# INVARIANTS OF THE QUARTIC BINARY FORM AND PROOFS OF CHEN'S CONJECTURES FOR PARTITION FUNCTION INEQUALITIES

KOUSTAV BANERJEE

Abstract. An extensive amount of studies have been done on inequalities for the partition function. In particular, the Turán inequality and the higher order Turán inequalities for $p(n)$ has been one of the more predominant themes. Recently, Griffin, Ono, Rolen, and Zagier proved that for every integer $d \geq 1$, there exists an integer $N(d)$ such that the Jensen polynomial of degree $d$ and shift $n$ associated with the partition function, denoted by $J_p^{d,n}(x)$, has only distinct real roots for all $n \geq N(d)$, conjectured by Chen, Jia, and Wang. Larson and Wagner have provided an estimate for $N(d)$. This implies that the discriminant of $J_p^{d,n}(x)$ is positive; i.e., $\mathrm{Disc}_x(J_p^{d,n}) > 0$. For $d = 2$, $\mathrm{Disc}_x(J_p^{d,n}) > 0$ when $n \geq N(d)$ is equivalent to the fact that $(p(n))_{n\geq 26}$ is log-concave. In 2017, Chen undertook a comprehensive investigation on inequalities for $p(n)$ through the lens of invariant theory of binary forms of degree $n$. Positivity of the invariant of a quadratic binary form (resp. cubic binary form) associated with $p(n)$ reflects that the sequence $(p(n))_{n\geq 26}$ satisfies the Turán inequalities (resp. $(p(n))_{n\geq 95}$ satisfies the higher order Turán inequalities). Chen further studied on the two invariants for a quartic binary form where its coefficients are shifted values of integer partitions and conjectured four inequalities for $p(n)$. In this paper, we confirm the conjectures of Chen.

**Keywords:** the partition function, higher order Turán inequalities, Hardy-Ramanujan-Rademacher formula, invariants of binary forms
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## Contents



## 1. Introduction

Throughout this paper, we consider only sequences of real numbers. A sequence $(a_n)_{n\geq 0}$ is said to satisfy the Turán inequlaities or to be log-concave, if

$$a_n^2 - a_{n-1}a_{n+1} \geq 0 \quad \text{for all} \quad n \geq 1, \tag{1.1}$$

see [41]. We say that a sequence $(a_n)_{n\geq 0}$ is said to satisfy the higher order Turán inequlaities if for all $n \geq 1$,

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2}) - (a_n a_{n+1} - a_{n-1}a_{n+2})^2 \geq 0. \tag{1.2}$$

The Turán inequalities and the higher order Turán inequalities are related to the Laguerre-Pólya class of real entire functions [13, 43]. A real entire function

$$\psi(x)=\sum_{k=0}^{\infty}a_k\frac{x^k}{k!} \tag{1.3}$$

is said to be in Laguerre-Pólya class, denoted by $\psi(x)\in\mathcal{LP}$, if it is of the form

$$\psi(x)=cx^m e^{-\alpha x^2+\beta x}\prod_{k=1}^{\infty}\Big(1+\frac{x}{x_k}\Big)e^{-\frac{x}{x_k}},$$

where $c,\beta,x_k$ are real numbers, $\alpha\geq 0$, $m\in\mathbb{Z}_{\geq 0}$, and $\sum_{k=1}^{\infty}x_k^{-2}$ converges. Any sequence of polynomials with only real zeroes, say $(P_n(x))_{n\geq 0}$, converges uniformly to a function $P(x)\in\mathcal{LP}$. For a more detailed study on the theory of the $\mathcal{LP}$ class, we refer to [38]. Jensen [21] proved that a real entire function $\psi(x)$ is in $\mathcal{LP}$ class if and only if for any $d\in\mathbb{Z}_{\geq 1}$, the Jensen polynomial of degree $d$ associated with a sequence $(a_n)_{n\geq 0}$:

$$J_a^d(x)=\sum_{k=0}^{d}\binom{d}{k}a_k x^k$$

has only real zeroes. Pólya and Schur [40] proved that for a real entire function $\psi(x)\in\mathcal{LP}$ and for any $n\geq\mathbb{Z}_{\geq 0}$, the $n$-th derivative $\psi^{(n)}(x)$ of $\psi(x)$ also belongs to the $\mathcal{LP}$ class, that is, the Jensen polynomial associated with $\psi^{(n)}(x)$

$$J_a^{d,n}(x)=\sum_{k=0}^{d}\binom{d}{k}a_{n+k}x^k$$

has only real zeroes. Observe that for $d=2$ and for all nonnegative integer $n$, the real-rootedness of $J_a^{d,n}(x)$ implies that the discriminant $4(a_{n+1}^2-a_na_{n+2})$ is nonnegative. Pólya's work [34] on $\mathcal{LP}$ class is closely connected with the Riemann hypothesis. He showed that the Riemann hypothesis is equivalent to the real rootedness of Jensen polynomial $J_a^{d,n}(x)$ for all nonnegative integers $d$ and $n$, where the coefficient sequence $\{a_n\}_{n\geq 0}$ is defined by

$$(-1+4z^2)\ \Lambda\Big(\frac{1}{2}+z\Big)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^{2n},$$

with $\Lambda(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)=\Lambda(1-s)$, where $\zeta$ denotes the Riemann zeta function and $\Gamma$ denotes the Gamma function. In 2019, Griffin, Ono, Rolen, and Zagier [17, Theorem 1] proved that for all $d\geq 1$, $J_a^{d,n}(x)$ has only real roots for all sufficiently large $n$.

Now we discuss in brief the inequalities of the partition function. A partition of a positive integer $n$ is a weakly decreasing sequence $(\lambda_1,\lambda_2,\ldots,\lambda_r)$ of positive integers such that $\lambda_1+\lambda_2+\cdots+\lambda_r=n$. Let $p(n)$ denote the number of partitions of $n$. Estimates on the partition function systematically began with the work of Hardy and Ramanujan [18] in 1918 and independently by Uspensky [44] in 1920:

$$p(n)\sim\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}\ \text{ as } n\to\infty. \tag{1.4}$$

Hardy and Ramanujan's proof involved an important tool called the Circle Method which has manifold applications in analytic number theory. For a well documented exposition on this collaboration, see [28]. During 1937-1943, Rademacher [35, 37, 36] improved the work of Hardy and Ramanujan and found a convergent series for $p(n)$ and Lehmer's [27, 26] considerations were on the estimation for the remainder term of the series for $p(n)$. The Hardy-Ramanujan-Rademacher formula reads

$$p(n) = \frac{\sqrt{12}}{24n-1}\sum_{k=1}^{N}\frac{A_k(n)}{\sqrt{k}}\left[\left(1-\frac{k}{\mu(n)}\right)e^{\mu(n)/k}+\left(1+\frac{k}{\mu(n)}\right)e^{-\mu(n)/k}\right]+R_2(n,N), \tag{1.5}$$

where

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}, \quad A_k(n) = \sum_{\substack{h \mod k\\(h,k)=1}} e^{-2\pi inh/k+\pi is(h,k)}$$

with

$$s(h,k) = \sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\left\lfloor\frac{\mu}{k}\right\rfloor-\frac{1}{2}\right)\left(\frac{h\mu}{k}-\left\lfloor\frac{h\mu}{k}\right\rfloor-\frac{1}{2}\right),$$

and

$$|R_2(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}}\left[\left(\frac{N}{\mu(n)}\right)^3\sinh\frac{\mu(n)}{N}+\frac{1}{6}-\left(\frac{N}{\mu(n)}\right)^2\right]. \tag{1.6}$$

Independently Nicolas [31] and DeSalvo and Pak [12, Theorem 1.1] proved that the partition function $(p(n))_{n\geq 26}$ is log-concave, conjectured by Chen [6]. DeSalvo and Pak [12, Theorem 4.1] also proved that for all $n \geq 2$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}, \tag{1.7}$$

conjectured by Chen [6]. Further, they improved the term $(1+\frac{1}{n})$ in (1.7) and proved that for all $n \geq 7$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{240}{(24n)^{3/2}}\right) > \frac{p(n)}{p(n+1)}, \tag{1.8}$$

see [12, p. 4.2]. DeSalvo and Pak [12] finally came up with the conjecture that the coefficient of $1/n^{3/2}$ in (1.8) can be improved to $\pi/\sqrt{24}$; i.e., for all $n \geq 45$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}, \tag{1.9}$$

which was proved by Chen, Wang and Xie [9, Sec. 2]. Paule, Radu, Zeng, and the author [4, Theorem 7.6] confirmed that the coefficient of $1/n^{3/2}$ is indeed $\pi/\sqrt{24}$, which is the optimal; i.e., they proved that for all $n \geq 120$,

$$p(n)^2 > \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{1}{n^2}\right)p(n-1)p(n+1). \tag{1.10}$$

Chen [7] conjectured that $p(n)$ satisfies the higher order Turán inequalities for all $n \geq 95$ which was proved by Chen, Jia, and Wang [8, Theorem 1.3] and analogous to the inequality (1.9), they conjectured that for all $n \geq 2$,

$$4(1-u_n)(1-u_{n+1}) < \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right)(1-u_nu_{n+1})^2 \text{ with } u_n := \frac{p(n+1)p(n-1)}{p(n)^2}, \tag{1.11}$$

settled by Larson and Wagner [25, Theorem 1.2]. In [8], Chen, Jia, and Wang conjectured[1] that for any integer $d \geq 1$ there exists an integer $N(d)$ such that the Jensen polynomial of degree $d$ and shift $n$ associated with $p(n)$ has only real roots which was settled by Griffin, Ono, Rolen, and Zagier [17, Theorem 5] and inspired by their work, Larson and Wagner [25, Theorem 1.3]

[1]Independently conjectured by K. Ono

proved that $N(d) \leq (3d)^{24d}(50d)^{3d^2}$. Proofs of the inequalities, stated before, primarily relies on the Hardy-Ramanujan-Rademacher formula (1.5) and Lehmer's error bound (1.6) but with different methodology.

While studying on higher order Turán inequality for $p(n)$, Chen [7] undertook a comprehensive study on inequalities pertaining to invariants of a binary form. A binary form $P(x,y)$ of degree $d$ is a homogeneous polynomial of degree $d$ in two variables $x$ and $y$ is defined by

$$P_d(x,y) := \sum_{i=0}^{d} \binom{n}{i} a_i x^i y^{n-i},$$

where $(a_i)_{1\leq i\leq n} \in \mathbb{C}^n$. But we restrict $a_i$ to be real numbers. The binary form $P_d(x,y)$ is transformed into a new binary form, say $Q(\overline{x},\overline{y})$ with

$$Q_d(\overline{x},\overline{y}) = \sum_{i=0}^{d} \binom{n}{i} c_i \overline{x}^i \overline{y}^{n-i}$$

under the action of $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in GL_2(\mathbb{R})$ as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}.$$

The transformed coefficients $(c_i)_{0\leq i\leq d}$ are polynomials in $(a_i)_{0\leq i\leq d}$ and entries of the matrix $M$. For $k \in \mathbb{Z}_{\geq 0}$, a polynomial $I(a_0, a_1, \ldots, a_d)$ in the coefficients $(a_i)_{0\leq i\leq d}$ is called an invariant of index of $k$ of the binary form $P_d(x,y)$ if for any $M \in GL_2(\mathbb{R})$,

$$I(\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_d) = (\det M)^k I(a_0, a_1, \ldots, a_n).$$

For a more detailed study on the theory of invariants, see, for example, Hilbert [19], Kung and Rota [24], and Sturmfels [42]. We observe that $I(a_0,a_1,a_2) = a_1^2 - a_0a_2$ is an invariant of the quadratic binary form

$$P_2(x,y) = a_2x^2 + 2a_1xy + a_0y^2$$

and the discriminant is $4I(a_0,a_1,a_2)$. For a sequence $(a_n)_{n\geq 0}$, define

$$I_{n-1}(a_0,a_1,a_2) := I(a_{n-1},a_n,a_{n+1}) = a_n^2 - a_{n-1}a_{n+1}.$$

Therefore, if we choose $a_n = p(n)$, then $I_{n-1}(p(0),p(1),p(2)) > 0$ for all $n \geq 26$ is the same thing as saying $(p(n))_{n\geq 26}$ is log-concave. For degree 3,

$$I(a_0,a_1,a_2,a_3) = 4(a_1^2 - a_0a_2)(a_2^2 - a_1a_3) - (a_1a_2 - a_0a_3)^2$$

is an invariant of the cubic binary form $P_3(x,y) = a_3x^3 + 3a_2x^2y + 3a_1xy^2 + a_0y^3$ and the discriminant is $27I(a_0,a_1,a_2,a_3)$. Similarly, setting $a_n = p(n)$, the positivity of $I_{n-1}(a_0,a_1,a_2,a_3)$ for all $n \geq 95$ is equivalent to state that $(p(n))_{n\geq 95}$ satisfies the higher order Turán inequality. Two invariants of the quartic binary form

$$P_4(x,y) = a_4x^4 + 4a_3x^3y + 6a_2x^2y^2 + 4a_1xy^3 + a_0y^4$$

are of the following form

$$\begin{aligned} A(a_0,a_1,a_2,a_3,a_4) &= a_0a_4 - 4a_1a_3 + 3a_2^2, \\ B(a_0,a_1,a_2,a_3,a_4) &= -a_0a_2a_4 + a_2^3 + a_0a_3^2 + a_1^2a_4 - 2a_1a_2a_3. \end{aligned}$$

Setting $a_n = p(n)$, Chen [7] conjectured that

$$A(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3}) > 0 \quad \text{and} \quad B(a_{n-1},a_n,a_{n+1},a_{n+2},a_{n+3}) > 0,$$

along with the associated companion inequalities in the spirit of (1.9) and (1.11). Here we list all the four conjectures with $a_n = p(n)$.

**Conjecture 1.1** (Eqn. (6.17), [7]).

$$a_{n-1}a_{n+3}+3a_{n+1}^2>4a_na_{n+2}\quad \textit{for all}\quad n\geq 185. \tag{1.12}$$

**Conjecture 1.2** (Conjecture 6.15, [7]). *We have*

$$4\Big(1+\frac{\pi^2}{16n^3}\Big)a_na_{n+2}>a_{n-1}a_{n+3}+3a_{n+1}^2\quad \textit{for all}\quad n\geq 218. \tag{1.13}$$

**Conjecture 1.3** (Eqn. (6.18), [7]).

$$a_{n+1}^3+a_{n-1}a_{n+2}^2+a_n^2a_{n+3}>2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3}\quad \textit{for all}\quad n\geq 221. \tag{1.14}$$

**Conjecture 1.4** (Conjecture 6.16, [7]). *We have*

$$\Big(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}\Big)(2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3})>a_{n+1}^3+a_{n-1}a_{n+2}^2+a_n^2a_{n+3}\quad \textit{for all}\quad n\geq 244. \tag{1.15}$$

We prove all the four conjectures along with the confirmation that the rate of decay $\pi^2/16n^3$ (resp. $\pi^3/72\sqrt{6}n^{9/2}$) in (1.2) (resp. in (1.4)) is the optimal one, as stated in Theorem 1.5 (resp. Theorem 1.7). We also ensure that the rate of decay is $\pi/\sqrt{24}n^{3/2}$ in context of (1.11) can not be improved further by proving Theorem 1.9.

A major part of this paper is devoted to obtain an infinite family of inequalities for $p(n-\ell)$ for a non-negative integer $\ell$, stated in Theorem 4.5, so that under a unified framework, we can prove inequalities for $p(n)$ stated below. Work done in Sections 3 and 4 incarnates the theme of work presented in [3].

Let $a_n:=p(n)$.

**Theorem 1.5.** *For all $n\geq 218$,*

$$4\Big(1+\frac{\pi^2}{16n^3}\Big)a_na_{n+2}>a_{n-1}a_{n+3}+3a_{n+1}^2>4\Big(1+\frac{\pi^2}{16n^3}-\frac{6}{n^{7/2}}\Big)a_na_{n+2}. \tag{1.16}$$

**Corollary 1.6.** *Conjecture 1.1 and 1.2 is true.*

**Theorem 1.7.** *For all $n\geq 244$,*

$$\begin{aligned}\Big(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}\Big)(2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3})&>a_{n+1}^3+a_{n-1}a_{n+2}^2+a_n^2a_{n+3}\\ &>\Big(1+\frac{\pi^3}{72\sqrt{6}n^{9/2}}-\frac{8}{n^5}\Big)(2a_na_{n+1}a_{n+2}+a_{n-1}a_{n+1}a_{n+3}).\end{aligned} \tag{1.17}$$

**Corollary 1.8.** *Conjecture 1.3 and 1.4 is true.*

**Theorem 1.9.** *For all $n\geq 115$,*

$$\begin{aligned}\Big(1+\frac{\pi}{\sqrt{24}n^{3/2}}\Big)(a_na_{n+1}-a_{n-1}a_{n+2})^2&>4(a_n^2-a_{n-1}a_{n+1})(a_{n+1}^2-a_na_{n+2})\\ &>\Big(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{3}{n^2}\Big)(a_na_{n+1}-a_{n-1}a_{n+2})^2.\end{aligned} \tag{1.18}$$

**Remark 1.10.** *We observe that Theorem 1.9 immediately implies the following three statements:*

(1) $(p(n))_{n\geq 95}$ *satisfies the higher order Turán inequalities [8, Theorem 1.3].*
(2) *For all $n\geq 2$, (1.11) holds [25, Theorem 1.2].*
(3) $\frac{\pi}{\sqrt{24}n^{3/2}}$ *is the optimal rate of decay of the quotient*

$$4(a_n^2-a_{n-1}a_{n+1})(a_{n+1}^2-a_na_{n+2})/(a_na_{n+1}-a_{n-1}a_{n+2})^2.$$

The rest of this paper is organized as follows. In Section 2, we shall present a couple of lemmas from [4, 3] that will be helpful in later sections. Following the work done by Paule, Radu, Schneider, and the aurhor [3], Section 3 prepares the set up by determining the coefficients in the asymptotic expansion of $p(n-\ell)$ along with its estimates. An infinite family of inequalities for $p(n-\ell)$ is presented in Section 4. Section 5 presents proofs of the Theorems 1.9, 1.5, and 1.7. We conclude this paper by a brief discussion on the future aspect of this work, given in Section 7.

## 2. Preliminaries

This section presents all the preliminary lemmas required for the proofs of the lemmas presented in subsequent sections.

**Lemma 2.1.** *[3, Lemma 3.3] For $j,\, k \in \mathbb{Z}_{\geq 0}$,*

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{i/2}{j} = \begin{cases} 1, & j=k=0 \\ (-1)^j 2^{k-2j}\frac{k}{j}\binom{2j-k-1}{j-k}, & \textit{otherwise} \end{cases}. \tag{2.1}$$

**Lemma 2.2.** *[3, Lemma 4.1] Let $x_1, x_2, \dots, x_n \leq 1$ and $y_1, \dots, y_1$ be non-negative real numbers. Then*

$$\frac{(1-x_1)(1-x_2)\cdots(1-x_n)}{(1+y_1)(1+y_2)\cdots(1+y_n)} \geq 1 - \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} y_j.$$

**Lemma 2.3.** *[3, Lemma 4.2] For $t \geq 1$ and non-negative integer $u \leq t$, we have*

$$\frac{1}{2t} \geq \frac{t(-t)_u(-1)^u}{(1+2t)(t+u)(t)_u} \geq \frac{1}{2t}\left(1 - \frac{u^2+\frac{1}{2}}{t}\right).$$

**Lemma 2.4.** *[3, Lemma 4.3] For $t \geq 1$ and non-negative integer $u \leq t$, we have*

$$\frac{2u+1}{2t} \geq \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i} \geq \frac{2u+1}{2t} - \frac{4u^3+6u^2+8u+3}{12t^2}.$$

Throughout the rest of this paper,

$$\alpha_\ell := \frac{\pi}{6}\sqrt{1+24\ell}.$$

**Lemma 2.5.** *We have*

$$\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} = \cosh(\alpha_\ell),\ \sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u)!} = \frac{1}{2}\alpha_\ell\sinh(\alpha_\ell),\ \sum_{u=0}^{\infty}\frac{u^2\alpha_\ell^{2u}}{(2u)!} = \frac{\alpha_\ell^2}{4}\cosh(\alpha_\ell) + \frac{\alpha_\ell}{4}\sinh(\alpha_\ell),$$

*and*

$$\sum_{u=0}^{\infty}\frac{u^3\alpha_\ell^{2u}}{(2u)!} = \frac{3\alpha_\ell^2}{8}\cosh(\alpha_\ell) + \frac{\alpha_\ell(\alpha_\ell^2+1)}{8}\sinh(\alpha_\ell).$$

**Lemma 2.6.** *[3, Lemma 4.5] Let $u \in \mathbb{Z}_{\geq 0}$. Assume that $a_{n+1}-a_n \geq b_{n+1}-b_n$ for all $n \geq u$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Then*

$$b_n \geq a_n \textit{ for all } n \geq u.$$

**Lemma 2.7.** *For $t \geq 1$ and $k \in \{0,1,2,3\}$ we have*

$$\sum_{u=t+1}^{\infty}\frac{u^k\alpha_\ell^{2u}}{(2u)!} \leq \frac{C_k(\ell)}{t^2},$$

*where*

$$C_k(\ell) = \begin{cases} C_k = \dfrac{\alpha_\ell^4 \cdot 2^k}{18}, & \ell = 0 \\ \dfrac{\lceil\sqrt{\ell}\rceil^2\left(1+\lceil\sqrt{\ell}\rceil\right)^{k+2}\alpha_\ell^{2(1+\lceil\sqrt{\ell}\rceil)}}{(1+2\lceil\sqrt{\ell}\rceil)(2+2\lceil\sqrt{\ell}\rceil)!}, & \ell \geq 1 \end{cases}.$$

*Proof.* Applying Lemma 2.6 with $a_n = \sum_{u=n+1}^{\infty} \frac{u^k \alpha_\ell^{2u}}{(2u)!}$ and $b_n = \frac{C_k(\ell)}{n^2}$, $b_{n+1} - b_n \leq a_{n+1} - a_n$ is equivalent to show that $f(n) := \frac{n^2(n+1)^{k+2}\alpha_\ell^{2n+2}}{(2n+1)(2n+2)!} \leq C_k(\ell)$. To prove $f(n) \leq C_k(\ell)$, it is sufficient to show that $f(m) \leq C_k(\ell)$ for a minimal $m$ such that $f(m)$ is maximal. In order to find such $m$, it is enough to that $\frac{f(n+1)}{f(n)} \leq 1$ for all $n \geq \max\{\lceil\sqrt{\ell}\rceil, 1\}$, and therefore, $\max\limits_{n\in\mathbb{Z}_{\geq 0}} f(n) = f(\lceil\sqrt{\ell}\rceil) = C_k(\ell)$ for all $\ell \geq 1$ and for $\ell = 0$, $\max\limits_{n\in\mathbb{Z}_{\geq 0}} f(n) = f(1) = C_k(0)$. Now, $\frac{f(n+1)}{f(n)} = \frac{\alpha_\ell^2(n+2)^{k+2}(2n+1)}{(2n+4)(2n+3)^2(n+1)^k n^2} \leq 1$ holds for all all $n \geq \max\{\lceil\sqrt{\ell}\rceil, 1\}$. □

**Lemma 2.8.** *[4, Equation 7.5, Lemma 7.3] For $n, k, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$ let*

$$b_{k,n}(s) := \frac{4\sqrt{s}}{\sqrt{s+k-1}}\binom{s+k-1}{s-1}\frac{1}{n^k},$$

*then*

$$0 < \sum_{t=k}^{\infty}\binom{-\frac{2s-1}{2}}{t}\frac{(-1)^k}{n^k} < b_{k,n}(s). \tag{2.2}$$

**Lemma 2.9.** *[4, Equation 7.9, Lemma 7.5] For $m, n, s \in \mathbb{Z}_{\geq 1}$ and $n > 2s$ let*

$$c_{m,n}(s) := \frac{2}{m}\frac{s^m}{n^m},$$

*then*

$$-\frac{c_{m,n}(s)}{\sqrt{m}} < \sum_{k=m}^{\infty}\binom{1/2}{k}\frac{(-1)^k s^k}{n^k} < 0. \tag{2.3}$$

**Lemma 2.10.** *[4, Equation 7.7, Lemma 7.4] For $n, s \in \mathbb{Z}_{\geq 1}$, $m \in \mathbb{N}$ and $n > 2s$ let*

$$\beta_{m,n}(s) := \frac{2}{n^m}\binom{s+m-1}{s-1},$$

*then*

$$0 < \sum_{k=m}^{\infty}\binom{-s}{k}\frac{(-1)^k}{n^k} < \beta_{m,n}(s). \tag{2.4}$$

## 3. Set Up

Using the Hardy-Ramanujan-Rademacher formula for $p(n)$ and Lehmer's error bound, we have the following inequality for $p(n)$ due to Chen, Jia, and Wang.

**Lemma 3.1.** *[8, Lemma 2.2 ] For all $n \geq 1206$,*

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu(n)^{10}}\right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu(n)^{10}}\right), \tag{3.1}$$

*where for $n \geq 1$, $\mu(n) := \frac{\pi}{6}\sqrt{24n-1}$.*

The definition of $\mu(n)$ is kept throughout this paper. Paule, Radu, Zeng, and the author extended Lemma 3.1 as follows.

**Theorem 3.2.** *[4, Theorem 4.4] For $k \in \mathbb{Z}_{\geq 2}$, define*

$$\widehat{g}(k) := \frac{1}{24}\left(\frac{36}{\pi^2}\cdot \nu(k)^2 + 1\right),$$

*where* $\nu(k) := 2\log 6 + (2\log 2)k + 2k\log k + 2k\log\log k + \dfrac{5k\log\log k}{\log k}$. *Then for all $k \in \mathbb{Z}_{\geq 2}$ and $n > \widehat{g}(k)$ such that $(n,k) \neq (6,2)$, we have*

$$\frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}-\frac{1}{\mu(n)^k}\right) < p(n) < \frac{\sqrt{12}e^{\mu(n)}}{24n-1}\left(1-\frac{1}{\mu(n)}+\frac{1}{\mu(n)^k}\right). \tag{3.2}$$

By making the shift $n-\ell$ in $p(n)$ for any $\ell \geq 0$, we obtain the following result.

**Theorem 3.3.** *Let $\ell \in \mathbb{Z}_{\geq 0}$. For $k \in \mathbb{Z}_{\geq 2}$, let $\widehat{g}(k)$ be as in Theorem 3.2. Then for all $k \in \mathbb{Z}_{\geq 2}$ and $n > \widehat{g}(k)+\ell$ such that $(n,k) \neq (6,2)$, we have*

$$\frac{\sqrt{12}e^{\mu(n-\ell)}}{24(n-\ell)-1}\left(1-\frac{1}{\mu(n-\ell)}-\frac{1}{\mu(n-\ell)^k}\right) < p(n-\ell) < \frac{\sqrt{12}e^{\mu(n-\ell)}}{24(n-\ell)-1}\left(1-\frac{1}{\mu(n-\ell)}+\frac{1}{\mu(n-\ell)^k}\right). \tag{3.3}$$

Rewrite the term $\dfrac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\dfrac{1}{\mu(n-\ell)}\Big)$ in the following way:

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big) = \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}\underbrace{e^{\pi\sqrt{2n/3}}\ \Big(\sqrt{1-\frac{1+24\ell}{24n}}-1\Big)}_{:=A_1(n,\ell)}\underbrace{\Big(1-\frac{1+24\ell}{24n}\Big)^{-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big)}_{:=A_2(n,\ell)}. \tag{3.4}$$

Now we compute the Taylor expansion of the residue parts of $A_1(n,\ell)$ and $A_2(n,\ell)$, defined in (3.4).

**Definition 3.4.** *For $t, \ell \in \mathbb{Z}_{\geq 0}$, define*

$$e_1(t,\ell) := \begin{cases} 1, & \text{if } t=0 \\ \dfrac{(-1)^t(1+24\ell)^t}{(24)^t}\dfrac{(1/2-t)_{t+1}}{t}\displaystyle\sum_{u=1}^{t}\frac{(-1)^u(-t)_u}{(t+u)!(2u-1)!}\alpha_\ell^{2u}, & \text{otherwise} \end{cases}, \tag{3.5}$$

*and*

$$E_1\Big(\frac{1}{\sqrt{n}},\ell\Big) := \sum_{t=0}^{\infty} e_1(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t},\ n \geq 1. \tag{3.6}$$

**Definition 3.5.** *For $t, \ell \in \mathbb{Z}_{\geq 0}$, define*

$$o_1(t,\ell) := -\frac{\pi}{12\sqrt{6}}(1+24\ell)\left(\frac{(-1)^t(1/2-t)_{t+1}(1+24\ell)^t}{(24)^t}\sum_{u=0}^{t}\frac{(-1)^u(-t)_u}{(t+u+1)!(2u)!}\alpha_\ell^{2u}\right) \tag{3.7}$$

*and*

$$O_1\Big(\frac{1}{\sqrt{n}},\ell\Big) := \sum_{t=0}^{\infty} o_1(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1},\ n \geq 1. \tag{3.8}$$

**Lemma 3.6.** *Let $A_1(n,\ell)$ be defined as in (3.4). Let $E_1(n,\ell)$ be as in Definition 3.4 and $O_1(n,\ell)$ as in Definition 3.5. Then*

$$A_1(n,\ell) = E_1\Big(\frac{1}{\sqrt{n}},\ell\Big) + O_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{3.9}$$

*Proof.* From (3.4), we get

$$\begin{aligned} A_1(n,\ell) &= e^{\pi\sqrt{2n/3}\,\left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)} \\ &= \sum_{k=0}^{\infty}\frac{(\pi\sqrt{2n/3})^k}{k!}\left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)^k \\ &= \sum_{k=0}^{\infty}\frac{(\pi\sqrt{2/3})^k}{k!}(\sqrt{n})^k\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(\sqrt{1-\frac{1+24\ell}{24n}}\right)^i \\ &= \sum_{k=0}^{\infty}\frac{(\pi\sqrt{2/3})^k}{k!}(\sqrt{n})^k\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=0}^{\infty}\binom{i/2}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j} \\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{\infty}\frac{(\pi\sqrt{2/3})^k}{k!}\frac{(-1)^{k-i+j}(1+24\ell)^j}{(24)^j}\binom{k}{i}\binom{i/2}{j}(\sqrt{n})^{k-2j}. \end{aligned} \tag{3.10}$$

Split $S := \left\{(k,i,j)\in\mathbb{Z}^3_{\geq 0} : 0\leq i\leq k\right\} := \bigcup_{t\in\mathbb{Z}_{\geq 0}} V(t)$, where for each $t\in\mathbb{Z}_{\geq 0}$,

$$V(2t) = \left\{(2u,i,u+t)\in\mathbb{Z}^3_{\geq 0} : 0\leq i\leq 2u\right\}$$

and

$$V(2t+1) = \left\{(2u+1,i,u+t+1)\in\mathbb{Z}^3_{\geq 0} : 0\leq i\leq 2u+1\right\}.$$

By Lemma 2.1, we have $\sum_{i=0}^{k}\binom{k}{i}\binom{i/2}{j} = 0$ for $k>j$. For $r=(k,i,j)\in S$, we define

$$S(r) := \frac{(\pi\sqrt{2/3})^k}{k!}\frac{(-1)^{k-i+j}(1+24\ell)^j}{(24)^j}\binom{k}{i}\binom{i/2}{j} \quad\text{and}\quad f(r) := k-2j.$$

Rewrite (3.10) as

$$A_1(n,\ell) = \sum_{t=0}^{\infty}\sum_{r\in V(2t)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \sum_{t=0}^{\infty}\sum_{r\in V(2t+1)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \tag{3.11}$$

Now

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} = \sum_{t=0}^{\infty}\frac{(-1)^t(1+24\ell)^t}{(24)^t}\left(\sum_{u=0}^{\infty}\frac{(-1)^u}{(2u)!}\alpha_\ell^{2u}\mathcal{E}_1(u,t)\right)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}, \tag{3.12}$$

where by Lemma 2.1,

$$\mathcal{E}_1(u,t) := \sum_{i=0}^{2u}(-1)^i\binom{2u}{i}\binom{i/2}{u+t} = \begin{cases} 1, & \text{if } u=t=0 \\ 0, & \text{if } u>t \\ \frac{2u(1/2-t)_{t+1}(-t)_u}{t(t+u)!}, & \text{otherwise} \end{cases}.$$

Consequently, we have

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} = E_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{3.13}$$

Simplifying,

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t+1)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}$$
$$= -\frac{\pi(1+24\ell)}{12\sqrt{6}}\sum_{t=0}^{\infty}\frac{(-1)^t(1+24\ell)^t}{(24)^t}\Bigg(\sum_{u=0}^{\infty}\frac{(-1)^u}{(2u+1)!}\alpha_\ell^{2u}\mathcal{O}_1(u,t)\Bigg)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}, \tag{3.14}$$

where by Lemma 2.1,

$$\mathcal{O}_1(u,t) := \sum_{i=0}^{2u+1}(-1)^i\binom{2u+1}{i}\binom{i/2}{u+t+1} = \begin{cases} 0, & \text{if } u>t \\ -\frac{(2u+1)(1/2-t)_{t+1}(-t)_u}{(t+u+1)!}, & \text{otherwise} \end{cases}.$$

Therefore, we have

$$\sum_{t=0}^{\infty}\sum_{r\in V(2t+1)} S(r)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} = O_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{3.15}$$

From (3.11), (3.13), and (3.15), we get (3.9). □

**Definition 3.7.** *For $t\in\mathbb{Z}_{\geq 0}$, define*

$$E_2\Big(\frac{1}{\sqrt{n}},\ell\Big) := \sum_{t=0}^{\infty} e_2(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \text{ with } e_2(t,\ell) := \frac{(1+24\ell)^t}{(24)^t}. \tag{3.16}$$

**Definition 3.8.** *For $t\in\mathbb{Z}_{\geq 0}$, define*

$$O_2\Big(\frac{1}{\sqrt{n}}\Big) := \sum_{t=0}^{\infty} o_2(t)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \text{ with } o_2(t) := -\frac{6}{\pi\sqrt{24}}\binom{-3/2}{t}\frac{(-1)^t(1+24\ell)^t}{(24)^t}. \tag{3.17}$$

**Lemma 3.9.** *Let $A_2(n,\ell)$ be defined as in* (3.4). *Let $E_2(n,\ell)$ be as in Definition 3.7 and $O_2(n,\ell)$ as in Definition 3.8. Then*

$$A_2(n,\ell) = E_2\Big(\frac{1}{\sqrt{n}},\ell\Big) + O_2\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{3.18}$$

*Proof.* Following the definition of $A_2(n,\ell)$ from (3.4) and expand it as follows:

$$\begin{aligned} A_2(n,\ell) &= \Big(1-\frac{1+24\ell}{24n}\Big)^{-1} - \frac{6}{\pi\sqrt{24}}\frac{1}{\sqrt{n}}\Big(1-\frac{1+24\ell}{24n}\Big)^{-3/2} \\ &= E_2\Big(\frac{1}{\sqrt{n}},\ell\Big) + O_2\Big(\frac{1}{\sqrt{n}},\ell\Big). \end{aligned} \tag{3.19}$$

This completes the proof of (3.18). □

**Definition 3.10.** *Following the Definitions 3.4-3.8, we define*

$$S_{e,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) := E_1\Big(\frac{1}{\sqrt{n}},\ell\Big)E_2\Big(\frac{1}{\sqrt{n}},\ell\Big), \tag{3.20}$$

$$S_{e,2}\Big(\frac{1}{\sqrt{n}},\ell\Big) := O_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big), \tag{3.21}$$

$$S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) := E_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big), \tag{3.22}$$

*and*

$$S_{o,2}\Big(\frac{1}{\sqrt{n}},\ell\Big) := E_2\Big(\frac{1}{\sqrt{n}},\ell\Big)O_1\Big(\frac{1}{\sqrt{n}},\ell\Big). \tag{3.23}$$

**Lemma 3.11.** *For each* $i \in \{1,2\}$*, let* $S_{e,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ *and* $S_{o,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ *be as in Definition 3.10. Then*

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big)=\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}\sum_{i=1}^{2}\Bigg(S_{e,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)+S_{o,i}\Big(\frac{1}{\sqrt{n}},\ell\Big)\Bigg). \tag{3.24}$$

*Proof.* The proof follows immediately by applying Lemmas 3.6 and 3.9 to (3.4). □

## 3.1. Coefficients in the asymptotic expansion of $p(n-\ell)$.

**Definition 3.12.** *For* $t,\ell \in \mathbb{Z}_{\geq 0}$*, define*

$$S_1(t,\ell):=\sum_{s=1}^{t}\frac{(-1)^s(1/2-s)_{s+1}}{s}\sum_{u=1}^{s}\frac{(-1)^u(-s)_u}{(s+u)!(2u-1)!}\alpha_\ell^{2u}, \tag{3.25}$$

*and*

$$g_{e,1}(t,\ell):=\frac{(1+24\ell)^t}{(24)^t}\Big(1+S_1(t,\ell)\Big). \tag{3.26}$$

**Lemma 3.13.** *Let* $S_{e,1}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ *be as in* (3.20)*. Let* $g_{e,1}(t,\ell)$ *be as in Definition 3.12. Then*

$$S_{e,1}\Big(\frac{1}{\sqrt{n}},\ell\Big)=\sum_{t=0}^{\infty}g_{e,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{3.27}$$

*Proof.* From (3.6), (3.16), and (3.20), we have

$$S_{e,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) = 1+\sum_{t=1}^{\infty}\Bigg(e_1(t,\ell)+e_2(t,\ell)+\sum_{s=1}^{t-1}e_1(s,\ell)e_2(t-s,\ell)\Bigg)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{3.28}$$

Combining (3.5) and (3.16), we obtain

$$e_1(t)+e_2(t)+\sum_{s=1}^{t-1}e_1(s)e_2(t-s) = \frac{(1+24\ell)^t}{(24)^t}\Big(1+S_1(t,\ell)\Big)=g_{e,1}(t,\ell), \tag{3.29}$$

which concludes the proof of (3.27). □

**Definition 3.14.** *For* $t \in \mathbb{Z}_{\geq 1}$ *and* $\ell \in \mathbb{Z}_{\geq 0}$*, define*

$$S_2(t,\ell):=\sum_{s=0}^{t-1}(1/2-s)_{s+1}\binom{-3/2}{t-s-1}\sum_{u=0}^{s}\frac{(-1)^u(-s)_u}{(s+u+1)!(2u)!}\alpha_\ell^{2u}, \tag{3.30}$$

*and*

$$g_{e,2}(t,\ell):=\frac{(-1)^{t-1}(1+24\ell)^t}{(24)^t}S_2(t,\ell). \tag{3.31}$$

**Lemma 3.15.** *Let* $S_{e,2}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ *as in* (3.21) *and* $g_{e,2}(t,\ell)$ *as in Definition 3.14. Then*

$$S_{e,2}\Big(\frac{1}{\sqrt{n}},\ell\Big)=\sum_{t=1}^{\infty}g_{e,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}. \tag{3.32}$$

*Proof.* From (3.8), (3.18) and (3.21), we have

$$\begin{aligned} S_{e,2}\Big(\frac{1}{\sqrt{n}},\ell\Big) &= O_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big) \\ &= \sum_{t=1}^{\infty}\Bigg(\sum_{s=0}^{t-1} o_1(s,\ell)o_2(t-s-1,\ell)\Bigg)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\ &= \sum_{t=1}^{\infty} g_{e,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \text{ (by (3.7) and (3.17)).} \end{aligned} \tag{3.33}$$

□

**Definition 3.16.** *For $t \in \mathbb{Z}_{\geq 2}$ and $\ell \in \mathbb{Z}_{\geq 0}$, define*

$$S_3(t,\ell) := \sum_{s=1}^{t} \frac{(1/2-s)_{s+1}\binom{-3/2}{t-s}}{s} \sum_{u=1}^{s} \frac{(-1)^u(-s)_u}{(s+u)!(2u-1)!}\alpha_\ell^{2u}, \tag{3.34}$$

*and*

$$g_{o,1}(t,\ell) := \begin{cases} -\dfrac{6}{\pi\sqrt{24}}\dfrac{(-1)^t(1+24\ell)^t}{(24)^t}\Big(\binom{-3/2}{t} + S_3(t)\Big), & \text{if } t \geq 2 \\ -\dfrac{432+(1+24\ell)\pi^2}{2304\sqrt{6}\pi}, & \text{if } t = 1\ . \\ -\dfrac{6}{\pi\sqrt{24}}, & \text{if } t = 0 \end{cases} \tag{3.35}$$

**Lemma 3.17.** *Let $S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ as in (3.22) and $g_{o,1}(t,\ell)$ be as in Definition 3.16. Then*

$$S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) = \sum_{t=0}^{\infty} g_{o,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \tag{3.36}$$

*Proof.* From (3.6), (3.17) and (3.22), it follows that

$$\begin{aligned} S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) &= E_1\Big(\frac{1}{\sqrt{n}},\ell\Big)O_2\Big(\frac{1}{\sqrt{n}},\ell\Big) \\ &= g_{o,1}(0,\ell)\frac{1}{\sqrt{n}} + g_{o,1}(1,\ell)\frac{1}{\sqrt{n}^3} + \sum_{t=2}^{\infty}\Bigg(o_2(t) + \sum_{s=1}^{t} e_1(s,\ell)o_2(t-s,\ell)\Bigg)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \\ &= g_{o,1}(0,\ell)\frac{1}{\sqrt{n}} + g_{o,1}(1,\ell)\frac{1}{\sqrt{n}^3} + \sum_{t=2}^{\infty} g_{o,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \text{(by (3.5) and (3.17)).} \end{aligned} \tag{3.37}$$

□

**Definition 3.18.** *For $t \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$, define*

$$S_4(t,\ell) := \sum_{s=0}^{t}(-1)^s(1/2-s)_{s+1} \sum_{u=0}^{s} \frac{(-1)^u(-s)_u}{(s+u+1)!(2u)!}\alpha_\ell^{2u}, \tag{3.38}$$

*and*

$$g_{o,2}(t,\ell) := -\frac{\pi(1+24\ell)}{12\sqrt{6}}\frac{(1+24\ell)^t}{(24)^t}S_4(t). \tag{3.39}$$

**Lemma 3.19.** *Let $S_{o,2}\Big(\frac{1}{\sqrt{n}},\ell\Big)$ be as in (3.23) and $g_{o,2}(t,\ell)$ be as in Definition 3.18. Then*

$$S_{o,2}\Big(\frac{1}{\sqrt{n}},\ell\Big)=\sum_{t=0}^{\infty}g_{o,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}. \tag{3.40}$$

*Proof.* From (3.8), (3.16) and (3.23), it follows that

$$\begin{aligned}S_{o,1}\Big(\frac{1}{\sqrt{n}},\ell\Big) &= O_1\Big(\frac{1}{\sqrt{n}},\ell\Big)E_2\Big(\frac{1}{\sqrt{n}},\ell\Big)\\ &= \sum_{t=0}^{\infty}\Bigg(\sum_{s=0}^{t}o_1(s,\ell)e_2(t-s,\ell)\Bigg)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\\ &= \sum_{t=0}^{\infty}g_{o,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}\ \text{(by (3.8) and (3.16)).}\end{aligned} \tag{3.41}$$

□

**Definition 3.20.** *For each $i\in\{1,2\}$, let $g_{e,i}(t,\ell)$ and $g_{o,i}(t,\ell)$ be as in Definitions 3.12-3.18. We define a power series*

$$G(n,\ell):=\sum_{t=0}^{\infty}g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{t}=\sum_{t=0}^{\infty}g(2t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}+\sum_{t=0}^{\infty}g(2t+1,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1},$$

*where*

$$g(2t,\ell):=g_{e,1}(t,\ell)+g_{e,2}(t,\ell)\quad\text{and}\quad g(2t+1,\ell):=g_{o,1}(t,\ell)+g_{o,2}(t,\ell). \tag{3.42}$$

**Lemma 3.21.** *Let $G(n,\ell)$ be as in Definition 3.20. Then*

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\Big(1-\frac{1}{\mu(n-\ell)}\Big)=\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}\cdot G(n,\ell). \tag{3.43}$$

*Proof.* Applying Lemmas 3.13-3.19 to Lemma 3.9, we have (3.43). □

**Remark 3.22.** *Using* `Sigma` *due to Schneider [39] and* `GeneratingFunctions` *due to* Mallinger *[29], we observe that for all $t\geq 0$,*

$$g(2t,\ell)=g_{e,1}(t,\ell)+g_{e,2}(t,\ell)=\omega_{2t,\ell}\quad\text{and}\quad g(2t+1,\ell)=g_{o,1}(t,\ell)+g_{o,2}(t,\ell)=\omega_{2t+1,\ell}, \tag{3.44}$$

*where*

$$g(t,\ell)=\omega_{t,\ell}=\frac{(1+24\ell)^t}{(-4\sqrt{6})^t}\sum_{k=0}^{\frac{t+1}{2}}\binom{t+1}{k}\frac{t+1-k}{(t+1-2k)!}\Big(\frac{\pi}{6}\Big)^{t-2k}\frac{1}{(1+24\ell)^k}. \tag{3.45}$$

*Note that for $\ell=0$, we retrieve $\omega_t$ as in O'Sullivan's [32, Proposition 4.4] work.*

3.2. **Estimation of $(S_i(t,\ell))$.** We present the Lemmas 3.24-3.30 which will be needed in the Subsection 3.3. A brief sketch of proofs of these lemmas are presented in the Section 6.

**Definition 3.23.** *Let $C_k(\ell)$ be as in Lemma 2.7. Define*

$$\begin{aligned}C_1^{\mathcal{L}}(\ell)&:=\frac{\cosh(\alpha_\ell)-1}{4}+C_0(\ell)+\frac{\alpha_\ell^2\cosh(\alpha_\ell)+\alpha_\ell\sinh(\alpha_\ell)}{8},\\ C_1^{\mathcal{U}}(\ell)&:=C_1(\ell)+\frac{\alpha_\ell^2+1}{4}\cosh(\alpha_\ell)+\frac{\alpha_\ell(\alpha_\ell^2+12)}{24}\sinh(\alpha_\ell).\end{aligned}$$

**Lemma 3.24.** *Let $S_1(t,\ell)$ be as in Definition 3.12 and $C_1^{\mathcal{L}}(\ell),C_1^{\mathcal{U}}(\ell)$ as in Definition 3.23. Then for all $t\geq 1$,*

$$-\frac{C_1^{\mathcal{L}}(\ell)}{t^2}<\frac{S_1(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}}-\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\big(\cosh(\alpha_\ell)-1\big)+\frac{1}{2t}\alpha_\ell\sinh(\alpha_\ell)<\frac{C_1^{\mathcal{U}}(\ell)}{t^2}. \tag{3.46}$$

**Definition 3.25.** *Let $C_k(\ell)$ be as in Lemma 2.7. Define*

$$\begin{aligned}
C^{\mathcal{L}}_{2,1}(\ell) &:= \frac{\cosh(\alpha_\ell)}{4} + \frac{\sinh(\alpha_\ell)}{4\alpha_\ell} + \frac{\alpha_\ell \sinh(\alpha_\ell)}{4} + \frac{2C_1(\ell)}{\alpha_\ell^2},\\
C^{\mathcal{U}}_{2,1}(\ell) &:= -\frac{\cosh(\alpha_\ell)}{2} + \frac{\sinh(\alpha_\ell)}{2\alpha_\ell} + \frac{2C_2(\ell)}{\alpha_\ell^2},\\
\operatorname{csh}(\ell) &:= \cosh(\alpha_\ell) + \alpha_\ell \sinh(\alpha_\ell),\\
C_{2,2}(\ell) &:= \frac{8C_3(\ell)}{\alpha_\ell^2} + \frac{(\alpha_\ell^2+1)\cosh(\alpha_\ell)}{4} + \frac{(\alpha_\ell^3 + 12\alpha_\ell)\sinh(\alpha_\ell)}{24},\\
C^{\mathcal{L}}_{2}(\ell) &:= C^{\mathcal{U}}_{2,1}(\ell) + \frac{\operatorname{csh}(\ell)}{2} + \frac{4C_2(\ell)}{\alpha_\ell^2},\\
C^{\mathcal{U}}_{2}(\ell) &:= C^{\mathcal{L}}_{2,1}(\ell) - \frac{\operatorname{csh}(\ell)}{2} + C_{2,2}(\ell).
\end{aligned}$$

**Lemma 3.26.** *Let $S_2(t,\ell)$ be as in Definition 3.14 and $C^{\mathcal{L}}_2(\ell), C^{\mathcal{U}}_2(\ell)$ as in Definition 3.25. Then for all $t \geq 1$,*

$$-\frac{C^{\mathcal{L}}_2(\ell)}{t} < \frac{S_2(t,\ell)}{\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha_\ell) + \frac{\sinh(\alpha_\ell)}{\alpha_\ell} < \frac{C^{\mathcal{U}}_2(\ell)}{t}. \tag{3.47}$$

**Definition 3.27.** *Let $C_k(\ell)$ be as in Lemma 2.7. Define*

$$\begin{aligned}
C_{3,1}(\ell) &:= \frac{3\alpha_\ell^2\cosh(\alpha_\ell) + 7\alpha_\ell\sinh(\alpha_\ell) + 2\cosh(\alpha_\ell) - 2}{8} + C_0(\ell),\\
C_{3,2}(\ell) &:= \frac{9\alpha_\ell^3\sinh(\alpha_\ell) + (\alpha_\ell^4 + 24\alpha_\ell^2)\cosh(\alpha_\ell) + 18\alpha_\ell\sinh(\alpha_\ell)}{24} + 2C_2(\ell) + C_1(\ell),\\
\operatorname{sch}(\ell) &:= \alpha_\ell^2\cosh(\alpha_\ell) + 2\alpha_\ell\sinh(\alpha_\ell),\\
C^{\mathcal{L}}_3(\ell) &:= C_{3,1}(\ell) + C_{3,2}(\ell) - \frac{\operatorname{sch}(\ell)}{2},\\
C^{\mathcal{U}}_3(\ell) &:= 3C_1(\ell) + \frac{\operatorname{sch}(\ell)}{2}.
\end{aligned}$$

**Lemma 3.28.** *Let $S_3(t,\ell)$ be as in Definition 3.16 and $C^{\mathcal{L}}_3(\ell), C^{\mathcal{U}}_3(\ell)$ as in Definition 3.27. Then for all $t \geq 2$,*

$$-\frac{C^{\mathcal{L}}_3(\ell)}{t} < \frac{S_3(t,\ell)}{\binom{-\frac{3}{2}}{t}} + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\alpha_\ell\sinh(\alpha_\ell) + 1 - \cosh(\alpha_\ell) < \frac{C^{\mathcal{U}}_3(\ell)}{t}. \tag{3.48}$$

**Definition 3.29.** *Let $C_k(\ell)$ be as in Lemma 2.7. Define*

$$\begin{aligned}
C_{4,1}(\ell) &:= \frac{\alpha_\ell^4}{72} + \frac{(\alpha_\ell^2+6)\cosh(\alpha_\ell) + 3\alpha_\ell\sinh(\alpha_\ell)}{16},\\
C^{\mathcal{L}}_4(\ell) &:= C_{4,1}(\ell) - \frac{\cosh(\alpha_\ell)}{4} + \frac{2C_0(\ell)}{3},\\
C^{\mathcal{U}}_4(\ell) &:= \frac{(\alpha_\ell^2+12)\cosh(\alpha_\ell) + 3\alpha_\ell\sinh(\alpha_\ell) + 12C_0(\ell)}{24}.
\end{aligned}$$

**Lemma 3.30.** *Let $S_4(t,\ell)$ be as in Definition 3.18 and $C^{\mathcal{L}}_4(\ell), C^{\mathcal{U}}_4(\ell)$ as in Definition 3.29. Then for $t \geq 1$,*

$$-\frac{C^{\mathcal{L}}_4(\ell)}{t^2} < \frac{S_4(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}} - \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{1}{2t}\cosh(\alpha_\ell) < \frac{C^{\mathcal{U}}_4(\ell)}{t^2}. \tag{3.49}$$

### 3.3. **Error bounds.**

**Lemma 3.31.** *For all* $k \in \mathbb{Z}_{\geq 1}$, $\ell \in \mathbb{Z}_{\geq 0}$, *and* $n \geq \ell + 1$,

$$\frac{(1+24\ell)^k}{(24n)^k} < \sum_{t=k}^{\infty} \frac{(1+24\ell)^t}{(24n)^t} \leq \frac{24(\ell+1)}{23} \frac{(1+24\ell)^k}{(24n)^k}. \tag{3.50}$$

*Proof.* Equation (3.50) follows from

$$\sum_{t=k}^{\infty} \frac{(1+24\ell)^t}{(24n)^t} = \frac{(1+24\ell)^k}{(24n)^k} \frac{24n}{24n-24\ell-1} \text{ and } 1 < \frac{24n}{24n-24\ell-1} \leq \frac{24(\ell+1)}{23} \text{ for all } n \geq \ell+1.$$

□

**Lemma 3.32.** *For all* $n, k, s \in \mathbb{Z}_{\geq 1}$, $\ell \in \mathbb{Z}_{\geq 0}$, *and* $n \geq \ell + 1$,

$$\frac{1}{(k+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^k}{(24n)^k} < \sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} < \frac{12(\ell+1)}{5(k+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^t}{(24n)^k}. \tag{3.51}$$

*Proof.* We observe that

$$\sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{1}{(24n)^t} = \sum_{t=k}^{\infty} \frac{\binom{2t+2}{t+1}}{4^t} \frac{t+1}{2t^s} \frac{(1+24\ell)^t}{(24n)^t}. \tag{3.52}$$

For all $t \geq 1$,

$$\frac{4^t}{2\sqrt{t}} \leq \binom{2t}{t} \leq \frac{4^t}{\sqrt{\pi t}}.$$

From (3.52) we obtain

$$\sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} \leq \sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{1}{(24n)^t} \leq \frac{4}{\sqrt{\pi}} \sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{2t^s} \frac{(1+24\ell)^t}{(24n)^t}. \tag{3.53}$$

For all $k \geq 1$,

$$\sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} \geq \sum_{t=k}^{\infty} \frac{\sqrt{t+1}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} > \frac{1}{(k+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^k}{(24n)^k} \tag{3.54}$$

and

$$\begin{aligned}
\sum_{t=k}^{\infty} \frac{(-1)^t \binom{-\frac{3}{2}}{t}}{t^s} \frac{(1+24\ell)^t}{(24n)^t} &< \frac{4}{\sqrt{\pi}} \sum_{t=k}^{\infty} \frac{1}{(t+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^t}{(24n)^t} \\
&\leq \frac{4}{\sqrt{\pi}(k+1)^{s-\frac{1}{2}}} \sum_{t=k}^{\infty} \frac{(1+24\ell)^t}{(24n)^t} \\
&< \frac{4 \cdot 24(\ell+1)}{23 \cdot \sqrt{\pi}} \frac{1}{(k+1)^{s-\frac{1}{2}}} \frac{(1+24\ell)^k}{(24n)^k} \text{ (by (3.50)).} \\
&< \frac{12}{5} \frac{(\ell+1)}{(k+1)^{s-\frac{1}{2}}} \frac{1}{(24n)^k}.
\end{aligned} \tag{3.55}$$

Equations (3.54) and (3.55) imply (3.51). □

**Lemma 3.33.** *For* $n \in \mathbb{Z}_{\geq 1}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, *and* $n \geq 4\ell + 1$,

$$0 < \sum_{t=k}^{\infty} \binom{-\frac{3}{2}}{t} \frac{(-1)^t (1+24\ell)^t}{(24n)^t} < 4\sqrt{2} \frac{\sqrt{k+1}(1+24\ell)^k}{(24n)^k}. \tag{3.56}$$

*Proof.* Setting $(n,s)\mapsto (\frac{24n}{24\ell+1},2)$ in (2.2), it follows that for all $n\geq 4\ell+1$,

$$0<\sum_{t=k}^{\infty}\binom{-\frac{3}{2}}{t}\frac{(-1)^t}{(24n)^t}<4\sqrt{2}\frac{\sqrt{k+1}(1+24\ell)^k}{(24n)^k}.$$

□

**Definition 3.34.** *Let $C_1^{\mathcal{L}}(\ell)$ and $C_1^{\mathcal{U}}(\ell)$ be as in Definition 3.23. Then for all $k\geq 1$ and $\ell\geq 0$, define*

$$L_1(k,\ell):=\left(\cosh(\alpha_\ell)-\frac{6\alpha_\ell\sinh(\alpha_\ell)(\ell+1)}{5\sqrt{k+1}}-\frac{12(\ell+1)}{5(k+1)^{3/2}}C_1^{\mathcal{L}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k}$$

*and*

$$U_1(k,\ell):=\left(\frac{24(\ell+1)\cosh(\alpha_\ell)}{23}-\frac{\alpha_\ell\sinh(\alpha_\ell)}{2\sqrt{k+1}}+\frac{12(\ell+1)}{5(k+1)^{3/2}}C_1^{\mathcal{U}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24n}}\right)^{2k}.$$

**Lemma 3.35.** *Let $L_1(k,\ell)$ and $U_1(k,\ell)$ be as in Definition 3.34. Let $g_{e,1}(t,\ell)$ be as in Definition 3.12. Then for all $k\in\mathbb{Z}_{\geq 1}$ and $n\geq 4\ell+1$,*

$$L_1(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k}<\sum_{t=k}^{\infty}g_{e,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}<U_1(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k}. \tag{3.57}$$

*Proof.* From (3.26) and (3.46), it follows that for $t\geq 1$,

$$\begin{aligned}\cosh(\alpha_\ell)-\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{2t}\alpha_\ell\sinh(\alpha_\ell)-\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}C_1^{\mathcal{L}}(\ell)&<\Big(\frac{24}{1+24\ell}\Big)^t g_{e,1}(t)=1+S_1(t,\ell)\\ &<\cosh(\alpha_\ell)-\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{2t}\alpha_\ell\sinh(\alpha_\ell)+\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}C_1^{\mathcal{U}}(\ell).\end{aligned} \tag{3.58}$$

Applying (3.50) and (3.51) with $s=1$ and 2, respectively, to (3.58), we obtain (3.57). □

**Definition 3.36.** *Let $C_2^{\mathcal{L}}(\ell)$ and $C_2^{\mathcal{U}}(\ell)$ be as in Definition 3.25. For all $k\geq 1$ and $\ell\geq 0$, define*

$$L_2(k,\ell):=\left(-\frac{24(\ell+1)\cosh(\alpha_\ell)}{23}-\frac{12(\ell+1)}{5\sqrt{k+1}}C_2^{\mathcal{U}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2k}$$

*and*

$$U_2(k,\ell):=\left(-\cosh(\alpha_\ell)+\frac{4\sqrt{2}\sinh(\alpha_\ell)}{\alpha_\ell}\sqrt{k+1}+\frac{12(\ell+1)}{5\sqrt{k+1}}C_2^{\mathcal{L}}(\ell)\right)\left(\sqrt{\frac{1+24\ell}{24}}\right)^{2k}.$$

**Lemma 3.37.** *Let $L_2(k,\ell)$ and $U_2(k,\ell)$ be as in Definition 3.36. Let $g_{e,2}(t,\ell)$ be as in Definition 3.14. Then for all $k\in\mathbb{Z}_{\geq 1}$ and $n\geq 4\ell+1$,*

$$L_2(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k}<\sum_{t=k}^{\infty}g_{e,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t}<U_2(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k}. \tag{3.59}$$

*Proof.* From (3.31) and (3.47), it follows that for $t \geq 1$,

$$-\cosh(\alpha_\ell) + (-1)^t \binom{-\frac{3}{2}}{t} \frac{\sinh(\alpha_\ell)}{\alpha_\ell} - \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t} C_2^{\mathcal{U}}(\ell) < \Big(\frac{1+24\ell}{24}\Big)^t g_{e,2}(t,\ell) = (-1)^{t-1} S_2(t,\ell)$$
$$< -\cosh(\alpha_\ell) + (-1)^t \binom{-\frac{3}{2}}{t} \frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t} C_2^{\mathcal{L}}(\ell). \tag{3.60}$$

Applying (3.50), (3.51) with $s = 1$ and (3.56) to (3.60), we get (3.59). □

**Definition 3.38.** *Let $C_3^{\mathcal{L}}(\ell)$ and $C_3^{\mathcal{U}}(\ell)$ be as in Definition 3.27. For all $k \geq 1$ and $\ell \geq 0$, define*

$$L_3(k,\ell) := -\Bigg(-\frac{6\alpha_\ell \sinh(\alpha_\ell)}{\pi\sqrt{1+24\ell}} + \frac{24\sqrt{2}\cosh(\alpha_\ell)\sqrt{k+1}}{\pi\sqrt{1+24\ell}} + \frac{72(\ell+1)}{5\pi\sqrt{1+24\ell}}\frac{C_3^{\mathcal{U}}(\ell)}{\sqrt{k+1}}\Bigg)\Bigg(\sqrt{\frac{1+24\ell}{24}}\Bigg)^{2k+1}$$

*and*

$$U_3(k,\ell) := \Bigg(\frac{6\cdot 24(\ell+1)}{23\pi\sqrt{1+24\ell}}\alpha_\ell \sinh(\alpha_\ell) + \frac{72(\ell+1)}{5\pi\sqrt{1+24\ell}}\frac{C_3^{\mathcal{L}}(\ell)}{\sqrt{k+1}}\Bigg)\Bigg(\sqrt{\frac{1+24\ell}{24}}\Bigg)^{2k+1}.$$

**Lemma 3.39.** *Let $L_3(k,\ell)$ and $U_3(k,\ell)$ be as in Definition 3.38. Let $g_{o,1}(t,\ell)$ be as in Definition 3.16. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell+1$,*

$$L_3(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1} < \sum_{t=k}^{\infty} g_{o,1}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} < U_3(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1}. \tag{3.61}$$

*Proof.* Define $c_1(t,\ell) := -\dfrac{6}{\pi\sqrt{1+24\ell}}(-1)^t\binom{-\frac{3}{2}}{t}$. From (3.35) and (3.48), it follows that for $t \geq 2$,

$$\frac{6\alpha_\ell \sinh(\alpha_\ell)}{\pi\sqrt{1+24\ell}} - \frac{6\cosh(\alpha_\ell)}{\pi\sqrt{1+24\ell}}(-1)^t\binom{-\frac{3}{2}}{t} - \frac{6C_3^{\mathcal{U}}(\ell)}{\pi\sqrt{1+24\ell}}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}$$
$$< \Bigg(\sqrt{\frac{24}{24\ell+1}}\Bigg)^{2t+1} g_{o,1}(t,\ell) = c_1(t,\ell)\Bigg(1 + \frac{S_3(t,\ell)}{\binom{-\frac{3}{2}}{t}}\Bigg) \tag{3.62}$$
$$< \frac{6\alpha_\ell \sinh(\alpha_\ell)}{\pi\sqrt{1+24\ell}} - \frac{6\cosh(\alpha_\ell)}{\pi\sqrt{1+24\ell}}(-1)^t\binom{-\frac{3}{2}}{t} + \frac{6C_3^{\mathcal{L}}(\ell)}{\pi\sqrt{1+24\ell}}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}.$$

We observe that (3.62) also holds for $t \in \{0,1\}$; see (3.35). Now, applying (3.50), (3.51) with $s = 1$, and (3.56) to (3.62), we conclude the proof. □

**Definition 3.40.** *Let $C_4^{\mathcal{L}}(\ell)$ and $C_4^{\mathcal{U}}(\ell)$ be as in Definition 3.29. For all $k \geq 1$ and $\ell \geq 0$, define*

$$L_4(k,\ell) := -\frac{\pi\sqrt{1+24\ell}}{6}\Bigg(-\frac{\cosh(\alpha_\ell)}{2\sqrt{k+1}} + \frac{24(\ell+1)\sinh(\alpha_\ell)}{23\alpha_\ell} + \frac{12(\ell+1)C_4^{\mathcal{U}}(\ell)}{5(k+1)^{3/2}}\Bigg)\Bigg(\sqrt{\frac{1+24\ell}{24}}\Bigg)^{2k+1}$$

*and*

$$U_4(k,\ell) := \frac{\pi\sqrt{1+24\ell}}{6}\Bigg(\frac{6(\ell+1)\cosh(\alpha_\ell)}{5\sqrt{k+1}} - \frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{12(\ell+1)C_4^{\mathcal{L}}(\ell)}{5(k+1)^{3/2}}\Bigg)\Bigg(\sqrt{\frac{1+24\ell}{24}}\Bigg)^{2k+1}.$$

**Lemma 3.41.** *Let $L_4(k,\ell)$ and $U_4(k,\ell)$ be as in Definition 3.40. Let $g_{o,2}(t,\ell)$ be as in Definition 3.18. Then for all $k \in \mathbb{Z}_{\geq 1}$ and $n \geq 4\ell+1$,*

$$L_4(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1} < \sum_{t=k}^{\infty} g_{o,2}(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} < U_4(k,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2k+1}. \tag{3.63}$$

*Proof.* Define $c_2(t,\ell) := -\frac{\pi\sqrt{1+24\ell}}{6}(-1)^t\binom{-\frac{3}{2}}{t}$. From (3.39) and (3.49), it follows that for $t\geq 1$,

$$\begin{aligned}&\frac{\pi\sqrt{1+24\ell}\cosh(\alpha_\ell)}{12}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}-\frac{\pi\sqrt{1+24\ell}\sinh(\alpha_\ell)}{6\alpha_\ell}-\frac{\pi\sqrt{1+24\ell}\ C_4^{\mathcal{U}}(\ell)}{6}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}\\&<\left(\sqrt{\frac{24}{24\ell+1}}\right)^{2t+1}g_{o,2}(t,\ell)=c_2(t,\ell)\frac{S_4(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}}\\&<\frac{\pi\sqrt{1+24\ell}\cosh(\alpha_\ell)}{12}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t}-\frac{\pi\sqrt{1+24\ell}\sinh(\alpha_\ell)}{6\alpha_\ell}+\frac{\pi\sqrt{1+24\ell}\ C_4^{\mathcal{L}}(\ell)}{6}\frac{(-1)^t\binom{-\frac{3}{2}}{t}}{t^2}.\end{aligned}\tag{3.64}$$

Now, applying (3.50) and (3.51) with $s=1$ and 2, respectively, to (3.64), we have (3.63). □

**Definition 3.42.** *For $k\geq 1$ and $\ell\geq 0$, define*

$$n_0(k,\ell) := \max_{k\geq 1,\ell\geq 0}\left\{\frac{(24\ell+1)^2}{16},\frac{(k+3)(24\ell+1)}{24}\right\}.$$

**Definition 3.43.** *Let $n_0(k,\ell)$ be as in Definition 3.42. For $k\geq 1$ and $\ell\geq 0$, define*

$$\widehat{L}_2(k,\ell) := \frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{4\sqrt{n_0(k,l)}}\right)\ \text{and}\ \ \widehat{U}_2(k,\ell) := \frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1+\frac{k(1+24\ell)}{3\cdot n_0(k,l)}\right).$$

**Lemma 3.44.** *Let $\widehat{L}_2(k,\ell)$, and $\widehat{U}_2(k,\ell)$ be as in Definition 3.43. Let $n_0(k,\ell)$ be as in Definition 3.42. Then for all $k\in\mathbb{Z}_{\geq 1}$ and $n>n_0(k,\ell)$,*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{L}_2(k,\ell)}{\sqrt{n}^k}<\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\frac{1}{\mu(n-\ell)^k}<\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{U}_2(k,\ell)}{\sqrt{n}^k}.\tag{3.65}$$

*Proof.* For all $k\geq 1$ and $\ell\geq 0$, define

$$\mathcal{E}(n,k,\ell) := \frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\frac{1}{\mu(n-\ell)^k},\ \mathcal{U}(n,k,\ell)=\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{1}{\sqrt{n}^k}$$

and

$$\mathcal{Q}(n,k,\ell) := \frac{\mathcal{E}(n,k,\ell)}{\mathcal{U}(n,k,\ell)}=\frac{e^{\pi\sqrt{\frac{2n}{3}}\left(\sqrt{1-\frac{1+24\ell}{24n}}-1\right)}}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}.$$

Using (2.3) with $(m,n,s)\mapsto(1,24n,24\ell+1)$, we obtain for all $n\geq 2\ell+1$,

$$-\frac{1+24\ell}{12n}<\sqrt{1-\frac{1}{24n}}-1=\sum_{m=1}^{\infty}\binom{1/2}{m}\frac{(-1)^m}{(24n)^m}<0,$$

and consequently for $n\geq n_0(k,\ell)$,

$$\left(1-\frac{1+24\ell}{4\sqrt{n_0(k,\ell)}}\right)<e^{-\frac{\pi(1+24\ell)}{6\sqrt{6n}}}<e^{\pi\sqrt{\frac{2n}{3}}\left(\sqrt{1-\frac{1}{24n}}-1\right)}<1.\tag{3.66}$$

Therefore

$$\frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}\left(1-\frac{1}{4\sqrt{n_0(k,\ell)}}\right)<\mathcal{Q}(n,k,\ell)<\frac{1}{\left(\alpha_0\sqrt{24}\right)^k}\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}.\tag{3.67}$$

We estimate $\left(1-\dfrac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}$ by splitting it into two cases depending on whether $k$ is even or odd.
For $k=2r$ with $r\in\mathbb{Z}_{\geq 0}$:

$$\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}=\left(1-\frac{1+24\ell}{24n}\right)^{-(r+1)}=1+\sum_{j=1}^{\infty}\binom{-(r+1)}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j}.$$

From (2.4) with $(m,s,n)\mapsto(1,r+1,\frac{24n}{24\ell+1})$, for all $n>\frac{(r+1)(1+24\ell)}{12}$, we get

$$0<\sum_{j=1}^{\infty}\binom{-(r+1)}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j}<\frac{(r+1)(24\ell+1)}{12n},$$

which is equivalent to

$$1<\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}<1+\frac{(k+2)(24\ell+1)}{24n}\quad\text{for all}\quad n>n_0(k,\ell). \tag{3.68}$$

For $k=2r+1$ with $r\in\mathbb{Z}_{\geq 0}$:

$$\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}=\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{2r+3}{2}}=1+\sum_{j=1}^{\infty}\binom{-\frac{2r+3}{2}}{j}\frac{(-1)^j(1+24\ell)^j}{(24n)^j}.$$

Using (2.2) with $(m,s,n)\mapsto(1,r+2,\dfrac{24n}{24\ell+1})$, for all $n>\frac{(r+2)(1+24\ell)}{12}$, we get

$$0<\sum_{j=1}^{\infty}\binom{-\frac{2\ell+3}{2}}{j}\frac{(-1)^j}{(24n)^j}<\frac{(r+2)(1+24\ell)}{6n}$$

which is equivalent to

$$1<\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}<1+\frac{k(1+24\ell)}{3n}\quad\text{for all}\quad n>n_0(k,\ell). \tag{3.69}$$

From (3.68) and (3.69), for all $n>n_0(k,\ell)$ it follows that

$$1<\left(1-\frac{1+24\ell}{24n}\right)^{-\frac{k+2}{2}}<1+\frac{k(1+24\ell)}{3\cdot n_0(k,\ell)}. \tag{3.70}$$

From (3.67) and (3.70), we conclude the proof.

□

## 4. Inequalities for $p(n-\ell)$

**Definition 4.1.** *Let $(L_i(k,\ell))_{1\leq i\leq 4}$ and $(U_i(k,\ell))_{1\leq i\leq 4}$ be as in Definitions 3.34-3.40. Let $\widehat{U}_2(k,\ell)$ be as in Definition 3.43. Then for all $w\in\mathbb{Z}_{\geq 1}$ with $\lceil w/2\rceil\geq 1$, define*

$$L(w,\ell):=L_1\Big(\Big\lceil\frac{w}{2}\Big\rceil,\ell\Big)+L_2\Big(\Big\lceil\frac{w}{2}\Big\rceil,\ell\Big)+L_3\Big(\Big\lfloor\frac{w}{2}\Big\rfloor,\ell\Big)+L_4\Big(\Big\lfloor\frac{w}{2}\Big\rfloor,\ell\Big)-\widehat{U}_2(w,\ell)$$

*and*

$$U(w,\ell):=U_1\Big(\Big\lceil\frac{w}{2}\Big\rceil,\ell\Big)+U_2\Big(\Big\lceil\frac{w}{2}\Big\rceil,\ell\Big)+U_3\Big(\Big\lfloor\frac{w}{2}\Big\rfloor,\ell\Big)+U_4\Big(\Big\lfloor\frac{w}{2}\Big\rfloor,\ell\Big)+\widehat{U}_2(w,\ell).$$

**Lemma 4.2.** *Let $\widehat{g}(k)$ be as in Theorem 3.2 and $n_0(k,\ell)$ as in Definition 3.42. Let $g(t,\ell)$ be as in (3.45). Let $L(w,\ell)$ and $U(w,\ell)$ be as in Definition 4.1. If $m\in\mathbb{Z}_{\geq 1}$ and $n>\max\{1,n_0(2m,\ell),\widehat{g}(2m)+\ell\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1}\frac{g(t,\ell)}{\sqrt{n}^{t}}+\frac{L(2m,\ell)}{\sqrt{n}^{2m}}\right)<p(n-\ell)<\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m-1}\frac{g(t,\ell)}{\sqrt{n}^{t}}+\frac{U(2m,\ell)}{\sqrt{n}^{2m}}\right).$$

*Proof.* Following Definition 3.20 and from Lemma 3.21, we have

$$\begin{aligned}\sum_{t=0}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t &= \sum_{t=0}^{2m-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=2m}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t \\ &= \sum_{t=0}^{2m-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=m}^{\infty} g(2t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t} + \sum_{t=m}^{\infty} g(2t+1,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1} \\ &= \sum_{t=0}^{2m-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \sum_{t=m}^{\infty}(g_{e,1}(t,\ell)+g_{e,2}(t,\ell))\Big(\frac{1}{\sqrt{n}}\Big)^{2t} \\ &\qquad + \sum_{t=m}^{\infty}(g_{o,1}(t,\ell)+g_{o,2}(t,\ell))\Big(\frac{1}{\sqrt{n}}\Big)^{2t+1}.\end{aligned} \tag{4.1}$$

Using Lemmas 3.35-3.41 by making the substitution $k \mapsto m$, it follows that

$$\begin{aligned}\frac{L_1(m,\ell)+L_2(m,\ell)}{\sqrt{n}^{2m}} + \frac{L_3(m,\ell)+L_4(m,\ell)}{\sqrt{n}^{2m+1}} &< \sum_{t=2m}^{\infty} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t \\ &< \frac{U_1(m,\ell)+U_2(m,\ell)}{\sqrt{n}^{2m}} + \frac{U_3(m,\ell)+U_4(m,\ell)}{\sqrt{n}^{2m+1}}.\end{aligned} \tag{4.2}$$

Moreover, by Lemma 3.44 with $k = 2m$, it follows that

$$\frac{\sqrt{12}\ e^{\mu(n-\ell)}}{24(n-\ell)-1}\frac{1}{\mu(n-\ell)^{2m}} < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\frac{\widehat{U}_2(2m,\ell)}{\sqrt{n}^{2m}}. \tag{4.3}$$

Combining (4.2) and (4.3), and applying to Theorem 3.2, we conclude the proof. □

**Lemma 4.3.** *Let $\widehat{g}(k)$ be as in Theorem 3.2 and $n_0(k,\ell)$ as in Definition 3.42. Let $g(t,\ell)$ be as in Equation* (3.45)*. Let $L(w,\ell)$ and $U(w,\ell)$ be as in Definition 4.1. If $m \in \mathbb{Z}_{\geq 0}$ and $n > \max\{1, n_0(2m+1,\ell), \widehat{g}(2m+1)+\ell\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m}\frac{g(t,\ell)}{\sqrt{n}^t} + \frac{L(2m+1,\ell)}{\sqrt{n}^{2m+1}}\right) < p(n-\ell) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\left(\sum_{t=0}^{2m}\frac{g(t,\ell)}{\sqrt{n}^t} + \frac{U(2m+1,\ell)}{\sqrt{n}^{2m+1}}\right).$$

*Proof.* The proof is analogous to the proof of Lemma 4.2. □

**Definition 4.4.** *Let $g(t,\ell)$ be as in* (3.45)*, $L(w,\ell), U(w,\ell)$ as in Definition 4.1. If $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$, define*

$$\mathcal{L}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{L(w,\ell)}{\sqrt{n}^w} \quad \text{and} \quad \mathcal{U}_n(w,\ell) := \sum_{t=0}^{w-1} g(t,\ell)\Big(\frac{1}{\sqrt{n}}\Big)^t + \frac{U(w,\ell)}{\sqrt{n}^w}.$$

**Theorem 4.5.** *Let $\widehat{g}(k)$ be as in Theorem 3.2 and $n_0(k,\ell)$ as in Definition 3.42. Let $\mathcal{L}_n(w,\ell)$ and $\mathcal{U}_n(w,\ell)$ be as in Definition 4.4. If $w \in \mathbb{Z}_{\geq 1}$ with $\lceil w/2 \rceil \geq 1$ and $n > \max\{\widehat{g}(w)+\ell, n_0(w,\ell)\}$, then*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{L}_n(w,\ell) < p(n-\ell) < \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\mathcal{L}_n(w,\ell). \tag{4.4}$$

*Proof.* Putting Lemmas 4.2 and 4.3 together, we obtain (4.4). □

## 5. Proof of Theorems 1.5, 1.7, and 1.9

*Proof of Theorem 1.5:* To prove the lower bound of (1.16), it is equivalent to show that

$$p(n-4)p(n)+3p(n-2)^2 > 4\left(1+\frac{\pi^2}{16(n-3)^3}-\frac{6}{(n-3)^{7/2}}\right)p(n-3)p(n-1). \tag{5.1}$$

Since $1+\frac{\pi^2}{16n^3}-\frac{5}{n^{7/2}} > 1+\frac{\pi^2}{16(n-3)^3}-\frac{6}{(n-3)^{7/2}}$ for all $n \geq 5$, it is enough to show that

$$p(n-4)p(n)+3p(n-2)^2 > 4\left(1+\frac{\pi^2}{16n^3}-\frac{5}{n^{7/2}}\right)p(n-3)p(n-1). \tag{5.2}$$

Choosing $w=12$ and applying Theorem 4.5, for all $n \geq 2329$, we have

$$p(n-4)p(n)+3p(n-2)^2 > \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{L}_n(12,4)\cdot\mathcal{L}_n(12,0)+3\,\mathcal{L}_n^2(12,2)\right), \tag{5.3}$$

and

$$p(n-3)p(n-1) < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{U}_n(12,3)\cdot\mathcal{U}_n(12,1)\right). \tag{5.4}$$

Therefore, it suffices to show that

$$\mathcal{L}_n(12,4)\cdot\mathcal{L}_n(12,0)+3\,\mathcal{L}_n^2(12,2) > 4\left(1+\frac{\pi^2}{16n^3}-\frac{5}{n^{7/2}}\right)\mathcal{U}_n(12,3)\cdot\mathcal{U}_n(12,1). \tag{5.5}$$

Using the `Reduce`[1] command within Mathematica, it can be easily checked that for all $n \geq 625$, (5.5) holds.

Similarly, to prove the upper bound of (1.16), it is equivalent to prove that

$$p(n-4)p(n)+3p(n-2)^2 < 4\left(1+\frac{\pi^2}{16(n-3)^3}\right)p(n-3)p(n-1). \tag{5.6}$$

Since $1+\frac{\pi^2}{16n^3} < 1+\frac{\pi^2}{16(n-3)^3}$ for all $n \geq 4$, it is enough to show that

$$p(n-4)p(n)+3p(n-2)^2 < 4\left(1+\frac{\pi^2}{16n^3}\right)p(n-3)p(n-1). \tag{5.7}$$

Choosing $w=12$ and applying Theorem 4.5, for all $n \geq 2329$, we have

$$p(n-4)p(n)+3p(n-2)^2 < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{U}_n(12,4)\cdot\mathcal{U}_n(12,0)+3\,\mathcal{U}_n^2(12,2)\right), \tag{5.8}$$

and

$$p(n-3)p(n-1) > \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^2\left(\mathcal{L}_n(12,3)\cdot\mathcal{L}_n(12,1)\right). \tag{5.9}$$

Therefore, it suffices to show that

$$\mathcal{U}_n(12,4)\cdot\mathcal{U}_n(12,0)+3\,\mathcal{U}_n^2(12,2) < 4\left(1+\frac{\pi^2}{16n^3}\right)\mathcal{L}_n(12,3)\cdot\mathcal{L}_n(12,1). \tag{5.10}$$

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[1]`Reduce` uses cylindrical algebraic decomposition for polynomials over real domains which is based on Collin's algorithm [10]. Cylindrical Algebraic Decomposition (CAD) is an algorithm which proves that a given polynomial in several variables is positive (non-negative).

In a similar way as stated before, it can be easily checked that for all $n \geq 784$, (5.5) holds. We conclude the proof of Theorem 1.5 by verifying the inequality (1.16) for all $218 \leq n \leq 2328$ with Mathematica. $\square$

*Proof of Theorem 1.7:* To prove the lower bound of (1.17), it is equivalent to show that

$$p(n-2)^3 + p(n-4)p(n-1)^2 + p(n-3)^2 p(n) > \left(1 + \frac{\pi^3}{72\sqrt{6}(n-3)^{9/2}} - \frac{8}{(n-3)^5}\right)\big(2p(n-3)p(n-2)p(n-1) + p(n-4)p(n-2)p(n)\big). \tag{5.11}$$

As $1 + \frac{\pi^3}{72\sqrt{6}n^{9/2}} - \frac{7}{n^5} > 1 + \frac{\pi^3}{72\sqrt{6}(n-3)^{9/2}} - \frac{8}{(n-3)^5}$ for all $n \geq 4$, it suffices to show that

$$p(n-2)^3 + p(n-4)p(n-1)^2 + p(n-3)^2 p(n) > \left(1 + \frac{\pi^3}{72\sqrt{6}n^{9/2}} - \frac{7}{n^5}\right)\big(2p(n-3)p(n-2)p(n-1) + p(n-4)p(n-2)p(n)\big). \tag{5.12}$$

Choosing $w = 15$ and applying Theorem 4.5, for all $n \geq 4047$, we have

$$p(n-2)^3 + p(n-4)p(n-1)^2 + p(n-3)^2 p(n) > \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^3 \Big(\mathcal{L}_n^3(15,2) + \mathcal{L}_n(15,4)\cdot\mathcal{L}_n^2(15,1) + \mathcal{L}_n^2(15,3)\cdot\mathcal{L}_n(15,0)\Big), \tag{5.13}$$

and

$$2p(n-3)p(n-2)p(n-1) + p(n-4)p(n-2)p(n) < \left(\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\right)^3 \Big(2\cdot\mathcal{U}_n(15,3)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,1) + \mathcal{U}_n(15,4)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,0)\Big). \tag{5.14}$$

Similar to the proof of (5.5), it can be easily checked that for all $n \geq 1444$,

$$\mathcal{L}_n^3(15,2) + \mathcal{L}_n(15,4)\cdot\mathcal{L}_n^2(15,1) + \mathcal{L}_n^2(15,3)\cdot\mathcal{L}_n(15,0) > \left(1 + \frac{\pi^3}{72\sqrt{6}n^{9/2}} - \frac{7}{n^5}\right)\Big(2\cdot\mathcal{U}_n(15,3)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,1) + \mathcal{U}_n(15,4)\cdot\mathcal{U}_n(15,2)\cdot\mathcal{U}_n(15,0)\Big) \tag{5.15}$$

Analogously, one can prove that for all $n \geq 2916$,

$$\mathcal{U}_n^3(15,2) + \mathcal{U}_n(15,4)\cdot\mathcal{U}_n^2(15,1) + \mathcal{U}_n^2(15,3)\cdot\mathcal{U}_n(15,0) < \left(1 + \frac{\pi^3}{72\sqrt{6}n^{9/2}}\right)\Big(2\cdot\mathcal{L}_n(15,3)\cdot\mathcal{L}_n(15,2)\cdot\mathcal{L}_n(15,1) + \mathcal{L}_n(15,4)\cdot\mathcal{L}_n(15,2)\cdot\mathcal{L}_n(15,0)\Big), \tag{5.16}$$

which is sufficient to prove the upper bound of (1.17). We conclude the proof of Theorem 1.7 by verifying the inequality (1.17) for all $244 \leq n \leq 4047$ with Mathematica. $\square$

*Proof of Theorem 1.9:* Corresponding to (1.18), we show

$$\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right)\Big(p(n-2)p(n-1) - p(n-3)p(n)\Big)^2 > 4\Big(p(n-2)^2 - p(n-3)p(n-1)\Big)\Big(p(n-1)^2 - p(n-2)p(n)\Big), \tag{5.17}$$

and

$$4\Big(p(n-2)^2-p(n-3)p(n-1)\Big)\Big(p(n-1)^2-p(n-2)p(n)\Big) > \left(1+\frac{\pi}{\sqrt{24}n^{3/2}}-\frac{2}{n^2}\right)\Big(p(n-2)p(n-1)-p(n-3)p(n)\Big)^2. \tag{5.18}$$

Applying Theorem 4.5 with $w=13$, and following the similar method worked out in the proof of Theorem 1.5, we obtain (1.18) for all $n \geq 2842$. For $115 \leq n \leq 2841$, we verified (1.18) numerically with Mathematica. ☐

## 6. Appendix

In the proofs of Lemmas 3.24-3.30, we follow the same notations and the proof strategy as in [3, Subsection 5.2].

*Proof of Lemma 3.24:* Following Definition 3.12, write $S_1(t,\ell)$ as follows:

$$\begin{aligned} S_1(t,\ell) &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\sum_{s=u}^{t}\frac{(-1)^s}{s}\Big(\frac{1}{2}-s\Big)_{s+1}\frac{(-s)_u}{(s+u)!} \\ &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\underbrace{\sum_{s=0}^{t-u}\frac{(-1)^{s+u}}{s+u}\Big(\frac{1}{2}-s-u\Big)_{s+u+1}\frac{(-s-u)_u}{(s+2u)!}}_{=:S_1(t,u)}. \end{aligned}$$

From [3, Eqn. (5.6)], we have

$$S_1(t,u) = (-1)^t\binom{-\frac{3}{2}}{t}\frac{(-1)^u}{2u}A_1(t,u), \tag{6.1}$$

where

$$A_1(t,u) = \frac{t(-t)_u(-1)^u}{(1+2t)(t+u)(t)_u} - \left(\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}+\frac{1}{(1+2t)}+\frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i}\right).$$

Now by Lemmas 2.3 and 2.4,

$$\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}-\frac{u}{t}-\frac{u^2}{2t^2} \leq A_1(t,u) \leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}+\frac{1}{4t^2}+u\Big(\frac{2}{3t^2}-\frac{1}{t}\Big)+\frac{u^2}{2t^2}+\frac{u^3}{3t^2}. \tag{6.2}$$

Equations (6.1) and (6.1), it follows that

$$S_1(t,\ell) = (-1)^t\binom{-\frac{3}{2}}{t}\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}A_1(t,u)}{(2u)!}. \tag{6.3}$$

Applying (6.2) to (6.3), we get the following lower bound of $S_1(t,\ell)$,

$$\begin{aligned} \frac{S_1(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}} &\geq \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}\right)\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u)!}-\frac{1}{t}\sum_{u=1}^{t}\frac{u\alpha_\ell^{2u}}{(2u)!}-\frac{1}{2t^2}\sum_{u=1}^{t}\frac{u^2\alpha_\ell^{2u}}{(2u)!} \\ &\geq \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}\right)\left(\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}-1-\sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}\right)-\frac{1}{t}\sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u)!}-\frac{1}{2t^2}\sum_{u=0}^{\infty}\frac{u^2\alpha_\ell^{2u}}{(2u)!}. \\ &> \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{1}{4t^2}\right)\left(\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}-1-\frac{C_0(\ell)}{t^2}\right)-\frac{1}{t}\sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u)!}-\frac{1}{2t^2}\sum_{u=0}^{\infty}\frac{u^2\alpha_\ell^{2u}}{(2u)!} \end{aligned}$$

$$\left(\text{by Lemma 2.7 and } \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} > \frac{1}{4t^2} \text{ for all } t\geq 1\right)$$

$$> \left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2}\right)\Bigl(\cosh(\alpha_\ell)-1\Bigr) - \frac{C_0(\ell)}{t^2} - \frac{\alpha_\ell\sinh(\alpha_\ell)}{2t}$$

$$-\frac{1}{2t^2}\Bigl(\frac{\alpha_\ell^2}{4}\cosh(\alpha_\ell)+\frac{\alpha_\ell}{4}\sinh(\alpha_\ell)\Bigr)$$

$$\left(\text{by Lemma 2.5 and } \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{4t^2} < 1 \text{ for all } t\geq 1\right)$$

$$= \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha_\ell)-1) - \frac{\alpha_\ell\sinh(\alpha_\ell)}{2t} - \frac{C_1^{\mathcal{L}}(\ell)}{2t^2} \quad \Bigl(\text{by Definition 3.23}\Bigr). \tag{6.4}$$

For the upper bound estimation, we have for all $t\geq 1$,

$$\frac{S_1(t,\ell)}{(-1)^t\binom{-\frac{3}{2}}{t}}$$

$$\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{t}\sum_{u=1}^{t}\frac{u\alpha_\ell^{2u}}{(2u)!} + \frac{1}{4t^2}\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u)!} + \frac{2}{3t^2}\sum_{u=1}^{t}\frac{u\alpha_\ell^{2u}}{(2u)!} + \frac{1}{2t^2}\sum_{u=1}^{t}\frac{u^2\alpha_\ell^{2u}}{(2u)!} + \frac{1}{3t^2}\sum_{u=1}^{t}\frac{u^3\alpha_\ell^{2u}}{(2u)!}$$

$$\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha_\ell)-1) - \frac{1}{2t}\alpha_\ell\sinh(\alpha_\ell) + \frac{C_1(\ell)}{t^3} + \frac{1}{4t^2}\cosh(\alpha_\ell) + \frac{1}{3t^2}\alpha_\ell\sinh(\alpha_\ell)$$

$$+\frac{1}{2t^2}\left(\frac{\alpha_\ell^2}{4}\cosh(\alpha_\ell)+\frac{\alpha_\ell}{4}\sinh(\alpha_\ell)\right) + \frac{1}{3t^2}\left(\frac{3\alpha_\ell^2}{8}\cosh(\alpha_\ell)+\frac{\alpha_\ell(\alpha_\ell^2+1)}{8}\sinh(\alpha_\ell)\right)$$

$$\Bigl(\text{by Lemmas 2.5 and } 2.7\Bigr)$$

$$\leq \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}(\cosh(\alpha_\ell)-1) - \frac{1}{2t}\alpha_\ell\sinh(\alpha_\ell) + \frac{C_1^{\mathcal{U}}(\ell)}{t^2} \quad \Bigl(\text{by Definition 3.23}\Bigr). \tag{6.5}$$

Combining (6.4) and (6.5), we arrive at (3.46) which concludes the proof. ☐

*Proof of Lemma 3.26:* Following Definition 3.14, write $S_2(t,\ell)$ as follows:

$$\begin{aligned} S_2(t,\ell) &= \sum_{u=0}^{t-1}\frac{(-1)^u\alpha_\ell^{2u}}{(2u)!}\sum_{s=u}^{t-1}\Bigl(\frac{1}{2}-s\Bigr)_{s+1}\binom{-\frac{3}{2}}{t-s-1}\frac{(-s)_u}{(s+u+1)!} \\ &= \sum_{u=0}^{t-1}\frac{(-1)^u\alpha_\ell^{2u}}{(2u)!}\underbrace{\sum_{s=0}^{t-u-1}\Bigl(\frac{1}{2}-s-u\Bigr)_{s+u+1}\binom{-\frac{3}{2}}{t-s-u-1}\frac{(-s-u)_u}{(s+2u+1)!}}_{=:S_2(t,u)}. \end{aligned} \tag{6.6}$$

From [3, Eqn. (5.13)], we have

$$S_2(t,u) = \binom{-\frac{3}{2}}{t}(-1)^{u+1}\Bigl(A_{2,1}(t,u)+A_{2,2}(t,u)\Bigr), \tag{6.7}$$

where

$$A_{2,1}(t,u) = \frac{2t(t-u)(-t)_u(-1)^u}{(1+2t)(1+2u)(t+u)(t)_u}$$

and

$$A_{2,2}(t,u) = \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} + \frac{1}{1+2t} + \frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-1)^i(-t)_i}{(t+i)(t)_i}.$$

Combining (6.6) and (6.7), we get

$$S_2(t,\ell) = -\binom{-\frac{3}{2}}{t}\Big(s_{2,1}(t,\ell)+s_{2,2}(t,\ell)\Big), \tag{6.8}$$

where

$$s_{2,1}(t,\ell) = \sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u)!}A_{2,1}(t,u) \ \text{ and } \ s_{2,2}(t,\ell) = \sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u)!}A_{2,2}(t,u). \tag{6.9}$$

By Lemma 2.3, we have

$$\frac{1}{1+2u} - \frac{u^2+u+\frac{1}{2}}{t(1+2u)} \le A_{2,1}(t,u) \le \frac{t-u}{t(1+2u)}. \tag{6.10}$$

Applying (6.10) into (6.9) we obtain

$$\sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \frac{1}{t}\sum_{u=0}^{t-1}\frac{u^2+u+\frac{1}{2}}{(2u+1)!}\alpha_\ell^{2u} \le s_{2,1}(t) \le \sum_{u=0}^{t-1}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \frac{1}{t}\sum_{u=0}^{t-1}\frac{u\alpha_\ell^{2u}}{(2u+1)!},$$

and consequently,

$$\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \frac{1}{t}\sum_{u=0}^{\infty}\frac{u^2+u+\frac{1}{2}}{(2u+1)!}\alpha_\ell^{2u} \le s_{2,1}(t,\ell) \le \sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!} - \frac{1}{t}\left(\sum_{u=0}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u+1)!} - \sum_{u=t}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u+1)!}\right). \tag{6.11}$$

By Lemma 2.7, it follows that

$$\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!} \le \frac{2C_1(\ell)}{\alpha_\ell^2\ t^2} \ \text{ and } \ \sum_{u=t}^{\infty}\frac{u\alpha_\ell^{2u}}{(2u+1)!} \le \frac{2C_2(\ell)}{\alpha_\ell^2 t^2}. \tag{6.12}$$

Applying (6.12) into (6.11) and by Lemma 2.5, we obtain

$$\frac{\sinh(\alpha_\ell)}{\alpha_\ell} - \frac{C_{2,1}^{\mathcal{L}}(\ell)}{t} \le s_{2,1}(t,\ell) \le \frac{\sinh(\alpha_\ell)}{\alpha_\ell} + \frac{C_{2,1}^{\mathcal{U}}(\ell)}{t}. \tag{6.13}$$

Next we apply Lemma 2.4 and get

$$\frac{2u+1}{2t} - \frac{4u^3+6u^2+8u+3}{12t^2} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}} \le A_{2,2}(t,u) \le \frac{2u+1}{2t} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}. \tag{6.14}$$

Plugging (6.14) into (6.9), we obtain

$$\frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!} - \frac{1}{2t}\sum_{u=t}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} - \frac{1}{12t^2}\sum_{u=0}^{\infty}\frac{p_3(u)\alpha_\ell^{2u}}{(2u)!} \le s_{2,2}(t,\ell) \le \frac{1}{2t}\sum_{u=0}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!} + \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} - \frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}, \tag{6.15}$$

where $p_3(u) = 4u^3+6u^2+8u+3$. By Lemma 2.7 we obtain

$$\sum_{u=t}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!} \le \frac{4C_2(\ell)}{\alpha_\ell^2\ t^2} \ \text{ and } \ \sum_{u=t}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u)!} \le \frac{8C_3(\ell)}{\alpha_\ell^2\ t^2}. \tag{6.16}$$

Note that for all $t \ge 1$,

$$\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} = \frac{2^{2t+1}}{t+1}\frac{1}{\binom{2t+2}{t+1}} < 1. \tag{6.17}$$

Combining (6.16) with (6.17) and applying Lemma 2.7 to (6.15), we obtain

$$\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha_\ell)+\frac{\operatorname{csh}(\alpha_\ell)}{2t}-\frac{C_{2,2}(\alpha_\ell)}{t^2}\le s_{2,2}(t,\ell)\le\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}\cosh(\alpha_\ell)+\frac{\operatorname{csh}(\alpha_\ell)}{2t}+\frac{4C_2(\ell)}{\alpha_\ell^2\,t^2}. \quad (6.18)$$

Applying (6.13) and (6.18) to (6.8), we obtain (3.47). □

*Proof of Lemma 3.28:* Recalling Definition 3.16, rewrite $S_3(t,\ell)$ as follows:

$$\begin{aligned} S_3(t,\ell) &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\sum_{s=u}^{t}\frac{1}{s}\Big(\frac{1}{2}-s\Big)_{s+1}\binom{-\frac{3}{2}}{t-s}\frac{(-s)_u}{(s+u)!}\\ &= \sum_{u=1}^{t}\frac{(-1)^u\alpha_\ell^{2u}}{(2u-1)!}\underbrace{\sum_{s=0}^{t-u}\frac{1}{s+u}\Big(\frac{1}{2}-s-u\Big)_{s+u+1}\binom{-\frac{3}{2}}{t-s-u}\frac{(-s-u)_u}{(s+2u)!}}_{=:S_3(t,u)}. \end{aligned} \quad (6.19)$$

From [3, Eqn. (5.34)], we have

$$S_3(t,u)=\binom{-\frac{3}{2}}{t}(-1)^u\Big(A_{3,1}(t,u)+A_{3,2}(t,u)\Big), \quad (6.20)$$

where

$$A_{3,1}(t,u)=\frac{t(1+2t-2u)(-t)_u(-1)^u}{2(1+2t)u(t+u)(t)_u}$$

and

$$A_{3,2}(t,u)=\frac{(-1)^{t+1}}{\binom{-\frac{3}{2}}{t}}+\frac{1}{1+2t}+\frac{2t}{1+2t}\sum_{i=1}^{u}\frac{(-t)_i(-1)^i}{(t+i)(t)_i}.$$

From (6.19) and (6.20), it follows that

$$S_3(t,\ell)=\binom{-\frac{3}{2}}{t}\Big(s_{3,1}(t)+s_{3,2}(t)\Big), \quad (6.21)$$

with

$$s_{3,1}(t,\ell)=\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u-1)!}A_{3,1}(t,u)\ \text{ and }\ s_{3,2}(t,\ell)=\sum_{u=1}^{t}\frac{\alpha_\ell^{2u}}{(2u-1)!}A_{3,2}(t,u). \quad (6.22)$$

By Lemma 2.3, we have

$$-\frac{3u^2+2u+\frac{1}{2}}{4ut}\le A_{3,1}(t,u)-\frac{1}{2u}\le 0. \quad (6.23)$$

Applying (6.23) into (6.22) and by Lemmas 2.7 and 2.5, we obtain

$$-\frac{C_{3,1}(\ell)}{t}\le s_{3,1}(t,\ell)+1-\cosh(\alpha_\ell)\le 0. \quad (6.24)$$

Now, by Lemma 2.4, we obtain

$$-\frac{4u^3+6u^2+8u+3}{12t^2}\le A_{3,2}(t,u)+\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{2u+1}{2t}\le 0. \quad (6.25)$$

Applying (6.25) to (6.22), it follows that

$$s_{3,2}(t,\ell)+\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u-1)!}-\frac{1}{2t}\sum_{u=1}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!}\le\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u-1)!}, \quad (6.26)$$

and

$$\begin{aligned} s_{3,2}(t,\ell) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \sum_{u=1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u-1)!} - \frac{1}{2t} \sum_{u=1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!} \geq \\ - \frac{1}{12t^2} \sum_{u=1}^{\infty} \frac{p_3(u)\alpha_\ell^{2u}}{(2u-1)!} - \frac{1}{2t} \sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!}, \end{aligned} \tag{6.27}$$

where $p_3(u) = 4u^3 + 6u^2 + 8u + 3$ is as in (6.15). By Lemma 2.7 we obtain

$$\sum_{u=t+1}^{\infty} \frac{\alpha_\ell^{2u}}{(2u-1)!} \leq \frac{2C_1(\ell)}{t^2} \quad \text{and} \quad \sum_{u=t+1}^{\infty} \frac{(2u+1)\alpha_\ell^{2u}}{(2u-1)!} \leq \frac{4C_2(\ell) + 2C_1(\ell)}{t^2}. \tag{6.28}$$

Applying (6.28) and Lemma Lemma 2.5 into (6.26) and (6.27), we have

$$-\frac{C_{3,2}(\ell)}{t^2} \leq s_{3,2}(t,\ell) + \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} \alpha_\ell \sinh(\alpha_\ell) - \frac{1}{2t} \mathrm{sch}(\alpha_\ell) \leq \frac{3C_1(\ell)}{t^2}. \tag{6.29}$$

Applying (6.24) and (6.29) into (6.21) we arrive at (3.48). □

*Proof of Lemma 3.30:* Following Definition 3.18, write $S_4(t,\ell)$ as follows:

$$\begin{aligned} S_4(t,\ell) &= \sum_{u=0}^{t} \frac{(-1)^u \alpha_\ell^{2u}}{(2u)!} \sum_{s=u}^{t} (-1)^s \Big(\frac{1}{2} - s\Big)_{s+1} \frac{(-s)_u}{(s+u+1)!} \\ &= \sum_{u=0}^{t} \frac{(-1)^u \alpha_\ell^{2u}}{(2u)!} \underbrace{\sum_{s=0}^{t-u} (-1)^{s+u} \Big(\frac{1}{2} - s - u\Big)_{s+u+1} \frac{(-s-u)_u}{(s+2u+1)!}}_{=:S_4(t,u)}. \end{aligned} \tag{6.30}$$

From [3, Eqn. (5.53)], we have

$$S_4(t,u) = \binom{-\frac{3}{2}}{t} (-1)^{u+t} \Big(A_{4,1}(t,u) + A_{4,2}(t,u)\Big), \tag{6.31}$$

where

$$A_{4,1}(t,u) = \frac{t(-t)_u(-1)^u}{2(1+2t)(t+u)(t+u+1)(t)_u}$$

and

$$A_{4,2}(t,u) = \frac{1}{1+2u} \left( \frac{(-1)^t}{\binom{-\frac{3}{2}}{t}} - \frac{1}{1+2t} - \frac{2t}{1+2t} \sum_{i=1}^{u} \frac{(-1)^i(-t)_i}{(t+i)(t)_i} \right).$$

From (6.30) and (6.31) it follows that

$$S_4(t,\ell) = (-1)^t \binom{-\frac{3}{2}}{t} \Big(s_{4,1}(t,\ell) + s_{4,2}(t,\ell)\Big), \tag{6.32}$$

where

$$s_{4,1}(t,\ell) = \sum_{u=0}^{t} \frac{\alpha_\ell^{2u}}{(2u)!} A_{4,1}(t,u) \quad \text{and} \quad s_{4,2}(t) := \sum_{u=0}^{t} \frac{\alpha_\ell^{2u}}{(2u)!} A_{4,2}(t). \tag{6.33}$$

Lemmas 2.2 and 2.3 imply that

$$\frac{1}{4t^2} \left( 1 - \frac{u^2 + u + \frac{3}{2}}{t} \right) \leq A_{4,1}(t,u) \leq \frac{1}{4t^2}. \tag{6.34}$$

From (6.34) and (6.33), we obtain

$$\frac{1}{4t^2}\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}-\frac{1}{4t^2}\sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}-\frac{1}{4t^3}\sum_{u=0}^{\infty}\frac{(u^2+u+\frac{3}{2})\alpha_\ell^{2u}}{(2u)!}\le s_{4,1}(t,\ell)\le\frac{1}{4t^2}\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}. \tag{6.35}$$

Applying Lemmas 2.7 and 2.5 to (6.35), it follows that

$$\frac{1}{4t^2}\cosh(\alpha_\ell)-\frac{C_{4,1}(\ell)}{t^3}\le s_{4,1}(t,\ell)\le\frac{1}{4t^2}\cosh(\alpha_\ell). \tag{6.36}$$

Now, by Lemma 2.4, we obtain

$$0\le A_{4,2}(t,u)-\frac{1}{1+2u}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{2u+1}{2t}\right)\le\frac{1}{1+2u}\frac{p_3(u)}{12t^2}, \tag{6.37}$$

where $p_3(u)$ is as in (6.15). Plugging (6.37) into (6.33), it follows that

$$\begin{aligned}-\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!}\le{}& s_{4,2}(t,\ell)-\sum_{u=0}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!}\left(\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}-\frac{2u+1}{2t}\right)\le\\ &\frac{1}{12t^2}\sum_{u=0}^{\infty}\frac{p_3(u)\alpha_\ell^{2u}}{(2u+1)!}+\frac{1}{2t}\sum_{u=t+1}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u+1)!}.\end{aligned} \tag{6.38}$$

Using Lemma 2.7, we get

$$\sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u+1)!}\le\frac{C_0(\ell)}{t^2}\quad\text{and}\quad\sum_{u=t+1}^{\infty}\frac{(2u+1)\alpha_\ell^{2u}}{(2u+1)!}=\sum_{u=t+1}^{\infty}\frac{\alpha_\ell^{2u}}{(2u)!}\le\frac{C_0(\ell)}{t^2}. \tag{6.39}$$

Plugging (6.39) to (6.38) and using Lemma 2.5, we finally obtain

$$-\frac{2C_0(\ell)}{3t^2}\le s_{4,2}(t,\ell)-\frac{(-1)^t}{\binom{-\frac{3}{2}}{t}}\frac{\sinh(\alpha_\ell)}{\alpha_\ell}+\frac{\cosh(\alpha_\ell)}{2t}\le\frac{(\alpha^2+6)\cosh(\alpha_\ell)+3\alpha_\ell\sinh(\alpha_\ell)+12C_0(\ell)}{24t^2}. \tag{6.40}$$

We conclude the proof by combining (6.36), (6.40), and (6.32). □

## 7. Conclusion

We conclude this paper with a list of possible ideas emerged from our work.

(1) Double Turán inequality (also known as 2-log-concavity) for the partition function has been studied independently in [22, Theorem 1.6] and [20, Page 128]. Similar to the proofs of Theorems 1.5-1.9, $(p(n))_{n\ge1873}$ is 2-log-concave follows directly from Theorem 4.5 by choosing $w = 11$ and with Mathematica, we confirm that $(p(n))_{n\ge221}$ is 2-log-concave.

(2) The partition function $p(n)$ satisfies shifted Laguerre-Pólya inequality of order $m$ if

$$L_m(p(n)) := \frac{1}{2}\sum_{k=0}^{2m}(-1)^{k+m}\binom{2m}{k}p(n+k)p(2m-k+n).$$

In [45], Wagner proved the $m$-th order shifted Laguerre-Pólya inequalities for the partition function as $n\to\infty$. He proposed a conjecture for the cut offs $(N(m))_{1\le m\le10}$ such that for all $n\ge N(m)$, $p(n)$ satisfies the $m$-th order shifted Laguerre-Pólya inequalities. Wang and Yang [46] settled the case $m = 2$. Dou and Wang [14] gave an explicit bounds for $(N(m))_{3\le m\le10}$ and consequently, Wagner's conjecture for $m = 3$ and 4 have been settled.

Applying Theorem 4.5, one can easily retrieve the result of Wang and Yang [46, Theorem 2.1]. Moreover, it seems to be possible that we can trace $N(m)$ for $3\le m\le10$ using our set up. In spite of having Wagner's proof on positivity of $L_m(p(n))$ as $n\to\infty$, it would be interesting to ask for the growth of $L_m(p(n))$ as $n\to\infty$.

(3) Recently, Gomez, Males, and Rolen [16] studied the positivity of $\Delta_j^2(p(n)) := p(n) - 2p(n-j) + p(n-2j)$ and consequently proved that $N_k(m,n) - N_k(m+1,n) > 0$, where the $k$-rank function $N_k(m,n)$ which counts the number of partitions of $n$ into at least $(k-1)$ successive Durfee squares with $k$-rank equal to $m$. One might retrieve their results from Theorem 4.5 by taking appropriate $w$. More generally, we believe that one can come up with the asymptotic expansion of $\Delta_j^r(p(n))$ for any positive integer $r$, which would finally complete Odlyzko's work [33] on $\Delta^r p(n)$ by not only proving the positivity phenomena but also shows its asymptotic growth.

(4) Partition inequalities arising from truncated theta series that has been documented in [1, 2, 15] among many research works done by Andrews, Guo, Merca, Yee, Zeng, to name a few. In spite of having combinatorial proofs of such inequalities for $p(n)$, it seems that no such inequalities have been traced via the analytic approach. Theorem 4.5 might play a key role in proving these inequalities. More generally, given non-trivial linear homogeneous partition inequalities considered by Merca and Katriel [23, 30], it would be nice to develop an algorithm by making an appropriate choice for $w$ and applying Theorem 4.5 so as to decide whether such a given inequality holds or not.

(5) Starting from the estimates of Dawsey and Masri [11] on Andrews' spt function, one can follow the similar method as worked out in this paper to settle all the conjectures on inequalities for spt function pertaining to the invariants of a quartic binary form given by Chen [7].

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Research Institute for Symbolic Computation, Johannes Kepler University, Altenberger Strasse 69, A-4040 Linz, Austria

*Email address*: `Koustav.Banerjee@risc.uni-linz.ac.at`