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Simple C^2-finite Sequences: a Computable Generalization of C-finite Sequences*

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Abstract

The class of C^2-finite sequences is a natural generalization of holonomic sequences and consists of sequences satisfying a linear recurrence with C-finite coefficients, i.e., coefficients satisfying a linear recurrence with constant coefficients themselves. Recently, we investigated computational properties of C^2-finite sequences: we showed that these sequences form a difference ring and provided methods to compute in this ring.

From an algorithmic point of view, some of these results were not as far reaching as we hoped for. In this paper, we define the class of *simple C^2-finite sequences* and show that it satisfies the same computational properties, but does not share the same technical issues. In particular, we are able to derive bounds for the asymptotic behavior, can compute closure properties more efficiently, and have a characterization via the generating function.

1 Introduction

Many interesting combinatorial objects or coefficient sequences of special functions satisfy linear recurrences with polynomial coefficients [5]. This class of sequences is also known under the names holonomic, D-finite, or P-recursive. It is well-known that holonomic sequences are closed under several operations such as addition, multiplication, taking subsequences, etc. The generating function of a holonomic sequence satisfies a linear differential equation with polynomial coefficients and there is a one-to-one correspondence between the sequence representation and the generating function representation. All these properties can be executed algorithmically [15] and they are implemented in different computer algebra systems [13].

If the recurrence coefficients are just constants, these sequences are also called C-finite or C-recursive. Recently, we have defined the class of C^2-finite sequences [12] as sequences satisfying a linear recurrence relation with C-finite coefficients. Holonomic and q-holonomic sequences are strictly contained in this set. The main computational issue when working with this more general class (compared to holonomic sequences) is the presence of zero divisors.

To our knowledge, C^2-finite sequences have first been introduced formally by Kotek and Makowsky [18] in the context of graph polynomials. Thanatipanonda and Zhang [24] gave an overview on different properties of polynomial, C-finite and holonomic sequences and considered an extension under the name X-recursive sequences. The setting in these articles is slightly

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different which leads to complications if one aims at developing an algorithmic approach. So far [12], we have shown that C^2-finite sequences form a difference ring with respect to termwise addition and termwise multiplication and presented a first step towards setting up the theory of C^2-finite sequences algorithmically. An implementation of these methods in SageMath [25] is under development and already available for download.[1]

Still, for C^2-finite sequences, it is not (always) clear a priori whether they are effectively computable. In this paper we define (in analogy to simple P-recursive sequences [17]) simple C^2-finite sequences as sequences satisfying linear recurrences with C-finite coefficients and constant leading coefficient. In fact, most C^2-finite sequences we studied are even simple C^2-finite. These include, for example, sparse subsequences of C-finite sequences.

We show that the class of simple C^2-finite sequences is a computable subring of the ring of C^2-finite sequences. When executing closure properties, it is possible to algorithmically compute recurrences with coefficients of smaller order compared to the ones obtained with the methods for general C^2-finite sequences. Additionally, we derive asymptotic bounds and obtain a characterization for the generating functions of simple C^2-finite functions.

In Section 2.2, we give the definition of simple C^2-finite sequences and derive the asymptotic bounds (Subsection 2.3). In Section 3, we provide the algebraic characterization of simple C^2-finite sequences that serves as the theoretical backbone. Next, in Section 4, we consider in full detail the closure properties addition and multiplication of two simple C^2-finite sequences. Finally, in Section 5, we show which type of functional equation the generating functions of (simple) C^2-finite functions satisfy and how they are related to the recurrence relation satisfied by the sequence.

2 Preliminaries

We introduce some notation and basic ideas that are used throughout the article. We call a ring (or field) computable if all elements admit a finite representation, the ring (or field) operations can be computed effectively and we can decide whether an element is zero. In the entire paper $\mathbb{Q} \subseteq \mathbb{K} \subset \overline{\mathbb{Q}}$ denotes a number field. In particular, $\mathbb{K}$ is a computable subfield of the field of algebraic numbers $\overline{\mathbb{Q}}$ (which is computable itself). Let $\mathbb{K}^{\mathbb{N}}$ denote the $\mathbb{K}$-algebra of sequences under termwise addition and termwise multiplication (the Hadamard product). We write $a(n)$ for a sequence $a = (a(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$. It will be always be clear from the context if we mean the sequence a or a specific term of the sequence. Furthermore, σ denotes the shift operator $\sigma(a(n))_{n\in\mathbb{N}} \coloneqq (a(n+1))_{n\in\mathbb{N}}$. A difference subring $R \subseteq \mathbb{K}^{\mathbb{N}}$ is a subring which is closed under shifts. The ring $R[\sigma]$ is in general non-commutative and an element $\mathcal{A} \coloneqq \sum_{i=0}^{r} c_i\sigma^i \in R[\sigma]$ acts on a sequence $a \in \mathbb{K}^{\mathbb{N}}$ in the natural way as $\mathcal{A}a = (\sum_{i=0}^{r} c_i(n)a(n+i))_{n\in\mathbb{N}}$. We call r the order of the operator $\mathcal{A}$.

2.1 C-finite sequences

A sequence $c \in \mathbb{K}^{\mathbb{N}}$ is called C-finite if there is a non-zero operator $\mathcal{C} \coloneqq \sum_{i=0}^{r} \gamma_i\sigma^i \in \mathbb{K}[\sigma]$ with $\mathcal{C}c = 0$. The order of c is the minimal order of such an annihilating operator. The sequence c can be described uniquely by the operator $\mathcal{C}$ and initial values $c(0), \dots, c(r-1)$. The set of C-finite sequences forms a difference ring under elementwise addition and multiplication. We denote this ring by $\mathcal{R}_C$. These sequences are also closed under taking subsequences at arithmetic progressions and interlacing. Also, the Cauchy product of two C-finite sequences is again C-finite. These are called *closure properties* of C-finite sequences [15].

[1] The package can be obtained from `https://github.com/PhilippNuspl/rec_sequences`.

Let $\mathcal{C} \coloneqq \sum_{i=0}^{r-1} \gamma_i \sigma^i + \sigma^r$ be the unique monic minimal annihilating operator of the C-finite sequence c. Then, $\chi_c(x) \coloneqq \sum_{i=0}^{r-1} \gamma_i x^i + x^r \in \mathbb{K}[x]$ is called the characteristic polynomial of c. The roots of c are called the eigenvalues of c and determine the asymptotic growth of c: Let $\mathbb{L} \supseteq \mathbb{K}$ be the splitting field of $\chi_c(x)$. Then, $\chi_c(x)$ factors as $\chi_c(x) = \prod_{i=1}^{m}(x-\lambda_i)^{d_i}$ where $\lambda_i \in \mathbb{L}$ are the pairwise different eigenvalues of c and d_i their respective multiplicities. Then, there is an index $n_0 \in \mathbb{N}$ and polynomials $p_1, \ldots, p_m \in \mathbb{L}[x]$ with $\deg(p_i) = d_i - 1$ for $i = 1, \ldots, m$ such that

$$c(n + n_0) = \sum_{i=1}^{m} p_i(n)\lambda_i^n, \quad \text{for all } n \geq 0.$$

We call this the closed form of c [15, 21]. The λ_i and polynomials p_i can be computed [9, 4]. The index n_0 can be chosen as the minimal i such that $\gamma_i \neq 0$. Let

$$B_c \coloneqq \left\{ \left(n^{d_i}\lambda_i^n\right)_{n\in\mathbb{N}} \mid i \in \{1, \ldots, m\}, d_i \leq \deg(p_i) \right\}. \tag{1}$$

Then, the sequence c is an $\mathbb{L}$-linear combination of sequences in B_c from n_0 on.

Using the closed form representation, it is clear that every C-finite sequence can be bounded by an exponential sequence. I.e., for every C-finite sequence c there is an $\alpha \in \mathbb{Q}$ such that $|c(n)| \leq \alpha^n$ for all $n \geq 1$ [6].

2.2 Simple C^2-finite sequences

We can now introduce C^2-finite sequences as sequences satisfying a linear recurrence with coefficients which are themselves C-finite.

Definition 2.1. A sequence $a \in \mathbb{K}^\mathbb{N}$ is called C^2-finite over $\mathbb{K}$ if there are C-finite sequences $c_0, \ldots, c_r$ over $\mathbb{K}$ with $c_r(n) \neq 0$ for all n such that

$$c_0(n)a(n) + \cdots + c_r(n)a(n + r) = 0, \qquad \text{for all } n \in \mathbb{N}. \tag{2}$$

As the leading coefficient has no zeros, such a C^2-finite sequence a can again be described by finite data, namely the coefficients $c_0, \ldots, c_r$ and the initial values $a(0), \ldots, a(r-1)$. In [12] it was shown that the set of C^2-finite sequences is a ring under elementwise addition and multiplication and some methods to compute in this ring were presented. For recognizing whether a recurrence of the form (2) indeed defines a C^2-finite sequence, we need to decide whether the leading coefficient c_r has any zeros. This is known as the Skolem-Problem and it is not known whether it is decidable in general [23].

In order to avoid this problem, we introduce sequences satisfying a recurrence of the form (2) where the leading coefficient c_r is just a non-zero constant in $\mathbb{K}$. In this case, we can multiply the entire equation (2) by $\frac{1}{c_r}$. Hence, we can equivalently assume that $c_r = 1$.

An analogous construction, called simple P-recursive sequences, was also introduced for holonomic sequences [17]. These are sequences satisfying a linear recurrence with polynomial coefficients with a constant leading coefficient. These simple P-recursive sequences are a proper subring of the ring of holonomic sequences as the sequence of Catalan numbers is holonomic but not simple P-recursive [17, Section 8.1.5].

Definition 2.2. A sequence $a \in \mathbb{K}^\mathbb{N}$ is called simple C^2-finite over $\mathbb{K}$ if there are C-finite sequences $c_0, \ldots, c_{r-1}$ over $\mathbb{K}$ such that

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n + r - 1) + a(n + r) = 0 \tag{3}$$

for all $n \in \mathbb{N}$.

Most C^2-finite sequences that are discussed in [18, 24, 12] are in fact simple C^2-finite.

Example 2.3. Let c be a C-finite sequence. Then, $a(n) = \prod_{i=1}^{n} c(i)$ defines a simple C^2-finite sequence satisfying

$$-c(n+1)a(n) + a(n+1) = 0, \quad \text{for all } n \in \mathbb{N}.$$

If c denotes the Fibonacci numbers, these are called Fibonorial or Fibonacci factorial numbers (A003266 in the OEIS [10]). If c denotes the Lucas numbers, they are called Lucastorial (A135407 in the OEIS).

Example 2.4. Let f denote the Fibonacci sequence. Then $b(n) = f(n^2)$ satisfies a C^2-finite recurrence of order 2 with coefficients having maximal order 2 [18]. This sequence is even simple C^2-finite and satisfies a recurrence of order 3 with coefficients having order at most 4:

$$-f(6n+11)b(n) - c_1(n)b(n+1) + f(6n+9)b(n+2) + b(n+3) = 0$$

with

$$c_1(n) - 54c_1(n+1) + 331c_1(n+2) - 54c_1(n+3) + c_1(n+4) = 0$$

and initial values

$$c_1(0) = 136, c_1(1) = 6710, c_1(2) = 317434, c_1(3) = 14927768.$$

This recurrence can be found using guessing and fixing the coefficients of the recurrence to only involve C-finite sequences which have certain powers of the golden ratio (and its conjugate) as roots. The recurrence can then be verified using closure properties of C^2-finite sequences. Using an algorithm for computing algebraic relations of C-finite sequences due to Kauers and Zimmermann [16], we can write c_1 in terms of the Fibonacci sequence as

$$c_1(n) = f(4n+6)(-1 - 2f(4n+4) + 3f(4n+6)).$$

The result of Example 2.4 holds more generally. The proof of Corollary 3.6 in [12] shows that for every C-finite sequence c, the subsequence $c(jn^2 + kn + l)$ is simple C^2-finite for every $j, k, l \in \mathbb{N}$.

2.3 Bounds

Lemma 5 in [18] states, without a proof, that all C^2-finite sequences with a leading coefficient c_r with $c_r(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$ are bounded by a sequence α^{n^2}. Hence, every simple C^2-finite sequence can be bounded in the same way. For the sake of completeness we include a proof here. The proof is analogous to the case of holonomic sequences [8, Proposition 1.2.1].

Lemma 2.5. *Let a be a simple C^2-finite sequence over $\mathbb{K}$. Then, there is an $\alpha \in \mathbb{Q}$ such that $|a(n)| \leq \alpha^{n^2}$ for all $n \geq 1$.*

Proof. Suppose a satisfies the recurrence

$$c_0(n)\, a(n) + \cdots + c_{r-1}(n)\, a(n+r-1) + a(n+r) = 0$$

for all $n \in \mathbb{N}$ with $c_0, \ldots, c_{r-1} \in \mathcal{R}_C$. Then, for all $c_i(n)$ with $i = 0, \ldots, r-1$ there exists an $\alpha_i \in \mathbb{Q}$ such that $|c_i(n)| \leq \alpha_i^n$ for all $n \geq 1$. Let $1 \leq \alpha \in \mathbb{Q}$ be large enough such that

$$\sum_{i=0}^{r-1} \alpha_i^n \leq r \left(\max_{i=0,\ldots,r-1} \alpha_i \right)^n \leq \alpha^n$$

for $n \geq 1$ and large enough such that $|a(n)| \leq \alpha^{n^2}$ holds for $n = 1, \ldots, r-1$. We show $|a(n)| \leq \alpha^{n^2}$ by induction on n. Suppose the inequality holds for all $a(i)$ with $i \leq n+r-1$. In the induction step we have

$$|a(n+r)| = \left| \sum_{i=0}^{r-1} c_i(n) a(n+i) \right| \leq \sum_{i=0}^{r-1} |c_i(n)| |a(n+i)| \leq \sum_{i=0}^{r-1} \alpha_i^n \alpha^{(n+i)^2}$$
$$\leq \alpha^{(n+r-1)^2} \sum_{i=0}^{r-1} \alpha_i^n \leq \alpha^{(n+r-1)^2} \alpha^n \leq \alpha^{(n+r)^2}.$$

□

The bound in Lemma 2.5 is exact as sequences α^{n^2} for $\alpha \in \mathbb{Q}$ are simple C^2-finite.

It is not clear whether the same bound holds for C^2-finite sequences in general. Let c be C-finite and non-zero everywhere. To generalize the proof of Lemma 2.5 to C^2-finite sequences there would have to exist a $0 \neq \alpha \in \mathbb{Q}$ such that $|c(n)| \geq \alpha^n$ for all $n \in \mathbb{N}$. We are only aware of such bounds for special cases where these bounds are used to decide the Skolem-Problem [22, 26].

Lemma 2.5 shows that the sequences $a(n) = 2^{n^3}$ and $b(n) = \prod_{i=0}^n i!$ are not simple C^2-finite. Nevertheless, both sequences satisfy a linear recurrence. In particular, the sequence a is C^3-finite and b is D^2-finite [11].

It is well known that the sequence n^n is not holonomic [7]. In [3], polynomial recursive and rational recursive sequences were introduced. A sequence is called polynomial (rational) recursive if it can be described by a certain system of polynomial (rational) difference equations. It was shown that n^n is neither polynomial nor rational recursive. As simple C^2-finite sequences are polynomial recursive and C^2-finite sequences are rational recursive, n^n is not (simple) C^2-finite over $\mathbb{Q}$. The Catalan numbers are holonomic but not polynomial recursive [3, Corollary 8] and not simple P-recursive [17, Section 8.1.5]. In particular, the Catalan numbers are not simple C^2-finite over $\mathbb{Q}$. This is due to the fact that simple P-recursive sequences are eventually periodic modulo a prime p whereas the Catalan numbers are not [1]. Hence, not all holonomic sequences are simple C^2-finite sequences.

3 Algebraic characterization

For sequences $c_0, \ldots, c_r \in \mathcal{R}_C$ we denote by $\mathbb{K}_\sigma[c_0, \ldots, c_r]$ the smallest $\mathbb{K}$-difference-algebra which contains the sequences $c_0, \ldots, c_r$. I.e., this is the smallest difference ring containing the sequences $c_0, \ldots, c_r$, their shifts and all constants. We show that such algebras are always Noetherian. The proof follows the argument from [12, Theorem 3.5].

Lemma 3.1. *Let $c_0, \ldots, c_r \in \mathcal{R}_C$. Then, $\mathbb{K}_\sigma[c_0, \ldots, c_r]$ is a Noetherian ring.*

Proof. All the $\mathbb{K}$-vector spaces $\langle \sigma^i c_j \mid i \in \mathbb{N} \rangle_{\mathbb{K}}$ are finitely generated. Hence, also the difference algebras $\mathbb{K}_\sigma[c_j]$ are finitely generated. Therefore, also $\mathbb{K}_\sigma[c_0, \ldots, c_r]$ is finitely generated and a Noetherian ring [2, Corollary 7.7]. □

Let $a \in \mathbb{K}^{\mathbb{N}}$ and let $Q(\mathcal{R}_C)$ denote the total ring of fractions of $\mathcal{R}_C$. [12, Theorem 3.3] shows that a is C^2-finite if and only if the module

$$\langle \sigma^i a \mid i \in \mathbb{N} \rangle_{Q(\mathcal{R}_C)}$$

is finitely generated. The same arguments can be used to show the following analogous theorem for simple C^2-finite sequences.

Theorem 3.2. *The following are equivalent:*

1. *The sequence a is simple C^2-finite.*
2. *There exists an operator $\mathcal{A} = \sum_{i=0}^{r-1} c_i\sigma^i + \sigma^r \in \mathcal{R}_C[\sigma]$ and a simple C^2-finite sequence b with $\mathcal{A}a = b$.*
3. *The module $\langle \sigma^i a \mid i \in \mathbb{N}\rangle_{\mathcal{R}_C}$ over the ring $\mathcal{R}_C$ is finitely generated.*

Using Lemma 3.1 and Theorem 3.2 we can show that the set of simple C^2-finite sequences is a difference ring.

Theorem 3.3. *The set of simple C^2-finite sequences is a difference ring under termwise addition and termwise multiplication.*

Proof. Let a, b be simple C^2-finite sequences and

$$\begin{aligned}\mathcal{A} &= c_0 + c_1\sigma + \cdots + c_{r_1-1}\sigma^{r_1-1} + \sigma^{r_1}\\ \mathcal{B} &= d_0 + d_1\sigma + \cdots + d_{r_2-1}\sigma^{r_2-1} + \sigma^{r_2}\end{aligned}$$

the corresponding annihilating operators. Let

$$R \coloneqq \mathbb{K}_\sigma[c_0, \ldots, c_{r_1-1}, d_0, \ldots, d_{r_2-1}] \subsetneq \mathcal{R}_C.$$

By the definition of a and b, the modules $\langle \sigma^i a \mid i \in \mathbb{N}\rangle_R$ and $\langle \sigma^i b \mid i \in \mathbb{N}\rangle_R$ are both finitely generated. With Lemma 3.1, R is a Noetherian ring. Hence,

$$\langle \sigma^i(a+b) \mid i \in \mathbb{N}\rangle_R \subseteq \langle \sigma^i a \mid i \in \mathbb{N}\rangle_R + \langle \sigma^i b \mid i \in \mathbb{N}\rangle_R$$

and

$$\langle \sigma^i(ab) \mid i \in \mathbb{N}\rangle_R \subseteq \langle \sigma^i(a)\sigma^j(b) \mid i, j \in \mathbb{N}\rangle_R$$

are finitely generated as they are submodules of finitely generated modules over a Noetherian ring. In particular, there is an $r \in \mathbb{N}$ such that $\sigma^r(a+b)$ is an R-linear combination of $\sigma^0(a+b), \ldots, \sigma^{r-1}(a+b)$ and there is an $s \in \mathbb{N}$ such that $\sigma^s(ab)$ is an R-linear combination of $\sigma^0(ab), \ldots, \sigma^{s-1}(ab)$. Hence, $a+b$ and ab are simple C^2-finite.

The operator

$$\tilde{\mathcal{A}} \coloneqq \sigma(c_0) + \sigma(c_1)\sigma + \cdots + \sigma(c_{r_1-1})\sigma^{r_1-1} + \sigma^{r_1} \in \mathcal{R}_C[\sigma]$$

annihilates σa as $\tilde{\mathcal{A}}(\sigma a) = (\sigma\mathcal{A})a = 0$. Hence, the ring of simple C^2-finite sequences is also closed under shifts. $\square$

4 Computable ring

For computing closure properties of C^2-finite sequences, the Skolem-Problem is a limiting factor. Hence, it is not known whether the ring of C^2-finite sequences is computable [12]. For simple C^2-finite sequences the situation is easier and we can show that the ring is in fact computable. The idea is that we can reduce the computation of closure properties to solving a linear system over a Noetherian subring of $\mathcal{R}_C$. Using the closed form of C-finite sequences, we can compute a solution of such a linear system.

4.1 Ansatz

Suppose a, b are simple C^2-finite sequences of order r_1, r_2, respectively, satisfying recurrences

$$c_0(n)a(n) + \dots + c_{r_1-1}(n)a(n+r_1-1) + a(n+r_1) = 0,$$
$$d_0(n)b(n) + \dots + d_{r_2-1}(n)a(n+r_2-1) + b(n+r_2) = 0.$$

Furthermore, we define the companion matrix M_a of the sequence a as

$$M_a := \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{r_1-1} \end{pmatrix} \in \mathcal{R}_C^{r_1 \times r_1}.$$

Using the recurrence of a, we can express higher order shifts $\sigma^i a$ as $\mathcal{R}_C$-linear combinations of the shifts $a, \sigma a, \dots, \sigma^{r_1-1}a$. In particular, there are vectors $u_i := (u_{i,0}, \dots, u_{i,r_1-1})^\top \in \mathcal{R}_C^{r_1}$ such that $\sigma^i a = \sum_{j=0}^{r_1-1} u_{i,j}\sigma^j a$ for all $i \in \mathbb{N}$. Clearly, u_0 is the 0-th unit vector $e_0 := (1, 0, \dots, 0)$. The other u_i can be iteratively computed as $u_{i+1} := M_a\sigma(u_i)$ [12, Lemma 4.2]. Let v_i be the corresponding vectors for b.

To compute a recurrence for the addition $a+b$ or the multiplication ab we can make an ansatz of some order s with undetermined coefficients $x_0, \dots, x_{s-1}$. Using the recurrences of a and b, this ansatz can be reduced to the linear system

$$(w_0, w_1, \dots, w_{s-1})x = -w_s$$

where $w_i = u_i \oplus v_i$ is the direct sum in the case of the addition and $w_i = u_i \otimes v_i$ is the Kronecker product in the case of multiplication [12]. To be more concise, we write $A_s x = -w_s$ for this system.

By construction, all components of w_i are contained in the ring $R := \mathbb{K}_\sigma[c_0, \dots, c_{r_1}, d_0, \dots, d_{r_2}]$. With Lemma 3.1, R is a Noetherian ring. In particular, if the ansatz s is chosen big enough, the linear system has a solution in R:

Lemma 4.1. *The ansatz for the addition and multiplication of simple C^2-finite sequences can be chosen big enough such that the corresponding linear system has a solution in R.*

Proof. By the construction of the A_s we have an increasing chain of modules

$$\operatorname{Im} A_0 \subseteq \operatorname{Im} A_1 \subseteq \operatorname{Im} A_2 \subseteq \dots \subseteq R^m.$$

Since R is Noetherian, this chain has to stabilize. In particular, there is an s such that $\operatorname{Im} A_s = \operatorname{Im} A_{s+1}$. Then, $w_s \in \operatorname{Im}_{s+1} = \operatorname{Im} A_s$. Therefore, $A_s x = -w_s$ has a solution $x \in R^s$. □

4.2 Solving linear systems

Now, we show how we can compute a solution $x \in R^s$ of a linear system $Ax = b$ where $A \in R^{m\times s}, b \in R^m$ for a Noetherian ring $R \subseteq \mathcal{R}_C$. We heavily use the closed form of C-finite sequences. Therefore, we assume that the base field is always the field of algebraic numbers $\overline{\mathbb{Q}}$. Note that every C-finite sequence over $\overline{\mathbb{Q}}$ has again a closed form as $\overline{\mathbb{Q}}$ is algebraically closed itself. First, we consider the special case, where we compute a constant solution of such a system.

Lemma 4.2. *We can compute all constant solutions $x \in \overline{\mathbb{Q}}^s$ of the linear system $Ax = b$ where $A \in \mathcal{R}_C^{m\times s}$ and $b \in \mathcal{R}_C^m$. In particular, we can decide whether such a solution exists.*

Proof. It is sufficient to consider one equation, i.e., $A \in \mathcal{R}_C^{1\times s}$. The set of constant solutions is an affine subspace of $\overline{\mathbb{Q}}^s$. For several equations we can compute the intersection of these affine subspaces to determine all solutions. Using the closed form of the sequences, we can rewrite the equation $Ax = b$ as

$$\sum_{k=1}^{l} \underbrace{\left(\sum_{i\in S_k} \epsilon_{k,i}x_i + \epsilon_k\right)}_{=:y_k} (n-n_0)^{d_k}\lambda_k^{n-n_0} = 0, \quad \text{for all } n \geq n_0 \tag{4}$$

with $n_0 \in \mathbb{N}$, and $\epsilon_{k,i}, \epsilon_k, \lambda_k \in \overline{\mathbb{Q}}, d_k \in \mathbb{N}$ and $S_k \subseteq \{1,\dots,s\}$ for all $k = 1,\dots,l$. Certainly, if $y_k = 0$ for all $k = 1,\dots,l$ we have a solution. On the other hand, evaluating this equation for $n = n_0, n_0+1,\dots,$ yields a linear system for the y_k. This linear system is a generalized Vandermonde matrix, in particular it is regular [19, 20]. Therefore, if equation (4) holds, then $y_k = 0$ for all $k = 1,\dots,l$. This yields a linear system over $\overline{\mathbb{Q}}$ which can be solved. For the initial terms $n = 0,1,\dots,n_0-1$ the equation $Ax = b$ can simply be solved over $\overline{\mathbb{Q}}$. The affine space of all solutions of the single equation is now given as the intersection of the affine subspace arising from solving equation (4) and the affine subspaces arising from the initial terms. ☐

Let $R := \mathbb{K}_\sigma[c_0,\dots,c_r]$ for C-finite sequences $c_0,\dots,c_r$. Suppose n_0 is the largest index such that each c_i is a $\overline{\mathbb{Q}}$-linear combination of sequences in B_{c_i}, as defined in (1), from n_0 on. We write

$$B := B_{c_0} \cup \cdots \cup B_{c_m} \cup \{1\}.$$

Then, for any sequence $c \in R$ there is an $N \in \mathbb{N}$ and $x_{d_1,\dots,d_N} \in \overline{\mathbb{Q}}$ such that

$$c(n) = \sum_{d_1,\dots,d_N\in B} x_{d_1,\dots,d_N} d_1(n)\cdots d_N(n), \quad \text{for all } n \geq n_0.$$

Lemma 4.3. *Let $A \in R^{m\times s}$ and $b \in R^m$. If $Ax = b$ has a solution $x \in R^s$, we can compute such a solution.*

Proof. For $N = 1,2,\dots$ we write

$$x_i = \sum_{d_1,\dots,d_N\in B} x_{i,d_1,\dots,d_N} d_1\cdots d_N$$

for unknown coefficients $x_{i,d_1,\dots,d_N} \in \overline{\mathbb{Q}}$. Then, $Ax = b$ can be written equivalently as linear system for these unknown coefficients $x_{i,d_1,\dots,d_N}$. With Lemma 4.2 we can check whether the linear system has a solution for these $x_{i,d_1,\dots,d_N}$. As we know that a solution x exists, this algorithm has to terminate. ☐

We can now combine the results from Section 4.1 and Lemma 4.3 to show the following main theorem:

Theorem 4.4. *The ring of simple C^2-finite sequences over $\overline{\mathbb{Q}}$ is computable.*

Proof. Suppose a, b are simple C^2-finite with annihilating operators $\sum_{i=0}^{r_1-1} c_i\sigma^i + \sigma^{r_1}$, $\sum_{i=0}^{r_2-1} d_i\sigma^i + \sigma^{r_2}$, respectively. Using the theory presented in Section 4.1, there is an order $s \in \mathbb{N}$ of the ansatz such that the corresponding linear system over the computable Noetherian ring $R := \overline{\mathbb{Q}}_\sigma[c_0,\dots,c_{r_1-1},d_0,\dots,d_{r_1-1}]$ has a solution. This linear system can be computed and a solution of the system can be obtained with Lemma 4.3. As we do not know a priori how big this order s is and how big the N in the proof of Lemma 4.3 has to be chosen, we can simultaneously increase s and N. Eventually, this algorithm terminates and any solution gives rise to a recurrence for $a + b$ or ab. ☐

C^2-finite sequences are also closed under taking differences, partial sums, subsequences at arithmetic progressions and interlacing [12, 11]. The same proofs carry over to simple C^2-finite sequences. Even more, as these operations can be reduced to solving linear systems, these closure properties can be computed effectively.

4.3 Example

Consider the sequences

$$2^n a(n) + a(n+1) = 0, \quad b(n) + b(n+1) = 0.$$

Both are simple C^2-finite. We want to compute a recurrence for $c = a + b$. An ansatz of order 3 yields the linear system

$$\begin{pmatrix} 1 & -2^n & 2 \cdot 4^n \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \cdot 8^n \\ 1 \end{pmatrix}.$$

This is the smallest system which has a solution. Using the generalized inverse method from [12] to compute the solution we get the recurrence

$$\begin{aligned} &\left(-2^{5n+4} + 2^{4n+2} + 2^{3n+3} - 2^{2n+1}\right) c(n) \\ &\quad + \left(2^{5n+4} - 2^{3n+3} - 2^{2n+1} + 1\right) c(n+2) \\ &\qquad + \left(2^{4n+2} - 2^{2n+2} + 1\right) c(n+3) = 0 \end{aligned}$$

if we use columns 0 and 2 of the matrix. Using columns 1 and 2 we get

$$\begin{aligned} &\left(2^{3n+4} - 3 \cdot 2^{2n+2} + 2^{n+1}\right) c(n+1) \\ &+ \left(2^{3n+4} - 2^{2n+3} - 2^{n+1} + 1\right) c(n+2) \\ &\quad + \left(2^{2n+2} - 2^{n+2} + 1\right) c(n+3) = 0. \end{aligned}$$

Both recurrences have coefficients with maximal order 4. By Theorem 3.3, we know that c also has to satisfy a recurrence with leading coefficient 1.

If we make an ansatz $x_i = x_{i,1} + x_{i,2} 2^n$, the corresponding linear system for the $x_{i,j} \in \mathbb{Q}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{1,1} \\ x_{1,2} \\ x_{2,1} \\ x_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 8 \\ 1 \\ 0 \end{pmatrix}.$$

This system has the unique solution

$$(x_{0,1}, x_{0,2}, x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) = (0, 2, 2, 6, 3, 4)$$

which gives rise to the recurrence

$$\begin{aligned} &(2 \cdot 2^n)\, c(n) + (2 + 6 \cdot 2^n) c(n+1) \\ &\quad + (3 + 4 \cdot 2^n)\, c(n+2) + c(n+3) = 0. \end{aligned}$$

Hence, the simple C^2-finite recurrence we have found using this new method is much shorter (i.e., the coefficients have lower order) than the ones computed with the original method.

5 Generating functions

Let $\mathbb{K}[\![x]\!]$ denote the ring of formal power series over a field $\mathbb{K}$. A common technique when working with sequences is to switch between the representation as a sequence $a(n) \in \mathbb{K}^{\mathbb{N}}$ and as its generating function $g(x) = \sum_{n\geq 0} a(n)x^n \in \mathbb{K}[\![x]\!]$. This is particularly useful for C-finite sequences (which have a one-to-one relation with rational functions) and holonomic sequences (which have a one-to-one relation with functions satisfying linear differential equations with polynomial coefficients). Hence, it is natural to investigate which properties the generating functions of (simple) C^2-finite sequences satisfy. Some first steps in this direction were already taken in [24].

5.1 Computing functional equation from recurrence

Let a be a C^2-finite sequence over a field $\mathbb{K}$ with annihilating operator $\mathcal{A} = c_0+\cdots+c_r\sigma^r \in \mathcal{R}_C[\sigma]$. Let $\mathbb{L} \supseteq \mathbb{K}$ be the smallest field which contains all splitting fields of the characteristic polynomials of $c_0, \dots, c_r$. We call $\mathbb{L}$ the *splitting field of* a.

For natural numbers $n \in \mathbb{N}$ we write

$$n^{\underline{k}} \coloneqq n(n-1)\cdots(n-k+1)$$

for the falling factorial. Let $\lambda \in \mathbb{L}$. Then, we write $g^{(d)}(\lambda x)$ for the d-th derivative of the formal power series $g(\lambda x)$, i.e.,

$$g^{(d)}(\lambda x) = \sum_{n\geq d} n^{\underline{d}} \lambda^n a(n) x^{n-d}.$$

Theorem 5.1. *Let a be a C^2-finite sequence over $\mathbb{K}$ with splitting field $\mathbb{L}$ and let $g(x) = \sum_{n\geq 0} a(n)x^n$ be its generating function. Then, $g(x)$ satisfies a functional equation of the form*

$$\sum_{k=1}^{m} p_k(x) g^{(d_k)}(\lambda_k x) = p(x) \tag{5}$$

for $p, p_1, \dots, p_m \in \mathbb{L}[x], d_1, \dots, d_m \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_m \in \mathbb{L}$.

Proof. Consider the defining recurrence of a:

$$c_0(n)a(n) + \cdots + c_r(n)a(n+r) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

Multiplying by x^n and summing over all $n \in \mathbb{N}$ yields

$$\sum_{n\geq 0} c_0(n)a(n)x^n + \cdots + \sum_{n\geq 0} c_r(n)a(n+r)x^n = 0. \tag{6}$$

The coefficients $c_0, \dots, c_r$ have some closed form for all $n \geq n_0$. Hence, the left-hand side of equation (6) is just an $\mathbb{L}$-linear combination of power series of the form

$$\tilde{h}(x) \coloneqq \sum_{n\geq n_0} n^j \lambda^n a(n+i) x^n$$

for $j \in \mathbb{N}, i \in \{0, \dots, r\}, \lambda \in \mathbb{L}$. The first terms $n = 0, \dots, n_0 - 1$ in (6) just yield some polynomial factors. Furthermore, it is sufficient to consider formal power series of the form

$h(x) := \sum_{n\geq 0} n^j \lambda^n a(n+i)x^n$ as $h(x) - \tilde{h}(x)$ is again polynomial. Hence, also these factors $h(x) - \tilde{h}(x)$ contribute to the right-hand side of (5).

Let $S(k,l)$ denote the Stirling numbers of the second kind. Then, $n^k = \sum_{l=0}^{k} S(k,l) n^{\underline{l}}$. Therefore,

$$\begin{aligned}
h(x) &= \sum_{n\geq i} (n-i)^j \lambda^{n-i} a(n) x^{n-i} = \sum_{n\geq i} \left(\sum_{k=0}^{j} \binom{j}{k} n^k (-i)^{j-k} \right) \lambda^{n-i} a(n) x^{n-i} \\
&= \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{j}{k} (-i)^{j-k} S(k,l) \sum_{n\geq i} n^{\underline{l}} \lambda^{n-i} a(n) x^{n-i} \\
&= \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{j}{k} (-i)^{j-k} S(k,l) \frac{x^{l-i}}{\lambda^i} \sum_{n\geq i} n^{\underline{l}} \lambda^n a(n) x^{n-l} \\
&= \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{j}{k} (-i)^{j-k} S(k,l) \frac{x^{l-i}}{\lambda^i} \left(g^{(l)}(\lambda x) + p_{k,l}(x) \right)
\end{aligned}$$

where $p_{k,l}(x) \in \mathbb{L}[x]$ is defined as

$$p_{k,l}(x) = \begin{cases} -\sum_{n=l}^{i-1} n^{\underline{l}} \lambda^n a(n) x^{n-l}, & \text{if } i > l, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, in particular $h(x) = \sum_{l=0}^{j} q_l(x) g^{(l)}(\lambda x) + q(x)$ with $q_0, \ldots, q_j, q \in \mathbb{L}(x)$. Using this in equation (6) and clearing the denominator x^r yields a functional equation of the desired form.

This functional equation is nontrivial, i.e., the right-hand side of the equation does not simplify to zero: Let $n^j \lambda^n$ be a term in $c_r(n)$ with j maximal. This term yields a nonzero term $x^{j-r} g^{(j)}(\lambda x)$ in the functional equation which cannot cancel because of the maximality of j. $\square$

The proof of Theorem 5.1 uses the closed form representation of the C-finite coefficients and generalizes the classical proof for holonomic sequences [15]. A close investigation of the proof shows the following bounds (in the special case of holonomic sequences, we get precisely the known bounds):

1. We have $\deg p_k \leq r + \max_i(\operatorname{ord} c_i)$.
2. The λ_k are exactly the roots of the polynomials χ_{c_i}.
3. The derivatives d_k are bounded by the highest multiplicity of the root λ_k in any χ_{c_i}. In particular, $\max_k d_k \leq \max_i(\operatorname{ord} c_i)$.
4. Let n_0 be minimal such that all $c_0, \ldots, c_r$ have closed forms from n_0 on. We have $\deg(p) < \max(r, n_0)$. If we differentiate the functional equation $\max(r, n_0)$ times, we get a homogeneous functional equation (i.e., $p = 0$). The functional equation then satisfies $\max_k d_k \leq \max_i(\operatorname{ord} c_i) + \max(r, n_0)$.
5. If we are given a functional equation as in equation (5) we can get a homogeneous functional equation of same degree and order (however, with the caveat that we might add more terms, i.e., increase m, and change the zeros λ_k): In (5) we can substitute $x \to \delta x$ and subtract this new equation form the original one. For suitable δ this decreases the degree of the right-hand side and iterating this process yields a homogeneous equation.

Theorem 5.1 also generalizes the result for q-holonomic sequences: Every q-holonomic sequence satisfies a q-shift equation [14]. In this case we would have $\lambda_k = q^k$.

Example 5.2. Let $a(n) = f(n^2)$ be the sparse subsequence of the Fibonacci sequence f. The generating function g of a satisfies the functional equation

$$\begin{aligned}\left(\phi^3 x^2 - \phi^{-3}\right) g\left(\phi^2 x\right) - \left(\psi^3 x^2 - \psi^{-3}\right) g\left(\psi^2 x\right)& \\ + xg\left(\psi^4 x\right) - xg\left(\psi^4 x\right) &= (\psi - \phi)x\end{aligned}$$

where $\phi \coloneqq \frac{1+\sqrt{5}}{2}$ denotes the golden ratio and $\psi \coloneqq \frac{1-\sqrt{5}}{2}$ its conjugate.

Example 5.3. Since $\frac{1}{n!}$ is C^2-finite (as it is D-finite), the coefficient sequence of the exponential generating function $\sum_{n\geq 0} \frac{a(n)}{n!} x^n$ of a C^2-finite sequence a is again C^2-finite. Let b be the coefficient sequence of the exponential generating function of the fibonorial numbers. Then, b satisfies

$$f(n+1)b(n) - (n+1)b(n+1) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

Let $h(x) = \sum_{n\geq 0} b(n)x^n$ be the generating function of b. Then, h satisfies

$$\phi\, h(\phi x) - \psi\, h(\psi x) - (\phi - \psi)\, h'(x) = 0$$

where ϕ, ψ are as in Example 5.2.

5.2 Computing recurrence from functional equation

We have seen that the generating functions of C^2-finite sequences satisfy a certain type of functional equations. Now, we want to understand, whether the converse holds as well. The question is if, given a functional equation of the form (5), the corresponding coefficient sequence is C^2-finite. We will see that this is not necessarily the case. The next theorem shows, however, that we always get a linear recurrence with C-finite coefficients. This recurrence can have a leading coefficient with infinitely many zeros. Sequences satisfying such recurrences are called X-recursive [24].

Theorem 5.4. *Let $g(x) = \sum_{n\geq 0} a(n)x^n$ satisfy a functional equation of the form*

$$\sum_{k=1}^{m} p_k(x) g^{(d_k)}(\lambda_k x) = p(x)$$

for $p, p_1, \ldots, p_m \in \mathbb{L}[x], d_1, \ldots, d_m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{L}$. Then, the coefficient sequence $(a(n))_{n\in\mathbb{N}}$ satisfies a linear recurrence with C-finite coefficients over $\mathbb{L}$.

Proof. The functional equation is an $\mathbb{L}$-linear combination of functions

$$x^j g^{(d)}(\lambda x) = x^j \sum_{n\geq d} n^{\underline{d}} \lambda^n a(n) x^{n-d} = \sum_{n\geq j} (n+d-j)^{\underline{d}} \lambda^{n+d-j} a(n+d-j) x^n.$$

We can compute this for every factor appearing in the functional equation. Comparing the coefficients yields a linear recurrence with C-finite coefficients. ☐

First, we give an example of a functional equation which gives rise to a simple C^2-finite recurrence for the coefficient sequence.

Example 5.5. Let $g(x) = \sum_{n\geq 0} a(n)x^n$ satisfy the equation

$$xg(2x) + g(x) = 1.$$

Then, $a(0) = 1$ and

$$2^n a(n) + a(n+1) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

However, there are more formal power series satisfying such functional equations than C^2-finite sequences. The equation satisfied by even and odd functions are of the form (5). A function $g(x) = \sum_{n\geq 0} a(n)x^n$ satisfies the equation $g(x) = g(-x)$ (i.e., is even) if and only if the coefficient sequence $a(n)$ satisfies $(1 - (-1)^n)a(n) = 0$ for all $n \in \mathbb{N}$ (i.e., $a(n) = 0$ for all odd $n \in \mathbb{N}$). By construction, C^2-finite sequences are uniquely defined by finitely many elements $\alpha \in \mathbb{K}$. This means in particular that there are only countably many C^2-finite sequences. On the other hand, there are uncountably many even functions. Hence, the coefficient sequences of functions satisfying a functional equation of the form (5) are usually not C^2-finite.

For simple C^2-finite we can, however, refine the condition on the functional equation to get an equivalent characterization for the generating function. A careful inspection of the proofs of Theorem 5.1 and Theorem 5.4 shows the following theorem:

Theorem 5.6. *The sequence $a \in \overline{\mathbb{Q}}^{\mathbb{N}}$ is simple C^2-finite if and only if its generating function $g(x) = \sum_{n\geq 0} a(n)x^n$ satisfies a functional equation of the form*

$$\sum_{k=1}^{m} \alpha_k x^{j_k} g^{(d_k)}(\lambda_k x) = p(x)$$

for

1. *$\alpha_1, \ldots, \alpha_k, \lambda_1, \ldots, \lambda_k \in \overline{\mathbb{Q}} \setminus \{0\}$,*
2. *$j_1, \ldots, j_m, d_1, \ldots, d_m \in \mathbb{N}$,*
3. *$p \in \overline{\mathbb{Q}}[x]$ and*
4. *let $s \coloneqq \max_{k=1,\ldots,m}(d_k - j_k)$, then for all $k = 1, \ldots, m$ with $d_k - j_k = s$ we have $j_k = 0$ and $\lambda_k = 1$.*

5.3 Cauchy product

For C-finite (or holonomic) sequences $a(n), b(n)$, also the Cauchy product

$$(a \odot b)(n) \coloneqq \sum_{i=0}^{n} a(i)b(n-i)$$

is C-finite (or holonomic). It is not known whether the same holds for (simple) C^2-finite sequences. Even for simple examples we were not able to find a recurrence:

Question 5.7. Let $a(n) = 2^{n^2}$ and $b(n) = 3^{n^2}$. Then, both sequences a, b are (simple) C^2-finite. Is the Cauchy product $a \odot b$ again (simple) C^2-finite?

However, if one of the sequences is C-finite, then the Cauchy product is again (simple) C^2-finite.

Lemma 5.8. *Let a be C^2-finite and b be C-finite over $\mathbb{K}$. Then, the Cauchy product $c \coloneqq a \odot b$ is again C^2-finite over the splitting field $\mathbb{L}$ of the characteristic polynomial of b.*

Proof. First, let $b(n) = n^d \lambda^n$ for all $n \in \mathbb{N}$ for some $k \in \mathbb{N}, \lambda \in \mathbb{L}$ and $c = a \odot b$. Furthermore, we denote $a_l(n) \coloneqq \sum_{i=0}^n a(i)(n-i)^l \lambda^{n-i}$ for $l = 0, \dots, d$. Then, for all $j \in \mathbb{N}, n \in \mathbb{N}$ we have

$$\begin{aligned}\sigma^j c(n) &= \sum_{i=0}^{n+j} a(i)(n+j-i)^d \lambda^{n+j-i} = \sum_{l=0}^{d} \lambda^j \binom{d}{l} j^{d-l} \sum_{i=0}^{n+j} a(i)(n-i)^l \lambda^{n-i} \\ &= \sum_{l=0}^{d} \lambda^j \binom{d}{l} j^{d-l} a_l(n) + \sum_{l=0}^{d} \lambda^j \binom{d}{l} j^{d-l} \sum_{i=1}^{j} a(n+i)(-i)^l \lambda^{-i}.\end{aligned}$$

Let $\mathcal{A} = c_0 + c_1\sigma + \cdots + c_r\sigma^r$ be an annihilating operator of a. With Lemma 3.1, the ring $R \coloneqq \mathbb{L}_\sigma[c_0, \dots, c_r]$ is Noetherian. The computation above shows

$$\langle \sigma^j c \mid j \in \mathbb{N} \rangle_{Q(R)} \subseteq \langle a_0, \dots, a_d \rangle_{Q(R)} + \langle \sigma^j a \mid j \in \mathbb{N} \rangle_{Q(R)}.$$

With [12, Lemma 3.1], the module on the right-hand side is finitely generated, hence also the module on the left-hand side is finitely generated. Therefore, with [12, Lemma 3.2], the sequence c is C^2-finite. As C^2-finite sequences are closed under elementwise addition and every C-finite sequence is just a linear combination of such exponential sequences from some term n_0 on, the Cauchy product of a C^2-finite sequence with a C-finite sequence is again C^2-finite. □

The proof of Lemma 5.8 works in the same way for simple C^2-finite sequences. Hence, the Cauchy product of a simple C^2-finite sequence with a C-finite sequence is again simple C^2-finite.

Example 5.9. Let $a(n) = 2^{n^2}, b(n) = 3^n, c = a \odot b$. Then, c is again C^2-finite and satisfies

$$4^n c(n) - (\tfrac{1}{3}4^n + \tfrac{1}{8})c(n+1) + \tfrac{1}{24}c(n+2) = 0, \qquad \text{for all } n \in \mathbb{N}$$

and $c(0) = 1, c(1) = 5$.

6 Conclusion

Summarizing, we have shown that simple C^2-finite sequences form a computable ring for which most of the closure properties satisfied by holonomic sequences carry over. Furthermore, we gave a characterization of the generating function of simple C^2-finite sequences as well as asymptotic bounds.

In our presentation, the theory is restricted to sequences defined over a number field. This includes for instance integer sequences arising from combinatorial problems. Some parts can, however, be generalized to other fields. For instance, Section 3 carries over immediately to arbitrary fields, i.e. the set of simple C^2-finite sequences over any field forms a ring.

One common technique to discover short recurrences for a given sequences is using guess-and-prove. The recurrences obtained for simple C^2-finite sequences are already smaller than the recurrences computed with the previous approach. Still, for many applications an efficient guessing routine would be desirable. This would come handy, for instance, for further investigation of the Cauchy-product of simple C^2-finite sequences.

Analogously to C^2-finite sequences, one can define D^2-finite (or P^2-recursive) sequences as sequences satisfying a linear recurrence with D-finite coefficients. Again, this class forms a ring [11]. One can also define simple D^2-finite sequences as D^2-finite sequences with a constant leading coefficient. The proof of Section 3 for simple C^2-finite sequences carries over immediately to this case showing that simple D^2-finite sequences form a ring. For proving that the ring of simple C^2-finite sequences is computable, we heavily relied on the closed form of C-finite sequences. Such a closed form does not exist for D-finite sequences. Hence, it is not clear whether this new ring is computable.

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References

[1] Ronald Alter and Kenneth Kubota. Prime and prime power divisibility of catalan numbers. *Journal of Combinatorial Theory, Series A*, 15(3):243–256, 1973.

[2] Michael F. Atiyah and Ian G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley-Longman, 1969.

[3] Michaël Cadilhac, Filip Mazowiecki, Charles Paperman, Michał Pilipczuk, and Géraud Sénizergues. On polynomial recursive sequences. *Theory of Computing Systems*, 2021.

[4] Henri Cohen. *A Course in Computational Algebraic Number Theory*. Graduate Texts in Mathematics. Springer Berlin Heidelberg, 2013.

[5] *NIST Digital Library of Mathematical Functions*. `http://dlmf.nist.gov`, Release 1.1.0 of 2020-12-15, 2021. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[6] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

[7] Stefan Gerhold. On Some Non-Holonomic Sequences. *The Electronic Journal of Combinatorics*, 11(1), 2004.

[8] Stefan Gerhold. *Combinatorial Sequences: Non-Holonomicity and Inequalities*. PhD thesis, Johannes Kepler University Linz, 2005.

[9] Xavier Gourdon and Bruno Salvy. Effective Asymptotics of Linear Recurrences with Rational Coefficients. In *FPSAC '93: Proceedings of the 5th conference on Formal power series and algebraic combinatorics*, page 145–163, 1996.

[10] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2021. `http://www.oeis.org`.

[11] Antonio Jiménez-Pastor, Philipp Nuspl, and Veronika Pillwein. An extension of holonomic sequences: C^2-finite sequences. *RISC Report Series*, 21(20), 2021. `https://doi.org/10.35011/risc.21-20`.

[12] Antonio Jiménez-Pastor, Philipp Nuspl, and Veronika Pillwein. On C^2-finite sequences. In *Proceedings of ISSAC 2021, Virtual Event Russian Federation, July 18–23, 2021*, pages 217–224, 2021.

[13] Manuel Kauers. The Holonomic Toolkit. In *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions*, Texts and Monographs in Symbolic Computation, pages 119–144. Springer, 2013.

[14] Manuel Kauers and Christoph Koutschan. A mathematica package for q-holonomic sequences and power series. *The Ramanujan Journal*, 19(2):137–150, 2009.

[15] Manuel Kauers and Peter Paule. *The Concrete Tetrahedron.* Texts and Monographs in Symbolic Computation. Springer, 2011.

[16] Manuel Kauers and Burkhard Zimmermann. Computing the Algebraic Relations of C-finite Sequences and Multisequences. *J. Symb. Comput*, 43(11):787–803, 2008.

[17] Tomer Kotek. *Definability of Combinatorial Functions.* PhD thesis, Technion – Israel Institute of Technology, 2012.

[18] Tomer Kotek and Johann A. Makowsky. Recurrence relations for graph polynomials on bi-iterative families of graphs. *Eur. J. Comb.*, 41:47–67, 2014.

[19] Hsuan-Chu Li and E. Tan. On a special generalized vandermonde matrix and its lu factorization. *Taiwanese Journal of Mathematics*, 12:1651–1666, 2008.

[20] Chung L. Liu. *Introduction to combinatorial mathematics.* Computer Science Series. McGraw-Hill, first edition, 1968.

[21] Stephen Melczer. *An Invitation to Analytic Combinatorics.* Texts and Monographs in Symbolic Computation. Springer, 2021.

[22] Maurice Mignotte, Tarlok Nath Shorey, and Robert Tijdeman. The distance between terms of an algebraic recurrence sequence. *Journal für die reine und angewandte Mathematik*, 1984(349):63–76, 1984.

[23] Joël Ouaknine and James Worrell. Decision Problems for Linear Recurrence Sequences. In *Lecture Notes in Computer Science*, pages 21–28. Springer, 2012.

[24] Thotsaporn Aek Thanatipanonda and Yi Zhang. Sequences: Polynomial, C-finite, Holonomic, ..., 2020. `https://arxiv.org/pdf/2004.01370`.

[25] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.4)*, 2022. `https://www.sagemath.org`.

[26] Nikolai Konstantinovich Vereshchagin. Occurrence of zero in a linear recursive sequence. *Mat. Zametki*, 38(2):609–615, 1985.