&\underset{\geq}{\Omega} \sum_{\substack{n_1,\ldots,n_{3m+1}\geq 0\\ 0\leq i_{3j+1}\leq n_{3j+1}\\ 0\leq j\leq m}} x_1^{n_1}x_2^{n_2}\ldots x_{3m+1}^{n_{3m+1}}\lambda_1^{n_1-n_2-i_1}\lambda_2^{n_1-n_3-i_1}\lambda_3^{n_2-n_4-i_4}\lambda_4^{n_3-n_4-i_4}\\
&\qquad\times \lambda_5^{n_4-n_5-i_4}\lambda_6^{n_4-n_6-i_4}\lambda_7^{n_5-n_7-i_7}\lambda_8^{n_6-n_7-i_7}\ldots\\
&\qquad\times \lambda_{4m-3}^{n_{3m-2}-n_{3m-1}-i_{3m-2}}\lambda_{4m-2}^{n_{3m-2}-n_{3m}-i_{3m-2}}\lambda_{4m-1}^{n_{3m-1}-n_{3m+1}-i_{3m+1}}\lambda_{4m}^{n_{3m}-n_{3m+1}-i_{3m+1}}\\
&= \sum_{\substack{n_1,\ldots,n_{3m+1}\geq 0\\ 0\leq i_{3j+1},0\leq j\leq m}} x_1^{n_1+i_1}x_2^{n_2}x_3^{n_3}x_4^{n_4+i_4}\ldots x_{3m-1}^{n_{3m-1}}x_{3m}^{n_{3m}}x_{3m+1}^{n_{3m+1}+i_{3m+1}}\\
&\qquad\times \lambda_1^{n_1-n_2}\lambda_2^{n_1-n_3}\lambda_3^{n_2-n_4-2i_4}\lambda_4^{n_3-n_4-2i_4}\lambda_5^{n_4-n_5}\lambda_6^{n_4-n_6}\lambda_7^{n_5-n_7-2i_7}\lambda_8^{n_6-n_7-2i_7}\ldots\\
&\qquad\times \lambda_{4m-3}^{n_{3m-2}-n_{3m-1}}\lambda_{4m-2}^{n_{3m-2}-n_{3m}}\lambda_{4m-1}^{n_{3m-1}-n_{3m+1}-2i_{3m+1}}\lambda_{4m}^{n_{3m}-n_{3m+1}-2i_{3m+1}},
\end{aligned}$$

where we shifted each n_{3j+1} to $n_{3j+1}+i_{3j+1}$. Using geometric series summation this reduces to

$$\begin{aligned}
S(m) &:= \frac{1}{1-x_1}\underset{\geq}{\Omega}\frac{1}{(1-x_1\lambda_1\lambda_2)}\frac{1}{\left(1-x_2\frac{\lambda_3}{\lambda_1}\right)\left(1-x_3\frac{\lambda_4}{\lambda_2}\right)\left(1-x_4\frac{\lambda_5\lambda_6}{\lambda_3\lambda_4}\right)\left(1-x_4\frac{1}{\lambda_3^2\lambda_4^2}\right)}\\
&\quad\times\frac{1}{\left(1-x_5\frac{\lambda_7}{\lambda_5}\right)\left(1-x_6\frac{\lambda_8}{\lambda_6}\right)\left(1-x_7\frac{\lambda_9\lambda_{10}}{\lambda_7\lambda_8}\right)\left(1-x_7\frac{1}{\lambda_7^2\lambda_8^2}\right)}\ldots\\
&\quad\times\frac{1}{\left(1-x_{3m-4}\frac{\lambda_{4m-5}}{\lambda_{4m-7}}\right)\left(1-x_{3m-3}\frac{\lambda_{4m-4}}{\lambda_{4m-6}}\right)\left(1-x_{3m-2}\frac{\lambda_{4m-3}\lambda_{4m-2}}{\lambda_{4m-5}\lambda_{4m-4}}\right)\left(1-x_{3m-2}\frac{1}{\lambda_{4m-5}^2\lambda_{4m-4}^2}\right)}\\
&\quad\times\frac{1}{\left(1-x_{3m-1}\frac{\lambda_{4m-1}}{\lambda_{4m-3}}\right)\left(1-x_{3m}\frac{\lambda_{4m}}{\lambda_{4m-2}}\right)\left(1-x_{3m+1}\frac{1}{\lambda_{4m-1}\lambda_{4m}}\right)\left(1-x_{3m+1}\frac{1}{\lambda_{4m-1}^2\lambda_{4m}^2}\right)}.
\end{aligned}$$

We shall prove Theorem 8 by mathematical induction on m. To this end, we rewrite the generating function as

$$\begin{aligned}
S(m) &= \frac{1}{1-x_1}\underset{\geq}{\Omega}\frac{1}{(1-x_1\lambda_1\lambda_2)}T_m(x_2,x_3,x_4,x_5,x_6,x_7,\ldots,x_{3m-4},x_{3m-3},x_{3m-2})\\
&\quad\times\frac{1}{\left(1-x_{3m-1}\frac{\lambda_{4m-1}}{\lambda_{4m-3}}\right)\left(1-x_{3m}\frac{\lambda_{4m}}{\lambda_{4m-2}}\right)\left(1-x_{3m+1}\frac{1}{\lambda_{4m-1}\lambda_{4m}}\right)\left(1-x_{3m+1}\frac{1}{\lambda_{4m-1}^2\lambda_{4m}^2}\right)}.
\end{aligned}$$

If $m=1$ then $T_1=1$, and the generating function is

$$S(1)=\frac{1}{1-x_1}\underset{\geq}{\Omega}\frac{1}{(1-x_1\lambda_1\lambda_2)}\frac{1}{\left(1-x_2\frac{\lambda_3}{\lambda_1}\right)\left(1-x_3\frac{\lambda_4}{\lambda_2}\right)\left(1-x_4\frac{1}{\lambda_3\lambda_4}\right)\left(1-x_4\frac{1}{\lambda_3^2\lambda_4^2}\right)}.$$

Using Lemma 4.1 with $z_i=x_i$ for $i=1,\ldots,4$ and $z_5=x_4$, this reduces to

$$S(1)=\frac{1}{1-x_1}\cdot\frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)(1-x_1^2x_2^2x_3^2x_4)},$$

which is (4.2) when $m = 1$.

For the induction step, using Lemma 4.1 with $z_1 = x_1, z_2 = x_2, z_3 = x_3, z_4 = x_4\lambda_5\lambda_6$, and $z_5 = x_4$, we eliminate $\lambda_1, \lambda_2, \lambda_3$, and λ_4 from $S(m)$ and obtain

$$S(m) = \frac{1}{1-x_1} \cdot \frac{1 - x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1^2x_2^2x_3^2x_4)}$$
$$\times \underset{\geq}{\Omega} \frac{1}{(1-x_1x_2x_3x_4\lambda_5\lambda_6)} T_{m-1}(x_5, x_6, x_7, \ldots, x_{3m-4}, x_{3m-3}, x_{3m-2})$$
$$\times \frac{1}{\left(1 - x_{3m-1}\frac{\lambda_{4m-1}}{\lambda_{4m-3}}\right)\left(1 - x_{3m}\frac{\lambda_{4m}}{\lambda_{4m-2}}\right)\left(1 - x_{3m+1}\frac{1}{\lambda_{4m-1}\lambda_{4m}}\right)\left(1 - x_{3m+1}\frac{1}{\lambda_{4m-1}^2\lambda_{4m}^2}\right)}.$$

The $\Omega_{\geq}$ expression is $(1 - x_1x_2x_3x_4)S(m-1)$ where in the $S(m-1)$ one replaces x_1 with $x_1x_2x_3x_4$ and x_i with x_{i+3} for $i \geq 2$. Substituting the induction hypothesis (i.e., the accordingly modified expression (4.2)) completes the proof of Theorem 8. ☐

Corollary 2. *The single variable generating function for partitions with $(n+1)$ copies of n on diamonds modified as in Theorem 8 with non-negative weighted difference between parts is*

$$\prod_{j=1}^{\infty} \frac{1+q^{3j-1}}{(1-q^j)(1-q^{6j-5})} = 1 + 2q + 5q^2 + 9q^3 + 16q^4 + 27q^5 + \ldots. \tag{4.4}$$

Remark. Figure 5 shows the 9 partitions of 3 in question. The 16 partitions of 4 in question are shown in Figure 6.

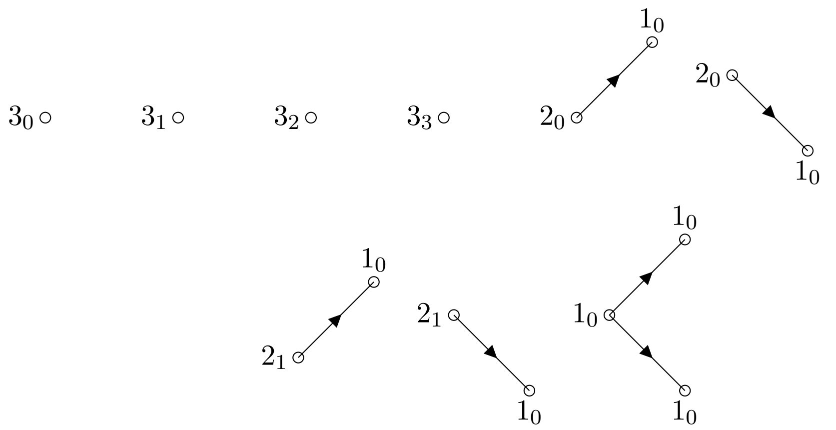

FIGURE 5. Nine partitions of 3 according to Corollary 2.

Proof of Corollary 2. The statement is obtained by setting the $x_i = q$ in Theorem 8; then let $m \to \infty$ which gives

$$\prod_{j=1}^{\infty} \frac{1-q^{6j-2}}{(1-q^j)(1-q^{6j-5})(1-q^{3j-1})} = \prod_{j=1}^{\infty} \frac{1+q^{3j-1}}{(1-q^j)(1-q^{6j-5})}.$$

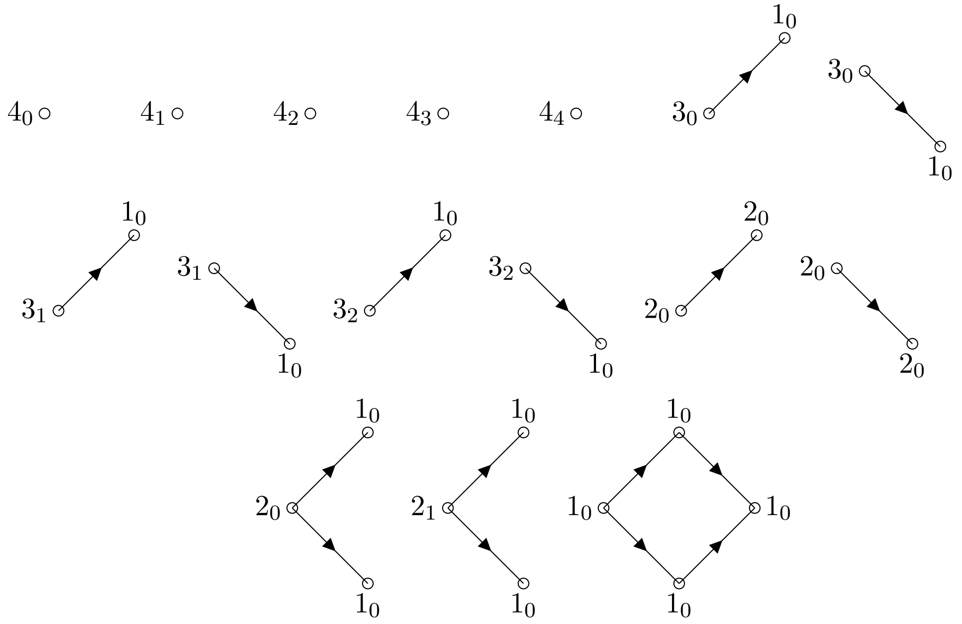

Figure 6. Sixteen partitions of 4 according to Corollary 2.

□

The infinite product in Corollary 2 is not a modular form, and thus we should not expect interesting arithmetic results for its coefficients. However, if we follow the lead of [6, Sec. 4], we obtain a lovely modular form.

Instead of summing all the summands in each position, we only sum $a_1 + a_4 + a_7 + \dots$. We called these modified partition diamonds with $(n+1)$ copies of n summed at the links, and the number of these that sum to N, we denote by PDN1(N).

Corollary 3.

$$\sum_{N\geq 0} \mathrm{PDN1}(N) q^N = \frac{(-q;q)_\infty^2}{(q;q)_\infty^3} = 1 + 5q + 18q^2 + 56q^3 + \dots. \tag{4.5}$$

Remark. The 18 partitions of 2 in question are shown in Figure 7.

Proof of Corollary 3. For $j \geq 0$ set $x_{3j+1} = q$ and all other $x_i = 1$ in Theorem 8, and let $m \to \infty$. The result is

$$\prod_{j=1}^{\infty} \frac{1-q^{2j}}{(1-q^j)^3(1-q^{2j-1})(1-q^j)} = \prod_{j=1}^{\infty} \frac{(1+q^j)^2}{(1-q^j)^3} \quad \text{(by [2, p. 5, eq. (1.2.5)])}.$$

□

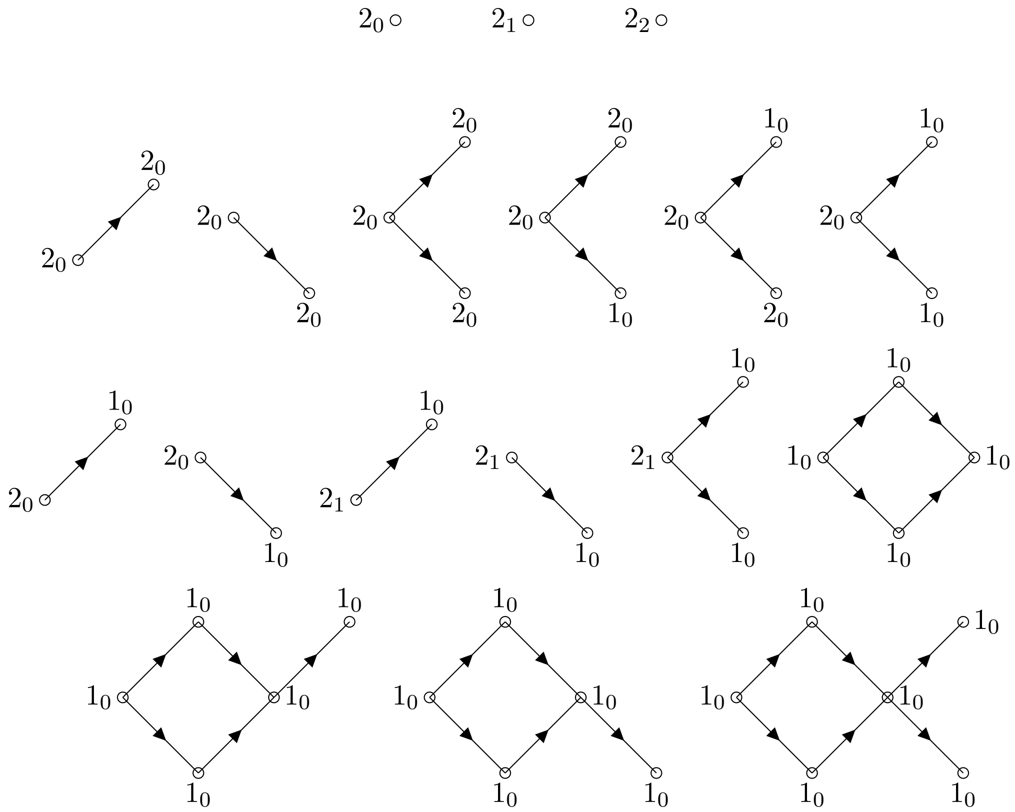

FIGURE 7. Eighteen partitions of 2 according to Corollary 3.

5. CONGRUENCES FOR PDN1(N)

There are various congruences for PDN1(N). We begin with one which we prove by a combination of elementary means and Radu's Ramanujan-Kolberg algorithm.

Theorem 9. *For* $N \geq 0$,

$$\mathrm{PDN1}(25N+24) \equiv 0 \pmod{5}. \tag{5.1}$$

Proof. First apply the relation $(1-x)^5 \equiv 1 - x^5 \,(\mathrm{mod}\, 5)$,

$$\sum_{N\geq 0} \mathrm{PDN1}(N)q^N = \frac{(-q;q)_\infty^2}{(q;q)_\infty^3} = \frac{(-q;q)_\infty^2 (q;q)_\infty^2}{(q;q)_\infty^5} \equiv \frac{(q^2;q^2)_\infty^2}{(q^5;q^5)_\infty} \pmod{5}. \tag{5.2}$$

Then let

$$\frac{(q^2;q^2)_\infty^2}{(q^5;q^5)_\infty} = \sum_{k=0} a(k)q^k.$$

Using the package RaduRK, one obtains a computer proof of the fact

$$\sum_{N\geq 0} a(25N+24)q^N = -5\frac{(q^2;q^2)_\infty^2\,(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{5.3}$$

The details of this RaduRK application are given in Section 6; see, in particular, Lemma 6.1.

Consequently, the coefficients of the powers q^{25N+24} in $\frac{(q^2;q^2)_\infty^2}{(q^5;q^5)_\infty}$ are divisible by 5, which implies

$$\mathrm{PDN1}(25N+24) \equiv 0 \pmod{5}.$$

$\square$

Remark. Using the RaduRK package, one can produce a direct proof of Theorem 9; see Theorem 10 below. In view of the size of the polynomial $p(t)$ in the witness identity (5.4), the proof based on human pre-processing in (5.2), combined with the remarkable identity (5.3), seems much preferable—at least to human standards. On the other hand, we acknowledge that identity (5.3) has been produced (and proved!) automatically using the package RaduRK which implements Radu's Ramanujan-Kolberg algorithm; further details are given in Section 6. Moreover, we want to stress as a remarkable fact that this huge polynomial $p(t)$, which arises in Theorem 10, shrinks significantly modulo 5; in this form it plays a crucial role in the proof of Theorem 17.

Theorem 10. *Let* PDN1(m) *be the number of partitions as in* (4.5). *Then*

$$f_1 \cdot \sum_{m=0}^{\infty} \mathrm{PDN1}(25m+24)q^m = p(t), \tag{5.4}$$

where

$$f_1 = \frac{1}{q^{44}}\frac{(q;q)_\infty^{115}(q^5;q^5)_\infty^{50}}{(q^2;q^2)_\infty^{22}(q^{10};q^{10})_\infty^{140}},\ t = \frac{1}{q}\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{10};q^{10})_\infty^5},$$

and $p(t) = 5tP(t)$ *with*

$$\begin{aligned}
&P(t)\\
&= 32459658256652793340860825 6 + 119503072847543353062939688960t\\
&+ 689622190840976629732090262 3232t^2 + 1533812850765002027699295202 83648t^3\\
&+ 182921846930805292869549295665 1520t^4 + 13819125710348893484341056396853248t^5\\
&+ 72971357560570615005925022116085760t^6 + 287199295372743533530494433069367296t^7\\
&+ 881346194614139082004384749478477 7824t^8 + 2180313233410380663445243315659735040t^9\\
&+ 4460970772899049133504095105652359168t^{10} + 7703065340325515530278640499388579840t^{11}\\
&+ 11409629778251399394140025877667250176t^{12} + 14687763409452739890760651458712961024t^{13}\\
&+ 16608609335697313347689892821671608320t^{14} + 16638911058122600685121502590204379136t^{15}\\
&+ 14869442236187928574382805868179619840t^{16} + 11916711798303140098466862359570808832t^{17}\\
&+ 8599098746081219357796950163878576128t^{18} + 5602894230658053193937556004352819200t^{19}
\end{aligned}$$

$$
\begin{aligned}
&+ 3302060814525530783442440164390993920t^{20} + 1761410314013856371267137936818176000t^{21}\\
&+ 850093690616930899306886917143920640t^{22} + 370637054941383822203500302706933760t^{23}\\
&+ 145587634974246956645505618359091200t^{24} + 51311239250489202655134309559566336t^{25}\\
&+ 16133387815338618133480714335682560t^{26} + 4490741688647323831089503395643392t^{27}\\
&+ 1095409165509054476024367114354688t^{28} + 231068016263146254880371030097920t^{29}\\
&+ 41433617578738390016198724026368t^{30} + 6178825282684139755638552002560t^{31}\\
&+ 745810437098929912506246004736t^{32} + 70567661190251551574232899584t^{33}\\
&+ 5048499353760053471298007040t^{34} + 262513445062472369716482048t^{35}\\
&+ 9499816320346226257952640t^{36} + 227617525868329328133536t^{37}\\
&+ 3396173595730773280064t^{38} + 29073363829111846545t^{39}\\
&+ 126681137724730556t^{40} + 231131520971565t^{41} + 121685404272t^{42} + 7157563t^{43}.
\end{aligned}
$$

Proof. Choosing $N = 10$, and $m = 25$ and $j = 24$ as the last two entries in the procedure call `RK[10,2,{−5,2},25,24]`, produces (5.4) as a Ramanujan type relation between modular functions for $\Gamma_0(10)$. □

6. Radu's Ramanujan-Kolberg algorithm

In the remaining part of this article we present results which were derived using the Ramanujan-Kolberg algorithm developed by Cristian-Silviu Radu [13]. For actual computations, we apply the Mathematica package RaduRK by Nicolas Smoot [18] which is very convenient to use.[1] To prepare for its usage, follow the installation instructions given in [18], and invoke it within a Mathematica session as follows:

```
In[1]:= << RaduRK`
```

```
math4ti2: Mathematica interface to 4ti2
(http://www.4ti2.de)
© 2017, Ralf Hemmecke <ralf@hemmecke.org>
© 2017, Silviu Radu <sradu@risc.jku.at>
```

```
RaduRK: Ramanujan–Kolberg Program Version 3.4
2021 written by Nicolas Smoot
<nicolas.smoot@risc.jku.at>
© Research Institute for Symbolic Computation (RISC),
Johannes Kepler University Linz
```

Before running the program, one needs to set the two global key variables q and t:

```
In[2]:= {SetVar1[q], SetVar2[t]}
```

---

[1] The package is freely available at `https://combinatorics.risc.jku.at/software` upon password request via email to the second named author.

```
Out[2]=  {q, t}
```

We illustrate the usage of the package by deriving and proving (6.1), a result which we applied in the proof of Theorem 9; see (5.3).

Lemma 6.1. *Let*

$$\frac{(q^2;q^2)_\infty^2}{(q^5;q^5)_\infty} = \sum_{k=0} a(k)q^k = 1 - 2q^2 - q^4 + q^5 + 2q^6 + \dots .$$

Then

$$\sum_{n\geq 0} a(25n+24)q^n = -5\frac{(q^2;q^2)_\infty^2\,(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{6.1}$$

Proof. Using the RaduRK package we derive and prove that

$$f_1(q)\cdot\sum_{n=0}^{\infty} a(25n+24)q^n = p(t), \tag{6.2}$$

with

$$f_1(q) = \frac{(q;q)_\infty^4\,(q^5;q^5)_\infty^5}{q^2\,(q^{10};q^{10})_\infty^{10}} \quad \text{and} \quad t = \frac{(q^2;q^2)_\infty\,(q^5;q^5)_\infty^5}{q(q;q)_\infty\,(q^{10};q^{10})_\infty^5}, \tag{6.3}$$

and where

$$p(t) = -5t^2. \tag{6.4}$$

The algorithmic proof of (6.2) is done with the procedure call

```
In[3]:= RK[10, 10, {0, 2, −1, 0}, 25, 24]
```

After a few seconds, Smoot's package delivers the proof in the form,

Out [3] =

N:	10
$\{M, (r_\delta)_{\delta\vert M}\}$:	$\{10, (0, 2, -1, 0)\}$
m:	25
$P_{m,r}(j)$:	$\{24\}$
$f_1(q)$:	$\frac{(q;q)_\infty^4\left(q^5;q^5\right)_\infty^5}{q^2\left(q^{10};q^{10}\right)_\infty^{10}}$
t:	$\frac{\left(q^2;q^2\right)_\infty\left(q^5;q^5\right)_\infty^5}{q(q;q)_\infty\left(q^{10};q^{10}\right)_\infty^5}$
AB:	$\{1\}$
$\{p_g(t)\colon\ g\in$AB$\}$	$\{-5t^2\}$
Common Factor:	5

The interpretation of the output is as follows:

- The assignment $\{M,(r_\delta)_{\delta|M}\}=\{10,(0,2,-1,0)\}$ comes from the second and third entry of the procedure call `RK[10,10,`$\{0,2,-1,0\}$`,25,24]`; this corresponds to specifying $M=10$ and $(r_\delta)_{\delta|10}=(r_1,r_2,r_5,r_{10})=(0,2,-1,0)$ such that

$$\sum_{n=0}^{\infty}a(n)q^n=\prod_{\delta|M}(q^\delta;q^\delta)_\infty^{r_\delta}=\frac{(q^2,q^2)_\infty^2}{(q^5,q^5)_\infty}.$$

- The last two entries in the procedure call `RK[10,10,`$\{0,2,-1,0\}$`,25,24]` correspond to the assignment $m=25$ and $j=24$, which means that we are interested in the generating function

$$\sum_{n=0}^{\infty}a(mn+j)q^n=\sum_{n=0}^{\infty}a(25n+24)q^n.$$

In the output expression $P_{m,r}(j)$ these parameters m and j are used; i.e., here $P_{m,r}(j)=P_{25,r}(1)$ with $r=(r_\delta)_{\delta|10}=(0,2,-1,0)$.

- The first entry in the procedure call `RK[10,10,`$\{0,2,-1,0\}$`,25,24]` corresponds to specifying $N=10$, which fixes the space of modular functions the program will work with:

$$M(\Gamma_0(N)):=\ \text{the algebra of modular functions for }\ \Gamma_0(N).$$

- The output $P_{m,r}(j)=P_{25,r}(24)=\{24\}$, where $r=(0,2,-1,0)$, means that there exists a q-product

$$f_1(q)=\frac{(q;q)_\infty^4\,(q^5;q^5)_\infty^5}{q^2\,(q^{10};q^{10})_\infty^{10}}$$

such that

$$f_1(q)\prod_{k\in P_{25,r}(24)}\sum_{n=0}^{\infty}a(25n+k)q^n=f_1(q)\sum_{n=0}^{\infty}a(25n+24)q^n\in M(\Gamma_0(N))\quad\text{with}\quad N=10.$$

Note. In general, the set $P_{m,r}(j)$ need not be a singleton. For example, $P_{m,r}(j)=\{0,3\}$ in the proof of Theorem 11.

- The output

$$t=\frac{(q^2;q^2)_\infty\,(q^5;q^5)_\infty^5}{q(q;q)_\infty\,(q^{10};q^{10})_\infty^5},\quad \mathrm{AB}=\{1\},\quad\text{and}\quad \{p_g(t):g\in\mathrm{AB}\}=\{-5t^2\}\tag{6.5}$$

presents a solution to the following task: find a modular function $t\in M(\Gamma_0(N))$ and polynomials $p_g(t)$ such that

$$f_1(q)\sum_{n=0}^{\infty}a(25n+24)q^n=\sum_{g\in\mathrm{AB}}p_g(t)\cdot g.\tag{6.6}$$

In general, the elements of the finite set AB constitute a $\mathbb{C}[t]$-module basis of $M(\Gamma_0(N))$, resp. of a large subspace of $M(\Gamma_0(N))$. The elements g of AB are $\mathbb{C}$-linear combinations of modular functions in $M(\Gamma_0(N))$ which are representable in q-product form such as $f_1(q)$

and t. In the specific case under consideration, the program delivers (6.5), which means, $p_1(t) = p(t) = -5t^2$ and

$$f_1(q) \sum_{n=0}^{\infty} a(25n+24)q^n = -5t^2 \cdot 1.$$

This completes the proof of (6.2) and also of the equivalent identity (6.1) in the statement of Lemma 6.1. □

Remark. For the definition of notions such as $\Gamma_0(N)$ or $M(\Gamma_0(N))$, together with a general introduction to Radu's Ramanujan-Kolberg algorithm, see [12]. For the correctness proof and details of the algorithm, resp. of the implementation, see [13], resp. [18].

In the remaining sections we shall present a variety of congruences related to the number $\mathrm{PDN1}(N)$ of partition diamonds under consideration.

7. Congruences on $5n + j$

In this section we consider partition diamonds on arithmetic subsequences $5n + j$, $j = 0, \dots, 4$.

Theorem 11. *Let* $\mathrm{PDN1}(N)$ *be the number of partitions as in* (4.5)*. Then*

$$f_1(q) \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(5n)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+3)q^n = u(t), \tag{7.1}$$

where

$$f_1(q) = \frac{(q;q)_\infty^{47}\,(q^5;q^5)_\infty^{15}}{q^{14}\,(q^2;q^2)_\infty^{11}\,(q^{10};q^{10})_\infty^{45}} \quad \text{and} \quad t = \frac{(q^2;q^2)_\infty\,(q^5;q^5)_\infty^5}{q(q;q)_\infty\,(q^{10};q^{10})_\infty^5}, \tag{7.2}$$

and

$$\begin{aligned} u(t) = {} & 55t^{14} + 21136t^{13} + 1168672t^{12} + 20559296t^{11} + 149214720t^{10} + 620602880t^9 \\ & + 1838579712t^8 + 4113465344t^7 + 7084900352t^6 + 9194373120t^5 + 8703180800t^4 \\ & + 5532286976t^3 + 1962934272t^2 + 268435456t. \end{aligned} \tag{7.3}$$

Proof. For the algorithmic proof of (7.1) we use Smoot's package. To this end, we choose $m = 5$ and $j = 0$ as the last two entries in the procedure call,

```
In[4]:= RK[10, 2,{−5, 2}, 5, 0]
```

The program produces the Ramanujan-Kolberg type identity (7.1) in the following form:

`Out[4]` =

N:	10
$\{M, (r_\delta)_{\delta\vert M}\}$:	$\{2, (-5, 2)\}$
m:	5
$P_{m,r}(j)$:	$\{0, 3\}$
$f_1(q)$:	$\langle$ as in (7.2)$\rangle$
t:	$\langle$ as in (7.2)$\rangle$
AB:	$\{1\}$
$\{p_g(t)$: $g \in$AB$\}$	$\{u(t)\ \langle$ as in (7.3)$\rangle\}$
Common Factor:	None

$\square$

Remark. Again the relation involves modular functions in $M(\Gamma_0(N))$ with $N = 10$. But now, according to the output $P_{m,r}(j) = \{0, 3\}$, the witness identity involves a product of generating functions,

$$f_1(q) \prod_{k\in P_{m,r}(j)} \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+k)q^n = f_1(q) \sum_{n=0}^{\infty} \mathrm{PDN1}(5n)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+3)q^n = u(t),$$

with the polynomial $u(t)$ as given in the output `Out[9]`; i.e., as in (7.3). Identities involving products in this form were first studied in systematic manner by Kolberg [10]. The entry "Common Factor" in the last output line refers to a possible common factor of all the integer coefficients of $u(t)$. Here this common factor is trivial (= 1), which is indicated by "None."

Corollary 4.

$$\sum_{n=0}^{\infty} \mathrm{PDN1}(5n)q^n \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+3)q^n \equiv \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^{11}} \pmod{2}. \tag{7.4}$$

Proof. Inspection of the coefficients of the polynomial $u(t)$ in (7.3) gives

$$f_1 \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(5n)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+3)q^n \equiv t^{14} \pmod{2}.$$

Now

$$t = \frac{(q^2;q^2)_\infty\,(q^5;q^5)_\infty^5}{q(q;q)_\infty\,(q^{10};q^{10})_\infty^5} \equiv \frac{1}{q}\frac{(q;q)_\infty}{(q^5;q^5)_\infty^5} \pmod{2},$$

where for the last relation we applied the identity $1-x^2 \equiv (1-x)^2 \pmod{2}$ twice. Similarly,

$$f_1 = \frac{(q;q)_\infty^{47}\,(q^5;q^5)_\infty^{15}}{q^{14}\,(q^2;q^2)_\infty^{11}\,(q^{10};q^{10})_\infty^{45}} \equiv \frac{1}{q^{14}}\frac{(q;q)_\infty^{25}}{(q^5;q^5)_\infty^{75}} \pmod{2}.$$

Considering the quotient $t^{14}/f_1(q) \pmod{2}$ using these reductions, completes the proof. $\square$

Theorem 12. *Let* PDN1(N) *be the number of partitions as in* (4.5)*. Then*

$$f_1 \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+1)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+2)q^n = v(t), \tag{7.5}$$

with f_1 *and* t *as in* (7.2)*, and where*

$$\begin{aligned} v(t) = {} & 90t^{14} + 19731t^{13} + 1192032t^{12} + 20350496t^{11} + 150273920t^{10} + 617881600t^9 \\ & + 1839153152t^8 + 4129521664t^7 + 7048855552t^6 + 9199288320t^5 + 8766095360t^4 \\ & + 5500829696t^3 + 1920991232t^2 + 268435456t. \end{aligned} \tag{7.6}$$

Proof. For the algorithmic proof of (7.1) we use Smoot's package. To this end, we choose $m = 5$ and $j = 1$ as the last two entries in the procedure call,

```
In[5]:= RK[10, 2,{−5, 2}, 5, 1]
```

the program produces the Ramanujan-Kolberg type identity (7.5) as follows:

Out[5] =

N:	10
$\{M, (r_\delta)_{\delta\mid M}\}$:	$\{2, (-5, 2)\}$
m:	5
$P_{m,r}(j)$:	$\{1, 2\}$
$f_1(q)$:	$\langle$ as in (7.2)$\rangle$
t:	$\langle$ as in (7.2)$\rangle$
AB:	$\{1\}$
$\{p_g(t)\colon\ g \in \mathrm{AB}\}$	$\{v(t)\,\langle$ as in (7.6)$\rangle\}$
Common Factor:	None

□

As in the proof of Corollary 4, the following fact is an immediate consequence of inspecting the coefficients of the polynomial $v(t)$ in (7.6).

Corollary 5.

$$\sum_{n=0}^{\infty} \mathrm{PDN1}(5n+1)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+2)q^n \equiv q \cdot \frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^{12}} \pmod{2}. \tag{7.7}$$

Another consequence of Theorem 11 and Theorem 12 is

Corollary 6.

$$\sum_{n=0}^{\infty} \mathrm{PDN1}(5n)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+3)q^n \equiv \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+1)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+2)q^n \pmod{5}.$$

Proof. Let f_1 and t be as in (7.2). Let $u(t)$ and $v(t)$ be the polynomials as in (7.3) and (7.6), respectively. Then (7.1) and (7.5) imply,

$$f_1 \cdot \left(\sum_{n=0}^{\infty} \mathrm{PDN1}(5n+1)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+2)q^n - \sum_{n=0}^{\infty} \mathrm{PDN1}(5n)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+3)q^n \right)$$
$$= v(t) - u(t) = 5(t-4)^8 t^2 (t+1)(7t^3 - 64t^2 - 224t - 128).$$

Owing to the common factor 5, this proves the statement. □

Remark. We want to remark that in the classical case (i.e., where $p(n)$ are the standard partition numbers) a similar relation holds up to a sign change,

$$\sum_{n=0}^{\infty} p(5n)q^n \sum_{n=0}^{\infty} p(5n+3)q^n \equiv -\sum_{n=0}^{\infty} p(5n+1)q^n \sum_{n=0}^{\infty} p(5n+2)q^n \pmod{5}. \tag{7.8}$$

Using the RaduRK package, this identity can proved analogously to Corollary 6. However, Kolberg in his pioneering work has derived explicitly the counterparts of the identities (7.1) and (7.5) for this classical case, namely, the Ramanujan-Kolberg relations (4.2) and (4.3) on page 83 in [10].

We conclude this section with the remark that, in contrast to Ramanujan's classical identity [15, eq. (17)],

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6},$$

for partition diamonds one only has

Theorem 13. *Let* PDN1(n) *be the number of partitions of* n *as in* (4.5)*. Then*

$$\sum_{n=0}^{\infty} \mathrm{PDN1}(5n+4)q^n \equiv 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} \pmod{2}. \tag{7.9}$$

Proof. Using the RaduRK package, with the procedure call

```
In[6]:= RK[10, 2,{−5, 2}, 5, 4]
```

one obtains

$$f_1 \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+4)q^n = w(t), \tag{7.10}$$

with

$$f_1 = \frac{(q;q)_\infty^{23}\,(q^5;q^5)_\infty^{10}}{q^7 (q^2;q^2)_\infty^5\,(q^{10};q^{10})_\infty^{25}} \quad \text{and} \quad t(q) = \frac{(q^2;q^2)_\infty\,(q^5;q^5)_\infty^5}{q(q;q)_\infty\,(q^{10};q^{10})_\infty^5},$$

and where

$$w(t) = 149t^7 + 3904t^6 + 17760t^5 + 36480t^4 + 66560t^3 + 57344t^2 + 16384t.$$

Now, observing that

$$\frac{t^7}{f_1} = \frac{(q^2;q^2)_\infty^{12}\,(q^5;q^5)_\infty^{25}}{(q;q)_\infty^{30}\,(q^{10};q^{10})_\infty^{10}} \equiv \frac{(q^5;q^5)_\infty^{5}}{(q;q)_\infty^{6}} \pmod 2,$$

and taking equation (7.10) mod 2, completes the proof of (7.9). □

8. Some more congruences

8.1. **Congruences modulo 7 and 7^2.** First we derive a Ramanujan-type congruence mod 7.

Theorem 14. *Let* PDN1(m) *be the number of partitions of m as in* (4.5). *Then*

$$f_1 \cdot \sum_{m=0}^{\infty} \mathrm{PDN1}(7m+5)q^m = g_1 \cdot p_1(t) + g_2 \cdot p_2(t), \tag{8.1}$$

where

$$f_1 = \frac{(q;q)_\infty^{33}\,(q^7;q^7)_\infty^{14}}{q^{15}\,(q^2;q^2)_\infty^{9}\,(q^{14};q^{14})_\infty^{35}}, \quad t = \frac{(q^2;q^2)_\infty\,(q^7;q^7)_\infty^{7}}{q^2(q;q)_\infty\,(q^{14};q^{14})_\infty^{7}},$$

and

$$g_1 = 1, \quad g_2 = \frac{(q^2;q^2)_\infty^{8}\,(q^7;q^7)_\infty^{4}}{q^3(q;q)_\infty^{4}\,(q^{14};q^{14})_\infty^{8}} - 4t,$$

and

$$\begin{aligned} p_1(t) &= 7(6600t^7 + 1215859t^6 + 16265680t^5 + 66910336t^4 + 119306240t^3 + 79962112t^2 \\ &\quad + 3014656t - 1048576), \\ p_2(t) &= 7(53t^6 + 139888t^5 + 3644288t^4 + 17618944t^3 + 30793728t^2 + 17956864t + 1048576). \end{aligned}$$

Proof. Choosing $N = 14$, and $m = 7$ and $j = 5$ as the last two entries in the procedure call `RK[14,2,{−5,2},7,5]`, produces (8.1) as a Ramanujan type relation between modular functions for the congruence subgroup $\Gamma_0(14)$. □

Owing to the common factor 7, Theorem 14 immediately implies

Corollary 7. *For $m \geq 0$,*

$$\mathrm{PDN1}(7m+5) \equiv 0 \pmod 7. \tag{8.2}$$

Continuing in the spirit of Ramanujan, we have

Theorem 15. *For $m \geq 0$,*

$$\mathrm{PDN1}(49m+47) \equiv 0 \pmod{49}. \tag{8.3}$$

Proof. The proof is a consequence of the relation

$$f_1 \cdot \sum_{m=0}^{\infty} \mathrm{PDN1}(49m+47)q^m = 49 \cdot (g_1 \cdot p_1(t) + g_2 \cdot p_2(t)), \tag{8.4}$$

where

$$f_1 = \frac{(q;q)_\infty^{229}\,(q^7;q^7)_\infty^{112}}{q^{125}\,(q^2;q^2)_\infty^{58}\,(q^{14};q^{14})_\infty^{280}},\ \ t = \frac{(q^2;q^2)_\infty\,(q^7;q^7)_\infty^7}{q^2(q;q)_\infty\,(q^{14};q^{14})_\infty^7},$$

and

$$g_1 = 1,\ \ g_2 = \frac{(q^2;q^2)_\infty^8\,(q^7;q^7)_\infty^4}{q^3(q;q)_\infty^4\,(q^{14};q^{14})_\infty^8} - 4t,$$

and where $p_1(t)$ and $p_2(t)$ are polynomials having (big) integer coefficients and being of degree 62 and degree 61, respectively. Using the package RaduRK, relation (8.4) together with the explicit forms of $f_1, t, p_1(t)$, and $p_2(t)$ can be derived with the procedure call `RK[14,2,{−5,2},49,47]` in about 420 seconds on a standard laptop. □

Analogously to Corollary 6 one can derive

Theorem 16.

$$\sum_{n=0}^{\infty} \mathrm{PDN1}(7n)q^n \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(7n+2)q^n \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(7n+6)q^n$$
$$\equiv -\sum_{n=0}^{\infty} \mathrm{PDN1}(7n+1)q^n \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(7n+3)q^n \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(7n+4)q^n \pmod 7.$$

8.2. **Further congruences modulo** 5 **and** 5^2**.** Before presenting Theorem (17), we prepare with two lemmas and their corollaries.

Lemma 8.1. *Let*

$$\beta(q) := (q;q)_\infty^3 \left(q^2;q^2\right)_\infty^{10} = \sum_{n=0}^{\infty} b(n)q^n = 1 - 3q - 10q^2 + 35q^3 + 35q^4 - 155q^5 + \dots. \tag{8.5}$$

Then

$$f_1 \cdot \sum_{m=0}^{\infty} b(5m+2)q^m \sum_{m=0}^{\infty} b(5m+4)q^m = p(t), \tag{8.6}$$

where

$$f_1 = \frac{(q^2;q^2)_\infty^7\,(q^5;q^5)_\infty^{29}}{q^{15}(q;q)_\infty^7\,(q^{10};q^{10})_\infty^{55}},\ \ t = \frac{(q^2;q^2)_\infty\,(q^5;q^5)_\infty^5}{q(q;q)_\infty\,(q^{10};q^{10})_\infty^5},$$

and

$$p(t) = -25(t-4)^2t^3(t+1)^4(14t^6 - 361t^5 - 490t^4 - 9695t^3 + 13160t^2 - 38416t + 1344).$$

Proof. Choosing $N = 10$, and $m = 5$ and $j = 2$ as the last two entries in the procedure call `RK[10,2,`$\{3, 10\}$`,5,2]`, produces (8.6) as a Ramanujan-Kolberg type relation between modular functions for the congruence subgroup $\Gamma_0(10)$. $\square$

Lemma 8.1 immediately implies

Corollary 8. *For $m \geq 0$ and $b(n)$ as in* (8.5),

$$b(5m+2) \equiv 0 \pmod 5 \tag{8.7}$$

and

$$b(5m+4) \equiv 0 \pmod 5. \tag{8.8}$$

Lemma 8.2. *Let*

$$\gamma(q) := \frac{(q^2;q^2)_\infty^{26}}{(q;q)_\infty^5} = \sum_{n=0}^{\infty} c(n)q^n = 1 + 5q - 6q^2 - 65q^3 - 31q^4 + 311q^5 + \cdots . \tag{8.9}$$

Then

$$f_1 \cdot \sum_{m=0}^{\infty} c(5m+1)q^m \sum_{m=0}^{\infty} c(5m+3)q^m = p(t), \tag{8.10}$$

where

$$f_1 = \frac{(q^2;q^2)_\infty^7 (q^5;q^5)_\infty^{45}}{q^{25}(q;q)_\infty^7 (q^{10};q^{10})_\infty^{87}}, \quad t = \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty^5}{q(q;q)_\infty (q^{10};q^{10})_\infty^5},$$

and

$$\begin{aligned} p(t) = {} & -25(t-4)^2t^2(t+1)^2(13t^{19} + 935t^{18} + 14804t^{17} - 62690t^{16} - 1022362t^{15} \\ & - 5356621t^{14} + 147806760t^{13} - 880790538t^{12} + 2763447750t^{11} - 6875765076t^{10} \\ & + 13913700244t^9 - 19042372685t^8 + 24101137912t^7 - 26597387330t^6 \\ & + 12726296084t^5 - 14930081247t^4 + 20927255240t^3 - 8305700496t^2 \\ & + 1323868480t - 47869952). \end{aligned}$$

Proof. Choosing $N = 10$, and $m = 5$ and $j = 1$ as the last two entries in the procedure call `RK[10,2,`$\{-5, 26\}$`,5,1]`, produces (8.10) as a Ramanujan-Kolberg type relation between modular functions for the congruence subgroup $\Gamma_0(10)$. $\square$

Lemma 8.2 immediately implies

Corollary 9. *For $m \geq 0$ and $c(n)$ as in* (8.9),

$$c(5m+1) \equiv 0 \pmod 5 \tag{8.11}$$

and

$$c(5m+3) \equiv 0 \pmod 5. \tag{8.12}$$

Theorem 17. *Let* PDN1(N) *be the number of partitions as in* (4.5)*. Then for* $m \geq 0$*,*

$$\mathrm{PDN1}(125m+74) \equiv 0 \pmod{25}, \tag{8.13}$$

and

$$\mathrm{PDN1}(125m+124) \equiv 0 \pmod{25}. \tag{8.14}$$

Proof. We begin the proof of Theorem 17 by recalling the relation (5.4),

$$f_1 \cdot \sum_{m=0}^{\infty} \mathrm{PDN1}(25n+24)q^n = 5tP(t), \tag{8.15}$$

where

$$f_1 = \frac{1}{q^{44}} \frac{(q;q)_\infty^{115}(q^5;q^5)_\infty^{50}}{(q^2;q^2)_\infty^{22}(q^{10};q^{10})_\infty^{140}},\ t = \frac{1}{q}\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{10};q^{10})_\infty^5},$$

and with $P(t)$ being the polynomial of degree 43 from Theorem 10. To proceed with this relation, the first important observation is that

$$P(t) \equiv 3(1+t)^{42}(2+t) \pmod 5, \tag{8.16}$$

which is easily verified with computer algebra. The second decisive fact is an observation made by Cristian-Silviu Radu:[2]

$$\varphi := 1+t = \frac{\left(q^2;q^2\right)_\infty^4 \left(q^5;q^5\right)_\infty^2}{q(q;q)_\infty^2 \left(q^{10};q^{10}\right)_\infty^4}, \tag{8.17}$$

which gives,

$$\frac{tP(t)}{f_1} \equiv \frac{3t}{f_1}(\varphi^{43}+\varphi^{42}) \pmod 5. \tag{8.18}$$

Now,

$$\frac{t}{f_1}\varphi^{43} = (q;q)_\infty^3 \left(q^2;q^2\right)_\infty^{10} = \beta(q)$$

and

$$\frac{t}{f_1}\varphi^{42} = q \cdot \frac{\left(q^2;q^2\right)_\infty^{26}}{(q;q)_\infty^5} = q\cdot\gamma(q),$$

where

$$\beta(q) = \sum_{n=0}^{\infty} b(n)q^n \quad \text{and} \quad \gamma(q) = \sum_{n=0}^{\infty} c(n)q^n$$

are as in (8.5) and (8.9), respectively. Consequently, the coefficient of q^{5m+2} in $\frac{t}{f_1}(\varphi^{43} + \varphi^{42}) = \beta(q) + q\cdot\gamma(q)$ is

$$b(5m+2)+c(5m+1),$$

which owing to (8.7) and (8.11) is divisible by 5. Similarly, the coefficient of q^{5m+4} in $\frac{t}{f_1}(\varphi^{43}+\varphi^{42}) = \beta(q)+q\cdot\gamma(q)$ is

$$b(5m+4)+c(5m+3),$$

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[2] Personal communication with Paule, October 3, 2022.

which owing to (8.8) and (8.12) is divisible by 5. Consequently, if $n = 5m + 2$ in (8.15) then

$$25 \mid \mathrm{PDN1}(25(5m+2)+24) = \mathrm{PDN1}(125m+74),$$

and if $n = 5m + 4$,

$$25 \mid \mathrm{PDN1}(25(5m+4)+24) = \mathrm{PDN1}(125m+124).$$

This completes the proof of Theorem 17. □

Remark. The crucial identity (8.17) together with a classical proof can be found in Shaun Cooper's monograph [8, eq. (10.6)]. We remark that such relations between q-products ("eta quotients") can be proven also by algorithmic methods, for example, the one presented in [14].

We conclude this article by revisiting Corollary 4 and Corollary 5 modulo 5.

Lemma 8.3.

$$\sum_{n=0}^{\infty} \mathrm{PDN1}(5n)q^n \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+3)q^n \equiv q \cdot \frac{(q^2;q^2)_\infty^{20}}{(q;q)_\infty^2} \pmod{5}. \tag{8.19}$$

Proof. Recall the relation (7.1),

$$f_1 \cdot \sum_{n=0}^{\infty} \mathrm{PDN1}(5n)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(5n+3)q^n = u(t),$$

where f_1, t and $u(t)$ are as in (7.2) and (7.3). Using computer algebra, one observes that

$$\begin{aligned} u(t) &\equiv t(1+t)^{12} \pmod{5} \\ &= t\varphi^{12}, \end{aligned}$$

with φ as in (8.17). Now, by repeated application of the identity $1 - x^5 \equiv (1-x)^5 \pmod{5}$,

$$t = \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty^5}{q(q;q)_\infty (q^{10};q^{10})_\infty^5} \equiv \frac{1}{q}\frac{(q;q)_\infty^{24}}{(q^2;q^2)_\infty^{24}} \pmod{5},$$

$$\varphi^{12} = \frac{1}{q^{12}}\frac{(q^2;q^2)_\infty^{48}(q^5;q^5)_\infty^{24}}{(q;q)_\infty^{24}(q^{10};q^{10})_\infty^{48}} \equiv \frac{1}{q^{12}}\frac{(q;q)_\infty^{96}}{(q^2;q^2)_\infty^{192}} \pmod{5},$$

and

$$f_1 = \frac{(q;q)_\infty^{47}(q^5;q^5)_\infty^{15}}{q^{14}(q^2;q^2)_\infty^{11}(q^{10};q^{10})_\infty^{45}} \equiv \frac{1}{q^{14}}\frac{(q;q)_\infty^{122}}{(q^2;q^2)_\infty^{236}} \pmod{5}.$$

Consequently,

$$\begin{aligned} \frac{u(t)}{f_1(q)} &\equiv \frac{t\varphi^{12}}{f_1} \pmod{5} \\ &= q \cdot \frac{(q^2;q^2)_\infty^{20}}{(q;q)_\infty^2} \pmod{5}, \end{aligned}$$

which proves (8.19). □

Lemma 8.4. *Let*

$$\delta(q) := \frac{(q^2;q^2)_\infty^{20}}{(q;q)_\infty^2} = \sum_{n=0}^\infty d(n)q^n = 1 + 2q - 15q^2 - 30q^3 + 90q^4 + 176q^5 + \dots. \tag{8.20}$$

Then

$$f_1 \cdot \sum_{m=0}^\infty d(5m+3)q^m = w(t), \tag{8.21}$$

where

$$f_1 = \frac{(q^2;q^2)_\infty^4 \, (q^5;q^5)_\infty^{22}}{q^{11}(q;q)_\infty^4 \, (q^{10};q^{10})_\infty^{40}}, \quad t = \frac{(q^2;q^2)_\infty \, (q^5;q^5)_\infty^5}{q(q;q)_\infty \, (q^{10};q^{10})_\infty^5},$$

and

$$\begin{aligned} w(t) = -5t(&-256 + 22592t + 30208t^2 + 51757t^3 - 38912t^4 + 800t^5 + 592t^6 \\ &- 4042t^7 + 1282t^8 - 62t^9 + 6t^{10}). \end{aligned}$$

Proof. Choosing $N = 10$, and $m = 5$ and $j = 3$ as the last two entries in the procedure call `RK[10,2,{−2,20},5,3]`, produces (8.6) as a Ramanujan-Kolberg type relation between modular functions for the congruence subgroup $\Gamma_0(10)$. □

An immediate consequence is

Corollary 10. *Let*

$$\frac{(q^2;q^2)_\infty^{20}}{(q;q)_\infty^2} = \sum_{n=0}^\infty d(n)q^n.$$

Then for all $m \geq 0$,

$$5 \mid d(5m+3). \tag{8.22}$$

Theorem 18. *Let* PDN1(N) *be the number of partitions as in* (4.5). *Then for* $m \geq 0$,

$$\sum_{j=0}^{5m+4} \mathrm{PDN1}(25m+20-5j)\,\mathrm{PDN1}(5j+3) \equiv 0 \pmod 5. \tag{8.23}$$

Proof. Recall relation (8.19),

$$\sum_{n=0}^\infty \mathrm{PDN1}(5n)q^n \cdot \sum_{n=0}^\infty \mathrm{PDN1}(5n+3)q^n \equiv q \cdot \delta(q) \pmod 5,$$

with $\delta(q)$ as in (8.20). By (8.22) the coefficient of q^{5m+4} in $q \cdot \delta(q)$ is divisible by 5. Hence

$$\sum_{\substack{i,j\geq 0 \\ i+j=5m+4}} \mathrm{PDN1}(5i)\,\mathrm{PDN1}(5j+3)$$

is divisible by 5 for all $m \geq 0$. This implies the statement (8.23). □

As a final observation, the mod 5 version of Corollary 5 is

Lemma 8.5.

$$\sum_{n=0}^{\infty}\mathrm{PDN1}(5n+1)q^n\cdot\sum_{n=0}^{\infty}\mathrm{PDN1}(5n+2)q^n\equiv q\cdot\frac{(q^2;q^2)_\infty^{20}}{(q;q)_\infty^2}\pmod{5}. \tag{8.24}$$

Proof. Recall the relation (7.5),

$$f_1\cdot\sum_{n=0}^{\infty}\mathrm{PDN1}(5n+1)q^n\sum_{n=0}^{\infty}\mathrm{PDN1}(5n+2)q^n=v(t),$$

where f_1, t and $v(t)$ are as in (7.2) and (7.6). Using computer algebra, one observes that

$$\begin{aligned}v(t)&\equiv t(1+t)^{12}\pmod{5}\\&=t\varphi^{12}.\end{aligned}$$

Hence the same argument as used in the proof of Lemma 8.3 also proves Lemma 8.5. □

Remark. The statements of Lemma 8.3 and Lemma 8.5 are refined versions of Corollary 6 which was proved differently. Moreover, Lemma 8.5 immediately implies the following counterpart of Theorem 18.

Theorem 19. *Let* PDN1(N) *be the number of partitions as in* (4.5). *Then for* $m\geq 0$,

$$\sum_{j=0}^{5m+4}\mathrm{PDN1}(25m+22-5j)\,\mathrm{PDN1}(5j+1)\equiv 0\pmod{5}. \tag{8.25}$$

9. CONCLUSION

This paper continues our project to find further natural arithmetic/combinatorial objects generated by modular forms. It will hopefully spur further efforts in this direction. Applications most often arise from the combinatorial side with subsequent important information being supplied by the fact that the generating functions are modular forms. The richness of results found from these few instances considered here and in [6] suggests that much awaits.

Concerning the congruence relations presented in this paper: As in [6], all these results were proven with the help of non-trivial computer algebra algorithms. Nevertheless, in order to obtain more substantial mathematical insight, classical proofs would be desirable. A different open question concerns the existence of other congruence relations than those presented here. Another task would be to determine infinite families of congruences similar to Ramanujan's classical $p(5^kn+d_k)\equiv 0\pmod{5^k}$ where $24d_k\equiv 1\pmod{5^k}$.

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