INEQUALITIES FOR THE MODIFIED BESSEL FUNCTION OF FIRST KIND OF NON-NEGATIVE ORDER

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ABSTRACT. Let $I_\nu(x)$ be the modified Bessel function of order ν with real argument x. We present explicit error bounds for the asymptotic expansion of $I_\nu(x)$ with $x \geq 1$. Two cases, ν an integer and ν a half-integer are considered separately. In addition, we present a short discussion on the error analysis for $I_\nu(x)$ where ν is any non-negative real number.

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Contents

1. Introduction

Consider Bessel's differential equation over the complex domain,

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0, \tag{1.1}$$

where ν is an arbitrary complex parameter. The solutions of this equation are termed as Bessel functions. In 1824, F. W. Bessel [2, 3] initiated a systematic rigorous analysis of such functions which was the starting point of a flourishing development along with a multitude of applications in connection with problems in number theory, integral transforms, differential equations, etc. The main object of this paper is $I_\nu(z)$, a solution of

$$z^2y'' + zy' - (z^2 + \nu^2)y = 0, \tag{1.2}$$

the so-called modified version of (1.1), with series representation

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2m}}{m!\Gamma(\nu + m + 1)}. \tag{1.3}$$

In 1854 Kirchhoff [10] established an asymptotic expansion of $I_\nu(z)$: for fixed $\nu \in \mathbb{C}$,

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left(1 - \frac{4\nu^2 - 1}{8z} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8z)^2} - \dots\right),\ |\arg z| < \frac{\pi}{2}.$$

Estimates for the error terms of asymptotic expansions of Bessel functions have been considered by Schläfli [18], Weber [21], Watson [20, p. 209-210], Meijer [12], Olver [15], Nemes [13], to name a few. For a more extensive study on the literature of the Bessel functions, we refer to [20].

This paper focuses on deriving a family of inequalities for $I_\nu(x)$ with ν a non-negative integer or a half-integer, and x a real number ≥ 1. Why it is necessary to get such inequalities for $I_\nu(x)$? We have already mentioned that the theory of Bessel functions often sprout out in problems related to number theory. For example, $I_\nu(x)$ with non-negative integral or half-integral order ν appears in Hardy-Ramanujan-Rademacher type series expansions for coefficients of certain classes of Dedekind eta quotients; see for example [5, Thm. 1.1] or [19, Thm. 1.1]. These coefficients are quite often entangled with combinatorial features that emerge from the question whether a real polynomial associated with such sequences has roots all real. For example, consider the Jensen polynomial of degree d and shift n for a sequence $\{\alpha(n)\}_{n\geq 0}$ of real numbers, defined as

$$J_\alpha^{d,n}(x) = \sum_{j=0}^{d} \binom{d}{j} \alpha(n+j)x^j.$$

Now, to prove log-concavity (resp. higher order Turán inequalities) of $\alpha(n)$, it is equivalent to prove that $J_\alpha^{2,n}(x)$ (resp. $J_\alpha^{d,n}(x)$ for $d \geq 3$) has roots all real for all $n > N(d)$ where $N(d)$ is a positive real number depending on the degree d.

To answer these problems for a sequence, say $a_f(n)$, arising from the Fourier expansion of a periodic meromorphic function, say a Dedekind eta quotient $f(q)$, we would like to estimate $a_f(n)$ by computing a precise estimation of the associated Hardy-Ramanujan-Rademacher type series, say S_f. Now, in order to provide such a precise estimate for the main term obtained after truncating the series S_f to a finite number of terms, inequalities for $I_{\nu(f)}(x)$ are needed, where the index $\nu(f)$ is depending on f. For instance, Griffin, Ono, Rolen and Zagier [9] proved the following theorem.

Theorem 1.1. [9, eqn. 9] *Let* $\{a_f(n)\}_{n\geq 0}$ *be a sequence of positive real numbers arising from the Fourier expansion of a periodic meromorphic function* f. *Suppose*

$$a_f(n) = A_f n^{\frac{k-1}{2}} I_{k-1}(4\pi\sqrt{mn}) + O\Big(n^C e^{2\pi\sqrt{mn}}\Big)$$

as $n \to \infty$ *for some non-zero real constants* $A_f, m,\ k,$ *and* C, *where* $I_\nu(x)$ *is the modified Bessel function of the first kind of order* ν. *Then for* $d \geq 1$, *the Jensen polynomial* $J_{a_f}^{d,n}(x)$ *associated to* $a_f(n)$ *has only real roots for all sufficiently large* n.

A concrete example with regard to log-concavity is this. In order to prove log-concavity of the colored partition function $p_k(n)$, conjectured by Chern, Fu and Tang [6, Conjecture 5.3], Bringmann et. al. estimated the error term by truncation of the asymptotic expansion of $I_\nu(x)$ at $N = 3$, which plays a key role in their proof of the conjecture [4, Conjecture 1]:

Theorem 1.2. [4, Lemma 2.2 (4)] *For* $\nu \geq 2$ *and* $x \geq \frac{1}{120}(\nu + \frac{7}{2})^6$,

$$\left| I_\nu(x)e^{-x}\sqrt{2\pi x} - 1 + \frac{4\nu^2-1}{8x} - \frac{(4\nu^2-1)(4\nu^2-9)}{128x^2} + \frac{(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)}{3072x^3} \right| < \frac{31\nu^8}{6x^4}. \tag{1.4}$$

Theorems 1.1 and 1.2 motivated us to study the inequalities for $I_\nu(x)$ by extending the truncation point to any positive integer N and estimating an error bound.

This paper is organized in the following way. First we will give some basic notations and definitions which we use throughout the paper. Section 2 presents lemmas, useful for the proofs given in later sections, followed by a brief illustration of the key features they possess. In Sections 3 and 4 we will discuss the method devised. Section 3 (resp. Section 4) presents the estimation of the error term of the asymptotic expansion of $I_\nu(x)$ with $\nu \in \mathbb{Z}_{\geq 0}$ (resp. $\nu \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$), and derives Theorem 3.9 with the Corollary 3.10 (resp. Theorem 4.6). Section 5 is devoted to the study of the error analysis for any non-negative real index ν. The Appendix, Section 6, is divided into two subsections: Subsection 6.1 presents the proofs of three lemmas (from Section 2) and in Subsection 6.2 a Mathematica computation is presented which is needed for the completion of the proof of Corollary 3.10.

Let $a^{\underline{k}}$ denote the falling factorial,

$$a^{\underline{k}} = \begin{cases} a(a-1)\dots(a-k+1), & \text{if } k \in \mathbb{Z}_{>0} \\ 1, & \text{if } k = 0 \end{cases},$$

and the binomial coefficient is defined by $\binom{a}{m} = \frac{a^{\underline{m}}}{m!}$. In this framework, we restrict ourselves to $a \in \mathbb{R}$. Similarly, the rising factorial is defined by

$$a^{\overline{k}} = \begin{cases} a(a+1)\dots(a+k-1), & \text{if } k \in \mathbb{Z}_{>0} \\ 1, & \text{if } k = 0 \end{cases};$$

nevertheless, we mostly prefer to use the classical notation $(a)_k = a^{\overline{k}}$. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, the gamma function is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt. \tag{1.5}$$

For $\mathrm{Re}(z) \leq 0$, $\Gamma(z)$ is defined by analytic continuation. It is a meromorphic function with no zeros, and with simple poles of residue $(-1)^n/n!$ at $z = -n$ when $n \in \mathbb{Z}_{\geq 0}$. Note that $(a)_n = \Gamma(a+n)/\Gamma(a)$ for $a \notin \mathbb{Z}_{\leq 0}$. For a brief survey on the gamma function, readers may consult [1, Ch. 6.1], [16, Ch. 2.1] and [11]. The incomplete gamma functions $\gamma(a,z)$ and $\Gamma(a,z)$ are defined by

$$\gamma(a,z) \quad = \quad \int_0^z e^{-t} t^{a-1}\, dt, \quad \mathrm{Re}(a) > 0, \tag{1.6}$$

and

$$\Gamma(a,z) \quad = \quad \int_z^\infty e^{-t} t^{a-1}\, dt; \tag{1.7}$$

moreover,

$$\gamma(a,z) + \Gamma(a,z) = \Gamma(a), \quad a \notin \mathbb{Z}_{\leq 0}. \tag{1.8}$$

For our purpose, we need to consider $I_\nu(x)$ only for $\nu \in \mathbb{R}_{\geq 2}$ and $x \in \mathbb{R}_{\geq 1}$. To this end, we shall use the following representation of $I_\nu(x)$ [20, Ch. VII, 7.25],

$$I_\nu(x) = \frac{(\frac{x}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi e^{x\cos\theta} \sin^{2\nu}\theta\, d\theta. \tag{1.9}$$

2. Preliminary Lemmas

This section presents all the preliminary facts needed for the proofs of the lemmas stated in Sections 3, 4 and 5. Lemma 2.1 helps us to estimate the integrand in $\gamma(a,x)$ and $\Gamma(a,x)$ for positive real numbers a and x. Using Lemmas 2.2 and 2.3 identifies the binomial coefficient $\binom{\nu-\frac{1}{2}}{m}$ with the standard binomial coefficients and as a consequence, we obtain an upper bound of the absolute value of $(-1)^m\binom{\nu-\frac{1}{2}}{m}$ in Lemma 2.4. The proofs of Lemmas 2.1 to 2.4 are presented in Subsection 6.1. Lemmas 2.6 and 2.7 illustrate the alternation of sign of the sums (2.5) and (2.6) depending on the parity of N for $\nu \in \mathbb{Z}_{\geq 0}$. Similar results are outlined in Lemmas 2.8 and 2.9 for $\nu \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$.

Lemma 2.1. *For all* $(x,y) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$,

$$e^{-x}\, x^y < \frac{1}{\sqrt{2\pi}}\, \Gamma(y)\sqrt{y}. \tag{2.1}$$

Lemma 2.2. *For* $(\nu, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,

$$\binom{\nu-\frac{1}{2}}{m} = \begin{cases} \frac{(-1)^{m-\nu}}{4^m} \frac{\binom{2\nu}{\nu}\binom{2m-2\nu}{m-\nu}}{\binom{m}{\nu}}, & \text{if } m > \nu \\ \frac{1}{4^m} \frac{\binom{2\nu}{\nu}\binom{\nu}{m}}{\binom{2\nu-2m}{\nu-m}}, & \text{if } m \leq \nu \end{cases}.$$

Lemma 2.3. *For $\nu \in \mathbb{R}$ and $(k, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,*

$$\sum_{m=k}^{N} (-1)^m \binom{\nu - \frac{1}{2}}{m} \binom{m}{k} = 2\ (-1)^{N+1}(N+1) \binom{\nu - \frac{1}{2}}{N+1} \frac{1}{2k - 2\nu + 1} \binom{N}{k}. \tag{2.2}$$

Lemma 2.4. *For $(\nu, m) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$,*

$$\left| (-1)^m \binom{\nu - \frac{1}{2}}{m} \right| \leq \begin{cases} \frac{1}{\pi\sqrt{\nu(m-\nu)}} \frac{1}{\binom{m}{\nu}}, & \text{if } m > \nu \\ \frac{2}{\sqrt{\pi}} \binom{\nu}{m}, & \text{if } m \leq \nu \end{cases}.$$

Lemma 2.5. *For $\alpha \in \mathbb{R}_{>1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,*

$$\sum_{m=0}^{N} \binom{\nu - \frac{1}{2}}{m} (-\alpha)^m = 2\ (-1)^{N+1}(N+1) \binom{\nu - \frac{1}{2}}{N+1} \sum_{m=0}^{N} \binom{N}{m} \frac{(\alpha-1)^m}{2m - 2\nu + 1}. \tag{2.3}$$

Proof. For $\beta := \alpha - 1$,

$$\begin{aligned} \sum_{m=0}^{N} \binom{\nu - \frac{1}{2}}{m} (-\alpha)^m &= \sum_{m=0}^{N} \sum_{k=0}^{m} (-1)^m \binom{\nu - \frac{1}{2}}{m} \binom{m}{k} \beta^k \\ &= \sum_{k=0}^{N} \sum_{m=k}^{N} (-1)^m \binom{\nu - \frac{1}{2}}{m} \binom{m}{k} \beta^k. \end{aligned} \tag{2.4}$$

From (2.4), using (2.2), the statement follows. □

Lemma 2.6. *For all $\alpha \in \mathbb{R}_{>1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 0}$ with $\nu \geq N + 1$,*

$$(-1)^N \sum_{m=0}^{N} \binom{\nu - \frac{1}{2}}{m} (-\alpha)^m > 0. \tag{2.5}$$

Proof. Multiplying $(-1)^N$ on both sides of (2.3) and the fact that $\nu \geq N + 1$ immediately implies (2.5). □

Lemma 2.7. *For all $\alpha \in (0, 1]$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 0}$ with $\nu \geq N + 1$,*

$$(-1)^N \sum_{m=N+1}^{\nu} \binom{\nu - \frac{1}{2}}{m} (-\alpha)^m < 0. \tag{2.6}$$

Proof. Let

$$S(\nu) := (-1)^N \sum_{m=N+1}^{\nu} \binom{\nu - \frac{1}{2}}{m} (-\alpha)^m < 0.$$

We prove (2.6) by induction on $\nu \geq N + 1$. For $\nu = N + 1$,

$$S(N+1) = -\alpha^{N+1} \binom{N + \frac{1}{2}}{N+1} < 0 \ \left(\text{by Lemma 2.2 with } m = \nu = N+1 \text{ and } \binom{N + \frac{1}{2}}{N+1} > 0\right).$$

Assuming $S(T)<0$ for $T\geq N+1$, we proceed to the case $\nu=T+1$. Using the Paule-Schorn [17] package fastZeil[1], after applying Lemma 2.2, we obtain

$$\begin{aligned} S(T+1) &= (1-\alpha)S(T)-\alpha^{N+1}\binom{T-\frac{1}{2}}{N}-(-1)^{T+N+1}\alpha^{T+1}\frac{4^{-T}}{2T+2}\binom{2T}{T} \\ &\leq -\alpha^{N+1}\binom{T-\frac{1}{2}}{N}-(-1)^{T+N+1}\alpha^{T+1}\frac{4^{-T}}{2T+2}\binom{2T}{T}. \end{aligned} \tag{2.7}$$

To see that the right hand side of (2.7) is strictly less than 0, we consider two cases depending on parity of T and N. If T and N have opposite parity; i.e., $T+N\equiv 1 (\mathrm{mod}\ 2)$, then $S(T+1)<0$, since

$$S(T+1)\leq -\alpha^{N+1}\binom{T-\frac{1}{2}}{N}-\alpha^{T+1}\frac{4^{-T}}{2T+2}\binom{2T}{T}<0.$$

Continuing with (2.7) in the case that $T\equiv N (\mathrm{mod}\ 2)$, it follows that

$$\begin{aligned} S(T+1) &\leq -\alpha^{N+1}\left(\binom{T-\frac{1}{2}}{N}-\frac{1}{2T+2}\frac{1}{4^T}\binom{2T}{T}\right) \quad (\text{as, } a\in(0,1] \text{ and } T>N) \\ &= -\alpha^{N+1}\left(\frac{1}{4^N}\frac{\binom{2T}{T}\binom{T}{N}}{\binom{2T-2N}{T-N}}-\frac{1}{2T+2}\frac{1}{4^T}\binom{2T}{T}\right) \\ &\qquad\qquad (\text{by Lemma 2.2 with } (m,\nu)\mapsto(N,T)) \\ &= -\alpha^{N+1}\frac{1}{4^T}\binom{2T}{T}\left(\frac{4^{T-N}}{\binom{2T-2N}{T-N}}\binom{T}{N}-\frac{1}{2T+2}\right) \\ &\leq -\alpha^{N+1}\frac{1}{4^T}\binom{2T}{T}\left(\binom{T}{N}\sqrt{\pi(T-N)}-\frac{1}{2T+2}\right) \quad (\text{by (6.2) with } n\mapsto T-N) \\ &< 0 \quad (\text{since } T>N). \end{aligned}$$

This finishes the proof of (2.6). □

Lemma 2.8. *For all $\alpha\in\mathbb{R}_{>1}$ and $(\nu,N)\in\mathbb{Z}_{\geq 2}\times\mathbb{Z}_{\geq 0}$ with $\nu\geq N$,*

$$(-1)^N\sum_{m=0}^{N}\binom{\nu}{m}(-\alpha)^m>0. \tag{2.8}$$

Proof. For $\nu=N$,

$$(-1)^N\sum_{m=0}^{N}\binom{\nu}{m}(-\alpha)^m=(\alpha-1)^N>0;$$

whereas for $\nu>N$, we apply Lemma (2.6) with the substitution $\nu\mapsto\nu+\frac{1}{2}$. □

Lemma 2.9. *For all $\alpha\in(0,1]$ and $(\nu,N)\in\mathbb{Z}_{\geq 2}\times\mathbb{Z}_{\geq 0}$ with $\nu\geq N$,*

$$(-1)^N\sum_{m=N+1}^{\nu}\binom{\nu}{m}(-\alpha)^m<0. \tag{2.9}$$

[1]The package is available at `https://combinatorics.risc.jku.at/software`.

Proof. Analogous to the proof of Lemma 2.7. □

The asymptotic expansion for $I_\nu(x)$ is well documented in the literature; see, for example, [1] or [14, 10.40.1]. Still, we recall it in brevity. Namely, in order to estimate the error term $E(\nu, N, x)$ in Lemma 3.1, the knowledge of both (2.10) and the variant (2.11) is required.

Lemma 2.10. ([20, Chapter VII, 7.25]) *For $x \in \mathbb{R}_{\geq 1}$ and $\nu \in \mathbb{R}_{>-\frac{1}{2}}$,*

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(2x)^m \Gamma(\nu + \frac{1}{2})} \int_0^{2x} e^{-t} t^{\nu+m-\frac{1}{2}} \, dt. \tag{2.10}$$

Now from (2.10) we can rephrase to the asymptotic expansion of $I_\nu(x)$ in the following way,

$$\begin{aligned}
\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(2x)^m \Gamma(\nu + \frac{1}{2})} \Big(\int_0^{\infty} e^{-t} t^{\nu+m-\frac{1}{2}} \, dt - \int_{2x}^{\infty} e^{-t} t^{\nu+m-\frac{1}{2}} \, dt \Big) \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(2x)^m \Gamma(\nu + \frac{1}{2})} \Gamma(\nu + m + \frac{1}{2}) - \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(2x)^m \Gamma(\nu + \frac{1}{2})} \int_{2x}^{\infty} e^{-t} t^{\nu+m-\frac{1}{2}} \, dt \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m} \left(\nu + \frac{1}{2}\right)_m}{(2x)^m} - \sum_{m=0}^{\infty} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}}{(2x)^m \Gamma(\nu + \frac{1}{2})} \int_{2x}^{\infty} e^{-t} t^{\nu+m-\frac{1}{2}} \, dt \\
&\underset{x\to\infty}{\sim} \sum_{m=0}^{\infty} (-1)^m \frac{\binom{\nu-\frac{1}{2}}{m} \left(\nu + \frac{1}{2}\right)_m}{(2x)^m}.
\end{aligned}$$

Summarizing,

$$I_\nu(x) \underset{x\to\infty}{\sim} \frac{e^x}{\sqrt{2\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m a_m(\nu)}{x^m} \quad \text{with} \quad a_m(\nu) = \frac{\binom{\nu-\frac{1}{2}}{m} \left(\nu + \frac{1}{2}\right)_m}{2^m}. \tag{2.11}$$

3. Inequalities for Modified Bessel function of integral order

In this section, we shall describe how one can obtain an infinite family of inequalities for $I_\nu(x)$, $\nu \in \mathbb{Z}_{\geq 0}$, as stated in Theorem (3.9). We first split the infinite series on the right hand side of (2.10). This results in the identity (3.2) where the left hand side presents the remainder term obtained from truncation of the asymptotic series expansion (2.11) of $\frac{\sqrt{2\pi x}}{e^x} I_\nu(x)$ after extracting its partial sum. In Lemma 3.1, we further dissect the remainder term $E(\nu, N, x)$ depending on $\nu \geq N + 1$ or $\nu \leq N$. Lemmas 3.2-3.5 (resp. Lemmas 3.6-3.8) deal with the error analysis for $\nu \geq N + 1$ (resp. for $\nu \leq N$).

For $\nu \geq N+1$, using Lemma 2.6 and 2.7, we obtain upper bounds for the absolute value of $E_{\nu,N,1}(x)$ and $E^{\nu}_{N,2}(x)$. In order to compute an upper bound of $|E^{\nu}_{N,3}(x)|$, we first estimate a bound for $|\psi^{\nu}_m(t; x)|$ using Lemma 2.1 and then estimate the sum by Lemma 2.4. Combining the upper bounds from Lemmas 3.2-3.4, we obtain the final bound for $|E(\nu, N, x)|$, as given in Lemma 3.5.

On the other hand, for $\nu \leq N$, the remainder term $E(\nu, N, x)$ is divided into two components, denoted by $E_{\nu,N,1}(x)$ and $E^{N}_{\nu,2}(x)$. Here we carry out a different method to obtain an upper bound for $|E_{\nu,N,1}(x)|$, since $\phi^{\nu}_{m}(t;x)$ for $\nu \leq N$ is different from the case $\nu \geq N+1$, see Lemma 2.2. Using Lemma 2.5, we shall finally get an upper bound for $|E_{\nu,N,1}(x)|$, as given in Lemma 3.6. Analogous to the computation for upper bound of $|E^{\nu}_{N,3}(x)|$, a similar estimation has been done for $|E^{N}_{\nu,2}(x)|$ to obtain (3.34). Lemmas 3.6 and 3.7 imply Lemma 3.8.

Finally, we state the main result of this paper, Theorem 3.9, as an immediate consequence of Lemmas 3.5 and 3.8. From Theorem 3.9, we get an analogous result to [4, Lemma 2.2 (4)] for $N = 3$ and $\nu \in \mathbb{Z}_{\geq 0}$, as documented in Corollary 3.10.
In this subsection,

$$a_m(\nu) = \frac{\binom{\nu-\frac{1}{2}}{m}\left(\nu+\frac{1}{2}\right)_m}{2^m}, \tag{A}$$

as in (2.11).

Define

$$\phi^{\nu}_{m}(t;x) = \frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{(2x)^m\Gamma(\nu+\frac{1}{2})}\int_{2x}^{\infty} e^{-t}t^{\nu+m-\frac{1}{2}}\,dt, \tag{PHI}$$

$$\psi^{\nu}_{m}(t;x) = \frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{(2x)^m\Gamma(\nu+\frac{1}{2})}\int_{0}^{2x} e^{-t}t^{\nu+m-\frac{1}{2}}\,dt, \tag{PSI}$$

$$E_{\nu,N,1}(x) = -\sum_{m=0}^{N}\phi^{\nu}_{m}(t;x), \tag{E1}$$

$$E^{\nu}_{N,2}(x) = \sum_{m=N+1}^{\nu}\psi^{\nu}_{m}(t;x), \tag{E2}$$

$$E^{\nu}_{N,3}(x) = \sum_{m=\nu+1}^{\infty}\psi^{\nu}_{m}(t;x), \tag{E3}$$

and

$$E^{N}_{\nu,2}(x) = \sum_{m=N+1}^{\infty}\psi^{\nu}_{m}(t;x). \tag{E0}$$

Lemma 3.1. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,*

$$\frac{\sqrt{2\pi x}}{e^x}I_\nu(x) - \sum_{m=0}^{N}\frac{(-1)^m a_m(\nu)}{x^m} = E(\nu, N, x),$$

with

$$E(\nu,N,x) = \begin{cases} E_{\nu,N,1}(x) + E^{\nu}_{N,2}(x) + E^{\nu}_{N,3}(x), & \text{if } \nu \geq N+1 \\ E_{\nu,N,1}(x) + E^{N}_{\nu,2}(x), & \text{if } \nu \leq N \end{cases}. \tag{3.1}$$

Proof. From (2.10) it follows that

$$\begin{aligned}\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) &= \sum_{m=0}^{\infty} \psi_m^\nu(t;x) \\ &= \sum_{m=0}^{N} \psi_m^\nu(t;x) + \sum_{m=N+1}^{\infty} \psi_m^\nu(t;x) \\ &= \sum_{m=0}^{N} \frac{(-1)^m \binom{\nu-\frac{1}{2}}{m}(\nu+\frac{1}{2})_m}{(2x)^m} - \sum_{m=0}^{N} \phi_m^\nu(t;x) + \sum_{m=N+1}^{\infty} \psi_m^\nu(t;x).\end{aligned}$$

Therefore

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} = -\sum_{m=0}^{N} \phi_m^\nu(t;x) + \sum_{m=N+1}^{\infty} \psi_m^\nu(t;x). \tag{3.2}$$

We split the right hand side of (3.2) according to $\nu \geq N+1$ and $\nu \leq N$ as follows. For $\nu \geq N+1$,

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} = -\sum_{m=0}^{N} \phi_m^\nu(t;x) + \sum_{m=N+1}^{\nu} \phi_m^\nu(t;x) + \sum_{m=\nu+1}^{\infty} \psi_m^\nu(t;x); \tag{3.3}$$

whereas, for $\nu \leq N$,

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} = -\sum_{m=0}^{N} \phi_m^\nu(t;x) + \sum_{m=N+1}^{\infty} \psi_m^\nu(t;x). \tag{3.4}$$

The relations (3.3) and (3.4) prove (3.1). □

Lemma 3.2. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\nu \geq N+1$,*

$$|E_{\nu,N,1}(x)| < \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}}\ \Gamma(\nu+N+\frac{3}{2}, 2x), \tag{3.5}$$

where Γ is the incomplete gamma function from (1.7).

Proof. From Lemma 3.1, for all $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\nu \geq N+1$ we have

$$\begin{aligned}E_{\nu,N,1}(x) = -\sum_{m=0}^{N} \phi_m^\nu(t;x) &= -\int_{2x}^{\infty} \frac{e^{-t} t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m} \Big(-\frac{t}{2x}\Big)^m\, dt \\ &= -\int_{2x}^{\infty} \frac{e^{-t} t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m} (-\theta)^m\, dt \quad \text{where } \theta := \frac{t}{2x}.\end{aligned} \tag{3.6}$$

Let N be even. Then using (2.5), we get

$$-\binom{\nu-\frac{1}{2}}{N+1}\theta^{N+1} < -\sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m}(-\theta)^m < 0 \quad \text{(by (2.5))} \tag{3.7}$$

Now from (3.6) and (3.7), by taking the integral, it follows that

$$-\phi^{\nu}_{N+1}(t;x) < -\sum_{m=0}^{N} \phi^{\nu}_{m}(t;x) < 0. \tag{3.8}$$

Similarly, for N odd,

$$\binom{\nu-\frac{1}{2}}{N+1}\theta^{N+1} > -\sum_{m=0}^{N}\binom{\nu-\frac{1}{2}}{m}(-\theta)^m > 0 \quad \text{(by (2.5))}. \tag{3.9}$$

Using (3.6) and (3.9), we have

$$0 < -\sum_{m=0}^{N} \phi^{\nu}_{m}(t;x) < \phi^{\nu}_{N+1}(t;x). \tag{3.10}$$

(3.8) and (3.10) together imply (3.5). □

Lemma 3.3. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\nu \geq N+1$,*

$$|E^{\nu}_{N,2}(x)| < \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}}\,\gamma(\nu+N+\frac{3}{2}, 2x), \tag{3.11}$$

where γ is the incomplete gamma function from (1.6).

Proof. For $\nu \geq N+1$, from Lemma 3.1, it follows that

$$\begin{aligned} E^{\nu}_{N,2}(x) &= \sum_{m=N+1}^{\nu} \psi^{\nu}_{m}(t;x) = \int_0^{2x} \frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=N+1}^{\nu}\binom{\nu-\frac{1}{2}}{m}\Big(-\frac{t}{2x}\Big)^m\, dt \\ &= \int_0^{2x} \frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=N+1}^{\nu}\binom{\nu-\frac{1}{2}}{m}(-\theta)^m dt \quad \text{where } \theta := \frac{t}{2x}. \end{aligned} \tag{3.12}$$

For N even,

$$-\binom{\nu-\frac{1}{2}}{N+1}\theta^{N+1} < \sum_{m=N+1}^{\nu}\binom{\nu-\frac{1}{2}}{m}(-\theta)^m < 0 \quad \text{(by (2.6))}. \tag{3.13}$$

From (3.12) and (3.13), we have

$$-\psi^{\nu}_{N+1}(t;x) < \sum_{m=N+1}^{\nu} \psi^{\nu}_{m}(t;x) < 0. \tag{3.14}$$

Likewise, for N odd,

$$0 < \sum_{m=N+1}^{\nu}\binom{\nu-\frac{1}{2}}{m}(-\theta)^m < \binom{\nu-\frac{1}{2}}{N+1}\theta^{N+1} \quad \text{(by (2.6))}, \tag{3.15}$$

and (3.12) and (3.15) together imply

$$0 < \sum_{m=N+1}^{\nu} \psi^{\nu}_{m}(t;x) < \psi^{\nu}_{N+1}(t;x). \tag{3.16}$$

Applying (3.14) and (3.16) concludes the proof. □

Define

$$E^{\nu}_{N+1} \quad = \quad \frac{\sqrt{2}}{\pi}\sqrt{2N+\frac{5}{2}}\Big(\sqrt{N+2}-1\Big) \tag{3.17}$$

and

$$E^{\nu}_{N+2} \quad = \quad \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{\nu+1}{\nu-N-1}}\sqrt{\nu+N+\frac{3}{2}}\Big(\sqrt{\frac{1}{\nu-N}}-\sqrt{\frac{1}{\nu+1}}\Big). \tag{3.18}$$

Lemma 3.4. *Let $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\nu \geq N+1$. Then with* (3.17) *and* (3.18) *one has,*

$$|E^{\nu}_{N,3}(x)| < E^{\nu}_{N,3}\frac{a_{N+1}(\nu)}{x^{N+1}},$$

with

$$E^{\nu}_{N,3} = \begin{cases} E^{\nu}_{N+2}, & \text{if } \nu \geq N+2 \\ E^{\nu}_{N+1}, & \text{if } \nu = N+1 \end{cases}. \tag{3.19}$$

Proof.

$$\begin{aligned}
|E^{\nu}_{N,3}(x)| \quad &= \quad \Big|\sum_{m=\nu+1}^{\infty}\psi^{\nu}_{m}(t;x)\Big| \quad \text{(by Lemma 3.1)} \\
&\leq \quad \sum_{m=\nu+1}^{\infty}\frac{\Big|(-1)^m\binom{\nu-\frac{1}{2}}{m}\Big|}{(2x)^m\Gamma(\nu+\frac{1}{2})}\int_0^{2x} e^{-t}t^{\nu+m-\frac{1}{2}}\ dt \\
&< \quad \sum_{m=\nu+1}^{\infty}\frac{\Big|(-1)^m\binom{\nu-\frac{1}{2}}{m}\Big|}{(2x)^m\Gamma(\nu+\frac{1}{2})}\frac{\Gamma(\nu+N+\frac{3}{2})\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}}\int_0^{2x} t^{m-N-2}\ dt \\
&\qquad \Big(\text{by } (x,y)\mapsto\Big(t,\nu+N+\frac{3}{2}\Big) \text{ in (2.1)}\Big) \\
&= \quad \frac{1}{\sqrt{2\pi}}\frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{(2x)^{N+1}}\binom{\nu-\frac{1}{2}}{N+1}\sum_{m=\nu+1}^{\infty}\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{\binom{\nu-\frac{1}{2}}{N+1}}\right|\frac{1}{m-N-1}.
\end{aligned} \tag{3.20}$$

Using Lemmas 2.2, 2.3, and (6.2) along with the fact that $\frac{1}{\binom{N}{k}} \leq \frac{k}{N}$ for all $N > k$ and $\sqrt{\frac{1}{1-\frac{\nu}{m}}} \leq \sqrt{\nu+1}$ for all $m \geq \nu+1$, it follows that

$$\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{\binom{\nu-\frac{1}{2}}{N+1}}\right| \quad \leq \quad \begin{cases} \frac{1}{\pi}(N+1)\sqrt{\frac{\nu+1}{\nu-N-1}}\frac{1}{m^{3/2}}, & \text{if } \nu \geq N+2 \\ \frac{1}{\sqrt{\pi}}(N+1)\sqrt{\nu+1}\frac{1}{m^{3/2}}, & \text{if } \nu = N+1 \end{cases}. \tag{3.21}$$

For $\nu \geq N+2$,

$$|E^{\nu}_{N,3}(x)| \quad < \quad \left(\frac{1}{\pi\sqrt{2\pi}}(N+1)\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\sum_{m=\nu+1}^{\infty}\frac{1}{m^{3/2}(m-N-1)}\right)\frac{a_{N+1}(\nu)}{x^{N+1}}$$

(by (3.20) and (3.21))

$$
\begin{aligned}
&< \left(\frac{1}{\pi\sqrt{2\pi}}(N+1)\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\int_{\nu+1}^{\infty}\frac{1}{t^{3/2}(t-N-1)}\ dt\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&= \left(\frac{1}{\pi\sqrt{2\pi}}(N+1)\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\int_{\nu-N}^{\infty}\frac{1}{(t+N+1)^{3/2}\ t}\ dt\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&= \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\left(\frac{\sinh^{-1}\left(\sqrt{\frac{N+1}{\nu-N}}\right)}{\sqrt{N+1}}-\sqrt{\frac{1}{\nu+1}}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&< \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{(\nu+1)(\nu+N+\frac{3}{2})}{(\nu-N-1)}}\left(\sqrt{\frac{1}{\nu-N}}-\sqrt{\frac{1}{\nu+1}}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \qquad (3.22) \\
&\qquad \Big(\text{since } \operatorname{arcsinh}\ x<x \text{ for all } x>0\Big).
\end{aligned}
$$

On the other hand, for $\nu=N+1$, it follows that

$$
\begin{aligned}
|E_{N,3}^{\nu}(x)| &< \left(\frac{1}{\pi\sqrt{2}}(N+1)\sqrt{(\nu+1)(\nu+N+\frac{3}{2})}\sum_{m=\nu+1}^{\infty}\frac{1}{m^{3/2}(m-N-1)}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&\qquad \text{(by (3.20) and (3.21))} \\
&= \left(\frac{1}{\pi\sqrt{2}}(N+1)\sqrt{(N+2)(2N+5/2)}\sum_{m=N+2}^{\infty}\frac{1}{m^{3/2}(m-N-1)}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&< \left(\frac{1}{\pi\sqrt{2}}(N+1)\sqrt{(N+2)(2N+5/2)}\int_{N+2}^{\infty}\frac{1}{t^{3/2}(t-N-1)}\ dt\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&= \left(\frac{1}{\pi\sqrt{2}}(N+1)\sqrt{(N+2)(2N+5/2)}\int_{1}^{\infty}\frac{1}{(t+N+1)^{3/2}\ t}\ dt\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&= \frac{\sqrt{2}}{\pi}\sqrt{(N+2)(2N+5/2)}\left(\frac{\operatorname{arcsinh}(\sqrt{N+1})}{\sqrt{N+1}}-\sqrt{\frac{1}{N+2}}\right)\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&< \frac{\sqrt{2}}{\pi}\sqrt{2N+5/2}\Big(\sqrt{N+2}-1\Big)\frac{a_{N+1}(\nu)}{x^{N+1}}. \qquad (3.23)
\end{aligned}
$$

Finally, (3.22) and (3.23) imply (3.19). □

Lemma 3.5. *Let $x\in\mathbb{R}_{\geq 1}$ and $(\nu,N)\in\mathbb{Z}_{\geq 2}\times\mathbb{Z}_{\geq 1}$ with $\nu\geq N+1$. Then with* (3.17) *and* (3.18) *one has,*

$$
|E(\nu,N,x)|<E_N^{\nu}\frac{a_{N+1}(\nu)}{x^{N+1}},
$$

with

$$
E_N^{\nu}=\begin{cases}1+E_{N+2}^{\nu}, & \text{if } \nu\geq N+2\\ 1+E_{N+1}^{\nu}, & \text{if } \nu=N+1\end{cases}. \qquad (3.24)
$$

Proof. For $\nu \geq N+1$,

$$\begin{aligned}
|E(\nu,N,x)| &\leq \Big|E_{\nu,N,1}(x)+E^{\nu}_{N,2}(x)+E^{\nu}_{N,3}(x)\Big| \quad \text{(by (3.1))} \\
&< \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}}\ \Big(\Gamma(\nu+N+\frac{3}{2},2x)+\gamma(\nu+N+\frac{3}{2},2x)\Big)+E^{\nu}_{N,3}\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&\qquad \text{(by (3.5), (3.11) and (3.19))} \\
&= \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}}\ \Gamma(\nu+N+\frac{3}{2})+E^{\nu}_{N,3}\frac{a_{N+1}(\nu)}{x^{N+1}} \\
&= (1+E^{\nu}_{N,3})\frac{a_{N+1}(\nu)}{x^{N+1}}. \qquad (3.25)
\end{aligned}$$

From (3.19), we get (3.24). □

Lemma 3.6. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu,N) \in \mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 1}$ with $\nu \leq N$,*

$$|E_{\nu,N,1}(x)| < \frac{1}{\sqrt{2\pi}}E^{N}_{\nu,1}\frac{|a_{N+1}(\nu)|}{x^{N+1}}\sqrt{\nu+N+\frac{3}{2}}\ \ln(N+1),$$

with

$$E^{N}_{\nu,1} = \left(1+\frac{(2\nu+1)(\nu+2)}{\ln(N+1)}+\frac{(2\nu+1)(\nu+2)}{N+2}\right). \qquad (3.26)$$

Proof. From Lemma 3.1, for all $x \in \mathbb{R}_{\geq 1}$ and $(\nu,N) \in \mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 1}$ with $\nu \leq N$, we have

$$\begin{aligned}
|E_{\nu,N,1}(x)| &= \Big|\sum_{m=0}^{N}\phi^{\nu}_{m}(t;x)\Big| = \left|\int_{2x}^{\infty}\frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\sum_{m=0}^{N}\binom{\nu-\frac{1}{2}}{m}(-\theta)^m\,dt\right| \quad \text{where } \theta := \frac{t}{2x} \\
&\leq \int_{2x}^{\infty}\frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\left|\sum_{m=0}^{N}\binom{\nu-\frac{1}{2}}{m}(-\theta)^m\right|dt \\
&= 2(N+1)\left|\binom{\nu-\frac{1}{2}}{N+1}\right|\int_{2x}^{\infty}\frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\left|\sum_{m=0}^{N}\frac{1}{2m-2\nu+1}\binom{N}{m}\vartheta^m\right|dt \quad \text{(by (2.3))}, \qquad (3.27)
\end{aligned}$$

where $\vartheta := \theta - 1$.
Define

$$\mathcal{M}_N(\vartheta) = \sum_{m=0}^{N}\frac{1}{2m-2\nu+1}\binom{N}{m}\vartheta^m$$

and

$$S_N(\vartheta) = \sum_{m=0}^{N}\frac{1}{2m+2}\binom{N}{m}\vartheta^m = \frac{(\vartheta+1)^{N+1}-1}{2\vartheta(N+1)}. \qquad (3.28)$$

Consequently,

$$\frac{\Big|\mathcal{M}_N(\vartheta)-S_N(\vartheta)\Big|}{S_N(\vartheta)} = \frac{1}{S_N(\vartheta)}(2\nu+1)\left|\sum_{m=0}^{N}\frac{1}{(2m-2\nu+1)(2m+2)}\binom{N}{m}\vartheta^m\right|$$

$$
\begin{aligned}
&\le \frac{1}{S_N(\vartheta)}\frac{(2\nu+1)(\nu+2)}{2}\sum_{m=0}^{N}\frac{1}{(m+1)(m+2)}\binom{N}{m}\vartheta^m \\
&\quad \left(\left|\frac{1}{2m-2\nu+1}\right| \le \frac{\nu+2}{m+2} \text{ for all integers } \nu, m \text{ with } 0\le \nu, m\le N\right) \\
&= \frac{(2\nu+1)(\nu+2)}{(\vartheta+1)^{N+1}-1}\left(\frac{(\vartheta+1)^{N+2}-\vartheta(N+2)-1}{\vartheta(N+2)}\right) \\
&\quad \left(\text{because } \sum_{m=0}^{N}\frac{1}{(m+1)(m+2)}\binom{N}{m}\vartheta^m = \frac{(\vartheta+1)^{N+2}-\vartheta(N+2)-1}{\vartheta^2(N+1)(N+2)}\right) \\
&= \frac{(2\nu+1)(\nu+2)}{N+2}\left(1+\frac{\sum_{i=0}^{N}\theta^i-(N+1)}{\theta^{N+1}-1}\right) \quad (\text{by replacing } \vartheta+1=\theta).
\end{aligned} \tag{3.29}
$$

Therefore,

$$
\begin{aligned}
\Big|\mathcal{M}_N(\vartheta)\Big| &= S_N(\vartheta)\frac{\Big|\mathcal{M}_N(\vartheta)-S_N(\vartheta)+S_N(\vartheta)\Big|}{S_N(\vartheta)} \\
&\le S_N(\vartheta)\left(1+\frac{\Big|\mathcal{M}_N(\vartheta)-S_N(\vartheta)\Big|}{S_N(\vartheta)}\right) \\
&= \frac{1}{2(N+1)}\frac{\theta^{N+1}-1}{\theta-1}\left(1+\frac{\Big|\mathcal{M}_N(\vartheta)-S_N(\vartheta)\Big|}{S_N(\vartheta)}\right) \\
&\le \frac{1}{2(N+1)}\frac{\theta^{N+1}-1}{\theta-1}\left(1+\frac{(2\nu+1)(\nu+2)}{N+2}\left(1+\frac{\sum_{i=0}^{N}\theta^i-(N+1)}{\theta^{N+1}-1}\right)\right) \quad (\text{by } (3.29)) \\
&= \frac{1}{2(N+1)}\left(1+\frac{(2\nu+1)(\nu+2)}{N+2}\right)\sum_{i=0}^{N}\theta^i + \frac{1}{2(N+1)}\frac{(2\nu+1)(\nu+2)}{N+2}\sum_{i=0}^{N-1}(N-i)\theta^i.
\end{aligned} \tag{3.30}
$$

Now,

$$
\begin{aligned}
|E_{\nu,N,1}(x)| &\le \left|\binom{\nu-\frac{1}{2}}{N+1}\right|\left(1+\frac{(2\nu+1)(\nu+2)}{N+2}\right)\int_{2x}^{\infty}\frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\sum_{i=0}^{N}\theta^i\ dt \\
&+ \left|\binom{\nu-\frac{1}{2}}{N+1}\right|\frac{(2\nu+1)(\nu+2)}{N+2}\int_{2x}^{\infty}\frac{e^{-t}t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\sum_{i=0}^{N-1}(N-i)\theta^i\ dt \quad (\text{by } (3.27) \text{ and } (3.30)) \\
&= \left|\binom{\nu-\frac{1}{2}}{N+1}\right|\left(1+\frac{(2\nu+1)(\nu+2)}{N+2}\right)\sum_{i=0}^{N}\frac{1}{\Gamma(\nu+\frac{1}{2})(2x)^i}\int_{2x}^{\infty}e^{-t}t^{\nu+i-\frac{1}{2}}\ dt \\
&+ \left|\binom{\nu-\frac{1}{2}}{N+1}\right|\frac{(2\nu+1)(\nu+2)}{N+2}\sum_{i=0}^{N-1}\frac{N-i}{\Gamma(\nu+\frac{1}{2})(2x)^i}\int_{2x}^{\infty}e^{-t}t^{\nu+i-\frac{1}{2}}\ dt,
\end{aligned} \tag{3.31}
$$

where $\theta = \frac{t}{2x}$.

In order to estimate the two sums with integrals on the right hand side of (3.31), define

$$I_1(\nu, N, x) \quad = \quad \left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right) \sum_{i=0}^{N} \frac{1}{\Gamma(\nu+\frac{1}{2})(2x)^i} \int_{2x}^{\infty} e^{-t} t^{\nu+i-\frac{1}{2}} \ dt$$

and

$$I_2(\nu, N, x) \quad = \quad \frac{(2\nu+1)(\nu+2)}{N+2} \sum_{i=0}^{N-1} \frac{N-i}{\Gamma(\nu+\frac{1}{2})(2x)^i} \int_{2x}^{\infty} e^{-t} t^{\nu+i-\frac{1}{2}} \ dt.$$

Applying the substitution $(x, y) \mapsto (t, \nu + N + \frac{3}{2})$ in (2.1), it follows that

$$\begin{aligned} I_1(\nu, N, x) \quad &< \quad \left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right) \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}} \sum_{i=0}^{N} \frac{1}{(2x)^i} \int_{2x}^{\infty} \frac{1}{t^{N-i+2}} \ dt \\ &= \quad \left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right) \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}\ (2x)^{N+1}} \sum_{i=0}^{N} \frac{1}{N-i+1} \\ &< \quad \left(1 + \frac{(2\nu+1)(\nu+2)}{N+2}\right) \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}\ (2x)^{N+1}} \ \ln(N+1), \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} I_2(\nu, N, x) \quad &< \quad \frac{(2\nu+1)(\nu+2)}{N+2} \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}} \sum_{i=0}^{N-1} \frac{N-i}{(2x)^i} \int_{2x}^{\infty} \frac{1}{t^{N-i+2}} \ dt \\ &= \quad \frac{(2\nu+1)(\nu+2)}{N+2} \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}\ (2x)^{N+1}} \sum_{i=0}^{N-1} \frac{N-i}{N-i+1} \\ &< \quad (2\nu+1)(\nu+2) \frac{(\nu+\frac{1}{2})_{N+1}\sqrt{\nu+N+\frac{3}{2}}}{\sqrt{2\pi}\ (2x)^{N+1}}. \end{aligned} \tag{3.33}$$

By (3.31), (3.32) and (3.33), we obtain

$$|E_{\nu,N,1}(x)| < \frac{1}{\sqrt{2\pi}} E_{\nu,1}^N \ \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}} \ \ln(N+1).$$

□

Lemma 3.7. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\nu \leq N$,*

$$|E_{\nu,2}^N(x)| < \left(\sqrt{2} + \frac{1}{\sqrt{\nu+N+\frac{3}{2}}}\right) \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu+N+\frac{3}{2}}. \tag{3.34}$$

Proof. From Lemma 3.1, we get

$$\begin{aligned}
\left|E_{\nu,2}^{N}(x)\right| &= \Big|\sum_{m=N+1}^{\infty}\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{(2x)^m\Gamma(\nu+\frac{1}{2})}\int_0^{2x} e^{-t}t^{\nu+m-\frac{1}{2}}\ dt\Big| \\
&\leq \frac{\left|\binom{\nu-\frac{1}{2}}{N+1}(-1)^{N+1}\right|}{(2x)^{N+1}\Gamma(\nu+\frac{1}{2})}\int_0^{2x} e^{-t}t^{\nu+N+\frac{1}{2}}\ dt+\sum_{m=N+2}^{\infty}\frac{\left|(-1)^m\binom{\nu-\frac{1}{2}}{m}\right|}{(2x)^m\Gamma(\nu+\frac{1}{2})}\int_0^{2x} e^{-t}t^{\nu+m-\frac{1}{2}}\ dt \\
&< \frac{|a_{N+1}(\nu)|}{x^{N+1}}+\frac{1}{\sqrt{2\pi}}\frac{|a_{N+1}(\nu)|}{x^{N+1}}\sqrt{\nu+N+\frac{3}{2}}\sum_{m=N+2}^{\infty}\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{(-1)^{N+1}\binom{\nu-\frac{1}{2}}{N+1}}\right|\frac{1}{m-N-1} \\
&\qquad \Big(\text{by the substitution } (x,y)\mapsto(t,\nu+N+\frac{3}{2}) \text{ in } (2.1)\Big),
\end{aligned} \tag{3.35}$$

and using Lemma 2.3, it follows that

$$\begin{aligned}
\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{(-1)^{N+1}\binom{\nu-\frac{1}{2}}{N+1}}\right| &= \frac{4^{N+1}}{4^m}\frac{\binom{2m-2\nu}{m-\nu}}{\binom{2N+2-2\nu}{N+1-\nu}}\frac{\binom{N+1}{\nu}}{\binom{m}{\nu}} \\
&\leq \frac{2}{\sqrt{\pi}}\sqrt{\frac{N+1-\nu}{m-\nu}}\frac{\binom{N+1}{\nu}}{\binom{m}{\nu}} \quad (\text{by } (6.2)) \\
&< \frac{2}{\sqrt{\pi}}\sqrt{\frac{N+1}{m}} \\
&\quad \left(\text{since } \binom{N+1}{\nu}<\binom{m}{\nu} \text{ and } \sqrt{\frac{1}{1-\frac{\nu}{m}}}\leq\sqrt{\frac{N+1}{N+1-\nu}} \text{ for all } m\geq N+2\right).
\end{aligned} \tag{3.36}$$

Using (3.35) and (3.36), we see that

$$\begin{aligned}
\left|E_{\nu,2}^{N}(x)\right| &< \frac{|a_{N+1}(\nu)|}{x^{N+1}}+\frac{\sqrt{2}}{\pi}\frac{|a_{N+1}(\nu)|}{x^{N+1}}\sqrt{\nu+N+\frac{3}{2}}\sqrt{N+1}\sum_{m=N+2}^{\infty}\frac{1}{\sqrt{m}\ (m-N-1)} \\
&< \frac{|a_{N+1}(\nu)|}{x^{N+1}}+\frac{\sqrt{2}}{\pi}\frac{|a_{N+1}(\nu)|}{x^{N+1}}\sqrt{\nu+N+\frac{3}{2}}\int_1^{\infty}\frac{\sqrt{N+1}}{t\ \sqrt{t+N+1}}\ dt \\
&= \frac{|a_{N+1}(\nu)|}{x^{N+1}}+\frac{2\sqrt{2}}{\pi}\frac{|a_{N+1}(\nu)|}{x^{N+1}}\sqrt{\nu+N+\frac{3}{2}}\left(\frac{\pi}{2}-\tan^{-1}\Big(\frac{1}{\sqrt{N+1}}\Big)\right) \\
&< \frac{|a_{N+1}(\nu)|}{x^{N+1}}+\sqrt{2}\frac{|a_{N+1}(\nu)|}{x^{N+1}}\sqrt{\nu+N+\frac{3}{2}} \\
&= \left(\sqrt{2}+\frac{1}{\sqrt{\nu+N+\frac{3}{2}}}\right)\frac{|a_{N+1}(\nu)|}{x^{N+1}}\sqrt{\nu+N+\frac{3}{2}}.
\end{aligned} \tag{3.37}$$

□

From (3.26) and (3.34), we have the final estimation for the error term with $\nu\leq N$, presented in the following lemma.

Lemma 3.8. *Let $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\nu \leq N$. Then with $E_{\nu,1}^N$ as in* (3.26),

$$|E(\nu, N, x)| < E_\nu^N \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}} \ \ln(N+1),$$

with

$$E_\nu^N = \frac{1}{\sqrt{2\pi}} E_{\nu,1}^N + \frac{1}{\ln(N+1)} \left(\sqrt{2} + \frac{1}{\sqrt{\nu + N + \frac{3}{2}}} \right). \tag{3.38}$$

Finally from Lemmas 3.5 and 3.8, we can bound the error term $E(\nu, N, x)$, given in (3.1), as follows.

Theorem 3.9. *Let $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. Then using definitions* (3.17)-(3.18) *and* (3.38),

$$\Big| \frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} \Big| < E(\nu, N) \frac{|a_{N+1}(\nu)|}{x^{N+1}},$$

with

$$E(\nu, N) = \begin{cases} 1 + E_{N+2}^\nu, & \text{if } \nu \geq N+2 \\ 1 + E_{N+1}^\nu, & \text{if } \nu = N+1 \\ E_\nu^N, & \text{if } \nu \leq N \end{cases}. \tag{3.39}$$

Corollary 3.10. *For $\nu \in \mathbb{Z}_{\geq 0}$, $N = 3$ and $x \in \mathbb{R}_{\geq 1}$,*

$$\Big| \frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{3} \frac{(-1)^m a_m(\nu)}{x^m} \Big| < E(\nu, 3, x),$$

with

$$E(\nu, 3, x) = \begin{cases} \frac{\nu^8}{382 x^4}, & \text{if } \nu \geq 4 \\ \frac{\nu^8}{86 x^4}, & \text{if } \nu = 3 \\ \frac{\nu^8}{25 x^4}, & \text{if } \nu = 2 \\ \frac{12\nu^8}{5 x^4}, & \text{if } \nu = 1 \\ \frac{1}{x^4}, & \text{if } \nu = 0 \end{cases}. \tag{3.40}$$

Proof. It suffices to estimate $E(\nu, N)|a_{N+1}(\nu)|$ for $N = 3$, defined in (3.39). For $\nu \in \{0, 1, 2, 3, 4\}$ and $N = 3$, by numerical checking in Mathematica, we confirm that

$$E(\nu, 3)|a_4(\nu)| < \begin{cases} \frac{\nu^8}{382}, & \text{if } \nu = 4 \\ \frac{\nu^8}{86}, & \text{if } \nu = 3 \\ \frac{\nu^8}{25}, & \text{if } \nu = 2 \\ \frac{12\nu^8}{5}, & \text{if } \nu = 1 \\ 1, & \text{if } \nu = 0 \end{cases}.$$

For the remaining case $\nu \geq 5$ we checked by Mathematica that $E(\nu, 3)|a_4(\nu)| < \frac{\nu^8}{382}$; see Subsection 6.2. □

4. Inequalities for Modified Bessel function of half-integral order

The section establishes inequalities for $I_{\nu+1/2}(x)$ with $\nu \in \mathbb{Z}_{\geq 2}$ and $x \in \mathbb{R}_{\geq 1}$. Again we use short hand notations from Section 3 as (A), (PHI), (PSI), etc. From (2.10) we obtain the asymptotic expansion of $\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x)$ in the form $\sum_{m=0}^{\infty} (-1)^m a_m(\nu + \frac{1}{2})/x^m$. Following a similar treatment as worked out in the proof of Lemma 3.1, we truncate the infinite series $\sum_{m=0}^{\infty} \psi_m^{\nu+1/2}(t;x)$ at some point $N > 0$ and consequently obtain two remainder terms, namely,

$$-\sum_{m=0}^{N} \phi_m^{\nu+1/2}(t;x) + \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x); \tag{4.1}$$

see also (4.2). Our next step is to obtain an upper bound of the absolute value of the remainder term by estimating the two finite sums (4.1). Using Lemma 2.8 (resp. 2.9), we obtain (4.5) (resp. (4.11)). Lemma 4.4 and 4.5 together imply Theorem 4.6 which introduces an infinite family of inequalities for $\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x)$.

Recalling (2.11), note that

$$a_m(\nu + 1/2) = \frac{\binom{\nu}{m}(\nu+1)_m}{2^m}.$$

Lemma 4.1. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$,*

$$\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu+1/2)}{(2x)^m} = -\sum_{m=0}^{N} \phi_m^{\nu+1/2}(t;x) + \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x). \tag{4.2}$$

Proof. From (2.10) it follows that

$$\begin{aligned}
\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) &= \sum_{m=0}^{\infty} \psi_m^{\nu+1/2}(t;x) \quad \Big(\text{by substitution } \nu \mapsto \nu + \frac{1}{2}\Big) \\
&= \sum_{m=0}^{\nu} \psi_m^{\nu+1/2}(t;x) \quad \Big(\text{as } \nu \in \mathbb{Z}_{\geq 2} \text{ and } \binom{\nu}{m} = 0 \text{ for } m > \nu\Big) \\
&= \sum_{m=0}^{N} \psi_m^{\nu+1/2}(t;x) + \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x) \\
&= \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu+1/2)}{(2x)^m} - \sum_{m=0}^{N} \phi_m^{\nu+1/2}(t;x) + \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x).
\end{aligned}$$

□

Remark 4.2. *From* (4.2)*, it is clear that throughout the rest of the section we have to consider the case $\nu \geq N$. This is because $\binom{\nu}{N} = 0$ for $\nu < N$ as pointed out in the proof of Lemma 4.1.*

We present identity (4.3) which serves for the error analysis for $N \in \mathbb{Z}_{\geq 1}$. To this end, following the Remark 4.2, we consider $\nu \in \mathbb{Z}_{\geq 2}$.

Lemma 4.3. *For $x \in \mathbb{R}_{\geq 1}$ and $\nu \in \{0, 1\}$,*

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x \quad \text{and} \quad I_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \Big(\cosh x - \frac{1}{x} \sinh x \Big). \tag{4.3}$$

Proof. We observe that

$$\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) = \sum_{m=0}^{\nu} \psi_m^{\nu+1/2}(t; x). \tag{4.4}$$

For $\nu = 0$ in (4.4), it follows that

$$I_{1/2}(x) = \frac{e^x}{\sqrt{2\pi x}} \int_0^{2x} e^{-t} \ dt = \sqrt{\frac{2}{\pi x}} \sinh x,$$

and for $\nu = 1$,

$$\begin{aligned} I_{3/2}(x) &= \frac{e^x}{\sqrt{2\pi x}} \Big(\int_0^{2x} e^{-t} t \ dt - \frac{1}{2x} \int_0^{2x} e^{-t} t^2 \ dt \Big) \\ &= \frac{e^x}{\sqrt{2\pi x}} \Big(\Big(1 - \frac{1}{x}\Big) + e^{-2x} \Big(1 + \frac{1}{x}\Big) \Big) = \sqrt{\frac{2}{\pi x}} \Big(\cosh x - \frac{1}{x} \sinh x \Big). \end{aligned}$$

□

Lemma 4.4. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$,*

$$\Big| - \sum_{m=0}^{N} \phi_m^{\nu+1/2}(t; x) \Big| < \frac{\binom{\nu}{N+1}}{\nu! (2x)^{N+1}} \ \Gamma(\nu + N + 2, 2x), \tag{4.5}$$

where Γ is the incomplete gamma function from (1.7).

Proof. From Lemma 4.1, for all $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ we have

$$\begin{aligned} - \sum_{m=0}^{N} \phi_m^{\nu+1/2}(t; x) &= - \int_{2x}^{\infty} \frac{e^{-t} t^{\nu}}{\nu!} \sum_{m=0}^{N} \binom{\nu}{m} \Big(- \frac{t}{2x} \Big)^m \ dt \\ &= - \int_{2x}^{\infty} \frac{e^{-t} t^{\nu}}{\nu!} \sum_{m=0}^{N} \binom{\nu}{m} (-\theta)^m \ dt \quad \text{where } \theta := \frac{t}{2x}. \end{aligned} \tag{4.6}$$

We first consider the case where N is an even positive integer. Then

$$- \binom{\nu}{N+1} \theta^{N+1} < - \sum_{m=0}^{N} \binom{\nu}{m} (-\theta)^m < 0 \quad \text{(by (2.8))}. \tag{4.7}$$

Now from (4.6) and (4.7), it follows that

$$- \phi_{N+1}^{\nu+1/2}(t; x) < - \sum_{m=0}^{N} \phi_m^{\nu+1/2}(t; x) < 0. \tag{4.8}$$

If N is an odd positive integer, it is immediate that

$$0 < - \sum_{m=0}^{N} \binom{\nu}{m} (-\theta)^m < \binom{\nu}{N+1} \theta^{N+1} \quad \text{(by (2.8))}. \tag{4.9}$$

By (4.6) and (4.9), we obtain

$$0 < -\sum_{m=0}^{N} \phi_m^{\nu+1/2}(t;x) < \phi_{N+1}^{\nu+1/2}(t;x). \tag{4.10}$$

(4.8) and (4.10) together imply (4.5). □

Lemma 4.5. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$,*

$$\Big|\sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x)\Big| < \frac{\binom{\nu}{N+1}}{\nu!(2x)^{N+1}}\ \gamma(\nu+N+2, 2x), \tag{4.11}$$

where γ is the incomplete gamma function from (1.6).

Proof. By Lemma 4.1, for all $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{>1} \times \mathbb{Z}_{>0}$, it follows that

$$\begin{aligned}\sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x) &= \int_0^{2x} \frac{e^{-t}t^{\nu}}{\nu!} \sum_{m=N+1}^{\nu} \binom{\nu}{m}\Big(-\frac{t}{2x}\Big)^m\, dt \\ &= \int_0^{2x} \frac{e^{-t}t^{\nu}}{\nu!} \sum_{m=N+1}^{\nu} \binom{\nu}{m}(-\theta)^m\, dt \ \text{ where } \theta := \frac{t}{2x}.\end{aligned} \tag{4.12}$$

If N is an even positive integer, then it follows that

$$-\binom{\nu}{N+1}\theta^{N+1} < \sum_{m=N+1}^{\nu} \binom{\nu}{N}(-\theta)^m < 0 \ \text{ (by (2.9))}. \tag{4.13}$$

Consequently, from (4.12) and (4.13) it is immediate that

$$-\psi_{N+1}^{\nu+1/2}(t;x) < \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x) < 0. \tag{4.14}$$

Similarly, if N is an odd positive integer we get

$$0 < \sum_{m=N+1}^{\nu} \binom{\nu}{N}(-\theta)^m < \binom{\nu}{N+1}\theta^{N+1} \ \text{ (by (2.9))}. \tag{4.15}$$

Equations (4.12) and (4.15) lead to the following inequality

$$0 < \sum_{m=N+1}^{\nu} \psi_m^{\nu+1/2}(t;x) < \psi_{N+1}^{\nu+1/2}(t;x). \tag{4.16}$$

Putting (4.14) and (4.16) together gives (4.11). □

Theorem 4.6. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$,*

$$\Big|\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu+1/2)}{(2x)^m}\Big| < \frac{a_{N+1}(\nu+\frac{1}{2})}{x^{N+1}}. \tag{4.17}$$

Proof. For $\nu = N$, we observe that the (4.2) of Lemma 4.1 reduces to

$$\frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu+1/2)}{(2x)^m} = -\sum_{m=0}^{N} \phi_m^{\nu+1/2}(t;x).$$

From (4.5), it follows that

$$\begin{aligned}\left| \frac{\sqrt{2\pi x}}{e^x} I_{\nu+1/2}(x) - \sum_{m=0}^{N} \frac{(-1)^m \binom{\nu}{m} (\nu+1)_m}{(2x)^m} \right| &< \frac{\binom{\nu}{N+1}}{\nu!(2x)^{N+1}}\ \Gamma(\nu+N+2, 2x) \\ &< \frac{\binom{\nu}{N+1}}{\nu!(2x)^{N+1}} \Gamma(\nu+N+2) \\ &= \frac{a_{N+1}(\nu+1/2)}{x^{N+1}}.\end{aligned}$$

Whereas for $\nu > N$, combining (4.5) and (4.11), we arrive at (4.17). □

5. Conclusion

We have studied the error analysis for $I_\nu(x)$, where ν either is a non-negative integer or a non-negative half integer. Our major results are the inequalities presented in Theorems 3.9 and 4.6. The main objective of this section is to carry out similar considerations as done in Section 3 but for $\nu \in \mathbb{R}_{\geq 0}$.

For $\nu \in \mathbb{R}_{\geq 0}$, define

$$\begin{aligned}\widetilde{E}_{\nu,N,1}(x) &= -\sum_{m=0}^{N} \phi_m^\nu(t;x), \\ \widetilde{E}^\nu_{N,2}(x) &= \sum_{m=N+1}^{\lfloor \nu \rfloor} \psi_m^\nu(t;x), \\ \widetilde{E}^\nu_{N,3}(x) &= \sum_{m=\lfloor \nu \rfloor+1}^{\infty} \psi_m^\nu(t;x),\end{aligned}$$

and

$$\widetilde{E}^N_{\nu,2}(x) = \sum_{m=N+1}^{\infty} \psi_m^\nu(t;x).$$

Throughout this section, for a given $x \in \mathbb{R}$, we follow the standard notation $\lfloor x \rfloor$ (resp. $\{x\}$) to denote integer part (resp. fractional part) of x. We split $E(\nu, N, x)$ depending on whether $\lfloor \nu \rfloor \geq N+1$ or $\lfloor \nu \rfloor \leq N$, as stated in Lemma 5.1. In (5.2), (5.3) and (5.4), we obtain upper bounds for $\widetilde{E}_{\nu,n,1}(x), \widetilde{E}^\nu_{N,2}(x)$, and $\widetilde{E}^\nu_{N,3}$ when $\lfloor \nu \rfloor \geq N+1$. From Lemmas 5.2, 5.3 and 5.4, we get an upper bound for $|\widetilde{E}(\nu, N, x)|$ in Lemma 5.5.

For $\lfloor \nu \rfloor \leq N$, we obtain an upper bound for $|E(\nu, N, x)|$ in Lemma 5.9 as a straightforward implication of Lemmas 5.6 and 5.8.

Lemmas 5.5 and 5.9 give rise to Theorem 5.10 for all $\nu \in \mathbb{R}_{\geq 0}$. Restricting $\nu \in \mathbb{Z}_{\geq 0}$ (resp. $\nu \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$) in (5.12), we retrieve Theorem 3.9 (resp. Theorem 4.6).

Lemma 5.1. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,*

$$\frac{\sqrt{2\pi x}}{e^x} I_\nu(x) - \sum_{m=0}^{N} \frac{(-1)^m a_m(\nu)}{x^m} = \widetilde{E}(\nu, N, x),$$

with

$$\widetilde{E}(\nu, N, x) = \begin{cases} \widetilde{E}_{\nu,N,1}(x) + \widetilde{E}^{\nu}_{N,2}(x) + \widetilde{E}^{\nu}_{N,3}(x), & \text{if } \lfloor \nu \rfloor \geq N+1 \\ \widetilde{E}_{\nu,N,1}(x) + \widetilde{E}^{N}_{\nu,2}(x), & \text{if } \lfloor \nu \rfloor \leq N \end{cases}, \tag{5.1}$$

and $a_m(\nu)$ be as in (2.11).

Lemma 5.2. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \geq N+1$,*

$$|\widetilde{E}_{\nu,N,1}(x)| < \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}}\ \Gamma(\nu+N+\frac{3}{2}, 2x), \tag{5.2}$$

where Γ is the incomplete gamma function from (1.7).

Proof. Analogous to the proof of Lemma 3.2, by Lemma 2.6, we have

$$(-1)^N \sum_{m=0}^{N} \binom{\nu-\frac{1}{2}}{m} (-\alpha)^m > 0,$$

which is also valid for $\nu \in \mathbb{R}_{\geq 2}$. This is due to the fact that Lemma 2.6 is an immediate implication of Lemma 2.5 which holds for all $\nu \in \mathbb{R}_{\geq 0}$. □

Lemma 5.3. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \geq N+1$,*

$$|\widetilde{E}^{\nu}_{N,2}(x)| < \frac{\binom{\nu-\frac{1}{2}}{N+1}}{\Gamma(\nu+\frac{1}{2})(2x)^{N+1}}\ \gamma(\nu+N+\frac{3}{2}, 2x), \tag{5.3}$$

where γ is the incomplete gamma function from (1.6).

Proof. For $\lfloor \nu \rfloor \geq N+1$,

$$\widetilde{E}^{\nu}_{N,2}(x) = \int_0^{2x} \frac{e^{-t} t^{\nu-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \sum_{m=N+1}^{\lfloor \nu \rfloor} \binom{\nu-\frac{1}{2}}{m} (-\theta)^m dt \quad \text{where } \theta := \frac{t}{2x}.$$

Following up the proof of Lemma 3.3, we observe that it remains to prove for all $\nu \in \mathbb{R}_{\geq 2}$ and $\theta \in (0, 1]$,

$$S(\lfloor \nu \rfloor) := (-1)^N \sum_{m=N+1}^{\lfloor \nu \rfloor} \binom{\nu-\frac{1}{2}}{m} (-\alpha)^m < 0.$$

Using the Paule-Schorn [17] package fastZeil, we obtain

$$S(\lfloor \nu \rfloor + 1) = (1-\theta) S(\lfloor \nu \rfloor) - \theta^{N+1} \binom{\nu-\frac{1}{2}}{N} - (-1)^{\lfloor \nu \rfloor + N} \theta^{N+1} \binom{\nu-\frac{1}{2}}{\lfloor \nu \rfloor + 1}.$$

The rest of the proof is analogous to the proof of Lemma 2.7. □

For the statements of Lemmas 5.4-5.9 and of Theorem 5.10 we use the following definitions,

$$\widetilde{E}^{\nu}_{N+1} = \frac{\sqrt{2}}{\pi}\sqrt{2N+\frac{5}{2}+\{\nu\}}\Big(\sqrt{N+2}-1\Big)\prod_{i=0}^{\lfloor\nu\rfloor-1}\left(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\right)$$

and

$$\widetilde{E}^{\nu}_{N+2} = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{\lfloor\nu\rfloor+1}{\lfloor\nu\rfloor-N-1}}\sqrt{\nu+N+\frac{3}{2}}\left(\sqrt{\frac{1}{\lfloor\nu\rfloor-N}}-\sqrt{\frac{1}{\lfloor\nu\rfloor+1}}\right)\prod_{i=0}^{\lfloor\nu\rfloor-1}\left(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\right).$$

Lemma 5.4. *For $x\in\mathbb{R}_{\geq 1}$ and $(\nu,N)\in\mathbb{R}_{\geq 2}\times\mathbb{Z}_{\geq 1}$ with $\lfloor\nu\rfloor\geq N+1$,*

$$|\widetilde{E}^{\nu}_{N,3}(x)| < \widetilde{E}^{\nu}_{N,3}\frac{a_{N+1}(\nu)}{x^{N+1}},$$

with

$$\widetilde{E}^{\nu}_{N,3} = \begin{cases}\widetilde{E}^{\nu}_{N+2}, & \textit{if } \lfloor\nu\rfloor\geq N+2\\ \widetilde{E}^{\nu}_{N+1}, & \textit{if } \lfloor\nu\rfloor = N+1\end{cases}. \tag{5.4}$$

Proof. Similar to (3.20) we get

$$|\widetilde{E}^{\nu}_{N,3}(x)| < \frac{1}{\sqrt{2\pi}}\frac{a_{N+1}(\nu)}{x^{N+1}}\sqrt{\nu+N+\frac{3}{2}}\sum_{m=\nu+1}^{\infty}\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{\binom{\nu-\frac{1}{2}}{N+1}}\right|\frac{1}{m-N-1}. \tag{5.5}$$

Now,

$$\begin{aligned}\left|\frac{(-1)^m\binom{\nu-\frac{1}{2}}{m}}{\binom{\nu-\frac{1}{2}}{N+1}}\right| &= \left|\frac{(-1)^m\binom{\lfloor\nu\rfloor-\frac{1}{2}}{m}}{\binom{\lfloor\nu\rfloor-\frac{1}{2}}{N+1}}\frac{\prod_{i=0}^{m-1}\Big(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\Big)}{\prod_{i=0}^{N}\Big(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\Big)}\right|\\ &= \left|\frac{(-1)^m\binom{\lfloor\nu\rfloor-\frac{1}{2}}{m}}{\binom{\lfloor\nu\rfloor-\frac{1}{2}}{N+1}}\right|\frac{\prod_{i=0}^{\lfloor\nu\rfloor-1}\Big(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\Big)\Big|(1-2\{\nu\})\Big|\prod_{i=1}^{m-\lfloor\nu\rfloor-1}\Big(1-\frac{\{\nu\}}{i+\frac{1}{2}}\Big)}{\prod_{i=0}^{N}\Big(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\Big)}\\ &< \left|\frac{(-1)^m\binom{\lfloor\nu\rfloor-\frac{1}{2}}{m}}{\binom{\lfloor\nu\rfloor-\frac{1}{2}}{N+1}}\right|\prod_{i=0}^{\lfloor\nu\rfloor-1}\left(1+\frac{\{\nu\}}{\lfloor\nu\rfloor-i-\frac{1}{2}}\right).\end{aligned} \tag{5.6}$$

Applying (3.21) with the substitution $\nu\mapsto\lfloor\nu\rfloor$, it follows that

$$\left|\frac{(-1)^m\binom{\lfloor\nu\rfloor-\frac{1}{2}}{m}}{\binom{\lfloor\nu\rfloor-\frac{1}{2}}{N+1}}\right| \leq \begin{cases}\frac{1}{\pi}(N+1)\sqrt{\frac{\lfloor\nu\rfloor+1}{\lfloor\nu\rfloor-N-1}}\frac{1}{m^{3/2}}, & \text{if } \lfloor\nu\rfloor\geq N+2\\ \frac{1}{\sqrt{\pi}}(N+1)\sqrt{\lfloor\nu\rfloor+1}\frac{1}{m^{3/2}}, & \text{if } \lfloor\nu\rfloor = N+1\end{cases}. \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.5) and proceeding analogously as for the estimation worked out in (3.22) and (3.23), we get (5.4). □

Lemma 5.5. *For $x\in\mathbb{R}_{\geq 1}$ and $(\nu,N)\in\mathbb{R}_{\geq 2}\times\mathbb{Z}_{\geq 1}$ with $\lfloor\nu\rfloor\geq N+1$,*

$$|\widetilde{E}(\nu,N,x)| < \widetilde{E}^{\nu}_{N}\frac{a_{N+1}(\nu)}{x^{N+1}},$$

with

$$\widetilde{E}^{\nu}_{N} = \begin{cases} 1 + \widetilde{E}^{\nu}_{N+2}, & \text{if } \lfloor \nu \rfloor \geq N+2 \\ 1 + \widetilde{E}^{\nu}_{N+1}, & \text{if } \lfloor \nu \rfloor = N+1 \end{cases}. \tag{5.8}$$

Lemma 5.6. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \leq N$,*

$$|\widetilde{E}_{\nu,N,1}(x)| < \frac{1}{\sqrt{2\pi}} \widetilde{E}^{N}_{\nu,1} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}}\ \ln(N+1),$$

with

$$\widetilde{E}^{N}_{\nu,1} = \left(1 + \left(\frac{1}{\ln(N+1)} + \frac{1}{N+2}\right) \frac{(2\nu+1)(\lceil \nu \rceil + 2)}{|(1-2\{\nu\})|}\right). \tag{5.9}$$

Proof. For all $\nu \in \mathbb{R}_{\geq 0}$ with $\lfloor \nu \rfloor \leq N$ and non-negative integers $0 \leq m \leq N$, it follows that

$$\Big|\frac{1}{2m - 2\nu + 1}\Big| \leq \frac{\lceil \nu \rceil + 2}{m+2} \Big|\frac{1}{1 - 2\{\nu\}}\Big|.$$

For the rest, one can follow the same line of arguments as presented in the proof of Lemma 3.6. □

Remark 5.7. *Observe that on the right hand side of* (5.9)*, the term $|(1-2\{\nu\})|$ is in the denominator. The factor $(1-2\{\nu\})$ makes trouble if and only if $\{\nu\} = 1/2$. But as we have already pointed out in the Remark 4.2 that for half-integral order one has to consider only the case $\nu \geq N$. In short, for the $\lfloor \nu \rfloor \leq N$ case, it is being understood that $\{\nu\} \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.*

Lemma 5.8. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \leq N$,*

$$|\widetilde{E}^{\nu}_{N,2}(x)| < \left(\sqrt{2}\left|\frac{\prod_{i=0}^{\lfloor \nu \rfloor}\left(1 + \frac{\{\nu\}}{i - \frac{1}{2}}\right)}{\prod_{i=0}^{N - \lfloor \nu \rfloor}\left(1 - \frac{\{\nu\}}{i + \frac{1}{2}}\right)}\right| + \frac{1}{\sqrt{\nu + N + \frac{3}{2}}}\right) \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}}. \tag{5.10}$$

Proof. Analogous to (3.35), it follows that

$$\left|\widetilde{E}^{N}_{\nu,2}(x)\right| < \frac{|a_{N+1}(\nu)|}{x^{N+1}} + \frac{1}{\sqrt{2\pi}} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}} \sum_{m=N+2}^{\infty} \left|\frac{(-1)^m \binom{\nu - \frac{1}{2}}{m}}{(-1)^{N+1} \binom{\nu - \frac{1}{2}}{N+1}}\right| \frac{1}{m - N - 1}.$$

Therefore, it is sufficient to estimate $\left|\frac{(-1)^m \binom{\nu - \frac{1}{2}}{m}}{(-1)^{N+1} \binom{\nu - \frac{1}{2}}{N+1}}\right|$ to get (5.10). Similar to (5.6), we see that

$$\left|\frac{(-1)^m \binom{\nu - \frac{1}{2}}{m}}{(-1)^{N+1} \binom{\nu - \frac{1}{2}}{N+1}}\right| < \left|\frac{(-1)^m \binom{\lfloor \nu \rfloor - \frac{1}{2}}{m}}{(-1)^{N+1} \binom{\lfloor \nu \rfloor - \frac{1}{2}}{N+1}}\right| \left|\frac{\prod_{i=0}^{\lfloor \nu \rfloor}\left(1 + \frac{\{\nu\}}{i - \frac{1}{2}}\right)}{\prod_{i=0}^{N - \lfloor \nu \rfloor}\left(1 - \frac{\{\nu\}}{i + \frac{1}{2}}\right)}\right|.$$

□

Lemma 5.9. *For $x \in \mathbb{R}_{\geq 1}$ and $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ with $\lfloor \nu \rfloor \leq N$,*

$$|\widetilde{E}(\nu, N, x)| < \widetilde{E}^{N}_{\nu} \frac{|a_{N+1}(\nu)|}{x^{N+1}} \sqrt{\nu + N + \frac{3}{2}}\ \ln(N+1),$$

with

$$\widetilde{E}_{\nu}^{N} = \frac{1}{\sqrt{2\pi}}\widetilde{E}_{\nu,1}^{N} + \frac{1}{\ln(N+1)}\left(\sqrt{2}\left|\frac{\prod_{i=0}^{\lfloor\nu\rfloor}\left(1+\frac{\{\nu\}}{i-\frac{1}{2}}\right)}{\prod_{i=0}^{N-\lfloor\nu\rfloor}\left(1-\frac{\{\nu\}}{i+\frac{1}{2}}\right)}\right| + \frac{1}{\sqrt{\nu+N+\frac{3}{2}}}\right). \tag{5.11}$$

Combining Lemmas 5.5 and 5.9, we arrive at the following theorem.

Theorem 5.10. *For* $x \in \mathbb{R}_{\geq 1}$ *and* $(\nu, N) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 1}$,

$$\left|\frac{\sqrt{2\pi x}}{e^x}I_\nu(x) - \sum_{m=0}^{N}\frac{(-1)^m a_m(\nu)}{x^m}\right| < \widetilde{E}(\nu, N)\frac{|a_{N+1}(\nu)|}{x^{N+1}},$$

with

$$\widetilde{E}(\nu, N) = \begin{cases} 1+\widetilde{E}_{N+2}^{\nu}, & \text{if } \lfloor\nu\rfloor \geq N+2 \\ 1+\widetilde{E}_{N+1}^{\nu}, & \text{if } \lfloor\nu\rfloor = N+1 \\ \widetilde{E}_{\nu}^{N}, & \text{if } \lfloor\nu\rfloor \leq N \end{cases}. \tag{5.12}$$

6. Appendix

6.1. **Proofs of some lemmas presented in Section 2.**

Proof of Lemma 2.1: We define $f(x) = \frac{x^y}{e^x}$. Now $f'(x) = \frac{yx^{y-1}-x^y}{e^x}$ and $f'(x) = 0$ at $x = y$. Note that $f''(x) = \frac{y(y-1)x^{y-2}-2yx^{y-1}+x^y}{e^x}$ and consequently, $f''(y) = -\frac{y^{y-1}}{e^y} < 0$ for all $y \in \mathbb{R}_{>0}$. So, $f(x)$ attains it's maximum at $x = y$; i.e., $f(x) \leq f(y) = (\frac{y}{e})^y$.
From [14, eq. 5.6.1], we have

$$1 < (2\pi)^{-1/2}\ x^{1/2-x}\ e^x\ \Gamma(x) \text{ for } \ x \in \mathbb{R}_{>0}. \tag{6.1}$$

By the substitution $x \mapsto y$ in (6.1) and using the maximum value of $f(x)$, it follows that

$$\frac{\Gamma(y)}{\sqrt{\frac{2\pi}{y}}} > (\frac{y}{e})^y \geq f(x),$$

which implies (2.1).

∎

Proof of Lemma 2.2: For $m > \nu$,

$$\begin{aligned}\binom{\nu-\frac{1}{2}}{m} &= \frac{(2\nu-1)(2\nu-3)\dots(2\nu-2m+1)}{2^m\ m!} \\ &= \frac{\{(2\nu-1)(2\nu-3)\dots(2\nu-(2\nu-1))\}\{(-1)(-3)\dots(-(2m-2\nu-1))\}}{2^m\ m!} \\ &= \frac{1}{2^m\ m!}\frac{\nu!}{2^\nu}\binom{2\nu}{\nu}(-1)^{m-\nu}\frac{(2m-2\nu)!}{2^{m-\nu}\ (m-\nu)!} = \frac{(-1)^{m-\nu}}{4^m}\frac{\binom{2\nu}{\nu}\binom{2m-2\nu}{m-\nu}}{\binom{m}{\nu}},\end{aligned}$$

and for $m \le \nu$,

$$\begin{aligned}\binom{\nu-\frac{1}{2}}{m} &= \frac{(2\nu-1)(2\nu-3)\dots(2\nu-2m+1)}{2^m\ m!}\\ &= \frac{1}{2^m\ m!}\frac{(2\nu-1)(2\nu-3)\dots(2\nu-2m+1)(2\nu-2m)!}{(2\nu-2m)!}\\ &= \frac{1}{2^m\ m!\ (2\nu-2m)!}\frac{2\nu(2\nu-1)(2\nu-2)\dots(2\nu-2m+2)(2\nu-2m+1)(2\nu-2m)!}{2^m\ \nu(\nu-1)\dots(\nu-m+1)}\\ &= \frac{1}{2^m\ m!}\frac{(2\nu)!}{(2\nu-2m)!}\frac{(\nu-m)!}{\nu!} = \frac{1}{4^m}\frac{\binom{2\nu}{\nu}\binom{\nu}{m}}{\binom{2\nu-2m}{\nu-m}}.\end{aligned}$$

□

Proof of Lemma 2.3:

$$\begin{aligned}\sum_{m=k}^{N}(-1)^m\binom{\nu-\frac{1}{2}}{m}\binom{m}{k} &= \sum_{m=k}^{N}(-1)^m\binom{\nu-\frac{1}{2}}{k}\binom{\nu-\frac{1}{2}-k}{m-k} \quad \text{(by [8, (5.21)])}\\ &= \binom{\nu-\frac{1}{2}}{k}(-1)^k\sum_{m=0}^{N-k}\binom{\nu-\frac{1}{2}-k}{m}(-1)^m\\ &= (-1)^N\binom{\nu-\frac{1}{2}}{k}\binom{\nu-\frac{3}{2}-k}{N-k} \quad \text{(by [8, (5.16)])},\end{aligned}$$

and

$$\begin{aligned}2(-1)^{N+1}(N+1)\binom{\nu-\frac{1}{2}}{N+1}\frac{1}{2k-2\nu+1}\binom{N}{k} &= 2(-1)^{N+1}\frac{N-k+1}{2k-2\nu+1}\binom{\nu-\frac{1}{2}}{N+1}\binom{N+1}{k}\\ &= 2(-1)^{N+1}\frac{N-k+1}{2k-2\nu+1}\binom{\nu-\frac{1}{2}}{k}\binom{\nu-\frac{1}{2}-k}{N+1-k}\\ &\qquad \text{(by [8, (5.21)])}\\ &= (-1)^N\binom{\nu-\frac{1}{2}}{k}\binom{\nu-\frac{3}{2}-k}{N-k}.\end{aligned}$$

□

Proof of Lemma 2.4: First, observe that for all $n \in \mathbb{Z}_{\ge 1}$,

$$\frac{4^n}{2\sqrt{n}} \le \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}}. \tag{6.2}$$

Now for $m > \nu$,

$$\begin{aligned}\left|(-1)^m\binom{\nu-\frac{1}{2}}{m}\right| &= \frac{1}{4^m}\frac{\binom{2\nu}{\nu}\binom{2m-2\nu}{m-\nu}}{\binom{m}{\nu}} \quad \text{(by Lemma 2.2)}\\ &\le \frac{1}{4^m}\frac{4^\nu}{\sqrt{\pi\nu}}\frac{4^{m-\nu}}{\sqrt{\pi(m-\nu)}}\frac{1}{\binom{m}{\nu}} \quad \text{(by (6.2))}\\ &= \frac{1}{\pi\sqrt{\nu(m-\nu)}}\frac{1}{\binom{m}{\nu}},\end{aligned}$$

and for $m \le \nu$,

$$\left|(-1)^m \binom{\nu-\frac{1}{2}}{m}\right| = \frac{1}{4^m}\frac{\binom{2\nu}{\nu}\binom{\nu}{m}}{\binom{2\nu-2m}{\nu-m}} \quad \text{(by Lemma 2.2)}$$
$$\le \frac{1}{4^m}\frac{4^\nu}{\sqrt{\pi\nu}}\frac{2\sqrt{(\nu-m)}}{4^{\nu-m}}\binom{\nu}{m} \quad \text{(by (6.2))}$$
$$\le \frac{2}{\sqrt{\pi}}\binom{\nu}{m} \quad \left(\text{since } \sqrt{\frac{\nu-m}{\nu}} \le 1\right).$$

□

6.2. **Mathematica computation for the proof of Corollary 3.10.** We complete the proof of Corollary 3.10 by checking $E(\nu,3)a_4(\nu) < \frac{\nu^8}{382}$ for all $\nu \ge 5$ with Mathematica using Cylindrical Algebraic Decomposition [7]. In order to do this computation, our input for $a_m(\nu)$ (resp. for $E(\nu,N)$ with $\nu > N+1$) is $a[v,m]$ (resp. $E1[v,N]$) in Mathematica.

In[1]:= $\mathbf{a[v,m] := \frac{\text{Binomial}[v-\frac{1}{2},m]\ \text{Pochhammer}[v+\frac{1}{2},m]}{2^m}}$

In[2]:= $\mathbf{E1[v,N] := \left(1+\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{v+1}{v-N-1}}\sqrt{v+N+\frac{3}{2}}\left(\sqrt{\frac{1}{v-N}}-\sqrt{\frac{1}{v+1}}\right)\right)a[v,N+1]}$

In[3]:= $\mathbf{\text{CylindricalDecomposition}[\{\frac{v^8}{382} > E1[v,3], v \ge 5\}, v]}$

Out[3]= $v \ge 5$

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