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Asymptotic analysis of a family of Sobolev orthogonal polynomials related to the generalized Charlier polynomials

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Abstract

In this paper we tackle the asymptotic behaviour of a family of orthogonal polynomials with respect to a nonstandard inner product involving the forward operator Δ. Concretely, we treat the generalized Charlier weights in the framework of Δ–Sobolev orthogonality. We obtain an asymptotic expansion for this orthogonal polynomials where the falling factorial polynomials play an important role.

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1 Introduction

Let $\mathbb{N}_0$ be the set of nonnegative integers

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

If $\mathfrak{L} : \mathbb{C}[x] \to \mathbb{C}$ is a linear functional, we say that a sequence $\{p_n\}_{n\geq 0}$, $\deg(p_n) = n$, is an *orthogonal polynomial sequence* with respect to $\mathfrak{L}$ if

$$\mathfrak{L}[p_k p_n] = h_n \delta_{k,n}, \quad k, n \in \mathbb{N}_0, \quad h_n \neq 0, \tag{1}$$

where $\delta_{k,n}$ denotes the Kronecker delta. If $h_n = 1$, then $\{p_n\}_{n\geq 0}$ is said to be an *orthonormal polynomial sequence*. We denote by $\{\mu_n\}_n \geq 0$ the *moment sequence* of the functional $\mathfrak{L}$ on the monomial basis,

$$\mu_n = \mathfrak{L}[x^n], \quad n \in \mathbb{N}_0.$$

Let $\{\mathrm{p}_n\}_{n\geq 0}$ be the sequence of **monic** polynomials, orthogonal with respect to $\mathfrak{L}$. From (1), we see that

$$\mathfrak{L}[x \mathrm{p}_n \mathrm{p}_k] = 0, \quad k \neq n, n \pm 1,$$

and therefore the polynomials $\mathrm{p}_n(x)$ satisfy the *three-term recurrence relation*

$$x\mathrm{p}_n = \mathrm{p}_{n+1} + \beta_n \mathrm{p}_n + \gamma_n \mathrm{p}_{n-1}, \quad n \in \mathbb{N}_0, \tag{2}$$

with initial values $\mathrm{p}_0(x) = 1, \ \ \mathrm{p}_1(x) = x - \beta_0$. Using (1), the coefficients β_n, γ_n are given by

$$\beta_n = \frac{\mathfrak{L}[x\mathrm{p}_n^2]}{h_n}, \quad \gamma_n = \frac{\mathfrak{L}[x\mathrm{p}_n \mathrm{p}_{n-1}]}{h_{n-1}}, \quad n \in \mathbb{N}, \tag{3}$$

with initial values

$$\beta_0 = \frac{\mu_1}{\mu_0}, \quad \gamma_0 = 0. \tag{4}$$

Note that (again using again (1)), we have

$$h_n = \mathfrak{L}[x^n \mathrm{p}_n] = \mathfrak{L}[x\mathrm{p}_n \mathrm{p}_{n-1}] = \gamma_n h_{n-1}, \quad n \in \mathbb{N},$$

and therefore

$$\gamma_n = \frac{h_n}{h_{n-1}}, \quad n \in \mathbb{N}. \tag{5}$$

The **monic** *Generalized Charlier polynomials*, $P_n(x;z)$ are orthogonal with respect to the linear functional

$$L[p]=\sum_{x=0}^{\infty}\frac{p(x)}{(b+1)_x}\frac{z^x}{x!},\quad p\in\mathbb{C}[x],\quad z>0, \tag{6}$$

where the *Pochhammer symbol* is defined by [17, 5.2.4]

$$(c)_n=\prod_{j=0}^{n-1}(c+j),\quad n\in\mathbb{N},\quad (c)_0=1. \tag{7}$$

In [13] Hounkonnou, Hounga, and Ronveaux studied the orthogonal polynomials associated with the linear functional

$$L_r[p]=\sum_{x=0}^{\infty}p(x)\frac{z^x}{(x!)^r},\quad p\in\mathbb{C}[x],\quad r\in\mathbb{N}. \tag{8}$$

When $r=2$, they derived nonlinear recurrences (known as the *Laguerre-Freud equations*) for the recurrence coefficients, and a second-order difference equation for the orthogonal polynomials associated with L_r. Note that the case $r=2$ is a particular example of (6) with $b=0$.

In [19] Van Assche and Foupouagnigni also considered (8) with $r=2$. They simplified the Laguerre-Freud equations obtained in [13], and obtained

$$u_{n+1}+u_{n-1}=\frac{1}{\sqrt{z}}\frac{nu_n}{1-u_n^2},\quad v_n=\sqrt{z}u_{n+1}u_n,$$

with $\gamma_n=z(1-u_n^2)$ and $\beta_n=v_n+n$. They showed that these equations are related to the discrete Painlevé II equation $\mathrm{dP_{II}}$. In [18], Smet and Van Assche studied the orthogonal polynomials associated with (6). They obtained the Laguerre-Freud equations

$$\begin{gathered}(\gamma_{n+1}-z)(\gamma_n-z)=z(\beta_n-n)(\beta_n-n+b),\\ \beta_n+\beta_{n-1}=n-1-b+\frac{nz}{\gamma_n},\end{gathered} \tag{9}$$

and showed that these equations are a limiting case of the discrete Painlevé IV equation $\mathrm{dP_{IV}}$.

We are interested in an inner product in the framework of Sobolev-type orthogonality. Concretely, a Δ–Sobolev inner product involving the linear operator L given in (6), i.e.

$$\langle p, q\rangle = L[pq] + \lambda L[\Delta p \Delta q], \quad p, q \in \mathbb{C}[x], \tag{10}$$

where $\lambda \in \mathbb{C}$ and the *finite difference operators* (in x) Δ, ∇ are defined by

$$\Delta[p] = p(x+1) - p(x), \quad \nabla[p] = p(x) - p(x-1).$$

We will denote by $S_n(x; \lambda, z)$ the **monic** polynomials orthogonal with respect to the inner product (10).

The study of Sobolev orthogonality, and corresponding orthogonal polynomials, is a relatively recent topic in the theory of orthogonal polynomials. The first seminal paper was written by Lewis in 1947 (see [15]) and other foundational articles were written in the sixties and seventies of the last century. However, the eclosion of investigations about this topic took place in the nineties. Sobolev orthogonal polynomials are attractive because they are not orthogonal in a standard way. For this reason nice properties of standard orthogonal polynomials such as the three-term recurrence relation, Christoffel–Darboux formula, etc. are lost. Therefore, it was necessary to construct a new (unfinished) theory. Originally, the Sobolev inner products involved the derivative operator. But, there is no reason to consider other operators. In this paper, as we have mentioned previously, we consider a Sobolev inner product involving the forward difference operator Δ, the so called Δ–Sobolev orthogonality in some papers (see, for example, [1], [2], [3], [16]).

The paper is organized as follows: in Section 2 we introduce some basic facts which are useful to establish the main result in this paper. The Δ–Charlier–Sobolev inner product is introduced in Section 3, where we obtain some properties of the corresponding orthogonal polynomials which allow us to obtain an asymptotic expansion for $S_n(x; \lambda, z)$.

2 Preliminary material

In this section, we review some of material that we will need in the rest of the paper.

Lemma 1 *If*

$$\phi(x) = x(x+b), \quad \psi(x) = z, \tag{11}$$

then the functional (6) satisfies the Pearson equation

$$L\left[\psi\mathfrak{S}p\right]=L\left[\phi p\right],\quad p\in\mathbb{C}\left[x\right],\tag{12}$$

where $\mathfrak{S}$ *denotes the shift operator (in* x*)*

$$\mathfrak{S}\left[p\right]=p\left(x+1\right).$$

Proof. We see from (6) that

$$\sum_{x=0}^{\infty}\frac{zp\left(x+1\right)}{\left(b+1\right)_{x}}\frac{z^{x}}{x!}=\sum_{x=1}^{\infty}\frac{p\left(x\right)}{\left(b+1\right)_{x-1}}\frac{z^{x}}{\left(x-1\right)!}$$
$$=\sum_{x=1}^{\infty}\frac{x\left(x+b\right)p\left(x\right)}{\left(b+1\right)_{x}}\frac{z^{x}}{x!}=\sum_{x=0}^{\infty}\frac{x\left(x+b\right)p\left(x\right)}{\left(b+1\right)_{x}}\frac{z^{x}}{x!},$$

and (12) follows. ■

In general, we say that a functional L satisfying the Pearson equation (12) where $\phi\left(x\right),\psi\left(x\right)$ are **fixed** polynomials is *discrete semiclassical.* Note that we can also write (12) as

$$L\left[\psi\Delta p\right]=L\left[\left(\phi-\psi\right)p\right],\quad p\in\mathbb{C}\left[x\right].$$

The *class* of the functional is defined by

$$s=\max\left\{\deg\left(\phi-\psi\right)-1,\deg\left(\phi\right)-2\right\},\tag{13}$$

and semiclassical functional of class $s=0$ are called *classical* [12]. Note that from (11) and (13) it follows that the generalized Charlier polynomials are discrete semiclassical of class $s=1$. In [10], we classified the discrete semiclassical orthogonal polynomials of class $s\leq1$ and in [11] we extended our results to $s\leq2$.

Proposition 2 *Let* $\mathrm{p}_{n}\left(x\right)$ *be the sequence of monic polynomials orthogonal with respect to a linear functional* L *satisfying the Pearson equation (12) with* $\deg\left(\phi\right)=r,\deg\left(\psi\right)=t$*.*

(i) The polynomials $\mathrm{p}_{n}\left(x\right)$ *satisfy the structure equation*

$$\psi\left(x\right)\mathrm{p}_{n}\left(x+1\right)=\sum_{k=-r}^{t}A_{k}\left(n\right)\mathrm{p}_{n+k}\left(x\right),\tag{14}$$

where the coefficients $A_k(n)$ *are solutions of the recurrence equation*

$$\begin{aligned} &\gamma_{n+k+1}A_{k+1}(n) - \gamma_n A_{k+1}(n-1) + A_{k-1}(n) - A_{k-1}(n+1) \\ &= (\beta_n - \beta_{n+k} - 1)\, A_k(n)\,, \end{aligned} \tag{15}$$

with

$$A_t(n) = z, \quad A_{-r}(n) = \gamma_n \gamma_{n-1} \cdots \gamma_{n-r+1}. \tag{16}$$

and $A_k(n) = 0,\ k \notin [-r, t]$.

(ii) The generalized Charlier polynomials $P_n(x; z)$ *satisfy*

$$\Delta P_n = nP_{n-1} + \xi_n P_{n-2}, \quad n \in \mathbb{N}_0, \tag{17}$$

where

$$\xi_n = \frac{\gamma_n \gamma_{n-1}}{z}, \quad n \in \mathbb{N}_0. \tag{18}$$

Proof. (i) See [5].

(ii) If $\phi(x), \psi(x)$ are given by (11), then $t = 0, r = 2$ and therefore

$$A_0(n) = z, \quad A_{-2}(n) = \gamma_n \gamma_{n-1}. \tag{19}$$

Setting $k = 0$ in (15), we get

$$A_{-1}(n+1) - A_{-1}(n) = z,$$

and we conclude that

$$A_{-1}(n) = nz. \tag{20}$$

Using (19) and (20) in (14) we obtain

$$zP_n(x+1) = zP_n(x) + nzP_{n-1}(x) + \gamma_n \gamma_{n-1} P_{n-2}(x)\,,$$

and (17) follows. ■

Remark 3 *If* $k = -1, -2$ *then (15) gives*

$$\begin{aligned} \gamma_n(\gamma_{n-1} - \gamma_{n+1}) &= nz(\beta_n - \beta_{n-1} - 1)\,, \\ nz\gamma_{n-1} - (n-1)\, z\gamma_n &= \gamma_n \gamma_{n-1}(\beta_n - \beta_{n-2} - 1)\,, \end{aligned}$$

from which the Laguerre-Freud equations (9) can be derived (see [18], equation 2.14 and beyond).

Equation (17) was derived in [18] using the method presented in [14]. For a different approach using infinite matrices, see [6].

Let $\varphi_n(x)$ denote the *falling factorial polynomials* defined by $\varphi_0(x) = 1$ and

$$\varphi_n(x) = \prod_{k=0}^{n-1} (x-k), \quad n \in \mathbb{N}. \tag{21}$$

Note that we can write

$$\varphi_n(x) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)} = n! \binom{x}{n}, \quad n \in \mathbb{N}_0, \tag{22}$$

where Γ denotes the gamma function [17, 5.2.1].

Proposition 4 *The moments of the functional L on the basis $\varphi_n(x)$ are given by*

$$\nu_n(z) = L[\varphi_n] = \frac{z^n}{(b+1)_n} \, {}_0F_1\left(\begin{matrix} - \\ b+n+1 \end{matrix} ; z \right), \tag{23}$$

where ${}_pF_q$ is the generalized hypergeometric function [17, 16.2.1].

Proof. Using (6) and (22), we have

$$L[\varphi_n] = \sum_{x=n}^{\infty} \frac{1}{(b+1)_x} \frac{z^x}{(x-n)!} = \sum_{x=0}^{\infty} \frac{1}{(b+1)_{x+n}} \frac{z^{x+n}}{x!},$$

and since

$$(c)_{n+m} = (c)_n (c+n)_m,$$

we obtain

$$L[\varphi_n] = \sum_{x=0}^{\infty} \frac{1}{(b+1)_n (b+n+1)_x} \frac{z^{x+n}}{x!},$$

and (23) follows. ■

Lemma 5 *The polynomials $\varphi_n(x)$ satisfy the connection (or linearization) formula*

$$\varphi_n(x) \varphi_m(x) = \sum_{k=0}^{\min\{n,m\}} \binom{n}{k} \binom{m}{k} k! \varphi_{n+m-k}(x). \tag{24}$$

Proof. From the definition of $\varphi_n(x)$, we see that

$$\varphi_{n+m}(x) = \varphi_n(x)\,\varphi_m(x-n). \tag{25}$$

Suppose that $m \leq n$. Using (25), we have

$$\sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} k!\varphi_{n+m-k}(x) = \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} k!\varphi_n(x)\,\varphi_{m-k}(x-n),$$

and using (22), we can write

$$\sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} k!\varphi_{m-k}(x-n) = m!\sum_{k=0}^{m} \binom{n}{k}\binom{x-n}{m-k}.$$

Using the Chu–Vandermonde identity, we have

$$\sum_{k=0}^{m} \binom{n}{k}\binom{x-n}{m-k} = \binom{x}{m},$$

and therefore

$$\sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} k!\varphi_{m-k}(x-n) = \varphi_m(x).$$

■

Corollary 6 *For all $m, n \in \mathbb{N}_0$, $m \leq n$, we have*

$$L[\varphi_n\varphi_m] = \binom{n}{m}\frac{m!}{(b+1)_n}z^n\left[1 + \frac{n+1}{(n-m+1)(n+b+1)}z + O\left(z^2\right)\right], \tag{26}$$

as $z \to 0$.

Proof. Using (24), we get

$$L[\varphi_n\varphi_m] = \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} k!\nu_{n+m-k}(z),$$

and (23) gives

$$\begin{aligned} L[\varphi_n\varphi_m] &= \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} k!\frac{z^{n+m-k}}{(b+1)_{n+m-k}}\left(1 + \frac{z}{n+m-k+b+1} + \cdots\right) \\ &= \binom{n}{m}\frac{m!}{(b+1)_n}z^n + \binom{n+1}{m}\frac{m!}{(b+1)_{n+1}}z^{n+1} + O\left(z^{n+2}\right), \quad z \to 0. \end{aligned}$$

■

3 Sobolev polynomials

Let $S_n(x;\lambda,z)$ be the **monic** polynomials orthogonal with respect to the inner product (10). Introducing the sequences

$$\mu_{i,j}(\lambda,z) = \langle \varphi_i, \varphi_j \rangle, \quad \nu_{i,j}(z) = L[\varphi_i \varphi_j],$$

and using the identity

$$\Delta \varphi_n = n\varphi_{n-1}, \quad n \in \mathbb{N}_0,$$

we have

$$\mu_{i,j} = \langle \varphi_i, \varphi_j \rangle = L[\varphi_i \varphi_j] + \lambda L[i\varphi_{i-1} j\varphi_{j-1}] = \nu_{i,j} + \lambda ij \nu_{i-1,j-1}, \tag{27}$$

for all $i,j \in \mathbb{N}_0$. Using (26) in (27), we have

$$\mu_{i,j}(z) = \frac{\lambda j}{(i-j)!} i! \frac{z^{i-1}}{(b+1)_{i-1}} + \frac{(\lambda i - 1) j + i + 1}{(i+1-j)!} i! \frac{z^i}{(b+1)_i} + O\left(z^{i+1}\right) \tag{28}$$

as $z \to 0$, with $j \leq i$.

In [7], we obtained power series solutions for the determinant of a square matrix whose entries are power series in z. Using (26) and (28), we see that

$$H_n(z) \sim z^{\binom{n}{2}} \prod_{k=1}^{n-1} \frac{k!}{(b+1)_k},$$

and

$$\widetilde{H}_n(\lambda,z) \sim \lambda^{n-1} z^{\binom{n-1}{2}} \prod_{k=1}^{n-2} \frac{(k+1)(k+1)!}{(b+1)_k},$$

as $z \to 0$, where $H_0 = \widetilde{H}_0 = 1$ and

$$H_n(z) = \det_{0 \leq i,j \leq n-1}(\nu_{i,j}), \quad \widetilde{H}_n(\lambda,z) = \det_{0 \leq i,j \leq n-1}(\mu_{i,j}), \quad n \in \mathbb{N}.$$

Since the determinants $H_n(z)$ and the norms of the polynomials are related by (see [4], Theorem 3.2)

$$h_n(z) = \frac{H_{n+1}(z)}{H_n(z)},$$

we get

$$h_n(z) = \frac{n!}{(b+1)_n} z^n + O\left(z^{n+1}\right), \quad n \in \mathbb{N}_0, \tag{29}$$

as $z \to 0$. Similarly, for all $n \in \mathbb{N}$

$$\widetilde{h}_n(\lambda, z) = \lambda \frac{nn! z^{n-1}}{(b+1)_{n-1}} + \frac{n!\,(n+b-1+b\lambda n)}{(n+b-1)\,(b+1)_n} z^n + O(z^n), \tag{30}$$

as $z \to 0$, where

$$\langle S_n, S_n \rangle = \widetilde{h}_n(\lambda, z) = \frac{\widetilde{H}_{n+1}(\lambda, z)}{\widetilde{H}_n(\lambda, z)}. \tag{31}$$

The polynomials $S_n(x; \lambda, z)$ and $P_n(x; z)$ are related by the following expression.

Theorem 7 *We have*

$$P_n(x; z) = S_n(x; \lambda, z) + a_n(\lambda, z)\, S_{n-1}(x; \lambda, z), \quad n \in \mathbb{N}, \tag{32}$$

where

$$a_n(\lambda, z) = \frac{(n-1)\lambda}{z} \frac{h_n(z)}{\widetilde{h}_{n-1}(z, \lambda)}, \quad n \in \mathbb{N}. \tag{33}$$

Proof. Since the polynomials $S_n(x; \lambda, z)$ are a basis of $\mathbb{C}[x]$ and $P_n(x; z), S_n(x; \lambda, z)$ are monic, it follows that

$$P_n = S_n + \sum_{k=0}^{n-1} c_{n,k} S_k.$$

Using orthogonality, we have

$$c_{n,k} = \frac{\langle P_n, S_k \rangle}{\widetilde{h}_k},$$

and using (10) we get

$$\widetilde{h}_k c_{n,k} = L[P_n S_k] + \lambda L[\Delta P_n \Delta S_k]. \tag{34}$$

Using (17) in (34), we obtain

$$\widetilde{h}_k c_{n,k} = L[P_n S_k] + \lambda n L[P_{n-1} \Delta S_k] + \lambda \xi_n L[P_{n-2} \Delta S_k] = 0, \quad 0 \leq k \leq n-2,$$

and therefore the only nonzero coefficient is

$$c_{n,n-1} = \lambda \frac{\xi_n}{\widetilde{h}_{n-1}} L\left[P_{n-2}\Delta S_{n-1}\right].$$

But since

$$\Delta S_{n-1} = (n-1)\,x^{n-2} + O\left(x^{n-3}\right) = (n-1)\,P_{n-2} + O\left(x^{n-3}\right),$$

we see that

$$L\left[P_{n-2}\Delta S_{n-1}\right] = (n-1)\,h_{n-2}.$$

Finally, we can use (5) and (18) to obtain

$$\xi_n h_{n-2} = \frac{\gamma_n \gamma_{n-1}}{z} h_{n-2} = \frac{h_n}{z}. \tag{35}$$

Thus, we conclude that

$$c_{n,n-1} = (n-1)\,\lambda \frac{\xi_n}{\widetilde{h}_{n-1}} h_{n-2} = \frac{(n-1)\,\lambda}{z} \frac{h_n}{\widetilde{h}_{n-1}}.$$

■

Remark 8 *If we use (30)-(29) in (33), we get*

$$a_n(\lambda, z) = \frac{nz}{(n+b)(n+b-1)} + O\left(z^2\right), \quad z \to 0, \quad n \geq 2. \tag{36}$$

Next, we shall find a recurrence for the Sobolev norms $\widetilde{h}_n(\lambda, z)$.

Theorem 9 *For all $n \in \mathbb{N}$, the functions $\widetilde{h}_n(\lambda, z)$ defined by (31) satisfy the nonlinear recurrence*

$$\widetilde{h}_n = \lambda n^2 h_{n-1} + \left(1 + \lambda \frac{\gamma_n \gamma_{n-1}}{z^2}\right) h_n - (n-1)^2 \frac{\lambda^2}{z^2} \frac{h_n^2}{\widetilde{h}_{n-1}}. \tag{37}$$

Proof. Using (10) and (31) we get

$$\widetilde{h}_n = \langle S_n, P_n \rangle = L\left[S_n P_n\right] + \lambda L\left[\Delta S_n \Delta P_n\right] = h_n + \lambda L\left[\Delta S_n \Delta P_n\right].$$

But from (32) we have

$$L\left[\Delta S_n \Delta P_n\right] = L\left[(\Delta P_n)^2\right] - a_n L\left[\Delta S_{n-1}\Delta P_n\right],$$

while (17) gives

$$L\left[\left(\Delta P_n\right)^2\right] = n^2 h_{n-1} + \xi_n^2 h_{n-2},$$

and

$$\begin{aligned} L\left[\Delta S_{n-1}\Delta P_n\right] &= nL\left[\Delta S_{n-1}P_{n-1}\right] + \xi_n L\left[\Delta S_{n-1}P_{n-2}\right] \\ &= 0 + (n-1)\,\xi_n h_{n-2}. \end{aligned}$$

Hence,

$$\widetilde{h}_n = h_n + \lambda n^2 h_{n-1} + \lambda\left[\xi_n^2 - a_n\,(n-1)\,\xi_n\right] h_{n-2},$$

or using (33) and (35), we conclude that

$$\widetilde{h}_n = h_n + \lambda n^2 h_{n-1} + \lambda\frac{\gamma_n\gamma_{n-1}}{z^2} h_n - (n-1)^2\,\frac{\lambda^2}{z^2}\frac{h_n^2}{\widetilde{h}_{n-1}}.$$

■

Since $\widetilde{h}_0 = h_0$ we know from (4) that $\gamma_0 = 0$, we can use (37) and obtain

$$\begin{aligned} \widetilde{h}_1 &= h_1 + \lambda h_0, \\ \widetilde{h}_2 &= h_2 + \left(4h_1 + \frac{\gamma_1\gamma_2 h_2}{z^2} - \frac{\lambda}{z^2}\frac{h_2^2}{h_1 + \lambda h_0}\right)\lambda. \end{aligned}$$

Using (33), it follows that

$$a_1 = 0, \quad a_2 = \frac{\lambda h_2}{z\,(h_1 + \lambda h_0)}.$$

Remark 10 *Note that using (33), we can rewrite (37) as*

$$\frac{n\lambda}{z}\frac{h_{n+1}}{a_{n+1}} = \lambda n^2 h_{n-1} + \left(1 + \lambda\frac{\gamma_n\gamma_{n-1}}{z^2} - (n-1)\,\frac{\lambda}{z}a_n\right) h_n,$$

or, using (5)

$$\frac{n\gamma_{n+1}}{z a_{n+1}} = \frac{n^2}{\gamma_n} + \frac{1}{\lambda} + \frac{\gamma_n\gamma_{n-1}}{z^2} - \frac{(n-1)\,a_n}{z}, \quad n\in\mathbb{N}. \tag{38}$$

4 Asymptotic analysis

In [8] we shown that the 3-term recurrence coefficients of the generalized Charlier polynomials have the asymptotic expansions

$$\beta_n(z) = n + \frac{bz}{n^2} - \frac{b(2b+1)z}{n^3} + O\left(n^{-4}\right),$$

and

$$\gamma_n(z) = z - zbn^{-1} + zb^2n^{-2} - bz\left(2z+b^2\right)n^{-3} + O\left(n^{-4}\right), \tag{39}$$

as $n \to \infty$. We continued our work in [9], where we obtained asymptotic expansions for all discrete semiclassical orthogonal polynomials.

Theorem 11 *Let the functions $a_n(\lambda, z)$ satisfy the nonlinear recurrence (41), with $a_n(\lambda, 0) = 0$. If we write*

$$a_n(\lambda, z) \sim z\sum_{k\geq 1}\alpha_k(\lambda, z)\, n^{-k}, \quad n \to \infty, \tag{40}$$

then

$$\alpha_1 = 1, \quad \alpha_2 = 1 - 2b, \quad \alpha_3 = 1 + 3b(b-1) - \frac{z}{\lambda}.$$

Proof. Let's start by rewriting (38) as

$$\left[n^2\frac{z}{\gamma_n} + \frac{z}{\lambda} + \frac{\gamma_n\gamma_{n-1}}{z} - (n-1)\,a_n\right]a_{n+1} - n\gamma_{n+1} = 0, \quad n \in \mathbb{N}, \tag{41}$$

and suppose that

$$a_n(\lambda, z) = \sum_{k=-N}^{N} u_k(\lambda, z)\, n^{-k}. \tag{42}$$

Using (39) and (42) in (41), we see that as $n \to \infty$

$$u_k = 0, \quad k \leq -2, \quad u_{-1}(u_{-1} - 1) = 0.$$

Thus, there are two solutions of (41), one with asymptotic behavior

$$a_n = n + b + 1 + \left(b + 1 + \frac{z}{\lambda}\right)n^{-1} + O\left(n^{-2}\right), \quad n \to \infty$$

and the other

$$a_n = zn^{-1} + (1-2b)\,zn^{-2} + \left(1 + 3b(b-1) - \frac{z}{\lambda}\right)zn^{-3} + O\left(n^{-4}\right), \quad n \to \infty.$$

Since from (36) we know that $a_n(\lambda, 0) = 0$, we must choose the second solution and (40) follows. ■

Remark 12 *Using (33) and (40) we deduce*

$$\frac{h_{n+1}(z)}{\widetilde{h}_n(z,\lambda)} \sim \frac{z^2}{(n+1)n\lambda} + O(n^{-3}).$$

In particular,

$$\lim_{n\to\infty} n^2 \frac{h_{n+1}(z)}{\widetilde{h}_n(z,\lambda)} = \frac{z^2}{\lambda}.$$

From (5) and (39), we obtain

$$\lim_{n\to\infty} \frac{\widetilde{h}_n(z,\lambda)}{n^2 h_n(z)} = \frac{\lambda}{z}.$$

The above asymptotic behaviour of the norms can also be obtained from Theorem 9 via Poincaré's Theorem. That technique has given fruitful results to obtain asymptotic properties in the context of Sobolev orthogonality.

In [9] we studied the asymptotic behavior of the generalized Charlier polynomials, and proved the following result.

Theorem 13 *The generalized Charlier polynomials satisfy*

$$\frac{P_n(x;z)}{\varphi_n(x)} \sim \sum_{k\geq 0} \omega_k(x;z)\, n^{-k}, \quad n\to\infty, \tag{43}$$

with

$$\omega_0 = 1, \quad \omega_1 = z, \quad \omega_2 = (x+1-b)\,z + \frac{z^2}{2},$$
$$\omega_3 = \left[(x+1)(x+1-b) + b^2\right] z + \left[2(x+1-b)+1\right]\frac{z^2}{2} + \frac{z^3}{6}.$$

We have now all the elements to state our main result.

Theorem 14 *Suppose that*

$$\frac{S_n(x;\lambda,z)}{\varphi_n(x)} \sim \sum_{k\geq 0} \sigma_k(x;z)\, n^{-k}, \quad n\to\infty. \tag{44}$$

Then,

$$\sigma_0 = 1, \quad \sigma_1 = z, \quad \sigma_2 = \omega_2 + z,$$
$$\sigma_3 = \omega_3 + \left[x + z + \alpha_2\right] z,$$
$$\sigma_4 = \omega_4 + \left[x^2 + (z+\alpha_2)\,x + z\,(2+\alpha_2) + \omega_2 + \alpha_3\right] z.$$

Proof. Using the binomial theorem in (44), we can see that

$$\frac{S_{n-1}}{\varphi_{n-1}(x)} \sim \sum_{k\geq 0}\left[\sum_{j=0}^{k-1}\binom{k-1}{j}\sigma_{j+1}\right] n^{-k}, \quad n\to\infty. \tag{45}$$

Using the recurrence

$$\varphi_n(x) = (x-n)\,\varphi_{n-1}(x)$$

in (32), we get

$$\frac{(x-n)(P_n - S_n)}{\varphi_n(x)} = a_n \frac{S_{n-1}}{\varphi_{n-1}(x)}. \tag{46}$$

Considering (43), (44) we have

$$\frac{(x-n)(P_n - S_n)}{\varphi_n(x)} \sim (\sigma_0-\omega_0)n + \sum_{k\geq 0}\left[x(\omega_k-\sigma_k) - (\omega_{k+1}-\sigma_{k+1})\right] n^{-k},$$

and from (40) and (45)

$$a_n \frac{S_{n-1}}{\varphi_{n-1}(x)} \sim z\sum_{k\geq 1}\left[\alpha_k\sigma_0 + \sum_{j=1}^{k-1}\alpha_{k-j}\sum_{i=0}^{j-1}\binom{j-1}{i}\sigma_{i+1}\right] n^{-k},$$

thus, from (46) we deduce

$$\sigma_0 = \omega_0 = 1,$$

$$x(\omega_0-\sigma_0) - (\omega_1-\sigma_1) = 0 \Rightarrow \sigma_1 = \omega_1 = z,$$

$$x(\omega_k-\sigma_k) - (\omega_{k+1}-\sigma_{k+1}) = z\left[\alpha_k + \sum_{j=1}^{k-1}\alpha_{k-j}\sum_{i=0}^{j-1}\binom{j-1}{i}\sigma_{i+1}\right], \quad k\geq 1.$$

To obtain σ_k with $k = 2,3,4$, it is enough to particularize the above expression. ■

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