COMPUTER ALGEBRA WITH THE FIFTH OPERATION: APPLICATIONS OF MODULAR FUNCTIONS TO PARTITION CONGRUENCES

by

NICOLAS ALLEN SMOOT

(Under the Direction of Peter Paule)

ABSTRACT

We give the implementation of an algorithm developed by Silviu Radu to compute examples of a wide variety of arithmetic identities originally studied by Ramanujan and Kolberg. Such identities employ certain finiteness conditions imposed by the theory of modular functions, and often yield interesting arithmetic information about the integer partition function $p(n)$, and other associated functions. We compute a large number of examples of such identities taken from contemporary research, often extending or improving existing results. We then use our implementation as a computational tool to help us achieve more theoretical results in the study of infinite congruence families. We finally describe a new method which extends the existing techniques for proving partition congruence families associated with a genus 0 modular curve.

Key Words: Integer Partitions, Modular Functions

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DEDICATION

Long before I began the writing of this dissertation, I had the distinction of taking a job as a file clerk. This was a very much unwelcome diversion in my career which I have the honor of sharing with both Albert Einstein and Srinivasa Ramanujan.

It left me to wonder: how many others are close to abandoning their careers due to their own hardships... and how many would contribute immeasurable value to the arts and sciences, if given the opportunity?

With this in mind, I wish to dedicate this thesis to every sculptor who has ever been forced to work in a rock quarry, to every historian who has been compelled to work in a department store, to every Shakespeare scholar who has taken a job in a bankruptcy firm, and indeed, to every other student of art or science who has been compelled to abandon their profession for a time, for the sake of food or shelter. Long may you fight in the ultimate service of your passion, and may your love of it never fail.

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As I complete this thesis, I have learned that the Austrian Science Fund has, in their exceeding generosity, chosen to support my research—largely in directions closely related to this thesis—for the next two years. No printed words will ever convey the happiness this has brought me, nor can I express my gratitude for the opportunity to continue this wonderful adventure with you all.

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TABLE OF CONTENTS

LIST OF TABLES

Table Page

LIST OF FIGURES

Figure | Page

LIST OF SYMBOLS

The following are used throughout the body of this paper:

$\mathbb{Z}$	Set of integers
$\mathbb{Z}_{\geq m}$	Set of integers greater than or equal to some m.
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{K}$	A field such that $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$
$\mathrm{SL}(2, \mathbb{Z})$	Group of 2×2 matrices with entries in $\mathbb{Z}$ and determinant 1, under matrix multiplication
ℓ	An arbitrary prime number

- $(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j) = (1 - a)(1 - aq)(1 - aq^2)...$
- Given $a, b \in \mathbb{Z}$, $a \mid b$ denotes that a divides b. Otherwise, $a \nmid b$.
- Given $a, b, m \in \mathbb{Z}$ with $m \geq 1$, $a \equiv b \pmod{m}$ denotes that $m \mid (a - b)$.
- Given some $k \in \mathbb{Z}_{\geq 1}$ the Laurent series

$$f = \frac{a(-k)}{q^k} + \frac{a(-k+1)}{q^{k-1}} + ... + \frac{a(-1)}{q} + a(0) + \sum_{n=1}^{\infty} a(n)q^n,$$

define

$$f^{(-)} := \frac{a(-k)}{q^k} + \frac{a(-k+1)}{q^{k-1}} + ... + \frac{a(-1)}{q} + a(0).$$

- Given some $k \in \mathbb{Z}$ and a function $f(\tau)$ which can be written in the form

$$f(\tau) = \sum_{n=k}^{\infty} a(n)e^{2\pi i n\tau},$$

define

$$\tilde{f}(q) := \sum_{n=k}^{\infty} a(n)q^n.$$

0.1 Prologue: Regarding the Title

Martin Eichler is said to have remarked that there are five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms. What follows is an exposition on the methods of computer algebra used to study a small part of the relationship between the first and the fifth of these important "operations" (the other three are also occasionally useful).

CHAPTER 1
INTRODUCTION

1.1 Background

The premise of our thesis is the study of additive number theory using the theory of modular functions and the methods of computer algebra, i.e., using computer algebra to study the relationship between the first and the fifth of Eichler's "operations."

We begin, however, by mentioning the study of the third of these operations: multiplication. The fundamental theorem of higher arithmetic states that every positive integer can be expressed as a unique product of primes—unique, at least up to the ordering of the factors. Thus, the number 6 possesses a single unique representation as a product of primes:

$$6 = 3 \cdot 2.$$

Due to the centrality of the primes in this representation, one might define the basic problem of multiplicative number theory as the study of the primes: their distribution, computation, and arithmetic properties, as well as the properties of their related functions, e.g., the prime counting function $\pi(x)$.

On the other hand, there exists no "additive equivalent" of the primes. The number 6 has no unique representation in addition—except perhaps as a trivial sum of 1s. Indeed, the number 6 has a total of 11 different additive representations in terms of other positive integers (again, up to the ordering of the terms):

$$\begin{aligned}
&6,\\
&5+1,\\
&4+2,\\
&4+1+1,\\
&3+3,\\
&3+2+1,\\
&3+1+1+1,\\
&2+2+2,\\
&2+2+1+1,\\
&2+1+1+1+1,\\
&1+1+1+1+1+1.
\end{aligned}$$

Definition 1.1. Given any nonnegative integer n, an *integer partition* of n is a representation of n as a sum of positive integers, called *parts,* in which ordering of parts is irrelevant. The number of integer partitions of n is the (unrestricted) *partition function,* denoted by $p(n)$. For a fixed n_0, $p(n_0)$ is sometimes referred to as the *partition number* of n_0.

Definition 1.2. The number of partitions of n under a given set of restrictions is a *restricted partition function.*

Remark 1.3. *We count a single-term sum as a distinct partition; thus,* 6 *counts as a 1-term partition.*

Remark 1.4. *For convenience of notation, we will denote* $p(0) = 1$*.*

Remark 1.5. *Without loss of generality, we will write partitions as a weakly decreasing sum of the parts, e.g.,* $3 + 2 + 1$*, rather than* $2 + 3 + 1$ *or any other permutation.*

It has been noted [106, 31:30] that, inasmuch as the study of the primes constitutes the basic problem of multiplicative number theory, the basic problem of additive number theory is the study of the partitions of a given integer—their number, their computation, the distribution and arithmetic properties of $p(n)$, and the properties of other functions which are closely associated with $p(n)$ (e.g., restricted partition functions, or functions which enumerate some important partition statistics).

We will begin with a discussion of the properties of $p(n)$, since much of the initial motivation in the theory of partitions stems from here.

The sequence of partition numbers $p(n)$ begins as follows:

$$1,\ 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 57, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, ... \tag{1.1}$$

Beyond the incidental (and short-lived) coincidence with the Fibonacci sequence, there is no immediately recognizable pattern to these values. On the other hand, the rapid growth of $p(n)$ suggests a roughly exponential asymptotic.

Euler studied $p(n)$ in 1748 [38]. He identified the generating function for $p(n)$, which we will denote as

$$\tilde{F}(q) := \prod_{m=1}^{\infty} \frac{1}{1 - q^m} = \sum_{n=0}^{\infty} p(n) q^n.$$

From this, numerous results for $p(n)$ were developed. However, the first useful formula for the computation of $p(n)$ did not come until the extremely precise asymptotic results of Hardy and Ramanujan in 1918 [46]. Some twenty years later, their methods were perfected by Rademacher [86], who provided the following exact formula for $p(n)$:

Theorem 1.6. *Let* $n \in \mathbb{Z}_{\geq 0}$*. Then*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} \cdot A_n(h, k) \frac{d}{dx} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)}\right)}{\sqrt{x - \frac{1}{24}}} \right) \Bigg|_{x=n}, \tag{1.2}$$

$$A_n(h,k) = \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \omega(h,k) e^{-2\pi i n h/k},$$

$$\omega(h,k) = \exp\left\{ \pi i \sum_{\mu=1}^{k-1} \frac{\mu}{k} \left(\frac{h\mu}{k} - \left\lfloor \frac{h\mu}{k} \right\rfloor - \frac{1}{2} \right) \right\}.$$

The existence of this formula is owed to the fact that the generating function of $p(n)$ is (with suitable adjustments) a modular form—a function defined on the upper half complex plane exhibiting powerful and surprising symmetric properties.

What is most astonishing about Theorem 1.6 is not its bizarre analytic structure, but its efficiency. The modularity of the generating function for $p(n)$ allows us to derive a formula which is efficient—indeed, very nearly *optimally* efficient—for direct computation [57].

To compare with the study of the prime numbers, it is well-known that the properties and symmetries of the Riemann ζ function allow us to extract information about the distribution of the primes. However, the resulting formulæ for the prime counting function $\pi(n)$ are notoriously difficult to compute. For example, the main asymptotic term given by the Prime Number Theorem only barely overpowers its error term [88, Chapter 7]; a more exact formula for $\pi(n)$ is possible, but it is not at all computationally efficient.

Of course, while Theorem 1.6 is remarkable from both a theoretical and a computational perspective, it only appears to exacerbate our confusion regarding the arithmetic properties of $p(n)$. It is not at all obvious, for example, when $p(n)$ is prime, composite, a perfect square or cube, even or odd. Indeed, reexamining (1.1), a first impression suggests that there are no interesting arithmetic properties. Perhaps, like the primes, there is a certain degree of randomness in these numbers, with no clear nontrivial arithmetic properties at all.

Only a century ago did we realize that the sequence (1.1) not only contains specific and nontrivial arithmetic properties; indeed, we realized almost overnight that $p(n)$ contains a considerably deep and magnificent arithmetic structure. We owe these discoveries to Srinivasa Ramanujan, who observed the following subsequence of (1.1) in which $n \equiv 4 \pmod 5$:

$$5, 30, 135, 490, 1575, 4565, 12310, 31185, 75175, 173525, 386155, 831820, 1741630, ... \tag{1.3}$$

From this he quickly guessed and proved that for all $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod 5. \tag{1.4}$$

Given the apparent lack of a pattern for (1.1) at first glance, this divisibility condition is remarkably simple and elegant. Similarly, Ramanujan was able to demonstrate that

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.5}$$
$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.6}$$

In these and all later examples, such congruences will hold for all nonnegative integers n.

Studying the relevant generating functions for these partition numbers, he discovered the following identities:

Theorem 1.7.

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \cdot \prod_{m=1}^{\infty} \frac{(1-q^{5m})^5}{(1-q^m)^6}, \tag{1.7}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 49q \prod_{m=1}^{\infty} \frac{(1-q^{7m})^7}{(1-q^m)^8} + 7 \prod_{m=1}^{\infty} \frac{(1-q^{7m})^3}{(1-q^m)^4}. \tag{1.8}$$

These identities are considered to be among Ramanujan's finest achievements, and they are central to the remainder of our dissertation.

Ramanujan did not stop with these results. He could not find similar results for primes other than 5, 7, and 11; however, noticing similar divisibility patterns for arithmetic progressions involving *powers* of 5, 7, 11, he made the following remarkable conjecture in 1918:

$$p\left(\ell^{\alpha}n + \lambda_{\ell,\alpha}\right) \equiv 0 \pmod{\ell^{\alpha}}, \tag{1.9}$$

with $\ell \in \{5, 7, 11\}$ and $\lambda_{\ell,\alpha}$ the minimal positive solution to $24x \equiv 1 \pmod{\ell^{\alpha}}$. This conjecture needed to be modified in the case that $\ell = 7$, but was otherwise exactly correct:

Theorem 1.8.

$$p\left(5^{\alpha}n + \lambda_{5,\alpha}\right) \equiv 0 \pmod{5^{\alpha}}, \tag{1.10}$$
$$p\left(7^{\alpha}n + \lambda_{7,\alpha}\right) \equiv 0 \pmod{7^{\lfloor \alpha/2 \rfloor + 1}}, \tag{1.11}$$
$$p\left(11^{\alpha}n + \lambda_{11,\alpha}\right) \equiv 0 \pmod{11^{\alpha}}. \tag{1.12}$$

The case $\alpha = 1$ for (1.10)-(1.12) is of course (1.4)-(1.6). The congruences for *all* $\alpha \in \mathbb{Z}_{\geq 0}$ are much more difficult to prove, and is the first known example of a considerably deep class of arithmetic results about partition numbers. Ramanujan's notebooks indicate that he had a proof of (1.10) [24], but the first published proof of (1.10)-(1.11) came from Watson in 1938 [114]. However, (1.12) proved to be much more challenging, and it was not proved until Atkin's work in 1967 [16].

These results were only the beginning of a century of progress on the arithmetic properties of $p(n)$ and its related functions. A large variety of other partition-related functions have since been found to exhibit similar congruence properties to those of $p(n)$. A quick (and not at all comprehensive) survey of the literature for proofs of such congruences can be found in Chapter 5 below.

Our understanding of these divisibility properties has advanced very rapidly, in various directions. Ahlgren and Boylan proved [3] that the *only* partition congruences of the form

$$p(\ell n + \lambda) \equiv 0 \pmod{\ell}$$

such that ℓ is prime and $0 \leq \lambda \leq \ell$ are Ramanujan's results (1.4)-(1.6).

However, a much larger variety of more intricate congruences have been found, i.e., congruences of the form

$$p(An + B) \equiv 0 \pmod{M}$$

with $A, B, M \in \mathbb{Z}_{\geq 0}$, $0 \leq B \leq A - 1$. A well known example by Atkin,

$$p(11^3 \cdot 13n + 237) \equiv 0 \pmod{13},$$

can be found in [17]. Other examples include results by Atkin and O'Brien [19], Hjelle and Klove [53], Klove [59], [60], and Newman [77].

In the last twenty years Ono [80], Ahlgren [2], and Ahlgren–Ono [7] have proved and carefully studied the existence an infinite number of congruence properties of $p(An + B)$ modulo primes greater than 5, and their powers. Additionally, Ono [81] and Radu [93] have demonstrated important properties of $p(An + B)$ modulo the primes 2 and 3. Much of this work was generalized by Treneer to include congruences for coefficients of weakly holomorphic modular forms [108]. More recently, Folsom, Kent, and Ono have studied the general ℓ-adic behavior of $p(n)$ [40].

Combinatorial interpretations of these congruence properties have been pioneered by Dyson [36], Garvan [42], and Andrews–Garvan [13]. More formal manipulations of integer partitions has led to an enormous variety of results by Andrews, Berndt, Hirschhorn, Garvan, and others.

At the same time, a more experimental approach to the study of these partition numbers has developed. Ramanujan himself worked heavily from example, computation, and experiment. Atkin has remarked [6, "Atkin's Examples"] that computational work is especially useful in the study of partitions. This fact has been confirmed in recent decades, as demonstrated by work at the Research Institute for Symbolic Computation (RISC). For example, Ono began a proof [81] of an important conjecture by Subbarao regarding the parity of $p(n)$; this proof was completed by Radu at RISC in 2012 [93]. Similarly, some important congruence conjectures by Sellers [100] were only resolved after nearly twenty years by Paule and Radu at RISC [82]. More recent progress on the meromorphic properties of modular functions has been made at RISC. For example Paule and Radu have also recently shown that the Weierstrass gap theorem may be proved without resorting to the Riemann–Roch theorem [84]. This has implications which affect not only the theory of partitions, but also the general theory of Riemann surfaces and algebraic geometry.

Each of these different approaches to the theory of partitions has contributed to an unexpected aspect of $p(n)$ and its associated functions: the dependence of their arithmetic properties on the same modular symmetries that underlie Theorem 1.6—the formula which appears at first sight only to obscure any understanding of the arithmetic of $p(n)$. The theory of modular forms—in an ever-more precise and deep form—continues to contribute to the forefront of the study of the arithmetic of partitions.

Moreover, just as it is useful for the *numerical* computation of $p(n)$ via Theorem 1.6, modularity is equally powerful for the *symbolic* computation of arithmetic information and identities associated with $p(n)$. We now understand the problem so well that we can use the underlying modular symmetries of $p(n)$ and related functions to create powerful algorithmic tools which are useful for deriving important relationships and identities. These identities can then be used for more experimental results from which deeper revelations may be derived.

For example, many different identities with a form resembling those of Theorem 1.7 have since been found for $p(n)$, as well as a variety of more restricted partition functions. For example, we have this result by Zuckerman [117]:

$$\begin{aligned}
&\sum_{n=0}^{\infty} p(13n+6)q^n \qquad (1.13)\\
&=11\prod_{m=1}^{\infty}\frac{(1-q^{13m})}{(1-q^m)^2}+468q\prod_{m=1}^{\infty}\frac{(1-q^{13m})^3}{(1-q^m)^4}+6422q^2\prod_{m=1}^{\infty}\frac{(1-q^{13m})^5}{(1-q^m)^6}\\
&\quad+43940q^3\prod_{m=1}^{\infty}\frac{(1-q^{13m})^7}{(1-q^m)^8}+171366q^4\prod_{m=1}^{\infty}\frac{(1-q^{13m})^9}{(1-q^m)^{10}}\\
&\quad+371293q^5\prod_{m=1}^{\infty}\frac{(1-q^{13m})^{11}}{(1-q^m)^{12}}+371293q^6\prod_{m=1}^{\infty}\frac{(1-q^{13m})^{13}}{(1-q^m)^{14}}.
\end{aligned}$$

Or consider the following results discovered by Kolberg [63], [62]:

$$\left(\sum_{n=0}^{\infty} p(5n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+2)q^n\right)=25q\prod_{m=1}^{\infty}\frac{(1-q^{5m})^{10}}{(1-q^m)^{12}}+2\prod_{m=1}^{\infty}\frac{(1-q^{5m})^4}{(1-q^m)^6}, \qquad (1.14)$$

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} p(7n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(7n+3)q^n\right)\left(\sum_{n=0}^{\infty} p(7n+4)q^n\right) \qquad (1.15)\\
&=117649q^4\prod_{m=1}^{\infty}\frac{(1-q^{7m})^{21}}{(1-q^m)^{24}}+50421q^3\prod_{m=1}^{\infty}\frac{(1-q^{7m})^{17}}{(1-q^m)^{20}}+8232q^2\prod_{m=1}^{\infty}\frac{(1-q^{7m})^{13}}{(1-q^m)^{16}}\\
&\quad+588q\prod_{m=1}^{\infty}\frac{(1-q^{7m})^9}{(1-q^m)^{12}}+15\prod_{m=1}^{\infty}\frac{(1-q^{7m})^5}{(1-q^m)^8},
\end{aligned}$$

$$\left(\sum_{n=0}^{\infty} p(2n)q^n\right)\left(\sum_{n=0}^{\infty} p(2n+1)q^n\right) = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^2(1-q^{8m})^2}{(1-q^m)^5(1-q^{4m})}. \tag{1.16}$$

Similar relationships can be found for a very large variety of restricted partition functions, as well as other functions which are closely associated with $p(n)$. These identities are generally referred to as *Ramanujan–Kolberg identities* (RK identities), after the two mathematicians who were principally responsible for the initial study of these results.

There exist many different proof techniques for these identities, including the more elementary manipulation of formal power series. Kolberg's methods are most notable in this respect [63]. However, these identities emerge very naturally from the theory of modular functions. Indeed, the techniques used to prove Theorem 1.7 by modularity may also be extended to include all of Kolberg's results.

Moreover, these techniques are so well-established, and based on sufficiently strong finiteness conditions, that they are capable of automation. This was demonstrated by Silviu Radu, who designed an algorithm [91] in 2014 to compute many more identities in the form above, for a broad class of arithmetic functions.

We have successfully implemented Radu's algorithm in Mathematica, and developed a freely available software package which can be used to examine a large number of contemporary results in the study of partitions. The associated identities are often far too lengthy to compute by hand; nevertheless, their computation is the result of a well-understood technique in the theory of modular functions, and their form often allows for the extraction of arithmetic information about the associated function.

The value of such computational tools to number theory is immense. In the first place, our software package very often enables us to extract optimal congruences for various partition functions. There are multiple instances in the recent mathematical literature of congruence results—usually proved by more elementary methods—which often fail to be optimal, and which our implementation can quickly improve. This is perhaps the most immediate application of our software package, and its utility may be recognized even by mathematicians who decry the use of computational techniques.

However, there exists another application of the automated computation of RK identities. Equations (1.7)-(1.8), along with the existence of similar identities for higher powers of 5 and 7, form the theoretical linchpin from which (1.10)-(1.11) of Theorem 1.8 are proved. Similarly, there exists a more complex sequence of identities which enable us to prove (1.12). The study of these congruence families is a far more ambitious problem than the computation of any single RK identity.

The theory of modular functions was certainly essential to the first proof of (1.10); more recent proofs can remove a direct reliance on modular functions. However, as we have already mentioned, over the last century a very large variety of similar infinite families of congruences have been found for other partition functions. These families vary wildly with respect to the difficulty in proving them.

The more difficult congruence families are often subject to complications which necessitate the use of modular functions, together with the underlying theory of compact Riemann surfaces. These proofs are generally inductive arguments which rely on a number of initial cases. These initial cases can often be proved using a modular cusp analysis which is necessary for the computation of Ramanujan–Kolberg identities with our software. This means that we can adapt much of the machinery for RK identities in order to prove the initial relations which are necessary for a complete proof of a given infinite congruence family.

1.2 Thesis Outline

The remainder of our work is as follows:

In Chapter 2 we will provide a very brief review of the underlying theory of modular functions. We begin with an exposition of the Riemann surface structure of the classical modular curves, and describe the impact of the topological properties of classical modular curves onto their associated spaces of modular functions. This will form the *theoretical* foundation for all of our later results. We will then define the Dedekind eta function, together with its modular symmetries, before developing the cusp analysis which allows us to represent and compare modular functions using computers.

In Chapters 3 and 4 we will demonstrate the application of modular functions to the computation of the class of identities studied by Ramanujan and Kolberg. We describe our Mathematica software implementation of Radu's algorithm, and give a number of interesting examples of the application of our implementation. We include new identities, and improvements on standing results. Radu's algorithm, together with our implementation, provides the *computational* foundation for our later results.

In the remaining chapters we utilize these tools, both theoretical and computational, to the end of more theoretical results, i.e., proving infinite families of partition congruences. In Chapter 5 we give a brief history of this subject. We also introduce a weighted partition function, here referred to as $A_1(n)$, connected with the celebrated Rogers–Ramanujan identities, for which an infinite family of congruences was conjectured by Choi, Kim, and Lovejoy. The difficulty for proving this infinite family, in contrast to more classical results of the form of Theorem 1.8, is that we have to contend with certain topological and analytic difficulties of the associated classical modular curve.

In Chapter 6 we first show how our implementation of Radu's algorithm may be adapted for the computation of multiple cases of the Choi–Kim–Lovejoy conjecture. This is a highly nontrivial problem, as the exponential growth of arithmetic progressions associated with infinite congruence families, together with the already subexponential growth of most restricted partition functions, very often makes the collection of compelling evidence for an infinite family difficult, even with a computer. Our methods allow for an explicit form of the Choi–Kim–Lovejoy congruence family to be stated, and for substantial evidence of its validity to be gathered.

In Chapters 7 and 8 we will give the complete proof using the method developed by Paule and Radu. We derive the necessary modular equation from which certain important recurrence relations may be derived, before confronting some of the problems involving the genus of the associated modular curve, and failure of universal 5-adic convergence. We build an induction argument, and use the cusp analysis to prove the initial cases, which will be found in the Appendix.

In the final portion of our work, we give the exposition of a new method of proving partition congruence families. Recent computational experiments, combined with some important theoretical results, have led us to suspect that the classical techniques for proving infinite families of congruences are incomplete. In Chapter 9 we give an example in a congruence family with respect to the smallest parts partition function spt_{ω}, associated with Ramanujan's third-order mock theta function. In theory, this congruence family should be amenable to a proof not dissimilar to those of (1.10)-(1.11) from Theorem 1.8: the underlying modular curve has a simple topology, whence the more classical techniques should be sufficient for a proof. However, in practice the techniques for handling modular curves with a positive genus were necessary for the very first proof by Liuquan Wang and Yifan Yang.

In trying to find a more classical proof for this family of congruences, we developed an approach which may be more complete and applicable to a wide variety of congruence families. Many interesting algebraic and arithmetic complexities arise, and the potential for further research is enormous. We finally give our proof of Wang and Yang's congruence family in Chapter 10.

CHAPTER 2
GENERAL THEORY

2.1 Introduction

In this chapter we will develop the necessary theory of modular functions. The subject is enormous, and we can only provide a brief introduction to it. We begin with an analytic motivation for the subject. Thereafter, we work in the context of the theory of Riemann surfaces. We will define the classical modular curves $\mathrm{X}_0(N)$, together with their underlying Riemann surface topology. We will prove some important topological properties, and introduce several extremely important results concerning functions over $\mathrm{X}_0(N)$, including a remarkable relationship between the set of functions over $\mathrm{X}_0(N)$ and the *genus* of $\mathrm{X}_0(N)$.

All of this may seem a little superfluous, but the advantages will be worthwhile. As we will show in later chapters, a good understanding of the topology of the Riemann surfaces associated with certain spaces of modular functions can inform our understanding of many arithmetic functions of interest to us, including $p(n)$.

Once we have established the relevant Riemann surface structure and properties of $\mathrm{X}_0(N)$, we will define our relevant classes of modular functions, and list some important properties. We will describe the remarkable properties of the Dedekind η function, which is closely related to the generating function $\tilde{F}(q)$ of $p(n)$. Thereafter we will discuss the modular cusp analysis, the means of using the properties of modularity to establish equivalence of any two modular functions by the comparison of a finite number of Fourier coefficients. We end with a brief description of the U_ℓ operator.

2.2 Conformal Mappings

We denote $\mathbb{H}$ as the upper half complex plane, and define

$$\begin{aligned}\hat{\mathbb{C}} &:= \mathbb{C} \cup \{\infty\},\\ \hat{\mathbb{H}} &:= \mathbb{H} \cup \mathbb{Q} \cup \{\infty\},\\ \hat{\mathbb{Q}} &:= \mathbb{Q} \cup \{\infty\},\end{aligned}$$

with $a/0 = \infty$ for $a \neq 0$.

We can ask for the set of all holomorphic mappings $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which are *automorphic*, i.e. bijective and conformal [105, Chapter 8, Section 2.1]. We denote the set of all such mappings as $\mathrm{Aut}(\hat{\mathbb{C}})$.

Automorphic mappings may be interpreted in a variety of ways. On a global scale, $\mathrm{Aut}(\hat{\mathbb{C}})$ may be thought of as the set of all functions which send circles to circles in the Riemann sphere (considering straight lines as circles which intersect with ∞) [66, II. 9A]. In terms of differential geometry, $\mathrm{Aut}(\hat{\mathbb{C}})$ is the set of all bijective functions on $\hat{\mathbb{C}}$ which preserve angles between intersecting curves [105, Chapter 8, Section 1, Problem 2].

Consider Möbius transformations, i.e., mappings of the form

$$\varphi : \tau \longmapsto \frac{a\tau + b}{c\tau + d},$$

with $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. All Möbius transformations are elements of $\mathrm{Aut}(\hat{\mathbb{C}})$ [66, II. 8D]. Notice that $\varphi(-d/c) = \infty$, and $\varphi(\infty) = a/c$.

Conversely, it turns out that every member of $\mathrm{Aut}(\hat{\mathbb{C}})$ can be expressed as a Möbius transformation [66, II. 8D]. Moreover, we may actually restrict the value of $ad - bc$ to 1 (since $ad - bc \neq 0$, we may always normalize by dividing the numerator and denominator by $ad - bc$, without changing the output of φ) [66, II. 9A]. We therefore have

$$\mathrm{Aut}(\hat{\mathbb{C}}) = \left\{ \varphi : \tau \mapsto \frac{a\tau + b}{c\tau + d} : a, b, c, d \in \mathbb{C},\ ad - bc = 1 \right\}.$$

Recall that

$$\mathrm{SL}(2, \mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ ad - bc = 1 \right\}.$$

Definition 2.1. We define the group action

$$\begin{aligned} \mathrm{SL}(2, \mathbb{C}) \times \hat{\mathbb{C}} &\longrightarrow \hat{\mathbb{C}}, \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) &\longmapsto \frac{a\tau + b}{c\tau + d}. \end{aligned}$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ and $\tau \in \hat{\mathbb{C}}$, then we write

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}.$$

We can express $\mathrm{Aut}(\hat{\mathbb{C}})$ as the group action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}$.

If we consider the set of automorphic mappings on $\mathbb{H}$, then we have a simple restriction on these transformations [101, Chapter VII, Section 1.1]:

$$\mathrm{Aut}(\hat{\mathbb{H}}) = \left\{ \varphi : \tau \mapsto \frac{a\tau + b}{c\tau + d} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

This set occurs as the group action of Definition 2.1 restricted to $\mathrm{SL}(2, \mathbb{R})$.

The set $\mathrm{Aut}(\hat{\mathbb{H}})$ is of interest to us in part because $\mathbb{H}$ is conformally equivalent to the open unit disk in $\mathbb{C}$ [105, Chapter 8, Theorem 1.2]. Notice that the open unit disk is the domain of $\tilde{F}(q)$ (and indeed, the domain of all of our generating functions of interest).

From a purely analytic standpoint, it is natural to ask whether there exist any functions over $\hat{\mathbb{H}}$ which are invariant (or nearly invariant) on orbits of $\mathrm{Aut}(\hat{\mathbb{H}})$; however, this is a massive set, and we are unlikely to find many interesting functions which exhibit symmetry for the entire set. On the other hand, we can find and functions which are invariant on orbits of certain discrete subgroups of $\mathrm{Aut}(\hat{\mathbb{H}})$.

We will be especially interested in subgroups of $\mathrm{Aut}(\hat{\mathbb{H}})$ determined by the group action of Definition 2.1 restricted to subgroups of $\mathrm{SL}(2,\mathbb{Z})$. A great variety of such subgroups has proven useful in the study of partition congruences, and the general theory of modular functions. We will concern ourselves with one specific class of subgroups (although much of the basic theory is very similar for other subgroups).

Definition 2.2. For any given $N \in \mathbb{Z}_{\geq 1}$, define the *congruence subgroup* $\Gamma_0(N)$ by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : N|c \right\}.$$

We will be working substantially with subgroups of this form; however, we will highlight an important subset of $\Gamma_0(N)$ which will prove useful in later computations:

Definition 2.3. For any given $N \in \mathbb{Z}_{\geq 1}$, define $\Gamma_0(N)^*$ by

$$\Gamma_0(N)^* := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : N|c,\ a > 0,\ c > 0,\ \gcd(a,6) = 1 \right\}.$$

This subset has the advantage of generating the entire subgroup.

Lemma 2.4. $\Gamma_0(N)^*$ *multiplicatively generates* $\Gamma_0(N)$.

For a proof, see [76].

Hereafter, when we speak of the group action of $\Gamma_0(N)$ on $\hat{\mathbb{H}}$, we mean the group action of Definition 2.1 restricted to $\Gamma_0(N)$.

Definition 2.5. For any subgroup $\Gamma \in \mathrm{SL}(2,\mathbb{Z})$ and $\tau \in \hat{\mathbb{H}}$, define the *stabilizer subgroup* Γ_τ by

$$\Gamma_\tau := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \frac{a\tau + b}{c\tau + d} = \tau \right\} / \{\pm I\}.$$

In particular,

$$\Gamma_\infty := \left\{ \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} \in \Gamma \right\} / \{\pm I\}.$$

An important point is the fact that $\Gamma_0(N)$ is a subgroup of finite index over $\mathrm{SL}(2,\mathbb{Z})$. In particular, we have [34, Problem 1.2.3.e]

Lemma 2.6. *For all $N \in \mathbb{Z}_{\geq 1}$,*

$$[\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0(N)] = N \prod_{\substack{p:\ prime,\\ p|N}} \left(1 + \frac{1}{p}\right).$$

Definition 2.7. Given $N \in \mathbb{Z}_{\geq 1}$, the *orbits* of the group action $\Gamma_0(N)$ on $\hat{\mathbb{H}}$ are denoted by

$$[\tau]_N := \{\gamma\tau : \gamma \in \Gamma_0(N)\}.$$

Because we are interested in functions which are invariant under transformations from $\Gamma_0(N)$, it is useful to consider identifying all points in a given orbit.

Definition 2.8. For any $N \in \mathbb{Z}_{\geq 1}$, we define the *classical modular curve of level N* as the set of all orbits of $\Gamma_0(N)$ applied to $\hat{\mathbb{H}}$:

$$\mathrm{X}_0(N) := \left\{[\tau]_N : \tau \in \hat{\mathbb{H}}\right\}$$

From a topological standpoint, the modular curves $\mathrm{X}_0(N)$ are the principal objects of interest to us. We can give the natural surjection

$$\begin{aligned} \pi : \hat{\mathbb{H}} &\longrightarrow \mathrm{X}_0(N) \\ : \tau &\longmapsto [\tau]_N. \end{aligned}$$

If our domain was only $\mathbb{H}$, this surjection would give a quotient topology to $\mathrm{X}_0(N)$, with a relatively straightforward notion of the neighborhood of a point. But because we also include $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, we have to be more careful about what constitutes a neighborhood. We will approach this in the next section.

For the moment, we list a few important facts about the behavior of rational points under our group action. Notice that, for all $\tau \in \mathbb{Q}$, $[\tau]_N \subseteq \mathbb{Q}$. Indeed, $\hat{\mathbb{Q}}$ is the disjoint union

$$\hat{\mathbb{Q}} = \bigsqcup_{\tau \in \mathcal{K}} [\tau]_N,$$

for some $\mathcal{K} \subseteq \hat{\mathbb{Q}}$.

Definition 2.9. The *cusps* of $\mathrm{X}_0(N)$ are the orbits of $\Gamma_0(N)$ acting on $\hat{\mathbb{Q}}$.

The number of cusps of $\mathrm{X}_0(N)$ is finite:

Lemma 2.10. *We denote the size of $\mathcal{K}$ by $\epsilon_\infty(\Gamma_0(N))$, and we have*

$$\epsilon_\infty(\Gamma_0(N)) = \sum_{\delta|N} \phi(\gcd(\delta, N/\delta)),$$

in which $\phi(n)$ is Euler's totient function.

See [34, Section 3.8].

In future chapters we will be especially interested in manipulating various representatives of cusps of a given modular curve. We therefore give some important theorems for comparing representatives.

The following theorem [34, Proposition 3.8.3] gives a condition for determining whether two elements of $\hat{\mathbb{Q}}$ represent the same cusp.

Theorem 2.11. *Let a/c, $a_1/c_1 \in \mathbb{Q} \cup \{\infty\}$ with $\gcd(a, c) = \gcd(a_1, c_1) = 1$. Then a_1/c_1 represents the same cusp over $\Gamma_0(N)$ as a/c if and only if there exist integers $m, n \in \mathbb{Z}$ such that*

$$\begin{aligned} ma_1 &\equiv a + nc \pmod{N}, \\ c_1 &\equiv mc \pmod{N}, \end{aligned}$$

with $\gcd(m, N) = 1$.

The proof can be found in [34, Section 3.8].

A useful matrix-based interpretation of these cusps may also be given. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Notice that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b + ak \\ c & d + ck \end{pmatrix}.$$

Indeed, *every* matrix with left terms a, c may be represented in this form. As such, we have a bijection from $\hat{\mathbb{Q}}$ to $\mathrm{SL}(2, \mathbb{Z})$ via the map

$$\frac{a}{c} \longmapsto \left\{ \begin{pmatrix} a & b + ak \\ c & d + ck \end{pmatrix} : k \in \mathbb{Z}; b, d \in \mathbb{Z} \text{ such that } ad - bc = 1 \right\}.$$

Knowing that representatives of a given cusp relate to each other through left multiplication by elements of $\Gamma_0(N)$, we have [34, Proposition 3.8.5]

Theorem 2.12. *The cusps of $\Gamma_0(N)$ may be represented by the double cosets*

$$\Gamma_0(N) \backslash \mathrm{SL}(2, \mathbb{Z}) / \mathrm{SL}(2, \mathbb{Z})_\infty = \{\Gamma_0(N) \gamma \mathrm{SL}(2, \mathbb{Z})_\infty : \gamma \in \mathrm{SL}(2, \mathbb{Z})\}.$$

2.3 Riemann Surface Structure of Modular Curves

2.3.1 A Topology for $\hat{\mathbb{H}}$

To build the complete topology for $\mathrm{X}_0(N)$, we begin with a topology for $\hat{\mathbb{H}}$. For $\tau \in \mathbb{H}$, a neighborhood is simply that defined for the topology on $\mathbb{H}$: a disk centered at τ,

$$\mathbb{D}_{\tau,r} := \{z : |z - \tau| < r\},$$

in which the radius r is small enough that $\mathbb{D}_{\tau,r} \subseteq \mathbb{H}$.

For convenience we imagine $\infty = i\infty$, i.e., a point at positive infinity along the imaginary axis. For any $R > 0$, we define a neighborhood of ∞ by the half plane

$$\mathcal{N}_R := \{z : \Im z > R\} \cup \{\infty\}.$$

Recall that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, the map $\gamma\tau$ is conformal, and sends circles and lines to circles and lines. One can easily show that the map imposed by γ will send $\mathcal{N}_R$ to the set

$$\gamma(\mathcal{N}_R) = \left\{ z : \left| z - \left(\frac{a}{c} + i\frac{1}{2Rc^2} \right) \right| < \frac{1}{2Rc^2} \right\} \cup \left\{ \frac{a}{c} \right\},$$

with ∞ mapping to a/c. We therefore define the neighborhood of any rational point $s \in \mathbb{Q}$ as any open disk (with interior in $\mathbb{H}$) which is tangent to $\mathbb{R}$ at s, together with s itself.

Taking these sets as a basis, we can define a topology on $\hat{\mathbb{H}}$. We then define the topology on $\mathrm{X}_0(N)$ as that imposed by the natural surjection

$$\begin{aligned} \pi : \hat{\mathbb{H}} &\longrightarrow \mathrm{X}_0(N) \\ : \tau &\longmapsto [\tau]_N, \end{aligned}$$

in which a subset $\mathcal{W} \subseteq \mathrm{X}_0(N)$ is defined to be open if $\pi^{-1}(\mathcal{W})$ is open in $\hat{\mathbb{H}}$.

With some difficulty, it can be shown that $\mathrm{X}_0(N)$ is Hausdorff. However, it is easier to demonstrate that $\mathrm{X}_0(N)$ is connected and compact under this topology. We will prove the latter two properties here, and direct the reader to Proposition 2.1.1, Corollary 2.1.2, and Proposition 2.4.2 of [34, Chapter 2] for a proof of the former.

Theorem 2.13. *For each N, $\mathrm{X}_0(N)$ is connected.*

Proof. We first show that $\hat{\mathbb{H}}$ is connected. Suppose that $\hat{\mathbb{H}} = U_1 \sqcup U_2$ is a disjoint union of open sets. Then $\mathbb{H} \subseteq U_1 \sqcup U_2$. Because $\mathbb{H}$ is connected, we may take $\mathbb{H} \subseteq U_1$ without loss of generality. Therefore, we must have $U_2 \subseteq \hat{\mathbb{Q}}$. But any neighborhood of a rational point will include points in $\mathbb{H}$. The only possibility is that $U_2 = \emptyset$. Therefore, $\hat{\mathbb{H}}$ is connected.

Because $\mathrm{X}_0(N)$ is the image of the corresponding map π, which is continuous, $\mathrm{X}_0(N)$ must be connected. $\square$

2.3.2 Example: $\Gamma_0(1) = \mathrm{SL}(2,\mathbb{Z})$

In order to prove compactness, we start by considering the example $N = 1$. To approach this, we consider two example elements of $\mathrm{SL}(2,\mathbb{Z})$, together with their corresponding automorphisms:

$$S := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ S\tau = \tau + 1$$
$$T := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T\tau = -1/\tau.$$

We also consider the set

$$\mathcal{D} := \left\{ \tau \in \mathbb{H} : |\Re(\tau)| \leq \frac{1}{2},\ |\tau| \geq 1 \right\}.$$

See the figure below.

Theorem 2.14. *The map $\pi : \tau \longmapsto [\tau]_1$ is a surjection with the domain of π restricted to $\mathcal{D} \cup \{\infty\}$. Moreover, any two points z, τ in $\mathcal{D}$ which lie in the same orbit must lie on the boundary of $\mathcal{D}$ with either $z = S^{\pm 1}\tau$ or $z = T\tau$.*

Corollary 2.15. $\mathrm{SL}(2,\mathbb{Z})$ *is generated by S and T.*

The proofs of these may be found in various books. We recommend [101, Chapter VII, Theorem 1 and Corollary, Theorem 2].

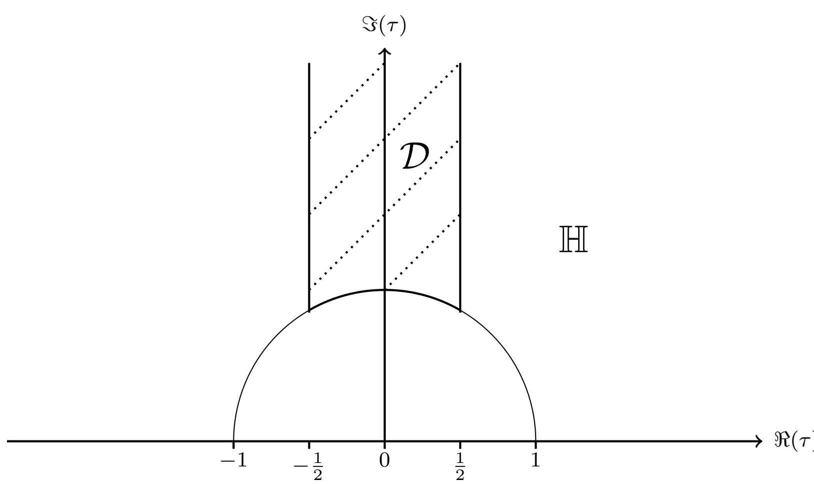

Figure 2.1: $\mathcal{D}$, a fundamental region for $\mathrm{SL}(2,\mathbb{Z}) = \Gamma_0(1)$

The figure $\mathcal{D}$ gives us a means of visualizing $\Gamma_0(N)$. Notice that the line $\Re z = -1/2$ and the line $\Re z = 1/2$ are equivalent to one another modulo the action of S. Thus, we may imagine gluing these lines together to make a cylinder.

Moreover, the arc of the unit circle for $\pi/3 \leq \theta \leq \pi/2$ is equivalent to the arc for $\pi/2 \leq \theta \leq 2\pi/3$ modulo the action of T. This allows us to glue the bottom of our cylinder together. Finally, the lines $\Re z = \pm 1/2$ intersect the line $\Re z = 0$ at $z = i\infty$. We may therefore take $\mathcal{D}$ to be topologically equivalent (i.e., homeomorphic) to a sphere with a single point missing modulo $\mathrm{SL}(2, \mathbb{Z})$. If we add this point (which is of course ∞), then $\mathcal{D} \cup \{\infty\}$ is a connected Hausdorff topological space homeomorphic to a sphere.

Theorem 2.16. *For each N, $\mathrm{X}_0(N)$ is compact.*

Proof. We first show that $\mathcal{D} \cup \{\infty\}$ is compact in the topology we have defined on $\hat{\mathbb{H}}$. Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a covering of $\mathcal{D} \cup \{\infty\}$. Because ∞ must be covered, there must exist some $\mu \in \Lambda$ and some $R > 0$ such that $\mathcal{N}_R \subseteq U_\mu$.

If we take $\mathcal{D}^* := \mathcal{D} \setminus U_\mu$, then $(\mathcal{D}^*)^c = \mathcal{D}^c \cup U_\mu$ is open. Therefore, $\mathcal{D}^*$ is closed.

Moreover, $\mathcal{D}$ is bounded by $|\Re(\tau)| \leq \pm 1/2$, and bounded below by $\Im(\tau) \geq \sqrt{3}/2$ by definition; and $\mathcal{D}^*$ is bounded above, since $\Im(\tau) \leq R$ for all $\tau \in \mathcal{D}^*$. So $\mathcal{D}^*$ must be closed and bounded, and therefore compact. Let $\mathcal{U}^* = \{U_{\lambda_1}, ..., U_{\lambda_k}\} \subseteq \mathcal{U}$ be a finite subcovering for $\mathcal{D}^*$. Then $\mathcal{U}^* \cup \{U_\mu\}$ is a finite subcovering of $\mathcal{D} \cup \{\infty\}$. This establishes that $\mathcal{D} \cup \{\infty\}$ is compact in the $\hat{\mathbb{H}}$ topology.

To show that $\mathrm{X}_0(1)$ is compact, notice that $\mathrm{X}_0(1) = \pi\left(\mathcal{D} \cup \{\infty\}\right)$, and that π is a continuous map. More generally, for any N, $\Gamma_0(N)$ is a subgroup of $\mathrm{SL}(2, \mathbb{Z})$ with finite index d. Then we can write

$$\hat{\mathbb{H}} = \mathrm{SL}(2, \mathbb{Z})\left(\mathcal{D} \cup \{\infty\}\right) = \bigcup_{j=1}^{d} \Gamma_0(N)\gamma_j\left(\mathcal{D} \cup \{\infty\}\right),$$

with $\{\gamma_j\}_{1 \leq j \leq d}$ a set of right coset representatives. Because $\Gamma_0(N)\tau = [\tau]_N$, we have

$$\mathrm{X}_0(N) = \bigcup_{j=1}^{d} \pi\left(\gamma_j\left(\mathcal{D} \cup \{\infty\}\right)\right).$$

The action of γ_j on τ is a continuous function, and $\mathcal{D} \cup \{\infty\}$ is compact. Therefore, $\gamma_j\left(\mathcal{D} \cup \{\infty\}\right)$ must be compact. Because π is continuous, $\pi\left(\gamma_j\left(\mathcal{D} \cup \{\infty\}\right)\right)$ must be compact. Finally, because d is finite, the union of these compact sets must also be compact. □

2.3.3 Riemann Surfaces

The fact that $\mathrm{X}_0(N)$ possesses a comparatively simple topology suggests that we may be able to treat $\mathrm{X}_0(N)$ as an object upon which we can do complex analysis. In this section, we will establish this fact in some detail.

Of course, π is continuous by definition. But we can prove still more:

Lemma 2.17. *For any $N \geq 1$, π is an open mapping.*

Proof. Let $\mathcal{U} \subseteq \hat{\mathbb{H}}$ be open. We want to show that $\pi(\mathcal{U})$ is open in $\mathrm{X}_0(N)$. To do this, we consider

$$\begin{aligned}\pi^{-1}(\pi(\mathcal{U})) &= \left\{\tau \in \hat{\mathbb{H}} : \pi(\tau) \in \pi(\mathcal{U})\right\} \\ &= \left\{\tau \in \hat{\mathbb{H}} : [\tau]_N = [z]_N, \text{ for some } z \in \mathcal{U}\right\} \\ &= \left\{\tau \in \hat{\mathbb{H}} : \tau = \gamma z, \text{ for some } z \in \mathcal{U} \text{ and some } \gamma \in \Gamma_0(N)\right\} \\ &= \{\gamma z : \gamma \in \Gamma_0(N), z \in \mathcal{U}\} \\ &= \bigcup_{\gamma \in \Gamma_0(N)} \gamma(\mathcal{U}).\end{aligned}$$

Each $\gamma(\mathcal{U})$ is open by the open mapping theorem [105, Theorem 4.4]. □

Definition 2.18. An *atlas* for a topological space X is a collection $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ of open sets, together with a set of mappings (*local coordinates*) in which

$$\bigcup_{\lambda \in \Lambda} U_\lambda = \mathrm{X}, \text{ and}$$

$$\varphi_\lambda : U_\lambda \longrightarrow \mathbb{C}$$

is a homeomorphism for each $\lambda \in \Lambda$. Each pair $(U_\lambda, \varphi_\lambda)$ constitutes a *local coordinate system* of X.

We now give the definition of a Riemann surface. For more details, see [72, Chapter I, Definition 1.17]

Definition 2.19. Let X be a connected Hausdorff topological space, together with an atlas $\mathcal{U} := \{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$, and the property that, for any $\mu, \nu \in \Lambda$, if $U_\mu \cap U_\nu \neq \emptyset$, then $\varphi_\mu \varphi_\nu^{-1}$ is holomorphic over $\varphi_\nu(U_\mu \cap U_\nu)$. Then $(\mathrm{X}, \mathcal{U})$ is a *Riemann surface*. The charts φ_λ are said to be *pairwise compatible*.

In short, a Riemann surface is a 1-dimensional complex manifold [72, Chapter I: "Real 2-Manifolds"]. The appeal of such a structure is that we can study many of the functions with domains over such surfaces using the tools of complex analysis, with little modification.

That $\mathrm{X}_0(N)$ has a Riemann surface structure is such a remarkable result—with such significant consequences—that we will give a brief sketch of the proof. For the complete details, see [34, Chapter 2].

To define a Riemann surface structure on $\mathrm{X}_0(N)$, we will need to define local coordinates at each point of $\mathrm{X}_0(N)$. Notice that each point $x \in \mathrm{X}_0(N)$ can be expressed as

$$x = \pi(\tau) = [\tau]_N,$$

for some $\tau \in \hat{\mathbb{H}}$. For most points $\tau \in \mathbb{H}$, π will have a local inverse for a sufficiently small neighborhood of x.

An exception holds for $\tau \in \mathbb{H}$ in which $\gamma\tau = \tau$ for $\gamma \in \Gamma_0(N) \setminus \{\pm I\}$, i.e., those τ for which $\Gamma_0(N)_\tau$ is nontrivial.

Definition 2.20. If $\tau \in \mathbb{H}$ such that $\gamma\tau = \tau$ for $\gamma \in \Gamma_0(N) \setminus \{\pm I\}$, then τ is an *elliptic point* of $\Gamma_0(N)$. If τ is an elliptic point of $\Gamma_0(N)$, then $[\tau]_N$ is an *elliptic point* of $\mathrm{X}_0(N)$.

If τ is an elliptic point of $\Gamma_0(N)$, we must have $\Gamma_0(N)_\tau \supsetneq \{\pm I\}$. Otherwise, $\Gamma_0(N)_\tau = \{\pm I\}$. We give an important theorem [34, Proposition 2.2.2] which we will need to extend the Riemann surface structure of $\mathrm{X}_0(N)$ to its elliptic points:

Theorem 2.21. *For every elliptic point τ of $\Gamma_0(N)$, $\Gamma_0(N)_\tau$ is finite cyclic.*

An additional problem holds for the cusps of $\mathrm{X}_0(N)$. Fortunately for us, we may recall from Lemma 2.10 that there are only a finite number of cusps. Moreover, there are only a finite number of each of these elliptic points for every $\mathrm{X}_0(N)$ [34, Corollary 2.3.5], and they can be properly accounted for.

Theorem 2.22. *For each N, the classical modular curve $\mathrm{X}_0(N)$ is a compact Riemann surface.*

Proof. We have already established that $\mathrm{X}_0(N)$ is Hausdorff, connected, and compact. We need to construct an atlas that covers $\mathrm{X}_0(N)$, such that the local coordinates are pairwise compatible.

For any $x \in \mathrm{X}_0(N)$ we have of course $x = \pi(\tau)$ for some $\tau \in \hat{\mathbb{H}}$. In this case, τ is either a member of $\mathbb{H}$, $\mathbb{Q}$, or $\{\infty\}$. We will build our associated neighborhoods and local coordinates of $\pi(\tau)$ in each of these cases, before checking compatibility.

Defining the Atlas Away From Cusps.

For points $\tau \in \mathbb{H}$ which are not elliptic, we would ordinarily take a neighborhood of τ which contains no elliptic points or members of $\hat{\mathbb{Q}}$, and then take the local inverse of π. However, to consider elliptic points, we will define our local coordinates a little more carefully.

An example of an elliptic point in $\hat{\mathbb{H}}$ is i under $\mathrm{SL}(2,\mathbb{Z})$. Notice that $Tz = z$ only has two solutions: $z = \pm i$. On the other hand, if we take some $z = a + bi$ such that $a^2 + b^2 = 1$, then $Tz = -a + bi$. For a close to 0 and b close to 1, both z and Tz are close to i. No matter how small we make the neighborhood of i, it will always contain equivalent points. A similar problem persists for the congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$.

To deal with this problem, we take two important facts into account. First, we note that elliptic points will not be dense in $\mathbb{H}$, by [34, Corollary 2.3.5]. Second, we note that for any $\tau \in \mathbb{H}$ there exists a neighborhood U_μ such that for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$,

$$\gamma(U_\mu) \cap U_\mu \neq \emptyset \implies \gamma \in \Gamma_0(N)_\tau \tag{2.1}$$

[34, Corollary 2.2.3]. These facts together allow the following:

Given some $x = \pi(\tau)$ with $\tau \in \mathbb{H}$, there exists a neighborhood U_μ of τ which lies entirely in $\mathbb{H}$, which contains no elliptic points, *with the possible exception of* τ itself, and with the additional condition that for any $z_1, z_2 \in U_\mu$, if $z_1 \in [z_2]_N$, we must have $z_1 = \gamma(z_2)$ for $\gamma \in \Gamma_0(N)_\tau$.

Let us take such a neighborhood U_μ for $x = \pi(\tau)$ with $\tau \in \mathbb{H}$. We now define the map

$$\begin{aligned} \delta_\tau : \hat{\mathbb{C}} &\longrightarrow \hat{\mathbb{C}}, \\ z &\longmapsto \frac{z-\tau}{z-\bar{\tau}}. \end{aligned}$$

Because $\tau \in \mathbb{H}$, the determinant of $\left(\begin{smallmatrix} 1 & -\tau \\ 1 & -\bar{\tau} \end{smallmatrix}\right)$ is $-\bar{\tau} + \tau = 2 \cdot \Im(\tau) > 0$. Therefore, δ_τ must be conformal, such that $\delta_\tau(\tau) = 0$, and $\delta_\tau(\bar{\tau}) = \infty$.

Now, consider the map $\delta_\tau \gamma$, for any $\gamma \in \Gamma_0(N)_\tau$. Notice that we have

$$\begin{aligned} \delta_\tau \gamma(\tau) &= \delta_\tau(\tau) = 0, \\ \delta_\tau \gamma(\bar{\tau}) &= \delta_\tau(\bar{\tau}) = \infty. \end{aligned}$$

So the map which sends $\delta_\tau(z)$ to $\delta_\tau \gamma(z)$, i.e., the map

$$\delta_\tau \gamma \delta_\tau^{-1} : \delta_\tau \left(U_\mu\right) \longrightarrow \hat{\mathbb{C}},$$

is a conformal map which sends 0 to 0, and ∞ to ∞. Recall that a conformal map will send circles to circles, with straight lines counted as circles which intersect ∞. Therefore, $\delta_\tau \gamma \delta_\tau^{-1}$ must send any given radial vector originating at 0 to another radial vector, also originating at 0. That is, it must be a rotation combined with a dilation about the point 0 in $\hat{\mathbb{C}}$.

Moreover, we know that $\Gamma_0(N)_\tau$ is finite cyclic by Theorem 2.21; and for any $a, b \in \mathbb{Z}$,

$$\delta_\tau \gamma^a \gamma^b \delta_\tau^{-1} = \left(\delta_\tau \gamma^a \delta_\tau^{-1}\right) \circ \left(\delta_\tau \gamma^b \delta_\tau^{-1}\right).$$

This makes the dilation trivial, and allows us to define an isomorphism between $\Gamma_0(N)_\tau$ and a certain group of rotations about 0 in $\hat{\mathbb{C}}$. Indeed, since the order of the generator for $\Gamma_0(N)_\tau$ must have order $|\Gamma_0(N)_\tau|$, the associated group of rotations must be generated by a rotation of the angle $2\pi / |\Gamma_0(N)_\tau|$.

Why is this important? Well, we consider two points $z_1, z_2 \in U_\mu$ such that

$$z_1 \in [z_2]_N. \tag{2.2}$$

Recall, by (2.1), that $z_1 = \gamma(z_2)$ for $\gamma \in \Gamma_0(N)_\tau$.

We know that $\delta_\tau(z_1), \delta_\tau(z_2)$ must lie on radial vectors from 0 which differ by an angle determined by γ, which must be an integer multiple of $2\pi / |\Gamma_0(N)_\tau|$.

This implies that no more than a single representative of every set of $\Gamma_0(N)$-equivalent points in $\delta_\tau \left(U_\mu\right)$ will be found in a sector Δ centered at 0 between two radial lines separated by the angle $2\pi / |\Gamma_0(N)_\tau|$. We can then include one of these radial lines, since a point on one radial line of Δ

will be equivalent to a point on the opposite radial line by the map $\delta_\tau \gamma \delta_\tau^{-1}$ corresponding to the rotation by $2\pi / |\Gamma_0(N)_\tau|$.

To identify the radial lines with one another, we consider the map

$$\begin{aligned} \rho_\mu : \mathbb{C} &\longrightarrow \mathbb{C}, \\ z &\longmapsto z^{|\Gamma_0(N)_\tau|}. \end{aligned}$$

Now if we take z_1, z_2 as in (2.2), then $\rho_\mu \delta_\tau(z_1), \rho_\mu \delta_\tau(z_2)$ are no longer separated by an angle: the power of ρ_μ cancels all fractional multiples of 2π.

Thus our key function for our chosen neighborhood U_μ of τ is

$$\begin{aligned} \psi_\mu : U_\mu &\longrightarrow \mathbb{C}, \\ z &\longmapsto \left(\frac{z - \tau}{z - \bar{\tau}} \right)^{|\Gamma_0(N)_\tau|}. \end{aligned}$$

Notice that ψ_μ is holomorphic, and therefore continuous and open, on U_μ. Moreover, ψ_μ bijectively maps the sector of $\Gamma_0(N)$-equivalent points to an open neighborhood V about 0.

Now, because π is also a continuous open map, the composition $\psi_\mu \pi^{-1}$ is a continuous open map, with π^{-1} defined on $\pi \delta_\tau^{-1}(\Delta)$. Finally, as Δ contains a single representative of every point in U_μ, it must be that $\psi_\mu \pi^{-1}$ is a bijection. We therefore have the neighborhood $\pi(U_\mu)$ and the homeomorphism

$$\begin{aligned} \varphi_\mu : \ \pi(U_\mu) &\longrightarrow V, \\ \pi(z) &\longmapsto \psi_\mu \pi^{-1}(z). \end{aligned}$$

Notice that in the event that τ is not elliptic, then U_μ contains no elliptic points, $\Gamma_0(N)_\tau$ is trivial, and our local coordinates are still suitable homeomorphisms.

Defining the Atlas Near Cusps.

Handling the cusps is similar. Suppose x is a cusp of $\mathrm{X}_0(N)$. We write $x = \pi(a/c)$ with $a, c \in \mathbb{Z}$ and $\gcd(a, c) = 1$. We can find some $b, d \in \mathbb{Z}$ such that $\delta := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$.

Recall that the map $\delta : z \mapsto \frac{az+b}{cz+d}$ is conformal, and therefore sends circles and lines to circles and lines. In particular, δ sends vertical lines to circles centered on $\mathbb{R}$ which intersect a/c.

Let $U_\kappa = \delta(\mathcal{N}_R)$ be a neighborhood of a/c in $\hat{\mathbb{H}}$ which contains no elliptic points. In particular, let us fix some $R \geq 2$.

Let us take two points, $\tau_1, \tau_2 \in U_\kappa$ with $\tau_1 \in [\tau_2]_N$. Suppose $\tau_1 = \gamma \tau_2$ with $\gamma \in \Gamma_0(N)$. We now apply the inverse mapping

$$\delta^{-1} : \tau \mapsto \frac{d\tau - b}{-c\tau + a}.$$

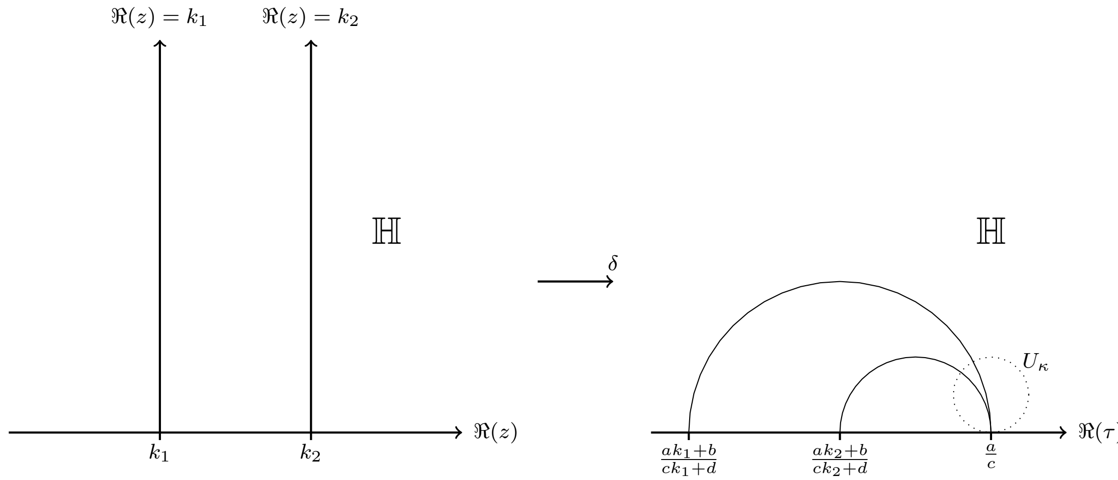

Figure 2.2: The map δ sends vertical lines to circles centered on the real axis which intersect a/c.

Then we must have

$$\delta^{-1}(\tau_1) = \delta^{-1}\gamma(\tau_2),$$
$$\delta^{-1}(\tau_1) = \delta^{-1}\gamma\delta(\delta^{-1}(\tau_2)).$$

Now of course, $\delta^{-1}\gamma\delta \in \mathrm{SL}(2,\mathbb{Z})$. And it can easily be shown that if $\gamma' \in \mathrm{SL}(2,\mathbb{Z})$, and both $\tau, \gamma'\tau \in \mathcal{N}_R$ for $R \geq 2$, then $\gamma' \in \mathrm{SL}(2,\mathbb{Z})_\infty$.

Since $\delta^{-1}(\tau_1), \delta^{-1}(\tau_2) \in \mathcal{N}_R$, we must have $\delta^{-1}\gamma\delta \in \mathrm{SL}(2,\mathbb{Z})_\infty$.

This means that $\delta^{-1}(z_2) = \delta^{-1}(z_1) + h$, for some $h \in \mathbb{Z}$. In particular, the complex numbers τ such that $\Re(\tau_1) - 1/2 < \Re(\tau) < \Re(\tau_1) + 1/2$ and $\Im(\tau) \geq R$ must be distinct with respect to $\Gamma_0(N)\delta$, since none of these numbers can be translations of each other.

Such a vertical strip has width 1. However, it might be possible that vertical strips of a greater width will be needed to contain a single equivalent representative of *every* point in U_κ. For the moment we will call the necessary width h. In that case, we define

$$\mathcal{S}_h := \{z \in \mathcal{N}_R : -h/2 \leq \Re z < h/2\} \cup \{\infty\}.$$

Then every point in U_κ will be equivalent to a single point in $\delta(\mathcal{S}_h) \subset U_\kappa$. As Figure 2.2 suggests, such a $\delta(\mathcal{S}_h)$ will take the form of a "triangular" sector (in which the sides of the triangle are circular arcs).

We can map $\hat{\mathbb{H}}$ to a neighborhood V of 0 (in the standard topology of $\mathbb{C}$) in the following way:

$$\begin{aligned} \rho_\kappa : \hat{\mathbb{H}} &\longrightarrow V, \\ z &\longmapsto \exp\left(2\pi i \, \frac{1}{h} z\right), \\ \infty &\longmapsto 0. \end{aligned}$$

Notice that ρ_κ is a bijection when restricted to $\mathcal{S}_h$.

If we now start with $\pi(U_\kappa)$, we can define

$$\pi^{-1} : \pi(U_\kappa) \longrightarrow \delta\left(\mathcal{S}_h\right).$$

As $\delta\left(\mathcal{S}_h\right)$ contains a single equivalent point to every point of U_κ, π^{-1} is well-defined (and of course bijective) on this domain. If one then composes π^{-1} with the bijection $\rho_\kappa \delta^{-1}$, one has the overall bijection

$$\begin{aligned} \varphi_\kappa :\ \pi\left(U_\kappa\right) &\longrightarrow V, \\ \pi(z) &\longmapsto \exp\left(2\pi i \, \frac{1}{h}\left(\frac{dz-b}{-cz+a}\right)\right), \\ \pi(a/c) &\longmapsto 0. \end{aligned} \tag{2.3}$$

Because π is a continuous open map, its inverse (when it exists over open sets) will also be open and continuous. Composed with $\rho_\kappa \delta^{-1}$, φ_κ is a homeomorphism from $\pi(U_\kappa)$ to V.

The appropriate value of h for $\Gamma_0(N)$ is $N/\gcd(c^2, N)$ (this is the *width* of the cusp $[a/c]_N$ of $\mathrm{X}_0(N)$ [99, Chapter 4, Section 8.3], [34, Section 2.4]).

Local Coordinates in Summary.

We have constructed an atlas for $\mathrm{X}_0(N)$, which has the form

$$\left\{\pi(U_\lambda), \varphi_\lambda\right\}_{\lambda \in \Lambda},$$

in which each U_λ is a neighborhood in $\hat{\mathbb{H}}$.

- If we are constructing local coordinates of a point x of $\mathrm{X}_0(N)$ which is not a cusp, our neighborhood $\pi(U_\mu)$ of x is constructed in which U_μ is an open set in $\mathbb{H}$ such that (2.1) applies.
- If we are constructing local coordinates of a cusp $[a/c]_N$ of $\mathrm{X}_0(N)$, our U_κ has the form $\delta(\mathcal{N}_R)$, in which $R \geq 2$ and $\delta \in \mathrm{SL}(2, \mathbb{Z})$ such that $\delta(\infty) \in [a/c]_N$.

Atlas Compatibility.

We now need to check that the charts in this atlas are pairwise compatible—that is, we need to check that $\varphi_\mu\varphi_\nu^{-1}$ is holomorphic on the domain $\varphi_\nu\left(\pi(U_\mu)\cap\pi(U_\nu)\right)$ whenever $\pi(U_\mu)\cap\pi(U_\nu)\neq\emptyset$.

Notice that—by definition—every point $z\in\varphi_\nu\left(\pi(U_\mu)\cap\pi(U_\nu)\right)$ can be expressed as $z=\varphi_\nu(x)$ for some $x\in\pi(U_\mu)\cap\pi(U_\nu)$. To prove that $\varphi_\mu\varphi_\nu^{-1}$ is holomorphic on $\varphi_\nu\left(\pi(U_\mu)\cap\pi(U_\nu)\right)$, it is sufficient to show that for every point $z=\varphi_\nu(x)\in\varphi_\nu\left(\pi(U_\mu)\cap\pi(U_\nu)\right)$, the mapping $\varphi_\mu\varphi_\nu^{-1}$ is holomorphic on some neighborhood V of z such that $V\subseteq\varphi_\nu\left(\pi(U_\mu)\cap\pi(U_\nu)\right)$.

Intersecting Neighborhoods Excluding Cusps.

We begin by considering intersecting neighborhoods of $\mathrm{X}_0(N)$ which do not include cusps. Suppose that we start with some $z=\varphi_\nu(x)$ for some $x\in\pi(U_\mu)\cap\pi(U_\nu)$. We know that

$$x=\pi(\tau_\mu)=\pi(\tau_\nu),$$

for $\tau_\mu\in U_\mu$, $\tau_\nu\in U_\nu$. More specifically, we will suppose that

$$\tau_\mu=\gamma(\tau_\nu),$$

for some $\gamma\in\Gamma_0(N)$. Define the set

$$U_{\mu,\nu}:=\gamma^{-1}\left(U_\mu\right)\cap U_\nu.$$

It is easy to see that $x\in\pi(U_{\mu,\nu})\subseteq\pi(U_\mu)\cap\pi(U_\nu)$, so that $z\in\varphi_\nu\left(\pi(U_{\mu,\nu})\right)$. We will show that $\varphi_\mu\varphi_\nu^{-1}$ is holomorphic on this set.

Case I: $\varphi_\nu(x)=0$.

We first suppose that $z=\varphi_\nu(x)=0$. We now take some $z'\in\varphi_\nu\left(\pi(U_{\mu,\nu})\right)$, such that $z'=\varphi_\nu(x')$ for some $x'\in\pi\left(U_{\mu,\nu}\right)$. Next, we let $x'=\pi(\tau')$ for $\tau'\in U_{\mu,\nu}$. We have

$$z'=\varphi_\nu(x')=\varphi_\nu(\pi(\tau'))=\left(\frac{\tau'-\tau_\nu}{\tau'-\bar{\tau}_\nu}\right)^{h(\tau_\nu)}.$$

We now want to examine $\varphi_\mu\varphi_\nu^{-1}(z')$. Notice that $\gamma\tau'\in U_\mu$, so that we can write

$$\varphi_\mu\varphi_\nu^{-1}(z')=\varphi_\mu(x')=\varphi_\mu(\pi(\gamma\tau'))=\left(\frac{\gamma\tau'-w}{\gamma\tau'-\bar{w}}\right)^{h(w)},$$

for the $w \in U_\mu$ such that $\varphi_\mu(\pi(w)) = 0$. Notice that, by the construction of the local coordinate φ_μ, w is the only possible elliptic point in U_μ. Continuing this calculation, we have

$$\begin{aligned}\varphi_\mu\varphi_\nu^{-1}(z') &= \left(\left(\begin{pmatrix}1 & -w\\ 1 & -\bar{w}\end{pmatrix}\gamma\begin{pmatrix}1 & -\tau_\nu\\ 1 & -\bar{\tau}_\nu\end{pmatrix}^{-1}\right)\left(\begin{pmatrix}1 & -\tau_\nu\\ 1 & -\bar{\tau}_\nu\end{pmatrix}\tau'\right)\right)^{h(w)}\\ &= \left(\left(\begin{pmatrix}1 & -w\\ 1 & -\bar{w}\end{pmatrix}\gamma\begin{pmatrix}1 & -\tau_\nu\\ 1 & -\bar{\tau}_\nu\end{pmatrix}^{-1}\right)\left(\varphi_\nu(x')^{1/h(\tau_\nu)}\right)\right)^{h(w)}\\ &= \left(\left(\begin{pmatrix}1 & -w\\ 1 & -\bar{w}\end{pmatrix}\gamma\begin{pmatrix}1 & -\tau_\nu\\ 1 & -\bar{\tau}_\nu\end{pmatrix}^{-1}\right)\left((z')^{1/h(\tau_\nu)}\right)\right)^{h(w)}\\ &= \Phi_{\mu,\nu}\left((z')^{1/h(\tau_\nu)}\right),\end{aligned}$$

in which $\Phi_{\mu,\nu}$ is a well-defined holomorphic map. Notice that if $h(\tau_\nu) = 1$, then $\varphi_\mu\varphi_\nu^{-1}(z')$ is holomorphic. Otherwise, τ_ν must be elliptic; and since $\tau_\mu = \gamma\tau_\nu$, $\tau_\mu \in U_\mu$ must also be elliptic, with $h(\tau_\mu) = h(\tau_\nu)$. But w is the *only* elliptic point in U_μ. Therefore, we must have $\tau_\mu = w$, and $h(w) = h(\tau_\mu) = h(\tau_\nu)$.

Finally, we must examine $\Phi_{\mu,\nu}$. Notice that

$$\begin{pmatrix}1 & -w\\ 1 & -\bar{w}\end{pmatrix}\gamma\begin{pmatrix}1 & -\tau_\nu\\ 1 & -\bar{\tau}_\nu\end{pmatrix}^{-1} = \begin{pmatrix}1 & -\tau_\mu\\ 1 & -\bar{\tau}_\mu\end{pmatrix}\gamma\begin{pmatrix}1 & -\tau_\nu\\ 1 & -\bar{\tau}_\nu\end{pmatrix}^{-1}.$$

If we denote this product of matrices as $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, then we have

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix} : \begin{cases} 0 \longmapsto 0,\\ \infty \longmapsto \infty.\end{cases}$$

We must therefore have $\frac{A(0)+B}{C(0)+D} = 0$ so that $B = 0$ and $D \neq 0$; and $\frac{A(\infty)}{C(\infty)+D} = \frac{A}{C} = \infty$, whence $C = 0$ and $A \neq 0$. We then have

$$\begin{aligned}\varphi_\mu\varphi_\nu^{-1}(z') = \Phi_{\mu,\nu}\left((z')^{1/h(\tau_\nu)}\right) &= \left(\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}(z')^{1/h(\tau_\nu)}\right)^{h(\tau_\nu)}\\ &= \frac{A^{h(\tau_\nu)}}{D^{h(\tau_\nu)}}\cdot z'\\ &= r\cdot z'\end{aligned}$$

for some $r \in \mathbb{C} - \{0\}$, and this is of course holomorphic.

Case II: $\varphi_\mu(x) = 0$.

We can apply a similar argument to Case I above for the reverse transition map: that is, we can prove that the transition map $\varphi_\nu\varphi_\mu^{-1}$ is holomorphic on $\varphi_\mu(\pi(U_\mu) \cap \pi(U_\nu))$. Since the maps φ_λ are all homeomorphisms, we can take the inverse

$$\left(\varphi_\nu\varphi_\mu^{-1}\right)^{-1} = \varphi_\mu\varphi_\nu^{-1}.$$

Finally, since the inverse of a holomorphic bijection must also be holomorphic, the map $\varphi_\mu\varphi_\nu^{-1}$ must be holomorphic.

Case III: $\varphi_\mu(x) \neq 0$**,** $\varphi_\nu(x) \neq 0$.

Recall that $x = \pi(\tau_\nu)$. We can always construct a neighborhood U_λ of τ_ν such that τ_ν is the only possible elliptic point of U_λ. In that case, by the rules of our atlas construction, we have the local coordinate

$$\begin{aligned} \varphi_\lambda : \mathrm{X}_0(N) &\longrightarrow V \subseteq \mathbb{C}, \\ \pi(\tau_\nu) &\longmapsto 0. \end{aligned}$$

We want to prove that $\varphi_\mu\varphi_\nu^{-1}$ must be holomorphic. To do this, we write

$$\varphi_\mu\varphi_\nu^{-1} = \left(\varphi_\mu\varphi_\lambda^{-1}\right)\left(\varphi_\lambda\varphi_\nu^{-1}\right).$$

Because $\varphi_\lambda(x) = 0$, Case I above shows that $\varphi_\mu\varphi_\lambda^{-1}$ is holomorphic, while Case II shows that $\varphi_\lambda\varphi_\nu^{-1}$ is holomorphic. The composition is therefore holomorphic.

A Neighborhood in $\mathbb{H}$ Intersecting a Neighborhood of a Cusp.

We now consider the neighborhood of a cusp $[a/c]_N$, intersecting a neighborhood which contains no cusp. We take U_μ to be an open neighborhood of a point $\tau_0 \in \mathbb{H}$, in which τ_0 is the only possible elliptic point. We also take U_κ to be an open neighborhood of $a/c \in \hat{\mathbb{Q}}$ as defined for our atlas, i.e., of the form

$$U_\kappa := \delta_\kappa\left(\mathcal{N}_R \cup \{\infty\}\right),$$

with $R \geq 2$ and $\delta_\kappa = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$. Notice that because $R \geq 2$, U_κ contains no elliptic points.

In that case, we are interested in the intersection $\pi(U_\mu) \cap \pi(U_\kappa)$.

Suppose that $x \in \pi(U_\mu) \cap \pi(U_\kappa)$, with $x = \pi(\tau_\mu) = \pi(\tau_\kappa)$, with $\tau_\mu \in U_\mu$ and $\tau_\kappa \in U_\kappa$. We know that $\tau_\kappa = \gamma(\tau_\mu)$ for some $\gamma \in \Gamma_0(N)$.

In similar manner to the case of neighborhoods with no cusps, we will define

$$U_{\mu,\kappa} := U_\mu \cap \gamma^{-1}(U_\kappa).$$

We have $x = \pi(\tau_\mu) \in \pi(U_{\mu,\kappa}) \subseteq \pi(U_\mu) \cap \pi(U_\kappa)$, and

$$z := \varphi_\mu(x) \in \varphi_\mu(\pi(U_{\mu,\kappa})) \subseteq \varphi_\mu(\pi(U_\mu) \cap \pi(U_\kappa)).$$

Let $z' \in \varphi_\mu(\pi(U_{\mu,\kappa}))$. We will show that $\varphi_\kappa\varphi_\mu^{-1}(z')$ is holomorphic.
We can represent z' as

$$z' = \varphi_\mu(x') = \left(\begin{pmatrix} 1 & -\tau_0 \\ 1 & -\bar{\tau}_0 \end{pmatrix}(\tau')\right)^{h_0},$$

for some $x' \in \pi(U_{\mu,\kappa})$ $\tau' \in U_{\mu,\kappa}$, and $h_0 = |\Gamma_0(N)_{\tau_0}|$. With this in mind, we write

$$\begin{aligned}
\varphi_\kappa\varphi_\mu^{-1}(z') = \varphi_\kappa(\pi(\gamma\tau')) &= \exp\left(2\pi i\frac{\gcd(c^2,N)}{N}\delta_\kappa^{-1}\gamma\tau'\right) \\
&= \exp\left(2\pi i\frac{\gcd(c^2,N)}{N}\delta_\kappa^{-1}\gamma\begin{pmatrix} 1 & -\tau_0 \\ 1 & -\bar{\tau}_0 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -\tau_0 \\ 1 & -\bar{\tau}_0 \end{pmatrix}\tau'\right) \\
&= \exp\left(2\pi i\frac{\gcd(c^2,N)}{N}\delta_\kappa^{-1}\gamma\begin{pmatrix} 1 & -\tau_0 \\ 1 & -\bar{\tau}_0 \end{pmatrix}^{-1}\varphi_\mu(x')^{1/h_0}\right) \\
&= \exp\left(2\pi i\frac{\gcd(c^2,N)}{N}\delta_\kappa^{-1}\gamma\begin{pmatrix} 1 & -\tau_0 \\ 1 & -\bar{\tau}_0 \end{pmatrix}^{-1}(z')^{1/h_0}\right).
\end{aligned}$$

If $h_0 = 1$, then this certainly is holomorphic. On the other hand, if $h_0 > 1$, then the map can still be considered holomorphic, provided that $z' \neq 0$. But if $z' = 0$, then $z' = \varphi_\mu(\tau_0)$, in which $\tau_0 \in U_{\mu,\kappa}$, and $\gamma\tau_0 \in U_\kappa$ are both elliptic. But by construction, U_κ excludes all elliptic points. So $0 \notin \varphi_\mu(\pi(U_{\mu,\kappa}))$, and our map is holomorphic.

Because $\varphi_\kappa\varphi_\mu^{-1}$ is also a bijection, its inverse $\varphi_\mu\varphi_\kappa^{-1}$ is also holomorphic. We have therefore proved compatibility between a neighborhood with a cusp and a neighborhood with no cusp—in both directions.

Neighborhoods of Two Cusps.

As a final case, we consider neighborhoods of rational points a_1/c_1, a_2/c_2. We let

$$\begin{aligned}
U_\kappa &:= \delta_\kappa(\mathcal{N}_R \cup \{\infty\}), \\
U_\chi &:= \delta_\chi(\mathcal{N}_R \cup \{\infty\})
\end{aligned}$$

with $R \geq 2$ and

$$\delta_\kappa := \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \delta_\chi := \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z}).$$

Before we examine the transition map, we should take note of an important fact. Supposing that $\pi(U_\kappa) \cap \pi(U_\chi) \neq \emptyset$, there must exist some $\gamma \in \Gamma_0(N)$ such that

$$U_\kappa \cap \gamma\left(U_\chi\right) \neq \emptyset,$$

i.e.,

$$\delta_\kappa^{-1} \gamma \delta_\chi(z) \in \mathcal{N}_R \cup \{\infty\}$$

for some $z \in \mathcal{N}_R \cup \{\infty\}$.

In constructing our atlas near cusps, we have already demonstrated that if $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ and $\tau, \gamma\tau \in \mathcal{N}_R$ for $R \geq 2$, then $\gamma \in \mathrm{SL}(2, \mathbb{Z})_\infty$, i.e., γ acts on τ by translation. Therefore, we have

$$\delta_\mu^{-1} \gamma \delta_\nu = S^k,$$
$$\gamma \delta_\nu S^{-k} = \delta_\mu,$$

for some $k \in \mathbb{Z}$. Therefore, $a_1/c_1 \in [a_2/c_2]_N$, and we are only near one cusp of $\mathrm{X}_0(N)$. Moreover, recall from Theorem 2.11 that we must have

$$c_2 \equiv mc_1 \pmod{N}$$

for some $m \in \mathbb{Z}$ with $\gcd(m, N) = 1$. Therefore, $c_2 = mc_1 + Nk$ for some $k \in \mathbb{Z}$, and

$$\gcd(c_2^2, N) = \gcd(m^2 c_1^2 + 2mc_1Nk + N^2k^2, N) = \gcd(m^2c_1^2, N) = \gcd(c_1^2, N).$$

With this in mind, we take some $x \in \pi(U_\kappa) \cap \pi(U_\chi)$, with $x = \pi(\tau_\kappa) = \pi(\tau_\chi)$, with $\tau_\kappa \in U_\kappa$, $\tau_\chi \in U_\chi$, and $\tau_\chi = \gamma(\tau_\kappa)$ for some $\gamma \in \Gamma_0(N)$.

We want to verify that the transition map $\varphi_\chi \varphi_\kappa^{-1}$ is holomorphic. To do this, we again define

$$U_{\kappa,\chi} := U_\kappa \cap \gamma\left(U_\chi\right),$$

and take some $x \in \pi(U_{\kappa,\chi})$. We then define $q = \varphi_\kappa(x) \in \varphi_\kappa\left(\pi(U_{\kappa,\chi})\right)$, and we have

$$q = \exp\left(2\pi i \cdot \frac{\gcd(c_1^2, N)}{N} \delta_\kappa^{-1}(\tau_\kappa)\right).$$

Notice that $\gamma\tau \in U_\chi$.

Our transition map, then, is

$$\begin{aligned}
\varphi_\chi\varphi_\kappa^{-1}(q) &= \exp\left(2\pi i \cdot \frac{\gcd(c_2^2, N)}{N}\delta_\chi^{-1}\gamma(\tau)\right) \\
&= \exp\left(2\pi i \cdot \frac{\gcd(c_2^2, N)}{N}\delta_\chi^{-1}\gamma\delta_\kappa\left(\delta_\kappa^{-1}\tau\right)\right) \\
&= \exp\left(2\pi i \cdot \frac{\gcd(c_1^2, N)}{N}\left(\delta_\kappa^{-1}\tau + k\right)\right) \\
&= e^{2\pi i k \cdot \gcd(c_2^2, N)/N} \exp\left(2\pi i \cdot \frac{\gcd(c_1^2, N)}{N}\left(\delta_\kappa^{-1}\tau\right)\right) \\
&= e^{2\pi i k \cdot \gcd(c_2^2, N)/N} q,
\end{aligned}$$

which is holomorphic in q.

We have verified that the local coordinates constructed for the neighborhoods of $\mathrm{X}_0(N)$ are pairwise compatible, and the Riemann surface structure of $\mathrm{X}_0(N)$ is established.

□

2.3.4 Some Important Theorems

Analytic Restrictions

The Riemann surface structure of $\mathrm{X}_0(N)$ allows the tools of complex analysis to be used to study the functions whose domain is $\mathrm{X}_0(N)$. Moreover, the compactness of $\mathrm{X}_0(N)$ imposes an especially strong condition on its associated holomorphic functions.

Definition 2.23. Let X be a Riemann surface. A function $\hat{f} : \mathrm{X} \longrightarrow \hat{\mathbb{C}}$ is *meromorphic* at $x \in \mathrm{X}$ if there exists a local coordinate (U, φ) at x, expressed as $q = \varphi(x')$ for $x' \in U$, with $\varphi(x) = 0$, such that for all $x' \in U$,

$$\hat{f}(x') = \sum_{n=n_0}^{\infty} \alpha(n) q^n$$

where $\alpha(n) \in \mathbb{C}$ for all $n \geq n_0$, $\alpha(n_0) \neq 0$, and $n_0 \in \mathbb{Z}$. If $n_0 \geq 0$, then $\hat{f}$ is *analytic* at x. The number n_0 is the *order* of $\hat{f}$ at x. If $n_0 > 0$, then $\hat{f}$ has a *zero* at x. If $n_0 < 0$, then $\hat{f}$ has a *pole* at x.

These are the Riemann surface analogues for meromorphic and analytic functions of complex analysis, and many of the important properties for analytic functions over $\mathbb{C}$ still apply.

Theorem 2.24. *[72, Chapter II, Theorem 1.37] Let* X *be a compact Riemann surface, and let* $\hat{f} : \mathrm{X} \longrightarrow \mathbb{C}$ *be analytic on all of* X. *Then* $\hat{f}$ *must be a constant function.*

Proof. Let $\hat{f} : \mathrm{X} \longrightarrow \mathbb{C}$ be analytic on all of X. Because X is compact, $|\hat{f}(x')|$ must attain a maximum at some point $x \in \mathrm{X}$. But $\hat{f}$ is holomorphic, and therefore an open mapping. Therefore, given a neighborhood U of x, the set $\hat{f}(U)$ is an open set containing $\hat{f}(x)$. □

Lehner has referred to this theorem as "the fundamental theorem of the subject [of modular functions]" [69, Chapter 1, Section 3]. Certainly from a computational standpoint, its importance cannot easily be overstated.

In the first place, it prohibits functions which are holomorphic over the whole of $\mathrm{X}_0(N)$. On the other hand, we shall see that $\mathrm{X}_0(N)$ admits meromorphic functions (this is a nontrivial fact, but we will demonstrate it by producing examples). Because of this, Theorem 2.24 is an extremely powerful tool. Supposing we have two functions, $\hat{f}, \hat{g}$ which are meromorphic over $\mathrm{X}_0(N)$, with matching poles $x_1, x_2, ..., x_m$. By hypothesis, each function is analytic everywhere else on $\mathrm{X}_0(N)$, and has a principal part at each x_j, which we may denote as

$$\left(\hat{f}(x_j')\right)^{(-)} := \sum_{n=n_j(\hat{f})}^{-1} \alpha_j(n) q_j^n,$$

$$\left(\hat{g}(x_j')\right)^{(-)} := \sum_{n=n_j(\hat{g})}^{-1} \beta_j(n) q_j^n.$$

for x_j' in a given neighborhood of x_j, and $q_j = \varphi(x_j')$ the corresponding local variable at $\varphi(x_j)$, $1 \leq j \leq m$.

Let us suppose that for each j, $n_j(\hat{f}) = n_j(\hat{g})$, and that $\alpha_j(n) = \beta_j(n)$ for $n < 0$. In this case, $\hat{f} - \hat{g}$ must have have no principal part at any x_j, and thus must be analytic at each x_j. Since $\hat{f}$, $\hat{g}$ are analytic everywhere else on $\mathrm{X}_0(N)$, and subtraction of analytic functions does not induce a pole, Theorem 2.24 compels that $\hat{f} - \hat{g} - c = 0$ for some $c \in \mathbb{C}$.

Now, the compactness of $\mathrm{X}_0(N)$, together with the meromorphicity of $\hat{f}$, $\hat{g}$, demand that $\hat{f}$, $\hat{g}$ can only have a finite number of poles. This fact, together with the fact that each principal part is finite in length, allows us to determine whether $\hat{f} = \hat{g}$ in only a finite number of steps.

Thus, the somewhat baroque theory of compact Riemann surfaces described above gives us (in principle, at least) a means of examining their corresponding meromorphic functions using a finite computational method. Indeed, this theory justifies the soundness of the sophisticated algorithmic tools that we will describe and utilize in Chapter 3, and whose significance will follow into the later chapters.

The Genus

Theorem 2.24 is certainly useful with respect to its algorithmic potential. However, we can go still further. The manifold structure of a Riemann surface implies the following proposition:

Proposition 2.25. *A compact Riemann surface* X *is an orientable path-connected 2-dimensional* $\mathcal{C}^\infty$ *real manifold, which is diffeomorphic to a* $\mathfrak{g}$*-handled torus for a unique* $\mathfrak{g} \in \mathbb{Z}_{\geq 0}$.

Consult [102] for a proof. We emphasize that in this context, a 0-handled torus is here defined as a sphere.

Definition 2.26. Let $\mathfrak{g} \in \mathbb{Z}_{\geq 0}$. A manifold X has *genus* $\mathfrak{g}$ if X is diffeomorphic to a torus with $\mathfrak{g}$ handles. The genus of X is often denoted as $\mathfrak{g}(\mathrm{X})$.

As a consequence of the proposition above, $\mathrm{X}_0(N)$ must be homeomorphic to a torus with a finite (possibly 0) number of handles.

For example, $\mathfrak{g}(\mathrm{X}_0(1)) = 0$. The genus will play a fundamental role in our later work. In particular, the genus affects the rank of the spaces of modular functions associated with a given family of partition congruences.

A more complete description of the impact of $\mathfrak{g}(\mathrm{X})$ on our ability to do analysis on X is embodied in the Riemann-Roch theorem [72, Chapter VI, Theorem 3.1]. However, we will not require the full theorem, and instead focus on one of its important corollaries:

Theorem 2.27 (Weierstrass)**.** *Let* X *be a compact Riemann surface, and let*

$$f : \mathrm{X} \longrightarrow \hat{\mathbb{C}}$$

be holomorphic over X*, except for a pole at a point* $p \in \mathrm{X}$*. Then the order of* f *at* p *can assume any negative integer, with exactly* $\mathfrak{g}(\mathrm{X})$ *exceptions, which must be members of the set* $\{1, 2, ..., 2\mathfrak{g} - 1\}$.

This is the *Lückensatz*, or *gap theorem*. Traditionally it may be proved after the Riemann–Roch theorem, but it has recently been proved as a consequence of the Riemann–Hurwitz formula [84].

Theorems 2.24 and 2.27 are enormously useful to us, in that they give important restrictions to the behavior of meromorphic functions defined on X.

We provide a formula which can be used to compute $\mathfrak{g}(\mathrm{X}_0(N))$ [34, Theorem 3.11, Corollary 3.7.2, Section 3.8].

Theorem 2.28.

$$\mathfrak{g}(\mathrm{X}_0(N)) = 1 + \frac{[\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0(N)]}{12} - \frac{\epsilon_2(\Gamma_0(N))}{4} - \frac{\epsilon_3(\Gamma_0(N))}{3} - \frac{\epsilon_\infty(\Gamma_0(N))}{2},$$

with

$$[\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right),$$

$$\epsilon_2(\Gamma_0(N)) = \begin{cases} \prod_{p|N} \left(1 + \left(\frac{-4}{p}\right)\right), & 4 \nmid N, \\ 0, & 4|N, \end{cases},$$

$$\epsilon_3(\Gamma_0(N)) = \begin{cases} \prod_{p|N} \left(1 + \left(\frac{-3}{p}\right)\right), & 9 \nmid N, \\ 0, & 9|N, \end{cases},$$

$$\epsilon_\infty(\Gamma_0(N)) = \sum_{\delta|N} \phi\left(\gcd(\delta, N/\delta)\right).$$

Here, $\phi(n)$ is Euler's totient function, $\left(\frac{\cdot}{p}\right)$ is the Legendre–Jacobi quadratic residue character, and any product over $p|N$ is taken only over the positive prime divisors of N. The product or sum over $\delta|N$ is taken over all of the positive divisors of N.

Remark 2.29 (Important!)**.** Many formula implementations for $\mathfrak{g}\left(\mathrm{X}_0(N)\right)$, e.g., that found in [99, Chapter 4, Theorem 15] do not always return the correct value. The reader is advised to use these formulæ with care. A useful precaution is to compare computational results of a given implementation of $\mathfrak{g}\left(\mathrm{X}_0(N)\right)$ with Sequence A001617 of Sloane's On-Line Encyclopedia of Integer Sequences [78].

Using this formula, we can establish that for $1 \leq N \leq 30$, we have

$$\mathfrak{g}\left(\mathrm{X}_0(N)\right) = \begin{cases} 0, & 1 \leq N \leq 10,\ N = 12, 13, 16, 18, 25, \\ 1, & N = 11, 14, 15, 17, 19, 20, 21, 24, 27 \\ 2, & N = 22, 23, 26, 28, 29 \\ 3, & N = 30. \end{cases}$$

2.4 Modular Functions

With this background in modular curves established, we take a meromorphic function $\hat{f} : \mathrm{X}_0(N) \longrightarrow \hat{\mathbb{C}}$. We may define a function $f : \hat{\mathbb{H}} \longrightarrow \hat{\mathbb{C}}$ with $f(\tau) := \hat{f}([\tau]_N)$. This function f exhibits the symmetry

$$f\left(\frac{a\tau + b}{Nc\tau + d}\right) = f(\tau),$$

for all $a, b, c, d \in \mathbb{Z}$ such that $ad - Nbc = 1$.

Hereafter, we let $q := q(\tau) = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$.

Definition 2.30. Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be holomorphic on $\mathbb{H}$. Then f is a *modular function* over $\Gamma_0(N)$ if the following properties are satisfied:

1. If $\tau_1, \tau_2 \in \mathbb{H}$ such that $\tau_2 \in [\tau_1]_N$, we have $f(\tau_2) = f(\tau_1)$;
2. For every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$, we have

$$f\left(\gamma\tau\right) = \sum_{n=n_\gamma}^{\infty} \alpha_\gamma(n) q^{n\,\gcd(c^2,N)/N},$$

with $n_\gamma \in \mathbb{Z}$, and $\alpha_\gamma(n_\gamma) \neq 0$. If $n_\gamma \geq 0$, then f is holomorphic at a/c. Otherwise, f is meromorphic with a pole of order n_γ, and principal part

$$\sum_{n=n_\gamma}^{-1} \alpha_\gamma(n) q^{n\,\gcd(c^2,N)/N}, \tag{2.4}$$

at a/c.

We refer to $\operatorname{ord}_{a/c}^{(N)}(f) := n_\gamma(f)$ as the order of f at a/c.

The notions of pole order and cusps of f used in Definition 2.30 have been constructed so as to coincide with the notions of Definition 2.23. Indeed, there is a bijection between the set of modular functions over $\Gamma_0(N)$ and the set of meromorphic functions for $\mathrm{X}_0(N)$ with poles only at the cusps [66, Chapter VI, Theorem 4A]. Moreover, many of the most important properties of either set of functions coincide.

In particular, (2.4) represents the principal part of $\hat{f}$ in a local coordinate near the cusp $[a/c]_N$. Notice the similarity to the local variable in the neighborhood of $\pi(a/c)$ in (2.3). Intuitively, we can see that as $\tau \to i\infty$, we must have $\gamma\tau \to a/c$, and $q \to 0$. Understanding the close relationship between f and $\hat{f}$, the reader might expect that the order of a modular function at a given cusp is unique, *regardless of the representative of the cusp.* This is correct:

Lemma 2.31. *Given two elements $a_1/c_1, a_2/c_2 \in \hat{\mathbb{Q}}$ which reside in the same orbit, and some function f which is modular over $\Gamma_0(N)$, we have $\operatorname{ord}_{a_1/c_1}^{(N)}(f) = \operatorname{ord}_{a_2/c_2}^{(N)}(f)$.*

Definition 2.32. For $N \geq 1$ we define a function $\hat{f} : \mathrm{X}_0(N) \to \mathbb{C}$ to be *induced* by a modular function f if for all $\tau \in \hat{\mathbb{H}}$ we have

$$\hat{f}([\tau]_N) = f(\tau).$$

The usefulness of this correspondence between meromorphic functions on $\mathrm{X}_0(N)$ and modular functions over $\Gamma_0(N)$ becomes clear when we remember that many important properties of the functions on $\mathrm{X}_0(N)$ are determined by its topology. Thus, the topology of Riemann surfaces can influence the form of the associated modular functions, which in turn can influence our understanding of partition congruences, as we will see in the sequel.

Remark 2.33. Some authors consider a modular function to have possible poles *anywhere in* $\hat{\mathbb{H}}$, rather than exclusively on $\hat{\mathbb{Q}}$ as we have defined. Other authors prefer the term *weight-0 automorphic form* for such functions [34, Section 3.2]. Many of the objects that we call modular functions, e.g., modular eta quotients, are sometimes referred to as *modular units* [65]. The reader is advised to be careful in consulting the technical literature, as conventions are not standardized.

We now define the relevant sets of all modular functions:

Definition 2.34. Denote the set of modular functions on $\Gamma_0(N)$ by $\mathcal{M}\left(\Gamma_0(N)\right)$. For any $a/c \in \hat{\mathbb{Q}}$, let $\mathcal{M}^{a/c}\left(\Gamma_0(N)\right) \subset \mathcal{M}\left(\Gamma_0(N)\right)$ denote the set of modular functions over $\Gamma_0(N)$ with a pole only at the cusp $[a/c]_N$. These are commutative algebras with 1, under standard addition and multiplication. Finally, if $\mathcal{M}^{a/c}\left(\Gamma_0(N)\right) = \mathbb{C}[t]$ for a function t, then t is a *principal modular function*, or *Hauptmodul*, for $\mathcal{M}^{a/c}\left(\Gamma_0(N)\right)$.

As an additional notational matter, for any set $\mathcal{S} \subseteq \mathcal{M}\left(\Gamma_0(N)\right)$, and any field $\mathbb{K} \subseteq \mathbb{C}$, define

$$\mathcal{S}_{\mathbb{K}} := \left\{ f \in \mathcal{S} : \alpha_I(n) \in \mathbb{K} \text{ for all } n \geq n_I(f) \right\},$$

in which the $\alpha_I(n)$ are the coefficients in the expansion of f at ∞ as in Definition 2.30. Also, for any set $\mathcal{S}$ of functions on $\mathbb{C}$, denote

$$\langle \mathcal{S} \rangle_{\mathbb{K}} := \left\{ \sum_{u=1}^{v} r_u \cdot g_u : g_u \in \mathcal{S}, r_u \in \mathbb{K} \right\}.$$

These notations will become useful later.

We now give a theorem which is a natural consequence of Theorem 2.24:

Theorem 2.35. *A modular function over $\Gamma_0(N)$ which is holomorphic at every cusp must be a constant.*

Proof. Let $f \in \mathcal{M}(\Gamma_0(N))$, and let $\hat{f} : \mathrm{X}_0(N) \longrightarrow \hat{\mathbb{C}}$ be the induced function over $\mathrm{X}_0(N)$. Suppose that f is holomorphic on $\mathbb{H}$ and at every cusp. Then $\hat{f}$ must be holomorphic on all of $\mathrm{X}_0(N)$. Therefore, $\hat{f}([\tau]_N) = f(\tau)$ must be a constant for all $[\tau]_N \in \mathrm{X}_0(N)$, and therefore for all $\tau \in \hat{\mathbb{H}}$. $\square$

This means that any nonconstant modular function must have a pole somewhere. Since we will take holomorphicity on $\mathbb{H}$ as a necessity, these poles must exist at $\hat{\mathbb{Q}}$.

We can give yet another restriction on the meromorphic behavior of modular functions, now imposed by Theorem 2.27:

Theorem 2.36. *Suppose that $N \in \mathbb{Z}_{\geq 1}$ and that $\mathfrak{g}(\mathrm{X}_0(N)) \geq 1$. Then there cannot exist an $f \in \mathcal{M}^{a/c}(\Gamma_0(N))$ such that $\mathrm{ord}_{a/c}^{(N)}(f) = -1$.*

Proof. Suppose that such a function f exists. Then f must induce the function $\hat{f}$ which is holomorphic on $\mathrm{X}_0(N)$ except for the cusp $[a/c]_N$, at which $\hat{f}$ has order -1. But then $\hat{f}^n$ will produce a function of order $-n$ for any negative integer $-n$ with 0 exceptions. This is a contradiction, as Theorem 2.27 demands that a single exception must exist. $\square$

Proving that modular functions of a given order over $\Gamma_0(N)$ actually exist is generally a more difficult problem. However, in Chapters 5-10, we will work with spaces of modular functions in which the underlying modular curve has genus 0 or 1, and in which a function of any order that does not conflict with Theorems 2.35 and 2.27 may be constructed.

For example, for $N = 5, 7, 10$, we will construct modular functions in $\mathcal{M}^0(\Gamma_0(N))$ and $\mathcal{M}^\infty(\Gamma_0(N))$ with order -1 (which is permitted, since the underlying modular curves have genus 0). On the other hand, for $N = 11, 20$, when the genus is 1, we will construct analogous functions with orders $-2, -3$, i.e., all orders that are not excluded by Theorem 2.27.

The means by which most of the modular functions relevant to us are actually constructed is revealed in the following section.

2.5 The Eta Function

2.5.1 Definition

The preceding theory does not address how to actually construct modular functions over a given congruence subgroup. The traditional theory generally begins with the Eisenstein series, and subsequent construction of the modular discriminant and the j invariant.

However, as the modular functions of interest to us will be expressed in terms of functions resembling $\tilde{F} = \prod_{m=1}^{\infty}(1-q^m)^{-1}$, we will begin by defining the eta (η) function of Dedekind [61, Chapter 3], and proceed to show how modular functions may be constructed using η.

Definition 2.37. For $\tau \in \mathbb{H}$,

$$\eta(\tau) := e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right).$$

The η function is not a modular function; it is a modular form of half-integral weight with a nontrivial multiplier system [61, Chapter 3, Theorem 10]. In particular, it exhibits near-symmetries with respect to the action of $\mathrm{SL}(2,\mathbb{Z})$ which are extraordinary, as well as useful.

Theorem 2.38. *Given* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$*, such that* $c \geq 0$*, we have*

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)(-i(c\tau+d))^{1/2}\eta(\tau),$$

and

$$\epsilon(a,b,c,d) := \begin{cases} \exp\left(\frac{\pi ib}{12}\right), & c=0,\\ \left(\frac{d}{c}\right) i^{(1-c)/2}\exp\left(\frac{\pi i}{12}\left(bd(1-c^2)+c(a+d)\right)\right), & c>0,\ 2\nmid c,\\ \left(\frac{c}{d}\right)\exp\left(\frac{\pi id}{4}+\frac{\pi i}{12}\left(ac(1-d^2)+d(b-c)\right)\right), & c>0,\ 2\nmid d. \end{cases}$$

Here, $z^{1/2}$ is taken along the principal branch, i.e., taking $z = re^{i\theta}$ with $r \geq 0$, $-\pi < \theta \leq \pi$.

Proofs can be found in [61, Chapters 3, 4], [88, Chapter 9], [10, Chapter 5, Examples 6-17], [23], and many others. None of the proofs are easy, but the proof in [61] is perhaps the most accessible.

We may construct modular functions of the form $\prod_{\delta|N}\eta(\delta\tau)^{r_\delta}$, for some integer-valued vector $(r_\delta)_{\delta|N}$.

For example, let us take the function

$$t(\tau) := \frac{\eta(5\tau)^6}{\eta(\tau)^6} = q\prod_{m=1}^{\infty}\left(\frac{1-q^{5m}}{1-q^m}\right). \tag{2.5}$$

Theorem 2.39. $t \in \mathcal{M}^0\left(\Gamma_0(5)\right)$*, and* $1/t \in \mathcal{M}^\infty\left(\Gamma_0(5)\right)$*.*

Proof. We will verify that t satisfies the properties of Definition 2.30.

Definition 2.30, Point 1.

We want to show that $t(\gamma\tau) = t(\tau)$ for $\tau \in \mathbb{H}$ and $\gamma \in \Gamma_0(5)$.
Let $\gamma = \left(\begin{smallmatrix} a & b \\ 5c & d \end{smallmatrix}\right) \in \Gamma_0(5)$ with $c \geq 0$. Notice that, because $\left(\begin{smallmatrix} a & 5b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, we can write

$$\begin{aligned} t\left(\frac{a\tau+b}{5c\tau+d}\right) =& \frac{\eta\left(5\cdot\frac{a\tau+b}{5c\tau+d}\right)^6}{\eta\left(\frac{a\tau+b}{5c\tau+d}\right)^6} = \frac{\eta\left(\frac{a(5\tau)+5b}{c(5\tau)+d}\right)^6}{\eta\left(\frac{a\tau+b}{5c\tau+d}\right)^6} \\ =& \frac{\epsilon(a,5b,c,d)^6}{\epsilon(a,b,5c,d)^6}\frac{(-i(c(5\tau)+d))^3}{(-i(5c\tau+d))^3}\frac{\eta(5\tau)^6}{\eta(\tau)^6} \\ =& \frac{\epsilon(a,5b,c,d)^6}{\epsilon(a,b,5c,d)^6}t(\tau). \end{aligned}$$

We need only determine that $\epsilon(a,5b,c,d)^6/\epsilon(a,b,5c,d)^6 = 1$.

For $c = 0$ we have

$$\frac{\epsilon(a,5b,c,d)^6}{\epsilon(a,b,5c,d)^6} = \exp\left(\frac{6\pi i}{12}(5b-b)\right) = \exp\left(\frac{\pi i}{2}(4b)\right) = 1.$$

For $c \neq 0$ and c odd, we have

$$\begin{aligned} \frac{\epsilon(a,5b,c,d)^6}{\epsilon(a,b,5c,d)^6} =& \frac{\left(\frac{d}{c}\right)^6 i^{3(1-c)}\exp\left(\frac{6\pi i}{12}\left(5bd(1-c^2)+c(a+d)\right)\right)}{\left(\frac{d}{5c}\right)^6 i^{3(1-5c)}\exp\left(\frac{6\pi i}{12}\left(bd(1-25c^2)+5c(a+d)\right)\right)} \\ =& i^{3(4c)}\exp\left(\frac{\pi i}{2}\left(5bd(1-c^2)+c(a+d)-bd(1-25c^2)-5c(a+d)\right)\right) \\ =& \exp\left(\frac{\pi i}{2}\left(5bd(1-c^2)+c(a+d)-bd(1-25c^2)-5c(a+d)\right)\right) \\ =& \exp\left(\frac{\pi i}{2}\left(4bd(1-25c^2)-4c(a+d)\right)\right) \\ =& 1. \end{aligned}$$

A similar result is achieved for $c \neq 0$ and d odd.

Definition 2.30, Point 2.

We now want to examine the behavior of t with respect to a $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) \setminus \Gamma_0(5)$. In this case, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} S^{-k}T^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ ck-d & * \end{pmatrix}.$$

Rearranging terms, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ ck-d & * \end{pmatrix} TS^k.$$

Since $\gamma \notin \Gamma_0(5)$, we have $\gcd(c, 5) = 1$. Therefore, as k cycles through the residues mod 5, so does ck, and $ck - d$. So for some k, we have $ck - d \equiv 0 \pmod 5$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(5) \backslash T / \mathrm{SL}(2, \mathbb{Z})_\infty.$$

Therefore, we can write

$$\gamma = \gamma_0 \cdot T \cdot S^k \tag{2.6}$$

for some $\gamma_0 \in \Gamma_0(5)$ and some $k \in \mathbb{Z}_{\geq 0}$. We therefore have

$$\begin{aligned} t(\gamma\tau) &= t\left(\gamma_0 \cdot T \cdot S^k \tau\right) && (2.7)\\ &= t\left(T \cdot S^k \tau\right) && (2.8)\\ &= t\left(T \cdot \tau'\right) && (2.9)\\ &= t\left(-1/\tau'\right) && (2.10)\\ &= \frac{\eta\left(5 \cdot -1/\tau'\right)^6}{\eta\left(-1/\tau'\right)^6} && (2.11)\\ &= \frac{\eta\left(-1/(\tau'/5)\right)^6}{\eta\left(-1/\tau'\right)^6} && (2.12) \end{aligned}$$

for $\tau' = \tau + k \in \mathbb{H}$. Examining the behavior of η with respect to $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we have

$$\eta\left(-1/\tau'\right)^6 = (-i\tau')^3 \eta(\tau')^6,$$

and therefore,

$$\begin{aligned}\frac{\eta\left(-1/(\tau'/5)\right)^6}{\eta\left(-1/\tau'\right)^6} &= \frac{(-i\tau/5)^3}{(-i\tau)^3}\frac{\eta\left(\tau'/5\right)^6}{\eta\left(\tau'\right)^6}\\ &= \frac{1}{125}\cdot\frac{1}{t(\tau'/5)}\\ &= \frac{1}{125}\cdot e^{-2\pi ik/5}\frac{1}{q^{1/5}}\prod_{m=1}^{\infty}\left(\frac{1-e^{2\pi ikm/5}q^{m/5}}{1-e^{2\pi ikm}q^m}\right)^6 \in q^{-1/5}\mathbb{C}[[q^{1/5}]],\end{aligned}$$

which is sufficient to us, since $\gcd(5,1^2)/5 = 1/5$. We therefore have $t \in \mathcal{M}\left(\Gamma_0(5)\right)$.

Notice that we have

$$\mathrm{ord}_{\infty}^{(5)}(t) = 1, \tag{2.13}$$
$$\alpha_{\infty}(1) = 1 \tag{2.14}$$

and

$$\mathrm{ord}_{0}^{(5)}(t) = -1, \tag{2.15}$$
$$\alpha_{0}(1) = 1/125. \tag{2.16}$$

We therefore have $t \in \mathcal{M}^0\left(\Gamma_0(5)\right)$, and $1/t \in \mathcal{M}^{\infty}\left(\Gamma_0(5)\right)$.

□

This process of checking membership in $\mathcal{M}\left(\Gamma_0(N)\right)$, while straightforward, is perhaps somewhat tedious. We need a more direct means of determining whether a given quotient of eta functions will exhibit modularity.

We now give two key theorems that will prove useful in checking the modularity of certain functions. The first is a theorem which gives us a means of constructing modular functions using η [76, Theorem 1], [90, Remark 2.36]:

Theorem 2.40. *Let* $f = \prod_{\delta|N}\eta(\delta\tau)^{r_\delta}$, *with* $\hat{r} = (r_\delta)_{\delta|N}$ *an integer-valued vector, for some* $N \in \mathbb{Z}_{\geq 1}$. *Then* $f \in \mathcal{M}\left(\Gamma_0(N)\right)$ *if and only if the following apply:*

1. $\sum_{\delta|N} r_\delta = 0$;
2. $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$;
3. $\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$;
4. $\prod_{\delta|N} \delta^{|r_\delta|}$ *is a perfect square.*

Taking $t(\tau) = \eta(5\tau)^6 \cdot \eta(\tau)^{-6}$, we have the vector $\hat{r} = (-6, 6)$. One can verify very quickly that each of the conditions of the theorem above are satisfied, and that therefore $t \in \mathcal{M}\left(\Gamma_0(5)\right)$.

To study the order of an eta quotient at a given cusp, we make use of a theorem that can be found in [91, Theorem 23], generally attributed to Ligozat:

Theorem 2.41. *If $f = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta} \in \mathcal{M}\left(\Gamma_0(N)\right)$, then the order of f at the cusp $[a/c]_N$ is given by the following:*

$$\mathrm{ord}_{a/c}^{(N)}(f) = \frac{N}{24\gcd\left(c^2, N\right)} \sum_{\delta|N} r_\delta \frac{\gcd\left(c, \delta\right)^2}{\delta}.$$

Examining t once more, we can quickly verify that $\mathrm{ord}_{\infty}^{(5)}(t) = 1$, $\mathrm{ord}_{0}^{(5)}(t) = -1$.

Definition 2.42. An eta quotient on $\Gamma_0(N)$ is an object of the form

$$\prod_{\lambda|N} \eta(\lambda\tau)^{s_\lambda} \in \mathcal{M}\left(\Gamma_0(N)\right).$$

Define $\mathcal{E}(N)$ as the set of all eta quotients on $\Gamma_0(N)$, and $\mathcal{E}^{a/c}(N) := \mathcal{E}(N) \cap \mathcal{M}^{a/c}\left(\Gamma_0(N)\right)$.

Thus, $t \in \mathcal{E}^0(5)$, and $1/t \in \mathcal{E}^\infty(5)$.

We finish with a useful simplification of the factor $\epsilon(a, b, c, d)$ in Theorem 2.38.

Theorem 2.43. *Consider the root of unity ϵ from Theorem 2.38. Given $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)^*$, then*

$$\epsilon(a, b, c, d) := \exp\left(-\frac{\pi i a}{12}(c - b - 3)\right).$$

This theorem is proved in [76].

2.5.2 Eta Algebra

It is easy to see that $\left\langle \mathcal{E}^{a/c}(N) \right\rangle_{\mathbb{K}}$ fulfills the conditions of a $\mathbb{K}$-algebra. Notice that, as in the case of t above, examining the order, principal part, and constant term of a function's expansion at ∞ is often easy, provided that t is given a representation in its Fourier variable q: one need only expand the function in terms of the nonpositive powers of q. Therefore, for the next few chapters we will place an emphasis on $\left\langle \mathcal{E}^\infty(N) \right\rangle_{\mathbb{K}}$.

Theorem 2.44. *For any $N \in \mathbb{Z}_{\geq 2}$, $\mathcal{E}^\infty(N)$ is a finitely generated monoid. Moreover, there exist functions $t, g_1, g_2, ..., g_{v-1} \in \mathcal{M}^\infty\left(\Gamma_0(N)\right)$ such that for $g_0 = 1$, we have*

$$\mathrm{ord}_{\infty}^{(N)}(t) = v-1, \tag{2.17}$$
$$\mathrm{ord}_{\infty}^{(N)}(g_i) < \mathrm{ord}_{\infty}^{(N)}(g_j),\ \textit{for}\ 1 \leq i < j \leq v-1, \tag{2.18}$$
$$\mathrm{ord}_{\infty}^{(N)}(g_i) \not\equiv \mathrm{ord}_{\infty}^{(N)}(g_j) \pmod{\mathrm{ord}_{\infty}^{(N)}(t)},\ \textit{for}\ 1 \leq i < j \leq v-1, \tag{2.19}$$
$$\mathrm{ord}_{\infty}^{(N)}(g_i) \not\equiv 0 \pmod{\mathrm{ord}_{\infty}^{(N)}(t)},\ \textit{for}\ 1 \leq i \leq v-1, \tag{2.20}$$
$$\langle \mathcal{E}^{\infty}(N)\rangle_{\mathbb{Q}} = \bigoplus_{j=0}^{v-1} g_j\mathbb{Q}[t]. \tag{2.21}$$

The proof can be found in [91, Sections 2.1, 2.2]. The structure given in (2.21) will prove extremely useful us. Moreover, there exist algorithms which can be used to compute the functions $t, g_1, g_2, ..., g_{v-1}$, as we will determine in the sequel.

2.6 Modular Cusp Analysis

By the term *cusp analysis*, we mean the computation and comparison of principal parts of modular functions at their cusps as a means of proving when any two such functions are equal, and determining representations. The subject is extremely powerful from a computational point of view, and it is based principally on Theorems 2.24 and 2.35 above.

Let us consider two functions $f_1, f_2 \in \mathcal{M}(\Gamma_0(N))$. We want to determine whether $f_1 = f_2$. Theorem 2.35 would require that we examine the principal parts of $f_1 - f_2$ at every cusp of $\Gamma_0(N)$. This is often unwieldy, since the principal parts of a given modular function at the cusps besides $[\infty]_N$ can often be difficult to precisely compute.

On the other hand, we will see that we can often compute bounds for the *order* of a function at a given cusp with relatively little work. Thus, we may apply our understanding of eta quotients to the problem, provided we have the following:

Theorem 2.45. *For every $N \in \mathbb{Z}_{\geq 2}$, there exists a function $\mu \in \mathcal{E}^{\infty}(N)$ which has positive order at every cusp of $\Gamma_0(N)$ except ∞.*

A proof can be found in [91, Lemma 20]. The significance of this theorem is that, for a large enough $k_1 \in \mathbb{Z}_{\geq 0}$, we must have $\mu^{k_1} \cdot f_1, \mu^{k_1} \cdot f_2 \in \mathcal{M}^{\infty}(\Gamma_0(N))$. If we have lower bounds for the orders of f_1, f_2 at their respective cusps, we may simply take k_1 to exceed the maximum magnitude of those bounds. In this case, we need only compare the principal parts of $\mu^{k_1} \cdot f_1, \mu^{k_1} \cdot f_2$.

For example, we will soon want to determine whether a given $f \in \mathcal{M}(\Gamma_0(N))$ can be expressed as a linear combination of eta quotients, i.e., whether $f \in \langle \mathcal{E}(N)\rangle_{\mathbb{Q}}$. To do this directly, we would be forced to have a complete set of generators for $\langle \mathcal{E}(N)\rangle_{\mathbb{Q}}$, and to study the behavior of f at each cusp of $\Gamma_0(N)$.

However, by Theorem 2.45, the condition of $f \in \langle \mathcal{E}(N)\rangle_{\mathbb{Q}}$ is equivalent to

$$\mu^{k_1} \cdot f \in \mathcal{M}^{\infty}(\Gamma_0(N))_{\mathbb{Q}} \cap \langle \mathcal{E}(N)\rangle_{\mathbb{Q}}$$

for a sufficiently large k_1. This requires only a single principal part to be examined.

Ideally, this membership condition would equate to checking whether

$$\mu^{k_1} \cdot f \in \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}} .$$

However, one theoretical problem persists. We know that

$$\mathcal{M}^{\infty} \left(\Gamma_0(N)\right)_{\mathbb{Q}} \cap \langle \mathcal{E}(N) \rangle_{\mathbb{Q}} \supseteq \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}} ,$$

but we have not yet established that

$$\mathcal{M}^{\infty} \left(\Gamma_0(N)\right)_{\mathbb{Q}} \cap \langle \mathcal{E}(N) \rangle_{\mathbb{Q}} \subseteq \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}} .$$

Current evidence suggests that the two sets are equal, and we strongly suspect that this is true. Unfortunately, we are as of yet unable to prove it. However, Radu was able [91, Lemma 28] to establish a weaker theorem:

Theorem 2.46. *Given some $N \in \mathbb{Z}_{\geq 2}$ and a $\mu \in \mathcal{E}^{\infty}(N)$ which has positive order at every cusp except ∞, there exists some $k_0 \in \mathbb{Z}_{\geq 0}$ such that*

$$\mu^{k_0} \cdot \left(\mathcal{M}^{\infty} \left(\Gamma_0(N)\right)_{\mathbb{Q}} \cap \langle \mathcal{E}(N) \rangle_{\mathbb{Q}}\right) \subseteq \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}} .$$

The ambiguity of whether $k_0 = 0$ will become important later. But what is important for the time being is that an upper bound for k_0 is at least computable [91, Proof of Lemma 28]. With the previous two theorems, in order to check whether $f \in \langle \mathcal{E}(N) \rangle_{\mathbb{Q}}$, we need to calculate a $\mu \in \mathcal{E}^{\infty}(N)$ which satisfies the conditions of Theorem 2.45, preferably with a minimal order at ∞; we can then compute k_0, k_1 and check whether

$$\mu^{k_0+k_1} \cdot f \in \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}} ,$$

by examining the single principal part of $\mu^{k_0+k_1} \cdot f$. The precise algebraic form of $\langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$ given in Theorem 2.44 will be used to build the proper algorithm for determining membership.

We now suppose that $f_0 = \mu^{k_0+k_1} \cdot f \in \mathcal{M}^{\infty} \left(\Gamma_0(N)\right)_{\mathbb{Q}}$, with μ, k_0, and k_1 defined as in Theorems 2.45 and 2.46.

We know that we can expand f_0 as the following:

$$f = \frac{c(-m_1)}{q^{m_1}} + \frac{c(-m_1+1)}{q^{m_1-1}} + ... + \frac{c(-1)}{q} + c(0) + \sum_{n=1}^{\infty} c(n) q^n, \tag{2.22}$$

with $c(-m_1) \neq 0$. Here we define $\left|\mathrm{ord}_{\infty}^{(N)}(f_0)\right| := m_1$, and $\mathrm{LC}(f_0) := c(-m_1)$.

To determine whether $f \in \langle \mathcal{E}(N) \rangle_{\mathbb{Q}}$, we need only determine whether $f_0 \in \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$. We construct an algebra basis $\{t, g_1, g_2, ..., g_{v-1}\}$ of the form described in Theorem 2.44; we want to know whether

$$f_0 \in \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}} = \bigoplus_{j=0}^{v-1} g_j \mathbb{Q}[t]. \tag{2.23}$$

Because the orders of the functions g_j give a complete set of representatives of the residue classes modulo v, we know that $m_1 \equiv \left|\mathrm{ord}_{\infty}^{(N)}(g_{j_1})\right| \pmod{v}$, for some j_1 with $1 \leq j_1 \leq v-1$.

Suppose first that $0 < m_1 < \left|\mathrm{ord}_{\infty}^{(N)}(g_{j_1})\right|$. In this case, no nonnegative power of g_{j_1} can reduce the order of f_0, and no other element in our basis can have a matching order modulo v. We must immediately conclude that the principal part of f_0 cannot be reduced in terms of the principal parts of $\{g_1, g_2, ..., g_{v-1}\}$. Of course, this implies that

$$f_0 \notin \bigoplus_{j=0}^{v-1} g_j \mathbb{Q}[t].$$

Now consider that $m_1 \geq \left|\mathrm{ord}_{\infty}^{(N)}(g_{j_1})\right|$. Let g_{j_1} have the expansion

$$g_{j_1} = \frac{b_1(-n_1)}{q^{n_1}} + \frac{b_1(-n_1+1)}{q^{n_1-1}} + ... + \frac{b_1(-1)}{q} + b_1(0) + \sum_{n=1}^{\infty} b_1(n) q^n, \tag{2.24}$$

with $b_1(-n_1) \neq 0$. Then we can write

$$f_1 = f_0 - \frac{c(-m_1)}{\mathrm{LC}\left(g_{j_1} \cdot t^{\frac{m_1-n_1}{v}}\right)} \cdot g_{j_1} \cdot t^{\frac{m_1-n_1}{v}}, \tag{2.25}$$

$$\left|\mathrm{ord}_{\infty}^{(N)}(f_1)\right| = m_2 < m_1. \tag{2.26}$$

Now let $m_2 \equiv \left|\mathrm{ord}_{\infty}^{(N)}(g_{j_2})\right| \pmod{v}$, with $1 \leq j_2 \leq v-1$. If $m_2 \geq \left|\mathrm{ord}_{\infty}^{(N)}(g_{j_2})\right|$, then we may construct f_2 in similar fashion as to f_1.

In this way, we may construct a sequence of functions $(f_l)_{l\geq 1}$, with $\left|\mathrm{ord}_{\infty}^{(N)}(f_l)\right| \geq \left|\mathrm{ord}_{\infty}^{(N)}(f_{l+1})\right|$ for each l. Since $\left|\mathrm{ord}_{\infty}^{(N)}(f_l)\right| \in \mathbb{Z}_{\geq 0}$ for each l, there are two possible outcomes. The first outcome is that for some $k \in \mathbb{Z}_{>1}$, we produce a function f_k such that $0 < \left|\mathrm{ord}_{\infty}^{(N)}(f_k)\right| = m_k < \left|\mathrm{ord}_{\infty}^{(N)}(g_{j_k})\right|$. At such a point, our sequence immediately terminates, and we must conclude that

$$f_0 \notin \bigoplus_{j=0}^{v-1} g_j \mathbb{Q}[t] = \langle \mathcal{E}^\infty(N) \rangle_{\mathbb{Q}}.$$

The second possible outcome is that for some $k \in \mathbb{Z}_{>1}$ we will have f_{k-1} with

$$f_{k-1} = c_{k-1}(0) + \sum_{n=1}^{\infty} c_{k-1}(n) q^n. \tag{2.27}$$

Of course, $c_{k-1}(0) \in \bigoplus_{j=0}^{v-1} g_j \mathbb{Q}[t]$, so that $f_k = f_{k-1} - c_{k-1}(0)$ has no principal part and no constant. In this case, we have shown that the principal part of f_0 can be constructed through combinations of the principal parts of monomials within our basis. Since we only need to match the principal parts and constants, we can conclude that

$$f_0 \in \bigoplus_{j=0}^{v-1} g_j \mathbb{Q}[t] = \langle \mathcal{E}^\infty(N) \rangle_{\mathbb{Q}},$$
$$f \in \langle \mathcal{E}(N) \rangle_{\mathbb{Q}}.$$

As we reduce the principal part of f_0, we can collect the terms

$$\frac{c(-m_l)}{\mathrm{LC}\left(g_{j_l} \cdot t^{\frac{m_l - n_l}{v}}\right)} \cdot g_{j_l} \cdot t^{\frac{m_l - n_l}{v}}$$

into a set $\mathcal{V}$ of v polynomials, each a sum of all the terms which use the same element g_{j_l}. In the event that we can completely reduce the principal part of f_0, $\mathcal{V}$ represents the basis decomposition of f_0 over $\bigoplus_{j=0}^{v-1} g_j \mathbb{Q}[t]$.

2.6.1 The Newman–Radu Condition

The utility and interest of the η function naturally compels the question of whether *all* modular functions over a given congruence subgroup may be expressed in terms of eta quotients. It is known that all automorphic functions over $\Gamma_0(N)$ may be written as rational polynomials in $j(\tau)$, $j(N\tau)$ [34, Proposition 7.5.1]. Because the j invariant can be written as a combination of eta functions, we may answer this general question in the affirmative.

However, if we focus exclusively on modular functions as we have defined them, and restrict to $\langle \mathcal{E}(N) \rangle_{\mathbb{C}}$, the question becomes more difficult, and our understanding less certain. In particular, as we shall see, the very natural equality

$$\mathcal{M}(\Gamma_0(N)) = \langle \mathcal{E}(N) \rangle_{\mathbb{C}}$$

does not apply *for all* N. For example, this equality fails for $N = 11$. Nevertheless, this relation applies for a large number of values of N.

Definition 2.47. Let $N \in \mathbb{Z}_{\geq 2}$. The space $\mathcal{M}(\Gamma_0(N))$ satisfies *the Newman–Radu condition* if

$$\mathcal{M}^{\infty}(\Gamma_0(N)) = \langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{C}}.$$

Notice that this implies that

$$\mathcal{M}(\Gamma_0(N)) = \langle \mathcal{E}(N) \rangle_{\mathbb{C}}.$$

The significance of this condition will be revealed in the sequel.

Conjecture 2.48. *For every $N \in \mathbb{Z}_{\geq 2}$ divisible by two distinct primes, the Newman-Radu condition applies.*

The original conjecture applied to all *composite* N. However, Radu found a counterexample for $N = 49$ [91, Section 3.3], and subsequently adjusted the conjecture.

2.7 U_ℓ Operator

We will finish with a review of the classical U_ℓ operator, which will play a significant role in our later chapters. Let $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$ as usual.

Definition 2.49. Let $\ell \in \mathbb{Z}_{>0}$ be a prime, and $f(\tau) = \sum_{m\geq M} a(m)q^m$. Then define

$$U_\ell(f(\tau)) := \sum_{\ell\cdot m\geq M} a(\ell\cdot m)q^m.$$

This operator is often enormously useful, because it gives us a means of constructing important sequences of modular functions over a given congruence subgroup.

We will list some key properties of U_ℓ in which ℓ is an arbitrary but fixed prime. These properties are standard to the theory of partition congruences, and proofs can be found in [10, Chapter 10] and [61, Chapter 8].

Lemma 2.50. *Given two functions*

$$f(\tau) = \sum_{m\geq M} a(m)q^m, \ \ g(\tau) = \sum_{m\geq N} b(m)q^m,$$

any $\alpha \in \mathbb{C}$, a primitive ℓ-th root of unity ζ, and the convention that $q^{1/\ell} = e^{2\pi i\tau/\ell}$, we have the following:

1. $U_\ell(\alpha \cdot f + g) = \alpha \cdot U_\ell(f) + U_\ell(g)$;
2. $U_\ell(f(\ell\tau)g(\tau)) = f(\tau)U_\ell(g(\tau))$;
3. $\ell \cdot U_\ell(f) = \sum_{r=0}^{\ell-1} f\left(\frac{\tau+r}{\ell}\right)$.

Finally, we give an important theorem on the stability of U_ℓ.

Theorem 2.51. *Let $N \in \mathbb{Z}_{\geq 1}$, with ℓ some prime number. Then the following are true:*

1. $U_\ell(f) \in \mathcal{M}(\Gamma_0(\ell))$ *for all* $f \in \mathcal{M}(\Gamma_0(1))$.
2. *If* $\ell | N$, *then* $U_\ell(f) \in \mathcal{M}(\Gamma_0(N))$ *for all* $f \in \mathcal{M}(\Gamma_0(N))$.
3. *If* $\ell^2 | N$, *then* $U_\ell(f) \in \mathcal{M}(\Gamma_0(N/\ell))$ *for all* $f \in \mathcal{M}(\Gamma_0(N))$.

For a proof, see [18, Lemma 7].

CHAPTER 3
RAMANUJAN–KOLBERG IDENTITIES

3.1 Introduction

In many respects, this chapter is the core of our dissertation. It applies the use of the theoretical machinery of the previous chapter to the more concrete question of computing identities of a certain class, for a broad range of arithmetic functions. There are many interesting aspects of such an application. In the first place, many of the identities of interest convey arithmetic information about the associated functions. In the sequel we will demonstrate how the class of identities of interest to us is useful for determining *optimal* congruences for restricted partition numbers over different arithmetic progressions. This chapter and the sequel are based largely on work which was published in [104].

Some, including the classic results of Ramanujan [95], are quite beautiful. Consider again (1.7)-(1.8):

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{m=1}^{\infty}\frac{(1-q^{5m})^5}{(1-q^m)^6},$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 49q\prod_{m=1}^{\infty}\frac{(1-q^{7m})^7}{(1-q^m)^8} + 7\prod_{m=1}^{\infty}\frac{(1-q^{7m})^3}{(1-q^m)^4}.$$

On the other hand, some identities of interest are quite cumbersome; these identities point to important subtleties in the structure of the relevant modular functions. For example, in the sequel we will examine a partition identity for $p(11n+6)$, and its relationship to the failue of the Newman-Radu condition over $\mathcal{M}\left(\Gamma_0(11)\right)$. This fact has led to an enormous amount of work over the past century, and there is sill more to discover. Thus, the "ugly" identities offer us even more than those which are considered "beautiful."

Moreover, beginning with Chapter 5, we will once again consider more abstract—and more ambitious—objectives. In so doing, much of the results of the previous chapter will become extremely important. However, our objects of study will become very difficult to closely examine, and we will witness a growing complexity which necessitates the use of computational machinery. Not only will the implemented algorithms presented below be extremely useful for manipulating and computing important auxiliary results, but much of our experimental methods will lead to important theoretical insights, e.g., techniques for proving ℓ-adic convergence of function sequences on a given modular curve.

3.1.1 Notation

In the previous chapter, we used j to refer to the principal modular function over $\mathrm{SL}(2,\mathbb{Z})$. However, as the j-invariant will not be frequently referenced for the rest of this dissertation, we will henceforth refer to j as the initial value of a linear progression with base $m \in \mathbb{Z}_{\geq 2}$.

We will also define

$$(q^a;q^b)_\infty := \prod_{m=0}^{\infty}(1-q^{a+bm}),$$

for any $a, b \in \mathbb{Z}$ with $b \geq 1$. In particular,

$$(q;q)_\infty = \prod_{m=1}^{\infty}(1-q^m).$$

3.1.2 Motivating Examples

Nearly 40 years after Ramanujan's classic results, Kolberg realized [63] that these identities of Ramanujan could, with a very slight generalization, be extended to include a much larger variety of similar identities for $p(5n+j)$, $p(7n+j)$, $p(3n+j)$, $p(2n+j)$, and others. For instance, Kolberg proved

$$\left(\sum_{n=0}^{\infty} p(5n)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+3)q^n\right) = 25q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^{12}} + 3\frac{(q^5;q^5)_\infty^{4}}{(q;q)_\infty^{6}}, \tag{3.1}$$

along with (1.14). He also proved

$$\begin{aligned}\left(\sum_{n=0}^{\infty} p(7n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(7n+3)q^n\right)\left(\sum_{n=0}^{\infty} p(7n+4)q^n\right)\\ =117649q^4\frac{(q^7;q^7)_\infty^{21}}{(q;q)_\infty^{24}} + 50421q^3\frac{(q^7;q^7)_\infty^{17}}{(q;q)_\infty^{20}} + 8232q^2\frac{(q^7;q^7)_\infty^{13}}{(q;q)_\infty^{16}}\\ + 588q\frac{(q^7;q^7)_\infty^{9}}{(q;q)_\infty^{12}} + 15\frac{(q^7;q^7)_\infty^{5}}{(q;q)_\infty^{8}}.\end{aligned}$$

In recent years a very large number of such identities have been produced. They often concern the coefficients of various q-Pochhammer quotients, many of which can be used to enumerate various restricted partitions. For example, if we define

$$\sum_{n=0}^{\infty} B_5(n)q^n = \frac{(q^5;q^5)_\infty^2}{(q;q)_\infty^2},$$

then we have the following identity:

$$\sum_{n=0}^{\infty} B_5(5n+3)q^n = 125q\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^{10}} + 10\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^4}. \tag{3.2}$$

This identity confirms that $B_5(5n+3) \equiv 0 \pmod 5$, a congruence discovered by Liuquan Wang [110].

Or consider a conjecture from Michael Schlosser of the University of Vienna [98]: if

$$\sum_{n=0}^{\infty} b(n)q^n := \frac{(q;q)_\infty^3}{(q^{11};q^{11})_\infty^3},$$

then $b(11n+4)$ is divisible by 11. Indeed, the following identity was found to confirm Schlosser's conjecture:

$$\sum_{n=0}^{\infty} b(11n+4)q^n = -11q\frac{(q^{11};q^{11})_\infty^3}{(q;q)_\infty^3}.$$

We will examine these latter two identities and many more in the sequel.

Examining the identities above, we emphasize the following:

- The arithmetic function of interest is enumerated by a generating function of the form $\prod_{\delta|M}(q^\delta;q^\delta)_{\delta|M}^{r_\delta}$, such that $M \in \mathbb{Z}_{\geq 1}$ and $r = (r_\delta)_{\delta|M}$ is an integer-valued vector;
- The left-hand side of a given identity consists of products involving our function of interest, taken over some set of linear progressions of the form $mn + j$, in which m is fixed, and j varies over some finite set such that $0 \leq j \leq m-1$;
- The right-hand side consists of linear combinations of eta quotients indexed over the divisors of some $N \in \mathbb{Z}_{\geq 2}$.

We will show that a given Ramanujan–Kolberg identity (hereafter referred to as an RK identity) is characterized by the 5-tuple (N, M, r, m, j).

3.2 Motivating Examples: $p(5n+j)$

To understand the key principles of Radu's algorithm, we will take some time to walk through a proof of (1.7), together with a proof of (1.14) and (3.1).

What follows is by no means the only proof of these identities. A more elegant proof of (1.7) can be found in [61, Chapter 8, Section 3] via a modular equation (although the modular equation itself is similarly difficult to prove). Ramanujan's original proof [95, Section 4] was based not on modularity, but depended instead on the manipulation of certain formal power series. Similar proofs of (1.7), (1.14), and (3.1) by formal power series manipulations have been given by Kolberg [63] and Hirschhorn [51, Chapters 5-6].

The proof we give is a reduction of the more generalized proof by Radu found in [89] and [91]. It is similar to that of Rademacher [87], and reduces to an application of the modular cusp analysis from the previous chapter. This is perhaps the longest and most technical proof of (1.7). Its appeal is that the steps are so well understood that they can be generalized and automated with relative ease. In contrast, the proof in [61] depends on the form of a specific modular equation, and therefore cannot be generalized.

On the other hand, the techniques in the proofs in [95], [63], and [51] can indeed be extended to other problems; however, these problems depend on evaluating and reducing the determinant of a certain large matrix, or in reducing a certain massive polynomial modulo a given ideal. For the moment, this technique, so much more elegant in proving (1.7) than Rademacher's method, appears far more cumbersome for more generalized problems.

It is interesting to consider an algorithmic approach to the methods of Ramanujan, Kolberg, and Hirschhorn by incorporation of a Gröbner basis manipulation. However, such an approach still requires a very careful study before it can be demonstrated as useful. For the time being, we will proceed on the assumption that these identities, in Rademacher's words, "belong to the theory of modular functions." [88, Chapter 13, Section 105].

3.2.1 Constructing the h_j

We begin by examining the generating function for $p(n)$. Let $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$. Moreover, let $q^{1/5} = e^{2\pi i\tau/5}$, $\rho = e^{2\pi i/5}$, and let $\kappa \in \mathbb{Z}$ with κ not divisible by 5. We write

$$\tilde{F}(q) = F(\tau) := \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{q^{1/24}}{\eta(\tau)}.$$

Now we consider taking a sum over F with a modified input:

$$\begin{aligned}
\sum_{\lambda=0}^{4} \tilde{F}\left(\rho^{\lambda} q^{1/5}\right) &= \sum_{\lambda=0}^{4} F\left(\frac{\tau+\lambda}{5}\right) \\
&= \sum_{\lambda=0}^{4} \sum_{n=0}^{\infty} p(n) e^{2\pi i n(\tau+\lambda)/5} \\
&= \sum_{n=0}^{\infty} p(n) e^{2\pi i n\tau/5} \sum_{\lambda=0}^{4} e^{2\pi i n\lambda/5} \\
&= 5 \sum_{n=0}^{\infty} p(5n) e^{2\pi i n\tau} \\
&= 5 \sum_{n=0}^{\infty} p(5n) q^{n}.
\end{aligned}$$

This is of course the manipulation which allows us to define U_ℓ in the previous chapter, with $\ell = 5$. We can use this manipulation to our advantage by introducing a root of unity and an additional q factor into the sum. Let us take some j with $0 \leq j \leq 4$:

$$\sum_{\lambda=0}^{4} \rho^{-j\kappa\lambda} q^{-j/5} \tilde{F}\left(\rho^{\kappa\lambda} q^{1/5}\right) = \sum_{\lambda=0}^{4} \rho^{-j\kappa\lambda} q^{-j/5} F\left(\frac{\tau+\kappa\lambda}{5}\right) \tag{3.3}$$

$$= \sum_{\lambda=0}^{4} \sum_{n=0}^{\infty} e^{-2\pi i\cdot j\kappa\lambda/5} e^{-2\pi i\tau\cdot j/5} p(n) e^{2\pi i n(\tau+\kappa\lambda)/5} \tag{3.4}$$

$$= \sum_{n=0}^{\infty} p(n) e^{2\pi i\tau(n-j)/5} \sum_{\lambda=0}^{4} e^{2\pi i\kappa\lambda(n-j)/5} \tag{3.5}$$

$$= 5 \sum_{n=0}^{\infty} p(5n+j) q^{n}. \tag{3.6}$$

This gives us a precise expression of $\sum_{n=0}^{\infty} p(5n+j)q^n$ that can be suitably manipulated. Indeed, rewriting (3.3) in terms of eta functions, we have

$$\sum_{\lambda=0}^{4} \rho^{-j\kappa\lambda} q^{-j/5} \tilde{F}\left(\rho^{\kappa\lambda} q^{1/5}\right) = \sum_{\lambda=0}^{4} \rho^{\lambda\kappa(-24j+1)/24} q^{(1-24j)/(24\cdot 5)} \eta\left(\frac{\tau+\kappa\lambda}{5}\right)^{-1},$$

so that

$$q^{(24j-1)/(24\cdot 5)} \sum_{n=0}^{\infty} p(5n+j) q^{n} = \frac{1}{5} \sum_{\lambda=0}^{4} \rho^{\lambda\kappa(j+1)/24} \eta\left(\frac{\tau+\kappa\lambda}{5}\right)^{-1}.$$

Let us take $\kappa = 24$, and define our expression as h_j:

$$h_j(\tau) := q^{(24j-1)/(24\cdot 5)} \sum_{n=0}^{\infty} p(5n+j)q^n = \frac{1}{5}\sum_{\lambda=0}^{4} \rho^{\lambda(j+1)} \eta\left(\frac{\tau+24\lambda}{5}\right)^{-1}.$$

3.2.2 Constructing Modular Symmetry

Let us consider $h_j(\gamma\tau)$ for $\gamma \in \Gamma_0(5)$. We want to build modular functions out of the h_j, i.e., functions invariant under the action of $\Gamma_0(5)$ with the additional meromorphic properties of the previous chapter.

Notice that it is sufficient to study $\gamma \in \Gamma_0(5)^*$, since $\Gamma_0(5)^*$ generates $\Gamma_0(5)$. Therefore we let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)^*$.

We first examine $\eta\left(\frac{\gamma\tau+24\lambda}{5}\right)$. Notice that there exist integers $x, y \in \mathbb{Z}$ such that

$$\begin{pmatrix} 1 & 24\lambda \\ 0 & 5 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+24\lambda c & -y \\ 5c & x \end{pmatrix}\begin{pmatrix} 1 & (b+24\lambda d)x+5dy \\ 0 & 5 \end{pmatrix},$$

with

$$\begin{pmatrix} a+24\lambda c & -y \\ 5c & x \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})^*.$$

Moreover, without loss of generality, we can assume that $y \equiv 0 \pmod{24^k}$, for k as large as we like. With this in mind, we can write

$$\begin{aligned}
\eta\left(\frac{\gamma\tau+24\lambda}{5}\right) &= (-i(c\tau+d))^{1/2}\epsilon(a+24\lambda c, -y, 5c, x)\eta\left(\frac{\tau+(b+24\lambda d)x+5dy}{5}\right) \\
&= (-i(c\tau+d))^{1/2}\epsilon(a+24\lambda c, -y, 5c, x)\cdot e^{5\pi i a' b/12}\eta\left(\frac{\tau+(b+24\lambda d)x-25a'b}{5}\right) \\
&= (-i(c\tau+d))^{1/2}\epsilon(a+24\lambda c, -y, 5c, x)\cdot e^{5\pi i a' b/12}\eta\left(\frac{\tau+24(\lambda dx+b(x-25a')/24)}{5}\right) \\
&= (-i(c\tau+d))^{1/2}\epsilon(a+24\lambda c, -y, 5c, x)\cdot e^{5\pi i a' b/12}\eta\left(\frac{\tau+24\mu}{5}\right), \qquad (3.7)
\end{aligned}$$

with $a'a \equiv 1 \pmod{24c}$, and

$$\mu = \lambda dx + b(x-25a')/24.$$

Notice that

$$ax \equiv 1 \pmod{24c}. \tag{3.8}$$

Therefore, we must have $x \equiv a' \pmod{24}$, and therefore $x - 25a' \equiv 0 \pmod{24}$, so that $\mu \in \mathbb{Z}$. Moreover, $\gcd(dx, 5) = 1$. Therefore, as λ ranges from 0 to 4, so too must μ.

Note that because $\gcd(a, 6) = 1$, we must have $a^2 \equiv 1 \pmod{24}$, and therefore $a \equiv a' \pmod{24}$. Therefore, we have

$$\eta\left(\frac{\gamma\tau + 24\lambda}{5}\right)^{-1} = (-i(c\tau+d))^{-1/2}\epsilon(a+24\lambda c, -y, 5c, x)^{-1}e^{-5\pi iab/12}\eta\left(\frac{\tau+24\mu}{5}\right)^{-1}.$$

We can express our root of unity using Theorem 2.43:

$$\begin{aligned}&\epsilon(a+24\lambda c, -y, 5c, x)^{-1}e^{-5\pi iab/12}\\&=\left(\frac{5c}{a+24\lambda c}\right)\cdot\exp\left((a+24\lambda c)\frac{\pi i}{12}(5c-3)-\frac{5\pi iab}{12}\right).\end{aligned}$$

Let us take $5c = 2^u \cdot v$, with $u \geq 0$ and v odd. Then we can write

$$\begin{aligned}&\left(\frac{5c}{a+24\lambda c}\right)\\&=\left(\frac{2^u}{a+24\lambda c}\right)\left(\frac{v}{a+24\lambda c}\right).\end{aligned}$$

Because both a and v are odd, we may invoke quadratic reciprocity:

$$\begin{aligned}\left(\frac{v}{a+24\lambda c}\right)&=\left(\frac{a+24\lambda c}{v}\right)(-1)^{\frac{v-1}{2}\frac{a+24\lambda c-1}{2}}\\&=\left(\frac{a+24\lambda c}{v}\right)(-1)^{\frac{v-1}{2}\frac{a-1}{2}}.\end{aligned}$$

Moreover, $a + 24\lambda c \equiv a \pmod{5c/2^u} \equiv a \pmod{v}$, so that

$$\begin{aligned}\left(\frac{v}{a+24\lambda c}\right)&=\left(\frac{a+24\lambda c}{v}\right)(-1)^{\frac{v-1}{2}\frac{a-1}{2}}\\&=\left(\frac{a}{v}\right)(-1)^{\frac{v-1}{2}\frac{a-1}{2}}\\&=\left(\frac{v}{a}\right),\end{aligned}$$

invoking reciprocity once more.

Next we examine the corresponding character for 2^u. If u is odd, then

$$\begin{aligned}\left(\frac{2^u}{a+24\lambda c}\right) &= \left(\frac{2}{a+24\lambda c}\right)\\ &= (-1)^{\frac{(a+24\lambda c)^2-1}{8}}\\ &= (-1)^{\frac{a^2-1}{8}}\\ &= \left(\frac{2}{a}\right)\\ &= \left(\frac{2^u}{a}\right).\end{aligned}$$

On the other hand, if u is even, then $\left(\frac{2^u}{a+24\lambda c}\right) = \left(\frac{2^u}{a}\right) = 1$. Either way, we may write

$$\left(\frac{2^u}{a+24\lambda c}\right) = \left(\frac{2^u}{a}\right),$$

whence we have

$$\begin{aligned}&\left(\frac{5c}{a+24\lambda c}\right)\\ &= \left(\frac{2^u}{a+24\lambda c}\right)\left(\frac{v}{a}\right)\\ &= \left(\frac{2^u}{a}\right)\left(\frac{v}{a}\right)\\ &= \left(\frac{5c}{a}\right),\end{aligned}$$

and

$$\begin{aligned}&\epsilon(a+24\lambda c, -y, 5c, x)^{-1}e^{-5\pi iab/12}\\ &= \left(\frac{5c}{a}\right)\cdot\exp\left((a+24\lambda c)\frac{\pi i}{12}(5c-3) - \frac{5\pi iab}{12}\right)\\ &= \left(\frac{5c}{a}\right)\cdot\exp\left(a\frac{\pi i}{12}(5c-3) - \frac{5\pi iab}{12}\right).\end{aligned}$$

We now have

$$h_j(\gamma\tau) = (-i(c\tau+d))^{-1/2}\left(\frac{5c}{a}\right)\cdot\exp\left(a\frac{\pi i}{12}(5c-3)-\frac{5\pi iab}{12}\right)\sum_{\lambda=0}^{4}\rho^{\lambda(j+1)}\eta\left(\frac{\tau+24\mu}{5}\right)^{-1}.$$

To write λ in terms of μ, we remember that $x \equiv a' \pmod{24c}$, so that

$$\begin{aligned}
24\mu &\equiv 24\lambda dx + bx - 25a'b \pmod{24c}\\
&\equiv 24\lambda dx + bx - 25bx \pmod{24c}\\
&\equiv 24\lambda dx - 24bx \pmod{24c}\\
&\equiv 24\left(\lambda dx - bx\right) \pmod{24c},
\end{aligned}$$

and

$$\mu \equiv \lambda dx - bx \pmod{c}.$$

Moreover, $ad \equiv ax \equiv 1 \pmod{c}$, so that $d \equiv x \pmod{c}$, and

$$\begin{aligned}
\mu &\equiv \lambda d^2 - bd \pmod{c},\\
\mu a^2 &\equiv \lambda a^2 d^2 - a^2 bd \pmod{c},\\
\mu a^2 &\equiv \lambda - ab \pmod{c},\\
\lambda &\equiv \mu a^2 + ab \pmod{c}.
\end{aligned}$$

We therefore have

$$\rho^{\lambda(j+1)} = \rho^{(\mu a^2+ab)(j+1)},$$

and

$$\begin{aligned}
h_j(\gamma\tau) &= (-i(c\tau+d))^{-1/2}\rho^{ab(j+1)}\left(\frac{5c}{a}\right)\cdot\exp\left(a\frac{\pi i}{12}(5c-3)-\frac{5\pi iab}{12}\right)\frac{1}{5}\sum_{\mu=0}^{4}\rho^{\mu a^2(j+1)}\eta\left(\frac{\tau+24\mu}{5}\right)^{-1}\\
&= (-i(c\tau+d))^{-1/2}\rho^{ab(j+1)}\left(\frac{5c}{a}\right)\cdot\exp\left(a\frac{\pi i}{12}(5c-3)-\frac{5\pi iab}{12}\right)\frac{1}{5}\sum_{\mu=0}^{4}\rho^{\mu(j'+1)}\eta\left(\frac{\tau+24\mu}{5}\right)^{-1},\\
&= (-i(c\tau+d))^{-1/2}\rho^{ab(j+1)}\left(\frac{5c}{a}\right)\cdot\exp\left(a\frac{\pi i}{12}(5c-3)-\frac{5\pi iab}{12}\right)h_{j'}(\tau), \qquad (3.9)
\end{aligned}$$

with

$$j' \equiv a^2(j+1) - 1 \pmod 5. \tag{3.10}$$

We have here some semblance of modular symmetry. Notice that

$$\text{If } j \equiv 4 \pmod 5, \text{ then } j' \equiv 4 \pmod 5, \tag{3.11}$$
$$\text{If } j \equiv 1,2 \pmod 5, \text{ then } j' \equiv 1,2 \pmod 5, \tag{3.12}$$
$$\text{If } j \equiv 0,3 \pmod 5, \text{ then } j' \equiv 0,3 \pmod 5. \tag{3.13}$$

Moreover, because $5 \nmid a$, we have $a^2 \equiv 1,4 \pmod 5$. If $a^2 \equiv 1 \pmod 5$, then $j' \equiv j \pmod 5$. On the other hand, if $a^2 \equiv 4 \pmod 5$, then $j' \equiv 4j+3 \pmod 5$. In the latter case, h_1 and h_2 interchange in their respective transformation equations, as do h_0 and h_3.

Therefore, buried in these tedious calculations, we have the forewarning of three significant identities. The case for $j \equiv 4 \pmod 5$ is the simplest, and we will give it the majority of our attention.

3.2.3 Ramanujan's Identity

Notice that (3.9) is true only for $\gamma \in \Gamma_0(5)^*$. For an arbitrary $\gamma \in \Gamma_0(5)$, the relationship between $h_j(\gamma\tau)$ and $h_j(\tau)$ will be more complicated. On the other hand, if we can construct a function using h_4 which is invariant under transformations of $\Gamma_0(5)^*$, then it will also be invariant under $\Gamma_0(5)$, as the former generates the latter. Let us reexamine $h_4(\gamma\tau)$. Notice that because ρ is a fifth root of unity, we must have

$$h_4(\gamma\tau) = (-i(c\tau+d))^{-1/2}\left(\frac{5c}{a}\right)\cdot \exp\left(a\frac{\pi i}{12}(5c-3) - \frac{5\pi i ab}{12}\right)h_4(\tau).$$

To construct an exact invariance, let us consider $\eta(5\tau)$. As we have already shown in Theorem 2.39,

$$\eta(5\gamma\tau) = \eta\left(5\frac{a\tau+b}{c\tau+d}\right) \tag{3.14}$$
$$= \eta\left(\frac{a(5\tau)+5b}{c(5\tau)/5+d}\right) \tag{3.15}$$
$$= \eta(\gamma'(5\tau)), \tag{3.16}$$

with $\gamma' = \left(\begin{smallmatrix} a & 5b \\ c/5 & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})^*$. We therefore have

$$\eta(5\gamma\tau) = (-i(c\tau+d))^{1/2}\epsilon(a,5b,c/5,d)\eta(5\tau), \tag{3.17}$$
$$= (-i(c\tau+d))^{1/2}\left(\frac{c/5}{a}\right)\exp\left(-\frac{a\pi i}{12}(c/5-5b-3)\right)\eta(5\tau). \tag{3.18}$$

Because

$$\begin{aligned}\left(\frac{c/5}{a}\right) &= \left(\frac{c/5}{a}\right)\left(\frac{5}{a}\right)^2 \\ &= \left(\frac{c/5}{a}\right)\left(\frac{5^2}{a}\right) \\ &= \left(\frac{5c}{a}\right),\end{aligned}$$

we have

$$\eta(5\gamma\tau) = (-i(c\tau+d))^{1/2}\left(\frac{5c}{a}\right)\exp\left(-\frac{a\pi i}{12}(c/5-5b-3)\right)\eta(5\tau). \tag{3.19}$$

Notice that

$$\eta(5\tau)\cdot h_4(\tau) = (q^5;q^5)_\infty \sum_{n=0}^{\infty} p(5n+4)q^{n+1},$$

and

$$\begin{aligned}\eta(5\gamma\tau)\cdot h_4(\gamma\tau) &= \left(\frac{5c}{a}\right)^2 \exp\left(-\frac{a\pi i}{12}(c/5-5b-3)\right)\cdot\exp\left(a\frac{\pi i}{12}(5c-3)-\frac{5\pi iab}{12}\right)\eta(5\tau)h_4(\tau) \\ &= \exp\left(-\frac{a\pi i}{12}(c/5-5b-3)+a\frac{\pi i}{12}(5c-3)-\frac{5\pi iab}{12}\right)\eta(5\tau)h_4(\tau).\end{aligned}$$

Now,

$$\begin{aligned}&-a(c/5-5b-3)+a(5c-3)-5ab \\ =&-ac/5+5ab+3a+5ac-3a-5ab \\ \equiv& ac/5(-1+25)\equiv 0 \pmod{24},\end{aligned}$$

so that

$$\eta(5\gamma\tau)h_4(\gamma\tau) = \eta(5\tau)h_4(\tau).$$

If we let $F_4(\tau) := \eta(5\tau)h_4(\tau)$, then we have established modular symmetry of $F_4(\tau)$ over $\Gamma_0(5)$.

This verifies Condition 1 of Definition 2.30. To verify that F_4 satisfies the second property, we need to verify the meromorphic behavior of F_4 over any element of $\mathrm{SL}(2,\mathbb{Z})$. For members of $\Gamma_0(5)$, notice that F_4 has the expansion

$$q \cdot (q^5;q^5)_\infty \cdot \sum_{n=0}^{\infty} p(5n+4)q^n,$$

and therefore has

$$\mathrm{ord}_\infty^{(5)}(F_4) = 1,$$

with leading coefficient $p(4) = 5$.

Next we examine the behavior of $F_4(\gamma\tau)$ in which $\gamma \in \mathrm{SL}(2,\mathbb{Z}) \setminus \Gamma_0(5)$. Recall from (2.6) that we can express any $\gamma \in \mathrm{SL}(2,\mathbb{Z}) \setminus \Gamma_0(5)$ as

$$\gamma = \gamma_0 \cdot T \cdot S^k,$$

with $\gamma_0 \in \Gamma_0(5)$ and $k \in \mathbb{Z}$. If we let $\tau' = \tau + k$, then we can write

$$\begin{aligned} F_4(\gamma\tau) &= F_4(\gamma_0 \cdot T \cdot S^k\tau) \\ &= F_4(T\tau') \\ &= F_4(-1/\tau'). \end{aligned}$$

Therefore, $F_4(\gamma\tau)$ has a Fourier expansion similar in form to $F_4(T\tau)$, and our problem is reduced to considering $F_4(T\tau)$.

As previously demonstrated in (2.7)-(2.12), we have

$$\eta(5 \cdot -1/\tau) = \eta(-1/(\tau/5)) = \frac{(-i\tau)^{1/2}}{\sqrt{5}}\eta(\tau/5).$$

Now we take $1 \le \lambda \le 4$, and we have

$$\begin{aligned} \eta\left(\frac{T\tau + 24\lambda}{5}\right) &= \eta\left(\frac{24\lambda\tau - 1}{5\tau}\right) \\ &= \eta\left(\frac{24\lambda z - y}{5z + x}\right), \end{aligned}$$

with $z = (\tau - x)/5$, and $x, y \in \mathbb{Z}$ such that $24\lambda x + 5y = 1$. Continuing, we have

$$\eta\left(\frac{T\tau+24\lambda}{5}\right)=(-i(5z+x))^{1/2}\epsilon(24\lambda,-y,5,x)\eta\left(\frac{\tau-x}{5}\right),$$
$$=(-i\tau)^{1/2}\epsilon(24\lambda,-y,5,x)\eta\left(\frac{\tau-x}{5}\right).$$

On the other hand, if $\lambda=0$, then we have

$$\eta\left(\frac{T\tau+24\lambda}{5}\right)=\eta\left(\frac{-1}{5\tau}\right)$$
$$=\sqrt{5}(-i\tau)^{1/2}\eta\left(5\tau\right).$$

We therefore have

$$F_4(T\tau)=\frac{1}{\sqrt{5}}(-i\tau)^{1/2}\eta(\tau/5)\left(\frac{1}{5\sqrt{5}}(-i\tau)^{-1/2}\eta(5\tau)^{-1}+\frac{1}{5}\sum_{\lambda=1}^{4}(-i\tau)^{-1/2}\epsilon(24\lambda,-y,5,x)\eta\left(\frac{\tau-x}{5}\right)^{-1}\right)$$
$$=\frac{1}{25}q^{1/5(24)}(q^{1/5};q^{1/5})_\infty\left(q^{-5/24}(1+...)+\sqrt{5}\sum_{\lambda=1}^{4}q^{-1/5(24)}\epsilon(24\lambda,-y,5,x)(1+...)\right)$$
$$=\frac{1}{25}q^{-1/5}(1+...).$$

Notice that

$$\mathrm{ord}_0^{(5)}(F_4)=-1,$$

with leading coefficient 1/25, at $[0]_5$.

We have established that $F_4(\tau)\in\mathcal{M}^0\left(\Gamma_0(5)\right)$. Recall the function t that we examined in Theorem 2.39:

$$t(\tau)=\frac{\eta(5\tau)^6}{\eta(\tau)^6}=q\prod_{m=1}^{\infty}\left(\frac{1-q^{5m}}{1-q^m}\right).$$

Notice from (2.15)-(2.16) that F_4 has a matching principal part to that of $5t$.

Therefore, $F_4-5t=c$ for some constant c. But we can quickly verify that

$$(q^5;q^5)_\infty\cdot\sum_{n=0}^{\infty}p(5n+4)q^{n+1}-5t$$

has constant term 0. With this, we have $c=0$, from which (1.7) follows.

3.2.4 Kolberg's Identities

On the other hand, if we consider $j \equiv 1 \pmod 5$, then matters are slightly more complicated. We see from (3.12) that the congruence classes $j \equiv 1, 2 \pmod 5$ are closely related. Multiplying $h_1(\gamma\tau)$ and $h_2(\gamma\tau)$ together, we have

$$h_1(\gamma\tau) \cdot h_2(\gamma\tau) = (-i(c\tau + d))^{-1} \exp\left(a\frac{\pi i}{6}(5c - 3) - \frac{5\pi iab}{6}\right) h_1(\tau)h_2(\tau).$$

The weight of this transformation suggests that $\eta(5\tau)$ is a sufficient prefactor. We see that

$$\begin{aligned}\eta(5\tau)^2 h_1(\tau)h_2(\tau) &= q^{10/24+23/24(5)+47/24(5)}(q^5;q^5)_\infty^2 \left(\sum_{n=0}^{\infty} p(5n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+2)q^n\right) \\ &= q(q^5;q^5)_\infty^2 \left(\sum_{n=0}^{\infty} p(5n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+2)q^n\right). \qquad (3.20)\end{aligned}$$

If we refer to this as $F_1(\tau)$, then

$$\begin{aligned}F_1(\gamma\tau) &= \exp\left(\frac{\pi i}{6}(5ac - 3a - 5ab - ac/5 + 5ab + 3a)\right) F_1(\tau) \\ &= \exp\left(\frac{\pi i}{6} ac/5(24)\right) F_1(\tau) \\ &= F_1(\tau).\end{aligned}$$

To establish the necessary meromorphic properties, we note from (3.20) that F_1 has a zero of order 1 at $[\infty]_5$.

For similar reasons to those of F_4, we need only examine $F_1(T\tau)$ to finish the proof that $F_1 \in \mathcal{M}^0(\Gamma_0(5))$. We have

$$\begin{aligned}F_1(T\tau) &= \frac{1}{5}(-i\tau)\eta(\tau/5)^2 \prod_{j=1}^{2}\left(\frac{1}{5\sqrt{5}}(-i\tau)^{-1/2}\eta(5\tau)^{-1} + \frac{1}{5}\sum_{\lambda=1}^{4}\rho^{\lambda(j+1)}(-i\tau)^{-1/2}\epsilon(24\lambda, -y, 5, x)\eta\left(\frac{\tau - x}{5}\right)^{-1}\right) \\ &= \frac{1}{5}q^{2/5(24)}(q^{1/5};q^{1/5})_\infty \left(\frac{1}{125}q^{-5/24}(1+...) + q^{-1/5(24)}\rho\epsilon(24\lambda, -y, 5, x)(1+...)\right) \\ &= \frac{1}{625}q^{-2/5}(1+...).\end{aligned}$$

Thus, F_1 has a pole of order -2 at $[0]_5$.

Notice that explicitly computing the principal part of F_1 is tedious. However, because we know from Theorem 2.39 that $1/t$ has a zero of order 1 at $[0]_5$, we can easily simplify the problem by multiplying F_1 by $1/t^2$:

$$t^{-2}F_1(\tau) = \frac{1}{q^2}\frac{(q;q)_\infty^{12}}{(q^5;q^5)_\infty^{10}}\left(\sum_{n=0}^{\infty} p(5n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+2)q^n\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(5)\right).$$

The single pole that $t^{-2}F_1$ has can now be very easily examined as

$$t^{-2}F_1(\tau) = \frac{2}{q} + 13 + q(1+...),$$
$$t^{-2}F_1(\tau) - 2t^{-1} = 25 + ...$$

We find that $t^{-2}F_1 - 2t^{-1} - 25 \in \mathcal{M}\left(\Gamma_0(5)\right)$ has no poles at any cusp, and no constant term. Therefore, it is identically 0. Rearranging, we find that

$$t^{-2}F_1(\tau) = 2t^{-1} + 25,$$
$$F_1(\tau) = 2t + 25t^2,$$

and we have proved (1.14).

In a similar manner, if we define

$$F_0(\tau) := \eta(5\tau)^2 h_0(\tau)h_3(\tau) = q(q^5;q^5)_\infty^2\left(\sum_{n=0}^{\infty} p(5n)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+3)q^n\right),$$

then we can prove that

$$F_0(\tau) = 3t + 25t^2.$$

3.3 General Setup

The previous section highlights the computational complexities of the problem. However, the reader may notice that the required computations, tedious though they may be, will not substantially vary for $p(mn+j)$ for any progression defined by $0 \le j \le m-1$. Indeed, the steps of verifying modular symmetry and appropriate meromorphic conditions are so straightforward that we may go further, and study $a(mn+j)$ with $a(n)$ a much more general arithmetic function. We will now give a brief overview of these steps.

Let us take $q^{1/m} = e^{2\pi i\tau/m}$, $\rho = e^{2\pi i/m}$ for some $m \ge 1$, $0 \le j \le m-1$, Then define

$$
\begin{aligned}
w(r) &:= \sum_{\delta|M} r_\delta, \\
\sigma_\infty(r) &:= \sum_{\delta|M} \delta r_\delta, \\
\sigma_0(r) &:= \sum_{\delta|M} \frac{M}{\delta} r_\delta, \text{ and} \\
\Pi(r) &:= \prod_{\delta|M} \delta^{|r_\delta|}.
\end{aligned}
$$

We will define an arithmetic function $a(n)$ by the generating function

$$F_r(\tau) := \sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = q^{-\sigma_\infty(r)/24} \prod_{\delta|M} \eta(\delta\tau)^{r_\delta},$$

with $M \geq 1$, and let $r := (r_\delta)_{\delta|M}$ be an integer-valued vector. Through a nearly identical process to the previous section, we can show that

$$\frac{1}{m} \sum_{\lambda=0}^{m} \rho^{-j\kappa\lambda} q^{-j/m} F\left(\frac{\tau + \kappa\lambda}{m}\right) = \sum_{n=0}^{\infty} a(mn+j)q^n,$$

with $\kappa = \gcd(m^2 - 1, 24)$. We will construct a function $h_{m,j}(\tau, r)$ in a similar fashion to the functions h_j in the previous section:

$$
\begin{aligned}
h_{m,j}(\tau) : = h_{m,j}(\tau, r) &= \frac{1}{m} \sum_{\lambda=0}^{m} \rho^{-j\kappa\lambda} \prod_{\delta|M} \eta\left(\delta \cdot \frac{\tau + \kappa\lambda}{m}\right)^{r_\delta} \\
&= q^{(24j+\sigma_\infty(r))/24m} \sum_{n=0}^{\infty} a(mn+j)q^n
\end{aligned}
$$

As before, we will be working primarily over $\Gamma_0(N)^*$. However, the matter of selecting the appropriate N is by no means a trivial task. We describe the following list of criteria that such an N will generally need to satisfy:

Definition 3.1. Define Δ^* as the set of 5-tuples (N, M, r, m, j) such that

- $N, M \in \mathbb{Z}_{\geq 1}$;
- $j, m \in \mathbb{Z}$ have $0 \leq j \leq m - 1$;

- $r = (r_\delta)_{\delta|M}$ is an integer-valued vector;
- for every prime p such that $p|m$, we have $p|N$;
- if $\delta|M$ and $r_\delta \neq 0$, then $\delta|mN$;
- For $\kappa := \gcd(m^2-1, 24)$:
  - $\kappa \cdot \frac{mN^2}{M}\sigma_0(r) \equiv 0 \pmod{24}$;
  - $\kappa \cdot Nw(r) \equiv 0 \pmod{8}$;
  - $N \equiv 0 \pmod{\frac{24m}{\gcd(\kappa\cdot(-24j-\sigma_\infty(r)),24m)}}$;
  - if $2|m$, $\Pi(r) = 2^\upsilon \cdot \omega$ with $\upsilon, \omega \in \mathbb{Z}_{\geq 0}$, $\omega \equiv 1 \pmod{2}$ then *at least one* of the following applies:
    - $\kappa \cdot N \equiv 0 \pmod{4}$ and $N\upsilon \equiv 0 \pmod{8}$,
    - $\upsilon \equiv 0 \pmod{2}$ and $N \cdot (1-\omega) \equiv 0 \pmod{8}$.

One can see, for example, that $(5, 1, (-1), 5, 4) \in \Delta^*$.

This is admittedly a long list of criteria. However, it can be shown [91, Section 3.1] that for any M, r, m, j, some N exists such that $(N, M, r, m, j) \in \Delta^*$. Moreover, these conditions can be checked rapidly by a computer.

It should also be noted that there are some RK identities in which an N can be chosen which does not satisfy Δ^*. This is incorporated into Radu's algorithm. We give one interesting example in the sequel. However, such identities are still closely related to other congruence levels which do satisfy Δ^*.

If an appropriate N has been selected, then we can construct a rough modular transformation equation as follows [89, Theorem 2.14]:

Theorem 3.2. *Let* $(N, M, r, m, j) \in \Delta^*$, *and* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$. *Then*

$$h_{m,j}(\gamma\tau) = \beta(\gamma, r) \cdot \exp\left(2\pi i \frac{ab(1-m^2)(24j+\sigma_\infty(r))}{24m}\right)(-i(c\tau+d))^{w(r)/2} \cdot h_{m,\gamma\odot_r j}(\tau),$$

with

$$\beta(\gamma, r) := \prod_{\delta|M}\left(\frac{mc\delta}{a}\right)^{|r_\delta|} \exp\left(-\frac{\pi i a}{12}\left(\frac{mc}{M}\sigma_0(r) - mb\sigma_\infty(r) - 3w(r)\right)\right),$$

$$\gamma \odot_r j :\equiv ja^2 + \frac{a^2-1}{24}\sigma_\infty(r) \pmod{m}.$$

As in the previous section, some semblance of modular symmetry is noticeable. We will define the collection of all possible $\gamma \odot_r j$ for a given (N, M, r, m, j):

Definition 3.3. Let

$$P_{m,r}(j) := \{\gamma \odot_r j : \gamma \in \Gamma_0(N)^*\}.$$

This set may be described in many different ways. We give a lemma which describes the set in a highly constructive manner [89, Lemma 3.11].

Lemma 3.4. *Let $\mathbb{Z}/n\mathbb{Z}^{\times 2}$ be the set of quadratic residues modulo n; and for any $s \in \mathbb{Z}$, let $[s]$ be the residue class of s mod $24m$. Given $(N, M, r, m, j) \in \Delta^*$, define*

$$P_{m,r}(j) := \left\{ js + \frac{s-1}{24} \cdot \sigma_\infty(r) : 0 \le s \le 24m-1,\ \gcd(s, 24m) = 1,\ [s] \in \mathbb{Z}/24m\mathbb{Z}^{\times 2} \right\}.$$

Thus, for $(5, 1, (-1), 5, 4) \in \Delta^*$, we have $P_{5,(-1)}(4) = \{4\}$. On the other hand, for $(5, 1, (-1), 5, 1) \in \Delta^*$, we have $P_{5,(-1)}(1) = \{1, 2\}$.

We now come to the principal theorem of Ramanujan–Kolberg identities which gives us the means of constructing a suitable algorithm for their construction [91, Theorem 45].

Theorem 3.5. *Let $(N, M, r, m, j) \in \Delta^*$, $s = (s_\delta)_{\delta|N}$ an integer-valued vector indexed over the divisors of N, and $\nu \in \mathbb{Z}$ such that*

$$\nu \equiv \sum_{j' \in P_{m,r}(j)} (1 - m^2)(24j' + \sigma_\infty(r))/m \pmod{24}.$$

Then

$$f(N, s, M, r, m, j) := f(N, s, M, r, m, j)(\tau) = \prod_{\delta|N} \eta(\delta\tau)^{s_\delta} \prod_{j' \in P_{m,r}(j)} h_{m,j'}(\tau) \in \mathcal{M}\left(\Gamma_0(N)\right)$$

if and only if the following conditions are met:

$$|P_{m,r}(j)| w(r) + w(s) = 0, \tag{3.21}$$

$$\nu + |P_{m,r}(j)| m \sigma_\infty(r) + \sigma_\infty(s) \equiv 0 \pmod{24}, \tag{3.22}$$

$$|P_{m,r}(j)| \frac{mN}{M} \sigma_0(r) + \sigma_0(s) \equiv 0 \pmod{24}, \tag{3.23}$$

$$\left(\prod_{\delta|M} (m\delta)^{|r_\delta|} \right)^{|P_{m,r}(j)|} \cdot \Pi(s) \in \mathbb{Z}^2. \tag{3.24}$$

For example, let us take $(5, 1, (-1), 5, 4) \in \Delta^*$. Then by the last theorem, we have $N = 5$, $M = 1$, $r = (-1)$, $m = 5$, $j = 4$. We already derived $P_{5,(-1)}(4) = \{4\}$. Next, we can define $h_{5,4}$ as

$$h_{5,4}(\tau,(-1)) = q^{19/24}\sum_{n=0}^{\infty} p(5n+4)q^n.$$

With a trivial product $P_{5,(-1)}(4)$, the product on the left-hand side of the corresponding RK identity will have the form

$$\prod_{j'\in P_{5,(-1)}(4)} h_{5,j'}(\tau) = h_{5,4}(\tau).$$

We have $N=5$. If we take $s := (0,1)$, then we have

$$\prod_{\delta|N} \eta(\delta\tau)^{s_\delta} = \eta(5\tau).$$

One can verify that $\nu \equiv 0 \pmod{24}$, and that

$$\begin{aligned}
|P_{5,(-1)}(4)|w((-1)) + w((0,1)) &= 1(-1)+1 = 0,\\
\nu + |P_{5,(-1)}(4)|\cdot 5\cdot \sigma_\infty((-1)) + \sigma_\infty((0,1)) &\equiv 5(-1)+5 \equiv 0 \pmod{24},\\
|P_{5,(-1)}(4)|\cdot 25/1\cdot \sigma_0((-1)) + \sigma_0((0,1)) &\equiv 25(-1)+1 \equiv 0 \pmod{24},\\
\left((5)^{-1}\right)^1 \cdot 1^0\cdot 5^1 &= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(5,(0,1),1,(-1),5,4) &= \eta(5\tau)\cdot q^{19/24}\sum_{n=0}^{\infty} p(5n+4)q^n\\
&= (q^5;q^5)_\infty \cdot \sum_{n=0}^{\infty} p(5n+4)q^{n+1} \in \mathcal{M}\left(\Gamma_0(5)\right).
\end{aligned}$$

As a final useful theorem, we give a means of establishing bounds for the order of f at the cusps of $\mathrm{X}_0(N)$ [91, Lemma 46]. This will help us to push all the poles of a given f to a single cusp later.

Theorem 3.6. *Let $(N,M,r,m,j)\in\Delta^*$, $\kappa = \gcd(m^2-1,24)$, and let $(s_\delta)_{\delta|N}$ be an integer-valued vector such that $f(N,s,M,r,m,j)\in\mathcal{M}\left(\Gamma_0(N)\right)$ as in Theorem 3.5, and let $\gamma\in\mathrm{SL}(2,\mathbb{Z})$. Then*

$$\mathrm{ord}_\gamma^{(N)}\left(f(N,s,M,r,m,j)\right) \geq \frac{N}{\gcd(c^2,N)}\left(|P_{m,r}(j)|\,p(\gamma,r) + p^*(\gamma,s)\right),$$

for

$$p(\gamma, r) := \min_{0\le\lambda\le m-1}\frac{1}{24}\sum_{\delta|M} r_\delta \frac{\gcd\left(\delta(a+\kappa\lambda c), mc\right)^2}{\delta m},$$
$$p^*(\gamma, s) := \frac{1}{24}\sum_{\delta|N} s_\delta \frac{\gcd\left(\delta, c\right)^2}{\delta}.$$

For example, for $(5,1,(-1),5,4)\in\Delta^*$, we have

$$\begin{aligned}
p(I,(-1)) :&= \min_{0\le\lambda\le 4}\frac{1}{24}(-1)\frac{\gcd\left(1,0\right)^2}{5},\\
&= -\frac{1}{120},\\
p(T,(-1)) :&= \min_{0\le\lambda\le 4}\frac{1}{120}(-1)\gcd\left(24\lambda,5\right)^2,\\
&= -\frac{5}{24},\\
p^*(I,(0,1)) :&= \frac{1}{120}\gcd\left(5,0\right)^2\\
&= \frac{5}{24},\\
p^*(T,(0,1)) :&= \frac{1}{120}\gcd\left(5,1\right)^2\\
&= \frac{1}{120}.
\end{aligned}$$

We now have enough information to give lower bounds to the order of $f(5,(0,1),1,(-1),5,4)$ at the cusps:

$$\begin{aligned}
\mathrm{ord}_I^{(5)}\left(f(5,(0,1),1,(-1),5,4)\right) &\ge \frac{5}{\gcd(0,5)}\left(p(I,(-1))+p^*(I,(0,1))\right)\\
&= -\frac{1}{120}+\frac{5}{24}=\frac{1}{5}\ge 0,\\
\mathrm{ord}_T^{(5)}\left(f(5,(0,1),1,(-1),5,4)\right) &\ge \frac{5}{\gcd(1,5)}\left(p(T,(-1))+p^*(T,(0,1))\right)\\
&= 5\left(-\frac{5}{24}+\frac{1}{120}\right)=-5\cdot\frac{1}{5}=-1.
\end{aligned}$$

3.3.1 Membership Algorithm

The previous subsection discusses what is in effect the left-hand side of a potential RK identity. To handle the right-hand side, we need to understand the space of eta quotients over $\Gamma_0(N)$. By the modular cusp analysis in the previous chapter, we will focus on the space $\langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$. Given any $N \in \mathbb{Z}_{\geq 2}$, the corresponding monoid generators of $\mathcal{E}^{\infty}(N)$ can be computed through a terminating algorithm [91, Lemma 25].

PROCEDURE: `etaGenerators` (Eta Monoid Generators)

INPUT:
$N \in \mathbb{Z}_{\geq 2}$

OUTPUT:
$\{g_1, g_2, ..., g_r\}$ such that $\left\{g_1^{k_1} \cdot g_2^{k_2} \cdot ... \cdot g_r^{k_r} : k_1, k_2, ..., k_r \in \mathbb{Z}_{\geq 0}\right\} = \mathcal{E}^{\infty}(N)$.

Similarly, the corresponding basis elements of $\langle \mathcal{E}^{\infty}(N) \rangle_{\mathbb{Q}}$ can be computed through a terminating algorithm [91, Theorem 16].

PROCEDURE: `AB` (Algebra Basis)

INPUT:
A set of modular functions $\mathcal{F} = \{f_1, f_2, ..., f_r\} \subseteq \mathcal{M}^{\infty}\left(\Gamma_0(N)\right)$

OUTPUT:
$t, g_1, g_2, ..., g_{v-1} \in \mathcal{M}^{\infty}\left(\Gamma_0(N)\right)$ such that conditions (2.17)-(2.20) are satisfied, and

$$\bigoplus_{j=0}^{v-1} g_j \mathbb{Q}[t] = \langle \mathcal{F}_{\mathrm{M}} \rangle_{\mathbb{Q}},$$

with $\mathcal{F}_{\mathrm{M}} := \left\{f_1^{k_1} \cdot f_2^{k_2} \cdot ... \cdot f_r^{k_r} : k_1, k_2, ..., k_r \in \mathbb{Z}_{\geq 0}\right\}$.

If we give the output of `etaGenerators[`N`]` as input for `AB`, then we will produce a generating set of functions satisfying the criteria of Theorem 2.44.

Below, let $f^{(-)}$ be the principal part of f (including its constant):

PROCEDURE: `MW` (Membership Witness)

INPUT:

- $N \in \mathbb{Z}_{\geq 2}$,
- $t, g_1, g_2, ..., g_{v-1} \in \mathcal{M}^{\infty}\left(\Gamma_0(N)\right)$ satisfying (2.17)-(2.21),
- $f^{(-)}$, for some $f \in \mathcal{M}^{\infty}\left(\Gamma_0(N)\right)$.

OUTPUT:

IF $f \in \langle \mathcal{E}^\infty(N) \rangle_{\mathbb{Q}}$, RETURN $\{p_0, p_1, ..., p_k\} \subseteq \mathbb{Q}[x]$ such that
$f = \sum_{k=0}^{v-1} g_k \cdot p_k(t)$ with $g_0 = 1$;
ELSE, PRINT “NO MEMBERSHIP”.

3.3.2 Main Procedure

Let us take an arithmetic function $a(n)$ with the generating function

$$F_r(\tau) = \sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta}, \tag{3.25}$$

with $r = (r_\delta)_{\delta|M}$ an integer-valued vector. Suppose we are interested in a possible RK identity for $a(mn+j)$, with $0 \leq j \leq m-1$, and that we wish for the eta quotients on the right-hand side to have factors indexed over the divisors of some $N \in \mathbb{Z}_{\geq 2}$.

With few exceptions, most RK identities of interest will occur for an N chosen such that $(N, M, r, m, j) \in \Delta^*$. We will therefore work with two distinct cases, depending on whether $(N, M, r, m, j) \in \Delta^*$. Let us first assume that this condition applies.

In this case, our first requirement is to compute $P_{m,r}(j)$. Then we must solve the system of equations (3.21)-(3.24) for an acceptable integer-valued vector $s = (s_\delta)_{\delta|N}$. Such a vector s satisfies

$$f(N, s, M, r, m, j) = \prod_{\delta|N} \eta(\delta\tau)^{s_\delta} \prod_{j' \in P_{m,r}(j)} h_{m,j'}(\tau) \in \mathcal{M}\left(\Gamma_0(N)\right).$$

We will in fact do more. Since we ultimately want a function with a pole only at ∞, we will compute the function $\mu \in \langle \mathcal{E}(N) \rangle_{\mathbb{Q}}$ with positive order at every cusp except ∞. We will then compute the order of $f(N, s, M, r, m, j)$ at every cusp, and therefore the minimal power k_1 such that

$$f_L := f_L(N, s, M, r, m, j) = \mu^{k_1} f(N, s, M, r, m, j) \in \mathcal{M}^\infty\left(\Gamma_0(N)\right).$$

Our next step is to determine whether $f_L(N, s, M, r, m, j) \in \langle \mathcal{E}^\infty(N) \rangle_{\mathbb{Q}}$.

We compute the set of possible solutions, and then select the optimal vector in the sense that f_L will have minimal order at ∞. This is why we incorporate μ^{k_1} into our s vector: doing so will greatly simplify our later calculations, since a smaller total order on the left hand side of our prospective identity ensures that less computation time will be needed to determine membership of f_L (we completely ignore μ^{k_0} for the time being).

We now define f_1 as our prefactor, together with the fractional powers of q taken in each $h_{m,j'}$. This gives us another way to write f_L:

$$f_1(N,s,M,r,m,j) = \prod_{\delta|N} \eta(\delta\tau)^{s_\delta} \cdot q^{\sum_{j'\in P_{m,r}(j)} \frac{24j'+\sum_{\delta|M}\delta\cdot r_\delta}{24m}}, \tag{3.26}$$

$$f_L(N,s,M,r,m,j) = \prod_{\delta|N} \eta(\delta\tau)^{s_\delta} \cdot \prod_{j'\in P_{m,r}(j)} h_{m,j'}(\tau) \tag{3.27}$$

$$= f_1(N,s,M,r,m,j) \cdot \prod_{j'\in P_{m,r}(j)} \left(\sum_{n=0}^{\infty} a(mn+j')q^n\right). \tag{3.28}$$

At last, we come to the question of how to program f_L into a computer. Of course, a computer cannot store an infinitely large generating function. However, owing to the finiteness conditions of modular functions, it does not have to. We have already established that f_L has only one pole over $\Gamma_0(N)$, and we will ultimately express f_L in terms of other modular functions with a single pole; we therefore only need to examine the principal part and constant of f_L.

Notice that f_1 has a principal part in q, and $\prod_{j'\in P_{m,r}(j)} \left(\sum_{n=0}^{\infty} a(mn+j')q^n\right)$ has no principal part in q. To take the full principal part and constant of f_L, we need only take the principal part of f_1, and every term of the form $a(mn+j')q^n$, with $n \leq \left|\mathrm{ord}_{\infty}^{(N)}(f_1)\right|$.

Let us take $\left|\mathrm{ord}_{\infty}^{(N)}(f_1)\right| = n_1$, and write

$$\begin{aligned} f_1 &= \sum_{n=-n_1}^{\infty} c(n)q^n \\ &= \frac{c(-n_1)}{q^{n_1}} + \frac{c(-n_1+1)}{q^{n_1-1}} + ... + \frac{c(1)}{q} + c(0) + \sum_{n=1}^{\infty} c(n)q^n, \\ f_1^{(-)} &= \frac{c(-n_1)}{q^{n_1}} + \frac{c(-n_1+1)}{q^{n_1-1}} + ... + \frac{c(1)}{q} + c(0). \end{aligned}$$

How many terms $a(n)$ of $F_r(\tau)$ do we need? We know that $0 \leq j' \leq m-1$; we also know that if $n_0 > \left|\mathrm{ord}_{\infty}^{(N)}(f_1)\right| = n_1$, then $a(mn_0+j')q^{n_0}$ cannot contribute to the principal part of f_L. Therefore, to have the principal part completely calculated, we need only take

$$mn + j' \leq m \cdot \left|\mathrm{ord}_{\infty}^{(N)}(f_1)\right| + j' < m\cdot n_1 + m = m(n_1+1).$$

We can compute and store

$$L := \sum_{n=0}^{m\cdot(n_1+1)} a(n)q^n.$$

We need not consider any larger values of $a(n)$.

Now we take

$$f_L^{(-)} = \left(f_1^{(-)} \cdot \prod_{j' \in P_{m,r}(j)} \left(\sum_{n=0}^{m} a(mn+j')q^n \right) \right)^{(-)}.$$

Of course, $f_L^{(-)}$ is a polynomial in q^{-1}. In particular, $f_L^{(-)}$ is finite, and can therefore be examined by a computer.

We can now define our main procedure. We want to determine whether our constructed $f_L \in \langle \mathcal{E}^\infty(N) \rangle_{\mathbb{Q}}$. We construct [91, Section 2.1] the functions $t, g_1, g_2, ..., g_{v-1} \in \mathcal{M}^\infty(\Gamma_0(N))$, satisfying conditions (2.17)-(2.21):

$$\langle \mathcal{E}^\infty(N) \rangle_{\mathbb{Q}} = \bigoplus_{u=0}^{v-1} g_u \mathbb{Q}[t].$$

We may now use our `MW` procedure to check whether $f_L \in \bigoplus_{u=0}^{v-1} g_u \mathbb{Q}[t]$ by examining $f_L^{(-)}$.

Notice that we cannot merely construct the principal parts of the functions t, g_u, and disregard the rest of each function. We reduce $f_L^{(-)}$ by subtracting monomials of the form $g_u \cdot t^n$; terms other than the principal parts of t, g_u will influence the overall principal part of the product. We must therefore be careful to construct the complete principal part of each $g_u \cdot t^n$.

If `MW` returns "NO MEMBERSHIP", then the suspected identity does not exist—at least over $\Gamma_0(N)$. One may attempt a different N to find an identity. Otherwise, `MW` will return

$$\{p_0, p_1, ..., p_{v-1}\} \subseteq \mathbb{Q}[x], \tag{3.29}$$

and we have the complete identity

$$f_1(N, s, M, r, m, j) \cdot \prod_{j' \in P_{m,r}(j)} \left(\sum_{n=0}^{\infty} a(mn+j')q^n \right) = \sum_{u=0}^{v-1} g_u \cdot p_u(t). \tag{3.30}$$

Finally, we make note of an application so ubiquitous that we include it in our main procedure. We will attempt to extract the GCD of all of the coefficients of the p_u. Mathematica has a GCD procedure. If all of the coefficients of the p_u are integers, the procedure returns the GCD, which we will denote here as $\mathcal{D}$. On the other hand, if there exists some $K \in \mathbb{Z}_{\geq 2}$ such that the coefficients are elements in $\frac{1}{K}\mathbb{Z}$, then the GCD procedure will return $\frac{1}{K}\mathcal{D}$, with $\mathcal{D}$ defined as the GCD of the coefficients with the factor $1/K$ removed.

Our procedure, `RK[`N, M, r, m, j`]`, takes as input an $N \in \mathbb{Z}_{\geq 2}$ which defines the congruence subgroup $\Gamma_0(N)$ to work over; a generating function (defined by M and r), an arithmetic progression $mn + j$, with $0 \leq j \leq m - 1$.

PROCEDURE: `RK` (Ramanujan–Kolberg Implementation)

INPUT:

$$N \in \mathbb{Z}_{\geq 2}, , \tag{3.31}$$
$$M \in \mathbb{Z}_{\geq 1}, \tag{3.32}$$
$$r = (r_\delta)_{\delta|M}, r_\delta \in \mathbb{Z} \tag{3.33}$$
$$m, j \in \mathbb{Z} \text{ such that } 0 \leq j \leq m-1. \tag{3.34}$$

OUTPUT:

$$\prod_{\delta|\mathtt{M}} (\mathtt{q}^\delta; \mathtt{q}^\delta)_\infty^{\mathtt{r}_\delta} = \sum_{\mathtt{n}=0}^{\infty} \mathtt{a}(\mathtt{n})\mathtt{q}^{\mathtt{n}} \tag{3.35}$$

$$\boxed{\mathtt{f}_1(\mathtt{q}) \cdot \prod_{\mathtt{j}' \in \mathtt{P}_{\mathtt{m},\mathtt{r}}(\mathtt{j})} \sum_{\mathtt{n}=0}^{\infty} \mathtt{a}(\mathtt{mn}+\mathtt{j}')\mathtt{q}^{\mathtt{n}} = \sum_{\mathtt{g} \in \mathrm{AB}} \mathtt{g} \cdot \mathtt{p}_{\mathtt{g}}(\mathtt{t})} \tag{3.36}$$

`Modular Curve:`	$\mathtt{X}_0(\mathtt{N})$	(3.37)
`N:`	N	(3.38)
$\{\mathtt{M},\ (\mathtt{r}_{\delta\vert\mathtt{M}})\}$:	$\{M, r\}$	(3.39)
`m :`	m	(3.40)
$\mathtt{P}_{\mathtt{m},\mathtt{r}}(\mathtt{j})$:	$P_{m,r}(j)$	(3.41)
$\mathtt{f}_1(\mathtt{q})$:	$f_1(q)$	(3.42)
`t:`	t	(3.43)
`AB:`	$\{1, g_1, g_2, ..., g_{v-1}\}$	(3.44)
$\{\mathtt{p}_{\mathtt{g}}(\mathtt{t}) : \mathtt{g} \in \mathtt{AB}\}$:	$\{p_1, p_{g_1}, ..., p_{g_{v-1}}\}$	(3.45)
`Common Factor:`	$\mathcal{D}$	(3.46)

Lines (3.35), (3.36), (3.37) are unsubstituted expressions which are printed before the remaining lines are computed. They are meant to serve as a guide for the remainder of the output. Lines (3.35), (3.36) indicate the form of a potential RK identity, while line (3.37) indicates the associated modular curve.

The remaining lines give the appropriate substitutions. First, (3.38), (3.39), (3.40) return N, M, r, m. Line (3.41) gives all the possible values for j', including the initial input j. If a vector s cannot be found, then line (3.42) will return

$\mathtt{f}_1(\mathtt{q})$: `Select Another N` indicating that we are unable to construct the necessary modular function on the given $\Gamma_0(N)$. Similarly, if $f_L \notin \langle \mathcal{E}^\infty(N) \rangle_\mathbb{Q}$, then line (3.45) will return

$$\{\mathtt{p_g(t):g} \in \mathtt{AB}\}\mathtt{:}\ \ \mathtt{No\ Membership}$$

Otherwise, the corresponding membership witness is returned.

Finally, if a greatest common factor exists and is greater than one, then $\mathcal{D}$ is returned in line (3.46); otherwise, the line will return

```
Common Factor:  None
```

3.3.3 Some Remarks

Delta

Reexamining the identities of the introduction—(1.7) and (1.8) in particular—one may naturally guess that for a given progression $mn + j$, we must work over $\langle \mathcal{E}^\infty(m) \rangle_\mathbb{Q}$, i.e., that the level N of the associated modular curve must be equal to m. In fact, while N and m are not always equal, they are usually closely related. As we shall see, determination of the correct value of N is an important problem for the computation of RK identities.

With a single exception, all of the examples found so far rely upon what Radu has called the Δ^* criterion. For a complete definition of this criterion, see [91, Definitions 34, 35]. We provide a procedure to check this criterion, in `Delta[N, M, r, m, j]`.

PROCEDURE: `Delta`

INPUT:

$$N \in \mathbb{Z}_{\geq 2} \tag{3.47}$$
$$M \in \mathbb{Z}_{\geq 1} \tag{3.48}$$
$$r = (r_\delta)_{\delta | M}, r_\delta \in \mathbb{Z} \tag{3.49}$$
$$m, j \in \mathbb{Z} \text{ such that } 0 \leq j \leq m - 1. \tag{3.50}$$

OUTPUT:

IF Δ^* IS SATISFIED, RETURN TRUE,
ELSE, RETURN FALSE

We provide an additional procedure, in `minN[M, r, m, j]`, which will compute the minimal N that satisfies the Δ^* criterion.
PROCEDURE: `minN`

INPUT:

$$M \in \mathbb{Z}_{\geq 1}$$
$$r = (r_\delta)_{\delta|M}, r_\delta \in \mathbb{Z}$$
$$m, j \in \mathbb{Z} \text{ such that } 0 \leq j \leq m-1.$$

OUTPUT:

$N \in \mathbb{Z}_{\geq 2}$ such that `Delta[N, M, r, m, j]=True`.

The `RK` algorithm works in two distinct cases: Case 1, in which the Δ^* criterion is satisfied, and Case 2, in which it fails [91, Section 3.1]. The great majority of identities we have found arise from the first case. We will provide one interesting example of an identity arising from Case 2. However, Case 1 is generally a faster algorithm, and we recommend that users compute an N for which the Δ^* criterion is satisfied.

At any rate, for any given $M, r = (r_\delta)_{\delta|M}, m, j$ with $0 \leq j \leq m-1$, there must exist an $N \in \mathbb{Z}_{\geq 2}$ such that the Δ^* criterion is satisfied [91, Section 3.1]. It is generally convenient to work with the smallest possible N that satisfies the criterion. However, we will see in subsequent examples that the minimum value of N is not always the most useful. We will therefore leave the criterion for establishing N as separate from the main algorithm, and define N as part of the input.

`RKMan`

We also include a slightly modified implementation that we refer to as `RKMan`. This procedure is nearly identical to that used for Radu's algorithm, except that the algebra basis is included in the input. This is often helpful because, as we will see in some examples, construction of the algebra basis for $\langle \mathcal{E}^\infty(N) \rangle_{\mathbb{Q}}$ is often inefficient. If we already have a suitable algebra basis calculated (perhaps from a database, or a general study of eta quotient spaces), and if we know the genus of the corresponding Riemann surface, we may be able to construct a basis by inspection. This can often easily shorten the computation time. See Section 4.1.5 in the sequel for an example.

`RKE`

Regarding the value of k_0 in Theorem 5, and considering Conjecture 2.48, we very strongly suspect that k_0 may always be set to 0, and that therefore

$$\mathcal{M}^{\infty}\left(\Gamma_0(N)\right)_{\mathbb{Q}} \cap \left\langle \mathcal{E}(N) \right\rangle_{\mathbb{Q}} = \left\langle \mathcal{E}^{\infty}(N) \right\rangle_{\mathbb{Q}}.$$

for all $N \in \mathbb{Z}_{\geq 2}$. This is important, because the computation of a bound for k_0 is costly, and increases the runtime of our package. We therefore include the procedure `RKE` in addition to `RK` command. The two commands are nearly identical, except that `RKE` includes the power μ^{k_0} in our prefactor. This often increases runtime. See our examples at `https://www3.risc.jku.at/people/nsmoot/RKAlg/RKSupplement1.nb`.

We finally include `RKManE`, which is identical to `RKMan`, except that it includes μ^{k_0}.

CHAPTER 4
EXAMPLES

4.1 Introduction

We now give an overview of applications of our package. Except for Sections 4.1.1-4.1.2, which cover the classical cases, each of our examples is chosen from contemporary work done in partition theory over the last ten years—in most cases, within the last five years. In many cases we give substantial improvements on previous results, and (with the notable exception of the identities found with respect to $\bar{p}(n)$) the necessary computations take a few minutes at most on a modest laptop.

4.1.1 Ramanujan's Classics

The most obvious examples to check are the classic identities of Ramanujan and Kolberg for $p(5n+4)$ and $p(7n+5)$.

The generating function for $p(n)$ is of course $1/(q;q)_\infty$, which can be described by setting $M = 1$, $r = (-1)$. If we now take $m = 5$, guess $N = 5$, and take $j = 4$, then we have

In[1] = $\mathtt{RK}[5, 1, \{-1\}, 5, 4]$

$$\prod_{\delta|M}(q^{\delta};q^{\delta})_\infty^{r_\delta} = \sum_{n=0}^{\infty} a(n)q^n$$

$$\boxed{f_1(q)\cdot \prod_{j'\in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn+j')q^n = \sum_{g\in AB} g\cdot p_g(t)}$$

Modular Curve: $X_0(N)$

Out[1] =

N:	5
{M, $(r_{\delta\|M})$}:	$\{1,\{-1\}\}$
m :	5
$P_{m,r}(j)$:	$\{4\}$
$f_1(q)$:	$\frac{((q;q)_\infty)^6}{((q^5;q^5)_\infty)^5}$
t:	$\frac{((q;q)_\infty)^6}{q((q^5;q^5)_\infty)^6}$
AB:	$\{1\}$
$\{p_g(t) : g \in AB\}$:	$\{5\}$
Common Factor:	5

We see that $P_{m,r}(j) = \{4\}$, indicating that our left hand side will only contain the series $\sum_{n\geq 0} p(5n+4)q^n$. With f_1, we have the left hand side of any possible identity as

$$f_L = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5} \sum_{n=0}^{\infty} p(5n+4)q^n \in \mathcal{M}^\infty\left(\Gamma_0(5)\right).$$

In this case our algebra basis is extremely simple:

$$\begin{aligned} \langle \mathcal{E}^\infty(5)\rangle_{\mathbb{Q}} &= \langle 1\rangle_{\mathbb{Q}[t]} = \mathbb{Q}[t], \\ t &= \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}. \end{aligned}$$

Because the basis contains only the identity, we only need a single polynomial in t. In this case, the polynomial is 5.

$$\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^5} \sum_{n=0}^{\infty} p(5n+4)q^n = 5.$$

A quick rearrangement gives us (1.7)

Similarly, taking $m = 7$, $j = 5$, and guessing $N = 7$, we have

In[2] = RK$[7,1,\{-1\},7,5]$

$$\prod_{\delta|\mathtt{M}}(\mathtt{q}^{\delta};\mathtt{q}^{\delta})_{\infty}^{\mathtt{r}_{\delta}}=\sum_{\mathtt{n}=0}^{\infty}\mathtt{a}(\mathtt{n})\mathtt{q}^{\mathtt{n}}$$

$$\boxed{\mathtt{f}_1(\mathtt{q})\cdot\prod_{\mathtt{j}'\in\mathtt{P}_{\mathtt{m},\mathtt{r}}(\mathtt{j})}\sum_{\mathtt{n}=0}^{\infty}\mathtt{a}(\mathtt{mn}+\mathtt{j}')\mathtt{q}^{\mathtt{n}}=\sum_{\mathtt{g}\in\mathrm{AB}}\mathtt{g}\cdot\mathtt{p}_{\mathtt{g}}(\mathtt{t})}$$

Modular Curve: $X_0(N)$

Out[2] =

N:	7
{M, ($r_{\delta\|M}$)}:	$\{1,\{-1\}\}$
m :	7
$P_{m,r}(j)$:	$\{5\}$
$f_1(q)$:	$\frac{((q;q)_\infty)^8}{q((q^7;q^7)_\infty)^7}$
t:	$\frac{((q;q)_\infty)^4}{q((q^7;q^7)_\infty)^4}$
AB:	$\{1\}$
$\{p_g(t):g\in AB\}$:	$\{49+7t\}$
Common Factor:	7

This gives us

$$\frac{(q;q)_\infty^8}{q(q^7;q^7)_\infty^7}\sum_{n=0}^{\infty}p(7n+5)q^n=49+7\frac{(q;q)_\infty^4}{q(q^7;q^7)_\infty^4},$$

which yields (1.8) on rearrangement.

In the following examples, we will omit the three printed lines, as well as the first three lines of output from each example for the sake of brevity.

4.1.2 Classic Identities by Kolberg and Zuckerman

A large number of classic analogues to Ramanujan's results have been found. We start with an identity discovered by Zuckerman [117] for $p(13n+6)$.

Theorem 4.1.

$$\sum_{n=0}^{\infty} p(13n+6)q^n = 11\frac{(q^{13};q^{13})_\infty}{(q;q)_\infty^2} + 468q\frac{(q^{13};q^{13})_\infty^3}{(q;q)_\infty^4} + 6422q^2\frac{(q^{13};q^{13})_\infty^5}{(q;q)_\infty^6}$$
$$+ 43940q^3\frac{(q^{13};q^{13})_\infty^7}{(q;q)_\infty^8} + 171366q^4\frac{(q^{13};q^{13})_\infty^9}{(q;q)_\infty^{10}}$$
$$+ 371293q^5\frac{(q^{13};q^{13})_\infty^{11}}{(q;q)_\infty^{12}} + 371293q^6\frac{(q^{13};q^{13})_\infty^{13}}{(q;q)_\infty^{14}}.$$

In[3] = RK$[13, 1, \{-1\}, 13, 6]$
Out[3] =

$P_{m,r}(j)$:	$\{6\}$
$f_1(q)$:	$\dfrac{((q;q)_\infty)^{14}}{q^6((q^{13};q^{13})_\infty)^{13}}$
t:	$\dfrac{((q;q)_\infty)^2}{q((q^{13};q^{13})_\infty)^2}$
AB:	$\{1\}$
$\{p_g(t) : g \in AB\}$:	$\{371293 + 371293t + 171366t^2 + 43940t^3 + 6422t^4 + 468t^5 + 11t^6\}$
Common Factor:	None

We will now use our algorithm to derive the identities which Kolberg found [63] for $p(5n+j)$, $p(7n+j)$, and $p(3n+j)$.

Starting with $p(5n+j)$ for $0 \le j \le 4$, if we take $N = 5$ once more, and set $j = 1$, [63, (4.2)] we have

In[4] = RK$[5, 1, \{-1\}, 5, 1]$
Out[4] =

$P_{m,r}(j)$:	$\{1, 2\}$
$f_1(q)$:	$\dfrac{((q;q)_\infty)^{12}}{((q^5;q^5)_\infty)^{10}}$
t:	$\dfrac{((q;q)_\infty)^6}{q((q^5;q^5)_\infty)^6}$
AB:	$\{1\}$
$\{p_g(t) : g \in AB\}$:	$\{25 + 2t\}$
Common Factor:	None

Working over the same congruence subgroup $\Gamma_0(5)$, we keep the same algebra basis and t. The most notable difference is that we have the product

$$\left(\sum_{n\geq 0} p(5n+1)q^n\right)\left(\sum_{n\geq 0} p(5n+2)q^n\right)$$

on the left hand side. Our right-hand side is given as a more complicated $25+2t$, and we have

$$\frac{(q;q)_\infty^{12}}{(q^5;q^5)_\infty^{10}}\left(\sum_{n=0}^{\infty} p(5n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+2)q^n\right) = 25 + 2\frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}.$$

We can similarly examine $j=3$ [63, (4.3)] and derive the identity

$$\frac{(q;q)_\infty^{12}}{(q^5;q^5)_\infty^{10}}\left(\sum_{n=0}^{\infty} p(5n+3)q^n\right)\left(\sum_{n=0}^{\infty} p(5n)q^n\right) = 25 + 3\frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}.$$

On the other hand, we can set $m=7, j=1, N=7$, [63, (5.2)] and we will derive

```
 In[5] = RK[7, 1, {−1}, 7, 1]
Out[5] =
```

$P_{m,r}(j)$:	$\{1,3,4\}$
$f_1(q)$:	$\frac{((q;q)_\infty)^{24}}{q((q^7;q^7)_\infty)^{21}}$
t:	$\frac{((q;q)_\infty)^{4}}{q((q^7;q^7)_\infty)^{4}}$
AB:	$\{1\}$
$\{p_g(t):g\in AB\}$:	$\{117649+50421t+8232t^2+588t^3+15t^4\}$
Common Factor:	None

and the identity

$$\frac{(q;q)_\infty^{24}}{q^4(q^7;q^7)_\infty^{21}}\left(\sum_{n=0}^{\infty} p(7n+1)q^n\right)\left(\sum_{n=0}^{\infty} p(7n+3)q^n\right)\left(\sum_{n=0}^{\infty} p(7n+4)q^n\right)$$
$$=117649 + 50421\frac{(q;q)_\infty^4}{q(q^7;q^7)_\infty^4} + 8232\frac{(q;q)_\infty^8}{q^2(q^7;q^7)_\infty^8} + 588\frac{(q;q)_\infty^{12}}{q^3(q^7;q^7)_\infty^{12}} + 15\frac{(q;q)_\infty^{16}}{q^4(q^7;q^7)_\infty^{16}}.$$

The corresponding identity for $p(7n+2)$ [63, (5.3)] can be easily found.

Finally, we set $m=3$, $j=1$, $N=9$, [63, (3.4)] and derive

```
 In[6] = RK[9, 1, {−1}, 3, 1]
Out[6] =
```

$\mathtt{P_{m,r}(j)}$:	$\{0, 1, 2\}$
$\mathtt{f_1(q)}$:	$\dfrac{((q;q)_\infty)^{10}}{q(q^3;q^3)_\infty((q^9;q^9)_\infty)^6}$
t:	$\dfrac{((q;q)_\infty)^3}{q((q^9;q^9)_\infty)^3}$
AB:	$\{1\}$
$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$:	$\{9+2t\}$
Common Factor:	None

And we have

$$\frac{(q;q)^{10}}{q(q^3;q^3)(q^9;q^9)^6}\left(\sum_{n=0}^{\infty}p(3n)q^n\right)\left(\sum_{n=0}^{\infty}p(3n+1)q^n\right)\left(\sum_{n=0}^{\infty}p(3n+2)q^n\right)$$
$$=9+2\frac{(q;q)^3}{q(q^9;q^9)^3}.$$

Finally, we give another result found by Kolberg [62, (2.4)]. We set $m=2$, $j=1$, $N=8$ and derive

```
 In[7] = RK[8, 1, {−1}, 2, 1]
Out[7] =
```

$\mathtt{P_{m,r}(j)}$:	$\{0, 1\}$
$\mathtt{f_1(q)}$:	$\dfrac{((q;q)_\infty)^5(q^4;q^4)_\infty}{((q^2;q^2)_\infty)^2((q^8;q^8)_\infty)^2}$
t:	$\dfrac{((q^4;q^4)_\infty)^{12}}{q((q^2;q^2)_\infty)^4((q^8;q^8)_\infty)^8}$
AB:	$\{1\}$
$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$:	$\{1\}$
Common Factor:	None

And we have

$$\frac{((q;q)_\infty)^5(q^4;q^4)_\infty}{((q^2;q^2)_\infty)^2((q^8;q^8)_\infty)^2}\left(\sum_{n=0}^{\infty}p(2n)q^n\right)\left(\sum_{n=0}^{\infty}p(2n+1)q^n\right)=1.$$

4.1.3 Radu's Identity for 11

A substantial amount of work has been done attempting to find a witness identity for $p(11n+6) \equiv 0 \pmod{11}$. We will show one interesting attempt by Radu, though we hasten to add that a great deal of work has been done by others on the problem (for an interesting approach, see [47]). If we attempt to find such an identity for $M = 1$, $r = (-1)$, $m = 11$, $N = 11$, $j = 6$, then our algorithm returns

In[8] = RK$[11, 1, \{-1\}, 11, 6]$

Out[8] =

$P_{m,r}(j)$:	$\{6\}$
$f_1(q)$:	$\dfrac{(q;q)_\infty^{12}}{q^4(q^{11};q^{11})_\infty^{11}}$
t:	$\dfrac{(q;q)_\infty^{12}}{q^5(q^{11};q^{11})_\infty^{12}}$
AB:	$\{1\}$
$\{p_g(t) : g \in AB\}$:	No Membership
Common Factor:	None

Our membership witness returns a null result, indicating that our constructed modular function does not lie within $\langle \mathcal{E}^\infty(11) \rangle_{\mathbb{Q}}$. If we take $N = 22$, however, we get

`In[9] = ` $\texttt{RK}[22, 1, \{-1\}, 11, 6]$

`Out[9] =`

$\mathtt{P_{m,r}(j)}$: $\{6\}$

$\mathtt{f_1(q)}$: $$\frac{(q;q)_\infty^{12}(q^2;q^2)_\infty^2(q^{11};q^{11})_\infty^{11}}{q^{14}(q^{22};q^{22})_\infty^{22}}$$

`t:` $$-\frac{1}{8}\frac{(q^2;q^2)_\infty(q^{11};q^{11})_\infty^{11}}{q^5(q;q)_\infty(q^{22};q^{22})_\infty^{11}}+\frac{1}{11}\frac{(q^2;q^2)_\infty^8(q^{11};q^{11})_\infty^4}{q^5(q;q)_\infty^4(q^{22};q^{22})_\infty^8}+\frac{3}{88}\frac{(q;q)_\infty^7(q^{11};q^{11})_\infty^3}{q^5(q^2;q^2)_\infty^3(q^{22};q^{22})_\infty^7}$$

`AB:` $$\{1,-\frac{1}{8}\frac{(q^2;q^2)_\infty(q^{11};q^{11})_\infty^{11}}{q^5(q;q)_\infty(q^{22};q^{22})_\infty^{11}}+\frac{2}{11}\frac{(q^2;q^2)_\infty^8(q^{11};q^{11})_\infty^4}{q^5(q;q)_\infty^4(q^{22};q^{22})_\infty^8}+\frac{5}{88}\frac{(q;q)_\infty^7(q^{11};q^{11})_\infty^3}{q^5(q^2;q^2)_\infty^3(q^{22};q^{22})_\infty^7},$$
$$\frac{5}{4}\frac{(q^2;q^2)_\infty(q^{11};q^{11})_\infty^{11}}{q^5(q;q)_\infty(q^{22};q^{22})_\infty^{11}}-\frac{3}{11}\frac{(q^2;q^2)_\infty^8(q^{11};q^{11})_\infty^4}{q^5(q;q)_\infty^4(q^{22};q^{22})_\infty^8}+\frac{1}{44}\frac{(q;q)_\infty^7(q^{11};q^{11})_\infty^3}{q^5(q^2;q^2)_\infty^3(q^{22};q^{22})_\infty^7}\}$$

$\{\mathtt{p_g(t)}:\mathtt{g}\in\mathtt{AB}\}$: $$\{6776+9427t+15477t^2+13332t^3+1078t^4,\ -9581+594t+5390t^2+187t^3,\ -6754+5368t+2761t^2+11t^3\}$$

`Common Factor:` 11

Our procedure returns a variation on a result that Radu already computed [91]. It serves as a witness identity for the divisibility of $p(11n+6)$ by 11, though it is not very satisfying. It has a form resembling the classic witness identities which Ramanujan discovered for his congruences of $p(5n+4), p(7n+5)$ by $5, 7$, respectively. In particular, the coefficients of t in the membership witness are all divisible by 11. Therefore, the result is a witness identity, provided one accepts that the functions of the algebra basis have integer power series expansions. This is true, but not obvious.

In particular, we find a prevalence of 11 throughout the denominators of each function in our basis. This is of course the one factor we would not want to find in the denominators! Peter Paule was the first to realize that it is necessary to prove that the expansions of the basis functions are in fact integral; he successfully did so in [85, Discussion, pp. 541-542].

4.1.4 An Identity for Broken 2-Diamond Partitions

Broken k-diamond partitions, denoted by $\Delta_k(n)$, were defined by Andrews and Paule in 2007 [14]. They conjectured that

Theorem 4.2. *For all $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\Delta_2(25n + 14) \equiv \Delta_2(25n + 24) \equiv 0 \pmod{5}.$$

This was subsequently proved in 2008 by Chan [26]. In 2015 Radu was able [91] to give a proof by studying another arithmetic function with a simpler generating function. Our complete implementation allows us to verify these congruences by directly examining the generating function for $\Delta_2(n)$.

We take $N = 10, M = 10, r = (-3, 1, 1, -1), m = 25, j = 14$. Our package returns

```
 In[10] = RK[10, 10, {-3, 1, 1, -1}, 25, 14]
Out[10] =
```

$\mathtt{P_{m,r}(j)}$:	$\{14, 24\}$
$\mathtt{f_1(q)}$:	$\dfrac{(q;q)_\infty^{126}(q^5;q^5)_\infty^{70}}{q^{58}(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^{190}}$
t:	$\dfrac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{q(q;q)_\infty(q^{10};q^{10})_\infty^5}$
AB:	$\{1\}$
$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$:	$\{...\}$
Common Factor:	25

The membership witness returns a lengthy result, with terms of the order of 10^{76}. However, the computation time is short—less than 40 seconds with a 2.6 GHz Intel Processor on a modest laptop. The complete witness is available, and easily computed, at `https://www3.risc.jku.at/people/nsmoot/RKAlg/RKSupplement1.nb`.

Each term in the membership witness is divisible by 25. By expanding the generating function for $\Delta_2(n)$, one determines that $\Delta_2(14) = 10445$, and that $\Delta_2(49) = 1022063815$.

Because each of these numbers is divisible by 5 but not by 25, it follows that $\sum_{n\geq 0} \Delta_2(25n + 14)q^n$, $\sum_{n\geq 0} \Delta_2(25n + 24)q^n$ must each be divisible by exactly one power of 5. This completes the proof.

4.1.5 Congruences with Overpartitions

An enormous amount of work has been published in recent years on the congruence properties of overpartition functions, and our package has a great deal of utility in this subject. We will examine three distinct problems here: two will involve the standard overpartition function $\bar{p}(n)$, and one will involve an overpartition function with additional restrictions $A_m(n)$. In each case, we are able to make substantial improvements to previously established results.

As a preliminary, an overpartition of n is a partition of n in which the first occurrence of a part may or may not be "marked." Generally, this "mark" is denoted with an overline (hence the term "overpartition"). For example, the number 3 has 8 overpartitions:

$$\begin{aligned}&3,\\&\bar{3},\\&2+1,\\&\bar{2}+1,\\&2+\bar{1},\\&\bar{2}+\bar{1},\\&1+1+1,\\&\bar{1}+1+1.\end{aligned}$$

We denote the number of overpartitions of n by $\bar{p}(n)$. The generating function for $\bar{p}(n)$ has the form

$$\sum_{n=0}^{\infty}\bar{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}$$

Part of the appeal of $\bar{p}(n)$ is the simplicity of the combinatorial interpretation, given the relative complexity of its generating function [33].

Congruences for $\bar{p}(n)$

We will begin by giving some remarkable improvements to previously established congruences for $\bar{p}(n)$. Moreover, we have the opportunity to apply our "manual" procedure, and use the connection of modular functions with the topology of associated Riemann surfaces in order to construct a suitable algebra basis.

In 2016 Dou and Lin showed [35] that

$$\bar{p}(80n+8) \equiv \bar{p}(80n+52) \equiv \bar{p}(80n+68) \equiv \bar{p}(80n+72) \equiv 0 \pmod 5. \tag{4.1}$$

Hirschhorn in 2016 [52], and Chern and Dastidar in 2018 [29] have studied these congruences as well, with the latter improving these congruences:

$$\bar{p}(80n+8) \equiv \bar{p}(80n+52) \equiv \bar{p}(80n+68) \equiv \bar{p}(80n+72) \equiv 0 \pmod{25}.$$

Chern and Dastidar go on to point out that

$$\bar{p}(135n+63) \equiv \bar{p}(135n+117) \equiv 0 \pmod 5.$$

However, a quick computation of each of these sequences of overpartition numbers reveals much more. For instance,

n	$\bar{p}(80n+8)$
0	100
1	8638130600
2	350865646632400
3	1512900775311002400
4	1919738036947929590800
5	1092453314947897908542800
6	348534368588210202093102600
7	71377855377904690816918291600
8	10261762697785410674339371853700

A very much stronger congruence clearly suggests itself. We are able to make the following substantial improvements in each case:

Theorem 4.3.

$$\bar{p}(80n+8) \equiv \bar{p}(80n+72) \equiv 0 \pmod{100},$$
$$\bar{p}(80n+52) \equiv \bar{p}(80n+68) \equiv 0 \pmod{200}.$$

Theorem 4.4.

$$\bar{p}(135n+63) \equiv \bar{p}(135n+117) \equiv 0 \pmod{40}.$$

Our package can be used to demonstrate each of these, though with some adjustments. In the case of $\bar{p}(80n+j)$, we are forced to work over the congruence subgroup $\Gamma_0(40)$.

Recall that we have an algorithm to calculate the generators of the monoid $\mathcal{E}^{\infty}(40)$ of monopolar eta quotients can be computed with relative ease using `etaGenerators` (Section 3.3.1). Let us order the set of generators by the order of the elements at ∞, and denote the resulting vector as $\mathcal{G}_0(40) = (\mathcal{G}_0(40)_n)_{1\leq n\leq n_{40}}$, for $n_{40} \in \mathbb{Z}_{\geq 1}$. This vector is extremely large, and our procedure to compute the algebra basis using `AB` would be extremely inefficient.

We can remedy the problem by taking advantage of the Weierstrass gap theorem, (see [115, Part 2, Section 17] for a classical introduction to the subject; see [84] for a more modern treatment of the theorem). We use [34, Theorem 3.1.1] to compute the genus of the corresponding modular curve $X_0(40)$ as 3, which implies that all modular functions with a pole only at ∞ on $\Gamma_0(40)$ must have order 4 or greater. Radu's refinement of Newman's conjecture (see Section 2.6.1) suggests that a suitable combination of eta quotients will yield functions in $\langle\mathcal{E}^{\infty}(40)\rangle_{\mathbb{Q}}$ with orders -4, -5, -6, -7. Such a set of functions would be a sufficient algebra basis for $\langle\mathcal{E}^{\infty}(40)\rangle_{\mathbb{Q}}$.

In this case, we are lucky, because a simple ordering of $\mathcal{G}_0(40)$ by the order of the elements at ∞ reveals that

$$\begin{aligned}
\mathcal{G}_0(40)_1 =& \frac{(q^4;q^4)_\infty^3(q^{20};q^{20})_\infty}{q^4(q^8;q^8)_\infty(q^{40};q^{40})_\infty^3},\\
\mathcal{G}_0(40)_4 =& \frac{(q^2;q^2)_\infty^3(q^5;q^5)_\infty(q^{20};q^{20})_\infty^2}{q^5(q;q)_\infty(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty^4},\\
\mathcal{G}_0(40)_7 =& \frac{(q^2;q^2)_\infty^6(q^5;q^5)_\infty^2(q^8;q^8)_\infty(q^{20};q^{20})_\infty^3}{q^6(q;q)_\infty^2(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty^2(q^{40};q^{40})_\infty^5},\\
\mathcal{G}_0(40)_{17} =& \frac{(q;q)_\infty^2(q^5;q^5)_\infty^2(q^8;q^8)_\infty^2(q^{20};q^{20})_\infty^3}{q^7(q^2;q^2)_\infty(q^4;q^4)_\infty(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty^6}.
\end{aligned}$$

We can then define our algebra basis as

$$\begin{aligned}
T &= \mathcal{G}_0(40)_1,\\
\mathrm{Ab40} &= \{T, \{1, \mathcal{G}_0(40)_4, \mathcal{G}_0(40)_7, \mathcal{G}_0(40)_{17}\}\}.
\end{aligned}$$

Since we computed our algebra basis separately, we may now employ the manual case of our package, `RKMan`:

```
 In[11] = RKMan[40, 2, {−2, 1}, 80, 8, Ab40]
Out[11] =
```

$\mathtt{P_{m,r}(j)}$:	$\{8, 72\}$
$\mathtt{f_1(q)}$:	$\frac{(q;q)_\infty^{333}(q^8;q^8)_\infty^{66}(q^{10};q^{10})_\infty^{36}(q^{20};q^{20})_\infty^{165}}{q^{400}(q^2;q^2)_\infty^{168}(q^4;q^4)_\infty^{31}(q^5;q^5)_\infty^{65}(q^{40};q^{40})_\infty^{334}}$
`t`:	$\frac{(q^4;q^4)_\infty^3(q^{20};q^{20})_\infty}{q^4(q^8;q^8)_\infty(q^{40};q^{40})_\infty^3}$
`AB`:	$\{1, \frac{(q^2;q^2)_\infty^3(q^5;q^5)_\infty(q^{20};q^{20})_\infty^2}{q^5(q;q)_\infty(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty^4}, \frac{(q^2;q^2)_\infty^6(q^5;q^5)_\infty^2(q^8;q^8)_\infty(q^{20};q^{20})_\infty^3}{q^6(q;q)_\infty^2(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty^2(q^{40};q^{40})_\infty^5}, \frac{(q;q)_\infty^2(q^5;q^5)_\infty^2(q^8;q^8)_\infty^2(q^{20};q^{20})_\infty^3}{q^7(q^2;q^2)_\infty(q^4;q^4)_\infty(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty^6}\}$
$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$:	$\{...\}$
`Common Factor`:	10000

The membership witness is too lengthy to present here. The complete output of the algorithm can be found at `https://www3.risc.jku.at/people/nsmoot/RKAlg/RKSupplement2.nb`. It is trivial to compute $\bar{p}(80n+\ 8)$, $\bar{p}(80n+72)$ for a handful of small n in order to demonstrate that neither is divisible by 2^3 or 5^3. Since the left hand side consists of a prefactor (with initial coefficient 1) and a product of the form

$$\left(\sum_{n\geq 0}\bar{p}(80n+8)q^n\right)\left(\sum_{n\geq 0}\bar{p}(80n+72)q^n\right),$$

with neither factor divisible by 2^3 or 5^3, the only remaining possibility is that each factor is divisible by $2^2\cdot 5^2=100$.

An almost identical output is produced for

In[11] = RKMan$[40,2,\{-2,1\},80,52,$Ab40$]$

but with a common factor of 40000 for congruences. This is also available at `https://www3.risc.jku.at/people/nsmoot/RKAlg/RKSupplement2.nb`. We may show that $\bar{p}(80n+52)$, $\bar{p}(80n+\ 68)$ are each divisible by 200, in a similar manner to the case of $\bar{p}(80n+8)$, $\bar{p}(80n+\ 72)$.

Finally, we consider the case of $\bar{p}(135n+63), \bar{p}(135n+117)$. We may similarly construct an algebra basis manually. In this case, the most convenient congruence subgroup is $\Gamma_0(30)$ ($N=30$). The genus of $X_0(30)$ is 3, but we are at a slight disadvantage: there are eta quotients in $\mathcal{E}^\infty(30)$ with orders -4, -6, and -7, but none with order -5. But we can construct a difference of eta quotients, each with order 6, to produce a function of order 5. If we define $\mathcal{G}_0(30)$ in a matter similar to $\mathcal{G}_0(40)$, i.e., by ordering the generators of $\mathcal{E}^\infty(30)$ by order at ∞, then

$$\begin{aligned}\mathcal{G}_0(30)[1] =&\frac{(q;q)_\infty(q^6;q^6)_\infty^6(q^{10};q^{10})_\infty^2(q^{15};q^{15})_\infty^3}{q^4(q^2;q^2)_\infty^2(q^3;q^3)_\infty^3(q^5;q^5)_\infty(q^{30};q^{30})_\infty^6},\\ \mathcal{G}_0(30)[4]-\mathcal{G}_0(30)[3] =&\frac{(q^2;q^2)_\infty^4(q^{10};q^{10})_\infty^4(q^{15};q^{15})_\infty^4}{q^6(q;q)_\infty^2(q^5;q^5)_\infty^2(q^{30};q^{30})_\infty^8}\\ &-\frac{(q;q)_\infty(q^6;q^6)_\infty^2(q^{10};q^{10})_\infty^{10}(q^{15};q^{15})_\infty^5}{q^6(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^5;q^5)_\infty^5(q^{30};q^{30})_\infty^{10}},\end{aligned}$$

$$\begin{aligned}\mathcal{G}_0(30)[2] =&\frac{(q;q)_\infty(q^2;q^2)_\infty(q^5;q^5)_\infty(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{15};q^{15})_\infty^3}{q^6(q^3;q^3)_\infty(q^{30};q^{30})_\infty^7},\\ \mathcal{G}_0(30)[6] =&\frac{(q;q)_\infty(q^5;q^5)_\infty^2(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{15};q^{15})_\infty^3}{q^7(q^{30};q^{30})_\infty^8}.\end{aligned}$$

The orders here are (respectively) $-4,-5,-6,-7$, again sufficient for an algebra basis:

$$\begin{aligned} T &= \mathcal{G}_0(30)_1, \\ G_1 &= \mathcal{G}_0(30)_4 - \mathcal{G}_0(30)_3, \\ G_2 &= \mathcal{G}_0(30)_2, \\ G_3 &= \mathcal{G}_0(30)_6, \\ \mathrm{Ab30} &= \{T, \{1, G_1, G_2, G_3\}\}. \end{aligned}$$

Employing `RKMan` once again, we get

`In[12] =` $\mathtt{RKMan}[30, 2, \{-2, 1\}, 135, 63, \mathtt{Ab30}]$

`Out[12] =`

$\mathtt{P_{m,r}(j)}$: $\{63, 117\}$

$\mathtt{f_1(q)}$: $$\frac{(q;q)_\infty^{653}(q^6;q^6)_\infty^{235}(q^{10};q^{10})_\infty^{272}(q^{15};q^{15})_\infty^{358}}{q^{507}(q^2;q^2)_\infty^{359}(q^3;q^3)_\infty^{275}(q^5;q^5)_\infty^{226}(q^{30};q^{30})_\infty^{656}}$$

`t:` $$\frac{(q;q)_\infty(q^6;q^6)_\infty^{6}(q^{10};q^{10})_\infty^{2}(q^{15};q^{15})_\infty^{3}}{q^4(q^2;q^2)_\infty^{2}(q^3;q^3)_\infty^{3}(q^5;q^5)_\infty(q^{30};q^{30})_\infty^{6}}$$

`AB:` $$\Big\{1, \frac{(q^2;q^2)_\infty^4(q^{10};q^{10})_\infty^4(q^{15};q^{15})_\infty^4}{q^6(q;q)_\infty^2(q^5;q^5)_\infty^2(q^{30};q^{30})_\infty^8} - \frac{(q;q)_\infty(q^6;q^6)_\infty^2(q^{10};q^{10})_\infty^{10}(q^{15};q^{15})_\infty^5}{q^6(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^5;q^5)_\infty^5(q^{30};q^{30})_\infty^{10}},$$
$$\frac{(q;q)_\infty(q^2;q^2)_\infty(q^5;q^5)_\infty(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{15};q^{15})_\infty^3}{q^6(q^3;q^3)_\infty(q^{30};q^{30})_\infty^7},$$
$$\frac{(q;q)_\infty(q^5;q^5)_\infty^2(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{15};q^{15})_\infty^3}{q^7(q^{30};q^{30})_\infty^8}\Big\}$$

$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$: $\{...\}$

`Common Factor:` $\frac{1600}{3}$

Once again, the membership witness is too large to present here. It can be found in its entirety at `https://www3.risc.jku.at/people/nsmoot/RKAlg/RKSupplement2.nb`. However, the fractional common factor emerges because each polynomial p_g in the witness has integer coefficients, except for p_{G_1}, which is a polynomial in $\frac{1}{3}\mathbb{Z}$. Because the remaining polynomials have integer coefficients (and all of the eta quotients involved have integer-coefficient expansions), we can conclude that G_1 has coefficients divisible by 3. At any rate, this makes no difference for congruences with respect to powers of 2 or 5.

We may again quickly demonstrate that $\bar{p}(135n + 63), \bar{p}(135n + 117)$ are not divisible by 2^4 or 5^2, indicating that they must each be divisible by $2^3 \cdot 5 = 40$.

A Congruence for $\bar{p}(n)$ Modulo 243

In 2017 Xia conjectured [116] that

$$\bar{p}(96n+76) \equiv 0 \pmod{3^5}$$

for all $n \in \mathbb{Z}_{\geq 0}$. This conjecture was recently proved by Huang and Yao [54].

We have extended the theorem further:

Theorem 4.5.

$$\bar{p}(96n+76) \equiv 0 \pmod{2^3 3^5}$$

for all $n \in \mathbb{Z}_{\geq 0}$.

The optimal congruence subgroup to work over in this case is $\Gamma_0(24)$. Setting $N = 24$, our software returns

```
 In[13] = RK[24, 2, {-2, 1}, 96, 76]
Out[13] =
```

$\mathtt{P_{m,r}(j)}$: $\{76\}$

$\mathtt{f_1(q)}$: $\dfrac{(q;q)_\infty^{213}(q^6;q^6)_\infty^{33}(q^8;q^8)_\infty^{77}(q^{12};q^{12})_\infty^{113}}{q^{150}(q^2;q^2)_\infty^{107}(q^3;q^3)_\infty^{64}(q^4;q^4)_\infty^{37}(q^{24};q^{24})_\infty^{227}}$

$\mathtt{t}$: $\dfrac{(q^6;q^6)_\infty^3(q^8;q^8)_\infty}{q^2(q^2;q^2)_\infty(q^{24};q^{24})_\infty^3}$

$\mathtt{AB}$: $\left\{1, \dfrac{(q^6;q^6)_\infty^3(q^8;q^8)_\infty}{q^2(q^2;q^2)_\infty(q^{24};q^{24})_\infty^3} + \dfrac{(q;q)_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty(q^4;q^4)_\infty^3}{q^3(q^2;q^2)_\infty^2(q^{24};q^{24})_\infty^4}\right\}$

$\{\mathtt{p_g(t)} : \mathtt{g} \in \mathtt{AB}\}$: $\{...\}$

Common Factor: 1944

The theorem is then established, since $1944 = 2^3 \cdot 3^5$. The full identity can be found at `https://www3.risc.jku.at/people/nsmoot/RKAlg/RKSupplement2.nb`.

A Restricted Overpartition Function

Let $A_m(n)$ be the number of overpartitions of n in which only the parts not divisible by m may be overlined. Then it can be shown [73] that

$$\sum_{n=0}^{\infty} A_m(n)q^n = \frac{(q^2;q^2)_\infty(q^m;q^m)_\infty}{(q;q)_\infty^2(q^{2m};q^{2m})_\infty}.$$

In 2014, Munagi and Sellers give a variety of interesting congruences for $A_m(n)$.

For instance, [73, Corollary 4.4, Theorem 4.5]:

Theorem 4.6.

$$A_3(3n+1) \equiv 0 \pmod{2},$$
$$A_3(3n+2) \equiv 0 \pmod{4}.$$

Both of these can be proved quickly with our package. For example, to prove $A_3(3n+1) \equiv 0 \pmod{2}$:

In[14] = RK$[6, 6, \{-2, 1, 1, -1\}, 3, 1]$

Out[14] =

$P_{m,r}(j)$:	$\{1\}$
$f_1(q)$:	$\frac{(q;q)_\infty^3 (q^2;q^2)_\infty (q^3;q^3)_\infty^6}{q(q^6;q^6)_\infty^9}$
t:	$\frac{(q;q)_\infty^5 (q^3;q^3)_\infty}{q(q^2;q^2)_\infty (q^6;q^6)_\infty^5}$
AB:	$\{1\}$
$\{p_g(t) : g \in AB\}$:	$\{16 + 2t\}$
Common Factor:	2

On the other hand, [73, Theorem 4.7, Theorem 4.9] $A_3(27n+26) \equiv 0 \pmod{3}$, and $A_9(27n+24) \equiv 0 \pmod{3}$. Using our package, we can prove more:

Theorem 4.7.

$$A_3(27n+26) \equiv 0 \pmod{12},$$
$$A_9(27n+24) \equiv 0 \pmod{24}.$$

For example, to show that $A_9(27n+24) \equiv 0 \pmod{24}$:

```
 In[15] = RK[6, 18, {−2, 1, 0, 0, 1, −1}, 27, 24]
Out[15] =
```

$\mathtt{P_{m,r}(j)}$:	$\{24\}$
$\mathtt{f_1(q)}$:	$\dfrac{(q;q)_\infty^{47}(q^3;q^3)_\infty^{12}}{q^9(q^2;q^2)_\infty^7(q^6;q^6)_\infty^{51}}$
t:	$\dfrac{(q;q)_\infty^{5}(q^3;q^3)_\infty}{q(q^2;q^2)_\infty(q^6;q^6)_\infty^{5}}$
AB:	$\{1\}$
$\{\mathtt{p_g(t)}:\mathtt{g}\in\mathtt{AB}\}$:	$\{7703510787293184 + 5456653474332672t + 1649478582927360t^2 + 276646783352832t^3 + 27989228519424t^4 + 1735943602176t^5 + 63885293568t^6 + 1269340416t^7 + 10941888t^8 + 22056t^9\}$
Common Factor:	24

We expect that a very large variety of other congruences and associated results for overpartition functions still await discovery, and that our package will prove extremely useful.

4.1.6 Some Identities by Baruah and Sarmah

For $r \in \mathbb{Z}$, define

$$\sum_{n=0}^{\infty} p_r(n)q^n = (q;q)_\infty^r.$$

In 2013 Baruah and Sarmah [22] gave a large variety of results for $p_r(n)$, all of which are accessible through our package. One especially interesting example, [22, Theorem 2.1, (2.10)] is not a congruence, but rather a simple identity:

Theorem 4.8.

$$p_8(3n+1) = 0.$$

We can verify this by taking $M = 1, r = (8), m = 4, j = 3, N = 4$:

```
 In[16] = RK[4, 1, {8}, 4, 3]
Out[16] =
```

$\mathtt{P_{m,r}(j)}$:	$\{3\}$
$\mathtt{f_1(q)}$:	$\frac{(q^2;q^2)_\infty^{12}}{q(q;q)_\infty^4(q^4;q^4)_\infty^{16}}$
t:	$\frac{(q;q)_\infty^8}{q(q^4;q^4)_\infty^8}$
AB:	$\{1\}$
$\{\mathtt{p_g(t)}:\mathtt{g}\in\mathtt{AB}\}$:	$\{0\}$
Common Factor:	0

Baruah and Sarmah list several congruences [22, Theorem 5.1] which may easily be proved. For example:

Theorem 4.9.

$$\begin{aligned} p_{-4}(4n+3) \equiv& 0 \pmod 8,\\ p_{-8}(4n+3) \equiv& 0 \pmod{64},\\ p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv& 0 \pmod 5,\\ p_{-4}(5n+3) \equiv p_{-4}(5n+4) \equiv& 0 \pmod 5. \end{aligned}$$

We prove the first case by setting $M=1, r=(-4), m=4, j=3, N=8$.

```
 In[17] = RK[8, 1, {−4}, 4, 3]
Out[17] =
```

$\mathtt{P_{m,r}(j)}$:	$\{3\}$
$\mathtt{f_1(q)}$:	$\frac{(q;q)_\infty^{19}(q^4;q^4)_\infty^{15}}{q^4(q^2;q^2)_\infty^8(q^8;q^8)_\infty^{22}}$
t:	$\frac{(q^4;q^4)_\infty^{12}}{q(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}$
AB:	$\{1\}$
$\{\mathtt{p_g(t)}:\mathtt{g}\in\mathtt{AB}\}$:	$\{512t+1408t^2+480t^3+40t^4\}$
Common Factor:	8

The other cases of this theorem can be proved similarly.

In another example, they prove [22, Theorem 5.1, (5.3)] that $p_{-8}(8n+7) \equiv 0 \pmod{2^9}$, but we prove even more:

Theorem 4.10.

$$p_{-8}(8n+7) \equiv 0 \pmod{2^{11}}.$$

We set $N = 4$:

```
 In[18] = RK[4, 1, {−8}, 8, 7]
Out[18] =
```

$P_{m,r}(j)$:	$\{7\}$
$f_1(q)$:	$\frac{(q;q)_\infty^{84}}{q^8(q^2;q^2)_\infty^4(q^4;q^4)_\infty^{72}}$
t:	$\frac{(q;q)_\infty^8}{q(q^4;q^4)_\infty^8}$
AB:	$\{1\}$
$\{p_g(t) : g \in AB\}$:	$\{576460752303423488 + 1621295865853337856t + 1871808595125862 4t^2 + 1139094046375936t^3 + 38970385760256t^4 + 737593524224t^5 + 7041187840t^6 + 27033600t^7 + 22528t^8\}$
Common Factor:	2048

4.1.7 5-Regular Bipartitions

In 2016 Liuquan Wang developed [110] a large class of interesting congruences for the 5-regular bipartition function $B_5(n)$, with the generator

$$\sum_{n=0}^{\infty} B_5(n)q^n = \frac{(q^5;q^5)_\infty^2}{(q;q)_\infty^2}.$$

Among many results were the following:

$$\begin{aligned} B_5(4n+3) &\equiv 0 \pmod 5, \\ B_5(5n+2) &\equiv B_5(5n+3) \equiv B_5(5n+4) \equiv 0 \pmod 5, \\ B_5(20n+7) &\equiv B_5(20n+19) \equiv 0 \pmod{25}. \end{aligned}$$

We are able to make the following improvements:

Theorem 4.11.

$$\begin{aligned} B_5(4n+3) &\equiv 0 \pmod{10}, \\ B_5(5n+2) &\equiv B_5(5n+3) \equiv B_5(5n+4) \equiv 0 \pmod 5, \\ B_5(20n+7) &\equiv B_5(20n+19) \equiv 0 \pmod{100}. \end{aligned}$$

In[19] = RK$[20,5,\{-2,2\},4,3]$
Out[19] =

$P_{m,r}(j)$:	$\{3\}$
$f_1(q)$:	$\dfrac{(q;q)_\infty^6(q^2;q^2)_\infty(q^4;q^4)_\infty(q^{10};q^{10})_\infty^7}{q^7(q^5;q^5)_\infty^2(q^{20};q^{20})_\infty^{13}}$
t:	$\dfrac{(q^4;q^4)_\infty^4(q^{10};q^{10})_\infty^2}{q^2(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty^4}$
AB:	$\{1,\dfrac{(q^4;q^4)_\infty(q^5;q^5)_\infty^5}{q^3(q;q)_\infty(q^{20};q^{20})_\infty^5}-\dfrac{(q^4;q^4)_\infty^4(q^{10};q^{10})_\infty^2}{q^2(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty^4}\}$
$\{p_g(t):g\in AB\}$:	$\{50-40t-50t^2+40t^3,-50+40t+10t^2\}$
Common Factor:	10

In[20] = RK$[5,5,\{-2,2\},5,2]$
Out[20] =

$P_{m,r}(j)$:	$\{2,4\}$
$f_1(q)$:	$\dfrac{(q;q)_\infty^{20}}{q^2(q^5;q^5)_\infty^{20}}$
t:	$\dfrac{((q;q)_\infty)^6}{q((q^5;q^5)_\infty)^6}$
AB:	$\{1\}$
$\{p_g(t):g\in AB\}$:	$\{15625+2500t+100t^2\}$
Common Factor:	25

In[21] = RK$[5,5,\{-2,2\},5,3]$
Out[21] =

$P_{m,r}(j)$:	$\{3\}$
$f_1(q)$:	$\dfrac{(q;q)_\infty^{10}}{q(q^5;q^5)_\infty^{10}}$
t:	$\dfrac{((q;q)_\infty)^6}{q((q^5;q^5)_\infty)^6}$
AB:	$\{1\}$
$\{p_g(t):g\in AB\}$:	$\{125+10t\}$
Common Factor:	5

```
 In[22] = RK[10, 5, {−2, 2}, 20, 7]
Out[22] =
```

$\mathtt{P_{m,r}(j)}$: $\{7, 19\}$

$\mathtt{f_1(q)}$: $\dfrac{(q;q)_\infty^{77}(q^5;q^5)_\infty^{31}}{q^{27}(q^2;q^2)_\infty^{21}(q^{10};q^{10})_\infty^{87}}$

t: $\dfrac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{q(q;q)_\infty(q^{10};q^{10})_\infty^5}$

AB: $\{1\}$

$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$:

$$\{7388718138654720000t^2 + 153008038121308160000t^3 + 1257731351012966400000t^4 + 5675499664745431040000t^5 + 16507857641427435520000t^6 + 34080767872618987520000t^7 + 53266856094927421440000t^8 + 65937188949118156800000t^9 + 66700597538020392960000t^{10} + 56314162511641313280000t^{11} + 40234227634725191680000t^{12} + 24527816166851215360000t^{13} + 12802067441385472000000t^{14} + 5714660420762992640000t^{15} + 2169098785981726720000t^{16} + 691839480120197120000t^{17} + 181850756413399040000t^{18} + 38175700204339200000t^{19} + 6075890734530560000t^{20} + 680092466755680000t^{21} + 49080942745680000t^{22} + 2083485921960000t^{23} + 46908276350000t^{24} + 483406090000t^{25} + 1812970000t^{26} + 1190000t^{27}\}$$

Common Factor: 10000

4.1.8 Some Congruences Related to the Tau Function

Please note that in this section we will assume $q = e^{2\pi i z}$ with $z \in \mathbb{H}$, to avoid confusion with τ, which will be used to identify a certain arithmetic function.

Ramanujan's tau function is defined by the following:

$$\Delta(z) := \sum_{n=1}^{\infty} \tau(n)q^n = q(q;q)^{24} = \eta(z)^{24}.$$

The functions $\Delta(z)$ and $\tau(n)$ are among the most studied objects in the theory of modular forms. In particular, numerous interesting congruences have been found. Many classic examples include the following, discovered by Ramanujan [96]:

Theorem 4.12.

$$\tau(7n+m) \equiv 0 \pmod 7$$

for $m \in \{0,3,5,6\}$.

Our algorithm can easily handle each of these cases. For example, we take the case of $\tau(7n)$ (notice that we study $(q;q)_\infty^{24}$, rather than $q(q;q)_\infty^{24}$; because of this, we need to examine the progression $7n+6$):

In[23] = RK$[7,1,\{24\},7,6]$
Out[23] =

$P_{m,r}(j)$:	$\{6\}$
$f_1(q)$:	$\dfrac{1}{q^6(q^7;q^7)_\infty^{24}}$
t:	$\dfrac{(q;q)_\infty^4}{q(q^7;q^7)_\infty^4}$
AB:	$\{1\}$
$\{p_g(t): g \in AB\}$:	$\{-1977326743 - 16744t^6\}$
Common Factor:	7

We will give a more recent example discovered by Koustav Banerjee [20]:

Theorem 4.13.

$$\tau(8(14n+k)) \equiv 0 \pmod{2^3 \cdot 3 \cdot 5 \cdot 11},$$

for all $n \in \mathbb{Z}_{\geq 0}$ *and* k *an odd integer mod 14.*

This may be broken up into three distinct RK identities. We give the case of $112n+56$ (here shifted to $112n+55$)

In[24] = RK$[14, 1, \{24\}, 112, 55]$

Out[24] =

$\mathtt{P_{m,r}(j)}$:	$\{55\}$
$\mathtt{f_1(q)}$:	$\dfrac{(q^2;q^2)_\infty^{12}(q^7;q^7)_\infty^{30}}{q^{25}(q;q)_\infty^6(q^{14};q^{14})_\infty^{60}}$
t:	$\dfrac{(q^2;q^2)_\infty(q^7;q^7)_\infty^7}{q^2(q;q)_\infty(q^{14};q^{14})_\infty^7}$
AB:	$\{1, \dfrac{(q^2;q^2)_\infty^8(q^7;q^7)_\infty^4}{q^3(q;q)_\infty^4(q^{14};q^{14})_\infty^8} - 4\dfrac{(q^2;q^2)_\infty(q^7;q^7)_\infty^7}{q^2(q;q)_\infty(q^{14};q^{14})_\infty^7}\}$
$\{\mathtt{p_g(t)}:\mathtt{g} \in \mathtt{AB}\}$:	$\{1483245480837120 + 22804899267870720t - 281353127146291200t^2 + 481330731305926656 0t^3 - 2117115491136307200t^4 - 3347863578673152000t^5 + 845098635118510080t^6 + 77358598094131200t^7 - 25371836549283840t^8 - 1132615297820160t^9 - 512964938787840t^{10} - 114993988032000t^{11} - 349389680640t^{12}, - 1483245480837120 - 6489198978662400t + 9909006840417484 80t^2 - 1517912267371315 20t^3 - 1234180893392240640t^4 + 461934380423577600t^5 - 65498418207129600t^6 + 2233732210913280t^7 + 170807954042880t^8 + 855016378191360t^9 - 4703322624000t^{10} - 1414533120t^{11}\}$
Common Factor:	591360

The congruence here is even stronger than in the more general case, since $591360 = 2^9{\cdot}3{\cdot}5{\cdot}7{\cdot}11$.

4.1.9 Conjectures of Schlosser

Michael Schlosser has recently [98] made a large number of conjectures regarding congruences and vanishing properties (similar to those seen in Section 4.1.6 above). We have begun to collaborate on possible extensions of Radu's algorithm which would prove these conjectures. We give one example which was already alluded to in Chapter 3. Schlosser conjectured that if

$$\sum_{n=0}^{\infty} b(n)q^n := \frac{(q;q)_\infty^3}{(q^{11};q^{11})_\infty^3},$$

then $b(11n+4)$ is divisible by 11. We found the following elegant identity which confirmed this conjecture:

$$\sum_{n=0}^{\infty} b(11n+4)q^n = -11q\frac{(q^{11};q^{11})_\infty^3}{(q;q)_\infty^3}.$$

```
 In[25] = RK[11, 11, {3, -3}, 11, 4]
Out[25] =
```

$P_{m,r}(j)$:	$\{4\}$
$f_1(q)$:	$\dfrac{(q;q)_\infty^3}{q(q^{11};q^{11})_\infty^3}$
t:	$\dfrac{((q;q)_\infty)^{12}}{q^5((q^{11};q^{11})_\infty)^{12}}$
AB:	$\{1\}$
$\{p_g(t) : g \in AB\}$:	$\{-11\}$
Common Factor:	11

We are confident that continued work with Dr. Schlosser will likely result in the discovery of additional interesting identities.

CHAPTER 5
AN INFINITE FAMILY OF CONGRUENCES (I)

5.1 Introduction

In the previous two chapters we showed how the theory of modular functions can give us a means to prove various partition identities (for $p(n)$ and many other functions), and in so doing, to retrieve important arithmetic information. Indeed, we understand these methods of proof so well that we can fully automate the procedure, as our software has shown.

We now turn to a problem which is more ambitious, and considerably "deeper" in nature. Functions such as $p(n)$ contain a far richer arithmetic structure than is revealed in the fact that, say, $5 \mid p(5n+4)$. The remaining chapters of our dissertation will focus on the problem of infinite families of congruences modulo powers of a prime ℓ. This chapter will begin with a brief history of the subject. We will then focus on the techniques developed by Paule and Radu [82] to prove congruence families which are associated with a modular curve in which piecewise ℓ-adic convergence fails. To show how these techniques may be applied, we will introduce a weighted partition function introduced by [64] which exhibits a family of congruences recently conjectured by Choi, Kim, and Lovejoy [31]. The following three chapters will discuss the examination of these congruences, as well as their proof, as published in [94] and [103].

5.2 History

5.2.1 Overview

Recall the remarkable congruences of Theorem 1.8:

$$\begin{aligned} p\left(5^{\alpha}n+\lambda_{5,\alpha}\right) &\equiv 0 \pmod{5^{\alpha}}, \\ p\left(7^{\alpha}n+\lambda_{7,\alpha}\right) &\equiv 0 \pmod{7^{\lfloor \alpha/2 \rfloor+1}}, \\ p\left(11^{\alpha}n+\lambda_{11,\alpha}\right) &\equiv 0 \pmod{11^{\alpha}}, \end{aligned}$$

with $\lambda_{\ell,\alpha}$ the least positive solution to $24x \equiv 1 \pmod{\ell^{\alpha}}$. Considering that no specific arithmetic properties about $p(n)$ were known *at all* prior to 1918, Ramanujan's discovery of the congruences in Theorem 1.8 was extraordinary and groundbreaking. This theorem has led to an extremely prolific century of work in the arithmetic theory of $p(n)$ and related functions. The theorem itself was proved gradually between 1919 and 1967. Ramanujan's notebooks suggest that he understood the proof in the case $\ell = 5$ [24], although he did not publish it before his death in 1920.

In 1934, Chowla showed that Ramanujan's original conjecture (1.9), fails for (1.11) when $\alpha = 3$ [32]. The corrected congruence family for (1.11) was proved by Watson in 1938, alongside the first published proof for (1.10) [114]. Indeed, both cases can be proved by a similar method, as we shall see.

The case for (1.12), on the other hand, is far more difficult, and a proof was not produced until Atkin's work in 1967 [16]. Since then, other proofs have been found, e.g., [85], but none are really easy.

Similar infinite families of congruences have been found for a very large variety of arithmetic functions. An extremely small sample includes [67], [68], [97], [45], [44], [25], [79], [41], [82], [22], [50], [111], [113], [30], [71].

For our purposes, we provide the following as a definition for an infinite family of congruences:

Definition 5.1. Given an arithmetic function $a(n)$, an *infinite congruence family*, or simply *congruence family*, for $a(n)$ with respect to a prime ℓ, is a collection of congruence relations of the form

$$a\left(\ell^{\alpha} n + \delta_{\ell,\alpha}\right) \equiv 0 \pmod{\ell^{\beta(\ell,\alpha)}},$$

in which $n \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{Z}_{\geq 1}$, $\beta(\ell, \alpha)$ is some invertible function in α, and $\delta_{\ell,\alpha}$ is the minimal positive solution to

$$C \cdot \delta_{\ell,\alpha} \equiv 1 \pmod{\ell^{\alpha}},$$

for some fixed $C \in \mathbb{Z}$.

In general, we will be interested in congruence families for functions $a(n)$ which are either coefficients of a quotient of q-Pochhammer rising factorials, or which are closely related to such coefficients.

The means by which a given congruence family is actually proved is as follows: for each $\alpha \geq 1$, one constructs a function $L_\alpha \in \mathcal{M}\left(\Gamma_0(N)\right)$ with the form

$$L_{\alpha} = \Phi_{\alpha} \cdot \sum_{C \cdot n \equiv 1 \bmod \ell^{\alpha}} a(n) q^{\lfloor n/\ell^{\alpha} \rfloor + 1},$$

in which Φ_α is a suitable prefactor (usually of the form given to Ramanujan–Kolberg identities in the previous two chapters), and $N \in \mathbb{Z}_{\geq 1}$. One then must determine whether the sequence

$$\mathcal{L} := \left(L_{\alpha}\right)_{\alpha \geq 1}$$

is ℓ-adically convergent, as defined below:

Definition 5.2. A given sequence $\mathcal{L} = \left(L_{\alpha}\right)_{\alpha \geq 1}$ of power series with integer coefficients is ℓ-adically convergent to 0 if, for any $M \in \mathbb{Z}_{\geq 0}$ there exists some $A \in \mathbb{Z}_{\geq 0}$ such that for all $\alpha \geq A$,

$$L_{\alpha} \equiv 0 \pmod{\ell^{M}}.$$

We will not give an exposition of ℓ-adic topology here, but we will briefly explain its analytic appeal in the final section of the sequel. In particular, we would want to determine that $A = \beta^{-1}(\ell, M)$ is sufficient.

The problem of proving ℓ-adic convergence varies enormously in difficulty, depending on a great variety of different factors. Complications include the genus $\mathfrak{g}\left(\mathrm{X}_0(N)\right)$, possible failure of the Radu–Newman condition over $\mathcal{M}(\Gamma_0(N))$, and certain extremely technical matters involving ℓ-adic convergence of the generating function elements of an appropriate subspace of $\mathcal{M}(\Gamma_0(N))$ (see Chapter 7 for a discussion of this kind of difficulty).

None of these complications arise in the cases of $\ell = 5, 7$ in Theorem 1.8. In these cases, $\mathfrak{g}(\mathrm{X}_0(\ell)) = 0$, and the spaces $\mathcal{M}^0(\Gamma_0(\ell))$ are isomorphic to the polynomial ring $\mathbb{C}[X]$. This means that most of the proof involves manipulating a single well-behaved Hauptmodul. In this case, most serious complications are not present. Moreover, the Newman–Radu condition applies, and we may represent this Hauptmodul as an appropriate eta quotient.

In the case of $\ell = 11$ we have $\mathfrak{g}(\mathrm{X}_0(11)) = 1$, instantly complicating matters. At the very best, we have

$$\mathcal{M}^0(\Gamma_0(11)) = \mathbb{C}[t] \oplus g_1\mathbb{C}[t],$$

for some $t, g_1 \in \mathcal{M}^0(\Gamma_0(11))$ such that

$$\begin{aligned}\mathrm{ord}_0^{(11)}(t) &= -2,\\ \mathrm{ord}_0^{(11)}(g_1) &= -3.\end{aligned}$$

Rather than working over a ring isomorphic to $\mathbb{Z}[t]$, our implied algebraic structure is a free rank 2 $\mathbb{Z}[t]$-module.

The implications of this are enormous. In the first place, the complexity of the space necessitates a larger number of computations. In addition, a larger number of generators of the space immediately raises the question of whether all relevant functions converge to 0 in the ℓ-adic sense. In the case of $\Gamma_0(11)$ this problem does not arise, but it will become important later.

Given the complications induced by the increase in the genus, it is perhaps understandable that to date every proof of Theorem 1.8 when $\ell = 11$ has been enormously technical, and has relied heavily on computer calculations (although the question of finding a proof which *minimizes* such calculations is still open). Atkin produced the first proof in 1967 [16].

The next breakthrough of importance to us came from a congruence family conjectured in 1994 by James Sellers [100] for a partition function first studied by George Andrews [8]. The generalized 2-color Frobenius partition function $c\phi_2(n)$ is the coefficient of the following generating function:

$$\mathrm{C}\Phi_2 := \sum_{n=0}^{\infty} c\phi_2(n)q^n = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4(q^4;q^4)_\infty} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^4(q^4;q^4)_\infty^2}. \tag{5.1}$$

Sellers conjectured that for all $\alpha \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$,

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha},$$

with λ_α the minimal positive integer such that

$$12\lambda_\alpha \equiv 1 \pmod{5^\alpha}.$$

One can show that the relevant space of modular functions is $\mathcal{M}\left(\Gamma_0(20)\right)$. From Theorem 2.28, we have $\mathfrak{g}\left(\mathrm{X}_0(20)\right) = 1$, and we therefore expect many of the complexities associated with Theorem 1.8 when $\ell = 11$.

On the other hand, the Newman–Radu condition applies, so that $\mathcal{M}\left(\Gamma_0(20)\right) = \mathcal{E}(20)$. This at least guarantees that any specific L_α can be represented in terms of eta quotients over $\Gamma_0(20)$. One would certainly expect that resolving this conjecture would be no more difficult than proving the case $\ell = 11$ of Theorem 1.8.

However, the conjecture proved to be far more resistant than expected. It was not until 2012 that a proof was finally given by Peter Paule and Cristian-Silviu Radu [82]. It is here that the question of piecewise ℓ-adic convergence becomes a serious concern. The method developed by Paule and Radu was not only sufficient to resolve the Andrews–Sellers conjecture, but other congruence families which present similar problems.

We devote the next few chapters to an exposition of the complication of piecewise ℓ-adic convergence, as well as the method devloped to overcome it. One could of course outline the proof of the Andrews–Sellers congruence family. However, in the interest of illustrating the breadth of the method to other problems, we will examine a different and quite interesting infinite family of congruences whose existence was originally conjectured by Youn-Seo Choi, Byungchan Kim, and Jeremy Lovejoy in [31].

5.3 The Rogers–Ramanujan Subpartition Function

For the remainder of this chapter, we will describe the arithmetic function studied by Kolitsch in [64], from which the Choi–Kim–Lovejoy conjecture arises. The function is quite attractive by itself, as it relates to the beautiful subject of the Rogers–Ramaujan identities. We give the identities below:

Theorem 5.3.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

Corollary 5.4. *Let $k \in \{1, 2\}$. The number of partitions of n with parts distinct, nonconsecutive, and greater than or equal to k are equal to the number of partitions of n with parts congruent to $\pm k \pmod 5$.*

The following definition is by Kolitsch [64]:

Definition 5.5. Let λ be a partition of m. The *Rogers–Ramanujan subpartition* of λ is the unique subpartition with a maximal number of parts, in which the parts are nonrepeating, nonconsecutive, and larger than the remaining parts of λ. More specifically, given the partition $\lambda : \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l \geq \lambda_{l+1} \geq ... \geq \lambda_k$, then the Rogers–Ramanujan subpartition of λ is the largest possible subpartition $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l$ with no repeated or consecutive parts, and with $\lambda_l > \lambda_{l+1}$ (If $l = k$, define $\lambda_{l+1} = 0$). Here, l is the length of the subpartition.

At times we will denote such a subpartition a R–R subpartition.

For instance, the Rogers–Ramanujan subpartition of

$$8 + 5 + 3 + 2 + 2 + 1 + 1 + 1$$

is $8 + 5 + 3$, with length 3. On the other hand, the Rogers–Ramanujan subpartition of

$$8 + 8 + 2 + 2 + 1 + 1 + 1$$

is simply the length-0 empty partition.

With this, we now define the partition functions $R_l(n)$, $A_1(n)$ as follows:

Definition 5.6. Let $R_l(n)$ be the number of partitions of n containing a Rogers–Ramanujan subpartition of length l, and

$$A_1(n) = \sum_{l \geq 0} l \cdot R_l(n). \tag{5.2}$$

For example, we consider $A_1(5)$. Here we give the 7 partitions of 5, with the corresponding R–R subpartitions:

$$\begin{aligned}
&(5) \supseteq (5),\\
&(4,1) \supseteq (4,1),\\
&(3,2) \supseteq (3),\\
&(3,1,1) \supseteq (3),\\
&(2,2,1),\\
&(2,1,1,1) \supseteq (2),\\
&(1,1,1,1,1).
\end{aligned}$$

So we find that of the seven partitions of 5, four of them contain a R–R subpartition of length 1, one partition contains a R–R subpartition of length 2, and two partitions contain no R–R subpartition. We therefore have

$$A_1(5) = 1 \cdot R_1(5) + 2 \cdot R_2(5) = 4 + 2 = 6.$$

As shown in [10, Chapter 7], the generating function for the number of partitions into exactly r parts, in which all parts are nonconsecutive and nonrepeating, is given by

$$\frac{q^{r^2}}{(q;q)_r}.$$

However, if we allow the denominator to grow to $(q;q)_\infty$, we now generate the number of partitions of m in which *the first r parts* are necessarily nonconsecutive and nonrepeating, and larger than all remaining parts.

Notice, however, that such a partition may indeed have a larger number of large, nonconsecutive, nonrepeating parts; that is, all partitions containing a Rogers–Ramanujan subpartition of length $l \geq r$ are also accounted for with $q^{r^2}/(q;q)_\infty$. In particular, if we sum from $r = 1$ to l, i.e.,

$$\sum_{r=1}^{l} \frac{q^{r^2}}{(q;q)_\infty} = \frac{1}{(q;q)_\infty} \sum_{r=1}^{l} q^{r^2},$$

the number of partitions of m containing a Rogers–Ramanujan subpartition of length l is accounted for a total of l times. Since of course, $(q;q)_\infty^{-1} \sum_{r=l+1}^{\infty} q^{r^2}$ will only account for subpartitions of length $> l$, we have

Theorem 5.7.

$$\frac{1}{(q;q)_\infty} \sum_{r=1}^{\infty} q^{r^2} = \sum_{n=1}^{\infty} \sum_{l \geq 0} l \cdot R_l(n) q^n = \sum_{n=1}^{\infty} A_1(n) q^n. \tag{5.3}$$

However, we can reduce the generating function to a simpler object. We see that

$$\sum_{n=1}^{\infty} A_1(n) q^n = \frac{1}{(q;q)_\infty} \left(\sum_{r=1}^{\infty} q^{r^2} \right) \tag{5.4}$$

$$= \frac{1}{2} \frac{1}{(q;q)_\infty} \left(\left(\sum_{r=-\infty}^{\infty} q^{r^2} \right) - 1 \right). \tag{5.5}$$

We can multiply both sides of (5.5) by 2:

$$2 \sum_{n=1}^{\infty} A_1(n) q^n = \frac{1}{(q;q)_\infty} \left(\sum_{r=-\infty}^{\infty} q^{r^2} \right) - \frac{1}{(q;q)_\infty}. \tag{5.6}$$

Now we bring Jacobi's triple product identity [61, Chapter 3, Theorem 3] to bear on $\sum_{r=-\infty}^{\infty} q^{r^2}$, and have

$$2\sum_{n=1}^{\infty} A_1(n)q^n = \frac{(q^2;q^2)_\infty(-q;q^2)_\infty^2}{(q;q)_\infty} - \frac{1}{(q;q)_\infty} \tag{5.7}$$

$$= \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^3(q^4;q^4)_\infty^2} - \frac{1}{(q;q)_\infty}. \tag{5.8}$$

Finally, we can express $1/(q;q)_\infty = \sum_{n=0}^{\infty} p(n)q^n$. If we let $a(n)$ represent the coefficient of q^n in our first term of (5.8), then we have

$$2 \cdot A_1(n) = a(n) - p(n). \tag{5.9}$$

5.3.1 The Choi–Kim–Lovejoy Conjecture

We see that $A_1(n)$ bears a close relationship to the functions $a(n)$ and $p(n)$. Notice that the generating function for $a(n)$ in (5.8) resembles that of $c\phi_2(n)$ in (5.1). Knowing that $c\phi_2(n)$ contains a family of congruences modulo powers of 5, Choi, Kim, and Lovejoy conjectured [31] a similar family of congruences might exist for $a(n)$. This fact, coupled with the well-known congruences of $p(n)$ in Theorem 1.8, could imply a family of congruences for $A_1(n)$.

Choi, Kim, and Lovejoy conjectured the existence of such a congruence family; but they did not give it a specific form. We will devote the next three chapters to an examination of this conjecture. In Chapter 6 we will provide an explicit statement of the existing congruence family, and give evidence for its validity. In Chapters 7 and 8 we will give a proof of this congruence family.

CHAPTER 6
AN INFINITE FAMILY OF CONGRUENCES (II)

This chapter is based on a collaboration with Cristian-Silviu Radu in [94]. We took significant inspiration from Archimedes' groundbreaking work, "The Method Treating of Mechanical Problems," [15] which emphasizes the importance of an experimental approach to mathematics—both to the solutions to specific problems and (in retrospect) to the development of new theoretical techniques.

6.1 Introduction

Recall the function $A_1(n)$ from the previous chapter. Choi, Kim, and Lovejoy proved in [31, Proposition 6.4] that

$$A_1(25n+9) \equiv A_1(25n+14) \equiv A_1(25n+24) \equiv 0 \pmod{5}. \tag{6.1}$$

They then suggest: "We remark that it appears from numerical computation that [the congruences in (6.1)] can be extended to a family of congruences modulo powers of 5."

As additional evidence, they conjecture the following congruences:

$$A_1(125n+74) \equiv A_1(125n+124) \equiv 0 \pmod{25}, \tag{6.2}$$
$$A_1(3125n+1849) \equiv A_1(3125n+3099) \equiv 0 \pmod{125}. \tag{6.3}$$

Notice, however, that the exact form of the congruence family is uncertain.

We have proved that Choi, Kim, and Lovejoy were indeed correct. However, it is interesting to describe the manner in which we arrived at the precise form of the congruence family.

In the first place, it may be checked that $A_1(5n+j)$ has no interesting congruence properties for $0 \leq j \leq 4$. On the other hand, we can verify using our software package that

$$\left(\sum_{n=0}^{\infty} a(25n+9)q^n\right)\left(\sum_{n=0}^{\infty} a(25n+14)q^n\right)$$

forms the left-hand side of a Ramanujan–Kolberg identity, while

$$\sum_{n=0}^{\infty} a(25n+24)q^n$$

forms a separate identity.

Let us take $M=4$, and $r=(-3,5,-2)$. We will examine the progressions $5n+j$, $25n+24$, $25n+9$, $25n+14$. For all of these $N=20$ is the most convenient.

In[26] = RK$[20,4,\{-3,5,-2\},5,4]$

Out[26] =

$P_{m,r}(j)$: $\{4\}$

$f_1(q)$: $\dfrac{(q;q)_\infty^7(q^4;q^4)_\infty^6(q^{10};q^{10})_\infty^6}{q^6(q^2;q^2)_\infty^7(q^{20};q^{20})_\infty^{12}}$

t: $\dfrac{(q^4;q^4)_\infty^4(q^{10};q^{10})_\infty^2}{q^2(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty^4}$

AB: $\{1, \dfrac{(q^4;q^4)_\infty(q^5;q^5)_\infty^5}{q^3(q;q)_\infty(q^{20};q^{20})_\infty^5} - \dfrac{(q^4;q^4)_\infty^4(q^{10};q^{10})_\infty^2}{q^2(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty^4}\}$

$\{p_g(t): g \in AB\}$: $\{-5-4t+13t^3, 5-t\}$

Common Factor: None

In[27] = RK$[20,4,\{-3,5,-2\},25,24]$

Out[27] =

$P_{m,r}(j)$: $\{24\}$

$f_1(q)$: $\dfrac{(q;q)_\infty^{35}(q^4;q^4)_\infty^{18}(q^{10};q^{10})_\infty^{30}}{q^{26}(q^2;q^2)_\infty^{27}(q^5;q^5)_\infty^8(q^{20};q^{20})_\infty^{48}}$

t: $\dfrac{(q^4;q^4)_\infty^4(q^{10};q^{10})_\infty^2}{q^2(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty^4}$

AB: $\{1, \dfrac{(q^4;q^4)_\infty(q^5;q^5)_\infty^5}{q^3(q;q)_\infty(q^{20};q^{20})_\infty^5} - \dfrac{(q^4;q^4)_\infty^4(q^{10};q^{10})_\infty^2}{q^2(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty^4}\}$

$\{p_g(t): g \in AB\}$:
$$\{126953125+74218750t-174609375t^2+25390625t^3$$
$$-1237031250t^4+1542084375t^5+3798876250t^6$$
$$-7568402750t^7+3755535625t^8+210440100t^9$$
$$-754603995t^{10}+190492925t^{11}+10649860t^{12}+5735t^{13},$$
$$-78125000+62500000t-46093750t^2+128906250t^3$$
$$+551875000t^4-1636475000t^5+430767500t^6$$
$$+1615951500t^7-1247744000t^8+145803400t^9$$
$$+72090170t^{10}+543930t^{11}\}$$

Common Factor: 5

In[28] = RK$[20, 4, \{-3, 5, -2\}, 25, 9]$
Out[28] =

P$_{m,r}$(j): $\{9, 14\}$

f$_1$(q): $\dfrac{(q;q)_\infty^{70}(q^4;q^4)_\infty^{36}(q^{10};q^{10})_\infty^{60}}{q^{53}(q^2;q^2)_\infty^{54}(q^5;q^5)_\infty^{16}(q^{20};q^{20})_\infty^{96}}$

t: $\dfrac{(q^4;q^4)_\infty^{4}(q^{10};q^{10})_\infty^{2}}{q^{2}(q^2;q^2)_\infty^{2}(q^{20};q^{20})_\infty^{4}}$

AB: $\{1, \dfrac{(q^4;q^4)_\infty(q^5;q^5)_\infty^{5}}{q^{3}(q;q)_\infty(q^{20};q^{20})_\infty^{5}} - \dfrac{(q^4;q^4)_\infty^{4}(q^{10};q^{10})_\infty^{2}}{q^{2}(q^2;q^2)_\infty^{2}(q^{20};q^{20})_\infty^{4}}\}$

{p$_g$(t):g $\in$ AB}:

$$\{10013580322265625 + 59127807617187500t$$
$$- 110969543457031250t^2 + 961914062500000000t^3$$
$$+ 1098876953125000000t^4 + 4741528320312500000t^5$$
$$- 308681793212890625 0t^6 + 36653533203125000000t^7$$
$$+ 15413320178222265625t^8 - 49289265947265625 00t^9$$
$$- 147282245984375000000t^{10} + 470157112375000000000t^{11}$$
$$- 173607091717968750000t^{12} - 822443421807500000000t^{13}$$
$$+ 128932571518142187500t^{14} - 661438408536830000000t^{15}$$
$$- 136103256772859531250t^{16} + 344538222774598625000t^{17}$$
$$- 172315185359567112500t^{18} + 249476899723256000000t^{19}$$
$$+ 10062440887227190000t^{20} - 4525047221487252000t^{21}$$
$$+ 295597542689803500t^{22} + 9249443740362400t^{23}$$
$$+ 193416533337075t^{24} + 417073480500t^{25}$$
$$+ 29806500t^{26},$$
$$- 7629394531250000 - 152587890625000000t$$
$$+ 470733642578125000t^2 - 6029510498046875 0t^3$$
$$- 6858215332031250 0t^4 + 3450347900390625 0t^5$$
$$+ 10226483154296875000t^6 - 20528072387695312500t^7$$
$$+ 77109506347656250 00t^8 + 181368631396484375 00t^9$$
$$+ 393194599882812500 00t^{10} - 222213167904296875 00t^{11}$$
$$+ 271274659717968750 00t^{12} + 80644763453906250 0t^{13}$$
$$- 297345876130406250 00t^{14} + 269733395977789375 00t^{15}$$
$$- 74468873479901250 00t^{16} - 30850427075086125 00t^{17}$$
$$+ 27887434598511425 00t^{18} - 58734228542786375 0t^{19}$$
$$- 68848607602826500t^{20} + 2989142953701445 0t^{21}$$
$$+ 1744979012801500t^{22} + 12418871762550t^{23}$$
$$+ 6315230100t^{24} + 39150t^{25}\}$$

Common Factor: 25

Here, as in the previous chapter, we have

$$2 \cdot A_1(n) = a(n) - p(n),$$

with

$$\mathrm{C}_R := \sum_{n=0}^{\infty} a(n)q^n = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^3 (q^4;q^4)_\infty^2}.$$

Because $25n + 9 \equiv 25n + 14 \equiv 25n + 24 \equiv 4 \pmod 5$, $p(5n + 4) \equiv 0 \pmod 5$, and because 2 and 5 are coprime, the RK identities above prove (6.1).

Indeed, divisibility by 5^α arises in (6.1), (6.2), (6.3) when $24n \equiv 1 \pmod{5^\alpha}$ and $\alpha = 1, 2$. The congruence properties of $p(n)$ therefore allow us to focus on the properties of $a(n)$.

Choi, Kim, and Lovejoy noted [31, Section 6] that C_R bears a similar form to $\mathrm{C}\Phi_2$ (see the previous chapter). This suggests a family of congruences for $a(n)$.

However, as we have already pointed out, no satisfying congruence exists for $a(5n + j)$. This alone shows that if our congruence family has the form given in Definition 5.1, then the corresponding power function $\beta(5, \alpha)$ cannot be simple equality. In order to specify the congruence family, we have to examine multiple cases.

At first sight, the matter of specifying a family of congruences might seem easy enough. Certainly, one could directly compute a list of the numerical values of $a(mn + j)$ for a fixed $m, j \in \mathbb{Z}_{\geq 0}$, as n varies over a large number of nonnegative integers. We could program a computer to check the greatest common divisor of this list.

We could also apply the machinery of Radu's algorithm from Chapters 3, 4 to $a(n)$ for various arithmetic progressions. However, for $24\lambda_\alpha \equiv 1 \pmod{5^\alpha}$, one can quickly show that

$$\lambda_{2\alpha-1} = \frac{19 \cdot 5^{2\alpha-1} + 1}{24},$$
$$\lambda_{2\alpha} = \frac{23 \cdot 5^{2\alpha} + 1}{24}.$$

This immediately implies that modest increases in α will drive even the smallest values of n to increase exponentially. Given that [10, Chapter 6] $a(n)$ already increases subexponentially with n, it is very clear that even the most powerful computers will not be able to check the explicit RK identities for

$$\sum_{n=0}^{\infty} a(5^{2\alpha}n + \lambda_{2\alpha})q^n$$

beyond the very smallest values of α.

This of course serves little concern for a conjecture already proven. However, the methods developed by Atkin, Paule, Radu, and others to actually prove a conjecture of this sort give comparatively little understanding of how these conjectures came to be inferred in the first place. We are currently proceeding without an exact form for the congruence family conjectured by Choi, Kim, and Lovejoy, so that we do not yet have an explicit theorem to prove.

This means that we will need to find a more efficient way to verify a family of congruences for many specific values of α.

In this chapter we will describe one such approach. In particular, we will show how to give an explicit form to Choi, Kim, and Lovejoy's congruence family, as in Theorem 6.3 below. We will prove this theorem in the next two chapters; but first we will show how substantial evidence for the congruence family may be gathered.

We will use the congruence family conjectured by Choi, Kim, and Lovejoy as our principal example, but we will also demonstrate that these techniques may be adapted with relatively little difficulty to many similar conjectures in which an arithmetic sequence has a generating function that is a q-rising factorial quotient with factors of the form $(q^m;q^m)_\infty$.

We outline the key algorithmic steps to check Theorem 6.3 for a large number of α. We construct a useful algebra basis for the space of eta quotients over $\Gamma_0(20)$. From here we show how the basis can be suitably modified to interact more carefully with the U_5 operator.

We then discuss how to apply our method in more generalized circumstances. In the sequel we explain why our approach, so useful in verifying a substantial number of cases of a conjecture, is not generally capable of providing a complete proof.

Remark 6.1. While our experimental method cannot yield a rigorous proof for a congruence family, its underlying intuitive approach *can* be made into a more theoretically sound technique. See the later chapters discussing the localization technique.

6.2 Gathering Evidence

Our observations in the previous section suggest that a congruence family for $a(n)$ will exist for n as a solution to $24x \equiv 1 \pmod{5^\alpha}$ for some values of α. The case for $\alpha = 1$ can readily be shown to have no congruences of interest, while the case $\alpha = 2$ was already checked in the affirmative by Choi, Kim, and Lovejoy. With some computational difficulty, it can be verified that $a(125n + 99)$ is not divisible by 25, showing that there is no congruence of interest in the case $\alpha = 3$. Examining even the very next case, the progression $625n + 599$, will be much more difficult.

But supposing that congruences of interest lie among solutions of $24x \equiv 1 \pmod{5^\alpha}$, let us determine how we might check successive cases. Our opening steps are not unlike the initial steps to a true proof. We can very quickly define a sequence of functions $\mathcal{L} = (L_\alpha)_{\alpha\geq 0}$ in which

$$\begin{aligned} L_0 &:= 1, \\ L_\alpha &:= \Phi_\alpha \cdot \sum_{24n\equiv 1 \bmod 5^\alpha} a(n) q^{\lfloor n/5^\alpha \rfloor}, \end{aligned}$$

$$\Phi_{2\alpha-1} = \frac{q}{\mathrm{C}_R(q^5)}, \text{ and } \Phi_{2\alpha} = \frac{q}{\mathrm{C}_R(q)}.$$

Writing each L_α more explicitly, we have

$$L_0 = 1, \tag{6.4}$$

$$L_{2\alpha-1} = \frac{(q^5;q^5)^3_\infty (q^{20};q^{20})^2_\infty}{(q^{10};q^{10})^5_\infty} \sum_{n=0}^{\infty} a(5^{2\alpha-1}n + \lambda_{2\alpha-1}) q^{n+1}, \tag{6.5}$$

$$L_{2\alpha} = \frac{(q;q)^3_\infty (q^4;q^4)^2_\infty}{(q^2;q^2)^5_\infty} \sum_{n=0}^{\infty} a(5^{2\alpha}n + \lambda_{2\alpha}) q^{n+1}. \tag{6.6}$$

Notice that the prefactors for each L_α can be expanded into integer power series in which the initial term has coefficient 1. This implies that no positive power of 5 can divide any Φ_α (that is, no positive power of 5 can divide every term of Φ_α). Therefore, if a given power of 5 divides L_α, then that given power of 5 must divide every term $a(n)$.

We define

$$A := q \cdot \frac{\mathrm{C}_R(q)}{\mathrm{C}_R(q^{25})},$$

and then the set of linear operators

$$U^{(0)}(f) := U_5 (A \cdot f),$$
$$U^{(1)}(f) := U_5 (f),$$
$$U^{(\alpha)}(f) := U^{(\alpha \bmod 2)}(f).$$

Notice that for two functions f and g, and any $\alpha \in \mathbb{C}$,

$$U^{(0)}(\alpha \cdot f + g)$$
$$= U_5 (A(q)(\alpha \cdot f + g)) = U_5 (\alpha \cdot A(q) \cdot f + A(q) \cdot g) \tag{6.7}$$
$$= \alpha \cdot U_5 (A(q) \cdot f) + U_5 (A(q) \cdot g) = \alpha \cdot U^{(0)}(f) + U^{(0)}(g). \tag{6.8}$$

Since we already know from Chapter 2 that $U^{(1)} = U_5$ is linear, we have thus established that $U^{(\alpha)}$ is linear for all $\alpha \geq 0$.

Moreover, we can quickly prove that A is an eta quotient for $\Gamma_0(100)$. This gives us the following important consequence of Theorem : for all $f \in \mathcal{M}(\Gamma_0(20))$,

$$U^{(\alpha)}(f) \in \mathcal{M}(\Gamma_0(20)). \tag{6.9}$$

This now gives us a means of connecting L_α with $L_{\alpha+1}$.

Theorem 6.2. *For all $n \in \mathbb{Z}_{>0}$,*

$$L_{\alpha+1} = U^{(\alpha)}\left(L_\alpha\right). \tag{6.10}$$

Proof. For a given $\alpha \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}
&U^{(1)}\left(L_{2\alpha-1}\right)\\
&= U_5\left(\frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty^2}{(q^{10};q^{10})_\infty^5}\sum_{n=0}^{\infty} a(5^{2\alpha-1}n+\lambda_{2\alpha-1})q^{n+1}\right) &(6.11)\\
&= \frac{(q;q)_\infty^3(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5} U_5\left(\sum_{n\geq 0} a(5^{2\alpha-1}n+\lambda_{2\alpha-1})q^{n+1}\right) &(6.12)\\
&= \frac{(q;q)_\infty^3(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5} U_5\left(\sum_{n\geq 1} a(5^{2\alpha-1}(n-1)+\lambda_{2\alpha-1})q^{n}\right) &(6.13)\\
&= \frac{(q;q)_\infty^3(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5} \sum_{5n\geq 1} a(5^{2\alpha-1}(5n-1)+\lambda_{2\alpha-1})q^{n} &(6.14)\\
&= \frac{(q;q)_\infty^3(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5} \sum_{n=0}^{\infty} a(5^{2\alpha}n+5^{2\alpha}-5^{2\alpha-1}+\lambda_{2\alpha-1})q^{n+1} &(6.15)\\
&= \frac{(q;q)_\infty^3(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5} \sum_{n=0}^{\infty} a(5^{2\alpha}n+\lambda_{2\alpha})q^{n+1}, &(6.16)
\end{aligned}$$

since

$$\begin{aligned}
5^{2\alpha}-5^{2\alpha-1}+\lambda_{2\alpha-1} &= 5^{2\alpha-1}(4)+\frac{19\cdot 5^{2\alpha-1}+1}{24} &(6.17)\\
&= \frac{5^{2\alpha-1}(5\cdot 23)+1}{24} = \frac{23\cdot 5^{2\alpha}+1}{24} = \lambda_{2\alpha}. &(6.18)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&U^{(0)}\left(L_{2\alpha}\right)\\
&= U_5\left(A\cdot\frac{(q;q)_\infty^3(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5}\sum_{n=0}^{\infty}a(5^{2\alpha}n+\lambda_{2\alpha})q^{n+1}\right) &(6.19)\\
&= U_5\left(\frac{(q^{25};q^{25})_\infty^3(q^{100};q^{100})_\infty^2}{(q^{50};q^{50})_\infty^5}\sum_{n=0}^{\infty}a(5^{2\alpha}n+\lambda_{2\alpha})q^{n+2}\right) &(6.20)\\
&= \frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty^2}{(q^{10};q^{10})_\infty^5}U_5\left(\sum_{n=0}^{\infty}a(5^{2\alpha}n+\lambda_{2\alpha})q^{n+2}\right) &(6.21)\\
&= \frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty^2}{(q^{10};q^{10})_\infty^5}U_5\left(\sum_{n\geq 2}a(5^{2\alpha}(n-2)+\lambda_{2\alpha})q^{n}\right) &(6.22)\\
&= \frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty^2}{(q^{10};q^{10})_\infty^5}\sum_{5n\geq 2}a(5^{2\alpha}(5n-2)+\lambda_{2\alpha})q^{n}. &(6.23)
\end{aligned}$$

Notice that $5n \geq 2$ implies that $n \geq 1$ for $n \in \mathbb{Z}$, so that

$$\begin{aligned}
&U^{(0)}\left(L_{2\alpha}\right)\\
&= \frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty^2}{(q^{10};q^{10})_\infty^5}\sum_{n\geq 1}a(5^{2\alpha+1}n-2\cdot 5^{2\alpha}+\lambda_{2\alpha})q^{n} &(6.24)\\
&= \frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty^2}{(q^{10};q^{10})_\infty^5}\sum_{n=0}^{\infty}a(5^{2\alpha+1}(n+1)-2\cdot 5^{2\alpha}+\lambda_{2\alpha})q^{n+1} &(6.25)\\
&= \frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty^2}{(q^{10};q^{10})_\infty^5}\sum_{n=0}^{\infty}a(5^{2\alpha+1}n+\lambda_{2\alpha+1})q^{n+1}, &(6.26)
\end{aligned}$$

since

$$\begin{aligned}
5^{2\alpha+1}-2\cdot 5^{2\alpha}+\lambda_{2\alpha} &= 5^{2\alpha}(3)+\frac{23\cdot 5^{2\alpha}+1}{24} &(6.27)\\
&= \frac{5^{2\alpha}(5\cdot 19)+1}{24} = \frac{19\cdot 5^{2\alpha+1}+1}{24} = \lambda_{2\alpha+1}. &(6.28)
\end{aligned}$$

□

We suspect that $\mathcal{L}_\alpha$ converges 5-adically to 0. We want to know its precise convergence rate. As an example, let us select L_1:

$$L_1 = \frac{(q^5;q^5)^3_\infty (q^{20};q^{20})^2_\infty}{(q^{10};q^{10})^5_\infty} \sum_{m=0}^\infty a(5m+4)q^{m+1}$$
$$= \frac{\eta(5\tau)^3\eta(20\tau)^2}{\eta(10\tau)^5} \cdot q^{1-5/24} \sum_{m=0}^\infty a(5m+4)q^m.$$

In keeping with the notation of Chapter 3, we have $M = 4, \hat{r} = (-3, 5, -2)$, $m = 25, t = 24$. The smallest possible value of N to satisfy the Δ^* criteria is $N = 20$.

Now, the vector $\hat{s} = (0, 0, 0, 3, -5, 2)$ satisfies the conditions of Theorem 3.5, and $P_{5,\hat{r}}(4) = \{4\}$ by Lemma 3.4. Finally, we have

$$\alpha = 1 - \frac{5}{24} = \frac{19}{24} = \frac{4}{5} + \frac{1}{120}(1(-3) + 2(5) + 4(-2)).$$

We have therefore shown that $L_1 \in \mathcal{M}(\Gamma_0(20))$.

Because $U^{(\alpha)}(f) \in \mathcal{M}(\Gamma_0(20))$ for all $f \in \mathcal{M}(\Gamma_0(20))$ by (6.9), we have that $\mathcal{L}$ forms a sequence of functions in $\mathcal{M}(\Gamma_0(20))$.

However, while $L_1 \in \mathcal{M}(\Gamma_0(20))$, it is not necessarily in $\mathcal{M}^\infty(\Gamma_0(20))$. We need some $\omega \in \mathcal{E}^\infty(20)$ that will overcome any other poles that L_1 has, i.e.,

$$\omega \cdot L_1 \in \mathcal{M}^\infty(\Gamma_0(20))_{\mathbb{Q}}.$$

We may take advantage of the fact that

$$\mathcal{M}^\infty(\Gamma_0(20))_{\mathbb{Q}} = \langle \mathcal{E}^\infty(20)\rangle_{\mathbb{Q}}. \tag{6.29}$$

In that case, $\omega \cdot L_1 \in \langle \mathcal{E}^\infty(20)\rangle_{\mathbb{Q}}$.

Let us define $\langle \mathcal{E}^\infty(20)\rangle_{\mathbb{Q}} = \langle 1, G_1, ..., G_v\rangle_{\mathbb{Q}[T]}$, in which $T, G_1, ..., G_v$ satisfy the conditions of Theorem 2.44.

We note that $\omega, T, T^{-1}, G_i \in \mathcal{E}(20)$. Let us suppose for the moment that each of these functions has integer coefficients in its q-expansion. We must therefore have polynomials $p_0, p_1, ..., p_v \in \mathbb{Z}[X]$ such that

$$\omega \cdot L_1 = p_0(T) + p_1(T)G_1 + ... + p_v(T)G_v, \tag{6.30}$$
$$L_1 = \frac{p_0(T)}{\omega} + \frac{p_1(T)}{\omega}G_1 + ... + \frac{p_v(T)}{\omega}G_v. \tag{6.31}$$

If we apply $U^{(1)} = U_5$ to both sides of (6.31), we then have an expression for L_2 in terms of $U^{(1)}(T^jG_k/\omega)$. If we were able to find appropriate expansions of these terms (e.g., expansions in terms of T, G_k), then we could apply U^{α} arbitrarily many times, and find expansions of L_{α}, no matter the size of α.

We can simplify matters enormously by imposing an additional condition to the necessary properties of our algebra basis. We know that L_1 has poles at various cusps of $\Gamma_0(20)$. If we were to choose T to have positive order at the corresponding poles of L_1, then we could make the substitution $\omega = T^l$, for $l \in \mathbb{Z}_{>0}$ sufficiently large:

$$\begin{aligned} T^l \cdot L_1 &= p_0(T) + p_1(T)G_1 + ... + p_v(T)G_v, \\ L_1 &\in \langle 1, G_1, ..., G_v \rangle_{\mathbb{Z}[T,T^{-1}]}. \end{aligned}$$

Now we have only to understand how to compute expansions for $U^{(i)}\left(T^j G_k\right)$, with $i \in \{0,1\}$, $j \in \mathbb{Z}$, and $k \in \{0, 1, ..., v\}$. Moreover, if we are careful to arrange so that T has positive order at every possible pole of the functions $A^i T^j G_k$ for all $i \in \{0,1\}$, $j \in \mathbb{Z}$, and $k \in \{0, 1, ..., v\}$, then we will have

$$U^{(i)}\left(T^j G_k\right) \in \langle 1, G_1, ..., G_v \rangle_{\mathbb{Z}[T,T^{-1}]}.$$

That is, $\langle 1, G_1, ..., G_v \rangle_{\mathbb{Z}[T,T^{-1}]}$ is closed under $U^{(\alpha)}$ for all $\alpha \in \mathbb{Z}_{\geq 0}$.

From this closure theorem, we can construct a relatively efficient algorithm for checking L_{α} for divisibility by powers of 5. Supposing we want to check our conjecture that $L_{2\alpha} \equiv 0 \pmod{5^{\alpha}}$, by examining $0 \leq \alpha \leq 2B$, for some $B \in \mathbb{Z}_{>0}$.

Noting that we can define $L_0 := 1$, we can begin by immediately establishing that $L_0 \in \langle 1, G_1, ..., G_v \rangle_{\mathbb{Z}[T,T^{-1}]}$.

From here, we compute

$$L_1 = U^{(0)}(1) = \sum_{\substack{j \in \mathbb{Z}, \\ 0 \leq k \leq v}} b_{1,j,k} T^j G_k.$$

However, as we apply $U^{(\alpha)}$ for increasing α, we will find the coefficients $b_{1,j,k}$ become very large. To resolve this, we reduce each coefficient to the least positive residue modulo 5^B:

$$L_1^{(B)} = \sum_{\substack{j \in \mathbb{Z}, \\ 0 \leq k \leq v}} c_{1,j,k} T^j G_k,$$

with $c_{1,j,k} \equiv b_{1,j,k} \pmod{5^B}$. We thus define the following sequence of functions:

$$\begin{aligned} L_0^{(B)} &:= 1, \\ L_{\alpha}^{(B)} &:= U_5^{(\alpha-1)}\left(L_{\alpha-1}^{(B)}\right) \pmod{5^B} = \sum_{\substack{j \in \mathbb{Z}, \\ 0 \leq k \leq v}} c_{\alpha,j,k} T^j G_k, \end{aligned}$$

with $0 \leq c_{\alpha,j,k} < 5^B$ for all α, j, k.

We now give the steps for checking this conjecture:

1. Begin with $\alpha = 0$, $v_0 = 0$, and $V = \{v_0\}$.
2. Expand $L_\alpha^{(B)}$ into $\langle 1, G_1, ..., G_v \rangle_{\mathbb{Z}[T,T^{-1}]}$: $L_\alpha^{(B)} = \sum_{\substack{j\in\mathbb{Z},\\ 0\leq k\leq v}} c_{\alpha,j,k} T^j G_k$.
3. Expand $U_5^{(\alpha)}\left(L_\alpha^{(B)}\right) = \sum_{\substack{j\in\mathbb{Z},\\ 0\leq k\leq v}} c_{\alpha,j,k} U_5^{(\alpha)}\left(T^j G_k\right)$.
4. Reduce $U_5^{(\alpha)}\left(L_\alpha^{(B)}\right) \pmod{5^B}$ to get $L_{\alpha+1}^{(B)} = \sum_{\substack{j\in\mathbb{Z},\\ 0\leq k\leq v}} c_{\alpha+1,j,k} T^j G_k$.
5. Let $v_{\alpha+1}$ be the maximal power of 5 (up to B) dividing each nonzero $c_{\alpha+1,j,k}$.
6. Set $V = V \cup \{v_{\alpha+1}\}$.
7. Set $\alpha = \alpha + 1$, and return to Step 2. Continue until $\alpha = B$.

Here, the growth of our coefficients $c_{\alpha,j,k}$ is limited by the size of 5^B. This bound grows exponentially with B, but it is far better than the sub-double-exponential coefficient growth that we would otherwise expect.

For example, setting $B = 10$ ensures that $L_\alpha^{(B)}$ will contain terms smaller than 5^{10}, of the order of 10^7. These numbers are small enough even for a modest laptop to manage, and we can now examine the divisibility of L_α for $1 \leq \alpha \leq 10$.

Doing so yields that

$$\begin{aligned}
L_1 &\equiv 0 \pmod{5^0},\\
L_2 &\equiv 0 \pmod{5^1},\\
L_3 &\equiv 0 \pmod{5^1},\\
L_4 &\equiv 0 \pmod{5^2},\\
L_5 &\equiv 0 \pmod{5^2},\\
L_6 &\equiv 0 \pmod{5^3},\\
L_7 &\equiv 0 \pmod{5^3},\\
L_8 &\equiv 0 \pmod{5^4},\\
L_9 &\equiv 0 \pmod{5^4},\\
L_{10} &\equiv 0 \pmod{5^5}.
\end{aligned}$$

Now we have some more compelling evidence from which to extract an explicit congruence family, which we may immediately express in terms of A_1:

Theorem 6.3. *If λ_α is the smallest positive integer such that $24\lambda_\alpha \equiv 1 \pmod{5^\alpha}$, we have*

$$A_1\left(5^{2\alpha}n + \lambda_{2\alpha}\right) \equiv 0 \pmod{5^\alpha}.$$

Notice that the experimental method above enabled us to formulate a precise conjecture while investing relatively little time or computation.

Moreover, our method begins with computation of a very precise algebra basis for $\langle \mathcal{E}^\infty(20)\rangle_{\mathbb{Q}}$. As is demonstrated in [103, Section 4.1], the functions in this basis are essential for actually completing the proof of the conjecture. This alone establishes that the full algorithm, with its relative efficiency and economy, may as well be brought to bear before attempting a proof.

We will prove Theorem 6.3 in the next two chapters. However, we will devote the rest of this chapter to a complete exposition of the method: namely, the procedure by which the functions $T, G_1, ..., G_v$ are computed. Finally, we will finish with a demonstration of why this method fails to generalize into a complete proof, and the failure of piecewise ℓ-adic convergence.

6.2.1 The Basis

We now discuss computation of the functions $T, G_1, ..., G_v$. The functions G_k may be computed using the algebra basis algorithm in [91], once the function T is known. However, T must be selected with care, so that its positive-order zeros correspond with any poles possessed by $U_5\left(A^i T^j G_k\right)$, for all $(i, j, k) \in \{0, 1\} \times \mathbb{Z} \times \{0, 1, ..., v\}$.

To begin with, we assume that T has the form

$$T = \prod_{\delta|20} \eta(\delta\tau)^{s_\delta}.$$

Since T is a modular function over $\Gamma_0(20)$, we know that $s = (s_\delta)_{\delta|20}$ must satisfy the conditions of Theorem 2.40:

$$\sum_{\delta|20} s_\delta = 0,$$
$$\sum_{\delta|20} \delta s_\delta + 24x_1 = 0,$$
$$\sum_{\delta|20} \frac{20}{\delta} s_\delta + 24x_2 = 0,$$
$$\prod_{\delta|20} \delta^{|s_\delta|} = x_3^2,$$

with $x_1, x_2, x_3 \in \mathbb{Z}$. What additional conditions are necessary for T?

Recall that

$$A := q \cdot \frac{\mathrm{C}_R(q)}{\mathrm{C}_R(q^{25})} = q \cdot \frac{(q^2;q^2)_\infty^5 (q^{25};q^{25})_\infty^3 (q^{100};q^{100})_\infty^2}{(q;q)_\infty^3 (q^4;q^4)_\infty^2 (q^{50};q^{50})_\infty^5}.$$

Notice that $A \in \mathcal{M}\left(\Gamma_0(100)\right)$, while $T, G_k \in \mathcal{M}\left(\Gamma_0(20)\right)$. Because $\mathcal{M}\left(\Gamma_0(20)\right) \subseteq \mathcal{M}\left(\Gamma_0(100)\right)$, we may take the product $A^i T^j G_k \in \mathcal{M}\left(\Gamma_0(100)\right)$. Then our U_5 operator maps $A^i T^j G_k \in \mathcal{M}\left(\Gamma_0(100)\right)$ to

$$\begin{aligned} f_k^{(i,j)} &= U_5\left(A^i T^j G_k\right) \\ &= \frac{1}{5}\sum_{r=0}^{4} A^i\left(\frac{\tau+r}{5}\right) T^j\left(\frac{\tau+r}{5}\right) G_k\left(\frac{\tau+r}{5}\right) \in \mathcal{M}\left(\Gamma_0(20)\right). \end{aligned}$$

We need to account for any possible poles of $f_k^{(i,j)}$, so that $T^m f_k^{(i,j)} \in \mathcal{M}^\infty\left(\Gamma_0(20)\right)$ for sufficiently large $m \in \mathbb{Z}_{>0}$. Let us begin by considering A alone.

We now give a set of representatives for the cusps of $\Gamma_0(20)$, and for those of $\Gamma_0(100)$. They may be calculated using Theorems 2.10 and 2.11:

$$\begin{aligned} \mathcal{C}(20) &= \left\{\frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \frac{1}{4}, \frac{1}{2}, 1\right\}, \\ \mathcal{C}(100) &= \left\{\frac{1}{100}, \frac{1}{50}, \frac{1}{25}, \frac{1}{20}, \frac{1}{10}, \frac{3}{20}, \frac{1}{5}, \frac{1}{4}, \frac{3}{10}, \frac{7}{20}, \frac{2}{5}, \frac{9}{20}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, 1\right\}. \end{aligned}$$

In the first place, we have an exact form for A, which allows us to compute its zeros and poles exactly, via Theorem 2.41. Doing so, and accounting for Theorem 2.11, reveals the following:

$$\begin{aligned} \mathrm{ord}_{1/100}^{(100)}(A) &= 1, \\ \mathrm{ord}_{1/50}^{(100)}(A) &= -5, \\ \mathrm{ord}_{1/25}^{(100)}(A) &= 4, \\ \mathrm{ord}_{1/4}^{(100)}(A) &= -1, \\ \mathrm{ord}_{1/2}^{(100)}(A) &= 5, \\ \mathrm{ord}_{1}^{(100)}(A) &= -4. \end{aligned}$$

In particular, A has negative order (i.e., poles) at $1/50, 1/4, 1$. Because U_5 sends A to $\frac{1}{5}\sum_{r=0}^{4} A^i\left((\tau+r)/5\right)$, we need to examine the possible rational numbers τ may approach so that $(\tau+r)/5$ approaches a rational number corresponding to the cusps at $1/50, 1/4, 1$.

In Table D.1 we take τ to approach an element of $\mathcal{C}(20)$. In the process, $(\tau+r)/5$ will tend to a rational number for $r = 0, 1, 2, 3, 4$. We then take the element in $\mathcal{C}(100)$ representing the same cusp as $(\tau + r)/5$ through use of Theorem 2.11. For example, as $\tau \to 1/10$, and for $r = 3$, $(\tau + r)/5 \to 31/50$. However, if we set $a_1/c_1 = 1/50 \in \mathcal{C}(100)$ and $a/c = 31/50$, and take $m = 31, n = 0$, then the congruences of Theorem 2.11 are satisfied, so that for $\tau \to 1/10$ and $r = 3$, we have the corresponding cusp $1/50$.

Notice that just three cusps over $\Gamma_0(20)$ (represented by $1, 1/2, 1/4$) correspond to 15 of the 18 cusps of $\Gamma_0(100)$. The remaining three cusps of $\Gamma_0(20)$ $(1/5, 1/10, 1/20)$ correspond bijectively to the remaining cusps over $\Gamma_0(100)$ $(1/25, 1/50, 1/100)$.

We see that for $(\tau + r)/5$ to approach the cusps $1/50, 1/4, 1$, τ must approach $1/10, 1/4, 1$, respectively.

In other words, $U_5(A)$ has possible poles at the cusps $1/10, 1/4, 1$. We therefore want our T to have positive order at these cusps. We therefore have the following system of inequalities that we know are necessary (but not yet sufficient) for $T = \prod_{\delta|20} \eta(\delta\tau)^{s_\delta}$:

$$\frac{1}{24}\sum_{\delta|20} \frac{\gcd(10,\delta)^2}{\delta} s_\delta \geq 1,$$

$$\frac{5}{24}\sum_{\delta|20} \frac{\gcd(4,\delta)^2}{\delta} s_\delta \geq 1,$$

$$\frac{5}{6}\sum_{\delta|20} \frac{\gcd(1,\delta)^2}{\delta} s_\delta \geq 1.$$

Next we consider G_k for $1 \leq k \leq v$. By our definition, we want $G_k \in \mathcal{M}^\infty(\Gamma_0(20))$, so that G_k only has a pole at the cusp at ∞ with respect to $\Gamma_0(20)$. Table D.2 below is analogous to Table D.1 but only considering the cusps of $\Gamma_0(20)$.

Notice that the cusp at ∞ is represented in $\Gamma_0(20)$ by $1/20$, which may be approached as τ approaches the cusps $1/20, 1/4$ over $\Gamma_0(20)$. Because we want T to have a pole at $1/20$, we therefore only need to account for the additional possible pole at $1/4$, which we already accounted for.

Therefore, a function T satisfying these three inequalities, together with the conditions of Theorem 2.40, will satisfy

$$T^m U_5\left(A^i T^j G_k\right) \in \mathcal{M}^\infty(\Gamma_0(20))$$

for $i = 0, 1, j \geq 0, 1 \leq k \leq v$, with sufficiently large m.

Finally, there is the question of negative powers of T. We know that because T must have positive order at $1/10, 1/4, 1$, therefore T^{-1} must have negative order at these cusps. This means of course that T must have positive order at any cusp representative a/c such that

$$\frac{a/c + r}{5} = \frac{a + cr}{5c} \in \left\{\frac{1}{10}, \frac{1}{4}, 1\right\}.$$

Examining our table above, it can quickly be seen that these values are approached as τ approaches the cusps at $1/10, 1/2, 1/4, 1$. This induces another constraint: T must have positive order at $1/2$.

We now have the additional inequality

$$\frac{5}{24}\sum_{\delta|20}\frac{\gcd(2,\delta)^2}{\delta}s_\delta \geq 1.$$

We now have conditions for the behavior of T at every cusp of $\Gamma_0(20)$ except for $\frac{1}{5}$. Since A, G_k do not have poles at $1/5$, we need only worry about T and T^{-1}. Suppose first that T has positive order at $1/5$. Then of course, T^{-1} must have negative order at $1/5$. Which cusps over $\Gamma_0(20)$ correspond to a potential pole at $1/5$? Examining our table above, we see that the only possible poles induced would occur at $1/5, 1$. Now T already has positive order at 1, as well as at $1/5$ by hypothesis.

Therefore, since the cusp at $1/5$ causes no problems whether T has positive or zero order there, we do not need to induce any specific condition at the cusp (besides the nonnegative order of T).

$$\frac{1}{6}\sum_{\delta|20}\frac{\gcd(5,\delta)^2}{\delta}s_\delta \geq 0.$$

We now have sufficient conditions from which to derive T, but we give one more mild condition for the sake of efficiency. We clearly want $|\mathrm{ord}^{(20)}_{1/20}(T)|$ to be as small as possible. We therefore take note of the fact that

$$\mathrm{ord}^{(20)}_{1/20}(T) = \frac{1}{24}\sum_{\delta|20}\delta s_\delta,$$

so that in our Newman system, $x_1 = -\mathrm{ord}^{(20)}_{1/20}(T)$. We therefore add the additional tentative condition to our system:

$$x_1 = 1,$$

to search for the possibility that there exists an acceptable T with $\mathrm{ord}^{(20)}_{1/20}(T) = -1$. If our system contains no solution, then we must reset $x_1 = 2$ and continue.

Our complete system, then, is

$$\sum_{\delta|20} s_\delta = 0,$$
$$\sum_{\delta|20} \delta s_\delta + 24x_1 = 0,$$
$$\sum_{\delta|20} \frac{20}{\delta} s_\delta + 24x_2 = 0,$$
$$\prod_{\delta|20} \delta^{|s_\delta|} = x_3^2,$$
$$\frac{1}{10}\sum_{\delta|20} \frac{\gcd(10,\delta)^2}{\delta} s_\delta \geq 1,$$
$$\frac{1}{6}\sum_{\delta|20} \frac{\gcd(5,\delta)^2}{\delta} s_\delta \geq 0,$$
$$\frac{5}{24}\sum_{\delta|20} \frac{\gcd(4,\delta)^2}{\delta} s_\delta \geq 1,$$
$$\frac{5}{24}\sum_{\delta|20} \frac{\gcd(2,\delta)^2}{\delta} s_\delta \geq 1,$$
$$\frac{1}{6}\sum_{\delta|20} \frac{\gcd(1,\delta)^2}{\delta} s_\delta \geq 1,$$
$$x_1 = -\mathrm{ord}_{1/20}^{(20)}(T),$$

with $x_1, x_2, x_3 \in \mathbb{Z}$.

This system allows us to obtain a vector s that is optimal with respect to x_1.

We made use of the software package 4ti2 [1] to solve this system, and discovered the solution vector $s = (2, 0, 2, -2, 8, -10)$, of minimal order $x_1 = 5$. This gives us the function

$$T = \frac{\eta(\tau)^2\eta(4\tau)^2\eta(10\tau)^8}{\eta(5\tau)^2\eta(20\tau)^{10}} = \frac{1}{q^5}\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^8}{(q^5;q^5)_\infty^2(q^{20};q^{20})_\infty^{10}}.$$

From here we may apply the algorithm `AB` of Section 3.3.1 to construct the functions G_k.

Theorem 6.4. *Let*

$$T := \frac{\eta(\tau)^2\eta(4\tau)^2\eta(10\tau)^8}{\eta(5\tau)^2\eta(20\tau)^{10}} = \frac{1}{q^5}\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^8}{(q^5;q^5)_\infty^2(q^{20};q^{20})_\infty^{10}} \tag{6.32}$$

$$H := \frac{\eta(4\tau)\eta(5\tau)^5}{\eta(\tau)\eta(20\tau)^5} = \frac{1}{q^3}\frac{(q^4;q^4)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{20};q^{20})_\infty^5}, \tag{6.33}$$

$$G := \frac{\eta(4\tau)^4\eta(10\tau)^2}{\eta(2\tau)^2\eta(20\tau)^4} = \frac{1}{q^2}\frac{(q^4;q^4)_\infty^4(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty^4}. \tag{6.34}$$

Then

$$\mathcal{M}^\infty\left(\Gamma_0(20)\right)_{\mathbb{Q}} = \langle 1, G_1, G_2, G_3, G_4\rangle_{\mathbb{Q}[T]}, \tag{6.35}$$

with

$$G_1 = G, \tag{6.36}$$

$$G_2 = H - G, \tag{6.37}$$

$$G_3 = G^2, \tag{6.38}$$

$$G_4 = (H-G)^2. \tag{6.39}$$

Moreover,

$$U_5\left(A^iT^jG_k\right) \in \langle 1, G_1, G_2, G_3, G_4\rangle_{\mathbb{Z}[T,T^{-1}]}, \tag{6.40}$$

for all $(i,j,k) \in \{0,1\}\times\mathbb{Z}\times\{0,1,2,3,4\}$.

With T derived, the algebra basis may be found with Radu's basis algorithm. We prove its validity using the properties of the corresponding modular curve $\mathrm{X}_0(20)$, together with the Weierstrass gap theorem.

Notice that we restrict our coefficients to rational numbers, but that our theorem applies equally if we extend our field to the whole of $\mathbb{C}$.

Proof. Condition (6.40) was verified in the construction of T. We are left to verify (6.35).

The conditions of Theorem 2.40 can be quickly checked with respect to G, H, and T, so that

$$\mathcal{M}^\infty\left(\Gamma_0(20)\right)_{\mathbb{Q}} \supseteq \langle 1, G_1, G_2, G_3, G_4\rangle_{\mathbb{Q}[T]}. \tag{6.41}$$

Let $f \in \mathcal{M}^\infty\left(\Gamma_0(20)\right)_{\mathbb{Q}}$. We want to prove that $f \in \langle 1, G_1, G_2, G_3, G_4\rangle_{\mathbb{Q}[T]}$.

With only one pole, f has an expansion

$$f = \frac{b(-m_0)}{q^{m_0}} + \frac{b(-m_0+1)}{q^{m_0-1}} + ... + \frac{b(-1)}{q} + b(0) + \sum_{n=0}^{\infty} b(n)q^n,$$

with $b(n) \in \mathbb{Q}$ for all $n \geq -m_0$, and $b(-m_0) \neq 0$.

We can now apply the MC algorithm given in Section 3.1. If we first assume that $m_0 \neq 1$, then there exist $a, b \in \mathbb{Z}_{\geq 0}$ such that $m_0 = 5a + b$, and
$b \in \{0, 2, 3, 4, 6\}$. Examining the orders of the functions T, G_k, we find that

$$\begin{aligned} -\mathrm{ord}_{\infty}^{(20)}(T) &= 5, \\ -\mathrm{ord}_{\infty}^{(20)}(G_1) &= 2, \\ -\mathrm{ord}_{\infty}^{(20)}(G_2) &= 3, \\ -\mathrm{ord}_{\infty}^{(20)}(G_3) &= 4, \\ -\mathrm{ord}_{\infty}^{(20)}(G_4) &= 6. \end{aligned}$$

We therefore have

$$-\mathrm{ord}_{\infty}^{(20)}(f_1) < m_0,$$

for

$$f_1 = f - \frac{b(m_0)}{\mathrm{LC}(T^a G_{k_1})} \cdot T^a G_{k_1} \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right)_{\mathbb{Q}},$$

with some $k_1 \in \{0, 1, 2, 3, 4\}$ (and taking $G_0 = 1$) such that $-\mathrm{ord}_{\infty}^{(20)}(G_{k_1}) = b$.

As described in the MC algorithm, we construct a sequence of functions $\mathcal{F} = \{f, f_1, f_2, ...\}$, each of which has a pole only at infinity, with $m_j := |\mathrm{ord}_{\infty}^{(20)}(f_j)|$, and $m_{j+1} < m_j$ for all $j \geq 0$. Membership is excluded if and only if within this sequence a function is produced with order exactly -1 at ∞. If we can prove that such a function can never be produced, then our sequence of functions must ultimately have order 0 at ∞, and membership is guaranteed.

Let us suppose that such a function does exist in our sequence, i.e., for some $M \in \mathbb{Z}_{\geq 0}$, $f_M \in \mathcal{F}$ has a pole only at ∞ and with order exactly -1. In that case, $(f_M)^n$ will have order $-n$ for all $n \in \mathbb{Z}_{>0}$. In other words, we can produce a function in $\mathcal{M}^{\infty}(\Gamma_0(20))$ with a pole only at ∞, and any order at that pole.

However, each of the functions of $\mathcal{M}^{\infty}(\Gamma_0(20))$ give rise to a function of the modular curve $\mathrm{X}_0(20)$ with a pole only at $[\infty]_{20}$, as we discussed in Section 2.4. This curve has genus 1, as can be demonstrated with (2.28), and Theorem 2.27 therefore requires that exactly one order must exist which cannot be assumed by any function over $\mathrm{X}_0(20)$ with a pole only at $[\infty]_{20}$.

But we just demonstrated that f_M taken to positive powers may assume any order at ∞, and that we can therefore construct functions over $\mathrm{X}_0(20)$ with a single pole of any order. We have a contradiction, and must therefore reject the hypothesis that such an f_M is ever produced.

Because this is the only possible case in which membership fails, we must conclude that we can complete our reduction of f, so that

$$f \in \langle 1, G_1, G_2, G_3, G_4 \rangle_{\mathbb{Q}[T]} .$$

We then have

$$\mathcal{M}^{\infty}\left(\Gamma_0(20)\right)_{\mathbb{Q}} \subseteq \langle 1, G_1, G_2, G_3, G_4 \rangle_{\mathbb{Q}[T]} , \tag{6.42}$$

which, with (6.41), yields equality. □

Corollary 6.5.

$$\mathcal{M}^{\infty}\left(\Gamma_0(20)\right)_{\mathbb{Q}} = \langle \mathcal{E}^{\infty}(20) \rangle_{\mathbb{Q}}$$

Proof.

$$\mathcal{M}^{\infty}\left(\Gamma_0(20)\right)_{\mathbb{Q}} = \langle 1, G_1, G_2, G_3, G_4 \rangle_{\mathbb{Q}[T]} \subseteq \langle \mathcal{E}^{\infty}(20) \rangle_{\mathbb{Q}} \subseteq \mathcal{M}^{\infty}\left(\Gamma_0(20)\right)_{\mathbb{Q}} .$$

□

Finally, we give the order of T at its poles and zeros through (2.41):

$$\begin{aligned}
\mathrm{ord}_{1/20}^{(20)}(T) &= -5, \\
\mathrm{ord}_{1/10}^{(20)}(T) &= 1, \\
\mathrm{ord}_{1/5}^{(20)}(T) &= 0, \\
\mathrm{ord}_{1/4}^{(20)}(T) &= 1, \\
\mathrm{ord}_{1/2}^{(20)}(T) &= 1, \\
\mathrm{ord}_{1}^{(20)}(T) &= 2.
\end{aligned}$$

6.2.2 Powers of T

We now give an outline for how to compute the powers $m \in \mathbb{Z}_{>0}$ so that

$$T^m U_5\left(A^i T^j G_k\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right).$$

It is clear, by our definition of T, that such a power must exist.

To begin, let us suppose that $h_1, h_2 \in \mathcal{M}\left(\Gamma_0(100)\right)$, and that $U_5\left(h_1\right), U_5\left(h_2\right)$ have poles which are canceled by the zeros of T. In this case, nonnegative integers m_1, m_2 must exist such that

$$T(\tau)^{m_1} \cdot U_5\left(h_1\right) = U_5\left(T(5\tau)^{m_1} h_1(\tau)\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right),$$
$$T(\tau)^{m_2} \cdot U_5\left(h_2\right) = U_5\left(T(5\tau)^{m_2} h_2(\tau)\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right).$$

Given $i \in \{1, 2\}$, one way of ensuring that $U_5\left(T(5\tau)^{m_i} h_i(\tau)\right)$ has no poles over $\Gamma_0(20)$ other than at $1/20$ is by ensuring that $T(5\tau)^{m_i} h_i(\tau)$ has no poles over $\Gamma_0(100)$ other than those which will manifest in $\Gamma_0(20)$ at $1/20$. That is, we need to ensure that $T(5\tau)^{\xi_i} h_i(\tau) \in \mathcal{M}^{\infty}\left(\Gamma_0(100)\right)$. Any cusp of $\Gamma_0(100)$ other than $1/100$ can be approached by $(\tau + r)/5$ as τ approaches a cusp not represented by $1/20$ (see Table D.1).

In this case, if we wish to examine $U_5\left(h_1 \cdot h_2\right)$, we may note that

$$\begin{aligned} T(\tau)^{m_1+m_2} \cdot U_5\left(h_1 \cdot h_2\right) &= U_5\left(T(5\tau)^{m_1+m_2} h_1(\tau) \cdot h_2(\tau)\right) \\ &= U_5\left(T(5\tau)^{m_1} h_1(\tau) \cdot T(5\tau)^{m_2} h_2(\tau)\right). \end{aligned}$$

Therefore, if $T(5\tau)^{m_1} h_1(\tau)$ and $T(5\tau)^{m_2} h_2(\tau)$ are both members of $\mathcal{M}^{\infty}\left(\Gamma_0(100)\right)$, then their product must be as well. But this means that

$$T(\tau)^{m_1+m_2} \cdot U_5\left(h_1 \cdot h_2\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right).$$

Therefore, if we have sufficient powers of T to push two functions $U_5(h_1), U_5(h_2)$ into $\mathcal{M}^{\infty}\left(\Gamma_0(20)\right)$, then we need only add the powers together to have a sufficient power of T to push $U_5(h_1 \cdot h_2)$ into $\mathcal{M}^{\infty}\left(\Gamma_0(20)\right)$.

So in order to work out sufficient powers of T to push $U_5\left(A^i T^j G_k\right)$ into $\mathcal{M}^{\infty}\left(\Gamma_0(20)\right)$, it is necessary only to know the sufficient powers of T for

$$U_5\left(A\right),\ U_5\left(T\right),\ U_5\left(T^{-1}\right),\ U_5\left(G_k\right),\ 1 \leq k \leq 4.$$

Let us suppose that the most optimal powers of T for this purpose are

$$m_A,\ m_{+t},\ m_{-t},\ m_k,\ 1 \leq k \leq 4,$$

respectively. In that case, each of these powers will correspond to the highest-order pole of the corresponding function over $\Gamma_0(100)$ (excluding the cusp at ∞, of course).

Notice that $G_3 = G^2$, $G_4 = G_2^2$, and $G_2 = H - G$. Therefore, the orders for $U_5(G_3), U_5(G_4)$, respectively, will simply be double the orders of $U_5(G), U_5(G_2)$, respectively. So we need only examine the orders of G, G_2. Also, we know that $G_2 = H - G$, so that we need to examine the orders of G and H. That is, we can compute m_2, m_3, m_4 using only the necessary powers for

$$U_5\left(G\right),\ U_5\left(H\right).$$

Let us refer to the necessary power for $U_5\left(H\right)$ as m_H. Then we have $m_2 = \max\{m_1, m_H\}$, $m_3 = 2m_1$, $m_4 = 2m_2$.

Finally, supposing that $m_t = m_{\mathrm{sign}(j)t}$, then for $U_5\left(A^iT^jG_k\right)$, we have

$$m(i,j,k) = i \cdot m_A + j \cdot m_t + m_k. \tag{6.43}$$

Powers for A, T, T^{-1}, G, H

We begin with m_A as our principal example. We know that for m_A sufficiently large, we have

$$T(\tau)^{m_A} \cdot U_5\left(A(\tau)\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right).$$

But notice that we can rewrite

$$T(\tau)^{m_A} \cdot U_5\left(A(\tau)\right) = U_5\left(T(5\tau)^{m_A}A(\tau)\right).$$

As covered in the beginning of the section, we need to ensure that $T(5\tau)^{m_A}A(\tau)$ only has a pole at the cusp represented by $1/100$.

Of course,

$$\mathrm{ord}_{a/c}^{(100)}\left(T(5\tau)^{m_A}\right) = m_A \cdot \mathrm{ord}_{a/c}^{(100)}\left(T(5\tau)\right).$$

With this in mind, in Table D.3 we examine the order of

$$T(5\tau)^{m_A}A(\tau),\ T(5\tau)^{m_{+t}}T(\tau),\ T(5\tau)^{m_{-t}}T(\tau)^{-1}$$

at the cusps over $\Gamma_0(100)$ using Theorem 2.41 once more. In Table D.4 we examine the orders of

$$T(5\tau)^{m_1}G(\tau),\ T(5\tau)^{m_H}H(\tau).$$

As before, we find possible poles at $1/100, 1/50, 1/4, 1$. We of course do not worry about the pole at $1/100$. For the cusps at $1/50, 1/4$, we only need $m_A \geq 1$. Finally, for the cusp at 1, we need $m_A \geq 2$. This gives us our best possible value: $m_A = 2$.

Similarly, we have $m_{+t} = 5$, $m_{-t} = 5$, and therefore $m_t = 5$.

We also have $m_1 = 2$, $m_H = 3$. Acknowledging that $m_2 = \max\{m_1, m_H\} = 3$, we finally have

$$m_2 = 3,\ m_3 = 4,\ m_4 = 6.$$

Complete Formula

Putting everything together, we now have the following formula:

Theorem 6.6. *For any* $(i, j, k) \in \{0, 1\} \times \mathbb{Z} \times \{0, 1, 2, 3, 4\}$*, we have*

$$T^{m(i,j,k)} U_5 \left(A^i T^j G_k \right) \in \mathcal{M}^{\infty} \left(\Gamma_0(20) \right),$$

with

$$m(i, j, k) = 2 \cdot i + 5 \cdot |j| + m_k.$$

and $m_1 = 2,\ m_2 = 3,\ m_3 = 4,\ m_4 = 6.$

6.2.3 The Andrews–Sellers Congruences

We briefly note that a similar technique may be brought to bear on the Andrews–Sellers congruence family [82]. Substantial direct evidence for its validity was not gathered before 2001, when Eichhorn and Sellers proved the first four cases. Their approach relied on recurrences given by a modular equation, and the total necessary calculations took place in 147 hours with a 600 MHz Pentium III Processor [37, Section 3]. Our approach allows us to check the first five cases with a 2.6 GHz Intel Processor in 1 hour, 45 minutes.

In this case we also work over $\Gamma_0(20)$ with an identical algebra basis to that constructed above.

6.3 A More General Algorithm

With these examples, we can now formulate a more general approach to our problems. This is by no means comprehensive, but serves rather as a guide for how families of congruences can be studied from a large class of generating functions.

We now fix an integer $M \in \mathbb{Z}_{>0}$, and an integer-valued vector $r = (r_\delta)_{\delta|M}$ indexed over the divisors of M. From this, we can define an arithmetic sequence with the generating function

$$\mathcal{G}(q) := \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n=0}^{\infty} a(n) q^n. \tag{6.44}$$

Let us take a prime $\ell > 3$. For simplicity, we will also take the assumption that

$$0 \leq -\sum_{\delta|M} \delta r_\delta \leq \frac{24}{\ell + 1}.$$

From here, define

$$A(q) := q^{(1-\ell^2)\sum_{\delta|M}\delta r_\delta/24}\frac{\mathcal{G}(q)}{\mathcal{G}(q^{\ell^2})}.$$

Let us set $N = \ell \cdot M$. In this case, $A(q)$ satisfies the conditions of Theorem 2.40, and

$$A(q) \in \mathcal{M}\left(\Gamma_0(\ell^2 \cdot M)\right) \supseteq \mathcal{M}\left(\Gamma_0(\ell \cdot M)\right).$$

Next, we make the assumption that the Newman–Radu condition applies:

$$\mathcal{M}\left(\Gamma_0(N)\right)_{\mathbb{Q}} = \langle \mathcal{E}(N) \rangle_{\mathbb{Q}}. \tag{6.45}$$

Very likely, our method may be extended to include modular curves in which some of the assumptions above fail. For the time being, we will take them as true.

From here, we define the operators

$$U^{(\alpha)} : \mathcal{M}\left(\Gamma_0(\ell \cdot M)\right) \to \mathcal{M}\left(\Gamma_0(N)\right),\ \alpha \in \mathbb{Z}_{\geq 0}$$

by

$$\begin{aligned} U^{(0)}(f) &:= U_\ell(A \cdot f), \\ U^{(1)}(f) &:= U_\ell(f), \\ U^{(\alpha)}(f) &:= U^{(\alpha \bmod 2)}(f), \end{aligned}$$

for $\alpha \geq 2$. If we also define

$$\Phi_{2\alpha-1} := \frac{q}{\mathcal{G}(q^\ell)},\ \text{and}\ \Phi_{2\alpha} := \frac{q}{\mathcal{G}(q)},$$

and set

$$L_0 := 1,$$

then we define a sequence of functions $\mathcal{L} = (L_\alpha)_{\alpha \geq 0}$ in which

$$L_{\alpha+1} = U^{(\alpha)}(L_\alpha),\ \text{and}$$

$$L_\alpha = \Phi_\alpha \cdot \sum_{n \in C_{\ell,\alpha}} a(n) q^{\lfloor n/\ell^\alpha \rfloor},$$

with $C_{\ell,\alpha}$ a set of arithmetic progressions, with bases of the form ℓ^α. In particular, for $\alpha = 1$, we have

$$C_{\ell,1} = \left\{ \ell \cdot n + \ell + \frac{(\ell^2 - 1)}{24} \sum_{\delta | M} \delta r_\delta : n \in \mathbb{Z}_{\geq 0} \right\}.$$

Suppose we suspect a family of congruences for $C_{\ell,\alpha}$. That is, we believe that $\mathcal{L}$ is ℓ-adically convergent to 0, and that we have a suspected pattern to the convergence.

From here, we define $\mathcal{C}(N)$ as a complete set of representatives for the cusps of $\Gamma_0(N)$, and similarly for $\mathcal{C}(\ell \cdot N)$. We now must construct an appropriate algebra basis,

$$\langle \mathcal{E}^\infty(N) \rangle_{\mathbb{Q}} = \langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t]},$$

such that

$$U_\ell \left(A^i t^j g_k \right) \in \langle 1, g_1, g_2, ..., g_v \rangle_{\mathbb{Q}[t,t^{-1}]},$$

for all $(i, j, k) \in \{0, 1\} \times \mathbb{Z} \times \{0, 1, ..., v\}$.

We begin with the derivation of t. As in the case of $\Gamma_0(20)$, we can give a system of equations and inequalities by which such a t can be derived. Let

$$t = \prod_{\delta | N} \eta(\delta\tau)^{w_\delta},$$

with $w := (w_\delta)_{\delta | N}$ an integer-valued vector. We begin again with Theorem 2.40:

$$\sum_{\delta | N} w_\delta = 0,$$

$$\sum_{\delta | N} \delta w_\delta + 24 x_1 = 0,$$

$$\sum_{\delta | N} \frac{N}{\delta} w_\delta + 24 x_2 = 0,$$

$$\prod_{\delta | N} \delta^{|w_\delta|} = x_3^2,$$

with $x_1, x_2, x_3 \in \mathbb{Z}$.

We now consider the poles of A. Define $\mathcal{P}_{\ell \cdot N}(A)$ as a set of representatives of cusps in $\mathcal{C}(\ell \cdot N)$ for which A possesses a pole. Then define $\mathcal{P}(A)$ as

$$\mathcal{P}(A) := \left\{ \frac{a}{c} \in \mathcal{C}(N) : \frac{a+cr}{c \cdot \ell} \in \Gamma_0(N)\frac{a'}{c'}, \text{ for some } \frac{a'}{c'} \in \mathcal{P}_{\ell \cdot N}(A), \; r \in \{0, 1, ..., \ell - 1\} \right\}.$$

We add to our system the inequalities

$$\frac{N}{24\mathrm{gcd}(c^2, N)} \sum_{\delta | N} \frac{\mathrm{gcd}(c, \delta)^2}{\delta} w_\delta > 0, \text{ for all } a/c \in \mathcal{P}(A).$$

Now we consider the poles of t, g_k, $1 \leq k \leq v$. Over $\Gamma_0(N)$, they only have a pole at $\frac{1}{N}$. Over $\Gamma_0(\ell \cdot N)$, however, t, g_k will have possible poles for any cusp represented by $\frac{a'}{N}$, $\gcd(a, N) = 1$.

Let $\mathcal{P}(g)$ be defined as

$$\mathcal{P}(g) := \left\{ \frac{a}{c} \in \mathcal{C}(N) : \frac{a+cr}{c \cdot \ell} = \frac{a'}{N}, \; \gcd(a', N) = 1, \text{for some } r \in \{0, 1, ..., \ell - 1\} \right\}.$$

We now have the additional set of inequalities

$$\frac{N}{24\mathrm{gcd}(c^2, N)} \sum_{\delta | N} \frac{\mathrm{gcd}(c, \delta)^2}{\delta} w_\delta > 0, \text{ for all } \frac{a}{c} \in \mathcal{P}(g).$$

Finally, we examine t^{-1}. Let

$$\mathcal{P}' = \mathcal{C}(N) \backslash (\mathcal{P}(A) \cup \mathcal{P}(g)),$$

and let $\frac{a}{c} \in \mathcal{P}'$. Consider

$$\mathcal{P}_{a,c} := \left\{ \frac{a'}{c'} \in \mathcal{C}(N) : \frac{a+cr}{c \cdot \ell} \in \Gamma_0(N)\frac{a'}{c'} \text{ for some } r \in \{0, 1, ..., \ell - 1\} \right\},$$

and define

$$\mathcal{P}'_0 := \left\{ \frac{a}{c} \in P' : \mathcal{P}_{a,c} \subseteq \mathcal{P}(A) \cup \mathcal{P}(g) \cup \left\{ \frac{a}{c} \right\} \right\},$$
$$\mathcal{P}'_1 := \mathcal{P}' \backslash \mathcal{P}'_0.$$

If $\frac{a}{c} \in \mathcal{P}_0'$, then we need establish no condition beyond

$$\frac{N}{24\gcd(c^2,N)}\sum_{\delta|N}\frac{\gcd(c,\delta)^2}{\delta}w_\delta \geq 0.$$

If $\frac{a}{c} \in \mathcal{P}_1'$, then it is not easy to determine at which cusps the order should be set to be 0 in order to give an optimal solution. Alternatively, we may simply set

$$\frac{N}{24\gcd(c^2,N)}\sum_{\delta|N}\frac{\gcd(c,\delta)^2}{\delta}w_\delta = 0.$$

This is not perfectly optimal, but gives us a complete set of equations and inequalities:

To summarize, we let $\mathcal{C}(N)$ be a complete set of representatives for the cusps of $\Gamma_0(N)$, and likewise for $\mathcal{C}(\ell \cdot N)$. Let $\mathcal{P}_{\ell\cdot N}(A) \subseteq \mathcal{C}(\ell\cdot N)$ be the set of representatives of cusps for which A possesses a pole. Let

$$\begin{aligned}
\mathcal{P}(A) &= \left\{\frac{a}{c} \in \mathcal{C}(N) : \frac{a+cr}{c\cdot\ell} \in \Gamma_0(N)\frac{a'}{c'},\ \text{for some}\ \frac{a'}{c'} \in \mathcal{P}_{\ell\cdot N}(A),\ r \in \{0,1,...,\ell-1\}\right\},\\
\mathcal{P}(g) &= \left\{\frac{a}{c} \in \mathcal{C}(N) : \frac{a+cr}{c\cdot\ell} = \frac{a'}{N},\ \gcd(a',N)=1, \text{for some}\ r \in \{0,1,...,\ell-1\}\right\},\\
\mathcal{P}' &= \mathcal{C}(N)\backslash(\mathcal{P}(A)\cup\mathcal{P}(g)),\\
\mathcal{P}_0' &= \left\{\frac{a}{c} \in P' : \mathcal{P}_{a,c} \subseteq \mathcal{P}(A)\cup\mathcal{P}(g)\cup\left\{\frac{a}{c}\right\}\right\},\\
\mathcal{P}_1' &= \mathcal{P}'\backslash\mathcal{P}_0'.
\end{aligned}$$

For some $n_0 \in \mathbb{Z}_{>0}$, define the system $W(n_0)$ by:

$$
\begin{aligned}
W(n_0):& \\
&\sum_{\delta|N} w_\delta = 0,\\
&\sum_{\delta|N} \delta w_\delta + 24x_1 = 0,\\
&\sum_{\delta|N} \frac{N}{\delta} w_\delta + 24x_2 = 0,\\
&\prod_{\delta|N} \delta^{|w_\delta|} = x_3^2,\\
&\frac{N}{24\mathrm{gcd}(c^2,N)} \sum_{\delta|N} \frac{\mathrm{gcd}(c,\delta)^2}{\delta} w_\delta > 0 \text{ for all } \frac{a}{c} \in \mathcal{P}(A),\\
&\frac{N}{24\mathrm{gcd}(c^2,N)} \sum_{\delta|N} \frac{\mathrm{gcd}(c,\delta)^2}{\delta} w_\delta > 0 \text{ for all } \frac{a}{c} \in \mathcal{P}(g),\\
&\frac{N}{24\mathrm{gcd}(c^2,N)} \sum_{\delta|N} \frac{\mathrm{gcd}(c,\delta)^2}{\delta} w_\delta \geq 0 \text{ for all } \frac{a}{c} \in \mathcal{P}_0',\\
&\frac{N}{24\mathrm{gcd}(c^2,N)} \sum_{\delta|N} \frac{\mathrm{gcd}(c,\delta)^2}{\delta} w_\delta = 0 \text{ for all } \frac{a}{c} \in \mathcal{P}_1',\\
&x_1 = n_0.
\end{aligned}
$$

We first attempt to solve $W(1)$, i.e., to find a satisfactory t such that $\mathrm{ord}_{1/20}^{(20)}\{t\} = -1$. If no solution exists, we take $n_0 = n_0 + 1$ and repeat, until a solution is found. This minimizes the order of t at ∞ with respect to our system.

With t defined, we use the basis algorithm of [91, Section 2, Algorithm AB] to produce the complete basis for $\langle \mathcal{E}^\infty(N)\rangle_{\mathbb{Q}}$, which finally yields

$$\mathcal{M}^\infty\left(\Gamma_0(N)\right) = \langle \mathcal{E}^\infty(N)\rangle_{\mathbb{Q}} = \langle 1, g_1, ..., g_v\rangle_{\mathbb{Q}[t]}.$$

We now use Theorem 2.41 to compute

$$\mathrm{ord}_{a/c}^{(\ell\cdot N)}\left(t\left(\ell\cdot\tau\right)^{m_f}\left(f(\tau)\right)\right) = m_f \cdot \mathrm{ord}_{a/c}^{(\ell\cdot N)}\left(t\left(\ell\cdot\tau\right)\right) + \mathrm{ord}_{a/c}^{(\ell\cdot N)}\left(\left(f(\tau)\right)\right),$$

with

$$f \in \left\{A, t, t^{-1}, g_1, ..., g_v\right\}.$$

We now compute the minimal m_f such that

$$m_f \cdot \mathrm{ord}_{a/c}^{(\ell\cdot N)}\left(t\left(\ell\cdot\tau\right)\right) + \mathrm{ord}_{a/c}^{(\ell\cdot N)}\left(\left(f(\tau)\right)\right) \geq 0.$$

Finally, we define

$$m_0 := m_0(j) = \begin{cases} m_t, & j>0 \\ -m_{t^{-1}}, & j<0 \end{cases},$$
$$m_k := m_{g_k},\ 1 \leq k \leq v.$$

This gives us the following:

$$t^{m(i,j,k)} \cdot U_\ell\left(A^i t^j g_k\right) \in \left\langle \mathcal{E}^\infty(N)\right\rangle_{\mathbb{Z}},$$

with

$$m(i,j,k) := i \cdot m_A + j \cdot m_0(j) + m_k.$$

Now for any $B \in \mathbb{Z}_{>0}$, define

$$\begin{aligned} L_0^{(B)} &:= 1, \\ L_1^{(B)} &:= \sum_{\substack{j\in\mathbb{Z},\\ 0\leq k\leq v}} c_{1,j,k} t^j g_k, \\ L_\alpha^{(B)} &:= U^{(\alpha-1)}\left(L_{\alpha-1}^{(B)}\right) = \sum_{\substack{j\in\mathbb{Z},\\ 0\leq k\leq v}} c_{\alpha,j,k} t^j g_k, \end{aligned}$$

where $c_{\alpha,j,k}$ is the coefficient of $t^j g_k$, reduced modulo ℓ^B.

We now give the steps for examining L_α for $0 \leq \alpha \leq B$ for possible divisibility by powers of ℓ (up to ℓ^B):

1. Begin with $\alpha = 0$, $v_0 = 0$, and $V = \{v_0\}$.
2. Expand $L_\alpha^{(B)}$ into $\langle 1, g_1, ..., g_v\rangle_{\mathbb{Z}[t,t^{-1}]}$: $L_\alpha^{(B)} = \sum_{\substack{j\in\mathbb{Z},\\ 0\leq k\leq v}} c_{\alpha,j,k} t^j g_k$.
3. Compute $U^{(\alpha)}\left(L_\alpha^{(B)}\right) = \sum_{\substack{j\in\mathbb{Z},\\ 0\leq k\leq v}} c_{\alpha,j,k} U^{(\alpha)}\{t^j g_k\}$.

4. Reduce $U^{(\alpha)}\left(L_\alpha^{(B)}\right) \pmod{\ell^B}$ to get $L_{\alpha+1}^{(B)} = \sum_{\substack{j\in\mathbb{Z},\\ 0\leq k\leq v}} c_{\alpha+1,j,k} t^j g_k$.
5. Let $v_{\alpha+1}$ be the maximal power of ℓ that divides each $c_{\alpha+1,j,k}$.
6. Set $V = V \cup \{v_{\alpha+1}\}$.
7. Set $\alpha = \alpha + 1$, and repeat.
8. Continue until $\alpha = B$.
9. The v_α will give the largest possible power of ℓ that divides L_α. We may either formulate a possible pattern, or check one already conjectured, for $0 \leq \alpha \leq B$.

CHAPTER 7
AN INFINITE FAMILY OF CONGRUENCES (III)

The first section of this chapter is based on a collaboration with Cristian-Silviu Radu in [94]. The remainder is based on work published in [103].

7.1 Failure of Piecewise Convergence

The previous chapter described a method by which we can examine a large variety of possible congruences for a class of arithmetic functions. In particular, it gave us enough to give a precise statement of the congruence family that was suggested to exist by Choi, Kim, and Lovejoy for $A_1(n)$. This method may of course be used to check as many specific cases of our congruence family as we please.

It is very tempting to believe that, given a conjectured family of congruences, a complete proof should be possible using the same techniques. As we will show in the final three chapters, the intuition behind this approach *can* be used to formulate complete proofs of a large variety of congruence families. However, with respect to the congruence family of Theorem 6.3, the situation is more complicated.

The traditional means of actually proving the existence of a given infinite family of partition congruences, with respect to powers of a prime ℓ, is the notion of ℓ-adic convergence for a sequence of generating functions for each given case. In our case, we have

$$L_0 = 1,$$
$$L_\alpha = \Phi_\alpha \cdot \sum_{24n\equiv 1 \bmod 5^\alpha} a(n) q^{\lfloor n/5^\alpha \rfloor}.$$

To prove that $a(n) \equiv 0 \pmod{5^\alpha}$ whenever $24n \equiv 1 \pmod{5^{2\alpha}}$, we need to prove the following:

Theorem 7.1. *The sequence $(L_\alpha)_{\alpha\geq 0}$ is 5-adically convergent to 0: For every $M \in \mathbb{Z}_{>0}$ there exists an $A \in \mathbb{Z}_{>0}$ such that for all $\alpha \geq A$,*

$$L_\alpha \equiv 0 \pmod{5^M}.$$

In particular, $A = \lfloor M/2 \rfloor$ will suffice.

This is done by very carefully constructing subspaces S_α of modular functions over $\Gamma_0(20)$, so that $L_\alpha \in S_\alpha$ for all $\alpha \geq 1$. Moreover, we need to construct the generators of S_α so that $U^{(\alpha)}(p_\alpha) \in S_{\alpha+1}$ for any $p_\alpha \in S_\alpha$.

For this to work, the generators of S_α must be very carefully selected, so that successive application of $U^{(\alpha)}$ will generate functions divisible by increasing powers of 5. While the algebra basis that we have used in the previous chapter is very powerful, it does not necessarily select functions t with the property that the sequence

$$\left(t,\ U^{(1)}\left(t\right),\ U^{(0)}\left(U^{(1)}\left(t\right)\right),...\right)$$

converges 5-adically to 0. Were we working with $\mathcal{M}\left(\Gamma_0(N)\right)$ with, say, $N = 5, 7, 10, 11$, this would not be a problem. However, we find that we may not be so careless for $N = 20$.

As an example, we take T as in the previous chapter:

$$T = \frac{\eta(\tau)^2\eta(4\tau)^2\eta(10\tau)^8}{\eta(5\tau)^2\eta(20\tau)^{10}} = \frac{1}{q^5}\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^8}{(q^5;q^5)_\infty^2(q^{20};q^{20})_\infty^{10}}.$$

Suppose we define $T_1 = U^{(1)}\left(T\right), T_\alpha = U^{(\alpha)}\left(T_{\alpha-1}\right)$, for $\alpha \geq 1$.

$$\begin{aligned}
T_1 &\equiv 4\frac{1}{T} + 2G_1\frac{1}{T} + G_3\frac{1}{T} + G_2\frac{1}{T} \pmod{5},\\
T_2 &\equiv 3G_1\frac{1}{T} + 2G_3\frac{1}{T} \pmod{5},\\
T_3 &\equiv 3G_1\frac{1}{T} + 2G_3\frac{1}{T} \pmod{5},\\
T_4 &\equiv 3\frac{1}{T} + 4G_1\frac{1}{T} + 2G_2\frac{1}{T} \pmod{5},\\
T_5 &\equiv 4G_1\frac{1}{T} + G_3\frac{1}{T} \pmod{5},\\
T_6 &\equiv 4\frac{1}{T} + 2G_1\frac{1}{T} + G_2\frac{1}{T} \pmod{5}, ...
\end{aligned}$$

Continuing this through to T_{14}, we eventually have

$$\begin{aligned}
T_{11} &\equiv 3G_1\frac{1}{T} + 2G_3\frac{1}{T} \pmod{5},\\
T_{12} &\equiv 3\frac{1}{T} + 4G_1\frac{1}{T} + 2G_2\frac{1}{T} \pmod{5},\\
T_{13} &\equiv 4G_1\frac{1}{T} + G_3\frac{1}{T} \pmod{5},\\
T_{14} &\equiv 4\frac{1}{T} + 2G_1\frac{1}{T} + G_2\frac{1}{T} \pmod{5}, ...
\end{aligned}$$

Notice the repetition: $T_{11} \equiv T_3 \pmod{5}$, $T_{12} \equiv T_4 \pmod{5}$, $T_{13} \equiv T_5 \pmod{5}$, and so on. This sequence settles into a repeated pattern modulo 5, so it can never become $0 \pmod{5}$, no matter how often we apply $U^{(\alpha)}$. In other words, the sequence $(T_\alpha)_{\alpha\geq 1}$ will not converge to 0 in the 5-adic sense.

This is the problem of the failure of piecewise ℓ-adic convergence discussed in Chapter 5. It is this complication which made the the Andrews–Sellers congruence family so resistant to proof, and it complicates the proof of Theorem 6.3 in a similar manner.

A good analogy can be found with the question of convergence in the standard topology. Suppose we have a sequence of functions

$$(L_\alpha)_{\alpha\geq 0},$$

and we suspect that

$$\lim_{\alpha\to\infty} L_\alpha = 0.$$

One way of proving this convergence is to find other sequences of functions, e.g. $(F_\alpha)_{\alpha\geq 0}, (G_\alpha)_{\alpha\geq 0}$ which can be put together in a given way to produce each L_α, e.g.,

$$L_\alpha = F_\alpha + G_\alpha.$$

Now, to prove that $\lim_{\alpha\to\infty} L_\alpha = 0$, it is certainly sufficient to prove that

$$\lim_{\alpha\to\infty} F_\alpha = \lim_{\alpha\to\infty} G_\alpha = 0.$$

However, it is not necessary at all—for instance, we might have

$$\lim_{\alpha\to\infty} F_\alpha = 1,\ \lim_{\alpha\to\infty} G_\alpha = -1.$$

If we want to prove convergence of L_α term-wise, it is clear that we need to carefully select our summands. A similar principle holds, if we replace the notion of convergence in the standard topology with that of 5-adic topology, with our sequences of modular functions $(L_\alpha)_{\alpha\geq 0}$.

7.2 Proof Setup

To overcome this problem, we need to build up the spaces S_α very carefully indeed. At present, much of the proof method depends on educated guesses as to the precise structures necessary. We know that $\mathfrak{g}\left(\mathrm{X}_0(20)\right) = 1$, so that we may count on manipulation of some free rank-2 $\mathbb{Z}[X]$-module.

Recall that $\mathcal{M}\left(\Gamma_0(5)\right) = \mathbb{C}[t]$ for $t = \eta(5\tau)^6/\eta(\tau)^6$. Given that proving Ramanujan's classic congruences for powers of 5 takes place in this space, we know that t is 5-adically convergent under repeated applications of the U_5 operator. Moreover, of course, $t \in \mathcal{M}\left(\Gamma_0(20)\right)$. So we elect to attempt a representation for each L_α in terms of t to as large an extent as possible.

7.2.1 $X^{(j)}$

We will now construct the spaces of modular functions in $\mathcal{M}(\Gamma_0(20))$ that are necessary for our purposes.

Let $q = e^{2\pi i\tau}$, with $\tau \in \mathbb{H}$, and define

$$t = \frac{\eta(5\tau)^6}{\eta(\tau)^6} = q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6}, \tag{7.1}$$

$$\rho = \frac{\eta(\tau)^2\eta(4\tau)^2\eta(10\tau)^8}{\eta(5\tau)^2\eta(20\tau)^{10}} = \frac{1}{q^5}\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^8}{(q^5;q^5)_\infty^2(q^{20};q^{20})_\infty^{10}} \tag{7.2}$$

$$\sigma = \frac{\eta(4\tau)^4\eta(10\tau)^2}{\eta(2\tau)^2\eta(20\tau)^4} = \frac{1}{q^2}\frac{(q^4;q^4)_\infty^4(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty^4}, \tag{7.3}$$

$$\mu = \frac{\eta(4\tau)\eta(5\tau)^5}{\eta(\tau)\eta(20\tau)^5} = \frac{1}{q^3}\frac{(q^4;q^4)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{20};q^{20})_\infty^5}. \tag{7.4}$$

One can verify that $L_1 \notin \mathbb{Z}[t]$. Indeed, we have the following:

$$\begin{aligned} L_1 =& 261\rho^{-1} + 126\sigma\rho^{-1} + 13\sigma^2\rho^{-1} - 960\rho^{-2} - 5120\sigma\rho^{-2} - 320\sigma^2\rho^{-2} \\ &+ 64\rho^{-1}\mu + 320\rho^{-2}\mu - 1280\sigma\rho^{-2}\mu + 640\rho^{-2}\mu^2. \end{aligned}$$

In the sequel we will prove the interesting result that

$$L_{2\alpha-1} \in \mathbb{Z}[t] \oplus L_1 \cdot \mathbb{Z}[t]$$

for all $\alpha \geq 1$. This rank 2 $\mathbb{Z}[t]$-module is therefore of interest to us.

However, $L_{2\alpha}$ is *not* a member of this module. To account for $L_{2\alpha}$ we will construct a *second* rank 2 $\mathbb{Z}[t]$-module, $\mathbb{Z}[t] \oplus p_0\mathbb{Z}[t]$. The exact form of p_0 was determined by experiment, based on studies of $U^{(1)}(L_1)$. It was determined (as is shown in Group IV of Appendix B) that $U^{(1)}(L_1/t) - 5^2t - 13$ yields a function which can nearly be used to represent $U^{(1)}(L_1t^n)$ for all $n \in \mathbb{Z}$. Indeed, we define p_0 as the function which satisfies

$$U^{(1)}(L_1/t) = 13 + 5^2t + 5p_0.$$

This function can be given an explicit form. We define

$$\begin{aligned} p_0 =& 31\rho^{-1} - 22\sigma\rho^{-1} - 9\sigma^2\rho^{-1} - 208\rho^{-2} - 96\sigma\rho^{-2} + 304\sigma^2\rho^{-2} \\ &- 32\rho^{-1}\mu + 416\rho^{-2}\mu + 416\sigma\rho^{-2}\mu - 208\rho^{-2}\mu^2, \end{aligned} \tag{7.5}$$

$$\begin{aligned} p_1 =& 261\rho^{-1} + 126\sigma\rho^{-1} + 13\sigma^2\rho^{-1} - 960\rho^{-2} - 5120\sigma\rho^{-2} - 320\sigma^2\rho^{-2} \\ &+ 64\rho^{-1}\mu + 320\rho^{-2}\mu - 1280\sigma\rho^{-2}\mu + 640\rho^{-2}\mu^2. \end{aligned} \tag{7.6}$$

Of course, $p_1 = L_1$. We also define

$$S_0 = \mathbb{Z}[t] \oplus p_0\mathbb{Z}[t], \tag{7.7}$$
$$S_1 = \mathbb{Z}[t] \oplus p_1\mathbb{Z}[t]. \tag{7.8}$$

That is, for $j = 0, 1$, S_j is the free rank 2 $\mathbb{Z}[t]$-module generated by 1 and p_j.
As the relations in Groups II and IV in the Appendix demonstrate,

$$L_2 = U^{(1)}(L_1) = U^{(1)}(p_1) \in S_0, \text{ and } U^{(0)}(p_0) \in S_1.$$

This, with the linearity of $U^{(j)}$, ensures that that for $\alpha \geq 0$,

$$L_{2\alpha} \in S_0, \tag{7.9}$$
$$L_{2\alpha-1} \in S_1. \tag{7.10}$$

We will work with specific subspaces of S_0, S_1.

Definition 7.2. A function $f : \mathbb{Z} \to \mathbb{Z}$ is discrete if it is nonzero for only finitely many integers. A function $h : \mathbb{Z}^k \to \mathbb{Z}$ is a discrete array if for any fixed $(m_1, m_2, ..., m_{k-1}) \in \mathbb{Z}^{k-1}$, $h(m_1, m_2, ..., m_{k-1}, n)$ is discrete with respect to n.

We now define our relevant subspaces:

$$X^{(0)} = \left\{ \sum_{n=0}^{\infty} r(n) 5^{\left\lfloor \frac{5n}{2} \right\rfloor} p_0 t^n + \sum_{n=1}^{\infty} s(n) 5^{\left\lfloor \frac{5n-3}{2} \right\rfloor} t^n : r, s \text{ discrete functions} \right\}, \tag{7.11}$$

$$X^{(1)} = \left\{ \sum_{n=0}^{\infty} r(n) 5^{\left\lfloor \frac{5n}{2} \right\rfloor} p_1 t^n + \sum_{n=1}^{\infty} s(n) 5^{\left\lfloor \frac{5n-1}{2} \right\rfloor} t^n : r, s \text{ discrete functions} \right\}. \tag{7.12}$$

Notice that for $j = 0, 1$, we have $X^{(j)} \subseteq S_j$. In particular, $L_1 = p_1 \in X^{(1)}$.

7.3 The Modular Equation

We have very carefully chosen the spaces $X^{(j)}$. Rather than working directly with L_n, we will show that $L_n \in X^{(r)}$, with r the residue of n (mod 2). We then study how $U^{(j)}$ changes the structure of an arbitrary $f \in X^{(j)}$.

To do this, we will need to know the effects of $U^{(j)}$ on $p_j t^n, t^n$. Our choice of $t = \eta(5\tau)^6/\eta(\tau)^6$ is especially convenient, as we have a powerful modular equation that can be brought to bear on the problem.

Theorem 7.3. *Let*

$$\begin{aligned}
a_0(\tau) &= -t,\\
a_1(\tau) &= -5^3t^2 - 6\cdot 5t,\\
a_2(\tau) &= -5^6t^3 - 6\cdot 5^4t^2 - 63\cdot 5t,\\
a_3(\tau) &= -5^9t^4 - 6\cdot 5^7t^3 - 63\cdot 5^4t^2 - 52\cdot 5^2t,\\
a_4(\tau) &= -5^{12}t^5 - 6\cdot 5^{10}t^4 - 63\cdot 5^7t^3 - 52\cdot 5^5t^2 - 63\cdot 5^2t.
\end{aligned}$$

Then

$$t(\tau)^5 + \sum_{j=0}^{4} a_j(5\tau)t(\tau)^j = 0. \tag{7.13}$$

A proof of this can be found in [82, Section 3].

The value of this equation becomes immediate when we consider the following theorem:

Lemma 7.4. *For any function $g : \mathbb{H} \to \mathbb{C}$, and any $n \in \mathbb{Z}$, we have*

$$U_5(g\cdot t^n) = -\sum_{j=0}^{4} a_j(\tau)U_5(g\cdot t^{n+j-5}). \tag{7.14}$$

Proof. With equation (7.13), we have

$$g(\tau)\cdot t(\tau)^n = -\sum_{j=0}^{4} a_j(5\tau)\cdot g(\tau)\cdot t(\tau)^{n+j-5}. \tag{7.15}$$

Taking the U_5 operator, and remembering that

$$U_5(a_j(5\tau)\cdot g(\tau)\cdot t(\tau)^{n+j-5}) = a_j(\tau)\cdot U_5(g(\tau)\cdot t(\tau)^{n+j-5}),$$

by Part 3 of Lemma 1, we find that

$$\begin{aligned}
U_5(g\cdot t(\tau)^n) &= -\sum_{j=0}^{4} U_5(a_j(5\tau)\cdot g\cdot t(\tau)^{n+j-5}) &(7.16)\\
&= -\sum_{j=0}^{4} a_j(\tau)\cdot U_5(g\cdot t(\tau)^{n+j-5}). &(7.17)
\end{aligned}$$

□

7.3.1 Lemmas

We now state and prove a key application of the modular equation for t. This lemma is given in the form of two lemmas in [82, Section 4], but we give the proof for the sake of completeness.

Lemma 7.5. *For any functions $g, y_0, y_1 : \mathbb{H} \to \mathbb{C}$, if there exist $u_0, u_1, v_0, v_1 \in \mathbb{Z}$, and discrete arrays $h_0(m,n), h_1(m,n)$ such that*

$$U_5(gt^n) = \sum_{m \geq \lceil \frac{n+u_0}{5} \rceil} h_0(m,n) 5^{\lfloor \frac{5m-n+v_0}{2} \rfloor} y_0 t^m + \sum_{m \geq \lceil \frac{n+u_1}{5} \rceil} h_1(m,n) 5^{\lfloor \frac{5m-n+v_1}{2} \rfloor} y_1 t^m \tag{7.18}$$

for five consecutive integers n, then such a relation holds for every larger integer.

Proof. Suppose that for specific functions $g,\ y_0,\ y_1$, discrete arrays $h_0,\ h_1$, and integers $u_0,\ u_1,\ v_0,\ v_1$, the given relation holds for five consecutive integers:

$$n_0,\ n_0+1,\ n_0+2,\ n_0+3,\ n_0+4.$$

We prove the lemma by induction.

Let $k \geq n_0 + 5$, and assume that the relation holds for all $j \in \mathbb{Z}$ such that $n_0 \leq j \leq k-1$. In particular, the relation holds for $j = k-5,\ k-4,\ ...,\ k-1$. We want to prove that the relation must hold for k. It can be quickly verified from the previous lemma that

$$a_j(\tau) = \sum_{l=1}^{5} s(j,l) 5^{\lfloor \frac{5l+j-4}{2} \rfloor} t^l, \tag{7.19}$$

for some unique function $s : \{0, ..., 4\} \times \{1, ..., 5\} \to \mathbb{Z}$. With this in mind, we have

$$\begin{aligned} U_5(gt^k) &= -\sum_{j=0}^{4} a_j(\tau) U_5(g \cdot t(\tau)^{k+j-5}) &(7.20)\\ &= -\sum_{j=0}^{4} a_j(\tau) \sum_{i=0,1} \sum_{m \geq \lceil \frac{k+j-5+u_i}{5} \rceil} h_i(m, k+j-5) 5^{\left\lfloor \frac{5m-(k+j-5)+v_i}{2} \right\rfloor} y_i t^m &(7.21)\\ &= -\sum_{i=0,1} \sum_{j=0}^{4} a_j(\tau) \sum_{m \geq \lceil \frac{k+u_i}{5} - \frac{5-j}{5} \rceil} h_i(m, k+j-5) 5^{\left\lfloor \frac{5m-(k+j-5)+v_i}{2} \right\rfloor} y_i t^m. &(7.22) \end{aligned}$$

Taking $m_{i,j} = \lceil \frac{k+u_i}{5} - \frac{5-j}{5} \rceil$, we have

$$U_5(gt^k) = -\sum_{\substack{i=0,1,\\0\le j\le 4,\\1\le l\le 5}} s(j,l)5^{\left\lfloor\frac{5l+j-4}{2}\right\rfloor}t^l \sum_{m\ge m_{i,j}} h_i(m,k+j-5)5^{\left\lfloor\frac{5m-(k+j-5)+v_i}{2}\right\rfloor}y_i t^m \tag{7.23}$$

$$= -\sum_{\substack{i=0,1,\\0\le j\le 4,\\1\le l\le 5}} \sum_{m\ge m_{i,j}} s(j,l)h_i(m,k+j-5)5^{\left\lfloor\frac{5m-(k+j-5)+v_i}{2}\right\rfloor+\left\lfloor\frac{5l+j-4}{2}\right\rfloor}y_i t^{m+l}. \tag{7.24}$$

Now, we note that for any $M_1, M_2 \in \mathbb{Z}$, we have $\left\lfloor\frac{M_1}{2}\right\rfloor + \left\lfloor\frac{M_2}{2}\right\rfloor \ge \left\lfloor\frac{M_1+M_2}{2} - \frac{1}{2}\right\rfloor$. Therefore,

$$\left\lfloor\frac{5m-(k+j-5)+v_i}{2}\right\rfloor + \left\lfloor\frac{5l+j-4}{2}\right\rfloor$$
$$\ge \left\lfloor\frac{5m-(k+j-5)+v_i}{2} + \frac{5l+j-4}{2} - \frac{1}{2}\right\rfloor = \left\lfloor\frac{5(m+l)-k+v_i}{2}\right\rfloor. \tag{7.25}$$

Now since $m_{i,j} = \left\lceil\frac{k+u_i}{5} - \frac{5-j}{5}\right\rceil \ge \left\lceil\frac{k+u_i}{5}\right\rceil - 1$, and since $l \ge 1$, we relabel our powers of t so that

$$U_5(gt^k) = -\sum_{\substack{i=0,1,\\0\le j\le 4,\\1\le l\le 5}} \sum_{m\ge\left\lceil\frac{k+u_i}{5}\right\rceil-1+l} s(j,l)h_i(m-l,k+j-5)5^{\left\lfloor\frac{5m-k+v_i}{2}\right\rfloor}y_i t^m. \tag{7.26}$$

Finally, defining the discrete function $H_i(m,k)$ by

$$H_i(m,k) = \begin{cases} -\sum_{j=0}^{4}\sum_{l=1}^{5} s(j,l)h_i(m-l,k+j-5), & m\ge l,\\ 0, & \text{otherwise,}\end{cases}$$

we have

$$U_5(gt^k) = \sum_{m\ge\left\lceil\frac{k+u_0}{5}\right\rceil} H_0(m,k)5^{\left\lfloor\frac{5m-k+v_0}{2}\right\rfloor}y_0 t^m + \sum_{m\ge\left\lceil\frac{k+u_1}{5}\right\rceil} H_1(m,k)5^{\left\lfloor\frac{5m-k+v_1}{2}\right\rfloor}y_1 t^m. \tag{7.27}$$

By induction, we have established the given relation for all $n \ge n_0$.

□

We can use this lemma to define a very useful "skeletal" structure for $U^{(j)}(p_j t^n), U^{(j)}(t^n)$ as follows:

Lemma 7.6. *There exist discrete arrays $a_j(m,n), b_j(m,n), c(m,n), d_j(m,n)$, with $j \in \{0,1\}$, such that for all nonnegative $n \in \mathbb{Z}$,*

$$U^{(0)}(t^n) = \sum_{m \geq \lceil \frac{n+1}{5} \rceil} a_0(m,n) 5^{\lfloor \frac{5m-n-1}{2} \rfloor} t^m + \sum_{m \geq \lceil \frac{n}{5} \rceil} a_1(m,n) 5^{\lfloor \frac{5m-n}{2} \rfloor} p_1 t^m, \tag{7.28}$$

$$U^{(0)}(p_0 t^n) = \sum_{m \geq \lceil \frac{n+2}{5} \rceil} b_0(m,n) 5^{\lfloor \frac{5m-n-1}{2} \rfloor} t^m + \sum_{m \geq \lceil \frac{n}{5} \rceil} b_1(m,n) 5^{\lfloor \frac{5m-n}{2} \rfloor} p_1 t^m, \tag{7.29}$$

$$U^{(1)}(t^n) = \sum_{m \geq \lceil \frac{n}{5} \rceil} c(m,n) 5^{\lfloor \frac{5m-n-1}{2} \rfloor} t^m, \tag{7.30}$$

$$U^{(1)}(p_1 t^n) = \sum_{m \geq \lceil \frac{n+1}{5} \rceil} d_0(m,n) 5^{\lfloor \frac{5m-n-1}{2} \rfloor} t^m + \sum_{m \geq \lceil \frac{n-1}{5} \rceil} d_1(m,n) 5^{\lfloor \frac{5m-n+2}{2} \rfloor} p_0 t^m, \tag{7.31}$$

Notice that we can set $a_0(m,n) = 0$ whenever $m < \lceil (n+1)/5 \rceil$. More generally, for $j = 0, 1$, we can define

$$a_j(m,n) = b_j(m,n) = c(m,n) = d_j(m,n) = 0 \tag{7.32}$$

if the corresponding inequalities for m, n in (7.28), (7.29), (7.30), (7.31) do not hold.

Proof. The previous lemma establishes that if these relations hold for $k-5, k-4, ...k-1$, then they will hold for all $n \geq k$. We therefore need twenty initial relations—relations for five consecutive values, in four categories.

The most obvious choices would be for $0 \leq n \leq 4$. However, these cases are relatively cumbersome to explicitly prove using the modular cusp analysis. For example, the expression for $U^{(0)}(t^4)$ has degree 20, in which the largest coefficient is on the order of 10^{33}. Instead, we take the relations for $-4 \leq n \leq 0$. These are smaller, and easier to prove. We can then use the modular equation to compute the relations for $1 \leq n \leq 4$ and examine the coefficients of each term using a computer to establish that (7.28)-(7.31) apply. □

In the following section, we discuss how to apply the modular cusp analysis to actually prove the cases for $-4 \leq n \leq 0$.

7.4 On the Initial Cases

The experimental work in the previous chapter is now enormously useful for us. Indeed, comparing (7.2)-(7.4) to Theorem 6.4, we have

$$T = \rho,$$
$$H = \mu,$$
$$G = \sigma.$$

As such, we have already established through Theorem 6.4 that $\rho, \mu, \sigma \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right)$. We also established that for $i = 0, 1$ and $j \in \mathbb{Z}$,

$$\rho^{2i+5|j|+2} U^{(i)}\left(\rho^j \sigma\right), \; \rho^{2i+5|j|+3} U^{(i)}\left(\rho^j \mu\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right).$$

Recall the definitions of p_0, p_1, in (7.5) and (7.6):

$$\begin{aligned} p_0 =& 31\rho^{-1} - 22\sigma\rho^{-1} - 9\sigma^2\rho^{-1} - 208\rho^{-2} - 96\sigma\rho^{-2} + 304\sigma^2\rho^{-2} \\ & - 32\rho^{-1}\mu + 416\rho^{-2}\mu + 416\sigma\rho^{-2}\mu - 208\rho^{-2}\mu^2, \\ p_1 =& 261\rho^{-1} + 126\sigma\rho^{-1} + 13\sigma^2\rho^{-1} - 960\rho^{-2} - 5120\sigma\rho^{-2} - 320\sigma^2\rho^{-2} \\ & + 64\rho^{-1}\mu + 320\rho^{-2}\mu - 1280\sigma\rho^{-2}\mu + 640\rho^{-2}\mu^2. \end{aligned}$$

We can also show that $\rho^2 t \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right)$; indeed,

$$\begin{aligned} \rho^2 t =& 1 - 5\sigma + 9\sigma^2 - 7\sigma^3 + 2\sigma^4 + \mu(-1 + 3\sigma - 3\sigma^2 + \sigma^3), \\ t =& \rho^{-2} - 5\rho^{-2}\sigma + 9\rho^{-2}\sigma^2 - 7\rho^{-2}\sigma^3 + 2\rho^{-2}\sigma^4 + \rho^{-2}\mu(-1 + 3\sigma - 3\sigma^2 + \sigma^3). \end{aligned}$$

Using the approach from Chapter 6, we can show that

$$\rho^{15} U^{(0)}\left(p_0 t^n\right), \; \rho^{15} U^{(1)}\left(p_1 t^n\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right)$$

for $-4 \leq n \leq 0$,

$$\rho^{15} U^{(1)}\left(p_1 t^n\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right)$$

for $-3 \leq n \leq 0$, and

$$t\rho^{15} U^{(1)}\left(p_1 t^{-4}\right) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right).$$

Because both sides of each of the twenty relations in Appendix B are members of $\mathcal{M}\left(\Gamma_0(20)\right)$, and a sufficiently large power of ρ can put both sides into $\mathcal{M}^{\infty}\left(\Gamma_0(20)\right)$, verification of each relation is merely a matter of comparing the principal parts at infinity of each side—a finite task that can easily be done by computer.

As an example, we choose the second relation of Group I. We have

$$\rho^2 U^{(0)}(t^{-1}) \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right). \tag{7.33}$$

Since $\rho^2 t \in \mathcal{M}^{\infty}\left(\Gamma_0(20)\right)$,

$$\rho^2(1 + 5^2 t - 5p_1) \in \mathcal{M}^\infty\left(\Gamma_0(20)\right), \tag{7.34}$$

we need only compare the principal parts and the constants of (7.33) and (7.34). We find that both expressions have the identical principal part and constant

$$\begin{aligned}&\frac{1}{q^{10}} - \frac{44}{q^9} - \frac{138}{q^8} - \frac{372}{q^7} - \frac{989}{q^6} - \frac{1584}{q^5} \\ &- \frac{2814}{q^4} - \frac{4356}{q^3} - \frac{5897}{q^2} - \frac{9508}{q} - 12696.\end{aligned} \tag{7.35}$$

As a result, both expressions must be equal:

$$\rho^2 U^{(0)}(t^{-1}) = \rho^2(1 + 5^2 t - 5p_1), \tag{7.36}$$

$$U^{(0)}(t^{-1}) = 1 + 5^2 t - 5p_1. \tag{7.37}$$

CHAPTER 8
AN INFINITE FAMILY OF CONGRUENCES (IV)

This chapter is based on work published in [103].

The “skeletal structure” in the last chapter enables us to conduct a careful study of the behavior of the functions in X^0, X^1 in (7.11)-(7.12) under the application of their respective U operators. The following lemma gives us the critical relationships that reveal the desirable 5-adic convergence which we need to finish the proof of Theorem 7.1.

8.1 The Main Lemma

Theorem 8.1. *If $f \in X^{(0)}$, then $U^{(0)}(f) \in X^{(1)}$. If $f \in X^{(1)}$, then $5^{-1}U^{(1)}(f) \in X^{(0)}$.*

Proof. Let $f \in X^{(0)}$. Then there exist discrete functions r, s such that

$$f = \sum_{n=0}^{\infty} r(n) 5^{\left\lfloor \frac{5n}{2} \right\rfloor} p_0 t^n + \sum_{n=1}^{\infty} s(n) 5^{\left\lfloor \frac{5n-3}{2} \right\rfloor} t^n. \tag{8.1}$$

We take $U^{(0)}(f)$. Using Lemma 7.6, with condition (7.32), we find that

$$\begin{aligned} U^{(0)}(f) &= \sum_{n=0}^{\infty} r(n) 5^{\left\lfloor \frac{5n}{2} \right\rfloor} U^{(0)}(p_0 t^n) + \sum_{n=1}^{\infty} s(n) 5^{\left\lfloor \frac{5n-3}{2} \right\rfloor} U^{(0)}(t^n) & (8.2)\\ &= \sum_{n=0}^{\infty} r(n) 5^{\left\lfloor \frac{5n}{2} \right\rfloor} \Bigg(\sum_{m \geq \left\lceil \frac{n+2}{5} \right\rceil} b_0(m,n) 5^{\left\lfloor \frac{5m-n-1}{2} \right\rfloor} t^m \\ &\qquad + \sum_{m \geq \left\lceil \frac{n}{5} \right\rceil} b_1(m,n) 5^{\left\lfloor \frac{5m-n}{2} \right\rfloor} p_1 t^m \Bigg) \\ &+ \sum_{n=1}^{\infty} s(n) 5^{\left\lfloor \frac{5n-3}{2} \right\rfloor} \Bigg(\sum_{m \geq \left\lceil \frac{n+1}{5} \right\rceil} a_0(m,n) 5^{\left\lfloor \frac{5m-n-1}{2} \right\rfloor} t^m \\ &\qquad + \sum_{m \geq \left\lceil \frac{n}{5} \right\rceil} a_1(m,n) 5^{\left\lfloor \frac{5m-n}{2} \right\rfloor} p_1 t^m \Bigg). & (8.3) \end{aligned}$$

Because $a_j(m,n), b_j(m,n), c(m,n), d_j(m,n)$ satisfy (7.32), we may rearrange our summands such that

$$U^{(0)}(f) = p_1 \sum_{m\geq 0}\sum_{n\geq 0} r(n)b_1(m,n)5^{\left\lfloor \frac{5n}{2}\right\rfloor + \left\lfloor \frac{5m-n}{2}\right\rfloor} t^m \tag{8.4}$$

$$+ p_1 \sum_{m\geq 1}\sum_{n\geq 1} s(n)a_1(m,n)5^{\left\lfloor \frac{5n-3}{2}\right\rfloor + \left\lfloor \frac{5m-n}{2}\right\rfloor} t^m \tag{8.5}$$

$$+ \sum_{m\geq 1}\sum_{n\geq 0} r(n)b_0(m,n)5^{\left\lfloor \frac{5n}{2}\right\rfloor + \left\lfloor \frac{5m-n-1}{2}\right\rfloor} t^m \tag{8.6}$$

$$+ \sum_{m\geq 1}\sum_{n\geq 1} s(n)a_0(m,n)5^{\left\lfloor \frac{5n-3}{2}\right\rfloor + \left\lfloor \frac{5m-n-1}{2}\right\rfloor} t^m. \tag{8.7}$$

Now, we simplify the powers of 5 in each double sum. For line (8.4), with $m, n \geq 0$, we have

$$\left\lfloor \frac{5n}{2}\right\rfloor + \left\lfloor \frac{5m-n}{2}\right\rfloor = \left\lfloor \frac{3n}{2}\right\rfloor + \left\lfloor \frac{5m+n}{2}\right\rfloor \geq \left\lfloor \frac{5m}{2}\right\rfloor. \tag{8.8}$$

For (8.5), notice that $m, n \geq 1$. So we have

$$\left\lfloor \frac{5n-3}{2}\right\rfloor + \left\lfloor \frac{5m-n}{2}\right\rfloor = \left\lfloor \frac{3n-3}{2}\right\rfloor + \left\lfloor \frac{5m+n}{2}\right\rfloor \geq \left\lfloor \frac{5m}{2}\right\rfloor. \tag{8.9}$$

Notice that $\left\lfloor \frac{5m}{2}\right\rfloor$ is the necessary power of 5 in the coefficient of $p_1 t^m$ for $X^{(1)}$ in (7.12).

For (8.6), with $m \geq 1$, $n \geq 0$, we have

$$\left\lfloor \frac{5n}{2}\right\rfloor + \left\lfloor \frac{5m-n-1}{2}\right\rfloor \geq \left\lfloor \frac{5m+n-1}{2}\right\rfloor \geq \left\lfloor \frac{5m-1}{2}\right\rfloor. \tag{8.10}$$

Finally, in (8.7), with $m, n \geq 1$, we have

$$\left\lfloor \frac{5n-3}{2}\right\rfloor + \left\lfloor \frac{5m-n-1}{2}\right\rfloor \geq \left\lfloor \frac{5m+n-1}{2}\right\rfloor \geq \left\lfloor \frac{5m-1}{2}\right\rfloor. \tag{8.11}$$

Since $\left\lfloor \frac{5m-1}{2}\right\rfloor$ is the 5-adic valuation of the coefficient of t^m for $X^{(1)}$, we have $U^{(0)}(f) \in X^{(1)}$.

To prove the second statement of our theorem, we let $f \in X^{(1)}$. We want $U^{(1)}(f) \in 5 \cdot X^{(0)}$. By hypothesis, we have

$$f = \sum_{n=0}^{\infty} r(n)5^{\left\lfloor \frac{5n}{2}\right\rfloor} p_1 t^n + \sum_{n=1}^{\infty} s(n)5^{\left\lfloor \frac{5n-1}{2}\right\rfloor} t^n. \tag{8.12}$$

We have

$$U^{(1)}(f) = \sum_{n=0}^{\infty} r(n) 5^{\left\lfloor \frac{5n}{2} \right\rfloor} U^{(1)}(p_1 t^n) + \sum_{n=1}^{\infty} s(n) 5^{\left\lfloor \frac{5n-1}{2} \right\rfloor} U^{(1)}(t^n) \tag{8.13}$$

$$= \sum_{n=0}^{\infty} r(n) 5^{\left\lfloor \frac{5n}{2} \right\rfloor} \left(\sum_{m \geq \left\lceil \frac{n+1}{5} \right\rceil} d_0(m,n) 5^{\left\lfloor \frac{5m-n-1}{2} \right\rfloor} t^m + \sum_{m \geq \left\lceil \frac{n-1}{5} \right\rceil} d_1(m,n) 5^{\left\lfloor \frac{5m-n+2}{2} \right\rfloor} p_0 t^m \right)$$

$$+ \sum_{n=1}^{\infty} s(n) 5^{\left\lfloor \frac{5n-1}{2} \right\rfloor} \left(\sum_{m \geq \left\lceil \frac{n}{5} \right\rceil} c(m,n) 5^{\left\lfloor \frac{5m-n-1}{2} \right\rfloor} t^m \right) \tag{8.14}$$

$$= p_0 \sum_{m \geq 0} \sum_{n \geq 0} r(n) d_1(m,n) 5^{\left\lfloor \frac{5n}{2} \right\rfloor + \left\lfloor \frac{5m-n+2}{2} \right\rfloor} t^m \tag{8.15}$$

$$+ \sum_{m \geq 1} \sum_{n \geq 0} r(n) d_0(m,n) 5^{\left\lfloor \frac{5n}{2} \right\rfloor + \left\lfloor \frac{5m-n-1}{2} \right\rfloor} t^m \tag{8.16}$$

$$+ \sum_{m \geq 1} \sum_{n \geq 1} s(n) c(m,n) 5^{\left\lfloor \frac{5n-1}{2} \right\rfloor + \left\lfloor \frac{5m-n-1}{2} \right\rfloor} t^m. \tag{8.17}$$

Examining the power of 5 in line (8.15), noting that $m, n \geq 0$, we find that

$$\left\lfloor \frac{5n}{2} \right\rfloor + \left\lfloor \frac{5m-n+2}{2} \right\rfloor = \left\lfloor \frac{3n}{2} \right\rfloor + \left\lfloor \frac{5m+n+2}{2} \right\rfloor \geq \left\lfloor \frac{5m+2}{2} \right\rfloor = \left\lfloor \frac{5m}{2} \right\rfloor + 1. \tag{8.18}$$

Similarly, we consider line (8.16), with $m \geq 1, n \geq 0$:

$$\left\lfloor \frac{5n}{2} \right\rfloor + \left\lfloor \frac{5m-n-1}{2} \right\rfloor \geq \left\lfloor \frac{5m+n-1}{2} \right\rfloor \geq \left\lfloor \frac{5m-1}{2} \right\rfloor = \left\lfloor \frac{5m-3}{2} \right\rfloor + 1. \tag{8.19}$$

Finally, for line (8.17), with $m, n \geq 1$:

$$\left\lfloor \frac{5n-1}{2} \right\rfloor + \left\lfloor \frac{5m-n-1}{2} \right\rfloor = \left\lfloor \frac{3n-1}{2} \right\rfloor + \left\lfloor \frac{5m+n-1}{2} \right\rfloor \geq 1 + \left\lfloor \frac{5m-3}{2} \right\rfloor. \tag{8.20}$$

We therefore have $U^{(1)}(f) = 5 \cdot g$, for some $g \in X^{(0)}$.

□

8.2 Completing the Proof

We have the following:

Theorem 8.2. *For every $n \in \mathbb{Z}_{\geq 1}$, there exist functions $g_{2n-1} \in X^{(1)}$ and $g_{2n} \in X^{(0)}$ such that*

$$L_{2n-1} = 5^{n-1} g_{2n-1},$$
$$L_{2n} = 5^{n} g_{2n}.$$

Proof. Since $L_1 = p_1 \in X^{(1)}$, we have

$$L_2 = U^{(1)}(L_1) = U^{(1)}(p_1) = 5g_1, \tag{8.21}$$

with $g_1 \in X^{(0)}$. Suppose that for some $k \in \mathbb{Z}_{>0}$, we have $L_{2k} = 5^k g_{2k}$, with $g_{2k} \in X^{(0)}$. Then

$$L_{2k+1} = U^{(0)}(L_{2k}) = U^{(0)}(5^k g_{2k}) = 5^k U^{(0)}(g_{2k}) = 5^k g_{2k+1}, \tag{8.22}$$

with $g_{2k+1} \in X^{(1)}$. Finally, we have

$$L_{2k+2} = U^{(1)}(5^k g_{2k+1}) = 5^k U^{(1)}(g_{2k+1}) = 5^k \cdot 5 \cdot g_{2k+2} = 5^{k+1} g_{2k+2}, \tag{8.23}$$

with $g_{2k+2} \in X^{(0)}$.

By induction, for every $n \in \mathbb{Z}_{>0}$, there must exist a $g_{2n} \in X^{(0)}$ such that $L_{2n} = 5^n g_{2n}$.

Since for every $n \in \mathbb{Z}_{\geq 1}$, we have

$$L_{2n+1} = U^{(0)}(5^n g_{2n}) = 5^n U^{(0)}(g_{2n}), \tag{8.24}$$

and since $L_1 = 5^0 p_1$, we immediately see that there must exist some $g_{2n-1} \in X^{(1)}$ such that

$$L_{2n-1} = 5^{n-1} g_{2n-1}. \tag{8.25}$$

□

Corollary 8.3. *For every $n \in \mathbb{Z}_{>0}$, $L_{2n} \equiv 0 \pmod{5^n}$.*

Proof. For every $n \in \mathbb{Z}_{>0}$, $L_{2n} = 5^n g_{2n}$ for some $g_{2n} \in X^{(0)}$. And the elements of $X^{(0)}$ have integer coefficients. □

With this, we have proven Theorem 7.1 and Theorem 6.3.

CHAPTER 9
CONGRUENCES AND GENUS (I)

The results of the previous four chapters indicate the methods developed by Paule and Radu which are useful to establish congruence families in which piecewise ℓ-adic convergence fails.

Notice that the Paule–Radu method shares much in common with the classical methods of proving congruence families. Notably, in all cases, the focus is on the manipulation of a free $\mathbb{Z}[X]$-module of a given rank. In particular, when the genus of the underlying modular curve is 0, this module is simply $\mathbb{Z}[X]$. As we have stated, this method was originally developed by Ramanujan and Watson.

For over a century, most proofs of congruence families made use of this method. The discovery that this approach is in fact insufficient for all congruence families over modular curves of genus 0 is extremely recent. The technique to overcome a striking exception to this classical approach has been developed only within the last 12 months.

This discovery is so remarkable, and this new technique so potentially prolific in its consequences, that we have chosen to present the background and the techniques involved in our final two chapters. We will first introduce the original problem as studied by Liuquan Wang and Yifan Yang, together with an outline of their "two-variable" approach. We will then give our central theorem which gives a representation of our relevant generating functions in terms of a "single-variable." We prove this result in the following chapter.

The results in these two chapters have been submitted for publication, in the manuscript available at `https://arxiv.org/abs/2004.03944`.

The intuition underlying this approach (based on the localization of $\mathbb{Z}[X]$ by powers of an appropriate polynomial in a given Hauptmodul) was developed largely from the experimental techniques already exposited in Chapter 6. If the reader is not already convinced, we emphasize: In this area of mathematics, experiment and theory are closely connected.

9.1 On the Relationship Between ℓ-adic Convergence and Genus

Let us recall the classical method for proving partition congruence families when the underlying modular curve has genus 0. Suppose that $a(n)$ is some integer sequence generated by a well-known function (usually a modular form up to an exponential factor), and that one wants to prove the congruence family

$$a\left(\ell^{\alpha}n + \delta_{\ell,\alpha}\right) \equiv 0 \pmod{\ell^{\alpha}},$$

for all $n \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{Z}_{\geq 1}$, in which $\delta_{\ell,\alpha}$ is the minimal positive solution to

$$C \cdot \delta_{\ell,\alpha} \equiv 1 \pmod{\ell^{\alpha}},$$

for some fixed $C \in \mathbb{Z}$. For simplicity we take the function $\beta(\ell, \alpha)$ from Definition 5.1 to be equal to α.

The way that this is generally done is that a sequence of functions

$$\mathcal{L} := (L_\alpha)_{\alpha \geq 1}$$

is constructed, in which L_α has the form

$$L_\alpha = \Phi_\alpha \cdot \sum_{C \cdot n \equiv 1 \bmod \ell^\alpha} a(n) q^{\lfloor n/\ell^\alpha \rfloor + 1}, \tag{9.1}$$

and Φ_α is a suitable prefactor (usually of the form given to Ramanujan–Kolberg identities in Chapters 3-4). With the right choice of Φ_α, we can usually select an $N \in \mathbb{Z}_{\geq 1}$ such that $L_\alpha \in \mathcal{M}(\Gamma_0(N))$ for all $\alpha \in \mathbb{Z}_{\geq 1}$.

Of course, the underlying Riemann surface of $\Gamma_0(N)$ is the classical modular curve $\mathrm{X}_0(N)$ that we discussed in Chapter 2. For the moment, we assume that $\mathfrak{g}(\mathrm{X}_0(N)) = 0$.

In the case that our functions L_α are modular over $\Gamma_0(\ell)$ with ℓ a prime number, we find a curious simplification. The number of cusps can be computed by Lemma 2.10:

$$\epsilon_\infty (\Gamma_0(\ell)) = 2\phi(\gcd(1, \ell)) = 2\phi(1) = 2.$$

Given that $\mathrm{ord}_\infty^{(N)}(L_\alpha) \geq 1$ by (9.1), we must have

$$\mathcal{L} \subseteq \mathcal{M}^0(\Gamma_0(\ell)).$$

Finally, because $\mathfrak{g}(\mathrm{X}_0(N)) = 0$ by hypothesis, we should have

$$\mathcal{M}^0(\Gamma_0(\ell)) = \mathbb{Z}[x],$$

for some $x \in \mathcal{M}^0(\Gamma_0(\ell))$. If we can construct such an x with the additional property that its coefficients $\alpha_I(n) \in \mathbb{Q}$ for all $n \geq 0$, then we will have

$$L_\alpha \in \mathbb{Q}[x].$$

Indeed, with the right choice of x, we may have

$$L_\alpha \in \mathbb{Z}[x].$$

As with the Ramanujan–Kolberg identities we have already explored, one need only examine the coefficients of x^n to determine divisibility properties of L_α.

However, in the case that N is composite, this is no longer necessarily true. It is possible that the functions L_α may have multiple poles and zeros at multiple cusps. This complicates the matter of representing L_α.

A very natural approach to this problem would be similar to our approach in Chapter 6: to find a function T such that $T^m \cdot L_\alpha \in \mathcal{M}^0(\Gamma_0(N)) = \mathbb{C}[x]$. However, as our work in Chapter 6 showed, such functions T are not always easy to work with; indeed, they often inflate the rank of the relevant $\mathbb{Z}[X]$-module that we end up working over. Our interest here is to keep the rank equal to 1.

Another possibility presents itself: Theorem 2.45 allows us to construct an eta quotient with positive order at every cusp except $[\infty]_N$. Of course, there is nothing special about $[\infty]_N$; Newman allows us to construct an eta quotient μ with positive order everywhere except $[0]_N$.

We know that $\mu \in \mathcal{M}^0(\Gamma_0(N))$, so that μ must be equal to some polynomial $f \in \mathbb{Z}[x]$, and therefore

$$f(x)^{\psi(\alpha)} \cdot L_\alpha \in \mathbb{Z}[x]$$

for an appropriate positive integer sequence $(\psi(\alpha))_{\alpha\geq 1}$. If we can also arrange so that $\ell \nmid f$, then we can demonstrate that $L_\alpha \equiv 0 \pmod{\ell^\alpha}$ by showing that the coefficients in the polynomial expansion of $f(x)^{\psi(\alpha)} \cdot L_\alpha$ are each divisible by ℓ^α.

What makes this a rather intimidating approach to the theory of partition congruences is that, in order to study L_α, one must work over the *localized* polynomial ring $\mathbb{Z}[X]_{\mathcal{S}}$, in which $\mathcal{S}$ is the multiplicatively closed set of positive multiples of $f(X)$. This can complicate the steps in proving a given congruence family.

The advantage of this approach is that it can potentially give a representation of the L_α in terms of an algebraic structure of minimal complexity, imposed by the topology of the underlying Riemann surface. In particular, we can often express L_α in terms of a Hauptmodul if the genus of the surface is 0.

To illustrate these ideas, we will give an example of this method of utilizing localized polynomial rings as applied to an infinite family of congruences originally studied and proved by Wang and Yang [113].

9.2 The ω Smallest Parts Function

The smallest parts function $\mathrm{spt}(n)$ was first studied by Andrews in [9].

Definition 9.1. The smallest parts function $\mathrm{spt}(n)$ gives the number of smallest parts in the partitions of n.

For example, let us list the partitions of 5:

$$\begin{aligned}&5,\\&4+1,\\&3+2,\\&3+1+1,\\&2+2+1,\\&2+1+1+1,\\&1+1+1+1+1,\end{aligned}$$

The partitions 5, $4+1$, $3+2$, $2+2+1$ each have a single smallest part (5, 1, 2, 1, respectively), and therefore each partition contributes 1. On the other hand, the partition $3+1+1$ has the smallest part 1 occurring twice, and thus contributes 2. Similarly, $2+1+1+1$ contributes 3, and $1+1+1+1+1$ contributes 5. Thus

$$\mathrm{spt}(5) = 1+1+1+1+2+3+5 = 14.$$

Andrews discovered that $\mathrm{spt}(n)$ satisfied congruences similar to those of $p(n)$, e.g. [9]

$$\mathrm{spt}(5n+4) \equiv 0 \pmod 5.$$

Garvan later discovered and proved [41] multiple congruence families for $\mathrm{spt}(n)$, e.g.,

$$\mathrm{spt}\left(5^{\alpha}n+\lambda_{\alpha}\right) \equiv 0 \pmod{5^{\lfloor(\alpha+1)/2\rfloor}}.$$

Analogues of $\mathrm{spt}(n)$ exist for various more restrictive partition functions. As an example, consider Ramanujan's order 3 mock theta function ω, defined by

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}. \tag{9.2}$$

The coefficient of q^n in the expansion of ω has a simple partition interpretation. We define the function $p_\omega(n)$ as follows:

Definition 9.2. The number of partitions of n in which the odd parts are less than twice the smallest part is denoted $p_\omega(n)$.

The following theorem is due to Andrews, Dixit, and Yee [12]:

Theorem 9.3. *We have*

$$\sum_{n=1}^{\infty} p_\omega(n)q^n = q\omega(q).$$

Finally, we can define the spt-analogue of interest to us:

Definition 9.4. Define $\mathrm{spt}_\omega(n)$ as the function which counts repetitions of the smallest parts in the partitions counted by $p_\omega(n)$.

9.3 Motivation

The theorem of interest to us was originally cojectured by Liuquan Wang [110] in 2017. It was proved the following year by Wang, together with Yifan Yang [113].

Theorem 9.5. *Let $\lambda_\alpha \in \mathbb{Z}$ be the minimal positive solution to $12x \equiv 1 \pmod{5^\alpha}$. Then*

$$\mathrm{spt}_\omega\left(2 \cdot 5^\alpha n + \lambda_\alpha\right) \equiv 0 \pmod{5^\alpha}. \tag{9.3}$$

Wang and Yang prove this theorem [113] by relating spt_ω to the spt functions for certain Bailey pairs $C1$, $C5$ studied by Garvan and Jennings–Shaffer [43], as well as the function $c(n)$, defined in terms of the normalized weight 2 Eisenstein series (disregarding the nonholomorphic term). We will recall some basic facts about Eisenstein series which can be found in [34, Section 1.1], [88, Chapter 8], [99, Chapter III]. Define the weight $2k$ Eisenstein series:

$$G_{2k}(\tau) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^{2k}}. \tag{9.4}$$

This is a holomorphic weight $2k$ modular form for $k > 1$, and can be written in the form

$$G_{2k}(\tau) = 2\zeta(2k) + (-1)^k \cdot 2\frac{(2\pi)^{2k}}{(2k-1)!}\sum_{m=1}^{\infty} m^{2k-1}\frac{x^m}{1-x^m}. \tag{9.5}$$

For $k = 1$, the series (9.4) is not absolutely convergent, and is not a holomorphic function. If we attempt to expand $G_2(\tau)$ by analytic continuation [34, Section 1.2], [88, Chapter 8, Section 63], [99, Chapter III.2], we compute

$$G_2(\tau) = \frac{\pi^2}{3} - 8\pi^2\sum_{m=1}^{\infty} m\frac{x^m}{1-x^m} - \frac{\pi}{\Im(\tau)}. \tag{9.6}$$

Notice that (9.6) matches (9.5) for $k = 1$, except for the nonholomorphic term $\frac{\pi}{\Im(\tau)}$.

If we disregard the holomorphic part and divide by $\pi^2/3$, then we have the normalized holomorphic part of G_2, denoted as E_2:

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

Notably, while $G_2(\tau)$ is not holomorphic, the nonholomorphic parts of $2G_2(2\tau) - G_2(\tau)$ cancel, so that $2G_2(2\tau) - G_2(\tau)$ (and therefore, $2E_2(2\tau) - E_2(\tau)$) is a weight 2 holomorphic modular form [34, Section 1.2].

We define $c(n)$ in the following manner:

$$\sum_{n=0}^{\infty} c(n)q^n := \frac{2E_2(2\tau) - E_2(\tau)}{(q^2;q^2)_\infty}, \tag{9.7}$$

Wang and Yang then show that Theorem 9.5 is a consequence of the following:

Theorem 9.6. *Let* $12n \equiv 1 \pmod{5^\alpha}$*. Then* $c(n) \equiv 0 \pmod{5^\alpha}$*.*

Wang and Yang employ a method similar in form to that used in Chapters 7-8. In particular, they develop a sequence of weakly holomorphic weight 2 modular forms $(L_\alpha)_{\alpha\geq 1}$ such that

$$L_\alpha = \Phi_\alpha \cdot \sum_{n=0}^{\infty} c\left(5^\alpha n + \lambda_\alpha\right) q^{n+1},$$

with Φ_α a certain integer power series which we will define in the next chapter, and λ_α the minimum positive solution to $12x \equiv 1 \pmod{5^\alpha}$ (see Section 2). They show that

$$\frac{L_\alpha}{5^\alpha \cdot F} = f_{0,\alpha}(t) + \rho \cdot f_{1,\alpha}(t), \tag{9.8}$$

in which $f_{i,\alpha} \in \mathbb{Z}[X]$,

$$F := F(\tau) = \frac{1}{24}\left(50E_2(10\tau) - 25E_2(5\tau) - 2E_2(2\tau) + E_2(\tau)\right)$$

is a weight 2 holomorphic modular form, and t, ρ are modular functions in $\Gamma_0(10)$ which take the form of eta quotients with integer expansions in the Fourier variable q.

This is standard to Paule and Radu's approach. Note the free rank 2 $\mathbb{Z}[X]$-module structure of (9.8), which is characteristic of the method.

However, Paule and Radu developed their method in order to overcome the complications which arise from a congruence family in which the associated modular curve has *nonzero genus*. The genus of $\mathrm{X}_0(10)$ is 0.

It is this extremely important and telling fact that drove us to attempt a more classical proof of Wang and Yang's theorem.

As an example, we take the first case of Theorem 9.6. Define

$$L_1 = (q^{10};q^{10})_\infty \sum_{n=0}^{\infty} c\left(5n+3\right) q^{n+1}, \tag{9.9}$$

in a manner standard to the theory (see Section 2). Wang and Yang prove that $L_1 \equiv 0 \pmod{5}$ by showing that

$$L_1 = F \cdot \left(\left(245t + 3750t^2 + 15625t^3\right) - \rho \cdot \left(125t + 3125t^2\right) \right), \tag{9.10}$$

with t and ρ defined as in (9.8). However, we were able to find a function $x \in \mathcal{M}\left(\Gamma_0(10)\right)$ with the following:

$$L_1 = \frac{F}{(1+5x)^3} \cdot \left(120x + 1805x^2 + 12050x^3 + 39500x^4 + 50000x^5\right). \tag{9.11}$$

If we note that x and F both expand into integer power series in the variable q with $5 \nmid F$, then we need only examine the remaining portion of the expression—a single-variable polynomial in x—in which divisibility by 5 very easily emerges.

An interesting complication emerges in the factor $(1+5x)^{-3}$. One might correctly guess that our relevant space of modular functions for all $\alpha \geq 1$ is isomorphic to a localization of $\mathbb{Z}[X]$, rather than to $\mathbb{Z}[X]$ itself. Indeed, we have the following remarkable result, which we consider to be the climax of our dissertation:

Theorem 9.7. *Let*

$$x = x(\tau) := q \prod_{m=1}^{\infty} \frac{(1-q^{2m})(1-q^{10m})^3}{(1-q^m)^3(1-q^{5m})}, \tag{9.12}$$

and

$$\psi(\alpha) := \left\lfloor \frac{5^{\alpha+1}}{12} \right\rfloor + 1. \tag{9.13}$$

For all $\alpha \geq 1$, we have

$$\frac{(1+5x)^{\psi(\alpha)}}{5^\alpha \cdot F} \cdot L_\alpha \in \mathbb{Z}[x]. \tag{9.14}$$

We are not aware of any other congruence families in which the proof necessitates such a ring structure, and it would be interesting to know whether any additional examples exist. We strongly believe that localized rings may yet prove to be an enormously productive environment in which to examine new arithmetic properties in partition theory, especially for situations in which more traditional methods fail.

Another interesting difficulty arises in the somewhat irregular 5-adic growth of each term of L_α under repeated application of the corresponding U_5 operators. In particular, in mapping $L_{2\alpha-1}$

to $L_{2\alpha}$, the individual components of the linear coefficient do not increase piecewise with respect to their 5-adic value—rather, the components must be shown *to sum to* the necessary multiple of 5 (see Definition 10.11 and Theorem 10.12).

Such a strange complication necessitates a very precise manipulation of the 5-adic convergence of our critical functions, together with a careful examination of the coefficients in our auxiliary functions modulo 5. As in the matter of localization, we are unaware of any other examples of congruence families which demand such a constructive method of verifying divisibility by 5.

A final complication emerges in the base cases of our key lemmas. We require the verification of 50 initial relations. However, we can show that these 50 are algebraically dependent, and that *a total of only 10 initial relations* need be directly established. This stands in contrast to the 20 that Wang and Yang require for their proof. From these 10 relations, the 50 relations necessary for our induction may be assembled and verified with relative ease through a computer algebra system. The computational complexity is striking; nevertheless, the reliance (in principle, at least) upon so few relations, together with the single-variable approach, is to be expected, given that the underlying genus is 0.

In total, these complications seem overwhelming, and it is understandable that a single-variable proof has not been found before now. It seems that the genus of the underlying modular curve alone is sufficient to compel a single-variable proof, in spite of the many considerable difficulties.

9.4 First Attempt at a Single-Variable Proof

Our initial attempt to describe L_α in terms of a certain Hauptmodul z failed almost immediately: one can verify that the function

$$z = z(\tau) := \prod_{m=1}^{\infty} \frac{(1-q^{2m})^5(1-q^{5m})}{(1-q^m)^5(1-q^{10m})}, \tag{9.15}$$

has zeros of positive order at all of the poles of L_1 except for a pole of order -1 at $[0]_{10}$. We therefore labored to produce a representation of L_1 in terms of z. Doing so results in the expression

$$\begin{aligned}\frac{L_1}{F} =& -\frac{624}{625z^3} - \frac{2487}{625z^2} + \frac{801}{625z} - \frac{422}{125} - \frac{3148z}{125}\\ &+ \frac{19904z^2}{625} + \frac{512z^3}{625} - \frac{256z^4}{625}.\end{aligned} \tag{9.16}$$

This clearly does not work.

However, at the advice of Silviu Radu, we attempted to adjust our function x. We discovered the appropriate substitution in the form of $z = 1 + 5x$. Substituting into (9.16) and simplifying, we derive (9.11).

The critical point is that we would prefer the function needed to annihilate the poles of L_α to be equal to (or a power of) the function used to describe the right-hand side of the witness identity. This is of course the intuition which underlies the techniques described in Chapter 6. It is far more probable that these functions are not equal. Nevertheless, if the function used on the right-hand side, e.g., x, is a hauptmodul, then we could still use this function to describe our prefactor function (e.g., $z = 1 + 5x$). This necessarily induces a localized ring.

In the following chapter we will give a proof of Theorem 9.7.

CHAPTER 10
CONGRUENCES AND GENUS (II)

In this chapter we will give a proof of Theorem 9.7. The complications which emerge in this problem ensure that many steps are quite tedious. However, much of it resembles the techniques described in Chapters 5-8.

We will begin by constructing our generating function sequence $(L_\alpha)_{\alpha\geq 0}$. We will then introduce the eta quotients z and x, show their relationship with one another, and give each function a useful modular equation.

In the following section we will define our necessary localized ring, together with our linear operators $U^{(i)}$. We then give the critical recurrence relation for elements of our localized ring, together with some very precise conditions on the associated auxiliary functions $h_i(m, n, r)$. The recurrence relation can be proved with an induction argument, of which the initial cases will be handled later.

We then begin to prove the 5-adic convergence of elements in our localized ring upon application of the $U^{(i)}$ operators. The case for $U^{(0)}$ can be done with relative ease; however, the case for $U^{(1)}$ is more difficult, and requires the previously established arithmetic properties of $h_i(m, n, r)$.

In the final section we prove the initial relations of our recurrence relation. The form of our induction argument requires 50 initial relations. However, these relations are algebraically dependent. We show that only 10 relations need actually be proved using the modular cusp analysis from Chapter 2; the 50 initial relations may be computed algebraically from these 10 relations. We finally prove the modular equations for z and x, as well as the representation for L_1 given in (9.11).

The results in this chapter have been submitted for publication (along with the results in the previous chapter), in the manuscript available at `https://arxiv.org/abs/2004.03944`.

10.1 Proof Setup

We begin by defining an important auxiliary function:

$$Z := Z(\tau) = \frac{\eta(50\tau)}{\eta(2\tau)}. \tag{10.1}$$

Here $Z(\tau)$ is a modular function over $\Gamma_0(10)$ (See the final section of this chapter).

We will now define our key generating functions, and their behavior under U_5.

10.1.1 Generating Functions

Our main generating functions L_α for each case of Theorem 9.6 are defined as follows:

$$L_0 := 2E_2(2\tau) - E_2(\tau), \tag{10.2}$$

$$L_{2\alpha-1} := (q^{10};q^{10})_\infty \sum_{n=0}^{\infty} c\left(5^{2\alpha-1}n + \lambda_{2\alpha-1}\right) q^{n+1}, \tag{10.3}$$

$$L_{2\alpha} := (q^2;q^2)_\infty \sum_{n=0}^{\infty} c\left(5^{2\alpha}n + \lambda_{2\alpha}\right) q^{n+1}, \tag{10.4}$$

with the λ_α defined as

$$\lambda_{2\alpha-1} := \frac{1 + 7 \cdot 5^{2\alpha-1}}{12}, \tag{10.5}$$

$$\lambda_{2\alpha} := \frac{1 + 11 \cdot 5^{2\alpha}}{12}. \tag{10.6}$$

In either case, $\lambda_\alpha \in \mathbb{Z}$ are the minimal positive solutions to

$$12x \equiv 1 \pmod{5^\alpha}.$$

Therefore, one can write L_α in the form

$$L_\alpha = \Phi_\alpha(q) \cdot \sum_{12n \equiv 1 \bmod 5^\alpha} c(n) q^{\left\lfloor \frac{n}{5^\alpha} \right\rfloor},$$

with

$$\Phi_{2\alpha-1} = q(q^{10};q^{10})_\infty,$$
$$\Phi_{2\alpha} = q(q^2;q^2)_\infty.$$

The U_5 operator provides us with a convenient means of accessing $L_{\alpha+1}$ from L_α, as the following lemma shows:

Lemma 10.1. *For all $\alpha \geq 0$, we have*

$$L_{2\alpha} = U_5\left(L_{2\alpha-1}\right), \tag{10.7}$$

$$L_{2\alpha+1} = U_5\left(Z \cdot L_{2\alpha}\right). \tag{10.8}$$

Proof. For any $\alpha \geq 1$, we have

$$\begin{aligned}
U_5\left(L_{2\alpha-1}\right) &= U_5\left((q^{10};q^{10})_\infty \sum_{n\geq 0} c\left(5^{2\alpha-1}n + \lambda_{2\alpha-1}\right) q^{n+1}\right)\\
&= (q^2;q^2)_\infty \cdot U_5\left(\sum_{n\geq 1} c\left(5^{2\alpha-1}(n-1) + \lambda_{2\alpha-1}\right) q^{n}\right)\\
&= (q^2;q^2)_\infty \cdot \sum_{5n\geq 1} c\left(5^{2\alpha-1}(5n-1) + \lambda_{2\alpha-1}\right) q^{n}\\
&= (q^2;q^2)_\infty \cdot \sum_{n\geq 1} c\left(5^{2\alpha}n - 5^{2\alpha-1} + \lambda_{2\alpha-1}\right) q^{n}\\
&= (q^2;q^2)_\infty \cdot \sum_{n\geq 0} c\left(5^{2\alpha}n + 5^{2\alpha} - 5^{2\alpha-1} + \lambda_{2\alpha-1}\right) q^{n+1}\\
&= (q^2;q^2)_\infty \cdot \sum_{n\geq 0} c\left(5^{2\alpha}n + \lambda_{2\alpha}\right) q^{n+1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
U_5\left(Z\cdot L_{2\alpha}\right) &= U_5\left(q^2\frac{(q^{50};q^{50})_\infty}{(q^2;q^2)_\infty}(q^2;q^2)_\infty \sum_{n\geq 0} c\left(5^{2\alpha}n + \lambda_{2\alpha}\right) q^{n+1}\right)\\
&= (q^{10};q^{10})_\infty \cdot U_5\left(\sum_{n\geq 3} c\left(5^{2\alpha}(n-3) + \lambda_{2\alpha}\right) q^{n}\right)\\
&= (q^{10};q^{10})_\infty \cdot \sum_{5n\geq 3} c\left(5^{2\alpha}(5n-3) + \lambda_{2\alpha}\right) q^{n}\\
&= (q^{10};q^{10})_\infty \cdot \sum_{n\geq 1} c\left(5^{2\alpha+1}n - 3\cdot 5^{2\alpha} + \lambda_{2\alpha}\right) q^{n}\\
&= (q^{10};q^{10})_\infty \cdot \sum_{n\geq 0} c\left(5^{2\alpha+1}(n+1) - 3\cdot 5^{2\alpha} + \lambda_{2\alpha}\right) q^{n+1}\\
&= (q^{10};q^{10})_\infty \cdot \sum_{n\geq 0} c\left(5^{2\alpha+1}n + \lambda_{2\alpha+1}\right) q^{n+1}.
\end{aligned}$$

□

10.2 The Modular Equation

We recall the functions $x, y \in \mathcal{M}^{(0)}\left(\Gamma_0(10)\right)$ that we introduced in the previous chapter:

$$z = z(\tau) := \prod_{m=1}^{\infty} \frac{(1-q^{2m})^5(1-q^{5m})}{(1-q^m)^5(1-q^{10m})},$$
$$x = x(\tau) := q\prod_{m=1}^{\infty} \frac{(1-q^{2m})(1-q^{10m})^3}{(1-q^m)^3(1-q^{5m})}.$$

Notice that, by the Freshman's Dream [55, Chapter III.1, Exercise 11],

$$(1-q^m)^5 \equiv 1-q^{5m} \pmod 5,$$
$$(1-q^{2m})^5 \equiv 1-q^{10m} \pmod 5.$$

This yields

$$\prod_{m=1}^{\infty} \frac{(1-q^{2m})^5(1-q^{5m})}{(1-q^m)^5(1-q^{10m})} \equiv 1 \pmod 5. \tag{10.9}$$

It is not difficult to verify that

$$\frac{z-1}{5} = q\prod_{m=1}^{\infty} \frac{(1-q^{2m})(1-q^{10m})^3}{(1-q^m)^3(1-q^{5m})}, \tag{10.10}$$

from which we obtain

$$z = 1+5x. \tag{10.11}$$

Theorem 10.2. *Define*

$$a_0(\tau) = -x - 5\cdot 4\cdot x^2 - 5^2\cdot 6\cdot x^3 - 5^3\cdot 4\cdot x^4 - 5^4\cdot x^5$$
$$a_1(\tau) = -5\cdot 3x - 5\cdot 61\cdot x^2 - 5^2\cdot 93\cdot x^3 - 5^3\cdot 63\cdot x^4 - 5^4\cdot 16\cdot x^5$$
$$a_2(\tau) = -5\cdot 17\cdot x - 5^3\cdot 14\cdot x^2 - 5^2\cdot 541\cdot x^3 - 5^3\cdot 372\cdot x^4 - 5^4\cdot 96\cdot x^5$$
$$a_3(\tau) = -5\cdot 43\cdot x - 5^2\cdot 179\cdot x^2 - 5^4\cdot 56\cdot x^3 - 5^3\cdot 976\cdot x^4 - 5^4\cdot 256\cdot x^5$$
$$a_4(\tau) = -5\cdot 41\cdot x - 5^2\cdot 172\cdot x^2 - 5^3\cdot 272\cdot x^3 - 5^4\cdot 192x^4 - 5^4\cdot 256\cdot x^5.$$

Then we have

$$x^5 + \sum_{j=0}^{4} a_j(5\tau)x^j = 0. \tag{10.12}$$

Proof. Because $x(5\tau)^{-1} \in \mathcal{M}^{\infty}(\Gamma_0(50))$ is a modular function with only one pole, we can prove this equation using cusp analysis. See the end of the chapter. $\square$

Theorem 10.3. *Define*

$$\begin{aligned}
b_0(\tau) &= -z^5 \\
b_1(\tau) &= 1 + 5z + 5z^2 + 5z^3 + 5z^4 - 16z^5 \\
b_2(\tau) &= -4 - 5\cdot 3\cdot z + 5\cdot 2\cdot z^2 + 5\cdot 7\cdot z^3 + 5\cdot 12\cdot z^4 - 96z^5 \\
b_3(\tau) &= 6 + 5\cdot 3\cdot z - 5\cdot 7z^2 + 5\cdot 8z^3 + 5\cdot 48\cdot z^4 - 256z^5 \\
b_4(\tau) &= -4 - 5z + 5\cdot 4\cdot z^2 - 5\cdot 16\cdot z^3 + 5\cdot 64\cdot z^4 - 256z^5.
\end{aligned}$$

Then we have

$$z^5 + \sum_{k=0}^{4} b_k(5\tau) z^k = 0. \tag{10.13}$$

Proof. Simply substitute $x = (z-1)/5$ into (10.12), and simplify. $\square$

For convenience of notation, in later sections we will define $b_5(\tau) := 1$.

10.3 Algebraic Structure

10.3.1 Localized Ring

We will begin to construct the algebra structure needed for our proof, beginning with the peculiar localization property. Define the multiplicatively closed set

$$\mathcal{S} := \{(1+5x)^n : n \in \mathbb{Z}_{n\geq 0}\}. \tag{10.14}$$

We will prove that for every $\alpha \geq 1$, L_α is a member of the localization of $\mathbb{Z}[x]$ at $\mathcal{S}$, which we will denote by $\mathbb{Z}[x]_{\mathcal{S}}$. Notice that because $1/z^n = 1/(1+5x)^n$ is an eta quotient with an integer power series expansion in $q = e^{2\pi i\tau}$ for every $n \geq 1$, we can expand every element of the localization into an integer power series in q, i.e., $\mathbb{Z}[x]_{\mathcal{S}} \subseteq \mathbb{Z}[[q]]$.

We need to define two general classes of subsets of $\mathbb{Z}[s]_{\mathcal{S}}$ to contain our L_α. Due to a somewhat irregular pattern of 5-adic convergence, we must build our 5-adic valuation function very carefully. We define the functions θ_0 and θ_1 by

$$\theta_0(m) := \begin{cases} \left\lfloor \frac{5m-5}{6} \right\rfloor, & 1 \leq m \leq 2, \\ \left\lfloor \frac{5m-5}{6} \right\rfloor - 1, & m \geq 3, \end{cases}$$

$$\theta_1(m) := \begin{cases} \left\lfloor \frac{5m-5}{6} \right\rfloor, & 1 \leq m \leq 3, \\ \left\lfloor \frac{5m-5}{6} \right\rfloor - 1, & m \geq 4. \end{cases}$$

Now we take an arbitrary $n \geq 1$, and define the following:

$$\mathcal{V}_n^{(0)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta_0(m)} \cdot x^m : s \text{ is discrete} \right\}, \tag{10.15}$$

$$\mathcal{V}_n^{(1)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta_1(m)} \cdot x^m : s \text{ is discrete} \right\}. \tag{10.16}$$

10.3.2 Recurrence Relation

We now define the following maps:

$$U^{(1-i)}(f) := \frac{U_5\left(F \cdot Z^i \cdot f\right)}{F} \tag{10.17}$$

for $i = 0, 1$.

We are ready to utilize our modular equations, together with our $U^{(i)}$ operators to build certain helpful recurrence relations.

Lemma 10.4. *For all $m, n \in \mathbb{Z}$, and $i \in \{0, 1\}$, we have*

$$U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right) = -\frac{1}{(1+5x)^5} \sum_{j=0}^{4} \sum_{k=1}^{5} a_j(\tau) b_k(\tau) \cdot U^{(i)}\left(\frac{x^{m+j-5}}{(1+5x)^{n-k}}\right). \tag{10.18}$$

Proof. We can write

$$\begin{aligned} b_0(5\tau) &= -\sum_{k=1}^{5} b_k(5\tau) z^k, \\ 1 &= -\frac{1}{b_0(5\tau)} \sum_{k=1}^{5} b_k(5\tau) z^k, \\ z^{-n} &= -\frac{1}{b_0(5\tau)} \sum_{k=1}^{5} b_k(5\tau) z^{-(n-k)}, \end{aligned} \tag{10.19}$$

for $n \geq 1$. Writing z in terms of x, we have

$$(1+5x)^{-n} = -\frac{1}{b_0(5\tau)} \sum_{k=1}^{5} b_k(5\tau)(1+5x)^{-(n-k)}. \tag{10.20}$$

If we multiply both sides by x^m for some $m \geq 1$, then

$$\frac{x^m}{(1+5x)^n} = -\frac{1}{b_0(5\tau)}\sum_{k=1}^{5} b_k(5\tau)\cdot\frac{x^m}{(1+5x)^{n-k}}$$
$$= -\frac{1}{(1+5x(5\tau))^5}\sum_{k=1}^{5} b_k(5\tau)\cdot\frac{x^m}{(1+5x)^{n-k}}. \tag{10.21}$$

We expand each power of x with its modular equation, and rearrange:

$$\frac{x^m}{(1+5x)^n} = -\frac{1}{b_0(5\tau)}\sum_{k=1}^{5} b_k(5\tau)\cdot\sum_{j=0}^{4} a_j(5\tau)\frac{x^{m+j-5}}{(1+5x)^{n-k}}$$
$$= -\frac{1}{(1+5x(5\tau))^5}\sum_{j=0}^{4}\sum_{k=1}^{5} a_j(5\tau)b_k(5\tau)\cdot\frac{x^{m+j-5}}{(1+5x)^{n-k}}. \tag{10.22}$$

Now multiply both sides by $F\cdot Z^{1-i}$:

$$F\cdot Z^{1-i}\cdot\frac{x^m}{(1+5x)^n} = -\frac{1}{b_0(5\tau)}\sum_{k=1}^{5} b_k(5\tau)\cdot\sum_{j=0}^{4} a_j(5\tau)\frac{x^{m+j-5}}{(1+5x)^{n-k}}$$
$$= -\frac{1}{(1+5x(5\tau))^5}\sum_{j=0}^{4}\sum_{k=1}^{5} a_j(5\tau)b_k(5\tau)\cdot F\cdot Z^{1-i}\cdot\frac{x^{m+j-5}}{(1+5x)^{n-k}}. \tag{10.23}$$

Recall that by Section 2.7, for any functions $f(\tau), g(\tau)$, we have

$$U_5(f(5\tau)\cdot g(\tau)) = f(\tau)\cdot U_5(g(\tau)).$$

This gives us

$$U_5\left(F\cdot Z^{1-i}\cdot\frac{x^m}{(1+5x)^n}\right)$$
$$= -\frac{1}{(1+5x)^5}\sum_{j=0}^{4}\sum_{k=1}^{5} a_j(\tau)b_k(\tau)\cdot U_5\left(F\cdot Z^{1-i}\cdot\frac{x^{m+j-5}}{(1+5x)^{n-k}}\right). \tag{10.24}$$

Dividing both sides by F, we achieve our formula.

□

10.3.3 General Relations

We need to provide certain general relations for $U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right)$. For this we will define the following:

$$\pi_1(m,r) := \begin{cases} 0, & 1 \le m \le 2, \text{ and } r = 1, \\ 3, & 1 \le m \le 2, \text{ and } r = 3, \\ \left\lfloor \frac{5r+1}{6} \right\rfloor, & 1 \le m \le 2, \text{ and } r \ge 2, \text{ and } r \ne 3, \\ 2, & m = 3, \text{ and } r = 2, \\ \left\lfloor \frac{5r-2}{6} \right\rfloor, & m = 3, \text{ and } r \ne 2, \\ \left\lfloor \frac{5r-m+1}{6} \right\rfloor, & m \ge 4; \end{cases}$$

$$\pi_0(m,r) := \begin{cases} \left\lfloor \frac{5r+1}{6} \right\rfloor, & m = 1, \\ \left\lfloor \frac{5r+1}{6} \right\rfloor, & m = 2, \text{ and } r \ne 3,4,5, \\ \left\lfloor \frac{5r-5}{6} \right\rfloor, & m = 2, \text{ and } 3 \le r \le 5, \\ \left\lfloor \frac{5r-m-2}{6} \right\rfloor, & m \ge 3. \end{cases}$$

Theorem 10.5. *There exist discrete arrays $h_1, h_0 : \mathbb{Z}^3 \to \mathbb{Z}$ such that*

$$U^{(1)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n-4}} \sum_{r \ge \lceil m/5 \rceil} h_1(m,n,r) \cdot 5^{\pi_1(m,r)} \cdot x^r, \tag{10.25}$$

$$U^{(0)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n-2}} \sum_{r \ge \lceil (m+2)/5 \rceil} h_0(m,n,r) \cdot 5^{\pi_0(m,r)} \cdot x^r. \tag{10.26}$$

Notice that

$$\pi_1(m,r) \ge \left\lfloor \frac{5r-m+1}{6} \right\rfloor, \tag{10.27}$$

$$\pi_0(m,r) \ge \left\lfloor \frac{5r-m-2}{6} \right\rfloor. \tag{10.28}$$

We will therefore begin by proving the following lemmas, which resemble Lemma 7.5 in form:

Lemma 10.6. *Let $\kappa, \delta, \mu \in \mathbb{Z}_{\ge 0}$ be fixed, and fix $i \in \{0,1\}$. If there exists a discrete array h_i such that*

$$U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n-\kappa}} \sum_{r \ge \left\lceil \frac{m+\delta}{5} \right\rceil} h_i(m,n,r) \cdot 5^{\left\lfloor \frac{5r-m+\mu}{6} \right\rfloor} \cdot x^r \tag{10.29}$$

for $1 \le m \le 5$, $1 \le n \le 5$, then such a relation can be made to hold for all $m \ge 1$, $n \ge 1$.

Lemma 10.7. *Let $\kappa, \delta \in \mathbb{Z}_{\geq 0}$ and $m_0 \in \mathbb{Z}_{\geq 1}$ be fixed, and fix $i \in \{0,1\}$. If there exists a discrete array h_i such that*

$$U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n-\kappa}} \sum_{r \geq \left\lceil \frac{m+\delta}{5} \right\rceil} h_i(m,n,r) \cdot 5^{\pi_i(m_0,r)} \cdot x^r \tag{10.30}$$

for $1 \leq n \leq 5$, then such a relation can be made to hold for all $n \geq 1$.

For $m \geq 5$, (10.27), (10.28) yield equality, and we may replace the floor functions of the powers of 5 in Lemma 10.6 with $\pi_i(m,r)$. However, for $1 \leq m \leq 4$, the functions $\pi_i(m,n)$ are more irregular. We therefore need Lemma 10.7 for $1 \leq m \leq 4$. If both lemmas are satisfied, and we can provide the initial relations for $1 \leq m \leq 5$, $1 \leq n \leq 5$, then we can construct the discrete arrays h_i such that Theorem 10.5 follows.

Proof of Lemma 10.6. We will use induction. Suppose that the relation holds for all positive integers strictly less than some $m_0, n_0 \in \mathbb{Z}_{\geq 6}$. We want to show that the relation can be made to hold for m_0 and n_0. We have

$$U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)$$

$$= -\frac{1}{(1+5x)^5} \sum_{j=0}^{4} \sum_{k=1}^{5} a_j(\tau) b_k(\tau) \cdot U^{(i)}\left(\frac{x^{m_0+j-5}}{(1+5x)^{n_0-k}}\right) \tag{10.31}$$

$$= -\frac{1}{(1+5x)^5} \sum_{j=0}^{4} \sum_{k=1}^{5} \frac{a_j(\tau) b_k(\tau)}{(1+5x)^{5(n_0-k)-\kappa}}$$

$$\times \sum_{r \geq \lceil (m_0+j-5+\delta)/5 \rceil} h_i(m_0+j-5, n_0-k, r) \cdot 5^{\left\lfloor \frac{5r-(m_0+j-5)+\mu}{6} \right\rfloor} \cdot x^r \tag{10.32}$$

$$= \frac{1}{(1+5x)^{5n_0-\kappa}} \sum_{j=0}^{4} \sum_{k=1}^{5} w(j,k)$$

$$\times \sum_{r \geq \lceil (m_0+j-5+\delta)/5 \rceil} h_i(m_0+j-5, n_0-k, r) \cdot 5^{\left\lfloor \frac{5r-(m_0+j-5)+\mu}{6} \right\rfloor} \cdot x^r, \tag{10.33}$$

with

$$w(j,k) := -\, a_j(\tau) b_k(\tau)(1+5x)^{5(k-1)}$$

$$= \sum_{l=1}^{25} v(j,k,l) \cdot 5^{\left\lfloor \frac{5l+j}{6} \right\rfloor} \cdot x^l. \tag{10.34}$$

Relation (10.34) can be demonstrated by expanding of $a_j(\tau) b_k(\tau)(1+5x)^{5(k-1)}$. Expanding $w(j,k)$, we have

$$\begin{aligned}
&U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)\\
=&\frac{1}{(1+5x)^{5n_0-\kappa}}\sum_{j=0}^{4}\sum_{k=1}^{5}\sum_{l=1}^{25}\sum_{r\geq\lceil(m_0+j-5+\delta)/5\rceil}\\
&v(j,k)\cdot h_i(m_0+j-5,n_0-k,r)\cdot 5^{\left\lfloor\frac{5r-(m_0+j-5)+\mu}{6}\right\rfloor+\left\lfloor\frac{5l+j}{6}\right\rfloor}\cdot x^{r+l}. \qquad (10.35)
\end{aligned}$$

Notice that for any $M, N \in \mathbb{Z}$, we have

$$\left\lfloor\frac{M}{6}\right\rfloor+\left\lfloor\frac{N}{6}\right\rfloor\geq\left\lfloor\frac{M+N-5}{6}\right\rfloor.$$

Because of this,

$$\left\lfloor\frac{5r-(m_0+j-5)+\mu}{6}\right\rfloor+\left\lfloor\frac{5l+j}{6}\right\rfloor\geq\left\lfloor\frac{5(r+l)-m_0+\mu}{6}\right\rfloor. \qquad (10.36)$$

And because

$$\begin{aligned}
r+l&\geq\left\lceil\frac{m_0+j-5+\delta}{5}\right\rceil+l\\
&\geq\left\lceil\frac{m_0+\delta}{5}-\frac{5-j}{5}\right\rceil+l\\
&\geq\left\lceil\frac{m_0+\delta}{5}\right\rceil-1+l\\
&\geq\left\lceil\frac{m_0+\delta}{5}\right\rceil,
\end{aligned}$$

we can relabel our powers of x so that

$$\begin{aligned}
U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)=&\frac{1}{(1+5x)^{5n_0-\kappa}}\sum_{\substack{0\leq j\leq 4,\\1\leq k\leq 5,\\1\leq l\leq 25}}\sum_{r\geq\left\lceil\frac{m_0+\delta}{5}\right\rceil}\\
&v(j,k)\cdot h_i(m_0+j-5,n_0-k,r-l)\cdot 5^{\left\lfloor\frac{5r-(m_0+j-5)+\mu}{6}\right\rfloor+\left\lfloor\frac{5l+j}{6}\right\rfloor}\cdot x^{r}. \qquad (10.37)
\end{aligned}$$

If we define the discrete array H_i by

$$H_i(m,n,r) := \begin{cases} \sum_{\substack{0\le j\le 4,\\ 1\le k\le 5,\\ 1\le l\le 25}} \quad \sum_{r\ge \lceil \frac{m+\delta}{5}\rceil -1+l} \hat{H}(i,j,k,l,r), \;\; r\ge l \\ 0, \;\; r<l \end{cases} \tag{10.38}$$

with

$$\hat{H}(i,j,k,l,r) := v(j,k)\cdot h_i(m+j-5, n-k, r-l)\cdot 5^{\epsilon(i,j,l,m,r)},$$

$$\epsilon(i,j,l,m,r) := \left\lfloor \frac{5(r-l)-(m+j-5)+\mu}{6} \right\rfloor + \left\lfloor \frac{5l+j}{6} \right\rfloor - \left\lfloor \frac{5r-m_0+\mu}{6} \right\rfloor,$$

then

$$U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right) = \frac{1}{(1+5x)^{5n_0-\kappa}} \times \sum_{r\ge\lceil \frac{m_0+\delta}{5}\rceil} H_i(m_0,n_0,r)\cdot 5^{\left\lfloor \frac{5r-m_0+\mu}{6}\right\rfloor}\cdot x^r. \tag{10.39}$$

□

This lemma nearly gives us the necessary relations for Theorem 10.5. However, we need to be more precise for $1\le m\le 4$. We therefore give the following lemma to provide for these cases.

Proof of Lemma 10.7.

$$\begin{aligned} &U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n}}\right) \\ &= -\frac{1}{(1+5x)^5}\sum_{k=1}^{5} b_k(\tau)\cdot U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n-k}}\right) && (10.40)\\ &= -\frac{1}{(1+5x)^5}\sum_{k=1}^{5}\frac{b_k(\tau)}{(1+5x)^{5(n-k)-\kappa}}\sum_{r\ge 1} h_i(m_0,n-k,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^r && (10.41)\\ &= \frac{1}{(1+5x)^{5n-\kappa}}\sum_{k=1}^{5}\hat{w}(k)\sum_{r\ge 1} h_i(m_0,n-k,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^r, && (10.42)\end{aligned}$$

with

$$\hat{w}(k) := -b_k(\tau)(1+5x)^{5(k-1)} = \begin{cases} \displaystyle\sum_{l=0}^{20} \hat{v}(k,l)\cdot 5^{\left\lfloor \frac{5l+10}{6}\right\rfloor}\cdot x^l, \ k<5 \\ \displaystyle 1+\sum_{l=1}^{20} \hat{v}(5,l)\cdot 5^{\left\lfloor \frac{5l+10}{6}\right\rfloor}\cdot x^l, \ k=5. \end{cases} \tag{10.43}$$

This can be demonstrated with a simple expansion of $\hat{w}(k)$. Expanding, we have

$$U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n-\kappa}}$$

$$\times\Bigg(\sum_{\substack{1\le k\le 4,\\ 0\le l\le 20,\\ r\ge \left\lceil \frac{m_0+\delta}{5}\right\rceil}} \hat{v}(k,l)\cdot h_i(m_0,n-k,r)\cdot 5^{\pi_i(m_0,r)+\left\lfloor \frac{5l+10}{6}\right\rfloor}\cdot x^{r+l} \tag{10.44}$$

$$+\sum_{\substack{1\le l\le 20,\\ r\ge \left\lceil \frac{m_0+\delta}{5}\right\rceil}} \hat{v}(5,l)\cdot h_i(m_0,n-5,r)\cdot 5^{\pi_i(m_0,r)+\left\lfloor \frac{5l+10}{6}\right\rfloor}\cdot x^{r+l} \tag{10.45}$$

$$+\sum_{r\ge \left\lceil \frac{m_0+\delta}{5}\right\rceil} h_i(m_0,n-5,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^r\Bigg). \tag{10.46}$$

With a change of index, we have

$$U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n-\kappa}} \tag{10.47}$$

$$\times\Bigg(\sum_{\substack{1\le k\le 4,\\ 0\le l\le 20,\\ r\ge l+\left\lceil \frac{m_0+\delta}{5}\right\rceil}} \hat{v}(k,l)\cdot h_i(m_0,n-k,r-l)\cdot 5^{\pi_i(m_0,r-l)+\left\lfloor \frac{5l+10}{6}\right\rfloor}\cdot x^{r} \tag{10.48}$$

$$+\sum_{\substack{1\le l\le 20,\\ r\ge l+\left\lceil \frac{m_0+\delta}{5}\right\rceil}} \hat{v}(5,l)\cdot h_i(m_0,n-5,r-l)\cdot 5^{\pi_i(m_0,r-l)+\left\lfloor \frac{5l+10}{6}\right\rfloor}\cdot x^{r} \tag{10.49}$$

$$+\sum_{r\ge \left\lceil \frac{m_0+\delta}{5}\right\rceil} h_i(m_0,n-5,r)\cdot 5^{\pi_i(m_0,r)}\cdot x^r\Bigg). \tag{10.50}$$

Now,

$$\pi_i(m_0, r-l) + \left\lfloor \frac{5l+10}{6} \right\rfloor \geq \pi_i(m_0, r). \tag{10.51}$$

This ensures that the 5-adic valuation of the terms in (10.48)-(10.49) is sufficient for us. We only need to examine the 5-adic valuation of (10.50):

$$\begin{aligned} \frac{1}{(1+5x)^{5n-\kappa}} & \sum_{r \geq \lceil \frac{m_0+\delta}{5} \rceil} h_i(m_0, n-5, r) \cdot 5^{\pi_i(m_0,r)} \cdot x^r \\ &= U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n-5}}\right). \end{aligned} \tag{10.52}$$

Therefore, if our relation in Lemma 10.7 is established for $1 \leq n \leq 5$, then it must be true for all $n \geq 6$ as well.

We may now rearrange the expression in (10.47)-(10.50) and define a new discrete array in a manner similar to (10.38) to finish the proof.

□

Proof of Theorem 10.5. These relations arise as consequences of Lemmas 10.6, 10.7 above, provided that the cases for $1 \leq m \leq 5$, $1 \leq n \leq 5$ are established. The computations needed to verify these relations are given at the end of the chapter. See our Mathematica supplement for the detailed computation.

□

As an additional consequence of Lemmas 10.6, 10.7, we have the following important result on the behavior of the coefficients in these expansions:

Corollary 10.8. *For all $n \in \mathbb{Z}_{\geq 1}$ we have:*

$$h_0(1, n, 1) \equiv 1 \pmod 5, \tag{10.53}$$
$$h_0(2, 5n-4, 1) \equiv 0 \pmod 5, \tag{10.54}$$
$$h_0(3, n, 1) \equiv 1 \pmod 5, \tag{10.55}$$
$$h_0(1, n, 2) \equiv 4 \pmod 5, \tag{10.56}$$
$$h_0(2, 5n-4, 2) \equiv 4 \pmod 5, \tag{10.57}$$
$$h_0(3, n, 2) \equiv 4 \pmod 5, \tag{10.58}$$
$$h_0(2, 5n-4, 3) \equiv 1 \pmod 5. \tag{10.59}$$

For all $n \in \mathbb{Z}_{\geq 1}$ and $1 \leq m \leq 3$ we have:

$$h_1(m, n, 1) \equiv 1 \pmod 5. \tag{10.60}$$

Proof. We will first prove (10.53)–(10.55). Let us reexamine (10.44), (10.45), (10.46). Notice that whenever (10.51) is strict, i.e.,

$$\pi_i(m_0, r-l) + \left\lfloor \frac{5l+10}{6} \right\rfloor > \pi_i(m_0, r), \tag{10.61}$$

we must have $h_i(m_0, n, r) \equiv h_i(m_0, n-5, r) \pmod 5$. We therefore need only establish that (10.61) is true in all relevant cases. Thereafter, we can simply compute the relevant coefficients for five consecutive values of n.

We note that in (10.45), $r \geq 1$ and $l \geq 1$. Because of this, $r+l \geq 2$, and (10.45) will contribute nothing to the linear coefficient. On the other hand, in (10.44), the only possibility is for $l = 0$ and $r = 1$. Because $\left\lfloor \frac{5(0)+10}{6} \right\rfloor > 1$, we easily get (10.61).

Therefore, we must have

$$\begin{aligned} h_0(1, n, 1) &\equiv h_0(1, n-5, 1) \pmod 5, \\ h_0(2, n, 1) &\equiv h_0(2, n-5, 1) \pmod 5, \\ h_0(3, n, 1) &\equiv h_0(3, n-5, 1) \pmod 5. \end{aligned}$$

These computations are too extensive to place here. We refer the reader to the online Mathematica supplement to the manuscript that this chapter is based on, at `https://www3.risc.jku.at/people/nsmoot/online7.nb`.

Our calculations find that $h_0(1, n, 1) \equiv h_0(3, n, 1) \equiv 1 \pmod 5$ for $1 \leq n \leq 5$. Therefore, (10.53)–(10.54) must be true for all n. On the other hand, $h_0(2, n, 1)$ does not have a constant residue class modulo 5, and $h_0(2, n, 1) \equiv 0 \pmod 5$ for $n \equiv 1 \pmod 5$.

To prove (10.56), (10.57), (10.58), we note that we may directly compute $\pi_0(m, r)$. Notice that the only way for $r + l = 2$ to be true is for $r = l = 1$ or $r = 2$ and $l = 0$. For the first case, we have

$$\pi_0(1, 2-1) + \left\lfloor \frac{5(1)+10}{6} \right\rfloor = 1 + 2 = 3 > 1 = \pi_0(1, 1), \tag{10.62}$$

$$\pi_0(2, 2-1) + \left\lfloor \frac{5(1)+10}{6} \right\rfloor = 1 + 2 = 3 > 1 = \pi_0(2, 1), \tag{10.63}$$

$$\pi_0(3, 2-1) + \left\lfloor \frac{5(1)+10}{6} \right\rfloor = 1 + 2 = 3 > 0 = \pi_0(3, 1). \tag{10.64}$$

Here, (10.51) is strict; for the second case, i.e., for $r = 2$ and $l = 0$, (10.51) follows immediately. We need only examine each case for five consecutive values of n.

To prove (10.59), we take into account that there are three different ways for $r + l = 3$ to be true. Either $r = 1$ and $l = 2$, or $r = 2$ and $l = 1$, or $r = 3$ and $l = 0$. We therefore have

$$\pi_0(2,3-2)+\left\lfloor\frac{5(2)+10}{6}\right\rfloor=1+3=4>1=\pi_0(2,1), \tag{10.65}$$

$$\pi_0(2,3-1)+\left\lfloor\frac{5(1)+10}{6}\right\rfloor=1+2=3>1=\pi_0(2,2), \tag{10.66}$$

and the inequality is again trivially true in the case that $l=0$. Because the inequality holds in both cases, we can again simply examine each case for five consecutive values of n.

Finally, to prove (10.60), we first note that for m fixed, we may use the same reasoning as was used to prove (10.54)–(10.55). To see how $h_1(m,n,1) \pmod 5$ varies with m, let us reexamine (10.35). Notice that for $m_0 \geq 6$, $U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)$ only depends on $U^{(i)}\left(\frac{x^{r+l}}{(1+5x)^{n_0-5}}\right)$, in which $r \geq 1$ and $l \geq 1$. In particular, for $m \geq 6$, the coefficient of $U^{(i)}\left(\frac{x^1}{(1+5x)^n}\right)$ cannot depend on $U^{(i)}\left(\frac{x^{m_0}}{(1+5x)^{n_0}}\right)$.

As only five values of m will contribute to the coefficient of $U^{(i)}\left(\frac{x^1}{(1+5x)^n}\right)$, we therefore only need to check (10.60) for $1 \leq m \leq 5$, and for $1 \leq n \leq 5$. □

10.4 5-adic Irregularities

10.4.1 Main Theorem

With the necessary relations established for $U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right)$, we can now work towards the main theorem. We begin with the following theorem:

Theorem 10.9.

$$\text{For every } f \in \mathcal{V}_n^{(0)}, \text{ we have } \frac{1}{5}\cdot U^{(0)}(f) \in \mathcal{V}_{5n-2}^{(1)}. \tag{10.67}$$

Proof. Let $f \in \mathcal{V}_n^{(0)}$. Then we can express f as

$$f=\frac{1}{(1+5x)^n}\sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)}\cdot x^m.$$

We write

$$U^{(0)}(f)=\sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)}\cdot U^{(0)}\left(\frac{x^m}{(1+5x)^n}\right) \tag{10.68}$$

$$=\frac{1}{(1+5x)^{5n-2}}\sum_{m\geq 1}\sum_{r\geq\lceil(m+2)/5\rceil} s(m)\cdot h_0(m,n,r)5^{\theta_1(m)+\pi_0(m,r)}\cdot x^r \tag{10.69}$$

$$=\frac{1}{(1+5x)^{5n-2}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_0(m,n,r)5^{\theta_1(m)+\pi_0(m,r)}\cdot x^r. \tag{10.70}$$

We examine $\theta_1(m) + \pi_0(m, r)$:

For $m = 1$:

$$\theta_1(1) + \pi_0(1, r) = 0 + \left\lfloor \frac{5r+1}{6} \right\rfloor \geq \theta_0(r) + 1.$$

For $m = 2$:

$$\theta_1(2) + \pi_0(2, r) = \begin{cases} 0 + \left\lfloor \frac{5r+1}{6} \right\rfloor, & 1 \leq r \leq 2 \text{ or } r \geq 6, \\ 0 + \left\lfloor \frac{5r-5}{6} \right\rfloor, & 3 \leq r \leq 5. \end{cases}$$

In both cases, $\theta_1(2) + \pi_0(2, r) \geq \theta_0(r) + 1$.

For $m = 3$:

$$\theta_1(3) + \pi_0(3, r) = 1 + \left\lfloor \frac{5r-5}{6} \right\rfloor \geq \theta_0(r) + 1.$$

For $m = 4$:

$$\theta_1(4) + \pi_0(4, r) = 1 + \left\lfloor \frac{5r-6}{6} \right\rfloor = \left\lfloor \frac{5r}{6} \right\rfloor \geq \theta_0(r) + 1$$

(recall that $m \geq 4$ cannot contribute to the coefficient of $U^{(0)}\left(\frac{x^1}{(1+5x)^n}\right)$ since $\lceil (4+2)/5 \rceil = 2$).

For $m \geq 5$:

$$\begin{aligned} \theta_1(m) + \pi_0(m, r) &= \left\lfloor \frac{5m-5}{6} \right\rfloor - 1 + \left\lfloor \frac{5r-m-2}{6} \right\rfloor \\ &\geq \left\lfloor \frac{5r-m-12}{6} \right\rfloor - 1 \\ &\geq \left\lfloor \frac{5r-5}{6} \right\rfloor + \left\lfloor \frac{4m-7}{6} \right\rfloor - 1 \\ &\geq \left\lfloor \frac{5r-5}{6} \right\rfloor + 2 - 1 \\ &\geq \left\lfloor \frac{5r-5}{6} \right\rfloor + 1 \\ &\geq \theta_0(r) + 1. \end{aligned}$$

Notice that, in all cases,

$$\theta_1(m) + \pi_0(m, r) \geq \theta_0(r) + 1,$$

so that $U^{(0)}(f) \in 5 \cdot \mathcal{V}_n^{(1)}$

$\square$

Theorem 10.10. *Let $f \in \mathcal{V}_n^{(1)}$ and write*

$$U^{(1)}(f) = \sum_{r\geq 1} \tilde{s}(r) \frac{x^r}{(1+5x)^{5n-4}}. \tag{10.71}$$

Then

$$\frac{1}{5}\left(U^{(1)}(f) - \tilde{s}(1)\frac{y}{(1+5x)^{5n-4}}\right) \in \mathcal{V}_{5n-4}^{(0)}. \tag{10.72}$$

Proof. Let $f \in \mathcal{V}_n^{(1)}$. Then we can express f as

$$f = \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta_0(m)} \cdot x^m.$$

We write

$$U^{(1)}(f) = \sum_{m\geq 1} s(m) \cdot 5^{\theta_0(m)} \cdot U^{(1)}\left(\frac{x^m}{(1+5x)^n}\right) \tag{10.73}$$

$$= \frac{1}{(1+5x)^{5n-4}} \sum_{m\geq 1} \sum_{r\geq \lceil m/5 \rceil} s(m) \cdot h_1(m,n,r) 5^{\theta_0(m)+\pi_1(m,r)} \cdot x^r \tag{10.74}$$

$$= \frac{1}{(1+5x)^{5n-4}} \sum_{r\geq 1} \sum_{m\geq 1} s(m) \cdot h_1(m,n,r) 5^{\theta_0(m)+\pi_1(m,r)} \cdot x^r. \tag{10.75}$$

Let us denote the coefficient of $\frac{x^r}{(1+5x)^{5n-4}}$ by $\tilde{s}(r)$. Now we examine the 5-adic valuation of each component

In the cases $1 \leq m \leq 2$, we have

$$\theta_0(1) + \pi_1(1,r) = \begin{cases} 0, & r = 1 \\ 3, & r = 3 \\ \left\lfloor \frac{5r+1}{6} \right\rfloor, & r = 2 \text{ or } r \geq 4. \end{cases} \tag{10.76}$$

With $m = 3$, we have

$$\theta_0(3) + \pi_1(3,r) = \begin{cases} 0, & r = 1 \\ 2, & r = 2 \\ \left\lfloor \frac{5r-2}{6} \right\rfloor, & r = 3 \text{ or } r \geq 4. \end{cases} \tag{10.77}$$

Notice that if $1 \leq m \leq 3$, then $\theta_0(m) + \pi_1(m,r) \geq \theta_1(r) + 1$ *except when* $r = 1$.

Finally, for $m \geq 4$, we have

$$\begin{aligned}
\theta_0(m) + \pi_1(m,r) &= \left\lfloor \frac{5m-5}{6} \right\rfloor - 1 + \left\lfloor \frac{5r-m+1}{6} \right\rfloor && (10.78)\\
&\geq \left\lfloor \frac{5r+4m-9}{6} \right\rfloor - 1 && (10.79)\\
&\geq \left\lfloor \frac{5r-5}{6} \right\rfloor + \left\lfloor \frac{4m-4}{6} \right\rfloor - 1 && (10.80)\\
&\geq \left\lfloor \frac{5r-5}{6} \right\rfloor + 1 && (10.81)\\
&\geq \theta_1(r) + 1. && (10.82)
\end{aligned}$$

We therefore have a 5-adic increase in the valuation of each component of $U^{(1)}(f)$ except for the coefficient of $\frac{y}{(1+5x)^{5n-4}}$. If we remove this component from $U^{(1)}(f)$ and then divide by 5, what remains is indeed a member of $\mathcal{V}^{(0)}_{5n-4}$.

□

10.5 Resolution

Our last two theorems are very nearly sufficient to give us the 5-adic convergence that we need, with the notable exception of the components which contribute to the coefficient of $\frac{y}{(1+5x)^{5n-4}}$. Indeed, the individual components need not be divisible by 5 at all. We therefore need to define a set under slightly more restrictive conditions than $\mathcal{V}^{(1)}_n$ for our purposes.

Definition 10.11.

$$\mathcal{W}^{(1)}_n := \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)} \cdot x^m \in \mathcal{V}^{(1)}_n : \sum_{m=1}^{3} s(m) \equiv 0 \bmod 5 \right\}. \tag{10.83}$$

This small additional condition is at last sufficient for our purposes.

Theorem 10.12. *Suppose that $f \in \mathcal{W}^{(1)}_n$. Then*

$$\frac{1}{5}\left(U^{(1)}(f)\right) \in \mathcal{V}^{(0)}_{5n-4}, \tag{10.84}$$

$$\frac{1}{5^2}\left(U^{(0)} \circ U^{(1)}(f)\right) \in \mathcal{W}^{(1)}_{25n-22}. \tag{10.85}$$

Proof. Let $f \in \mathcal{W}^{(1)}_n$ be written as

$$f = \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)} \cdot x^m.$$

We then have

$$U^{(1)}(f) = \frac{1}{(1+5x)^{5n-4}} \sum_{m\geq 1} \sum_{r\geq \lceil m/5 \rceil} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_0(m)+\pi_1(m,r)}\cdot x^r$$
$$= \frac{1}{(1+5x)^{5n-4}} t(1)\cdot 5^{\theta_1(1)}\cdot y + \frac{1}{(1+5x)^{5n-4}} \sum_{r\geq 2} t(r)\cdot 5^{\theta_1(r)+1}\cdot x^r,$$

with

$$t(r) = \begin{cases} \sum_{1\leq m\leq 5} s(m)\cdot h_1(m,n,1)\cdot 5^{\theta_0(m)+\pi_1(m,1)-\theta_1(r)}, & r=1 \\ \sum_{1\leq m\leq 5r} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_0(m)+\pi_1(m,r)-\theta_1(r)-1}, & r\geq 2. \end{cases}$$

We first prove (10.84). Notice that

$$t(1) = \sum_{m=1}^{5} s(m)\cdot h_1(m,n,1)\cdot 5^{\theta_0(m)+\pi_1(m,1)},$$

since $\theta_1(1) = 0$. Moreover, $\theta_0(4), \theta_0(5) \geq 1$, and $\theta_0(m) + \pi_1(m,1) = 0$ for $1 \leq m \leq 3$, so that

$$t(1) \equiv \sum_{m=1}^{3} s(m)\cdot h_1(m,n,1) \pmod 5.$$

Taking advantage of (10.60), we have

$$t(1) \equiv \sum_{m=1}^{3} s(m) \equiv 0 \pmod 5.$$

Writing

$$\tilde{t}(r) := \begin{cases} \frac{1}{5}\cdot t(1)\in\mathbb{Z},\ r=1 \\ t(r),\ r\neq 1, \end{cases}$$

we have

$$U^{(1)}(f) = \frac{1}{(1+5x)^{5n-4}} \sum_{r\geq 1} \tilde{t}(r)\cdot 5^{\theta_1(r)+1}\cdot x^r,$$

so that

$$\frac{1}{5}\left(U^{(1)}(f)\right) \in \mathcal{V}^{(0)}_{5n-4}.$$

We now prove (10.85). Dividing by 5^2, we find that

$$\begin{aligned}\frac{1}{5^2}\cdot\left(U^{(0)}\circ U^{(1)}(f)\right) =& \frac{5^{-2}}{(1+5x)^{25n-22}}\\ &\times\sum_{r\geq 1}\sum_{w\geq\lceil (r+2)/5\rceil}\tilde{t}(r)\cdot h_0(r,5n-4,w)\cdot 5^{\pi_0(r,w)+\theta_1(r)}\cdot x^w\\ =&\frac{1}{(1+5x)^{25n-22}}\sum_{w\geq 1} q(r)\cdot 5^{\theta_0(w)}x^w,\end{aligned}$$

with

$$\begin{aligned}q(w) &= \sum_{r=1}^{5w-2}\tilde{t}(r)\cdot h_0(r,5n-4,w)\cdot 5^{\pi_0(r,w)-\theta_0(w)-2}\\ &=\sum_{r=1}^{5w-2}\sum_{m=1}^{5r} s(m)\cdot h_1(m,n,r)\cdot h_0(r,5n-4,w)\cdot 5^{\theta_0(m)+\pi_1(m,r)+\pi_0(r,w)-\theta_0(w)-2}.\end{aligned}$$

In particular, since $\theta_0(w)=0$ for $1\leq w\leq 3$, we have

$$\begin{aligned}q(1) &= \sum_{r=1}^{3}\sum_{m=1}^{5r} s(m)\cdot h_1(m,n,r)\cdot h_0(r,5n-4,1)\cdot 5^{\theta_0(m)+\pi_1(m,r)+\pi_0(r,1)-2},\\ q(2) &= \sum_{r=1}^{8}\sum_{m=1}^{5r} s(m)\cdot h_1(m,n,r)\cdot h_0(r,5n-4,2)\cdot 5^{\theta_0(m)+\pi_1(m,r)+\pi_0(r,2)-2},\\ q(3) &= \sum_{r=1}^{13}\sum_{m=1}^{5r} s(m)\cdot h_1(m,n,r)\cdot h_0(r,5n-4,3)\cdot 5^{\theta_0(m)+\pi_1(m,r)+\pi_0(r,3)-2}.\end{aligned}$$

We want to show that $q(1)+q(2)+q(3)\equiv 0 \pmod 5$. First, we may remove all cases in which $\theta_0(m)+\pi_1(m,r)+\pi_0(r,w)-2\geq 1$. A quick estimate shows that for $1\leq w\leq 3$, we have

$$\begin{aligned}
&\theta_0(m) + \pi_1(m,r) + \pi_0(r,w) - 2 \\
&\geq \left\lfloor \frac{5m-5}{6} \right\rfloor - 1 + \left\lfloor \frac{5r-m+1}{6} \right\rfloor + \left\lfloor \frac{5r-w-2}{6} \right\rfloor - 2 \\
&\geq \left\lfloor \frac{5r+4m-9}{6} \right\rfloor + \left\lfloor \frac{5r-w-2}{6} \right\rfloor - 3 \\
&\geq 2\left\lfloor \frac{5r-5}{6} \right\rfloor + \left\lfloor \frac{4m-4}{6} \right\rfloor - 3 \\
&\geq 1,
\end{aligned}$$

for $m \geq 7$ or $r \geq 4$. We therefore need to examine the cases when $1 \leq m \leq 6$, $1 \leq r \leq 3$, and $1 \leq w \leq 3$. We provide three tables in Appendix D which show $\theta_0(m) + \pi_1(m,r) + \pi_0(r,w) - 2$ over this range.

Examining Table D.5, we see that we get a value of 0 for

$$(r,m) = (1,4), (2,1), (2,2), (3,3).$$

Moreover, we get a value of -1 for

$$(r,m) = (1,1), (1,2), (1,3).$$

Examining Table D.6, we see that we get a value of 0 for

$$(r,m) = (1,4), (2,1), (2,2), (3,3),$$

and a value of -1 for

$$(r,m) = (1,1), (1,2), (1,3).$$

Finally, examining Table D.7, we see that we get a value of 0 for

$$(r,m) = (1,1), (1,2), (1,3), (2,1), (2,2),$$

and no negative values.

We therefore have

$$
\begin{aligned}
&q(1)+q(2)+q(3)\\
&\quad\equiv\frac{1}{5}\cdot\sum_{m=1}^{3}s(m)\cdot h_1(m,n,1)\cdot h_0(1,5n-4,1)+s(4)\cdot h_1(4,n,1)\cdot h_0(1,5n+4,1)\\
&\quad+\sum_{m=1}^{2}s(m)\cdot h_1(m,n,2)\cdot h_0(2,5n-4,1)+s(3)\cdot h_1(3,n,3)\cdot h_0(3,5n-4,1)\\
&\quad+\frac{1}{5}\cdot\sum_{m=1}^{3}s(m)\cdot h_1(m,n,1)\cdot h_0(1,5n-4,2)+s(4)\cdot h_1(4,n,1)\cdot h_0(1,5n+4,2)\\
&\quad+\sum_{m=1}^{2}s(m)\cdot h_1(m,n,2)\cdot h_0(2,5n-4,2)+s(3)\cdot h_1(3,n,3)\cdot h_0(3,5n-4,2)\\
&\quad+\sum_{m=1}^{3}s(m)\cdot h_1(m,n,1)\cdot h_0(1,5n-4,3)+\sum_{m=1}^{2}s(m)\cdot h_1(m,n,2)\cdot h_0(2,5n-4,3)\pmod{5}.
\end{aligned}
$$

We want to prove that the right-hand side is integral, and divisible by 5. Rearranging, we have

$$
\begin{aligned}
&q(1)+q(2)+q(3)\\
&\quad\equiv\frac{1}{5}\cdot\left(\sum_{j=1}^{2}h_0(1,5n-4,j)\right)\cdot\left(\sum_{m=1}^{3}s(m)\cdot h_1(m,n,1)\right) &(10.86)\\
&\quad+h_0(1,5n-4,3)\cdot\left(\sum_{m=1}^{3}s(m)\cdot h_1(m,n,1)\right) &(10.87)\\
&\quad+\left(\sum_{j=1}^{2}h_0(1,5n-4,j)\right)\cdot s(4)\cdot h_1(4,n,1) &(10.88)\\
&\quad+\left(\sum_{j=1}^{3}h_0(2,5n-4,j)\right)\cdot\sum_{m=1}^{2}s(m)\cdot h_1(m,n,2) &(10.89)\\
&\quad+\left(\sum_{j=1}^{2}h_0(3,5n-4,j)\right)\cdot s(3)\cdot h_1(3,n,3)\pmod{5}. &(10.90)
\end{aligned}
$$

It now remains to demonstrate that this expression is $0 \pmod 5$.

We are going to show that each sum within parentheses is divisible by 5. In the first place, $h_1(m,n,1)\equiv 1 \pmod 5$ by (10.60). Therefore,

$$\sum_{m=1}^{3}s(m)\cdot h_1(m,n,1)\equiv\sum_{m=1}^{3}s(m)\equiv 0\pmod{5},$$

since $f \in \mathcal{W}_n^{(1)}$. Moreover,

$$\sum_{j=1}^{2} h_0(1, 5n-4, j) \equiv 0 \pmod 5$$

by (10.53), (10.56). Therefore, (10.86) is $0 \pmod 5$.

In like manner, we have the parenthesized sums in (10.87) congruent to $0 \pmod 5$ by (10.60); (10.88) congruent to $0 \pmod 5$ by (10.53) and (10.56); (10.89) congruent to $0 \pmod 5$ by (10.54), (10.57), and (10.59); (10.90) congruent to $0 \pmod 5$ by (10.55) and (10.58).

We then have

$$\frac{1}{5^2} \cdot \left(U^{(0)} \circ U^{(1)}(f)\right) = \frac{1}{(1+5x)^{25n-22}} \sum_{w \geq 1} q(r) \cdot 5^{\theta_0(w)} x^w \in \mathcal{V}_{25n-22}^{(1)},$$

with $q(1) + q(2) + q(3) \equiv 0 \pmod 5$, i.e.,

$$\frac{1}{5^2} \cdot \left(U^{(0)} \circ U^{(1)}(f)\right) \in \mathcal{W}_{25n-22}^{(1)}. \tag{10.91}$$

□

10.6 Completion of the Proof

At last, we have enough to prove Theorem 9.7:

Proof of Theorem 9.7. In the next section we will demonstrate that

$$\begin{aligned} L_1 =& \frac{F}{(1+5y)^3} \cdot \left(120y + 1805x^2 + 12050x^3 + 39500x^4 + 50000x^5\right) \\ =& \frac{5 \cdot F}{(1+5y)^3} \cdot \left(24y + 361x^2 + 2410x^3 + 7900x^4 + 10000x^5\right). \end{aligned}$$

Notice that

$$\frac{1}{5 \cdot F} \cdot L_1 = f_1 \in \mathcal{W}_3^{(1)}.$$

Suppose that for some $\alpha \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 1}$, we have

$$\frac{1}{5^{2\alpha-1} \cdot F} \cdot L_{2\alpha-1} \in \mathcal{W}_n^{(1)}.$$

Then

$$L_{2\alpha-1} = F \cdot 5^{2\alpha-1} \cdot f_{2\alpha-1}, \tag{10.92}$$

with $f_{2\alpha-1} \in \mathcal{W}_n^{(1)}$. Now,

$$L_{2\alpha} = U_5\left(L_{2\alpha-1}\right) = U_5\left(F \cdot 5^{2\alpha-1} \cdot f_{2\alpha-1}\right) = F \cdot 5^{2\alpha-1} \cdot U^{(1)}\left(f_{2\alpha-1}\right). \tag{10.93}$$

By (10.84) of Theorem 10.12, we know that there exists some $f_{2\alpha} \in \mathcal{V}_{5n-4}^{(0)}$ such that

$$U^{(1)}\left(f_{2\alpha-1}\right) = 5 \cdot f_{2\alpha}. \tag{10.94}$$

Therefore

$$L_{2\alpha} = F \cdot 5^{2\alpha} \cdot f_{2\alpha}. \tag{10.95}$$

Moreover,

$$L_{2\alpha+1} = U_5\left(Z \cdot L_{2\alpha}\right) = U_5\left(F \cdot 5^{2\alpha} \cdot Z \cdot f_{2\alpha}\right) = F \cdot 5^{2\alpha} \cdot U^{(0)}\left(f_{2\alpha}\right). \tag{10.96}$$

By (10.85) of Theorem 10.12, we know that there exists some $f_{2\alpha+1} \in \mathcal{W}_{5n-2}^{(1)}$ such that

$$U^{(0)}\left(f_{2\alpha}\right) = 5 \cdot f_{2\alpha+1}. \tag{10.97}$$

Therefore,

$$L_{2\alpha+1} = F \cdot 5^{2\alpha+1} \cdot f_{2\alpha+1}. \tag{10.98}$$

We briefly show that the power of our localizing factor for L_α matches with $\psi(\alpha)$ from (9.13), i.e., that

$$\frac{L_{2\alpha-1}}{5^{2\alpha-1} \cdot F} \in \mathcal{W}_{\psi(2\alpha-1)}^{(1)},$$
$$\frac{L_{2\alpha}}{5^{2\alpha} \cdot F} \in \mathcal{V}_{\psi(2\alpha)}^{(0)}.$$

It is a fact of elementary number theory that for all $\alpha \geq 1$,

$$\begin{aligned} 5^{2\alpha-1} &\equiv 5 \pmod{12}, \\ 5^{2\alpha} &\equiv 1 \pmod{12}, \end{aligned}$$

and therefore that

$$\begin{aligned} \left\lfloor \frac{5^{2\alpha-1}}{12} \right\rfloor &= \frac{5^{2\alpha-1}}{12} - \frac{5}{12}, \\ \left\lfloor \frac{5^{2\alpha}}{12} \right\rfloor &= \frac{5^{2\alpha}}{12} - \frac{1}{12}. \end{aligned}$$

With this, we have

$$\begin{aligned} 5 \cdot \psi(2\alpha - 1) - 4 &= 5 \cdot \left(\left\lfloor \frac{5^{2\alpha}}{12} \right\rfloor + 1 \right) - 4 \\ &= 5 \cdot \left(\frac{5^{2\alpha}}{12} - \frac{1}{12} + 1 \right) - 4 \\ &= \frac{5^{2\alpha+1}}{12} - \frac{5}{12} + 1 \\ &= \left\lfloor \frac{5^{2\alpha+1}}{12} \right\rfloor + 1 \\ &= \psi(2\alpha). \end{aligned}$$

In similar fashion, it can be proved that

$$5 \cdot \psi(2\alpha) - 2 = \psi(2\alpha + 1).$$

This is compatible with the increase in the localizing powers in Theorem 10.5. Finally, $\psi(1) = 3$ is the localizing power for L_1.

□

10.7 Initial Relations

For i fixed, our theorem for expanding $U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right)$ requires 25 initial relations to be justified. However, these relations are ultimately dependent on far fewer relations, since one can very quickly verify that

$$U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right) =\frac{1}{5^m}\cdot U^{(i)}\left(\frac{(z-1)^m}{z^n}\right) \tag{10.99}$$

$$=\frac{1}{5^m}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\cdot U^{(i)}\left(z^{r-n}\right) \tag{10.100}$$

$$=\frac{1}{5^m}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\cdot U^{(i)}\left((1+5x)^{r-n}\right). \tag{10.101}$$

We can very quickly compute any value of $U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right)$, provided we have exact expressions for $U^{(i)}\left((1+5x)^r\right)$ for $-n \leq r \leq m-n$.

To compute $U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right)$ for $1 \leq m, n \leq 5$, we need to have expressions for $U^{(i)}\left((1+5x)^r\right)$ for $-5 \leq r \leq 4$. However, we have the degree 5 modular equation for $z = 1+5x$, which yields

$$U^{(i)}\left((1+5x)^n\right) = -\sum_{k=0}^{4} b_k(\tau)\cdot U^{(i)}\left((1+5x)^{k+n-5}\right). \tag{10.102}$$

Moreover, for $n \geq 0$ we obviously have

$$U^{(i)}\left((1+5x)^n\right) = \sum_{k=0}^{n}\binom{n}{k}\cdot 5^k\cdot U^{(i)}\left(x^k\right), \tag{10.103}$$

and x is the solution to a degree 5 modular equation, (10.12).

Therefore, in order to determine $U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right)$ for any $m, n \in \mathbb{Z}$ and $i \in \{0, 1\}$ fixed, we only need to determine relations for $U^{(i)}\left(x^k\right)$ for five consecutive values of k. With two values of i, this gives us 10 relations to establish. Once these relations are established, verification of the 50 initial relations follows as a relatively simple, if somewhat tedious, computational exercise.

Theorem 10.13. *The relations from Theorem 10.5, together with the congruence conditions of Corollary 10.8, hold for* $1 \leq m \leq 5$, *and* $1 \leq n \leq 5$.

The calculation is straightforward, but we detail it in a Mathematica supplement which can be found online at `https://www3.risc.jku.at/people/nsmoot/online7.nb`.

10.7.1 Computing the Initial Cases

Here we compute the relations which can be found in Appendix C, together with the modular equations (10.12), (10.13), and our representation for L_1, (9.11).

Our initial relations consist of

$$U^{(i)}\left(x^l\right) = p_{i,l}(y) \in \mathbb{Z}[y], \tag{10.104}$$

for $1 \leq l \leq 4$, and

$$(1+5x) \cdot U^{(i)}(1) = p_{i,0}(y) \in \mathbb{Z}[y] \tag{10.105}$$

(in both cases, $0 \leq i \leq 1$).

We can use Theorem 2.40 to verify that $x \in \mathcal{M}(\Gamma_0(10))$, and Theorem 2.41 to show that

$$\begin{aligned}
\mathrm{ord}_{\infty}^{(10)}(x^{-1}) &= -1,\\
\mathrm{ord}_{1/5}^{(10)}(x^{-1}) &= 0,\\
\mathrm{ord}_{1/2}^{(10)}(x^{-1}) &= 0,\\
\mathrm{ord}_{0}^{(10)}(x^{-1}) &= 1.
\end{aligned}$$

This proves that $1/y \in \mathcal{M}^{\infty}(\Gamma_0(10))$. Therefore, if we denote m as the degree of $p_{i,l}$, then multiplying both sides of the proposed relations above by $1/x^m$, our relations take on the form

$$\frac{1}{x^m} \cdot U^{(i)}\left(x^l\right) \in \mathbb{Z}[x^{-1}] \subseteq \mathcal{M}^{\infty}(\Gamma_0(10)), \tag{10.106}$$

$$\frac{1}{x^m} \cdot (1+5x) \cdot U^{(i)}(1) \in \mathbb{Z}[x^{-1}] \subseteq \mathcal{M}^{\infty}(\Gamma_0(10)). \tag{10.107}$$

Here all that remains is to verify that the left-hand sides of each prospective relation are elements of $\mathcal{M}^{\infty}(\Gamma_0(10))$. Then we may compare the principal parts and constants of both sides: equality of these parts implies equality overall.

We will begin with the relations of the form (10.106). If we recall the definition of $U^{(i)}$, then our left-hand side takes the form

$$\begin{aligned}
&\frac{1}{x(\tau)^m} \cdot \frac{1}{F(\tau)} \cdot U_5\left(F(\tau) \cdot Z(\tau)^{1-i} \cdot x(\tau)^l\right)\\
&= U_5\left(\frac{F(\tau)}{F(5\tau)} \cdot \frac{Z(\tau)^{1-i} \cdot x(\tau)^l}{x(5\tau)^m}\right).
\end{aligned}$$

Now, it is well-known (e.g., [41, Corollary 2.3]) that

$$F(\tau) \in \mathcal{M}_2(\Gamma_0(2)) \subseteq \mathcal{M}_2(\Gamma_0(10)) \subseteq \mathcal{M}_2(\Gamma_0(50)).$$

Moreover,

$$F(5\tau) \in \mathcal{M}_2\left(\Gamma_0(50)\right).$$

This implies that

$$\frac{F(\tau)}{F(5\tau)} \in \mathcal{M}\left(\Gamma_0(50)\right).$$

One can directly compute that

$$\begin{aligned}
\mathrm{ord}_{\infty}^{(50)}\left(F\left(\tau\right)\right) &= 1, & \mathrm{ord}_{\infty}^{(50)}\left(F\left(5\tau\right)\right) &= 5,\\
\mathrm{ord}_{1/25}^{(50)}\left(F\left(\tau\right)\right) &= 0, & \mathrm{ord}_{1/25}^{(50)}\left(F\left(5\tau\right)\right) &= 0,\\
\mathrm{ord}_{1/2}^{(50)}\left(F\left(\tau\right)\right) &= 5, & \mathrm{ord}_{1/2}^{(50)}\left(F\left(5\tau\right)\right) &= 1,\\
\mathrm{ord}_{0}^{(50)}\left(F\left(\tau\right)\right) &= 5, & \mathrm{ord}_{0}^{(50)}\left(F\left(5\tau\right)\right) &= 1,\\
\mathrm{ord}_{k/5}^{(50)}\left(F\left(\tau\right)\right) &= 1, & \mathrm{ord}_{k/5}^{(50)}\left(F\left(5\tau\right)\right) &= 1\ (k=1,2,3,4),\\
\mathrm{ord}_{k/10}^{(50)}\left(F\left(\tau\right)\right) &= 1, & \mathrm{ord}_{k/10}^{(50)}\left(F\left(5\tau\right)\right) &= 1\ (k=1,3,7,9).
\end{aligned}$$

From this, we have

$$\begin{aligned}
\mathrm{ord}_{\infty}^{(50)}\left(\frac{F\left(\tau\right)}{F(5\tau)}\right) &= -4,\\
\mathrm{ord}_{1/25}^{(50)}\left(\frac{F\left(\tau\right)}{F(5\tau)}\right) &= 0,\\
\mathrm{ord}_{1/2}^{(50)}\left(\frac{F\left(\tau\right)}{F(5\tau)}\right) &= 4,\\
\mathrm{ord}_{0}^{(50)}\left(\frac{F\left(\tau\right)}{F(5\tau)}\right) &= 4,\\
\mathrm{ord}_{k/5}^{(50)}\left(\frac{F\left(\tau\right)}{F(5\tau)}\right) &= 0,\ \ (k=1,2,3,4),\\
\mathrm{ord}_{k/10}^{(50)}\left(\frac{F\left(\tau\right)}{F(5\tau)}\right) &= 0,\ \ (k=1,3,7,9).
\end{aligned}$$

Therefore, we have

$$\frac{F(\tau)}{F(5\tau)} \in \mathcal{M}^{\infty}(\Gamma_0(50)),$$

with a zero of order 4 at the cusp $[1/2]_{50}$. On the other hand, if we take

$$W_{i,l,m}(\tau) := \frac{Z(\tau)^{1-i} \cdot x(\tau)^l}{x(5\tau)^m},$$

we can use Theorem 2.41 to show that

$$\begin{aligned}
\mathrm{ord}_0^{(50)}\left(W_{i,l,m}(\tau)\right) &= -(1-i) - 5l + m, \\
\mathrm{ord}_{1/2}^{(50)}\left(W_{i,l,m}(\tau)\right) &= -2(1-i), \\
\mathrm{ord}_{k/5}^{(50)}\left(W_{i,l,m}(\tau)\right) &= m \ (k = 1,2,3,4), \\
\mathrm{ord}_{k/10}^{(50)}\left(W_{i,l,m}(\tau)\right) &= l \ (k = 1,3,7,9).
\end{aligned}$$

Therefore, if we let

$$G_{i,l,m}(\tau) := \frac{F(\tau) W_{i,l,m}(\tau)}{F(5\tau)},$$

then we have

$$\begin{aligned}
\mathrm{ord}_0^{(50)}\left(G_{i,l,m}(\tau)\right) &= 4 - (1-i) - 5l + m, \\
\mathrm{ord}_{1/2}^{(50)}\left(G_{i,l,m}(\tau)\right) &= 4 - 2(1-i), \\
\mathrm{ord}_{k/5}^{(50)}\left(G_{i,l,m}(\tau)\right) &= m \ (k = 1,2,3,4), \\
\mathrm{ord}_{k/10}^{(50)}\left(G_{i,l,m}(\tau)\right) &= l \ (k = 1,3,7,9).
\end{aligned}$$

Inspection of the relations in Appendix C quickly reveals that $4 - (1-i) - 5l + m = 0$ for each relation of the form (10.106). Moreover, $4 - 2(1-i)$ is clearly positive for $i = 0, 1$. The final two orders are positive, as $m, l > 0$.

For relations of the form (10.107), our right-hand side has the form

$$\begin{aligned}
&\frac{1}{x(\tau)^2} \cdot z(\tau) \cdot \frac{1}{F(\tau)} \cdot U_5\left(F(\tau) \cdot Z(\tau)^{1-i}\right) \\
&= U_5\left(\frac{F(\tau)}{F(5\tau)} \cdot \frac{Z(\tau)^{1-i} \cdot z(5\tau)}{x(5\tau)^2}\right).
\end{aligned}$$

If we take

$$W_i(\tau) := \frac{Z(\tau)^{1-i} \cdot z(5\tau)}{x(5\tau)^2},$$

we can use Theorem 2.41 theorem to show that

$$\begin{aligned}
\mathrm{ord}_0^{(50)}\left(W_i(\tau)\right) &= 1-i,\\
\mathrm{ord}_{1/2}^{(50)}\left(W_i(\tau)\right) &= 1-2i,\\
\mathrm{ord}_{k/5}^{(50)}\left(W_i(\tau)\right) &= 1 \ (k=1,2,3,4),\\
\mathrm{ord}_{k/10}^{(50)}\left(W_i(\tau)\right) &= 1 \ (k=1,3,7,9).
\end{aligned}$$

Therefore, if we let

$$G_i(\tau) := \frac{F(\tau)W_{i,l,m}(\tau)}{F(5\tau)},$$

then we have

$$\begin{aligned}
\mathrm{ord}_0^{(50)}\left(G_i(\tau)\right) &= 5-i,\\
\mathrm{ord}_{1/2}^{(50)}\left(G_i(\tau)\right) &= 5-2i,\\
\mathrm{ord}_{k/5}^{(50)}\left(G_i(\tau)\right) &= 1 \ (k=1,2,3,4),\\
\mathrm{ord}_{k/10}^{(50)}\left(G_i(\tau)\right) &= 1 \ (k=1,3,7,9).
\end{aligned}$$

These orders are all nonnegative.

The functions inside the U_5 operators on the right hand sides of each of our prospective relations are each elements of $\mathcal{M}^\infty(\Gamma_0(50))$. Therefore, By Theorem 6.2, the U_5 operator pushes each to an element of $\mathcal{M}^\infty(\Gamma_0(50/5)) = \mathcal{M}^\infty(\Gamma_0(10))$.

We have verified that the left hand side of each of our relation can be be put into a relation of the form (10.106) or (10.107), in which either side is an element of $\mathcal{M}^\infty(\Gamma_0(10))$. All that remains is to examine the principal parts and constants of each of these relations.

This approach can also be used to prove (9.11). In this case, we want to prove that

$$\begin{aligned}
&U_5\left(\frac{L_0(\tau)}{F(5\tau)}\cdot\frac{Z(\tau)\cdot z(5\tau)^3}{x(5\tau)^5}\right)\\
&\quad=\left(120x^{-4}+1805x^{-3}+12050x^{-2}+39500x^{-1}+50000\right). \qquad (10.108)
\end{aligned}$$

If we define

$$W_y := \frac{Z(\tau)\cdot z(5\tau)^3}{x(5\tau)^5},$$

then

$$\begin{aligned}
\mathrm{ord}_{0}^{(50)}\left(W_y(\tau)\right) &= 1,\\
\mathrm{ord}_{1/2}^{(50)}\left(W_y(\tau)\right) &= 1,\\
\mathrm{ord}_{k/5}^{(50)}\left(W_y(\tau)\right) &= 2\ (k=1,2,3,4),\\
\mathrm{ord}_{k/10}^{(50)}\left(W_y(\tau)\right) &= 3\ (k=1,3,7,9).
\end{aligned}$$

If we let

$$G_y(\tau) := \frac{W_y(\tau)}{F(5\tau)},$$

then

$$\begin{aligned}
\mathrm{ord}_{0}^{(50)}\left(G_y(\tau)\right) &\geq -1+1,\\
\mathrm{ord}_{1/2}^{(50)}\left(G_y(\tau)\right) &\geq -1+1,\\
\mathrm{ord}_{k/5}^{(50)}\left(G_y(\tau)\right) &\geq -1+2\ (k=1,2,3,4),\\
\mathrm{ord}_{k/10}^{(50)}\left(G_y(\tau)\right) &\geq -1+2\ (k=1,3,7,9).
\end{aligned}$$

These orders are again all nonnegative. Because $L_0 \in \mathcal{M}_2(\Gamma_0(10))$, it will contribute no poles, and we need not examine it. In this case, the principal part on either side of (10.108) takes the form

$$\frac{120}{q^4} + \frac{365}{q^3} + \frac{2765}{q^2} + \frac{5030}{q} + 9375.$$

As a final application, we consider the proof of (10.12). We can use Theorem 2.41 to determine that $x(5\tau)^{-1} \in \mathcal{M}^{\infty}\left(\Gamma_0(50)\right)$, and that $x(5\tau)^{-5}\cdot x(\tau) \in \mathcal{M}^{\infty}\left(\Gamma_0(50)\right)$. As such, the principal part and constant of

$$x(5\tau)^{-25}\cdot\left(x^5 + \sum_{j=0}^{4} a_j(5\tau)x^j\right) \tag{10.109}$$

can quickly be verified to be 0, thus giving us (10.12).

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Appendix A
IMPLEMENTATION OF RADU'S ALGORITHM

A.1 Accessibility

Our software package is freely available as `RaduRK.m` via `https://www3.risc.jku.at/people/nsmoot/RKAlg/RaduRK.m`. The implementation uses Mathematica, and requires installation of a Diophantine software package called `4ti2` [1]. In particular, we used the interface `math4ti2.m` developed by Ralf Hemmecke and Silviu Radu. We will also make our software available on the Computer Algebra for Combinatorics section of the RISC webpage `https://risc.jku.at/research_topic/computer-algebra-for-combinatorics/`.

A demonstration of the software can be found at `https://www3.risc.jku.at/people/nsmoot/RKAlg/RKSupplement1.nb`, in which most of the examples in Chapter 4 are computed; and `https://www3.risc.jku.at/people/nsmoot/RKAlg/RKSupplement2.nb`, in which the overpartitions examples in Chapter 4 are computed.

Because `4ti2` is a Linux program, some additional steps are necessary in order to properly install our software onto an Apple or Windows operating system. We provide the necessary steps for installation onto Apple and Windows at `https://www3.risc.jku.at/people/nsmoot/RKAlg/4ti2installationinstructions.rtf`. All difficulties in installation should be communicated immediately to the author's email `nicolas.smoot@risc.jku.at`.

Appendix B
INITIAL RELATIONS (I)

The initial relations for our proof of the Choi–Kim–Lovejoy congruence family are as follows:

B.1 Group I

$$U^{(0)}(1) = p_1, \tag{B.1}$$

$$U^{(0)}(t^{-1}) = 1 + 5^2 t - 5p_1, \tag{B.2}$$

$$U^{(0)}(t^{-2}) = -9 + 5^5 t^2 + 9 \cdot 5p_1, \tag{B.3}$$

$$U^{(0)}(t^{-3}) = 17 \cdot 5 + 5^8 t^3 - 17 \cdot 5^2 p_1, \tag{B.4}$$

$$U^{(0)}(t^{-4}) = -161 \cdot 5 + 5^{11} t^4 + 161 \cdot 5^2 p_1. \tag{B.5}$$

B.2 Group II

$$\begin{aligned} U^{(0)}(p_0) =& - 63 \cdot 5^2 t - 104 \cdot 5^5 t^2 - 189 \cdot 5^7 t^3 - 24 \cdot 5^{10} t^4 - 5^{13} t^5 \\ & + p_1\big(1 - 63 \cdot 5^2 t - 104 \cdot 5^5 t^2 - 189 \cdot 5^7 t^3 - 24 \cdot 5^{10} t^4 \\ & - 5^{13} t^5\big), \end{aligned} \tag{B.6}$$

$$U^{(0)}(p_0 t^{-1}) = 5^2 t - 6p_1, \tag{B.7}$$

$$U^{(0)}(p_0 t^{-2}) = - 9 - 5^3 t + 5^5 t^2 + p_1(9 \cdot 5 - 5^3 t), \tag{B.8}$$

$$U^{(0)}(p_0 t^{-3}) = 17 \cdot 5 - 5^6 t^2 + 5^8 t^3 - p_1(17 \cdot 5^2 - 5^6 t^2), \tag{B.9}$$

$$U^{(0)}(p_0 t^{-4}) = - 161 \cdot 5 - 5^9 t^3 + 5^{11} t^4 + p_1(161 \cdot 5^2 - 5^9 t^3). \tag{B.10}$$

B.3 Group III

$$U^{(1)}(1) = 1, \tag{B.11}$$

$$U^{(1)}(t^{-1}) = -6 - 5^2 t, \tag{B.12}$$

$$U^{(1)}(t^{-2}) = 54 - 5^5 t^2, \tag{B.13}$$

$$U^{(1)}(t^{-3}) = -102 \cdot 5 - 5^8 t^3, \tag{B.14}$$

$$U^{(1)}(t^{-4}) = 966 \cdot 5 - 5^{11} t^4. \tag{B.15}$$

B.4 Group IV

$$\begin{aligned}
U^{(1)}(p_1) =& 233 \cdot 5^2 t + 1188 \cdot 5^4 t^2 + 317 \cdot 5^7 t^3 + 31 \cdot 5^{10} t^4 + 5^{13} t^5 \\
&+ p_0(2 \cdot 5 + 44 \cdot 5^3 t + 14 \cdot 5^6 t^2 + 5^9 t^3), && \text{(B.16)}\\
U^{(1)}(p_1 t^{-1}) =& 13 + 5^2 t + 5p_0, && \text{(B.17)}\\
U^{(1)}(p_1 t^{-2}) =& -66 - 5^4 t + 5^5 t^2 + 5^4 t p_0, && \text{(B.18)}\\
U^{(1)}(p_1 t^{-3}) =& 114 \cdot 5 - 5^7 t^2 + 5^8 t^3 + 5^7 t^2 p_0, && \text{(B.19)}\\
U^{(1)}(p_1 t^{-4}) =& -1037 \cdot 5 + 82 \cdot 5^4 t + 112 \cdot 5^6 t^2 - 7 \cdot 5^9 t^3 - 4 \cdot 5^{11} t^4 \\
&+ p_0 \left(t^{-1} - 2 \cdot 5^3 - 44 \cdot 5^5 t - 14 \cdot 5^8 t^2 - 4 \cdot 5^{10} t^3 \right). && \text{(B.20)}
\end{aligned}$$

Appendix C
INITIAL RELATIONS (II)

Below we list the ten fundamental relations that are justified using our cusp analysis in Chapter 10. For the complete derivation of the 50 relations, see our Mathematica supplement online at `https://www3.risc.jku.at/people/nsmoot/online3.nb`.

C.1 Group I

$$U^{(1)}(1) = \frac{1}{1+5y}\left(1 + 5^2 y + 16\cdot 5\cdot y^2\right) \tag{C.1}$$

$$U^{(1)}(y) = y \tag{C.2}$$

$$\begin{aligned} U^{(1)}\left(y^2\right) &= 51y + 471\cdot 5\cdot y^2 + 1364\cdot 5^2\cdot y^3 + 1776\cdot 5^3\cdot y^4 \\ &+ 1088\cdot 5^4\cdot y^5 + 256\cdot 5^5\cdot y^6 \end{aligned} \tag{C.3}$$

$$\begin{aligned} U^{(1)}\left(y^3\right) &= 41y + 2474\cdot 5\cdot y^2 + 29193\cdot 5^2\cdot y^3 + 152248\cdot 5^3\cdot y^4 \\ &+ 2231024\cdot 5^3\cdot y^5 + 814336\cdot 5^5\cdot y^6 + 4833536\cdot 5^5\cdot y^7 \\ &+ 3753984\cdot 5^6\cdot y^8 + 1847296\cdot 5^7\cdot y^9 + 524288\cdot 5^8\cdot y^{10} \\ &+ 65536\cdot 5^9\cdot y^{11} \end{aligned} \tag{C.4}$$

$$\begin{aligned} U^{(1)}\left(y^4\right) &= 11y + 3981\cdot 5\cdot y^2 + 138181\cdot 5^2\cdot y^3 + 8956203\cdot 5^2\cdot y^4 \\ &+ 62033852\cdot 5^3\cdot y^5 + 53739872\cdot 5^5\cdot y^6 + 791357952\cdot 5^5\cdot y^7 \\ &+ 1662808832\cdot 5^6\cdot y^8 + 2561985536\cdot 5^7\cdot y^9 \\ &+ 14663327744\cdot 5^7\cdot y^{10} + 2496888832\cdot 5^9\cdot y^{11} \\ &+ 7817854976\cdot 5^9\cdot y^{12} + 3503816704\cdot 5^{10}\cdot y^{13} \\ &+ 1065353216\cdot 5^{11}\cdot y^{14} + 197132288\cdot 5^{12}\cdot y^{15} \\ &+ 16777216\cdot 5^{13} y^{16} \end{aligned} \tag{C.5}$$

C.2 Group II

$$U^{(0)}(1) = \frac{1}{1+5y}\left(-5y - 4\cdot 5\cdot y^2\right) \tag{C.6}$$

$$U^{(0)}(y) = 5y + 4\cdot 5\cdot y^2 \tag{C.7}$$

$$\begin{aligned} U^{(0)}\left(y^2\right) &= 5y + 153\cdot 5\cdot y^2 + 3956\cdot 5\cdot y^3 + 8528\cdot 5^2\cdot y^4 + 9152\cdot 5^3\cdot y^5 \\ &+ 4864\cdot 5^4\cdot y^6 + 1024\cdot 5^5\cdot y^7 \end{aligned} \tag{C.8}$$

$$\begin{aligned} U^{(0)}\left(y^3\right) &= y + 1874y^2 + 40101\cdot 5\cdot y^3 + 309864\cdot 5^2\cdot y^4 \\ &+ 1252624\cdot 5^3\cdot y^5 + 3071232\cdot 5^4\cdot y^6 + 4892928\cdot 5^5\cdot y^7 \\ &+ 26039296\cdot 5^5\cdot y^8 + 18464768\cdot 5^6\cdot y^9 + 8404992\cdot 5^7\cdot y^{10} \\ &+ 2228224\cdot 5^8\cdot y^{11} + 262144\cdot 5^9\cdot y^{12} \end{aligned} \tag{C.9}$$

$$\begin{aligned} U^{(0)}\left(y^4\right) &= 329\cdot 5\cdot y^2 + 116926\cdot 5\cdot y^3 + 2285653\cdot 5^2\cdot y^4 \\ &+ 21410212\cdot 5^3\cdot y^5 + 119101984\cdot 5^4\cdot y^6 + 438497152\cdot 5^5\cdot y^7 \\ &+ 45458688\cdot 5^8\cdot y^8 + 2150618112\cdot 5^7\cdot y^9 + 3033554944\cdot 5^8\cdot y^{10} \\ &+ 3217784832\cdot 5^9\cdot y^{11} + 12811829248\cdot 5^9\cdot y^{12} \\ &+ 37793038336\cdot 5^9\cdot y^{13} + 16051601408\cdot 5^{10}\cdot y^{14} \\ &+ 4647288832\cdot 5^{11}\cdot y^{15} + 822083584\cdot 5^{12}\cdot y^{16} \\ &+ 67108864\cdot 5^{13}\cdot y^{17} \end{aligned} \tag{C.10}$$

Appendix D
MISCELLANEOUS

D.1 Tables From Chapter 6

	r				
Elements a/c of $\mathcal{C}(20)$ Approached by τ	0	1	2	3	4
$\frac{1}{20}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$
$\frac{1}{10}$	$\frac{1}{50}$	$\frac{1}{50}$	$\frac{1}{50}$	$\frac{1}{50}$	$\frac{1}{50}$
$\frac{1}{5}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$
$\frac{1}{4}$	$\frac{1}{20}$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{3}{20}$	$\frac{7}{20}$
$\frac{1}{2}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{7}{10}$	$\frac{9}{10}$
1	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1

Table D.1: Elements of $\mathcal{C}(100)$ Approached by $\frac{\tau+r}{5}$

	r				
Elements a/c of $\mathcal{C}(20)$ Approached by τ	0	1	2	3	4
$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$
$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
$\frac{1}{4}$	$\frac{1}{20}$	$\frac{1}{4}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$
$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{10}$
1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1

Table D.2: Elements of $\mathcal{C}(20)$ Approached by $\frac{\tau+r}{5}$

	f		
$\frac{a}{c} \in \mathcal{C}(100)$	$T(5\tau)^{m_A}A(\tau)$	$T(5\tau)^{m_{+t}}T(\tau)$	$T(5\tau)^{m_{-t}}T(\tau)^{-1}$
$\frac{1}{100}$	$1-25m_A$	$-5-25m_{+t}$	$5-25m_{-t}$
$\frac{1}{50}$	$-5+5m_A$	$1+5m_{+t}$	$-1+5m_{-t}$
$\frac{1}{25}$	4	0	0
$\frac{1}{20}$	m_A	$-5+m_{+t}$	$5+m_{-t}$
$\frac{1}{10}$	m_A	$1+m_{+t}$	$-1+m_{-t}$
$\frac{3}{20}$	m_A	$-5+m_{+t}$	$5+m_{-t}$
$\frac{1}{5}$	$2m_A$	$2m_{+t}$	$2m_{-t}$
$\frac{1}{4}$	$-1+m_A$	$5+m_{+t}$	$-5+m_{-t}$
$\frac{3}{10}$	m_A	$1+m_{+t}$	$-1+m_{-t}$
$\frac{7}{20}$	m_A	$-5+m_{+t}$	$5+m_{-t}$
$\frac{2}{5}$	$2m_A$	$2m_{+t}$	$2m_{-t}$
$\frac{9}{20}$	m_A	$-5+m_{+t}$	$5+m_{-t}$
$\frac{1}{2}$	$5+m_A$	$5+m_{+t}$	$-5+m_{-t}$
$\frac{3}{5}$	$2m_A$	$2m_{+t}$	$2m_{-t}$
$\frac{7}{10}$	m_A	$1+m_{+t}$	$-1+m_{-t}$
$\frac{4}{5}$	$2m_A$	$2m_{+t}$	$2m_{-t}$
$\frac{9}{10}$	m_A	$1+m_{+t}$	$-1+m_{-t}$
1	$-4+2m_A$	$10+2m_{+t}$	$-10+2m_{-t}$

Table D.3: $\mathrm{ord}_{a/c}^{(100)}(f)$ for $a/c \in \mathcal{C}(100)$

$\mathrm{ord}_{a/c}^{(100)}(f)$	f	
$\frac{a}{c} \in \mathcal{C}(100)$	$T(5\tau)^{m_1}G(\tau)$	$T(5\tau)^{m_H}H(\tau)$
$\frac{1}{100}$	$-2-25m_1$	$-3-25m_2$
$\frac{1}{50}$	$5m_1$	$5m_2$
$\frac{1}{25}$	0	3
$\frac{1}{20}$	$-2+m_1$	$-3+m_2$
$\frac{1}{10}$	m_1	m_2
$\frac{3}{20}$	$-2+m_1$	$-3+m_2$
$\frac{1}{5}$	$2m_1$	$3+2m_2$
$\frac{1}{4}$	$10+m_1$	m_2
$\frac{3}{10}$	m_1	m_2
$\frac{7}{20}$	$-2+m_1$	$-3+m_2$
$\frac{2}{5}$	$2m_1$	$3+2m_2$
$\frac{9}{20}$	$-2+m_1$	$-3+m_2$
$\frac{1}{2}$	m_1	m_2
$\frac{3}{5}$	$2m_1$	$3+2m_2$
$\frac{7}{10}$	m_1	m_2
$\frac{4}{5}$	$2m_1$	$3+2m_2$
$\frac{9}{10}$	m_1	m_2
1	$2m_1$	$2m_2$

Table D.4: $\mathrm{ord}_{a/c}^{(100)}(f)$ for $a/c \in \mathcal{C}(100)$

D.2 Tables From Chapter 10

We provide the tables used in Chapter 10. These can easily be constructed by hand. We provide additional details on this and other computations in our Mathematica supplement at `https://www3.risc.jku.at/people/nsmoot/online3.nb`.

m	$r=1$	$r=2$	$r=3$
1	-1	0	1
2	-1	0	1
3	-1	1	0
4	0	1	1
5	1	2	1
6		2	2

Table D.5: Value of $\theta(m) + \pi_1(m, r) + \pi_0(r, 1) - 2$ with $1 \leq m \leq 6$, $1 \leq r \leq 3$

m	$r=1$	$r=2$	$r=3$
1	-1	0	1
2	-1	0	1
3	-1	1	0
4	0	1	1
5	1	2	1
6		2	2

Table D.6: Value of $\theta(m) + \pi_1(m, r) + \pi_0(r, 2) - 2$ with $1 \leq m \leq 6$, $1 \leq r \leq 3$

m	$r=1$	$r=2$	$r=3$
1	0	0	2
2	0	0	2
3	0	1	1
4	1	1	2
5	2	2	2
6		2	3

Table D.7: Value of $\theta(m) + \pi_1(m, r) + \pi_0(r, 3) - 2$ with $1 \leq m \leq 6$, $1 \leq r \leq 3$