JOHANNES KEPLER
UNIVERSITY LINZ

Submitted by
Evans Doe Ocansey

Submitted at
Research Institute for Symbolic Computation – (RISC)

Supervisor and
First Examiner
Assoc. Prof. DI Dr. Carsten Schneider (Betreuung)

Second Examiner
Prof. Dr. Marko Petkovšek

Co-Supervisor
Univ. Prof. Dr. Peter Paule

October 2019

Difference Ring Algorithms for Nested Products

Doctoral Thesis
to obtain the academic degree of
Doktor der technischen Wissenschaften
in the Doctoral Program
Technische Wissenschaften

JOHANNES KEPLER
UNIVERSITY LINZ
Altenbergerstraße 69
4040 Linz, Österreich
www.jku.at
DVR 0093696

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Linz, October, 2019. Evans Doe Ocansey.

Kurzfassung

Der allgemeine Rahmen für den Umgang mit symbolischer Summation, nämlich $\Pi\Sigma$-Körper, wurde von Michael Karr eingeführt. Er entwickelte algorithmisch, wie indefinit verschachtelte Summen und Produkte als transzendente Erweiterungen über einen berechenbaren Grundkörper $\mathbb{K}$ dargestellt werden können. Er stellte auch einen Algorithmus vor, der das parametrisierte Teleskopproblem und als Spezialfall das (kreative) Teleskopproblem innerhalb eines gegebenen $\Pi\Sigma$-Körpers, löst.

In den letzten Jahren wurde Karrs Differenzenkörpertheorie von C. Schneider auf die sogenannten $\mathrm{R}\Pi\Sigma^*$-Erweiterungen verallgemeinert. In solchen $\mathrm{R}\Pi\Sigma^*$-Erweiterungen kann man nicht nur indefinit verschachtelte Summen und Produkte, die durch transzendente Ringerweiterungen ausgedrückt werden, darstellen, sondern es können auch algebraische Produkte der Form α^n behandelt werden, wobei α eine primitive Einheitswurzel repräsentiert. Darüber hinaus wurden allgemeine Algorithmen entwickelt, um die parametrisierten Teleskopgleichungen innerhalb eines so konstruierten Differenzenrings zu lösen. Dieser Mechanismus, der im Mathematica-Summationspaket `Sigma` implementiert ist, funktioniert automatisch für indefinit verschachtelte Summen, definiert über hypergeometrischen Produkten welche sich zu Folgen in $\mathbb{Q}$ oder in einem rationalen Funktionenkörper über $\mathbb{Q}$ auswerten. Ansonsten funktioniert dieser Mechanismus nur, wenn man bereits weiß, wie die entstehenden Produkte innerhalb einer $\mathrm{R}\Pi\Sigma^*$-Ring-Erweiterung umformuliert werden können.

In dieser Dissertation ergänzen wir diese Summationstheorie substanziell um den folgenden Baustein. Wir stellen neue Algorithmen zur Verfügung, mit denen man eine endliche Anzahl hypergeometrischer oder gemischt $(q_1, \ldots, q_e)$-multibasiger hypergeometrischer Produkte in einem solchen Differenzenring darstellen kann. Genauer gesagt, sei eine endliche Anzahl hypergeometrischer Produkte mit beliebiger Verschachtlungstiefe oder eine endliche Anzahl von gemischt $(q_1, \ldots, q_e)$-mehrbasigen hypergeometrischen Produkten mit Verschachtlungstiefe-1, die sich im berechenbaren Körper $\mathbb{K}$ auswerten, gegeben. Dann können eine $\mathrm{R}\Pi$-Erweiterung (wobei der Konstantenkörper $\mathbb{K}$ eventuell durch algebraische Erweiterungen erweitert werden muss) und Elemente in einem solchen Differenzenring mit der Eigenschaft, dass sie genau die Input-Produkte modellieren, konstruiert werden. Diese neue Erkenntnis bietet einen vollständigen Summationsmechanismus, mit dessen Hilfe solche allgemeine Produkte und auch indefinit verschachtelte Summen, die über solche Produkte definiert sind, voll automatisch in $\mathrm{R}\Pi\Sigma^*$-Erweiterungen formuliert werden können.

Als Nebenprodukt erhält man kompakte Produktausdrücke, die untereinander algebraisch unabhängig sind. Desweiteren kann man für solche Produkte das Nullerkennungsproblem lösen. Die Algorithmen sind im Mathematica-Paket `NestedProducts` implementiert.

Abstract

The general framework for handling symbolic summation, namely $\Pi\Sigma$-fields was introduced by Michael Karr. He developed algorithmically how indefinite nested sums and products can be represented as transcendental extensions over a computable ground field $\mathbb{K}$. He also presented an algorithm that solves the parameterized telescoping problem, and as special case the telescoping and creative telescoping problem within a given $\Pi\Sigma$-field.

In recent years, Karr's difference field theory has been extended by C. Schneider to the so-called $\mathrm{R}\Pi\Sigma^*$-extensions in which one can represent not only indefinite nested sums and products that can be expressed by transcendental ring extensions, but algebraic products of the form α^n where α is a primitive root of unity can also be treated. In addition, general algorithms have been worked out to solve the parameterised telescoping equations within such a constructed difference ring. This machinery which is implemented in the Mathematica summation package `Sigma`, works in full automatically for indefinite nested sums defined over hypergeometric products that evaluate to sequences in $\mathbb{Q}$ or a rational function field over $\mathbb{Q}$. Otherwise it works only if one knows in advance how the arising products can be rephrased within an $\mathrm{R}\Pi\Sigma^*$-ring extension.

In this thesis we supplement this summation theory substantially by providing new algorithms that enable one to represent a finite number of hypergeometric or mixed $(q_1, \ldots, q_e)$-multibasic hypergeometric products in such a difference ring. More precisely, given a finite number of hypergeometric products of arbitrary nested depth, or a finite number of mixed $(q_1, \ldots, q_e)$-multibasic hypergeometric products of nested depth-1 which evaluate to the computable field $\mathbb{K}$, we can construct an $\mathrm{R}\Pi$-extension (where the constant field $\mathbb{K}$ might be extended by algebraic number extensions) and elements in such a difference ring with the property that they model precisely the input products. This new insight provides a complete summation machinery that enables one to formulate such general products and indefinite nested sums defined over such products in $\mathrm{R}\Pi\Sigma^*$-extensions fully automatically.

As a by-product, one obtains compactified product expressions where the products are algebraically independent among each other. Furthermore, one can solve the zero-recognition problem for such products. The algorithms are implemented in the Mathematica package `NestedProducts`.

Acknowledgement

My heartfelt gratitude goes to God for the strength, wisdom and direction in this journey of completing this work. All glory to God.

I also want to express my sincere appreciation to my advisor Carsten Schneider for mentoring me in this Doctoral program. I am most grateful for the opportunity to be under your supervision. Thank you again for your valuable contributions during all our discussions. I can still recollect some of your statements you made: "I am sure the proof is in your code so look at that" or "try to program it and you will see the proof". These hints were indeed my guiding light to not be fishy but concrete. I am most grateful for all you did within your capacity to make sure that my studies and research at RISC was a good one.

Thank you Peter Paule for not hesitating to be the co-supervisor of this work. I am grateful for your advise anytime I gave a talk at the algorithmic combinatorics seminar.

Many thanks to the past and current members of the algorithmic combinatorics group. I specially thank Jakob Ablinger, Ralf Hemmecke, Manuel Kauers and Christoph Koutschan. I appreciate all of you for your enormous support. Not forgetting the entire RISC community for the wonderful scientific environment to work. Many thanks to Wolfgang Windsteiger and Franz Winkler. Thank you Marko Petkovšek for your willingness to be the second examiner for this thesis.

I want to also thank the people at the Grace International Fellowship for the warm atmosphere during this period.

Finally, I want to express my profound gratitude to my family. All I can say is I am grateful for the emotional support, patience, prayers and love in these years. You have been my pillar and I can't thank you enough. Not forgetting my mother and my siblings. Many thanks to you all.

This research was funded by the Austrian Science Fund (FWF): Grant SFB F50, project F5009-N15.

Dedication

To my wife Mary.

Soli Deo Gloria.

Contents

List of Problem Specifications

List of Mathematica Sessions

List of Notations

$\mathrm{sconst}_G(\mathbb{A}, \sigma)$	The set of semi-constants of $(\mathbb{A}, \sigma)$ over a group G. 126
$\mathrm{ProdSum}(\mathbb{X})$	The set of (indefinite nested) product-sum expressions. 23
$\mathrm{Prod}(\mathbb{K})$	The set of geometric products. 26
$\mathrm{Prod}(\mathbb{K}(n))$	The set of hypergeometric products. 25
$\mathrm{Prod}(\mathbb{K}(n, \mathbf{q}^n))$	The set of mixed $(q_1, \ldots, q_e)$-multibasic hypergeometric products. 25
$\mathrm{Prod}(\mathbb{K}(\mathbf{q}^n))$	The set of $(q_1, \ldots, q_e)$-multibasic hypergeometric products. 25
$f \nsim_\sigma h$	f and h are shift co-prime. 58
$f \sim_\sigma h$	f and h are σ-equivalent. 58
$f_{(k,\sigma)}$	k-th σ-factorial. 100
$\deg(f)$	The degree of a (Laurent) polynomial f. 35
$\mathrm{ldeg}(f)$	The order of a (Laurent) polynomial f. 35
$\mathrm{ProdE}(\mathbb{S}_\mathbb{K})$	The set of evaluated product expressions with $\mathbb{S}_\mathbb{K} \in \{\mathbb{K}, \mathbb{K}(n), \mathbb{K}(\mathbf{q}^n), \mathbb{K}(n, \mathbf{q}^n)\}$. 26
$\mathbb{Z}$	The set of integers. 43
$\mathbb{N}$	The set of non-negative integers. 26
$\mathbb{Q}$	The set of rational numbers. 23
$\mathcal{M}((f_1, \ldots, f_s), \mathbb{F})$	A submodule of $\mathbb{Z}^s$. 61
0_s	The vector $(0, \ldots, 0)$ with s zeros. 61

Chapter 1

Introduction

In his pioneering work (Karr, 1981, 1985), Michael Karr introduced the so-called $\Pi\Sigma^*$-fields that provide a general framework for handling problems in symbolic summation. In general, a $\Pi\Sigma^*$-field is a difference field $(\mathbb{F}, \sigma)$ where $\mathbb{F}$ is a field whose elements represent the summation objects and $\sigma : \mathbb{F} \to \mathbb{F}$ is a field automorphism that models the shift operator acting on the summation objects. More precisely in Karr's setting, the field $\mathbb{F}$ is constructed by a tower of transcendental field extensions whose generators either represent sums (resp. products) where the summands (resp. the multiplicands) are elements from the field below. In particular, the following problem has been solved.

Problem T: Telescoping Problem for $(\mathbb{F}, \sigma)$.

Given a $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ and given $f \in \mathbb{F}^*$. *Find*, if possible a $g \in \mathbb{F}$ such that the telescoping equation

$$f = \sigma(g) - g \tag{1.1}$$

holds.

Hence if f represents an expression $F(k)$ in terms of indefinite nested sums and products and if one rephrases the generator t_i as indefinite nested sums and products yielding the expression $G(k)$, one obtains the telescoping relation

$$F(k) = G(k+1) - G(k) \tag{1.2}$$

that holds within a certain range $a \leqslant k \leqslant b$. Then summing this telescoping equation over the valid range, one gets the identity

$$\sum_{k=a}^{b} F(k) = G(b+1) - G(a).$$

The multiplicative version of Problem T is as follows:

Problem MT: Multiplicative Telescoping Problem for $(\mathbb{F}, \sigma)$.

Given a $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ and given $f \in \mathbb{F}$. *Find*, if possible a $g \in \mathbb{F}^*$ such that the multiplicative telescoping equation

$$f = \frac{\sigma(g)}{g} \tag{1.3}$$

holds.

If f represents an expression $F(k)$ in terms of indefinite nested products and if one rephrases the t_i as indefinite nested products yielding the expression $G(k)$, one obtains the multiplicative telescoping relation

$$F(k) = \frac{G(k+1)}{G(k)} \tag{1.4}$$

that holds within a certain range $a \leqslant k \leqslant b$. Then taking the product over this multiplicative telescoping equation over the valid range, one gets the identity

$$\prod_{k=a}^{b} F(k) = \frac{G(b+1)}{G(a)}$$

with $G(a) \neq 0$. In a nutshell, the following strategy can be applied:

(1) Construct a $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ in which a given summand (resp. multiplicand) $F(k)$ in terms of indefinite nested sums (resp. products) is rephrased by $f \in \mathbb{F}$;

(2) Compute $g \in \mathbb{F}^*$ such that (1.1) (resp. (1.3)) holds;

(3) Rephrase $g \in \mathbb{F}^*$ to an expression $G(k)$ such that (1.2) (resp. (1.4)) holds.

In recent years new algorithms and further improvements of Karr's difference field theory have been worked out in order to turn the difference field approach into an automatic simplification toolkit for nested sum (resp. product) expressions. Here the key observation is that a sum (resp. product) can be either expressed in the existing difference field $(\mathbb{F}, \sigma)$ by solving Problem T (resp. Problem MT), or the sum (and in parts the product) can be adjoined as a new extension on top of the already constructed field $\mathbb{F}$; see Theorem 2.3.51 (resp. Theorem 2.3.42) below. Exploiting this machinery it is now possible to simplify sum expressions such that the nesting depth is minimized (Schneider, 2008), the number of summation objects arising in summands is optimized (Schneider, 2015), or the degrees in the numerator and denominator (Schneider, 2007a) are as small as possible. All these algorithms are implemented in the summation package `Sigma` (Schneider, 2007b, 2013).

On the other hand, representing indefinite nested products in $\Pi\Sigma^*$-fields can cause much troubles. For instance, suppose that we have already adjoined 2^n to the field and later on we discover that we also need to treat $\left(\sqrt{2}\right)^n$, then obviously we must deal with the relation $2^n = \left(\left(\sqrt{2}\right)^n\right)^2$. Hence, if we want to stay in the $\Pi\Sigma^*$-field setting (i.e., dealing only with transcendental extensions), then we are forced to redesign the previously constructed field by adjoining $\left(\sqrt{2}\right)^n$ first and then representing 2^n by $\left(\left(\sqrt{2}\right)^n\right)^2$. Even worse, we may have to work with product expressions like

$$\prod_{k=1}^{n} k \quad \text{and} \quad \prod_{k=1}^{n} -k.$$

Then, we obtain the relation

$$\left(\prod_{k=1}^{n} k\right)^2 = \left(\prod_{k=1}^{n} -k\right)^2$$

which cannot be resolved in a field. On the other hand, writing

$$\prod_{k=1}^{n} -k = \left(\prod_{k=1}^{n} -1\right)\left(\prod_{k=1}^{n} k\right) = (-1)^n \left(\prod_{k=1}^{n} k\right)$$

we observe that the alternating sign $(-1)^n$ is the troublemaker which can be only represented in a ring with zero divisors, i.e., having relations such as

$$\left(1-(-1)^n\right)\left(1+(-1)^n\right)=0.$$

In Schneider (2005) and a streamlined version worked out in Schneider (2014), this troublemaker has been resolved for the class of hypergeometric products of the form

$$\prod_{i=\ell}^{n} f(i)$$

with $\ell \in \mathbb{N}$ and $f(n) \in \mathbb{K}(n)$ where $\mathbb{K}$ is, for example, a rational-function field over the field of rational numbers: namely, a finite number of such products, can always be represented in a $\Pi\Sigma^*$-field adjoined with an algebraic extension which handles the element $(-1)^n$. In particular, the construction of nested sums over such a difference ring together with the related telescoping and creative telescoping algorithms can be applied in the so-called $\mathrm{R}\Pi\Sigma^*$-extensions (Schneider, 2016, 2017), which are difference rings built by transcendental ring extensions (polynomial ring extensions for sums and Laurent polynomial extensions for products), and extensions of the form α^n where α is a primitive root of unity. For many problems coming, e.g., from combinatorics or particle physics (for the newest applications see Schneider and Sulzgruber (2016) or Ablinger et al. (2016)) this difference ring toolkit works perfectly well. But in general, if one solves, e.g., linear recurrence relations in terms of d'Alembertian solutions (Petkovšek, 1992; Abramov and Petkovšek, 1994; Abramov and Zima, 1996; Van Hoeij, 1999), or Liouvillian solutions (Van Der Put and Singer, 2006; Petkovšek and Zakrajšek, 2013), one is faced with nested sums in terms of hypergeometric products which are defined over the algebraic closure $\overline{\mathbb{Q}}$ of the rational numbers (or an algebraic number field built by a finite extension over the field of rational numbers $\mathbb{Q}$).

In this thesis we will generalise the existing product algorithms in Schneider (2005, 2014) for the following class of expressions given in Definition 2.1.10. Loosely speaking, this is the class of indefinite nested products where the products arise only in the numerator. For the treatment of the special class of products of nesting depth 1, which are also called single nested products, we refer to Ocansey and Schneider (2018). The terminologies used in defining these classes of product expressions were borrowed from Bauer and Petkovšek (1999). For these classes where $\mathbb{K}$ itself can be a rational function field over an algebraic number field, we will solve the Problem RPE over such a field $\mathbb{K}$, which is specified in Chapter 2. Product identities that do not belong to the class of expressions in Definition 2.1.10 are not treated in this thesis. For example, the product formula for counting spanning trees and perfect matchings on a lattices in Ocansey (2013, Chapter 3 and 4).

The strategy we adopt internally is to factor the multiplicands of the product expression into monic and irreducible factors. The shift-equivalent factors are then rewritten in terms of one of these factors; compare Schneider (2005); Abramov (1971); Paule (1995); Abramov and Petkovšek (2010); Chen et al. (2011). Then using results of Schneider (2010a); Hardouin and Singer (2008), we can conclude that products defined over these irreducible monic factors can be rephrased as transcendental difference ring extensions. Using a similar scheme, one can treat the content coming from the monic irreducible polynomials, and obtains finally an $\mathrm{R}\Pi\Sigma^*$-extension in which the products can be rephrased. We remark that the normal forms presented in Chen et al. (2011) are closely related to this representation and enable one to check, e.g., if the given products are algebraically independent. Moreover, there is an algorithm in Kauers and Zimmermann (2008) that can compute all algebraic relations for C-finite sequences, i.e., for expressions from $\mathrm{ProdE}(\mathbb{K})$. Recently this have been extended in parts by Schneider (2019) for nesting depth 1 mixed $(q_1, \ldots, q_e)$-multibasic products. In addition, algorithms have been developed in Schneider (2005) and Abramov and Petkovšek (2010) to find an optimal representation of nesting depth 1 products such that the degrees of the numerator and denominator of the multiplicands are minimal.

In this thesis, our main focus is different. We will compute alternative products which are by construction algebraically independent among each other, and which enable one to express the given products in terms of the algebraic independent products. In particular, we will make this algebraic independence statement (see property (2) of Problem RPE on page 27) very precise by embedding the constructed $R\Pi$-ring extension explicitly into the ring of sequences (Petkovšek et al., 1996) by using results from Schneider (2017). In particular, we will solve the zero-recognition problem for our class of product expressions (see property (3) of Problem RPE). The underlying algorithms for handling these product expressions have been implemented in my Mathematica package `NestedProducts`.

The outline of the thesis is as follows. In Chapter 2 we begin with a formal description of the class of product expressions that will be treated in this thesis. These products are also considered as elements of the ring of $\mathbb{K}$-sequences (Petkovšek et al., 1996). These will equip us with the necessary ingredients needed to formulate the main problem, Problem RPE, that this thesis solves. We then discuss briefly difference ring theory (Schneider, 2016, 2017) by introducing APS-/$R\Pi\Sigma^*$-ring extensions in which these classes of product expressions can be modelled. The chapter ends with the crucial result of embedding of $R\Pi\Sigma^*$-ring extensions into the ring of $\mathbb{K}$-sequences.

In Chapter 3, we present a solution to Problem RPE for the class of product expressions of nesting depth 1. In particular, we reformulate Problem RPE in the notions introduced in Chapter 2 and then present the basic strategy utilised in tackling this problem.

In Chapter 4, we work out the algorithmic properties that are required to turn our results to algorithms.

Chapter 5 gives a detailed solution to Problem RPE for the class of hypergeometric and mixed multibasic products of nesting depth 1. This result has also been published in Ocansey and Schneider (2018). For these classes of product expressions, we will demonstrate using my Mathematica package `NestedProducts`, how they can be reduced to obtain compactified product expressions that are algebraically independent among each other. The flowchart in Figure 1.1 below gives a summary of how Problem RPE is solved for the class of nesting depth 1 mixed $(q_1, \dots, q_e)$-multibasic hypergeometric products, $(q_1, \dots, q_e)$-multibasic hypergeometric products, hypergeometric products and geometric products. In particular, it presents a general overview of how all the arising sub-problems are handled with the Mathematica package `NestedProducts`.

Here are two examples of the class of product expressions considered in this chapter.

Example 1.1.1.
Given the hypergeometric product expression

$$P = \prod_{k=1}^{n} -k\sqrt{147} + \prod_{k=1}^{n} \frac{28\sqrt{-5}}{k\left(\mathbb{i}+\sqrt{3}\right)^2} + \prod_{k=1}^{n} \frac{\sqrt{-2}}{6\,(k+2)} + \prod_{k=1}^{n} \frac{-10541350400\sqrt{10}}{\left(\mathbb{i}+\sqrt{3}\right)^{10} k^2 \left(k^2+2\,k\right)} \in \mathrm{ProdE}(\mathbb{K}(n))^1$$

where $\mathbb{K} = \mathbb{Q}\left(e^{\frac{\pi \mathbb{i}}{6}}, \sqrt{-2}, \sqrt{3}, \sqrt{-5}\right)$, its output is

$$Q = \left(\left((-1)^{\frac{1}{6}}\right)^n\right)^6 \left(\sqrt{3}\right)^n 7^n\, n! + \frac{\left((-1)^{\frac{1}{6}}\right)^n \left(\sqrt{5}\right)^n 7^n}{n!} + \frac{2}{(n+1)(n+2)} \frac{\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^3}{\left(\sqrt{2}\right)^n \left(\left(\sqrt{3}\right)^n\right)^2 n!}$$
$$+ \frac{2}{(n+1)(n+2)} \frac{\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^8 \left(\left(\sqrt{5}\right)^n\right)^5 \left(7^n\right)^7}{\left(\sqrt{2}\right)^n (n!)^4} \in \mathrm{ProdE}(\tilde{\mathbb{K}}(n))$$

[1] See Definition 2.1.10 for the definition of $\mathrm{ProdE}(\mathbb{K}(n))$.

where $\tilde{\mathbb{K}} = \mathbb{Q}(e^{\frac{\pi i}{6}}, \sqrt{2}, \sqrt{3}, \sqrt{5})$ and the 12-th root of unity $(-1)^{\frac{1}{6}}$ is the complex number $e^{\frac{\pi i}{6}}$. In particular,

$$P(n) = Q(n) \quad \forall n \in \mathbb{N}$$

holds. Further, Q can be given in the Laurent polynomial ring

$$\tilde{\mathbb{K}}(n)\big[\big((-1)^{\frac{1}{6}}\big)^n\big]\Big[\big(\sqrt{2}\big)^n, \tfrac{1}{(\sqrt{2})^n}\Big]\Big[\big(\sqrt{3}\big)^n, \tfrac{1}{(\sqrt{3})^n}\Big]\Big[\big(\sqrt{5}\big)^n, \tfrac{1}{(\sqrt{5})^n}\Big]\big[7^n, \tfrac{1}{7^n}\big]\big[n!, \tfrac{1}{n!}\big]$$

defined over the ring $\mathbb{K}(n)[((-1)^{\frac{1}{6}})^n]$ where the generators $(\sqrt{2})^n$, $(\sqrt{3})^n$, $(\sqrt{5})^n$, 7^n and $n!$ are transcendental. That is, the sequences generated by these generators satisfy no algebraic relation; see Definition 2.2.5. ⋆

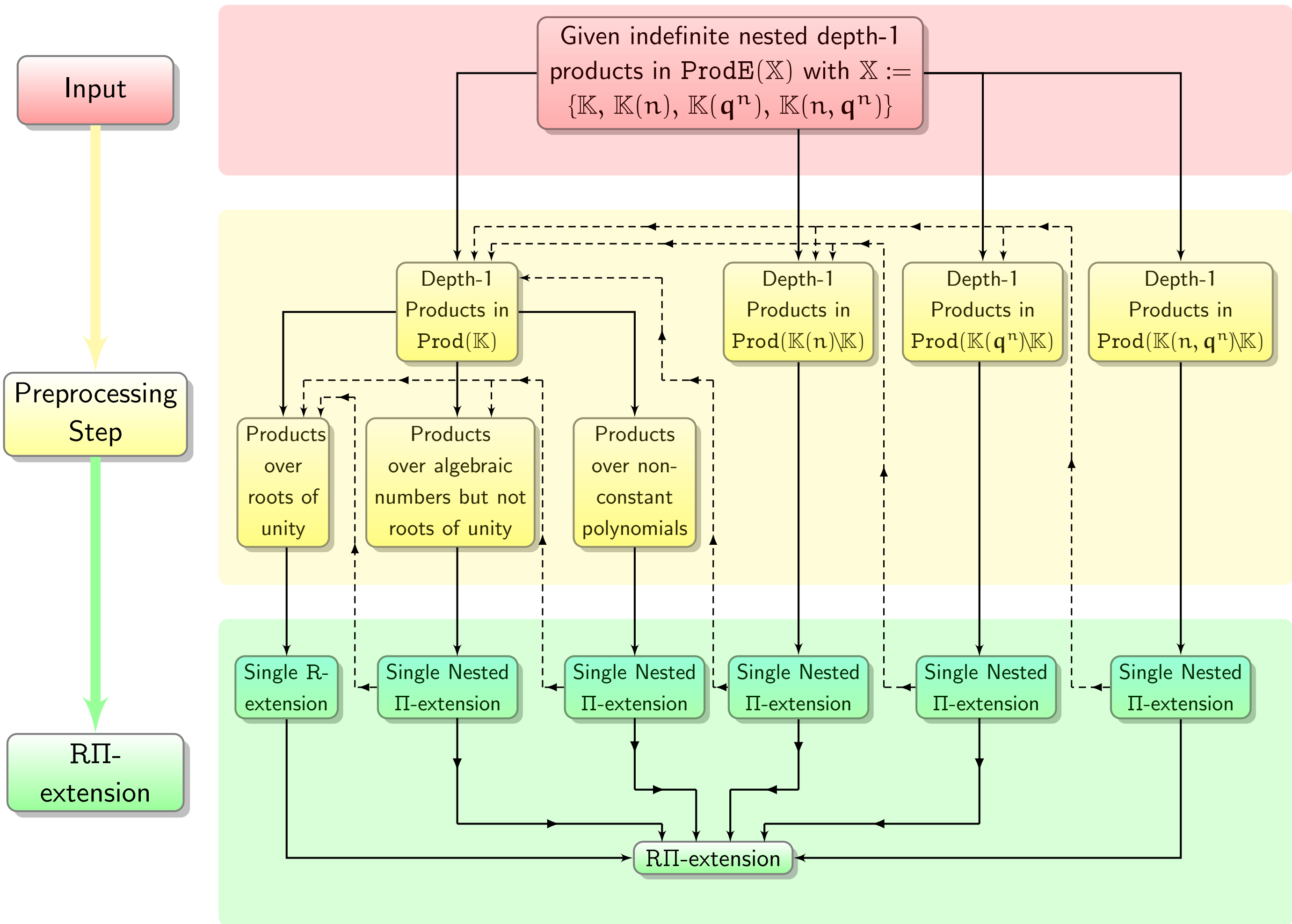


Figure 1.1: A general overview of the machinery for solving Problem RPE for nesting depth-1 products.

Example 1.1.2.
Given the mixed $\mathbf{q} = (q_1, q_2)$-multibasic hypergeometric product expression

$$F = \prod_{k=1}^{n} \frac{\sqrt{-13}\,(k\,q_1^k + 1)}{k^2\,(q_1^{k+1}\,q_2^{k+1} + k + 1)} + \prod_{k=1}^{n} \frac{k^2\,(k + q_1^k\,q_2^k)^2}{\sqrt{-3}\,(k+1)^2} + \prod_{k=1}^{n} \frac{169\,(k\,q_1^k\,q_2^k + q_2^k + k\,q_1^k + 1)}{(k\,q_1^{k+2} + 2\,q_1^{k+2} + 1)\,k^2} \in \mathrm{ProdE}(\mathbb{K}(n))$$

where $\mathbb{K} = K(q_1, q_2)$, is a rational function field over the algebraic number field $K = \mathbb{Q}(\sqrt{-3}, \sqrt{-13})$. The alternative representation

$$G = \frac{(q_2\, q_1 + 1)}{\left(q_2^{n+1}\, q_1^{n+1} + n + 1\right)} \left(\mathbb{i}\right)^n \left(\sqrt{13}\right)^n \frac{1}{(n!)^2} \left(\prod_{k=1}^{n} \left(k\, q_1^k + 1\right)\right) \left(\prod_{k=1}^{n} \frac{1}{\left(q_2^k\, q_1^k + k\right)}\right)$$
$$+ \frac{1}{(n+1)^2} \left(\left(\mathbb{i}\right)^n\right)^3 \left(\left(\sqrt{3}\right)^n\right)^{-1} \left(\prod_{k=1}^{n} \left(q_2^k\, q_1^k + k\right)\right)^2$$
$$+ \frac{(q_1 + 1)\left(2\, q_1^2 + 1\right)}{\left((n+1)\, q_1^{n+1} + 1\right)\left((n+2)\, q_1^{n+2} + 1\right)} \left(\left(\sqrt{13}\right)^n\right)^4 \frac{1}{(n!)^2} \left(\prod_{k=1}^{n} \left(q_2^k + 1\right)\right) \in \mathrm{ProdE}(\tilde{\mathbb{K}}(n))$$

is such that

$$F(n) = G(n) \quad \forall n \in \mathbb{N}$$

holds. Here, $\tilde{\mathbb{K}} = \mathbb{Q}\left(\mathbb{i}, \sqrt{3}, \sqrt{13}\right)\left(q_1, q_2\right)$. The sequences generated by

$$\left(\sqrt{3}\right)^n,\ \left(\sqrt{13}\right)^n,\ n!,\ \prod_{k=1}^{n} \left(k\, q_1^k + 1\right),\ \prod_{k=1}^{n} \left(q_2^k + 1\right),\ \prod_{k=1}^{n} \left(q_2^k\, q_1^k + k\right)$$

are algebraically independent among each other over the ring $\tilde{\mathbb{K}}(n)[(\mathbb{i})^n]$. ⋆

Finally in Chapter 6, we solve Problem RPE for the class of indefinite hypergeometric products of arbitrary but finite nesting depth. In addition, we will also demonstrate how this class of product expressions can be reduced using our Mathematica package `NestedProducts` to obtain compactified product expressions that are algebraically independent among each other. The flowchart in Figure 1.2 below illustrates a summary of how Problem RPE is solved for this class of indefinite product expressions. Again it presents a general overview of how all the arising sub-problems are handled with the Mathematica package `NestedProducts`.

Here are some examples of the class of product expressions considered in this chapter.

Example 1.1.3.
Given the nesting depth 2 geometric product

$$A = \prod_{k=1}^{n} 2\, \frac{\prod_{i=1}^{k} \frac{15}{\sqrt{6}}}{3} \in \mathrm{Prod}(\mathbb{K})$$

where $\mathbb{K} = \mathbb{Q}(\sqrt{6})$, the alternative product expression

$$B = \frac{\left(\left(\sqrt{2}\right)^n\right)^2 \left(\sqrt{3}\right)^{\binom{n+1}{2}} 5^{\binom{n+1}{2}}}{\left(\left(\sqrt{3}\right)^n\right)^2 \left(\sqrt{2}\right)^{\binom{n+1}{2}}} \in \mathrm{Prod}(\tilde{\mathbb{K}})$$

with $\tilde{\mathbb{K}} = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, is calculated. In particular, we have that,

$$A(n) = B(n) \quad \forall n \in \mathbb{N}$$

holds and the generators

$$\left(\sqrt{2}\right)^n,\ \left(\sqrt{3}\right)^n,\ \left(\sqrt{2}\right)^{\binom{n+1}{2}},\ \left(\sqrt{3}\right)^{\binom{n+1}{2}},\ \left(\sqrt{5}\right)^{\binom{n+1}{2}}$$

are algebraically independent over $\tilde{\mathbb{K}}(n)$. ⋆

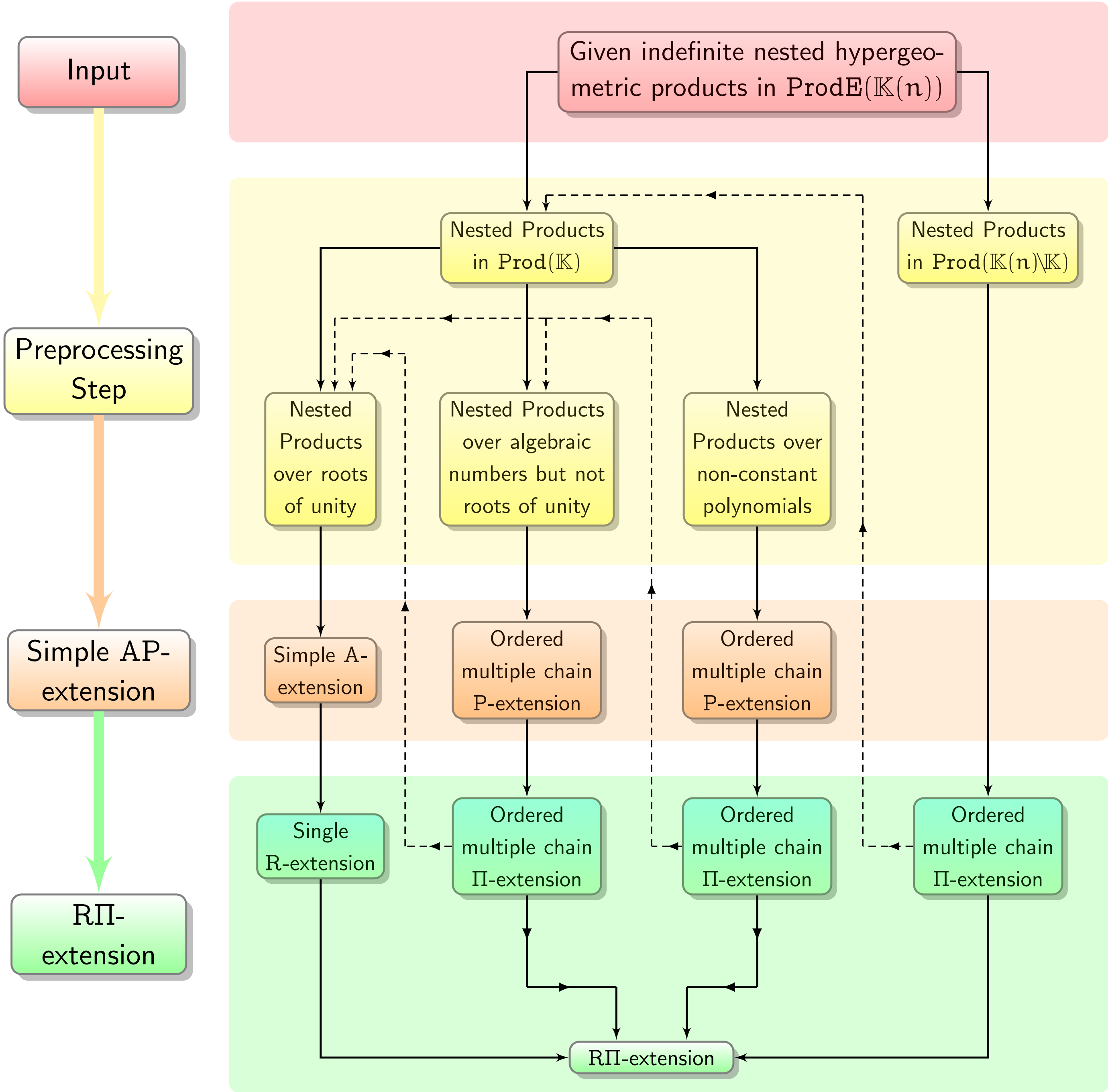


Figure 1.2: A general overview of the machinery for solving Problem RPE for products in $\mathrm{ProdE}(\mathbb{K}(n))$.

Example 1.1.4.
Given the nesting depth 2 hypergeometric product

$$P = \prod_{k=1}^{n} \frac{k \prod_{i=1}^{k} -\frac{\sqrt{3}}{i}}{3\,(k+1)^2} + \prod_{k=1}^{n} \frac{k \prod_{i=1}^{k} (i+1)}{(2\,k+5)} + \prod_{k=1}^{n} \frac{\prod_{i=1}^{k} \left(\frac{i+3}{i}\right)}{(2\,k+1)} \in \mathrm{ProdE}(\mathbb{K}(n))$$

where $\mathbb{K} = \mathbb{Q}(\sqrt{3})$, we can find the alternative expression

$$
\begin{aligned}
Q = {} & \frac{1 - \mathbb{i}}{2\,(n+1)^2}\,(\mathbb{i})^n \left((\mathbb{i} + ((\mathbb{i})^n)^2\right) \left((\sqrt{3})^n\right)^{-2} (\sqrt{3})^{\binom{n+1}{2}} \frac{1}{n!} \left(\prod_{k=1}^{n}\prod_{i=1}^{k}\frac{1}{i}\right) + \\
& \frac{15\,(n+1)}{(n+3)\,(n+5)}\,(2^n)^{-1}\,(n!)^2 \left(\prod_{k=1}^{n}\frac{1}{(k+\frac{1}{2})}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k} i\right) + \\
& \frac{(n+1)^3\,(n+2)^2\,(n+3)}{12}\,(2^n)^{-2}\left((\sqrt{3})^n\right)^{-2}(n!)^3 \left(\prod_{k=1}^{n}\frac{1}{(k+\frac{1}{2})}\right) \in \mathrm{ProdE}(\tilde{\mathbb{K}}(n))
\end{aligned}
$$

where $\tilde{\mathbb{K}} = \mathbb{Q}(\mathbb{i}, \sqrt{3})$ such that

$$P(n) = Q(n) \quad \forall\, n \in \mathbb{N}$$

holds and the sequence generated by

$$2^n,\ (\sqrt{3})^n,\ n!,\ \prod_{k=1}^{n}(k+\tfrac{1}{2}),\ (\sqrt{3})^{\binom{n+1}{2}},\ \prod_{k=1}^{n}\prod_{i=1}^{k} i$$

are algebraically independent over the ring $\tilde{\mathbb{K}}(n)[(\mathbb{i})^n]$. $\star$

Further examples arising from combinatorial problems (Mills et al., 1983; Zeilberger, 1996a; Krattenthaler, 2001; Fischer, 2007; Kauers, 2018) will be presented at the end of Chapter 6.

Chapter 2

Algebraic preliminaries

In this chapter, we will set up the algebraic framework necessary to specify the problems we want to solve. We will also introduce more formally, the objects we will be working with throughout this thesis.

2.1 Formal description of indefinite product expressions.

The ideas discussed in this section were inspired by Buchberger and Loos (1983), Paule and Nemes (1997) and Schneider (2010b). Let $\mathbb{X}$ be a set of expression (i.e., terms of certain types), $\mathbb{K}$ be a field and ev be an evaluation function

$$\mathrm{ev} : \mathbb{X} \times \mathbb{N} \to \mathbb{K}$$

with $(f, k) \mapsto f(k)$. Here, we consider the evaluation function ev as a procedure that effectively computes $f(k)$ for a given $f \in \mathbb{X}$ and $k \in \mathbb{N}$ in a finite number of steps.

Example 2.1.1.
Let $\mathbb{X} = \mathbb{K}(x)$ be a rational function field. Then for $f = \frac{g}{h} \in \mathbb{K}(x)$ with $g, h \in \mathbb{K}[x]$ where $h \neq 0$ and g and h are co-prime, we define the evaluation function

$$\mathrm{ev}(f, k) := \begin{cases} 0 & \text{if } h(k) = 0, \\ \frac{g(k)}{h(k)} & \text{if } h(k) \neq 0. \end{cases} \tag{2.1}$$

Here, $g(k)$ and $h(k)$ are the usual polynomial evaluation at some natural number k. We call $(\mathbb{K}(x), \mathrm{ev})$ a *rational sequence domain*. ⋆

Example 2.1.2.
Let $\mathbb{K} = K(q_1, \ldots, q_e)$ be a rational function field extension over a field K. Let $\mathbb{X} := \mathbb{K}(x, t_1, \ldots, t_e)$ be a rational function field over $\mathbb{K}$. Then for $f = \frac{g}{h} \in \mathbb{X}$, with $g, h \in \mathbb{K}[x, t_1, \ldots, t_e]$ where $h \neq 0$ and g, h are co-prime, we define

$$\mathrm{ev}(f, k) := \begin{cases} 0 & \text{if } h(k, q_1^k, \ldots, q_e^k) = 0, \\ \frac{g(k, q_1^k, \ldots, q_e^k)}{h(k, q_1^k, \ldots, q_e^k)} & \text{if } h(k, q_1^k, \ldots, q_e^k) \neq 0. \end{cases} \tag{2.2}$$

Subsequently, for any $k \in \mathbb{Z}$, we write $q_1^k, \ldots, q_e^k$ or $(q_1^k, \ldots, q_e^k)$ as $\mathbf{q}^k$ and $t_1, \ldots, t_e$ or $(t_1, \ldots, t_e)$ as $\mathbf{t}$. We call $(\mathbb{K}(x, \mathbf{t}), \mathrm{ev})$ a *mixed* $\mathbf{q}$*-multibasic rational sequence domain*. If $e = 0$, then we are back to the rational sequence domain. If $\mathbb{X} = \mathbb{K}(\mathbf{t})$ which is free of x, then $(\mathbb{K}(\mathbf{t}), \mathrm{ev})$ is called a $\mathbf{q}$*-multibasic rational sequence domain*. If $e = 1$, then we call $(\mathbb{K}(t), \mathrm{ev})$ a *basic* or q*-rational sequence domain* where $t = t_1$ and $q = q_1$. ⋆

For $\mathbb{X} \in \{\mathbb{K}, \mathbb{K}(x), \mathbb{K}(\mathbf{t}), \mathbb{K}(x, \mathbf{t})\}$ we follow the construction from Schneider (2010b). The set of (*indefinite nested*) *product-sum expressions*, denoted by $\mathrm{ProdSum}(\mathbb{X})$ is the freely generated algebraic structure over the signature:

$$\begin{aligned} \oplus: &\quad \mathrm{ProdSum}(\mathbb{X}) \times \mathrm{ProdSum}(\mathbb{X}) &\to&\quad \mathrm{ProdSum}(\mathbb{X}) \\ \otimes: &\quad \mathrm{ProdSum}(\mathbb{X}) \times \mathrm{ProdSum}(\mathbb{X}) &\to&\quad \mathrm{ProdSum}(\mathbb{X}) \\ \mathrm{Prod}: &\quad \mathbb{N} \times \mathrm{ProdSum}(\mathbb{X}) &\to&\quad \mathrm{ProdSum}(\mathbb{X}) \\ \mathrm{Sum}: &\quad \mathbb{N} \times \mathrm{ProdSum}(\mathbb{X}) &\to&\quad \mathrm{ProdSum}(\mathbb{X}). \end{aligned}$$

Observe that the term algebra $\mathrm{ProdSum}(\mathbb{X}) \supseteq \mathbb{X}$ is the smallest set satisfying the following rules:

(1) For any $p, r \in \mathrm{ProdSum}(\mathbb{X})$, $p \oplus r \in \mathrm{ProdSum}(\mathbb{X})$ and $p \otimes r \in \mathrm{ProdSum}(\mathbb{X})$.

(2) For any $p \in \mathrm{ProdSum}(\mathbb{X})$ and any $m \in \mathbb{N}$, $\mathrm{Prod}(m, p) \in \mathrm{ProdSum}(\mathbb{X})$ and $\mathrm{Sum}(m, p) \in \mathrm{ProdSum}(\mathbb{X})$.

Example 2.1.3 (Cont. Example 2.1.1).
Given $(\mathbb{X}, \mathrm{ev})$ from Example 2.1.1 with $\mathbb{X} = \mathbb{K}(x)$ where $\mathbb{K} = \mathbb{Q}\big(\mathbb{i} + \sqrt{3}, \sqrt{-13}\big)$, the following product expressions are in $\mathrm{ProdSum}(\mathbb{X})$:

$$P_1 = \mathrm{Prod}\left(1, \frac{-13\sqrt{-13}}{x}\right),\ P_2 = \mathrm{Prod}\left(1, \frac{-784\,x}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^4\,(x+2)^2}\right),$$
$$P_3 = \mathrm{Prod}\left(1, \frac{-17210368\,x}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^{10}\,(x+2)^5}\right)$$

and

$$P = P_1 \oplus P_2 \oplus P_3. \qquad \star$$

Example 2.1.4 (Cont. Example 2.1.1).
Let $(\mathbb{X}, \mathrm{ev})$ from Example 2.1.1 with $\mathbb{X} = \mathbb{K}(x)$ where $\mathbb{K} = \mathbb{Q}\big(\sqrt{3}\big)$. Then

$$H = \mathrm{Prod}\left(1, \frac{x}{3\,(x+1)^2} \otimes \left(\mathrm{Prod}\left(1, -\frac{\sqrt{3}}{x}\right)\right) \otimes \left(\mathrm{Prod}\left(1, \frac{x}{x+2}\right)\right)\right) \in \mathrm{ProdSum}(\mathbb{X}). \qquad \star$$

Finally, we extend the evaluation function ev from $\mathbb{X}$ to $\mathrm{ev}' : \mathrm{ProdSum}(\mathbb{X}) \times \mathbb{N} \to \mathbb{K}$ with $(p, n) \mapsto p(n)$ as follows.

(1) For $f \in \mathbb{X}$, set $\mathrm{ev}'(f, k) := \mathrm{ev}(f, k)$ for $k \in \mathbb{N}$.

(2) For $f, g \in \mathrm{ProdSum}(\mathbb{X})$, set

$$\mathrm{ev}'(f \oplus g, k) := \mathrm{ev}'(f, k) + \mathrm{ev}'(g, k) \quad \text{and} \quad \mathrm{ev}'(f \otimes g, k) := \mathrm{ev}'(f, k)\,\mathrm{ev}'(g, k) \quad \text{for } k \in \mathbb{N}.$$

Here the operations on the right hand side are from the field $\mathbb{K}$.

(3) For $f \in \mathrm{ProdSum}(\mathbb{X})$ and $i \in \mathbb{N}$ we define

$$\mathrm{ev}'\big(\mathrm{Prod}(i, f), n\big) = \prod_{k=i}^{n} \mathrm{ev}'(f, k) \quad \text{and} \quad \mathrm{ev}'\big(\mathrm{Sum}(i, f), n\big) = \sum_{k=i}^{n} \mathrm{ev}'(f, k).$$

Subsequently, we do not distinguish any longer between ev' and ev since they agree on $\mathbb{X}$.

In this thesis, we will only consider (*indefinite nested*) *product expressions*. That is, the subclass of $\mathrm{ProdSum}(\mathbb{X})$ which are free of Sum or the summation quantifier. Thus, we introduce the following definition.

Definition 2.1.5.
Let $\mathbb{S} \in \{\mathbb{K}, \mathbb{K}(n), \mathbb{K}(\mathbf{q}^n), \mathbb{K}(n, \mathbf{q}^n)\}$. We call

$$\mathrm{Prod}(\mathbb{S}) = \{f \in \mathrm{ProdSum}(\mathbb{S}) \,|\, f \text{ is free of function symbol } \mathrm{Sum}\}$$

the *nested product sequence domain over* $\mathbb{S}$. $*$

Example 2.1.6 (Cont. Example 2.1.3).
The product expressions in Example 2.1.3 are evaluated as

$$\begin{aligned}
\mathrm{ev}(P_1, n) = P_1(n) &= \prod_{k=1}^{n} \mathrm{ev}\Big(\frac{-13\sqrt{-13}}{x}, k\Big) = \prod_{k=1}^{n} \frac{-13\sqrt{-13}}{k} \in \mathrm{Prod}(\mathbb{K}(n)),\\
\mathrm{ev}(P_2, n) = P_2(n) &= \prod_{k=1}^{n} \mathrm{ev}\Big(\frac{-784\,x}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^4\,(x+2)^2}, k\Big) = \prod_{k=1}^{n} \frac{-784\,k}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^4\,(k+2)^2} \in \mathrm{Prod}(\mathbb{K}(n)),\\
\mathrm{ev}(P_3, n) = P_3(k) &= \prod_{k=1}^{n} \mathrm{ev}\Big(\frac{-17210368\,x}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^{10}\,(x+2)^5}, k\Big) = \prod_{k=1}^{n} \frac{-17210368\,k}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^{10}\,(k+2)^5} \in \mathrm{Prod}(\mathbb{K}(n)). \star
\end{aligned}$$

Example 2.1.7 (Cont. Example 2.1.4).
The product expression in Example 2.1.4 evaluates as

$$\begin{aligned}
\mathrm{ev}(H, n) = H(n) &= \prod_{k=1}^{n} \mathrm{ev}\left(\frac{x}{3\,(x+2)^2}, k\right) \otimes \left(\prod_{i=1}^{k} \mathrm{ev}\left(-\frac{\sqrt{3}}{x}, i\right)\right) \otimes \left(\prod_{i=1}^{k} \mathrm{ev}\left(\frac{x}{x+2}, i\right)\right)\\
&= \prod_{k=1}^{n} k \frac{\prod_{i=1}^{k} \frac{-\sqrt{3}}{i} \prod_{i=1}^{k} \frac{i}{i+2}}{3\,(k+1)^2} \in \mathrm{Prod}(\mathbb{K}(n)).
\end{aligned} \qquad \star$$

From Example 2.1.7, the following remarks are in place.

Remark 2.1.8.

(1) For a given product expression say

$$F = u \oplus v \otimes \mathrm{Prod}(\ell_1, w \otimes \mathrm{Prod}(\ell_2, \frac{1}{x})) \in \mathrm{Prod}(\mathbb{X})$$

with $u, v, w \in \mathbb{X}$, and $\ell_1, \ell_2 \in \mathbb{N}$ we write F in the form

$$F' = \mathrm{ev}(u, n) \oplus \mathrm{ev}(v, n) \otimes \prod_{i=\ell_1}^{n} \mathrm{ev}(w, i) \otimes \prod_{j=\ell_2}^{\ell_1} \mathrm{ev}(\tfrac{1}{x}, j)$$

for a symbolic variable n. Clearly by fixing n we can transform any $F \in \mathrm{ProdSum}(\mathbb{X})$ to the form F' and vice-versa.

(2) Furthermore, we use the usual field operations in $\mathbb{K}$ instead of $\oplus$ and $\otimes$. This abuse of notation results in the evaluation mechanism: $\mathrm{ev}(F, k) = F(k)$ for a concrete non-negative integer $k \in \mathbb{N}$ is produced by substituting in F' the variable n with the concrete non-negative integer $k \in \mathbb{N}$.

(3) Finally, whenever possible, the evaluation $\mathrm{ev}(f, n)$ for some $f \in \mathbb{X}$ is expressed by some well known function. For example,

$$\mathrm{ev}(\tfrac{1}{x}, n) = \frac{1}{n}, \; \prod_{k=1}^{n} \mathrm{ev}(x, k) = n!, \; \mathrm{ev}(t_i, n) = q_i^n \text{ for } 1 \leqslant i \leqslant e.$$

♠

Lemma 2.1.9.
Let $f \in \mathbb{K}[x, \mathbf{t}] \setminus \{0\}$. Then the set

$$\{n \in \mathbb{N} \,|\, f(n, \mathbf{q}^n) = 0\}$$

is finite. In particular, one can compute its maximal value or one can decide if the set is empty. ♦

Proof:
See Bauer and Petkovšek (1999, Section 3). ■

We will now give a precise definition (compare Bauer and Petkovšek (1999)) of the different classes of indefinite product expressions that we will consider in this thesis.

Definition 2.1.10.
Let $\mathbb{K} = K(q_1, \dots, q_e)$ be a rational function field over a field K and let $\mathbb{F} = \mathbb{K}(x, \mathbf{t})$ be a rational function field over $\mathbb{K}$. The indefinite product expression

$$\prod_{k_1=l_1}^{n} f_1(k_1, \mathbf{q}^{k_1}) \cdots \prod_{k_m=l_m}^{k_{m-1}} f_m(k_m, \mathbf{q}^{k_m}) \in \mathrm{Prod}(\mathbb{K}(n, \mathbf{q}^n)) \tag{2.3}$$

is called a *mixed* $(q_1, \dots, q_e)$*-multibasic hypergeometric product* in n of nesting depth m, if $\ell \in \mathbb{N}$ is chosen big enough such that for all $1 \leqslant i \leqslant m$ with $\ell \geqslant l_i$, $f_i(\ell, q_1^\ell, \dots, q_e^\ell) \in \mathbb{F}$ has no pole and is non-zero. Note that given $f_1, \dots, f_m$ such $\ell_1, \dots, \ell_m$ can always be chosen by Lemma 2.1.9. If for all $1 \leqslant i \leqslant m$, $f_i(t_1, \dots, t_e) \in \mathbb{F}^*$ which are all free of x, then the product expression

$$\prod_{k_1=l_1}^{n} f_1(\mathbf{q}^{k_1}) \cdots \prod_{k_m=l_m}^{k_{m-1}} f_m(\mathbf{q}^{k_m}) \in \mathrm{Prod}(\mathbb{K}(\mathbf{q}^n)) \tag{2.4}$$

is called a $(q_1, \dots, q_e)$*-multibasic hypergeometric product* in n of nesting depth m. If $e = 1$, then it is called a *basic* or q*-hypergeometric product* in n of nesting depth m where $q = q_1$. If $e = 0$ and $f_i \in \mathbb{K}(x)^*$ for all $1 \leqslant i \leqslant m$, then the product expression

$$\prod_{k_1=l_1}^{n} f_1(k_1) \cdots \prod_{k_m=l_m}^{k_{m-1}} f_m(k_m) \in \mathrm{Prod}(\mathbb{K}(n)) \tag{2.5}$$

is called a *hypergeometric product* in n of nesting depth m. Finally, if $f_i \in \mathbb{K}^*$ for all $1 \leqslant i \leqslant m$, then we call the product expression

$$\prod_{k_1=l_1}^{n} f_1 \cdots \prod_{k_m=l_m}^{k_{m-1}} f_m \in \mathrm{Prod}(\mathbb{K}) \tag{2.6}$$

a *constant or geometric product* in n of nesting depth m.

In addition, let $\mathbb{S}_{\mathbb{K}} \in \{\mathbb{K}, \mathbb{K}(n), \mathbb{K}(\mathbf{q}^n), \mathbb{K}(n, \mathbf{q}^n)\}$. We define the set of evaluated product expressions $\mathrm{ProdE}(\mathbb{S}_{\mathbb{K}})$ as the set of all elements

$$\sum_{(\nu_1,\ldots,\nu_m)\in S} a_{(\nu_1,\ldots,\nu_m)}(n)\, P_1(n)^{\nu_1} \cdots P_m(n)^{\nu_m} \tag{2.7}$$

with $m \in \mathbb{N}$, $S \subseteq \mathbb{Z}^m$ finite, $a_{(\nu_1,\ldots,\nu_m)}(n) \in \mathbb{S}_{\mathbb{K}}$ and $P_1(n), \ldots, P_m(n) \in \mathrm{Prod}(\mathbb{S}_{\mathbb{K}})$. $*$

For convenience we introduce the following notations. If it is clear from the context, we denote $f(k, \mathbf{q}^k)$ by $f[k]$, if $f \in \mathbb{K}(x, \mathbf{t}) \setminus \mathbb{K}$. $f[k]$ denotes $f(\mathbf{q}^k)$, if $f \in \mathbb{K}(\mathbf{t}) \setminus \mathbb{K}$ and finally $f[k]$ denotes $f(k)$, if $f \in \mathbb{K}(x)$.

Remark 2.1.11.
Let $f_1, \ldots, f_m \in \mathbb{X}$ where $\mathbb{X} \in \{\mathbb{K}, \mathbb{K}(x), \mathbb{K}(\mathbf{t}), \mathbb{K}(x, \mathbf{t})\}$. Then the equality

$$\prod_{k_1=\ell_1}^{n} f_1[k_1] \cdots \prod_{k_m=\ell_m}^{k_{m-1}} f_m[k_m] = \left(\prod_{k_1=\ell_1}^{n} f_1[k_1]\right)\left(\prod_{k_1=\ell_1}^{n}\prod_{k_2=\ell_2}^{k_1} f_2[k_2]\right)\cdots\left(\prod_{k_1=\ell_1}^{n}\prod_{k_2=\ell_2}^{k_1}\cdots\prod_{k_m=\ell_m}^{k_{m-1}} f_m[k_m]\right) \tag{2.8}$$

holds. We call the right hand side of (2.8) a *product factored form* of the left hand side. Note that if $f_1, \ldots, f_m \in \mathbb{X}$ are irreducible, then the product factored form is unique. Otherwise one can always factor the expressions $f_1, \ldots, f_m \in \mathbb{X}$ and expand the product quantifiers over each factor to obtain a unique product factored form. ♠

Proposition 2.1.12.
Let $\mathbb{S} \in \left\{\mathbb{K}, \mathbb{K}(n), \mathbb{K}(\mathbf{q}^n), \mathbb{K}(n, \mathbf{q}^n)\right\}$. Every indefinite product expression in $\mathrm{Prod}(\mathbb{S})$ *has a product factored form.* □

2.2 Ring of sequences

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{N}$ be the set of non-negative integers. We denote by $\mathbb{K}^{\mathbb{N}}$ the set of all sequences

$$\langle a(n) \rangle_{n\geqslant 0} = \langle a(0), a(1), a(2), \ldots \rangle \tag{2.9}$$

whose terms are in $\mathbb{K}$. With component-wise addition and multiplication, $\mathbb{K}^{\mathbb{N}}$ forms a commutative ring. The field $\mathbb{K}$ can be naturally embedded into $\mathbb{K}^{\mathbb{N}}$ as a subring, by identifying $c \in \mathbb{K}$ with the constant sequence $\langle c, c, c, \ldots \rangle \in \mathbb{K}^{\mathbb{N}}$. Following the construction in Petkovšek et al. (1996, Section 8.2), we turn the shift operator

$$S : \begin{cases} \mathbb{K}^{\mathbb{N}} & \to \mathbb{K}^{\mathbb{N}} \\ \langle a(0), a(1), a(2), \ldots \rangle & \mapsto \langle a(1), a(2), a(3), \ldots \rangle \end{cases}$$

into a ring automorphism by introducing an equivalence relation $\sim$ on sequences in $\mathbb{K}^{\mathbb{N}}$. Two sequences $\langle a(n)\rangle_{n\geqslant 0}$ and $\langle b(n)\rangle_{n\geqslant 0}$ are said to be equivalent if and only if there exists a natural number δ such that

$$\forall n \geqslant \delta : a(n) = b(n).$$

The set of equivalence classes form a ring again with component-wise addition and multiplication which we will denote by $\mathcal{S}(\mathbb{K}) := {}^{\mathbb{K}^{\mathbb{N}}}\!/\!_{\sim}$. For simplicity, we denote the elements of $\mathcal{S}(\mathbb{K})$ by the usual sequence notation as in (2.9) above. Now it is obvious that $S : \mathcal{S}(\mathbb{K}) \to \mathcal{S}(\mathbb{K})$ is a ring automorphism. We call $(\mathcal{S}(\mathbb{K}), S)$ the *difference ring of sequences over* $\mathbb{K}$.

Example 2.2.1.
Let $\mathbb{K} = \mathbb{Q}(\mathbb{i}, (-1)^{\frac{1}{6}})$. Then the geometric products over roots of unity in the expression

$$\prod_{k=1}^{n} (-1)^{\frac{1}{6}} + \prod_{k=1}^{n} (-1)^{\frac{1}{2}} \in \mathrm{ProdE}(\mathbb{K}) \tag{2.10}$$

yield the sequences:

$$\left\langle \left((-1)^{\frac{1}{6}}\right)^{n} \right\rangle_{n\geqslant 0}, \left\langle (\mathbb{i})^{n} \right\rangle_{n\geqslant 0} \in \mathcal{S}(\mathbb{K})$$

with

$$S_n \left\langle \left((-1)^{\frac{1}{6}}\right)^{n} \right\rangle_{n\geqslant 0} = \left\langle S_n \left((-1)^{\frac{1}{6}}\right)^{n} \right\rangle_{n\geqslant 0} = \left\langle \left((-1)^{\frac{1}{6}}\right)^{n+1} \right\rangle_{n\geqslant 0} = (-1)^{\frac{1}{6}} \left\langle \left((-1)^{\frac{1}{6}}\right)^{n} \right\rangle_{n\geqslant 0}, \tag{2.11}$$

$$S_n \left\langle (\mathbb{i})^{n} \right\rangle_{n\geqslant 0} = \left\langle S_n (\mathbb{i})^{n} \right\rangle_{n\geqslant 0} = \left\langle (\mathbb{i})^{n+1} \right\rangle_{n\geqslant 0} = \mathbb{i} \left\langle (\mathbb{i})^{n} \right\rangle_{n\geqslant 0}. \tag{2.12}$$

Here, $\mathbb{i}$ is the complex unit which we also write as $(-1)^{\frac{1}{2}}$ and $(-1)^{\frac{1}{6}}$ is the 12-th root of unity $\mathbb{e}^{\frac{\pi \mathbb{i}}{6}}$. ⋆

Example 2.2.2.
Let $\mathbb{K} = \mathbb{Q}((-1)^{\frac{2}{3}})$. Then the geometric products in the expression

$$\prod_{k=1}^{n} (-1) + \prod_{k=1}^{n} (-1)^{\frac{2}{3}} \in \mathrm{ProdE}(\mathbb{K}) \tag{2.13}$$

yield the sequences:

$$\left\langle (-1)^{n} \right\rangle_{n\geqslant 0}, \left\langle \left((-1)^{\frac{2}{3}}\right)^{n} \right\rangle_{n\geqslant 0} \in \mathcal{S}(\mathbb{K})$$

with

$$S_n \left\langle (-1)^{n} \right\rangle_{n\geqslant 0} = \left\langle S_n (-1)^{n} \right\rangle_{n\geqslant 0} = \left\langle -(-1)^{n} \right\rangle_{n\geqslant 0} = -\left\langle (-1)^{n} \right\rangle_{n\geqslant 0}, \tag{2.14}$$

$$S_n \left\langle \left((-1)^{\frac{2}{3}}\right)^{n} \right\rangle_{n\geqslant 0} = \left\langle S_n \left((-1)^{\frac{2}{3}}\right)^{n} \right\rangle_{n\geqslant 0} = \left\langle \left((-1)^{\frac{2}{3}}\right)^{n+1} \right\rangle_{n\geqslant 0} = (-1)^{\frac{2}{3}} \left\langle \left((-1)^{\frac{2}{3}}\right)^{n} \right\rangle_{n\geqslant 0}. \tag{2.15}$$

Here $(-1)^{\frac{2}{3}}$ is the 3-rd root of unity $\mathbb{e}^{\frac{2\pi \mathbb{i}}{3}}$. ⋆

Example 2.2.3.
Let $\mathbb{K} = \mathbb{Q}(\sqrt{13})$. The geometric products in the expression

$$\prod_{k=1}^{n} \sqrt{13} + \prod_{k=1}^{n} 7 + \prod_{k=1}^{n} 169 \in \mathrm{ProdE}(\mathbb{K}) \tag{2.16}$$

generate the sequences:

$$\langle (\sqrt{13})^n \rangle_{n\geqslant 0},\ \langle 7^n \rangle_{n\geqslant 0},\ \langle 169^n \rangle_{n\geqslant 0} \in \mathcal{S}(\mathbb{K})$$

with

$$\begin{aligned} S_n\langle (\sqrt{13})^n \rangle_{n\geqslant 0} &= \langle S_n (\sqrt{13})^n \rangle_{n\geqslant 0} &&= \langle (\sqrt{13})^{n+1} \rangle_{n\geqslant 0} &&= \sqrt{13}\,\langle (\sqrt{13})^n \rangle_{n\geqslant 0}, \\ S_n\Big\langle \frac{1}{(\sqrt{13})^n} \Big\rangle_{n\geqslant 0} &= \Big\langle S_n \frac{1}{(\sqrt{13})^n} \Big\rangle_{n\geqslant 0} &&= \Big\langle \frac{1}{(\sqrt{13})^{n+1}} \Big\rangle_{n\geqslant 0} &&= \frac{1}{\sqrt{13}} \Big\langle \frac{1}{(\sqrt{13})^n} \Big\rangle_{n\geqslant 0}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} S_n\langle 7^n \rangle_{n\geqslant 0} &= \langle S_n 7^n \rangle_{n\geqslant 0} &&= \langle 7^{n+1} \rangle_{n\geqslant 0} &&= 7\,\langle 7^n \rangle_{n\geqslant 0}, \\ S_n\Big\langle \frac{1}{7^n} \Big\rangle_{n\geqslant 0} &= \Big\langle S_n \frac{1}{7^n} \Big\rangle_{n\geqslant 0} &&= \Big\langle \frac{1}{7^{n+1}} \Big\rangle_{n\geqslant 0} &&= \frac{1}{7} \Big\langle \frac{1}{7^n} \Big\rangle_{n\geqslant 0}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} S_n\langle 169^n \rangle_{n\geqslant 0} &= \langle S_n 169^n \rangle_{n\geqslant 0} &&= \langle 169^{n+1} \rangle_{n\geqslant 0} &&= 169\,\langle 169^n \rangle_{n\geqslant 0}, \\ S_n\Big\langle \frac{1}{169^n} \Big\rangle_{n\geqslant 0} &= \Big\langle S_n \frac{1}{169^n} \Big\rangle_{n\geqslant 0} &&= \Big\langle \frac{1}{169^{n+1}} \Big\rangle_{n\geqslant 0} &&= \frac{1}{169} \Big\langle \frac{1}{169^n} \Big\rangle_{n\geqslant 0}. \end{aligned} \tag{2.19}$$

$\star$

Example 2.2.4.
The hypergeometric products in the expression

$$P(n) = P_1(n) + P_2(n) \in \mathrm{ProdE}(\mathbb{Q}(n)) \tag{2.20}$$

where

$$P_1(n) = \prod_{k=1}^{n} k \ \text{ and } \ P_2(n) = \prod_{k=1}^{n} (k+2)$$

yield the sequences:

$$\langle \prod_{k=1}^{n} k \rangle_{n\geqslant 0},\ \langle \prod_{k=1}^{n} k+2 \rangle_{n\geqslant 0} \in \mathcal{S}(\mathbb{Q})$$

with

$$\begin{aligned} S_n\langle P_1(n) \rangle_{n\geqslant 0} &= \langle S_n P_1(n) \rangle_{n\geqslant 0} &&= \langle P_1(n+1) \rangle_{n\geqslant 0} &&= (n+1)\,\langle P_1(n) \rangle_{n\geqslant 0}, \\ S_n\Big\langle \frac{1}{P_1(n)} \Big\rangle_{n\geqslant 0} &= \Big\langle S_n \frac{1}{P_1(n)} \Big\rangle_{n\geqslant 0} &&= \Big\langle \frac{1}{P_1(n+1)} \Big\rangle_{n\geqslant 0} &&= \frac{1}{n+1} \Big\langle \frac{1}{P_1(n)} \Big\rangle_{n\geqslant 0}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} S_n\langle P_2(n) \rangle_{n\geqslant 0} &= \langle S_n P_2(n) \rangle_{n\geqslant 0} &&= \langle P_2(n+1) \rangle_{n\geqslant 0} &&= (n+3)\,\langle P_2(n) \rangle_{n\geqslant 0}, \\ S_n\Big\langle \frac{1}{P_2(n)} \Big\rangle_{n\geqslant 0} &= \Big\langle S_n \frac{1}{P_2(n)} \Big\rangle_{n\geqslant 0} &&= \Big\langle \frac{1}{P_2(n+1)} \Big\rangle_{n\geqslant 0} &&= \frac{1}{n+3} \Big\langle \frac{1}{P_2(n)} \Big\rangle_{n\geqslant 0}. \end{aligned} \tag{2.22}$$

$\star$

Definition 2.2.5.
Let $\mathbb{S} \in \{\mathbb{K},\, \mathbb{K}(n),\, \mathbb{K}(\mathbf{q}^n),\, \mathbb{K}(n, \mathbf{q}^n)\}$. The sequences:

$$\langle P_1(n) \rangle_{n \geqslant 0}, \ldots, \langle P_e(n) \rangle_{n \geqslant 0}$$

generated by $P_1(n), \ldots, P_e(n) \in \mathrm{Prod}(\mathbb{S})$ are said to be *algebraically independent* if there is no polynomial $F(x_1, \ldots, x_e) \in \mathbb{S}[x_1, \ldots, x_e] \setminus \{0\}$ such that

$$F(P_1(n), \ldots, P_e(n)) = 0$$

for all $n \in \mathbb{N}$ with $n \geqslant \delta$ for some $\delta \in \mathbb{N}$. $*$

Throughout this thesis, our ground field $\mathbb{K}$ can be a rational function field defined over an algebraic number field K. For such a ground field we will consider the following problem.

Problem RPE: Representation of Product Expressions.

Let $\mathbb{S}_{\mathbb{K}} \in \{\mathbb{K}, \mathbb{K}(n), \mathbb{K}(\mathbf{q}^n), \mathbb{K}(n, \mathbf{q}^n)\}$. *Given* $P(n) \in \mathrm{ProdE}(\mathbb{S}_{\mathbb{K}})$;
find $Q(n) \in \mathrm{ProdE}(\mathbb{S}_{\mathbb{K}'})$ with $\mathbb{K}'$ a finite algebraic field extension[a] of $\mathbb{K}$ and a natural number δ with the following properties:

(1) $P(n) = Q(n)$ for all $n \in \mathbb{N}$ with $n \geqslant \delta$;

(2) The product expressions in $Q(n)$ (apart from products over roots of unity) are algebraically independent among each other.

(3) The zero-recognition property holds, i.e., $P(n) = 0$ holds for all n from a certain point on if and only if $Q(n)$ is the zero-expression.

[a] If $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)(q_1, \ldots, q_e)$ is a rational function field over an algebraic number field K, then in worst case $\mathbb{K}$ is extended to $\mathbb{K}' = K'(\kappa_1, \ldots, \kappa_u)(q_1, \ldots, q_e)$ where K' is an algebraic field extension of K. Subsequently, all algebraic field extensions are finite.

For nesting depth 1 products which are also called single nested products, Problem RPE will be solved completely; see also Ocansey and Schneider (2018). More generally, the problem will be tackled for products of arbitrary but finite nesting depth for $\mathbb{S}_{\mathbb{K}} \in \{\mathbb{K},\, \mathbb{K}(n)\}$. In the future, we intend to solve the problem for $\mathbb{S}_{\mathbb{K}} \in \{\mathbb{K}(\mathbf{q}^n),\, \mathbb{K}(n, \mathbf{q}^n)\}$.

2.3 Difference fields and difference rings

In this section, we discuss the algebraic setting of difference rings (resp. fields) and their connection to the ring of sequences as elaborated in Karr (1981) and Schneider (2016, 2017). In particular, we demonstrate how sequences generated by product expressions in $\mathrm{ProdE}(\mathbb{S})$ with $\mathbb{S} \in \{\mathbb{K},\, \mathbb{K}(n),\, \mathbb{K}(\mathbf{q}^n),\, \mathbb{K}(n, \mathbf{q}^n)\}$ can be modelled in this algebraic framework.

Let $f \in \mathrm{ProdSum}(\mathbb{S})$ and $i \in \mathbb{N}$. For a product expression $P \in \mathrm{ProdSum}(\mathbb{S})$ we have that

$$P(n) = \prod_{k=i}^{n} f(i) \quad \text{and} \quad P(n+1) = \prod_{k=i}^{n+1} f(k) = f(n+1) \prod_{k=i}^{n} f(k) = f(n+1)\, P(n). \tag{2.23}$$

Similarly, for a sum expression $S \in \mathrm{ProdSum}(\mathbb{S})$ we have that

$$S(n) = \sum_{k=i}^{n} f(k) \quad \text{and} \quad S(n+1) = \sum_{k=i}^{n+1} f(k) = \sum_{k=i}^{n} f(k) + f(n+1) = S(n) + f(n+1). \tag{2.24}$$

From (2.23) and (2.24), it is clear that for any given $f \in \mathrm{ProdSum}(\mathbb{S})$ and $i \in \mathbb{N}$, the product (resp. sum) expression P (resp. S) satisfies the recurrence relation

$$P(n+1) = f(n+1)\,P(n) \;\text{ and }\; S(n+1) = S(n) + f(n+1) \quad \forall n \geqslant i$$

respectively. As a consequence of the above, we can define a shift operator acting on the evaluation of the product expression $P(n)$ (resp. sum expression $S(n)$) in an algebraic framework.

Definition 2.3.1.
A difference ring[1] (resp. field) denoted $(\mathbb{A}, \sigma)$ is a ring (resp. field) $\mathbb{A}$ together with a ring (resp. field) automorphism $\sigma : \mathbb{A} \to \mathbb{A}$. $*$

Subsequently, all rings (resp. fields) contain the set of rational numbers $\mathbb{Q}$, as a subring (resp. subfield). The multiplicative group of units of a ring (resp. field) $\mathbb{A}$ is denoted by $\mathbb{A}^*$. A ring (resp. field) is computable if all of its operations are computable.

Definition 2.3.2.
Let $(\mathbb{A}, \sigma)$ be a difference ring (resp. field). The set of constants of $(\mathbb{A}, \sigma)$ is defined by

$$\mathrm{const}(\mathbb{A}, \sigma) = \{c \in \mathbb{A} \mid \sigma(c) = c\}. \qquad *$$

Note that $\mathrm{const}(\mathbb{A}, \sigma)$ is a subring (resp. subfield) of $\mathbb{A}$ and contains $\mathbb{Q}$ as a subring (resp. subfield). For any difference ring (resp. field), we shall denote the set of constants by $\mathbb{K}$ which is often also called the ring (resp. field) of constants. A difference ring (resp. field) $(\mathbb{A}, \sigma)$ is computable if $\mathbb{A}$ and σ are both computable. Thus, given a computable difference ring (resp. field), one can decide if $\sigma(c) = c$.

Definition 2.3.3.
A difference ring $(\tilde{\mathbb{A}}, \tilde{\sigma})$ is said to be a *difference ring extension* of a difference ring $(\mathbb{A}, \sigma)$ if $\mathbb{A}$ is a subring of $\tilde{\mathbb{A}}$ which we write as $\mathbb{A} \leqslant \tilde{\mathbb{A}}$ and for all $a \in \mathbb{A}$, $\tilde{\sigma}(a) = \sigma(a)$ (i.e., $\tilde{\sigma}|_{\mathbb{A}} = \sigma$). The definition of a *difference field extension* is the same by only replacing the word ring with field. In short for difference ring (resp. field) extensions, we write $(\mathbb{A}, \sigma) \leqslant (\tilde{\mathbb{A}}, \tilde{\sigma})$ and if the context is clear, we do not distinguish any more between σ and $\tilde{\sigma}$. $*$

2.3.1 APS-extensions

APS-extensions introduced in Schneider (2017) are difference ring (resp. field) extensions where indefinite sums and products can be represented in a naive way. In this thesis, we are interested in a subclass of these extensions called AP-extensions. We remark that the extensions introduced here have also been discussed in Schneider (2017, Section 2).

[1] Subsequently, all rings (resp. fields) are commutative with unity and have characteristics 0.

2.3.1.1 S-EXTENSIONS

S-extensions are used to model indefinite sum expressions in $\mathrm{ProdSum}(\mathbb{S})$ with

$$\mathbb{S} \in \{\mathbb{K},\ \mathbb{K}(n),\ \mathbb{K}(\mathbf{q}^n),\ \mathbb{K}(n, \mathbf{q}^n)\}.$$

Definition 2.3.4.
Let $(\mathbb{A}, \sigma)$ be a difference ring, $\beta \in \mathbb{A}$ and $\mathbb{A}[x]$ be a polynomial ring (i.e., x is transcendental over $\mathbb{A}$). A difference ring $(\mathbb{A}[x], \sigma)$ with $\sigma(x) = x + \beta$ is called a *sum-extension* (in short an S-*extension*) of $(\mathbb{A}, \sigma)$. We call x an S-*monomial*. Suppose that $\mathbb{A}$ is a field and $\mathbb{A}(x)$ is a rational function field (i.e., x is transcendental over $\mathbb{A}$). Let $\beta \in \mathbb{A}$. A difference field $(\mathbb{A}(x), \sigma)$ with $\sigma(x) = x + \beta$ is called an S-*field extension* of $(\mathbb{A}, \sigma)$. We call x an S-*field monomial*. $*$

Remark 2.3.5.
Observe from Definition 2.3.4 that we have the chain of extensions $(\mathbb{A}, \sigma) \leqslant (\mathbb{A}[x], \sigma) \leqslant (\mathbb{A}(x), \sigma)$. Throughout this thesis, we allow only a single S-field extension in our base field. ♠

Example 2.3.6.
Let $\mathbb{K}(x)$ be a rational function field and define the field automorphism $\sigma : \mathbb{K}(x) \to \mathbb{K}(x)$ with $\sigma(x) = x+1$. We call $(\mathbb{K}(x), \sigma)$ the *rational difference field* over $\mathbb{K}$. $\star$

2.3.1.2 P-EXTENSIONS

P-extensions are used to model indefinite product expressions in $\mathrm{Prod}(\mathbb{S})$ with

$$\mathbb{S} \in \{\mathbb{K},\ \mathbb{K}(n),\ \mathbb{K}(\mathbf{q}^n),\ \mathbb{K}(n, \mathbf{q}^n)\}.$$

This thesis focuses mainly on this class of extensions as well as the extensions discussed in the next Section 2.3.1.3.

Definition 2.3.7.
Let $(\mathbb{A}, \sigma)$ be a difference ring, $\alpha \in \mathbb{A}^*$ be a unit and consider the ring of Laurent polynomials $\mathbb{A}[t, t^{-1}]$ (i.e., t is transcendental over $\mathbb{A}$). A difference ring $(\mathbb{A}[t, t^{-1}], \sigma)$ with $\sigma(t) = \alpha\, t$ is called a product-extension (in short P-*extension*) of $(\mathbb{A}, \sigma)$. The generator t is called a P-*monomial*. Suppose that $\mathbb{A}$ is a field and $\mathbb{A}(t)$ is a rational function field (i.e., t is transcendental over $\mathbb{A}$). Let $\alpha \in \mathbb{A}^*$. A difference field $(\mathbb{A}(t), \sigma)$ with $\sigma(t) = \alpha\, t$ is called an P-*field extension* of $(\mathbb{A}, \sigma)$. We call t an P-*field monomial*. $*$

Remark 2.3.8.
From Definition 2.3.7, we also get the chain of extensions $(\mathbb{A}, \sigma) \leqslant (\mathbb{A}[t, t^{-1}], \sigma) \leqslant (\mathbb{A}(t), \sigma)$. ♠

Subsequently, we introduce nested P-extensions.

Definition 2.3.9.
Let $(\mathbb{A}, \sigma)$ be a difference ring. We call $(\mathbb{A}[t_1, t_1^{-1}] \dots [t_e, t_e^{-1}], \sigma)$ a *(nested)* P*-extension* of $(\mathbb{A}, \sigma)$ if it is a tower of P-extensions. That is, $(\mathbb{A}[t_1, t_1^{-1}] \dots [t_i, t_i^{-1}], \sigma)$ is a P-extension of $(\mathbb{A}[t_1, t_1^{-1}] \dots [t_{i-1}, t_{i-1}^{-1}], \sigma)$ for $1 \leqslant i \leqslant e$. Similarly, we call $(\mathbb{A}(t_1) \dots (t_e), \sigma)$ a *(nested)* P*-field extension* of $(\mathbb{A}, \sigma)$ if it is a tower of P-field extensions. That is, $(\mathbb{A}(t_1) \dots (t_i), \sigma)$ is a P-field extension of $(\mathbb{A}(t_1) \dots (t_{i-1}), \sigma)$ for $1 \leqslant i \leqslant e$.$*$

An important subclasses of difference fields covered in this thesis are the mixed $\mathbf{q}$-multibasic difference field and the $\mathbf{q}$-multibasic difference field. These difference fields are a generalisation of the rational difference field introduced in Example 2.3.6. The terminologies used here have been carefully introduced by Bauer and Petkovšek (1999).

Example 2.3.10.
Let $\mathbb{K} = \mathsf{K}(q_1, \dots, q_e)$ be a rational function field (i.e., the q_i are transcendental among each other over the field K and let $(\mathbb{K}(x), \sigma)$ be the rational difference field over $\mathbb{K}$. Consider a (nested) P-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $\mathbb{E} = \mathbb{K}(x)[t_1, \frac{1}{t_1}] \dots [t_e, \frac{1}{t_e}]$ and $\sigma(t_i) = q_i\, t_i$ for $1 \leqslant i \leqslant e$. Now consider the field of fractions $\mathbb{F} = Q(\mathbb{E}) = \mathbb{K}(x)(t_1) \dots (t_e)$. We also use the shortcut $\mathbf{t} = (t_1, \dots, t_e)$ and write $\mathbb{F} = \mathbb{K}(x)(\mathbf{t}) = \mathbb{K}(x, \mathbf{t})$. Then $(\mathbb{K}(x, \mathbf{t}), \sigma)$ is a (nested) P-field extension of the difference field $(\mathbb{K}(x), \sigma)$. It is also called the *mixed* $\mathbf{q}$*-multibasic difference field* over $\mathbb{K}$. If $\mathbb{F} = \mathbb{K}(t_1) \dots (t_e) = \mathbb{K}(\mathbf{t})$ which is free of x, then $(\mathbb{K}(\mathbf{t}), \sigma)$ is called the $\mathbf{q}$*-multibasic difference field* over $\mathbb{K}$. Finally, if $e = 1$, then $\mathbb{F} = \mathbb{K}(t_1)$ and $(\mathbb{K}(t_1), \sigma)$ is called a q- or a *basic difference field* over $\mathbb{K}$; compare Bauer and Petkovšek (1999). $\star$

2.3.1.3 A-extensions

Let $(\mathbb{A}, \sigma)$ be a difference ring (or field) with constant field $\mathbb{K}$ and let $\zeta \in \mathbb{K}^*$ be a primitive λ-th root of unity. That is, $\zeta^\lambda = 1$ and $\lambda > 1$ and minimal. Our goal is to construct a difference ring extension in which the object ζ^k can be represented. In particular, we will rephrase the following properties, namely $(\zeta^n)^\lambda = 1$ and $\zeta^{n+1} = \zeta\,(\zeta)^n$ algebraically in this difference ring extension.

Definition 2.3.11.
Let $(\mathbb{A}, \sigma)$ be a difference ring. We say $\mathcal{I}$ is a difference ideal of $(\mathbb{A}, \sigma)$ if and only if

(1) $\mathcal{I} \subseteq \mathbb{A}$ and

(2) $f \in \mathcal{I} \Longrightarrow \sigma(f) \in \mathcal{I}$.

If $\sigma(f) \in \mathcal{I} \Longrightarrow f \in \mathcal{I}$ then we call $\mathcal{I}$ a reflexive difference ideal (Cohn, 1965, page 70-71). $*$

Proposition 2.3.12.
Let $\mathbb{A}[y]$ be a polynomial ring with y being transcendental over a ring $\mathbb{A}$ and let $(\mathbb{A}[y], \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$ with $\sigma(y) = \zeta\, y$ where $\zeta \in \mathbb{K}^*$ is a primitive λ-th root of unity, i.e., $\zeta^\lambda = 1$. Then the ideal $\mathcal{I} := \langle y^\lambda - 1 \rangle$ is a reflexive difference ideal. $\square$

PROOF:
Clearly, $\mathcal{J} := \langle y^\lambda - 1\rangle = \{h\,(y^\lambda - 1)\,|\,h \in \mathbb{A}[y]\} \subseteq \mathbb{A}[y]$. If $f \in \mathcal{J}$, then $f = h\,(y^\lambda - 1)$ for some $h \in \mathbb{A}[y]$ and

$$\sigma(f) = \sigma(h\,(y^\lambda - 1)) = \sigma(h)\,\sigma(y^\lambda - 1) = \sigma(h)\,(\zeta^\lambda\, y^\lambda - 1) = \sigma(h)\,(y^\lambda - 1) \in \mathcal{J}.$$

This proves that $\mathcal{J}$ is a difference ideal. It remains to prove that $\mathcal{J}$ is closed under σ^{-1}. If $\sigma(f) = h\,(y^\lambda - 1) \in \mathcal{J}$, then

$$\sigma^{-1}(h\,(y^\lambda - 1)) = \sigma^{-1}(h)\,\sigma^{-1}(y^\lambda - 1) = \sigma^{-1}(h)\,\Big(\big(\tfrac{y}{\zeta}\big)^\lambda - 1\Big) = \sigma^{-1}(h)\,(y^\lambda - 1) \in \mathcal{J}$$

since, $\sigma^{-1}(h) \in \mathbb{A}$. Thus, $f \in \mathcal{J}$ which completes the prove. ■

Proposition 2.3.13.
Let $\mathbb{A}[y]$ be a polynomial ring with y being transcendental over a ring $\mathbb{A}$ and let $(\mathbb{A}[y], \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$ with $\sigma(y) = \zeta\, y$ where $\zeta \in \mathbb{K}^*$ is a primitive λ-th root of unity, i.e., $\zeta^\lambda = 1$ with $\lambda > 1$. Let $\mathcal{J} := \langle y^\lambda - 1\rangle$ be an ideal of $\mathbb{A}[y]$ and $\mathbb{E} := {}^{\mathbb{A}[y]}\!/_{\mathcal{J}}$ be a quotient ring. Then the canonical ring homomorphism $\sigma_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}$ defined by $\sigma_{\mathbb{E}}(f + \mathcal{J}) = \sigma(f) + \mathcal{J}$ is a ring automorphism. Furthermore, $(\mathbb{E}, \sigma)$ is a difference ring extension of $(\mathbb{A}, \sigma)$. □

PROOF:
We first prove that $\sigma_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}$ is a ring homomorphism. If $(f + \mathcal{J}), (g + \mathcal{J}) \in \mathbb{E}$, then

$$\sigma_{\mathbb{E}}(f+g)+\mathcal{J} = \sigma_{\mathbb{E}}((f+g)+\mathcal{J}) = \sigma_{\mathbb{E}}\big((f+\mathcal{J})+(g+\mathcal{J})\big) = (\sigma_{\mathbb{E}}(f)+\mathcal{J})+(\sigma_{\mathbb{E}}(g)+\mathcal{J}) = \big(\sigma_{\mathbb{E}}(f)+\sigma(g)\big)+\mathcal{J}.$$

Also,

$$\sigma_{\mathbb{E}}(f\,g) + \mathcal{J} = \sigma_{\mathbb{E}}((f\,g) + \mathcal{J}) = \sigma_{\mathbb{E}}\big((f + \mathcal{J})\,(g + \mathcal{J})\big) = (\sigma_{\mathbb{E}}(f) + \mathcal{J})\,(\sigma_{\mathbb{E}}(g) + \mathcal{J}) = \big(\sigma_{\mathbb{E}}(f)\,\sigma_{\mathbb{E}}(g)\big) + \mathcal{J}.$$

Suppose that $\sigma_{\mathbb{E}}(f) + \mathcal{J} = \sigma_{\mathbb{E}}(g) + \mathcal{J}$. Then $\sigma_{\mathbb{E}}(f) - \sigma_{\mathbb{E}}(g) \in \mathcal{J}$. Hence $f - g \in \mathcal{J}$ and consequently $f + \mathcal{J} = g + \mathcal{J}$. Therefore $\sigma_{\mathbb{E}}$ is injective. Also, $\sigma_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}$ is surjective by definition. Therefore, $\sigma_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}$ is a ring automorphism. Clearly, $\mathbb{A} \leqslant \mathbb{E}$ and $\sigma_{\mathbb{E}}|_{\mathbb{A}} = \sigma_{\mathbb{A}}$ by identifying $a \in \mathbb{A}$ with $a + \mathcal{J} \in \mathbb{E}$ under the natural embedding $\mathbb{A} \hookrightarrow \mathbb{E}$. Hence $(\mathbb{E}, \sigma_{\mathbb{E}})$ is a difference ring extension of $(\mathbb{A}, \sigma)$. ■

Note: As earlier we do not distinguish any more between $\sigma_{\mathbb{E}}$ and σ since they agree on $\mathbb{A}$.

Definition 2.3.14.
Let $(\mathbb{A}, \sigma)$ and $(\mathbb{A}', \sigma')$ be difference rings. The map $\tau : \mathbb{A} \to \mathbb{A}'$ is called a *difference ring-homomorphism* if τ is a ring homomorphism and

$$\forall\, f \in \mathbb{A},\ \tau(\sigma(f)) = \sigma'(\tau(f)).$$

If τ is injective, then it is called a *difference ring monomorphism* or a *difference ring embedding*. In this case $(\tau(\mathbb{A}), \sigma)$ is a sub-difference ring of $(\mathbb{A}', \sigma')$ where $(\mathbb{A}, \sigma)$ and $(\tau(\mathbb{A}), \sigma)$ are the same up to renaming with respect to τ. If τ is a bijection, then it is a *difference ring isomorphism* and we say that $(\mathbb{A}, \sigma)$ and $(\mathbb{A}', \sigma')$ are isomorphic; we write $(\mathbb{A}, \sigma) \simeq (\mathbb{A}', \sigma)$. Let $(\mathbb{E}, \sigma)$ and $(\tilde{\mathbb{E}}, \tilde{\sigma})$ be difference ring extensions of $(\mathbb{A}, \sigma)$. Then a difference ring-homomorphism/isomorphism/monomorphism $\tau : \mathbb{E} \to \tilde{\mathbb{E}}$ is called an $\mathbb{A}$-*homomorphism*/$\mathbb{A}$-*isomorphism*/$\mathbb{A}$-*monomorphism*, if $\tau|_{\mathbb{A}} = \mathrm{id}$. *

Lemma 2.3.15.
Let $(\mathbb{A}, \sigma)$ *be a difference ring and let* $\zeta \in \mathbb{K}^*$ *be a primitive* λ*-th root of unity, i.e.,* $\zeta^\lambda = 1$ *with* $\lambda > 1$*. Then there is (up to an* $\mathbb{A}$*-isomorphism) a unique difference ring extension* $(\mathbb{A}[\vartheta], \sigma)$ *of* $(\mathbb{A}, \sigma)$ *with* $\vartheta \notin \mathbb{A}$ *and subject to the relations* $\vartheta^\lambda = 1$ *and* $\sigma(\vartheta) = \zeta\,\vartheta$*.* ♦

Proof:
Consider the difference ring extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ in Proposition 2.3.13 and define $\vartheta := y + \mathcal{I}$. Then

$$\sigma(\vartheta) = \sigma(y + \mathcal{I}) = \sigma(y) + \mathcal{I} = \zeta\, y + \mathcal{I} = \zeta(y + \mathcal{I}) = \zeta\,\vartheta$$

and

$$\vartheta^\lambda = (y + \mathcal{I})^\lambda = y^\lambda + \mathcal{I} = 1 + \mathcal{I} = 1.$$

With $\mathbb{E} = \{\sum_{i=0}^{\lambda-1} a_i\,\vartheta^i \,|\, a_i \in \mathbb{A}\}$, we have a difference ring extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$. Next we prove that the difference ring extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ is unique up to a difference ring isomorphism. Suppose $(\mathbb{A}[\vartheta'], \sigma')$ is a difference ring extension of $(\mathbb{A}, \sigma)$ with $\vartheta' \notin \mathbb{A}$ and subject to the relations ${\vartheta'}^\lambda = 1$ and $\sigma'(\vartheta') = \zeta\,\vartheta'$. Define the map $\tau : \mathbb{E} \to \mathbb{A}[\vartheta']$ with

$$\tau\left(\sum_{i=0}^{\lambda-1} f_i\,\vartheta^i\right) = \sum_{i=0}^{\lambda-1} f_i\,{\vartheta'}^i.$$

We show that τ is an $\mathbb{A}$-isomomorphism. τ is obviously surjective ring homomorphism. Suppose that

$$\tau\left(\sum_{i=0}^{\lambda-1} f_i\,\vartheta^i\right) = \tau\left(\sum_{i=0}^{\lambda-1} f'_i\,{\vartheta'}^i\right)$$

Then

$$\tau\left(\sum_{i=0}^{\lambda-1} (f_i - f'_i)\,\vartheta^i\right) = \sum_{i=0}^{\lambda-1} (f_i - f'_i)\,{\vartheta'}^i = 0.$$

Since $\mathbb{A}[\vartheta']$ is an $\mathbb{A}$-module with basis 1, ϑ', ${\vartheta'}^2, \ldots, {\vartheta'}^{\lambda-1}$, $f_i = f'_i$ for all i. Thus τ is a ring isomorphism and by definition $\tau|_{\mathbb{A}} = \mathrm{id}$. It remains to show that $\tau\,\sigma = \sigma'\,\tau$.

$$\tau(\sigma(\vartheta)) = \tau(\zeta\,\vartheta) = \tau(\zeta)\,\tau(\vartheta) = \zeta\,\vartheta' = \sigma'\,(\vartheta') = \sigma'\,(\tau(\vartheta)).$$

Thus it follows that $\tau(\sigma(f)) = \sigma'\,(\tau(f))$ for all $f \in \mathbb{A}[\vartheta]$ and τ is an $\mathbb{A}$-isomorphism; which completes the proof. ■

For further consideration, we introduce the order function.

Definition 2.3.16.
Let $\mathbb{A}$ be a ring and let $\alpha \in \mathbb{A}$. Then the order of α is defined by

$$\mathrm{ord}(\alpha) = \begin{cases} 0, & \text{if } \nexists\, n > 0 \text{ with } \alpha^n = 1. \\ \min\{n > 0 \,|\, \alpha^n = 1\}, & \text{otherwise.} \end{cases}$$

*

Definition 2.3.17.
Let $(\mathbb{A},\sigma)$ be a difference ring and let $\zeta \in \mathbb{K}^*$ be a primitive λ-th root of unity. The difference ring extension $(\mathbb{A}[\vartheta],\sigma)$ of $(\mathbb{A},\sigma)$ with $\vartheta \notin \mathbb{A}$ and subject to the relations $\vartheta^\lambda = 1$ and $\sigma(\vartheta) = \zeta\vartheta$ is called an algebraic extension (in short A-extension) of order λ. The generator ϑ is called an A-monomial and we define $\operatorname{ord}(\vartheta) := \operatorname{ord}(\zeta) = \lambda$ as its order. $*$

Definition 2.3.18.
Let $(\mathbb{A},\sigma)$ be a difference ring. We call $(\mathbb{A}[\vartheta_1]\dots[\vartheta_e],\sigma)$ a *(nested)* A-*extension* of $(\mathbb{A},\sigma)$ if it is a tower of A-extensions. That is, $(\mathbb{A}[\vartheta_1]\dots[\vartheta_i],\sigma)$ is an A-extension of $(\mathbb{A}[\vartheta_1]\dots[\vartheta_{i-1}],\sigma)$ for $1 \leqslant i \leqslant e$. $*$

2.3.1.4 APS-extensions

We introduce the following notations for convenience. Let $(\mathbb{E},\sigma)$ be a difference ring extension of $(\mathbb{A},\sigma)$ with $t \in \mathbb{E}$. $\mathbb{A}\langle t\rangle$ denotes the ring of Laurent polynomials $\mathbb{A}[t,\frac{1}{t}]$ (i.e., t is transcendental over $\mathbb{A}$) if $(\mathbb{A}[t,\frac{1}{t}],\sigma)$ is a P-extension of $(\mathbb{A},\sigma)$. Lastly, $\mathbb{A}\langle t\rangle$ denotes the ring $\mathbb{A}[t]$ with $t \notin \mathbb{A}$ but subject to the relation $t^\lambda = 1$ if $(\mathbb{A}[t],\sigma)$ is an A-extension of $(\mathbb{A},\sigma)$ of order λ. We say that the difference ring extension $(\mathbb{A}\langle t\rangle,\sigma)$ of $(\mathbb{A},\sigma)$ is an AP-extension (and t is an AP-monomial) if it is an A- or a P-extension. Further, we call $(\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle,\sigma)$ a (nested) AP-extension/A-extension/P-extension of $(\mathbb{A},\sigma)$ if it is built by a tower of such extensions. Finally, following Schneider (2008), we introduce the depth function $\mathfrak{d}$ for (nested) AP-extensions.

Definition 2.3.19.
Let $\mathbb{A}\langle t\rangle$ be a ring of (Laurent) polynomials. For $f = \sum_i f_i\, t^i$ we define

$$\deg(f) = \begin{cases} \max\{i \,|\, f_i \neq 0\} & \text{if } f \neq 0 \\ -\infty & \text{if } f = 0 \end{cases} \quad \text{and} \quad \operatorname{ldeg}(f) = \begin{cases} \min\{i \,|\, f_i \neq 0\} & \text{if } f \neq 0 \\ \infty & \text{if } f = 0. \end{cases} \qquad *$$

Definition 2.3.20.
Let $(\mathbb{E},\sigma)$ be a (nested) AP-extension of $(\mathbb{A},\sigma)$ with $\mathbb{E} = \mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle$ where $\sigma(t_i) = \alpha_i\, t_i$ for $1 \leqslant i \leqslant e$. We define the *depth* of elements of $\mathbb{E}$ over $\mathbb{A}$, $\mathfrak{d}_{\mathbb{A}} : \mathbb{E} \to \mathbb{N}$ as follows:

(1) For any $h \in \mathbb{A}$, $\mathfrak{d}_{\mathbb{A}}(h) = 0$.

(2) If $\mathfrak{d}_{\mathbb{A}}$ is defined for $(\mathbb{A}\langle t_1\rangle\dots\langle t_{i-1}\rangle,\sigma)$ with $i > 1$, then we define $\mathfrak{d}_{\mathbb{A}}(t_i) := \mathfrak{d}_{\mathbb{A}}(\alpha_i) + 1$ and for $f \in \mathbb{A}\langle t_1\rangle\dots\langle t_i\rangle$, we define

$$\mathfrak{d}_{\mathbb{A}}(f) := \max\big(\{\mathfrak{d}_{\mathbb{A}}(t_i) \,|\, t_i \text{ occurs in } f\} \cup \{0\}\big). \tag{2.25}$$

For $\mathbf{f} = (f_1,\dots,f_r) \in \mathbb{E}^r$, we have that

$$\mathfrak{d}_{\mathbb{A}}(\mathbf{f}) = \max_{1\leqslant i\leqslant r}(\mathfrak{d}_{\mathbb{A}}(f_i)).$$

The depth of $(\mathbb{E},\sigma)$ is given by

$$\mathfrak{d}_{\mathbb{A}}(\mathbb{E}) := \big(0, \mathfrak{d}_{\mathbb{A}}(t_1),\dots,\mathfrak{d}_{\mathbb{A}}(t_e)\big).$$

Similarly, we define the *extension depth* of an AP-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ with $\mathbb{G} = \mathbb{E}\langle x_1\rangle \dots \langle x_s\rangle$ as

$$\mathfrak{d}_{\mathbb{A}}(\mathbb{G}) := \big(0, \mathfrak{d}_{\mathbb{A}}(x_1), \dots, \mathfrak{d}_{\mathbb{A}}(x_s)\big).$$

We call such an extension an *ordered* AP-*extension*, if $\mathfrak{d}_{\mathbb{A}}(x_1) \leqslant \mathfrak{d}_{\mathbb{A}}(x_2) \leqslant \cdots \leqslant \mathfrak{d}_{\mathbb{A}}(x_s)$. We say $\mathbb{G}$ is of *monomial depth* m if $m = \max(\mathfrak{d}_{\mathbb{A}}(\mathbb{G}))$. If the ground field is clear from the context, we write $\mathfrak{d}_{\mathbb{A}}$ as $\mathfrak{d}$.∗

Remark 2.3.21.
Completely analogously one can define the depth function for a PS-field extension $(\mathbb{A}(t_1)\dots(t_e), \sigma)$ of a difference field $(\mathbb{A}, \sigma)$ by replacing (2.25) with

$$\mathfrak{d}_{\mathbb{A}}(f) := \max\big(\{\mathfrak{d}_{\mathbb{A}}(t_i) \,|\, t_i \text{ occurs in } h \text{ or } g\} \cup \{0\}\big).$$

where $f = \frac{h}{g} \in \mathbb{A}\langle t_1\rangle \dots \langle t_i\rangle$ with h and g being coprime. ♠

Example 2.3.22.
The depth of the rational difference field $(\mathbb{K}(x), \sigma)$ in Example 2.3.6 is

$$\mathfrak{d}(\mathbb{K}(x)) = (0, \mathfrak{d}(x)) = (0, 1)$$

and the depth of the mixed $\mathbf{q}$-multibasic difference field $(\mathbb{K}(x, \mathbf{t}), \sigma)$ in Example 2.3.10 is

$$\mathfrak{d}(\mathbb{K}(x, \mathbf{t})) = (0, \mathfrak{d}(x), \mathfrak{d}(t_1), \dots, \mathfrak{d}(t_e)) = (0, 1, 1, \dots, 1).$$
⋆

We will follow the convention introduced in Paule and Schneider (2019) to illustrate how the products covered in this thesis are modelled by expressions in a difference ring.

Definition 2.3.23.
Let $(\mathbb{A}, \sigma)$ be a difference ring with a constant field $\mathbb{K} = \operatorname{const}(\mathbb{A}, \sigma)$. An evaluation function $\operatorname{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ is a function which satisfies the following three properties:

(i) for all $c \in \mathbb{K}$, there is a natural number $\delta \geqslant 0$ such that

$$\forall n \geqslant \delta \,:\, \operatorname{ev}(c, n) = c; \tag{2.26}$$

(ii) for all $f, g \in \mathbb{A}$ there is a natural number $\delta \geqslant 0$ such that

$$\begin{aligned} \forall n \geqslant \delta \,&:\, \operatorname{ev}(f\,g, n) = \operatorname{ev}(f, n)\ \operatorname{ev}(g, n), \\ \forall n \geqslant \delta \,&:\, \operatorname{ev}(f + g, n) = \operatorname{ev}(f, n) + \operatorname{ev}(g, n); \end{aligned} \tag{2.27}$$

(iii) for all $f \in \mathbb{A}$ and $i \in \mathbb{Z}$, there is a natural number $\delta \geqslant 0$ such that

$$\forall n \geqslant \delta \,:\, \operatorname{ev}(\sigma^i(f), n) = \operatorname{ev}(f, n + i). \tag{2.28}$$

We say a sequence $\langle F(n)\rangle_{n\geqslant 0} \in \mathcal{S}(\mathbb{K})$ is *modelled* by $f \in \mathbb{A}$ in the difference ring $(\mathbb{A}, \sigma)$, if there is an evaluation function ev such that

$$F(k) = \operatorname{ev}(f, k)$$

holds for all $k \in \mathbb{N}$ from a certain point on. ∗

Example 2.3.24 (Cont. Examples 2.3.6 and 2.3.10).
The map $\mathrm{ev} : \mathbb{K}(x) \times \mathbb{N} \to \mathbb{K}$ with (2.1) is an evaluation function of the rational difference field $(\mathbb{K}(x), \sigma)$; see Example 2.3.6. Similarly, let $\mathbb{F} = \mathbb{K}(x, t_1, \dots, t_e)$ be a rational function field over $\mathbb{K}$. Then the map $\mathrm{ev} : \mathbb{F} \times \mathbb{N} \to \mathbb{K}$ with (2.2) is an evaluation function of the mixed $\mathbf{q}$-multibasic difference field $(\mathbb{F}, \sigma)$; see Example 2.3.10. ⋆

Example 2.3.25.
Obeserve that in the A-extension $(\mathbb{A}[\vartheta], \sigma)$ of $(\mathbb{A}, \sigma)$, the A-monomial ϑ, with the relations

$$\vartheta^{\lambda} = 1 \quad \text{and} \quad \sigma(\vartheta) = \zeta\,\vartheta$$

and evaluation function $\mathrm{ev} : \mathbb{A}[\vartheta] \times \mathbb{N} \to \mathbb{K}$ with $\mathrm{ev}(\vartheta, n) = \zeta^n$, models the sequence generated by ζ^n with the relations

$$(\zeta^n)^{\lambda} = 1 \quad \text{and} \quad \zeta^{n+1} = \zeta\,\zeta^n.$$

In addition, the ring $\mathbb{A}[\vartheta]$ is not an integral domain (i.e., it has zero-divisors) since

$$(\vartheta - 1)\,(\vartheta^{\lambda-1} + \cdots + \vartheta + 1) = 0$$

but

$$(\vartheta - 1) \neq 0 \neq (\vartheta^{\lambda-1} + \cdots + \vartheta + 1).$$

⋆

Lemma 2.3.26.
Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$ and let $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ be an evaluation function. Let $(\mathbb{A}\langle t\rangle, \sigma)$ be an AP-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\,t$ where $\alpha \in \mathbb{A}^*$. Let δ be large enough such that $\mathrm{ev}(\alpha, k) \neq 0$ for all $k \geqslant \delta$. Let $u \in \mathbb{K}^*$ where $u^{\lambda} = 1$, if $\lambda = \mathrm{ord}(t) > 0$. Consider the map $\mathrm{ev}' : \mathbb{A}\langle t\rangle \times \mathbb{N} \to \mathbb{K}$ defined by

$$\mathrm{ev}'\Big(\sum_i h_i\, t^i, n\Big) = \sum_i \mathrm{ev}(h_i, n)\ \mathrm{ev}'(t, n)^i$$

with

$$\mathrm{ev}'(t, n) = u \prod_{k=\delta}^{n} \mathrm{ev}(\alpha, k - 1).$$

Then ev' is an evaluation function for the difference ring $(\mathbb{A}\langle t\rangle, \sigma)$. ♦

Proof:
By Schneider (2017, Lemma 5.4) the statement follows. ■

Throughout this thesis, the classes of extensions introduced in Example 2.3.6 and Example 2.3.10 will be our ground field. Based on these ground fields we will construct our (nested) AP-ring (respectively field) extensions to model product expressions whose multiplicands are from these ground fields. We will begin in Chapter 5 where we restrict to nesting depth one product expressions whose multiplicands are from these ground fields.

Example 2.3.27.
Take the rational difference field $(\mathbb{K}(x), \sigma)$ with $\mathbb{K} = \mathbb{Q}\big(\mathbb{i},\ (-1)^{\frac{1}{6}}\big)$ and consider the A-extension $(\mathbb{K}(x)[\vartheta_1], \sigma)$ of $(\mathbb{K}(x), \sigma)$ with

$$\sigma(\vartheta_1) = (-1)^{\frac{1}{6}}\,\vartheta_1 \tag{2.29}$$

of order 12. By Lemma 2.3.26 we extend the evaluation function (2.1) to $\text{ev} : \mathbb{K}(x)[\vartheta_1] \times \mathbb{N} \to \mathbb{K}$ for the difference ring $(\mathbb{K}(x)[\vartheta_1], \sigma)$ with

$$\text{ev}(\vartheta_1, n) = \prod_{k=1}^{n} (-1)^{\frac{1}{6}} = \left((-1)^{\frac{1}{6}}\right)^n. \tag{2.30}$$

Then the sequence generated by the product expression $\left((-1)^{\frac{1}{6}}\right)^n$ is modelled by the A-monomial ϑ_1. Observe that the shift behaviour (2.29) corresponds to the shift behaviour (2.11) in the sequence setting. Further, $(\mathbb{K}(x)[\vartheta_1][\vartheta_2], \sigma)$ with

$$\sigma(\vartheta_2) = \mathbb{i}\,\vartheta_2 \tag{2.31}$$

is also an A-extension of $(\mathbb{K}(x)[\vartheta_1], \sigma)$ of order 4. Utilising Lemma 2.3.26 we extend the definition of the evaluation function to $\text{ev} : \mathbb{K}(x)[\vartheta_1][\vartheta_2] \times \mathbb{N} \to \mathbb{K}$ for the difference ring $(\mathbb{K}(x)[\vartheta_1][\vartheta_2], \sigma)$ with

$$\text{ev}(\vartheta_2, n) = \prod_{k=1}^{n} \mathbb{i} = (\mathbb{i})^n. \tag{2.32}$$

Then the A-monomial ϑ_2 models the sequence generated by the product expression $(\mathbb{i})^n$. Again the shift behaviour (2.31) corresponds to (2.12). Observe that $\mathfrak{d}(x) = \mathfrak{d}(\vartheta_1) = \mathfrak{d}(\vartheta_2) = 1$ and $\mathfrak{d}(\mathbb{K}(x)[\vartheta_1][\vartheta_2]) = (0, 1, 1, 1)$ is the depth of $(\mathbb{K}(x)[\vartheta_1][\vartheta_2], \sigma)$. $\star$

Example 2.3.28.
Let $\mathbb{K} = \mathbb{Q}\big((-1)^{\frac{2}{3}}\big)$ and let $(\mathbb{K}(x), \sigma)$ be the rational difference field. Take the A-extension $(\mathbb{K}(x)[\vartheta_1], \sigma)$ of $(\mathbb{K}(x), \sigma)$ with

$$\sigma(\vartheta_1) = -\,\vartheta_1 \tag{2.33}$$

of order 2. Let $\text{ev} : \mathbb{K}(x)[\vartheta_1] \times \mathbb{N} \to \mathbb{K}$ be an extension of the evaluation function (2.1) with

$$\text{ev}(\vartheta_1, n) = \prod_{k=1}^{n} (-1) = (-1)^n. \tag{2.34}$$

Then the A-monomial ϑ_1 models the sequence generated by the product expression $(-1)^n$. The shift behaviour (2.33) of ϑ_1 corresponds to the shift behaviour (2.14). Further, $(\mathbb{K}(x)[\vartheta_1][\vartheta_2], \sigma)$ is an A-extension of $(\mathbb{K}(x)[\vartheta_1], \sigma)$ with

$$\sigma(\vartheta_2) = (-1)^{\frac{2}{3}}\,\vartheta_2 \tag{2.35}$$

of order 3. Extending the evaluation function to $\text{ev} : \mathbb{K}(x)[\vartheta_1][\vartheta_2] \times \mathbb{N} \to \mathbb{K}$ with

$$\text{ev}(\vartheta_2, n) = \prod_{k=1}^{n} (-1)^{\frac{2}{3}} = \left((-1)^{\frac{2}{3}}\right)^n, \tag{2.36}$$

the A-monomial ϑ_2 models the sequence generated by $\left((-1)^{\frac{2}{3}}\right)^n$. The shift behaviour (2.35) corresponds to the shift behaviour (2.15) in the sequence setting. The depth of $(\mathbb{K}(x)[\vartheta_1][\vartheta_2], \sigma)$ is $\mathfrak{d}(\mathbb{K}(x)[\vartheta_1][\vartheta_2]) = (0, 1, 1, 1)$. $\star$

Example 2.3.29.
Let $(\mathbb{K}(x), \sigma)$ be the rational difference field with $\mathbb{K} = \mathbb{Q}(\sqrt{13})$

(1) Consider the P-extension $(\mathbb{A}_1, \sigma)$ of the rational difference field $(\mathbb{K}(x), \sigma)$ with $\mathbb{A}_1 = \mathbb{K}(x)[y_1, \frac{1}{y_1}]$ and automorphism

$$\sigma(y_1) = (\sqrt{13})\, y_1, \qquad \sigma\Big(\frac{1}{y_1}\Big) = \frac{1}{\sqrt{13}}\frac{1}{y_1}. \tag{2.37}$$

Extend the evaluation function (2.1) to $\mathrm{ev} : \mathbb{A}_1 \times \mathbb{N} \to \mathbb{K}$ with

$$\mathrm{ev}(y_1, n) = \prod_{k=1}^{n} \sqrt{13} = \big(\sqrt{13}\big)^n, \qquad \mathrm{ev}\left(\frac{1}{y_1}, n\right) = \prod_{k=1}^{n} \frac{1}{\sqrt{13}} = \big(\sqrt{13}\big)^{-n}. \tag{2.38}$$

Then the P-monomial y_1 and its inverse ${y_1}^{-1}$ model the sequences generated by $\big(\sqrt{13}\big)^n$ and $\big(\sqrt{13}\big)^{-n}$ respectively. The shift behaviour (2.37) corresponds to the shift behaviour (2.17) in the sequence setting.

(2) Constructing the P-extension $(\mathbb{A}_2, \sigma)$ of $(\mathbb{A}_1, \sigma)$ with $\mathbb{A}_2 = \mathbb{A}_1[y_2, \frac{1}{y_2}]$,

$$\sigma(y_2) = 7\, y_2, \qquad \sigma\Big(\frac{1}{y_2}\Big) = \frac{1}{7}\frac{1}{y_2} \tag{2.39}$$

and extending the evaluation function in the previous construction to $\mathrm{ev} : \mathbb{A}_2 \times \mathbb{N} \to \mathbb{K}$ with

$$\mathrm{ev}(y_2, n) = \prod_{k=1}^{n} 7 = 7^n, \qquad \mathrm{ev}\left(\frac{1}{y_2}, n\right) = \prod_{k=1}^{n} \frac{1}{7} = 7^{-n}, \tag{2.40}$$

we are able to model the sequences generated by 7^n and 7^{-n} with the P-monomial y_2 and its inverse ${y_2}^{-1}$. Here, the automorphism (2.39) corresponds to (2.18) in the sequence setting.

(3) Introducing the P-extension $(\mathbb{A}_3, \sigma)$ of $(\mathbb{A}_2, \sigma)$ with $\mathbb{A}_3 = \mathbb{A}_2[y_3, \frac{1}{y_3}]$,

$$\sigma(y_3) = 169\, y_3, \qquad \sigma\Big(\frac{1}{y_3}\Big) = \frac{1}{169}\frac{1}{y_3} \tag{2.41}$$

and the evaluation function $\mathrm{ev} : \mathbb{A}_3 \times \mathbb{N} \to \mathbb{K}$ with

$$\mathrm{ev}(y_3, n) = \prod_{k=1}^{n} 169 = 169^n, \qquad \mathrm{ev}\left(\frac{1}{y_3}, n\right) = \prod_{k=1}^{n} \frac{1}{169} = 169^{-n}, \tag{2.42}$$

we can model the sequences generated by 169^n and 169^{-n} with the P-monomial y_3 and its inverse ${y_3}^{-1}$. The automorphism (2.41) corresponds to (2.19) in the sequence setting.

The depth of $(\mathbb{A}_3, \sigma)$ is $\mathfrak{d}(\mathbb{A}_3) = \big(0, \mathfrak{d}(x), \mathfrak{d}(y_1), \mathfrak{d}(y_2), \mathfrak{d}(y_3)\big) = (0, 1, 1, 1, 1)$. ⋆

Example 2.3.30.
Let $(\mathbb{Q}(x)\langle z_1\rangle, \sigma)$ be a P-extension of $(\mathbb{Q}(x), \sigma)$ with

$$\sigma(z_1) = (x + 1)\, z_1, \qquad \sigma\Big(\frac{1}{z_1}\Big) = \frac{1}{(x+1)}\frac{1}{z_1}. \tag{2.43}$$

Extend the evaluation function (2.1) to $\text{ev} : \mathbb{Q}(x)\langle \varkappa_1 \rangle \times \mathbb{N} \to \mathbb{Q}$ with

$$\text{ev}(\varkappa_1, n) = \prod_{k=1}^{n} k = n!, \qquad \text{ev}\left(\frac{1}{\varkappa_1}, n\right) = \prod_{k=1}^{n} \frac{1}{k} = \frac{1}{n!}. \tag{2.44}$$

Then the P-monomial $\varkappa_1$ and its inverse ${\varkappa_1}^{-1}$ model the sequences generated by $n!$ and $\frac{1}{n!}$ respectively. In particular, the automorphism (2.43) corresponds to the automorphism (2.21) in the sequence setting.

Finally, taking the P-extension $(\mathbb{Q}(x)\langle \varkappa_1 \rangle\langle \varkappa_2 \rangle, \sigma)$ of $(\mathbb{Q}(x)\langle \varkappa_1 \rangle, \sigma)$ with

$$\sigma(\varkappa_2) = (x+3)\, \varkappa_2, \qquad \sigma\Big(\frac{1}{\varkappa_2}\Big) = \frac{1}{(x+3)}\,\frac{1}{z_3} \tag{2.45}$$

and extending the evaluation function in the previous construction to $\text{ev} : \mathbb{Q}(x)\langle \varkappa_1 \rangle\langle \varkappa_2 \rangle \times \mathbb{N} \to \mathbb{Q}$ with

$$\text{ev}(\varkappa_2, n) = \prod_{k=1}^{n} (k+2) = \frac{(n+2)!}{2}, \qquad \text{ev}\left(\frac{1}{\varkappa_2}, n\right) = \prod_{k=1}^{n} \frac{1}{(k+2)} = \frac{2}{(n+2)!} \tag{2.46}$$

we can model sequences generated by $(n+2)!$ and $\frac{1}{(n+2)!}$ with the P-monomial $\varkappa_2$ and its inverse ${\varkappa_2}^{-1}$. The depth of $(\mathbb{Q}(x)\,\langle \varkappa_1 \rangle\,\langle \varkappa_2 \rangle, \sigma)$ is $(0, \mathfrak{d}(x), \mathfrak{d}(\varkappa_1), \mathfrak{d}(\varkappa_2)) = (0, 1, 2, 2)$. ⋆

In order to solve Problem RPE, we rely on a refined construction of AP-extensions. More precisely, we are interested in those AP-extensions that do not change the set of constants. These subclasses of AP-extension are the so-called RΠ-extensions introduced and explored in Schneider (2014, 2016, 2017).

2.3.2 RΠΣ-extensions

In this subsection, we discuss the so-called RΠΣ-extensions. These are a subclass of APS-extensions discussed in the previous subsection 2.3.1 with the extra property that their construction do not extend the ring of constants. The main results including the proofs are taken from Schneider (2016).

2.3.2.1 R-extensions

R-extensions are a subclass of A-extensions with the property that the constant ring remains unchanged in their construction.

Definition 2.3.31.
Let $\zeta \in \mathbb{K}^*$ be a primitive λ-th root of unity and let difference ring $(\mathbb{A}[\vartheta], \sigma)$ be an A-extension of order λ. Then $(\mathbb{A}[\vartheta], \sigma)$ is called an R-extension of $(\mathbb{A}, \sigma)$ of order λ, if $\text{const}(\mathbb{A}[\vartheta], \sigma) = \text{const}(\mathbb{A}, \sigma) = \mathbb{K}$. We call the generator ϑ an R-monomial. ∗

Subsequently, we introduce nested R-extensions.

Definition 2.3.32.
Let $(\mathbb{A}, \sigma)$ be a difference ring. We call a nested A-extension $(\mathbb{A}[\vartheta_1] \dots [\vartheta_e], \sigma)$ of $(\mathbb{A}, \sigma)$ a *(nested)* R-*extension* of $(\mathbb{A}, \sigma)$ if it is a tower of R-extensions. That is, $(\mathbb{A}[\vartheta_1] \dots [\vartheta_i], \sigma)$ is an R-extension of $(\mathbb{A}[\vartheta_1] \dots [\vartheta_{i-1}], \sigma)$ for $1 \leqslant i \leqslant e$. ∗

The following result gives a characterization of R-extensions.

Theorem 2.3.33 (Schneider (2016), Theorem 2.12(3)).
Let $(\mathbb{A}[\vartheta],\sigma)$ *be an* A*-extension of* $(\mathbb{A},\sigma)$ *of order* $\lambda > 1$ *with* $\sigma(\vartheta) = \zeta\,\vartheta$ *where* $\zeta \in \mathbb{A}^*$*. Then* $(\mathbb{A}[\vartheta],\sigma)$ *is a* R*-extension of* $(\mathbb{A},\sigma)$ *(i.e.,* $\operatorname{const}(\mathbb{A}[\vartheta],\sigma) = \operatorname{const}(\mathbb{A},\sigma)$*) if and only if there is no* $g \in \mathbb{A}\setminus\{0\}$ *and* $m \in \{1,\dots,\lambda-1\}$ *with* $\sigma(g) = \zeta^m g$*. If* $(\mathbb{A}[\vartheta],\sigma)$ *is an* R*-extension of* $(\mathbb{A},\sigma)$*, then* ζ *is primitive, i.e.,* $\operatorname{ord}(\zeta) = \lambda$*.* ♢

Proof:
" $\Longleftarrow$ " Let $g \in \mathbb{A}\setminus\{0\}$ and $m \in \{1,\dots,\lambda-1\}$ such that $\sigma(g) = \zeta^m\,g$. From $\sigma(\vartheta) = \zeta\,\vartheta$ we have that $\sigma(\vartheta^m) = \zeta^m\,\vartheta^m$. Take $g\,\vartheta^{\lambda-m} \in \mathbb{A}[\vartheta]$ then,

$$\sigma(g\,\vartheta^{\lambda-m}) = \sigma(g)\,\sigma(\vartheta^{\lambda-m}) = \zeta^m\,g\,(\zeta\,\vartheta)^{\lambda-m} = \zeta^m\,g\,\zeta^{\lambda-m}\,\vartheta^{\lambda-m} = g\,\vartheta^{\lambda-m} \in \operatorname{const}(\mathbb{A}[\vartheta],\sigma).$$

Clearly $g\,\vartheta^{\lambda-m} \notin \mathbb{A}$ thus, $g\,\vartheta^{\lambda-m} \notin \operatorname{const}(\mathbb{A},\sigma)$.

" $\Longrightarrow$ " Let $g = \sum_{i=0}^{\lambda-1} g_i\,\vartheta^i \in \mathbb{A}[\vartheta]\setminus\mathbb{A}$ with $\sigma(g) = g$. Then for some $e \in \{1,\dots,\lambda-1\}$, $g_e \neq 0$. Comparing coefficients in $\sigma(g) = g$ we get $\sigma(g_e) = \zeta^{\lambda-e}\,g_e$ with $\lambda - e \in \{1,\dots,\lambda-1\}$.
Let ϑ be an R-monomial with $m = \operatorname{ord}(\zeta) < \lambda$. Then with $g = 1 \in \mathbb{A}\setminus\{0\}$ we have $\sigma(g) = 1 = \zeta^m 1 = \zeta^m\,g$ which contradicts the first statement by our choice of g. ■

We give further examples and non-examples of R-extensions.

Example 2.3.34 (Cont. Example 2.3.27).
In Example 2.3.27, the A-extension $(\mathbb{K}(x)[\vartheta_1],\sigma)$ is an R-extension of $(\mathbb{K}(x),\sigma)$ of order 12 since there are no $g \in \mathbb{K}(x)^*$ and $\nu \in \{1,\dots,11\}$ with $\sigma(g) = \left((-1)^{1/6}\right)^{\nu} g$. However, the A-extension $(\mathbb{K}(x)[\vartheta_1][\vartheta_2],\sigma)$ is not an R-extension of $(\mathbb{K}(x)[\vartheta_1],\sigma)$ since with $g = \vartheta_1^3 \in \mathbb{K}(x)[\vartheta_1]$ and $\nu = 1$ we have $\sigma(g) = \mathbb{i}\,g$. In particular, we have that

$$\forall\, c \in \left\{\vartheta_1^3\,\vartheta_2^3,\ \vartheta_1^6\,\vartheta_2^2,\ \vartheta_1^9\,\vartheta_2\right\},\ \ c \in \operatorname{const}(\mathbb{K}(x)[\vartheta_1][\vartheta_2],\sigma)\setminus\operatorname{const}(\mathbb{K}(x)[\vartheta_1],\sigma). \qquad \star$$

Example 2.3.35 (Cont. Example 2.3.28).
In Example 2.3.28, the A-extension $(\mathbb{K}(x)[\vartheta_1],\sigma)$ is an R-extension of $(\mathbb{K}(x),\sigma)$ of order 2 since there are no $g \in \mathbb{K}(x)^*$ and $\nu \in \{1\}$ with $\sigma(g) = (-1)^{\nu}\,g$. Further, the A-extension $(\mathbb{K}(x)[\vartheta_1][\vartheta_2],\sigma)$ is an R-extension of $(\mathbb{K}(x)[\vartheta_1],\sigma)$ of order 3 since there are no $g \in \mathbb{K}(x)[\vartheta_1]\setminus\{0\}$ and $\nu \in \{1,2\}$, with $\sigma(g) = \left((-1)^{2/3}\right)^{\nu} g$. ⋆

We will rely on the following notions introduced in Schneider (2016).

Definition 2.3.36.
A difference ring $(\mathbb{A},\sigma)$ with constant field $\mathbb{K}$ is called *constant-stable* if $\operatorname{const}(\mathbb{A},\sigma^k) = \mathbb{K}$ for all $k \geqslant 1$. It is called *strong constant-stable*, if it is constant-stable and any root of unity of $\mathbb{A}$ is in $\mathbb{K}$. ∗

We obtain a simple characterization under the assumption that the ground difference field is constant-stable.

Proposition 2.3.37.
Let $(\mathbb{F}, \sigma)$ be a constant-stable difference field and let $(\mathbb{F}[\vartheta], \sigma)$ be an A*-extension of $(\mathbb{F}, \sigma)$ of order λ with $\zeta = \frac{\sigma(\vartheta)}{\vartheta} \in \mathrm{const}(\mathbb{F}, \sigma)^*$. Then $(\mathbb{F}[\vartheta], \sigma)$ is an* R*-extension of $(\mathbb{F}, \sigma)$ if and only if ζ is a primitive λ-th root of unity.* □

Proof:
" $\Longrightarrow$ " Suppose ζ is not a primitive λ-th root of unity. Then for some $m \in \mathbb{N}$ with $1 \leqslant m \leqslant \lambda - 1$ we have $\zeta^m = 1$. Furthermore, $\sigma(\vartheta^m) = \sigma(\vartheta)^m = (\zeta^m \vartheta^m) = \vartheta^m \in \mathrm{const}(\mathbb{F}[\vartheta], \sigma)$. Since $\vartheta^\lambda = 1$ is the defining relation, $\vartheta^m \notin \mathbb{F}$. Thus $\mathrm{const}(\mathbb{F}[\vartheta], \sigma) \supsetneq \mathrm{const}(\mathbb{F}, \sigma)$ and ϑ is not an R-monomial.
" $\Longleftarrow$ " Suppose that there is a $g = \sum_{i=1}^{\lambda-1} g_i \vartheta^i \in \mathbb{F}[\vartheta] \setminus \mathbb{F}$ with $\sigma(g) = g$. Take a natural number n with $1 \leqslant n \leqslant \lambda - 1$ such that $g_n \in \mathbb{F}^*$. By comparing the n-th coefficient in $\sigma(g) = g$ we have $\sigma(g_n) = \zeta^{-n} g_n$. Since ζ is a constant, $\sigma^\lambda(g_n) = (\zeta^{-n})^\lambda g_n = g_n$. Hence $g_n \in \mathrm{const}(\mathbb{F}, \sigma)^*$ since $(\mathbb{F}, \sigma)$ is constant stable. On the other hand, $g_n = \sigma(g_n) = \zeta^{-n} g_n$ implies that $\zeta^n = 1$. Therefore, ζ is not a primitive root of unity of order λ. ■

2.3.2.2 Π-extensions

The extensions we discuss here are a subclass of P-extensions with the property that ring of constants is not extended in their construction.

Definition 2.3.38.
Let $(\mathbb{A}, \sigma)$ be a difference ring. We call a P-extension $(\mathbb{A}[t, t^{-1}], \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ and $\sigma(t^{-1}) = \alpha^{-1} t^{-1}$ for $\alpha \in \mathbb{A}^*$ a Π-*extension* of $(\mathbb{A}, \sigma)$, if $\mathrm{const}(\mathbb{A}[t, t^{-1}], \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$. The generator t, is called a Π-*monomial*. Furthermore, a nested P-extension $(\mathbb{A}\langle t_1 \rangle \dots \langle t_e \rangle, \sigma)$ is a *(nested)* Π-*extension* of $(\mathbb{A}, \sigma)$, if it is a tower of Π-extensions, i.e., $(\mathbb{A}\langle t_1 \rangle \dots \langle t_i \rangle, \sigma)$ is a Π-extension of $(\mathbb{A}\langle t_1 \rangle \dots \langle t_{i-1} \rangle, \sigma)$ for $1 \leqslant i \leqslant e$. Similarly, a P-field extension $(\mathbb{A}(t), \sigma)$ of a difference field $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ for $\alpha \in \mathbb{A}^*$ is called a Π-*field extension* of $(\mathbb{A}, \sigma)$, if $\mathrm{const}(\mathbb{A}(t), \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$. Furthermore, a nested P-field extension $(\mathbb{A}(t_1) \dots (t_e), \sigma)$ is a *(nested)* Π-*field extension* of $(\mathbb{A}, \sigma)$, if it is a tower of Π-field extensions, i.e., $(\mathbb{A}(t_1) \dots (t_i), \sigma)$ is a Π-field extension of $(\mathbb{A}(t_1) \dots (t_{i-1}), \sigma)$ for $1 \leqslant i \leqslant e$. *

Lemma 2.3.39.
Let $(\mathbb{A}\langle t \rangle, \sigma)$ be a P*-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ where $\alpha \in \mathbb{A}^*$. Let $u \in \mathbb{A}$ and $g = \sum_{i=0}^{n} g_i\, t^i \in \mathbb{A}\langle t \rangle$. If $\sigma(g) = u\, g$, then $\sigma(g_i) = u\, \alpha^{-i} g_i$ for all $0 \leqslant i \leqslant n$.* ♦

Proof:
Let $g = \sum_{i=0}^{n} g_i\, t^i \in \mathbb{A}\langle t \rangle$, $u \in \mathbb{A}$, $\sigma(t) = \alpha\, t$ and $\sigma(g) = u\, g$. We prove that $\sigma(g_i) = u\, \alpha^{-i} g_i$ for $0 \leqslant i \leqslant n$. From $\sigma(g) = u\, g$ we have

$$\sigma\Big(\sum_{i=0}^{n} g_i\, t^i\Big) = \sum_{i=0}^{n} \sigma(g_i)\, \sigma(t^i) = \sum_{i=0}^{n} \sigma(g_i)\, (\sigma(t))^i = \sum_{i=0}^{n} \sigma(g_i)\, \alpha^i\, t^i = \sum_{i=0}^{n} u\, g_i\, t^i.$$

By comparing the coefficients of the t^i we get

$$\sigma(g_i)\, \alpha^i = u\, g_i \iff \sigma(g_i) = u\, \alpha^{-i} g_i \quad \forall i : 0 \leqslant i \leqslant n.$$ ■

The theorem below gives a criterion for a P-extension to be called a Π-extension.

Theorem 2.3.40 (Schneider (2016), Theorem 2.12(2)).
Let $(\mathbb{A}\langle t\rangle, \sigma)$ be a P-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ where $\alpha \in \mathbb{A}^*$. Then this is a Π-extension (i.e., $\operatorname{const}(\mathbb{A}\langle t\rangle, \sigma) = \operatorname{const}(\mathbb{A}, \sigma) = \mathbb{K}$) if and only if there is no $g \in \mathbb{A} \setminus \{0\}$ and $\nu \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = \alpha^\nu\, g$. If $(\mathbb{A}\langle t\rangle, \sigma)$ is a Π-extension of $(\mathbb{A}, \sigma)$ then $\operatorname{ord}(\alpha) = 0$. $\diamondsuit$

Proof:
“ $\Longleftarrow$ ” Suppose there is a $g \in \mathbb{A} \setminus \{0\}$ and $\nu \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = \alpha^\nu\, g$. We prove $\operatorname{const}(\mathbb{A}\langle t\rangle, \sigma) \supsetneq \operatorname{const}(\mathbb{A}, \sigma)$. With $\sigma(t) = \alpha\, t$ and $\nu \in \mathbb{Z} \setminus \{0\}$ we have that $\sigma(t^\nu) = \alpha^\nu\, t^\nu$. Take $h := \frac{g}{t^\nu} \in \mathbb{A}\langle t\rangle$ then,

$$\sigma(h) = \sigma\left(\frac{g}{t^\nu}\right) = \frac{\sigma(g)}{\sigma(t^\nu)} = \frac{\alpha^\nu\, g}{\alpha^\nu\, t^\nu} = \frac{g}{t^\nu} =: h \in \operatorname{const}(\mathbb{A}\langle t\rangle, \sigma).$$

Clearly, $h := \frac{g}{t^\nu} \notin \mathbb{A} \Longrightarrow h \notin \operatorname{const}(\mathbb{A}, \sigma)$.
“ $\Longrightarrow$ ” Suppose $\operatorname{const}(\mathbb{A}\langle t\rangle, \sigma) \supsetneq \operatorname{const}(\mathbb{A}, \sigma)$. We prove that there is a $g \in \mathbb{A} \setminus \{0\}$ and $\nu \in \mathbb{Z} \setminus \{0\}$ such that $\sigma(g) = \alpha^\nu\, g$. Let $g = \sum_i g_i\, t^i \in \mathbb{A}\langle t\rangle \setminus \mathbb{A}$ with $\sigma(g) = g$. Then $\mathbb{A} \ni g_\nu \neq 0$ for some $\nu \neq 0$. By Lemma 2.3.39 with $u = 1$, we have that $\sigma(g_\nu) = \alpha^{-\nu}\, g_\nu$. Suppose t is a Π-monomial but $\operatorname{ord}(\alpha) = n > 0$. Then $\sigma(t^n) = \alpha^n t^n = t^n$. But this is a contradiction to the assumption that $(\mathbb{A}\langle t\rangle, \sigma)$ is a difference ring extension of $(\mathbb{A}, \sigma)$ as t is transcendental by definition. ∎

Lemma 2.3.41.
Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ and let $f, h \in \mathbb{F}[t] \setminus \{0\}$ with $\gcd(f, h) = 1$. Then the following holds:

(1) If $f \mid h$ then for $r \in \mathbb{Z}$, $\sigma^r(f) \mid \sigma^r(h)$.

(2) For $r \in \mathbb{Z}$, $\gcd(\sigma^r(f), \sigma^r(h)) = 1$.

(3) $\frac{\sigma(f/h)}{f/h} \in \mathbb{F}$ if and only if $\frac{\sigma(f)}{f} \in \mathbb{F}$ and $\frac{\sigma(h)}{h} \in \mathbb{F}$. ♦

Proof:
(1) If $f \mid h$ then there exist $w \in \mathbb{F}[t] \setminus \{0\}$ such that $f\, w = h$. Thus for $r \in \mathbb{Z}$ we get $\sigma^r(f)\, \sigma^r(w) = \sigma^r(h)$ and thus $\sigma^r(f) \mid \sigma^r(h)$.

(2) Assume $f, h \in \mathbb{F}[t] \setminus \{0\}$ with $\gcd(f, h) = 1$ and suppose $1 \neq \gcd(\sigma^r(f), \sigma^r(h)) =: \omega \in \mathbb{F}[t] \setminus \mathbb{F}$. From $\omega := \gcd(\sigma^r(f), \sigma^r(h))$, we know that $\omega \mid \sigma^r(f)$ and $\omega \mid \sigma^r(h)$. Since $\omega \in \mathbb{F}[t] \setminus \mathbb{F}$, it follows that $\sigma^{-r}(\omega) \in \mathbb{F}[t] \setminus \mathbb{F}$. Therefore $\sigma^{-r}(\omega) \mid f$ and $\sigma^{-r}(\omega) \mid h$. Thus $\gcd(f, h) \neq 1$ which is a contradiction to the assumption that $\gcd(f, h) = 1$.

(3) “ $\Longleftarrow$ ” Since $\mathbb{F}$ is a field, it follows that $\frac{1}{\sigma(h)/h} \in \mathbb{F}$ and

$$\frac{\sigma(f)}{f}\, \frac{h}{\sigma(h)} \iff \frac{\sigma(f)}{\sigma(h)}\, \frac{h}{f} = \frac{\sigma(f/h)}{f/h} \in \mathbb{F}.$$

“ $\Longrightarrow$ ” Let $\nu := \frac{\sigma(f/h)}{f/h} \in \mathbb{F}$. Then

$$\sigma(f)\, h = \nu\, f\, \sigma(h).$$

Since $\gcd(f, h) = 1$ implies that $\gcd(\sigma^k(f), \sigma^k(h)) = 1$ by part (2) of the Lemma, it follows that $f \mid \sigma(f)$ and $\sigma(f) \mid f$. Therefore $\frac{\sigma(f)}{f} \in \mathbb{F}$. A similar argument for h implies that $\frac{\sigma(h)}{h} \in \mathbb{F}$. ∎

Lastly, using Lemma 2.3.41 we are able to rediscover Karr's field version for Π-extension; see Karr (1985, Theorem 2.2); again we follow the proof from Schneider (2016, Theorem 3.18) which was inspired by (Karr, 1985).

Theorem 2.3.42.
Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ with $\alpha = \frac{\sigma(t)}{t} \in \mathbb{F}^*$. Then $(\mathbb{F}(t), \sigma)$ is a Π-extension of $(\mathbb{F}, \sigma)$ if and only if there does not exist a $g \in \mathbb{F}^*$ and a positive integer m such that $\sigma(g) = \alpha^m g$ holds. $\diamondsuit$

PROOF:
" $\Longrightarrow$ " Suppose that $\mathrm{const}(\mathbb{F}(t), \sigma) \supsetneq \mathrm{const}(\mathbb{F}, \sigma)$. We prove there is a $g \in \mathbb{F}^*$ and a positive integer m with $\sigma(g) = \alpha^m g$. Let $g \in \mathbb{F}(t) \setminus \mathbb{F}$ with $\sigma(g) = g$. Write $g = \frac{p}{q}$ with $p \in \mathbb{F}[t]$, $q \in \mathbb{F}[t] \setminus \{0\}$ and p and q are co-prime. W.l.o.g assume that $\deg(p) \leqslant \deg(q)$ (otherwise take $\frac{1}{g}$ instead of g). Since $\frac{\sigma(g)}{g} \in \mathbb{F}$, it follows by statement (3) of Lemma 2.3.41 that $\frac{\sigma(p)}{p}, \frac{\sigma(q)}{q} \in \mathbb{F}$. We consider two cases:

Case 1: If $p \in \mathbb{F}^*$ and $q = t^m$ with $m > 0$. Then

$$\frac{p}{t^m} = g = \sigma(g) = \frac{\sigma(p)}{\alpha^m t^m}.$$

Thus $\sigma(p) = \alpha^m p$.

Case 2: Suppose $p \notin \mathbb{F}$ or $q \neq t^m$ with $m > 0$ and define

$$v = \begin{cases} p & \text{if } q = t^m \text{ for some } m > 0 \\ q & \text{otherwise.} \end{cases}$$

Then the following holds.

(1) $v \in \mathbb{F}[t] \setminus \mathbb{F}$: There are two cases here. Namely, if $v = q$ then $q \in \mathbb{F}[t] \setminus \mathbb{F}$ with $\deg(p) \leqslant \deg(q)$ and $\frac{p}{q} \notin \mathbb{F}$. Otherwise, if $v = p$ then $q = t^m$ with $\deg(p) \leqslant \deg(q)$.

(2) By Lemma 2.3.41(3), it follows that $u := \frac{\sigma(v)}{v} \in \mathbb{F}^*$.

(3) $v \neq u\, t^m$ for all $u \in \mathbb{F}^*$ and $m > 0$: v could only be of this form if $q = t^m$ for some $m > 0$. Thus $v = p$ and since $\gcd(p, q) = 1$ it follows that $t \nmid p$.

It follows by properties (1) and (3) that $v = \sum_{i=k}^{n} v_i\, t^i$ with $v_k \neq 0 \neq v_n$ where $n > k \geqslant 0$. By property (2) and Lemma 2.3.39, we have that $\sigma(v_k) = \frac{u}{\alpha^k} v_k$ and $\sigma(v_n) = \frac{u}{\alpha^n} v_n$. Using these information, we have

$$\sigma\left(\frac{v_k}{v_n}\right) = \alpha^{n-k} \frac{v_k}{v_n}.$$

Since $\frac{v_k}{v_n} \in \mathbb{F}^*$ and $n - k > 0$, this proves the implication " $\Longrightarrow$ ".
" $\Longleftarrow$ " This direction of the proof follows by the first part of Theorem 2.3.40 since any Π-field extension is automatically a Π-ring extension. ∎

We give further examples and non-examples of Π-extensions.

Example 2.3.43 (Cont. Example 2.3.29).
(1) The P-extension $(\mathbb{A}_1, \sigma)$ of $(\mathbb{K}(x), \sigma)$ in item (1) of Example 2.3.29 with $\sigma(y_1) = \sqrt{13}\, y_1$ is a Π-extension of $(\mathbb{K}(x), \sigma)$ as there are no $g \in \mathbb{K}(x)^*$ and $v \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = \left(\sqrt{13}\right)^v g$.

(2) Similarly, the P-extension $(\mathbb{A}_2, \sigma)$ of $(\mathbb{A}_1, \sigma)$ in item (2) of Example 2.3.29 with $\sigma(y_2) = 7\, y_2$ is also a Π-extension since there does not exist a $g \in \mathbb{A}_1 \setminus \{0\}$ and a $\nu \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = 7^{\nu}\, g$.

(3) However, the P-extension $(\mathbb{A}_3, \sigma)$ of $(\mathbb{A}_2, \sigma)$ in item (3) of Example 2.3.29 is not a Π-extension because with $g = {y_1}^4 \in \mathbb{A}_2$ and $\nu = 1$ we have $\sigma(g) = (169)\, g$. In particular,

$$w = g^{-1} y_3 \in \mathrm{const}(\mathbb{K}(x)\langle y_1\rangle\langle y_2\rangle\langle y_3\rangle, \sigma) \setminus \mathrm{const}(\mathbb{K}(x)\langle y_1\rangle\langle y_2\rangle, \sigma). \qquad \star$$

Example 2.3.44 (Cont. Example 2.3.30).

(1) The P-extension $(\mathbb{Q}(x)\,\langle z_1\rangle, \sigma)$ of $(\mathbb{Q}(x), \sigma)$ with $\sigma(z_1) = (x+1)\, z_1$ is a Π-extension since there are no $g \in \mathbb{Q}(x)^*$ and $\nu \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = (x+1)^{\nu}\, g$.

(2) But the P-extension $(\mathbb{Q}(x)\,\langle z_1\rangle\,\langle z_2\rangle, \sigma)$ of $(\mathbb{Q}(x)\,\langle z_1\rangle, \sigma)$ with $\sigma(z_2) = (x+3)\, z_2$ is not a Π-extension since with $g = (x+2)\,(x+1)\, z_1$ and $\nu = 1$ we have $\sigma(g) = (x+3)\, g$. In particular, we get

$$c = g\, {z_2}^{-1} \in \mathrm{const}(\mathbb{Q}(x)\,\langle z_1\rangle\,\langle z_2\rangle, \sigma) \setminus \mathrm{const}(\mathbb{Q}(x)\,\langle z_1\rangle, \sigma). \qquad \star$$

Example 2.3.45 (Cont. Example 2.3.10).

The mixed $\boldsymbol{q}$-multibasic difference field $(\mathbb{F}, \sigma)$ with $\mathbb{F} = \mathbb{K}(x)(t_1)\ldots(t_e)$ in Example 2.3.10 is a nested Π-field extension of the rational difference field $(\mathbb{K}(x), \sigma)$. As a consequence, we have that $\mathrm{const}(\mathbb{F}, \sigma) = \mathrm{const}(\mathbb{K}(x), \sigma) = \mathbb{K}$. Consequently, the P-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $\mathbb{E} = \mathbb{K}(x)[t_1, \frac{1}{t_1}]\ldots[t_e, \frac{1}{t_e}]$ in Example 2.3.10 is a Π-extension. See Corollary 5.4.2 below for the proof. $\star$

2.3.2.3 Σ-extensions

Although we concentrate mainly on product extensions in this thesis, we still need to handle the very special case of the rational difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x+1$ in Example 2.3.6 or the mixed $\boldsymbol{q}$-multibasic difference field $(\mathbb{K}(x, \boldsymbol{t}), \sigma)$ with $\sigma(x) = x+1$ and $\sigma(t_1) = q_i\, t_i$ for $1 \leqslant i \leqslant e$ in Example 2.3.10. Thus it will be convenient to introduce also the field version of Σ-extensions (Karr, 1981; Schneider, 2001). We will start with the ring version introduced in Schneider (2016).

Definition 2.3.46.

Let $(\mathbb{A}, \sigma)$ be a difference ring. We call an S-extension $(\mathbb{A}[x], \sigma)$ of $(\mathbb{A}, \sigma)$ with $\sigma(x) = x + \beta$ for some $\beta \in \mathbb{A}$ a Σ-extension of $(\mathbb{A}, \sigma)$ if $\mathrm{const}(\mathbb{A}[x], \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$. The generator x, is called a Σ-monomial. Further, if $(\mathbb{A}(x), \sigma)$ is an S-field extension of a difference field $(\mathbb{A}, \sigma)$ with $\sigma(x) = x + \beta$ where $\beta \in \mathbb{A}$, then it is called a Σ-field extension of $(\mathbb{A}, \sigma)$ if $\mathrm{const}(\mathbb{A}(x), \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$. We call the monomial x a Σ-field monomial. $\ast$

We need a special case of Schneider (2016, Lemma 3.7) inspired by (Karr, 1985).

Lemma 2.3.47.

Let $(\mathbb{A}[x], \sigma)$ be an S-extension of $(\mathbb{A}, \sigma)$ with $\sigma(x) = x + \beta$ for some $\beta \in \mathbb{A}$ where $\mathbb{K} = \mathrm{const}(\mathbb{A}, \sigma)$ is a field. Let $g \in \mathbb{A}[x]$ with $\deg(g) \geqslant 1$ such that $\deg(\sigma(g) - g) < \deg(g) - 1$. Then there is a $\gamma \in \mathbb{F}$ such that $\sigma(\gamma) - \gamma = \beta$. ♦

Proof:
Let $g = \sum_{i=1}^{n} g_i\, x^i$ with $\sigma(x) = x+\beta$ such that $\deg(\sigma(g)-g) < \deg(g)-1$. Define $f := \sigma(g)-g \in \mathbb{F}[x]$. Then $\deg(f) \leqslant n-2$. Thus for $f = \sum_{i=0}^{n-2} f_i\, x^i$ we have

$$\begin{aligned}\sum_{i=0}^{n-2} f_i\, x^i = f = \sigma(g) - g = \sigma\Big(\sum_{i=1}^{n} g_i\, x^i\Big) - \sum_{i=1}^{n} g_i\, x^i \\ &= \sum_{i=0}^{n} \sigma(g_i)\,(\sigma(x))^i - \sum_{i=1}^{n} g_i\, x^i \\ &= \sum_{i=0}^{n} \sigma(g_i)\,(x+\beta)^i - \sum_{i=1}^{n} g_i\, x^i \\ &= \sum_{i=0}^{n} \sigma(g_i) \sum_{j=0}^{i} \binom{i}{j} x^{i-j}\, \beta^j - \sum_{i=1}^{n} g_i\, x^i.\end{aligned}$$

Comparing the n-th and $(n-1)$-th coefficients we get

$$0 = \sigma(g_n) - g_n \tag{2.47}$$

and

$$0 = \sigma(g_n)\binom{n}{1}\beta + \sigma(g_{n-1}) - g_{n-1} \tag{2.48}$$

respectively. From equation (2.47) $\sigma(g_n) = g_n \Longrightarrow g_n \in \mathbb{K}^*$. Substituting (2.47) into (2.48) we get

$$\begin{aligned}\sigma(g_{n-1}) - g_{n-1} &= -\sigma(g_n)\, n\, \beta \\ \sigma\left(-\frac{g_{n-1}}{n\, g_n}\right) + \frac{g_{n-1}}{n\, g_n} &= \beta\end{aligned}$$

Take $\gamma = -\frac{g_{n-1}}{n\, g_n} \in \mathbb{F}$, then $\sigma(\gamma) - \gamma = \beta$. This completes the proof. ■

Furthermore, Schneider (2016, Lemma 3.8) can be specialised to

Lemma 2.3.48.
Let $(\mathbb{A}[x], \sigma)$ be an S-extension of $(\mathbb{A}, \sigma)$ with $\sigma(x) = x + \beta$ for some $\beta \in \mathbb{A}$ where $\mathbb{K} = \mathrm{const}(\mathbb{A}, \sigma)$ is a field. Then the following statements are equivalent.

(1) There exist a $g \in \mathbb{A}[x] \setminus \mathbb{A}$ with $\sigma(g) = g$.

(2) There exist a $\gamma \in \mathbb{A}$ such that $\sigma(\gamma) = \gamma + \beta$.

(3) $\mathrm{const}(\mathbb{A}, \sigma) \subsetneq \mathrm{const}(\mathbb{A}[x], \sigma)$. ♦

Proof:
(1) $\Longrightarrow$ (2): Since $g \in \mathbb{A}[x] \setminus \mathbb{A}$, $n := \deg(g) \geqslant 1$ and $\deg(\sigma(g) - g) < 0 \leqslant n-1$. Therefore it follows by Lemma 2.3.47 that, there is a $\gamma \in \mathbb{A}$ such that $\sigma(\gamma) = \gamma + \beta$.
(2) $\Longrightarrow$ (3): Let $\gamma \in \mathbb{A}$ with $\sigma(\gamma) = \gamma + \beta$. Then with $g := (x - \gamma) \in \mathbb{A}[x] \setminus \mathbb{A}$ we have that $\sigma(g) = g$. Thus $\mathrm{const}(\mathbb{A}, \sigma) \subsetneq \mathrm{const}(\mathbb{A}[x], \sigma)$.
(3) $\Longrightarrow$ (1): If $\mathrm{const}(\mathbb{A}, \sigma) \subsetneq \mathrm{const}(\mathbb{A}[x], \sigma)$, then it follows by definition that there is a $g \in \mathbb{A}[x] \setminus \mathbb{A}$ with $\sigma(g) = g$. This completes the proof. ■

The next Theorem gives a characterization for an S-extension to be a Σ-extension.

Theorem 2.3.49 (Schneider (2016), Theorem 2.12(1)).
Let $(\mathbb{A}[x], \sigma)$ be an S*-extension of $(\mathbb{A}, \sigma)$ with $\sigma(x) = x + \beta$ where $\beta \in \mathbb{A}$ and $\mathbb{K} = \mathrm{const}(\mathbb{A}, \sigma)$ is a field. Then $(\mathbb{A}[x], \sigma)$ is a Σ-extension (i.e., $\mathrm{const}(\mathbb{A}[x], \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$) if and only if there does not exist a $g \in \mathbb{A}$ such $\sigma(g) = g + \beta$* $\diamondsuit$

Proof:
The proof of the Theorem follows from the proof of (2) $\Longleftrightarrow$ (3) of Lemma 2.3.48. $\blacksquare$

We also rediscover Karr's field version (Karr, 1981, 1985) by extending Lemma 2.3.48 to the difference field setting and utilizing statement (3) of Lemma 2.3.41; compare Schneider (2016).

Lemma 2.3.50.
Let $(\mathbb{F}(x), \sigma)$ be an S*-field extension of $(\mathbb{F}, \sigma)$ with $\sigma(x) = x + \beta$ for some $\beta \in \mathbb{F}$. Then the following statements are equivalent.*

(1) There is a $g \in \mathbb{F}(x) \setminus \mathbb{F}$ such that $\frac{\sigma(g)}{g} \in \mathbb{F}$.

(2) There is a $\gamma \in \mathbb{F}$ such that $\sigma(\gamma) - \gamma = \beta$.

(3) $\mathrm{const}(\mathbb{F}, \sigma) \subsetneq \mathrm{const}(\mathbb{F}(x), \sigma)$. $\blacklozenge$

Proof:
Let $g \in \mathbb{F}(x) \setminus \mathbb{F}$ and write $g = \frac{f}{h}$ with $f, h \in \mathbb{F}[x] \setminus \{0\}$ and $\gcd(f, h) = 1$.
(1) $\Longrightarrow$ (2): By statement (3) of Lemma 2.3.41 we have that

$$\frac{\sigma(f)}{f} \in \mathbb{F} \text{ and } \frac{\sigma(h)}{h} \in \mathbb{F}.$$

Since $g \notin \mathbb{F}$, it follows that $f \notin \mathbb{F}$ and $h \notin \mathbb{F}$. Thus there is a $g' \in \mathbb{F}[x] \setminus \mathbb{F}$ such that $\deg(\sigma(g') - g') = \deg(0) = -\infty < 0 \leqslant \deg(g') - 1$. Therefore by Lemma 2.3.47, there is a $\gamma \in \mathbb{F}$ such that $\sigma(\gamma) = \gamma + \beta$. The proofs of the statements (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are analogous to the proof of the same statements in Lemma 2.3.48. $\blacksquare$

Lemma 2.3.50 is contained in Karr's work by combining Theorems 2.1 and 2.3 of Karr (1985). As a consequence, we obtain a criterion for an S-field extension to be a Σ-field extension.

Theorem 2.3.51.
Let $(\mathbb{F}(x), \sigma)$ be an S*-field extension of $(\mathbb{F}, \sigma)$ with $\sigma(x) = x + \beta$ for some $\beta \in \mathbb{F}$. Then $(\mathbb{F}(x), \sigma)$ is a Σ-field extension of $(\mathbb{F}, \sigma)$ if and only if there does not exit a $g \in \mathbb{F}$ with $\sigma(g) = g + \beta$.* $\diamondsuit$

Example 2.3.52 (Cont. Example 2.3.6).
The rational difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x+1$ in Example 2.3.6 is a Σ-field extension of $(\mathbb{K}, \sigma)$.$\star$

2.3.2.4 $R\Pi\Sigma$-EXTENSIONS

For further considerations, we introduce the following terminologies.

Definition 2.3.53.
Let $(\mathbb{A}\langle t\rangle, \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$. We call such an extension an $R\Pi$-/$R\Sigma$-/$\Pi\Sigma$-/$R\Pi\Sigma$-extension, if it is an AP-/AS-/PS-/APS-extension and $\mathrm{const}(\mathbb{A}\langle t\rangle, \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$ respectively. Depending on the type of extension, we call the generator t an R-/ Π-/ Σ-/ $R\Pi$-/ $R\Sigma$-/ $\Pi\Sigma$-/ $R\Pi\Sigma$-monomial respectively. In addition, an APS-extension $(\mathbb{A}\langle t_1\rangle, \dots, \langle t_e\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ is called an $R\Pi\Sigma$-extension (resp. R-, Π-, Σ-, $R\Pi$-, $R\Sigma$-, $\Pi\Sigma$-extension) if it is a tower of such extensions. $*$

Definition 2.3.54 (Cont. Definition 2.3.20).
Let $(\mathbb{E}, \sigma)$ and $(\mathbb{G}, \sigma)$ be the difference ring extension defined in Definition 2.3.20. If $\mathrm{const}(\mathbb{G}, \sigma) = \mathrm{const}(\mathbb{E}, \sigma)$ and $\mathfrak{d}_{\mathbb{A}}(x_1) \leqslant \mathfrak{d}_{\mathbb{A}}(x_2) \leqslant \cdots \leqslant \mathfrak{d}_{\mathbb{A}}(x_s)$, then $(\mathbb{G}, \sigma)$ is called an *ordered nested* $R\Pi$*-extension* of $(\mathbb{E}, \sigma)$. $*$

Example 2.3.55 (Cont. Example 2.3.10).
The mixed $\mathbf{q}$-multibasic difference ring $(\mathbb{K}(x)[t_1, \frac{1}{t_1}] \dots [t_e, \frac{1}{t_e}], \sigma)$ (resp. mixed $\mathbf{q}$-multibasic difference field $(\mathbb{K}(x)(t_1) \dots (t_e), \sigma)$) in Example 2.3.10 is a $\Pi\Sigma$-ring (resp. $\Pi\Sigma$-field). $\star$

Remark 2.3.56.
In Karr (1981) and Schneider (2016) algorithms have been developed to check if an already designed difference ring (resp. field) is built by properly chosen $R\Pi\Sigma$-extensions. Therefore with these algorithms, one can decided whether or not the constructed difference rings in Examples 2.3.34, 2.3.35, 2.3.43, 2.3.44, 2.3.45, 2.3.52, and 2.3.55 are $R\Pi\Sigma$-extensions. However, in this thesis we are more ambitious. We will carefully construct AP-extensions over the base difference rings (resp. fields) in Examples 2.3.6 and 2.3.10 such that they are automatically $R\Pi$-extensions of these base rings (resp. fields) and such that the given products can be rephrased within these extensions straightforwardly. In this regard, we will utilize the following lemma. ♠

Lemma 2.3.57.
Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field extension of $(\mathbb{K}, \sigma)$ with $\mathrm{const}(\mathbb{K}, \sigma) = \mathbb{K}$. Then the A-extension $(\mathbb{F}[\vartheta], \sigma)$ of $(\mathbb{F}, \sigma)$ with order $\lambda > 1$ is an R-extension. ♦

PROOF:
By Karr (1985, Lemma 3.5) we have $\mathrm{const}(\mathbb{F}, \sigma^k) = \mathrm{const}(\mathbb{F}, \sigma)$ for all $k \in \mathbb{N} \setminus \{0\}$. Thus with Proposition 2.3.37, $(\mathbb{F}[\vartheta], \sigma)$ is an R-extension of $(\mathbb{F}, \sigma)$. ∎

Corollary 2.3.58.
Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}=\text{const}(\mathbb{F},\sigma)$ and let $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{F}\langle t_1\rangle\dots\langle t_e\rangle$ be a Π-extension of $(\mathbb{F},\sigma)$ where

$$\sigma(t_k)=\alpha_k\,t_k \tag{2.49}$$

with $\alpha_k\in\mathbb{F}^*$ for $1\leqslant k\leqslant e$. Then the A-extension $(\mathbb{E}[\vartheta],\sigma)$ of $(\mathbb{E},\sigma)$ with order $\lambda>1$ is an R-extension.$\diamond$

Proof:
Note that the quotient field $\mathbb{D}=\mathbb{F}(t_1)\dots(t_e)$ together with (2.49) forms a Π-extension of $(\mathbb{F},\sigma)$ by iterative application of Schneider (2017, Corollary 2.6). Consequently also,

$$\mathbb{K}=\text{const}(\mathbb{F},\sigma)=\text{const}(\mathbb{D}[\vartheta],\sigma)=\text{const}(\mathbb{E}[\vartheta],\sigma)$$

and thus $(\mathbb{E}[\vartheta],\sigma)$ is an R-extension of $(\mathbb{E},\sigma)$. ∎

2.4 Difference fields, difference rings and difference ring of sequences

In this section we will discuss the connection between difference rings and the difference ring of sequences as included already in Examples 2.3.27, 2.3.28, 2.3.29 and 2.3.30 to Examples 2.2.4, 2.2.2, 2.2.3 and 2.2.4 respectively. More precisely, we elaborate how $R\Pi$-extensions can be embedded into the difference ring of sequences Schneider (2017); compare also Van Der Put and Singer (2006). Precisely this feature will enable us to handle condition (2) of Problem RPE.

Definition 2.4.1 (Cont. Definition 2.3.14).
Let $(\mathbb{A},\sigma)$ be a difference ring with constant field $\mathbb{K}$. A difference ring homomorphism (resp. monomorphism) $\tau:\mathbb{A}\to\mathcal{S}(\mathbb{K})$ is called $\mathbb{K}$-*homomorphism* (resp. $\mathbb{K}$-*monomorphism*) if

$$\forall\, c\in\mathbb{K},\ \tau(c)=\mathbf{c}:=\langle c,c,c,\dots\rangle\,. \qquad *$$

The following lemma is the key tool to embed difference rings constructed by $R\Pi$-extensions into the difference ring of sequences; compare Schneider (2017, 2010a).

Lemma 2.4.2.
Let $(\mathbb{A},\sigma)$ be a difference ring with constant field $\mathbb{K}$. Then the map $\tau:\mathbb{A}\to\mathcal{S}(\mathbb{K})$ is a $\mathbb{K}$-homomorphism if and only if there is an evaluation function $\text{ev}:\mathbb{A}\times\mathbb{N}\to\mathbb{K}$ for $(\mathbb{A},\sigma)$ (see Definition 2.3.23) with

$$\tau(f)=\langle \text{ev}(f,0),\ \text{ev}(f,1),\dots\rangle. \qquad \blacklozenge$$

Proof:
The proof follows by Schneider (2001, Lemma 2.5.1). ∎

Lemma 2.4.3.
Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$. Then the following statements hold:

(1) Let $(\mathbb{A}\langle t\rangle, \sigma)$ be an AP-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ ($\alpha \in \mathbb{A}^*$) and suppose that $\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K})$ as given in Lemma 2.4.2 is a $\mathbb{K}$-homomorphism. Suppose that there is a $\delta \in \mathbb{N}$ such that $\mathrm{ev}(\alpha, n) \neq 0$ for all $n \geqslant \delta$. Further, take $u \in \mathbb{K}^*$; if $t^\lambda = 1$ for some $\lambda > 1$, we further assume that $u^\lambda = 1$ holds. Consider the map $\tau' : \mathbb{A}\langle t\rangle \to \mathcal{S}(\mathbb{K})$ with $\tau'(f) = \langle \mathrm{ev}'(f, n)\rangle_{n\geqslant 0}$ where the evaluation function $\mathrm{ev}' : \mathbb{A}\langle t\rangle \times \mathbb{N} \to \mathbb{K}$ is defined by

$$\mathrm{ev}'(\sum_i h_i\, t^i, n) = \sum_i \mathrm{ev}(h_i, n)\ \mathrm{ev}'(t, n)^i$$

with

$$\mathrm{ev}'(t, n) = u \prod_{k=\delta}^{n} \mathrm{ev}(\alpha, k-1).$$

Then τ' is a $\mathbb{K}$-homomorphism.

(2) If $(\mathbb{A}, \sigma)$ is a difference field and $(\mathbb{E}, \sigma)$ is a (nested) $\mathrm{R}\Pi$-extension of $(\mathbb{A}, \sigma)$, then any $\mathbb{K}$-homomorphism $\tau : \mathbb{E} \to \mathcal{S}(\mathbb{K})$ is injective. ♦

Proof:
(1) The proof follows by Schneider (2017, Lemma 5.4(1)).

(2) By Schneider (2017, Theorem 3.3) $(\mathbb{E}, \sigma)$ is simple that is, any ideal of $\mathbb{E}$ which is closed under σ is either $\mathbb{E}$ or $\{0\}$. Thus by Schneider (2017, Lemma 5.8) τ' is injective.

In this thesis, we will apply statement (1) of Lemma 2.4.3 iteratively. As base case, we will use the following difference fields that can be embedded into the ring of sequences.

Example 2.4.4 (Cont. Example 2.3.6).
Take the rational difference field $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}$ defined in Example 2.3.6 and consider the map $\tau : \mathbb{K}(x) \to \mathcal{S}(\mathbb{K})$ defined by $\tau(\frac{a}{b}) = \langle \mathrm{ev}\left(\frac{a}{b}, n\right)\rangle_{n\geqslant 0}$ with $a, b \in \mathbb{K}[x]$ and $b \neq 0$ where the evaluation map $\mathrm{ev} : \mathbb{K}(x) \to \mathbb{K}$ is as defined in (2.1). Then by Lemma 2.4.2 it follows that $\tau : \mathbb{K}(x) \to \mathcal{S}(\mathbb{K})$ is a $\mathbb{K}$-homomorphism. We can define the function:

$$Z(p) = \max\left(\{k \in \mathbb{N} \mid p(k) = 0\}\right) + 1 \quad \text{for any } p \in \mathbb{K}[x] \tag{2.50}$$

with $\max(\varnothing) = -1$. Now let $f = \frac{a}{b} \in \mathbb{K}(x)$ where $a, b \in \mathbb{K}[x]$, $b \neq 0$. Since $a(x)$, $b(x)$ have only finitely many roots, it follows that $\tau(\frac{a}{b}) = \mathbf{0}$ if and only if $\frac{a}{b} = 0$. Hence $\ker(\tau) = \{0\}$ and thus τ is injective. Summarizing, we have constructed a $\mathbb{K}$-embedding, $\tau : \mathbb{K}(x) \to \mathcal{S}(\mathbb{K})$ where the difference field $(\mathbb{K}(x), \sigma)$ is embedded in the difference ring of $\mathbb{K}$-sequences $(\mathcal{S}(\mathbb{K}), S)$ as the sub-difference ring of $\mathbb{K}$-sequences $(\tau(\mathbb{K}(x)), S)$. We call $(\tau(\mathbb{K}(x)), S)$ the *difference field of rational sequences*. ⋆

Example 2.4.5 (Cont. Example 2.3.10).
Take the mixed $\mathbf{q}$-multibasic difference field $(\mathbb{F}, \sigma)$ with $\mathbb{F} = \mathbb{K}(x, \mathbf{t})$ defined in Example 2.3.10. Then, $\tau : \mathbb{F} \to \mathcal{S}(\mathbb{K})$ defined by $\tau(\frac{a}{b}) = \langle \text{ev}\left(\frac{a}{b}, n\right)\rangle_{n \geqslant 0}$ with $a, b \in \mathbb{K}[x, \mathbf{t}]$ and $b \neq 0$ where the evaluation function $\text{ev} : \mathbb{F} \to \mathbb{K}$ is as defined in (2.2) is a $\mathbb{K}$-homomorphism. By Lemma 2.1.9 based on Bauer and Petkovšek (1999, Section 3.2) we can define for $p \in \mathbb{K}[x, \mathbf{t}] \setminus \{0\}$ the function

$$Z(p) = \max\left(\{k \in \mathbb{N} \,|\, p(k, \mathbf{q}^k) = 0\}\right) + 1 \in \mathbb{N} \tag{2.51}$$

with $\max(\varnothing) = -1$. For any rational function, $f = \frac{g}{h} \in \mathbb{F} \setminus \{0\}$ with $g, h \in \mathbb{K}[x, \mathbf{t}]$, let

$$\delta = \max(\{Z(g),\, Z(h)\}) \in \mathbb{N}.$$

Then $f(n) \neq 0$ for all $n \geqslant \delta$ and thus $\tau(f) \neq \mathbf{0}$. Hence $\ker(\tau) = \{0\}$ and thus τ is injective. In summary, we have constructed a $\mathbb{K}$-embedding $\tau : \mathbb{F} \to \mathcal{S}(\mathbb{K})$ where the difference field $(\mathbb{F}, \sigma)$ is embedded in $(\mathcal{S}(\mathbb{K}), S)$ as $(\tau(\mathbb{F}), S)$ which we call the *difference field of mixed $\mathbf{q}$-multibasic rational sequences*. ⋆

Example 2.4.6 (Cont. Examples 2.3.27, 2.3.28, 2.3.29 and 2.3.30).
Since the evaluation functions (2.30), (2.32), (2.34), (2.36), (2.38), (2.40), (2.42), (2.44) and (2.46) satisfy conditions (2.26), (2.27) and (2.28), we get:

(1) Polynomial expressions in $\left((-1)^{\frac{1}{6}}\right)^n$ and $(\mathbb{i})^n$ with shift behaviours

$$S_n\left((-1)^{\frac{1}{6}}\right)^n = \left((-1)^{\frac{1}{6}}\right)^{n+1} = (-1)^{\frac{1}{6}}\left((-1)^{\frac{1}{6}}\right)^n \qquad \text{and} \qquad S_n(\mathbb{i})^n = (\mathbb{i})^{n+1} = \mathbb{i}\,\mathbb{i}^n$$

can be modelled by polynomial expressions in the nested A-extension $(\mathbb{K}(x)[\vartheta_1][\vartheta_2], \sigma)$ of $(\mathbb{K}(x), \sigma)$ constructed in Example 2.3.27.

(2) Polynomial expressions in $(-1)^n$ and $\left((-1)^{\frac{2}{3}}\right)^n$ with shift behaviours

$$S_n(-1)^n = (-1)^{n+1} = -(-1)^n \qquad \text{and} \qquad S_n\left((-1)^{\frac{2}{3}}\right)^n = \left((-1)^{\frac{2}{3}}\right)^{n+1} = (-1)^{\frac{2}{3}}\left((-1)^{\frac{2}{3}}\right)^n$$

can be modelled by polynomial expressions in the nested A-extension $(\mathbb{K}(x)[\vartheta_1][\vartheta_2], \sigma)$ of $(\mathbb{K}(x), \sigma)$ constructed in Example 2.3.28.

(3) Polynomial expressions in $\left(\sqrt{13}\right)^n$ and $\left(\sqrt{13}\right)^{-n}$, 7^n and 7^{-n}, 169^n and 169^{-n} with shift behaviours

$$S_n\left(\sqrt{13}\right)^n = \left(\sqrt{13}\right)^{n+1} = \sqrt{13}\left(\sqrt{13}\right)^n, \qquad S_n\frac{1}{(\sqrt{13})^n} = \frac{1}{(\sqrt{13})^{n+1}} = \frac{1}{\sqrt{13}}\,\frac{1}{(\sqrt{13})^n},$$
$$S_n 7^n = 7^{n+1} = 7\,7^n, \qquad S_n\frac{1}{7^n} = \frac{1}{7^{n+1}} = \frac{1}{7}\,\frac{1}{7^n},$$
$$S_n 169^n = 169^{n+1} = 169\,169^n, \qquad S_n\frac{1}{169^n} = \frac{1}{169^{n+1}} = \frac{1}{169}\,\frac{1}{169^n},$$

can be modelled by polynomial expressions in the nested P-extension, $(\mathbb{K}(x)\langle y_1\rangle\langle y_2\rangle\langle y_3\rangle, \sigma)$ of $(\mathbb{K}(x), \sigma)$ constructed in Example 2.3.29.

(4) Polynomial expressions in $n!$ and $\frac{1}{n!}$, $(n+2)!$ and $\frac{1}{(n+2)!}$ with shift behaviours

$$S_n n! = (n+1)! = (n+1)\,n!, \qquad S_n\frac{1}{n!} = \frac{1}{(n+1)!} = \frac{1}{(n+1)}\,\frac{1}{n!},$$
$$S_n(n+2)! = (n+3)! = (n+3)\,(n+2)!, \qquad S_n\frac{1}{(n+2)!} = \frac{1}{(n+3)!} = \frac{1}{(n+3)}\,\frac{1}{(n+2)!},$$

can be modelled by polynomial expressions in the nested P-extension, $(\mathbb{Q}(x)\langle z_1\rangle\langle z_2\rangle, \sigma)$ of $(\mathbb{Q}(x), \sigma)$ constructed in Example 2.3.30.

Let $(\mathbb{A}, \sigma)$ with constant field $\mathbb{K}$ be any of the difference rings in items (1), (2), (3) and (4) and let $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ be the corresponding evaluation function. Then $\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K})$ with $\tau(f) = \langle \mathrm{ev}(f, n) \rangle_{n \geqslant 0}$ is a difference ringhomomorphism by Lemma 2.4.3. $\star$

Chapter 3

Main result for nesting depth 1 product expressions

In this chapter we discuss the main results for nesting depth 1 product expressions, i.e., those expressions in $\mathrm{Prod}(\mathbb{S})$ with only one product quantifier where $\mathbb{S} = \{\mathbb{K}, \mathbb{K}(n), \mathbb{K}(\mathbf{q}^n), \mathbb{K}(n, \mathbf{q}^n)\}$. We shall solve Problem RPE algorithmically by proving the following main result in Theorem 3.1.1. Here the specialization $e = 0$ covers the hypergeometric case. Similarly, taking $\mathbf{q}$-multibasic hypergeometric products in (3.1) and suppressing x yield the multibasic case. Further, setting $e = 1$ provides the q-hypergeometric case. Most of the results presented here and in the subsequent two chapters are from our article (Ocansey and Schneider, 2018, Section 3, 4,and 5).

Theorem 3.1.1.
Let $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)(q_1, \dots, q_e)$ be a rational function field over a field K and consider the mixed $\mathbf{q}$-multibasic hypergeometric products

$$P_1(n) = \prod_{k=\ell_1}^{n} h_1(k, \mathbf{q}^k), \ \ \dots, \ \ P_m(n) = \prod_{k=\ell_m}^{n} h_m(k, \mathbf{q}^k) \ \in \ \mathrm{Prod}(\mathbb{K}(n, \mathbf{q}^n)) \tag{3.1}$$

with $\ell_i \in \mathbb{N}$ and $h_i(x, \mathbf{t}) \in \mathbb{K}(x, \mathbf{t})$ such that $h_i(k, \mathbf{q}^k)$ has no pole and is non-zero for $k \geqslant \ell_i$.
Then there exist irreducible monic polynomials $f_1, \dots, f_s \in \mathbb{K}[x, \mathbf{t}] \setminus \mathbb{K}$, non-negative integers $\ell'_1, \dots, \ell'_s$ and a finite algebraic field extension K' of K with a λ-th root of unity $\zeta \in K'$ and elements $\alpha_1, \dots, \alpha_w \in K'^*$ which are not roots of unity with the following properties.
One can choose natural numbers $\mu_i, \delta_i \in \mathbb{N}$ for $1 \leqslant i \leqslant m$, integers $u_{i,j}$ with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant w$, integers $v_{i,j}$ with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant s$ and rational functions $r_i \in \mathbb{K}(x, \mathbf{t})^*$ for $1 \leqslant i \leqslant m$ such that the following holds:

(1) For all $n \in \mathbb{N}$ with $n \geqslant \delta_i$,

$$P_i(n) = (\zeta^n)^{\mu_i} (\alpha_1^n)^{u_{i,1}} \cdots (\alpha_w^n)^{u_{i,w}} r_i(n, \mathbf{q}^n) \left(\prod_{k=\ell'_1}^{n} f_1(k, \mathbf{q}^k)\right)^{v_{i,1}} \cdots \left(\prod_{k=\ell'_s}^{n} f_s(k, \mathbf{q}^k)\right)^{v_{i,s}}. \tag{3.2}$$

(2) The sequences with entries from the field $\mathbb{K}' = K'(\kappa_1, \dots, \kappa_u)(q_1, \dots, q_e)$,

$$\langle \alpha_1^n \rangle_{n\geqslant 0}, \dots, \langle \alpha_w^n \rangle_{n\geqslant 0}, \langle \prod_{k=\ell'_1}^{n} f_1(k, \mathbf{q}^k) \rangle_{n\geqslant 0}, \dots, \langle \prod_{k=\ell'_s}^{n} f_s(k, \mathbf{q}^k) \rangle_{n\geqslant 0}, \tag{3.3}$$

are among each other algebraically independent over $\tau(\mathbb{K}'(x, \mathbf{t}))[\langle \zeta^n \rangle_{n\geqslant 0}]$; here $\tau : \mathbb{K}'(x, \mathbf{t}) \to \mathcal{S}(\mathbb{K}')$ is a difference ring monomorphism where $\tau(\frac{a}{b}) = \langle \mathrm{ev}\,(\frac{a}{b}, n) \rangle_{n\geqslant 0}$ for $a, b \in \mathbb{K}'[x, \mathbf{t}]$ is defined by (2.2).

If K *is a strongly* σ*-computable field (see Definition 4.1.1 below), then the components in* (3.2) *are computable.* $\diamondsuit$

Namely, Theorem 3.1.1 provides a solution to Problem RPE as follows. Let $P(n) \in \mathrm{ProdE}(\mathbb{K}(n, \mathbf{q}^n))$ be defined as in (2.7) with $S \subseteq \mathbb{Z}^m$ finite, $a_{(\nu_1,\dots,\nu_m)}(n) \in \mathbb{K}(n, \mathbf{q}^n)$ and where the products $P_i(n)$ are given as in (3.1). Now assume that we have computed all the components as stated in Theorem 3.1.1. Then determine[1] $\lambda \in \mathbb{N}$ such that all $a_{(n_1,\dots,n_m)}(n)$ have no pole for $n \geqslant \lambda$, and set $\delta = \max(\lambda, \delta_1, \dots, \delta_m)$. Moreover, replace all P_i with $1 \leqslant i \leqslant m$ by their right-hand sides of (3.2) in the expression $P(n)$ yielding the expression $Q(n) \in \mathrm{ProdE}(\mathbb{K}'(n, \mathbf{q}^n))$. Then by this construction we have

$$P(n) = Q(n) \quad \text{for all } n \geqslant \delta.$$

Furthermore, statement (2) of Theorem 3.1.1 shows statement (2) of Problem RPE.

Finally, we look at the zero-recognition statement which is part (3) of Problem RPE. If $Q = 0$, then $P(n) = 0$ for all $n \geqslant \delta$ by statement (1) of Problem RPE. Conversely, if $P(n) = 0$ for all n from a certain point on, then also $Q(n) = 0$ holds for all n from a certain point on by part statement (1). Since the sequences (3.3) are algebraically independent over $\tau(\mathbb{K}'(x, \mathbf{t}))[\langle \zeta^n \rangle_{n \geqslant 0}]$, the expression $Q(n)$ must be free of these products. Consider the mixed $\mathbf{q}$-multibasic difference field $(\mathbb{K}'(x, \mathbf{t}), \sigma)$ and the A-extension $(\mathbb{K}'(x, \mathbf{t})[\vartheta], \sigma)$ of $(\mathbb{K}'(x, \mathbf{t}), \sigma)$ of order λ with $\sigma(\vartheta) = \zeta\,\vartheta$. By Corollary 5.4.2 below it follows that the mixed $\mathbf{q}$-multibasic difference field $(\mathbb{K}'(x, \mathbf{t}), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{K}', \sigma)$ with $\mathrm{const}(\mathbb{K}', \sigma) = \mathbb{K}'$. Thus by Lemma 2.3.57 it follows that the A-extension is an R-extension. In particular, it follows by Lemma 2.4.3 that the homomorphic extension of τ from $(\mathbb{K}'(x, \mathbf{t}), \sigma)$ to $(\mathbb{K}'(x, \mathbf{t})[\vartheta], \sigma)$ with $\tau(\vartheta) = \langle \zeta^n \rangle_{n \geqslant 0}$ is a $\mathbb{K}'$-embedding. Since $Q(n)$ is a polynomial expression in ζ^n with coefficients from $\mathbb{K}'(n, \mathbf{q}^n)$ (ζ^n comes from (3.2)), we can find an $h(x, \mathbf{t}, \vartheta) \in \mathbb{K}'(x, \mathbf{t})[\vartheta]$ such that the expression $Q(n)$ equals $h(n, \mathbf{q}^n, \zeta^n)$. Further observe that $\tau(h)$ and the produced sequence of $Q(n)$ agree from a certain point on. Thus $\tau(h) = \mathbf{0}$ and since τ is a $\mathbb{K}'$-embedding, $h = 0$. Consequently, $Q(n)$ must be the zero-expression.

We will provide a proof (and an underlying algorithm) for Theorem 3.1.1 by tackling the following sub-problem formulated in the difference ring setting.

Problem RΠ-RC: Construction of RΠ-ring extension $(\mathbb{A}, \sigma)$ of $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$.

Given a mixed $\mathbf{q}$-multibasic difference field $(\mathbb{F}, \sigma)$ with $\mathbb{F} = \mathbb{K}(x)(t_1)\dots(t_e)$ where $\sigma(x) = x + 1$ and $\sigma(t_\ell) = q_\ell\, t_\ell$ for $1 \leqslant \ell \leqslant e$; given $h_1, \dots, h_m \in \mathbb{F}^*$. *Find* an R$\Pi$-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}'(x)(t_1)\dots(t_e), \sigma)$ where $\mathbb{K}'$ is an algebraic field extension of $\mathbb{K}$ and $g_1, \dots, g_m \in \mathbb{A} \setminus \{0\}$ where

$$\sigma(g_i) = \sigma(h_i)\, g_i, \quad \text{for } 1 \leqslant i \leqslant m.$$

Namely, taking the special case $\mathbb{F} = \mathbb{K}(x)$ with $\sigma(x) = x + 1$, we will tackle the above problem in Theorem 5.1.1, and we will derive the general case in Theorem 5.4.6. Then based on the particular choice of the g_i this will lead us directly to Theorem 3.1.1.

We will now give a concrete example of the above strategy for nesting depth 1 hypergeometric products expressions in $\mathrm{ProdE}(\mathbb{K}(n))$. An example for the mixed $\mathbf{q}$-multibasic situation is given in Example 5.5.5.

[1] Note that λ can be determined by using the Z-function defined in Example 2.4.5

Example 3.1.2.
Take the rational function field $\mathbb{K} = \mathsf{K}(\kappa)$ defined over the algebraic number field $\mathsf{K} = \mathbb{Q}\big((\mathbb{i}+\sqrt{3}), \sqrt{-13}\big)$ and take the rational function field $\mathbb{K}(x)$ defined over $\mathbb{K}$. Now consider the hypergeometric product expressions

$$P(n) = \prod_{k=1}^{n} h_1(k) + \prod_{k=1}^{n} h_2(k) + \prod_{k=1}^{n} h_3(k) \in \mathrm{ProdE}(\mathbb{K}(n)) \tag{3.4}$$

with

$$h_1(x) = \frac{-13\sqrt{-13}\,\kappa}{x},\ h_2(x) = \frac{-784\,(\kappa+1)^2\,x}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^4\,\kappa\,(x+2)^2},\ h_3(x) = \frac{-17210368\,(\kappa+1)^5\,x}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^{10}\,\kappa\,(x+2)^5} \tag{3.5}$$

where $h_1, h_2, h_3 \in \mathbb{K}(x)$. With our algorithm (see Theorem 5.4.6 below) we construct the algebraic field extension $\mathsf{K}' = \mathbb{Q}((-1)^{\frac{1}{6}}, \sqrt{13})$ of K, take the rational function field $\mathbb{K}' = \mathsf{K}'(\kappa)$ and define on top the rational difference field $(\mathbb{K}'(x), \sigma)$ with $\sigma(x) = x+1$. Based on this, we obtain the RΠ-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}'(x), \sigma)$ with

$$\mathbb{A} = \mathbb{K}'(x)[\vartheta][y_1, {y_1}^{-1}][y_2, {y_2}^{-1}][y_3, {y_3}^{-1}][y_4, {y_4}^{-1}][z, z^{-1}] \tag{3.6}$$

and the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\begin{aligned} &\sigma(\vartheta) = (-1)^{\frac{1}{6}}\,\vartheta, && \sigma(y_1) = \sqrt{13}\,y_1, && \sigma(y_2) = 7\,y_2,\\ &\sigma(y_3) = \kappa\,y_3, && \sigma(y_4) = (\kappa+1)\,y_4, && \sigma(z) = (x+1)\,z; \end{aligned}$$

note that $\mathrm{const}(\mathbb{A}, \sigma) = \mathbb{K}'$. Now consider the difference ring homomorphism $\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K}')$ which we define as follows. For the base field $(\mathbb{K}'(x), \sigma)$ we take the difference ring embedding $\tau(\frac{a}{b}) = \langle \mathrm{ev}\left(\frac{a}{b}, n\right)\rangle_{n\geqslant 0}$ for $a, b \in \mathbb{K}'[x]$ where ev is defined in (2.1). Further, applying iteratively part (1) of Lemma 2.4.3 we obtain the difference ring homomorphism $\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K}')$ determined by

$$\begin{aligned} &\tau(\vartheta) = \langle\big((-1)^{\frac{1}{6}}\big)^n\rangle_{n\geqslant 0}, && \tau(y_1) = \langle\big(\sqrt{13}\big)^n\rangle_{n\geqslant 0}, && \tau(y_2) = \langle 7^n\rangle_{n\geqslant 0},\\ &\tau(y_3) = \langle\kappa^n\rangle_{n\geqslant 0}, && \tau(y_4) = \langle(\kappa+1)^n\rangle_{n\geqslant 0}, && \tau(z) = \langle n!\rangle_{n\geqslant 0}. \end{aligned}$$

In addition, since $(\mathbb{A}, \sigma)$ is an RΠ-extension of $(\mathbb{K}'(x), \sigma)$, it follows by part (3) of Lemma 2.4.3 that τ is a $\mathbb{K}'$-embedding. This implies that

$$\tau(\mathbb{K}'(x))[\tau(\vartheta)][\tau(y_1), \tau({y_1}^{-1})][\tau(y_2), \tau({y_2}^{-1})][\tau(y_3), \tau({y_3}^{-1})][\tau(y_4), \tau({y_4}^{-1})][\tau(z), \tau(z^{-1})]$$

is a Laurent polynomial ring over the ring $\tau(\mathbb{K}'(x))[\tau(\vartheta)]$. Further, we find

$$Q' = \underbrace{\frac{\vartheta^9\,y_1^3\,y_3}{z}}_{=:\,g_1} + 4\,\underbrace{\frac{\vartheta^{11}\,y_2^2\,y_4^2}{(x+1)^2\,(x+2)^2\,y_1^3\,y_3\,z}}_{=:\,g_2} + 32\,\underbrace{\frac{\vartheta^5\,y_2^5\,y_4^5}{(x+1)^5\,(x+2)^5\,y_1^3\,y_3\,z^4}}_{=:\,g_3} \in \mathbb{A} \tag{3.7}$$

where

$$\sigma(g_i) = \sigma(h_i)\,g_i,$$

for $i = 1, 2, 3$. Thus the g_i model the shift behaviours of the hypergeometric products with the multiplicands $h_i \in \mathbb{K}(x)$. In particular, we have defined Q' such that

$$\tau(Q') = \langle P(n)\rangle_{n\geqslant 0}$$

holds. Rephrasing

$$x \leftrightarrow n, \quad \vartheta \leftrightarrow \big((-1)^{\frac{1}{6}}\big)^n, \quad y_1 \leftrightarrow \big(\sqrt{13}\big)^n, \quad y_2 \leftrightarrow 7^n, \quad y_3 \leftrightarrow \kappa^n, \quad y_4 \leftrightarrow \big(\kappa+1\big)^n, \quad z \leftrightarrow n!$$

in (3.7) we get

$$Q(n) = \frac{\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^9 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n}{n!} + 4\,\frac{\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^{11} (7^n)^2 \left((\kappa+1)^n\right)^2}{(n+1)^2 (n+2)^2 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n\, n!} + 32\,\frac{\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^5 (7^n)^5 \left((\kappa+1)^n\right)^5}{(n+1)^5 (n+2)^5 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n\, (n!)^4} \tag{3.8}$$

in $\mathrm{ProdE}(\mathbb{K}'(n))$. Note: $n!$ and a^n with $a \in \mathbb{K}'^*$ are just shortcuts for $\prod_{k=1}^n k$ and $\prod_{k=1}^n a$, respectively. Based on the corresponding proof of Theorem 3.1.1 at the end of Section 5.4 we can ensure that

$$P(n) = Q(n)$$

holds for all $n \in \mathbb{N}$ with $n \geqslant 1$. Further details on the computation steps can be found in Examples 5.4.7 and 5.4.8 below. ⋆

Chapter 4

Algorithmic preliminaries: strongly σ-computable fields

In Karr's algorithm (Karr, 1981) and all the improvements (Schneider, 2008, 2015, 2007a, 2016, 2017; Abramov and Petkovšek, 2010; Kauers and Schneider, 2006) one relies on certain algorithmic properties of the constant field $\mathbb{K}$. Among those, one needs to solve the following problem.

Problem GO for $\alpha_1, \dots, \alpha_w \in K^*$

Given a field K and $\alpha_1, \dots, \alpha_w \in K^*$. *Compute* a basis of the submodule

$$\mathbb{V} := \Big\{(u_1, \dots, u_w) \in \mathbb{Z}^w \,\Big|\, \prod_{i=1}^{w} {\alpha_i}^{u_i} = 1\Big\} \text{ of } \mathbb{Z}^w \text{ over } \mathbb{Z}.$$

Note that Problem GO is a generalisation of Problem O. In particular, if $w = 1$, then Problem GO reduces to computing the order of α in K^* (see Definition 2.3.16) which is precisely Problem O stated as follows.

Problem O for $\alpha \in K^*$

Given a field K and $\alpha \in K^*$. *Find* $\operatorname{ord}(\alpha)$.

In Schneider (2005) it has been worked out that Problem GO is solvable in any rational function field $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ provided that one can solve Problem GO in K and that one can factor multivariate polynomials over K. In this thesis we require the following stronger assumption: Problem GO can be solved not only in K (K with this property was called σ-computable in Schneider (2005); Kauers and Schneider (2006)) but also in any algebraic extension of it. So far, the class of σ-computable fields treated in the Mathematica package `Sigma` (Schneider, 2007b, 2013) are quotient field of the unique factorisation domains. Among them $\mathbb{K}$ can be a rational function field over the rational numbers $\mathbb{Q}$. Moreover, as suggested in Schneider (2005, page 89), an implementation of the algorithms in Ge (1993a,b) to extend the computational capabilities `Sigma` will be an important contribution.

Definition 4.1.1.
A field K is strongly σ-computable if the standard operations in K can be performed, multivariate polynomials can be factored over K and Problem GO can be solved for K and any finite algebraic field extension of K. $*$

Note that Ge's algorithm (Ge, 1993a) or (Kauers, 2005, Algorithm 7.16, page 84), solves Problem GO over an arbitrary number field K. Since any finite algebraic extension of an algebraic number field is again an algebraic number field, it follows with Ge's algorithm, that any number field K with a finite number of generators is σ-computable.

Summarizing, we can turn our theoretical results to algorithmic versions, if we assume that $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ is a rational function field over a field K which is strongly σ-computable. In particular, the underlying algorithms are implemented in the Mathematica package `NestedProducts` for the case that K is a finite algebraic field extension of $\mathbb{Q}$.

Besides these fundamental properties of the constant field, we rely on further (algorithmic) properties that can be ensured by difference ring theory. Let $(\mathbb{F}[t], \sigma)$ be a difference ring over the field $\mathbb{F}$ with t transcendental over $\mathbb{F}$ and $\sigma(t) = \alpha\, t + \beta$ where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Note that for any $h \in \mathbb{F}[t]$ and any $k \in \mathbb{Z}$ we have $\sigma^k(h) \in \mathbb{F}[t]$. Furthermore, if h is irreducible, then also $\sigma^k(h)$ is irreducible.

Definition 4.1.2.
Two polynomials $f, h \in \mathbb{F}[t] \setminus \{0\}$ are said to be *shift co-prime*, also denoted by $\gcd_\sigma(f, h) = 1$, if for all $k \in \mathbb{Z}$ we have that $\gcd(f, \sigma^k(h)) = 1$. Furthermore, we say that f and h are *shift-equivalent*, denoted by $f \sim_\sigma h$, if there is a $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{F}$. If there is no such k, then we also write $f \nsim_\sigma h$. ∗

It is immediate that $\sim_\sigma$ is an equivalence relation.

Definition 4.1.3.
Let $(\mathbb{F}[t], \sigma)$ be a difference ring over $\mathbb{F}$ with $\sigma(t) = \alpha\, t+\beta$ where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Let $\mathcal{D} = \{f_1, \dots, f_e\}$ be a finite set of polynomials in $\mathbb{F}[t]$ for some $e \in \mathbb{N}$ that are shift equivalent among each other. We call $f_i \in \mathcal{D}$ for some i with $i \in \{1, 2, \dots, e\}$ the *leftmost polynomial* in $\mathcal{D}$ if there is a non-negative integer $m \geqslant 0$ such that for all $h \in \mathcal{D}$, $\sigma^m(f_i) = h$. ∗

In the following we will focus mainly on irreducible polynomials $f, h \in \mathbb{F}[t]$. Then observe that $f \sim_\sigma h$ holds if and only if $\gcd_\sigma(f, h) \neq 1$ holds. In the following it will be important to determine such a k. Here we utilize the following property of ΠΣ-extensions whose proof can be found in Karr (1981, Theorem 4), Bronstein (2000, Corollary 1,2) or Schneider (2001, Theorem 2.2.4).

Lemma 4.1.4.
Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ and $f \in \mathbb{F}(t)^*$. Then $\frac{\sigma^k(f)}{f} \in \mathbb{F}$ for some $k \neq 0$ iff $\frac{\sigma(t)}{t} \in \mathbb{F}$ and $f = u\, t^m$ with $u \in \mathbb{F}^*$ and $m \in \mathbb{Z}$. ♦

Namely, using this result one can deduce when such a k is unique.

Lemma 4.1.5.
Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ for $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Let $f, h \in \mathbb{F}[t]$ be irreducible with $f \sim_\sigma h$. Then there is a unique $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{F}^*$ iff $\frac{\sigma(t)}{t} \notin \mathbb{F}$ or $f \neq a\, t$ or $h \neq b\, t$ for some $a, b \in \mathbb{F}^*$. ♦

PROOF:
" $\Longrightarrow$ " Suppose that $\frac{\sigma(t)}{t} \in \mathbb{F}$, (i.e., $\beta = 0$) and $f = a\,t$ and $h = b\,t$ for some $a, b \in \mathbb{F}^*$. Then $\frac{\sigma^k(f)}{h} \in \mathbb{F}^*$ for all $k \in \mathbb{Z}$ and thus k is not unique.
" $\Longleftarrow$ " Conversely, suppose that k is not unique. Then there are $k_1, k_2 \in \mathbb{Z}$ with $k_1 > k_2$ such that $\sigma^{k_1}(f) = u\,h$ and $\sigma^{k_2}(f) = v\,h$. Then $\frac{\sigma^{k_1-k_2}(f)}{f} = \frac{u}{v} \in \mathbb{F}^*$. Thus by Lemma 4.1.4, $\frac{\sigma(t)}{t} \in \mathbb{F}$ and $f = a\,t$ for some $a \in \mathbb{F}^*$. Thus also $h = b\,t$ for some $b \in \mathbb{F}^*$. ∎

Consider the rational difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x + 1$. Note that x is a Σ-monomial. Let $f, h \in \mathbb{K}[x] \setminus \mathbb{K}$ be irreducible polynomials. If $f \sim_\sigma h$, then there is a unique $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{K}$. Similarly for the mixed $\mathbf{q}$-multibasic difference field $(\mathbb{K}(x)(t_1)\dots(t_e), \sigma)$ with $\sigma(x) = x + 1$ and $\sigma(t_i) = q_i\,t_i$ for $1 \leqslant i \leqslant e$ we note that the t_i are Π-monomials; see Corollary 5.4.2 below. For $1 \leqslant i \leqslant e$ and $\mathbb{E} = \mathbb{K}(x)(t_1)\dots(t_{i-1})$, let $f, h \in \mathbb{E}[t_i]$ be monic irreducible polynomials. If $f \sim_\sigma h$, then there is a unique $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{E}$ if and only if $f \neq t_i \neq h$. In both cases, such a unique k can be computed if one can perform the usual operations in $\mathbb{K}$; see Karr (1981) or Kauers and Schneider (2006, Theorem 1). Optimized algorithms for these cases can be found in Bauer and Petkovšek (1999, Section 3). In addition, the function Z given in (2.50) or in (2.51) can be computed due to Bauer and Petkovšek (1999). Summarizing, the following properties hold.

Lemma 4.1.6.
Let $(\mathbb{F}, \sigma)$ be the rational or mixed q-multibasic difference field over $\mathbb{K}$ as defined in Examples 2.3.6 and 2.3.10. Suppose that the usual operations[1] *in $\mathbb{K}$ are computable. Then one compute*

(1) the Z-functions given in (2.50) or in (2.51);

(2) one can compute for shift-equivalent irreducible polynomials f, h in $\mathbb{K}[x]$ (or in $\mathbb{K}(x)(t_1, \dots, t_{i-1})[t_i]$) a $k \in \mathbb{Z}$ with $\frac{\sigma^k(f)}{h} \in \mathbb{K}$ (or $\frac{\sigma^k(f)}{h} \in \mathbb{K}(x)(t_1, \dots, t_{i-1})$). ♦

For further considerations, we introduce the following Lemma which gives a relation between two polynomials that are shift-equivalent. This proof is a corrected version of the proof of the corresponding Lemma 4.4 in Ocansey and Schneider (2018).

Lemma 4.1.7.
Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ with t transcendental over $\mathbb{F}$ and $\sigma(t) = \alpha\,t + \beta$ ($\alpha \in \mathbb{F}^*$ and $\beta = 0$ or $\alpha = 1$ and $\beta \in \mathbb{F}$). Let $f, h \in \mathbb{F}[t] \setminus \mathbb{F}$ be monic and $f \sim_\sigma h$. Then there is a $g \in \mathbb{F}(t)^*$ with

$$h = \frac{\sigma(g)}{g}\,f. \tag{4.1}$$

If f and h are in addition irreducible, all the irreducible factors in g with the exception of t if $\beta = 0$, are shift equivalent to f (resp. h). ♦

[1]This is the case if $\mathbb{K}$ is strongly σ-computable, or if $\mathbb{K}$ is a rational function field over a strongly σ-computable field.

PROOF:
Since $f \sim_\sigma h$, there is a $k \in \mathbb{Z}$ and $u \in \mathbb{F}^*$ with $\sigma^k(f) = u\,h$. In particular $u = 1$, if $\alpha = 1$ and $\beta \in \mathbb{F}^*$. Note that $\deg(f) = \deg(h) = m$. If $\beta = 0$, then by comparing coefficients of the leading terms and using the fact that f, h are monic, we get $u\,t^m = \sigma^k(t^m)$. If $k \geqslant 0$ and $\beta = 0$, set

$$g := \prod_{i=0}^{k-1} \sigma^i(t^{-m}\, f).$$

Then

$$\frac{\sigma(g)}{g} = \frac{\sigma^k(t^{-m}f)}{t^{-m}f} = \frac{\sigma^k(f)\, t^m}{f\,\sigma^k(t^m)} = \frac{h\,u\,t^m}{f\,\sigma^k(t^m)} = \frac{h}{f}.$$

If $k \geqslant 0$ and $\alpha = 1$ and $\beta \in \mathbb{F}^*$, set

$$g := \prod_{i=0}^{k-1} \sigma^i(f).$$

Then

$$\frac{\sigma(g)}{g} = \frac{\sigma^k(f)}{f} = \frac{h}{f}.$$

Thus (4.1) holds in both cases. On the other hand, if $k < 0$ and $\beta = 0$, set

$$g := \prod_{i=1}^{-k} \sigma^{-i}\left(\frac{t^m}{f}\right).$$

Then

$$\frac{\sigma(g)}{g} = \frac{t^m\, f^{-1}}{\sigma^k(t^m\, f^{-1})} = \frac{t^m\, \sigma^k(f)}{\sigma^k(t^m)\, f} = \frac{h\,u\,t^m}{f\,\sigma^k(t^m)} = \frac{h}{f}.$$

If $k < 0$ and $\alpha = 1$ and $\beta \in \mathbb{F}^*$, set

$$g := \prod_{i=1}^{-k} \sigma^{-i}\left(\frac{1}{f}\right).$$

Then

$$\frac{\sigma(g)}{g} = \frac{f^{-1}}{\sigma^k(f^{-1})} = \frac{\sigma^k(f)}{f} = \frac{h}{f}.$$

Hence again (4.1) holds which completes the proof. ∎

Chapter 5

Algorithmic construction of RΠ-extensions for nesting depth 1 expressions in $\mathrm{ProdE}(\mathbb{K}(n, q^n))$.

In this chapter we will provide a proof for Theorem 3.1.1 for the case $\mathrm{ProdE}(\mathbb{K}(n))$ which is also discussed briefly in Ocansey and Schneider (2017). Afterwards, this proof strategy will be generalized for the case $\mathrm{ProdE}(\mathbb{K}(n, q^n))$ in Section 5.5; compare Ocansey and Schneider (2018). In both cases, we will need the following set from Karr (1981, Definition 21).

Definition 5.0.1.
For a difference field $(\mathbb{F}, \sigma)$ and $\mathbf{f} = (f_1, \dots, f_s) \in (\mathbb{F}^*)^s$ we define

$$\mathcal{M}(\mathbf{f}, \mathbb{F}) := \left\{(\nu_1, \dots, \nu_s) \in \mathbb{Z}^s \mid \tfrac{\sigma(g)}{g} = f_1^{\nu_1} \cdots f_s^{\nu_s} \text{ for some } g \in \mathbb{F}^*\right\}. \tag{5.1}$$

$*$

Note that $\mathcal{M}(\mathbf{f}, \mathbb{F})$ is a $\mathbb{Z}$-submodule of $\mathbb{Z}^s$ which has finite rank. We observe further that for the special case $\mathrm{const}(\mathbb{A}, \sigma) = \mathbb{A}$, we have that $\frac{\sigma(g)}{g} = 1$ for all $g \in \mathbb{A}^*$. Thus

$$\mathcal{M}(\mathbf{f}, \mathbb{A}) := \{(\nu_1, \dots, \nu_s) \in \mathbb{Z}^s \mid f_1^{\nu_1} \cdots f_s^{\nu_s} = 1\}$$

which is nothing else but the set in Problem GO.
Finally, we will heavily rely on the following Lemma that ensures if a P-extension forms a Π-extension. This result is also related to ideas from Hardouin and Singer (2008).

Lemma 5.0.2.
Let $(\mathbb{F}, \sigma)$ be a difference field and let $\mathbf{f} = (f_1, \dots, f_s) \in (\mathbb{F}^*)^s$. Then the following statements are equivalent.

(1) There are no $(\nu_1, \dots, \nu_s) \in \mathbb{Z}^s \setminus \{0_s\}$ and $g \in \mathbb{F}^*$ with (5.25), i.e., $\mathcal{M}(\mathbf{f}, \mathbb{F}) = \{0_s\}$.

(2) One can construct a Π-field extension $(\mathbb{F}(z_1) \dots (z_s), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(z_i) = f_i\, z_i$, for $1 \leqslant i \leqslant s$.

(3) One can construct a Π-extension $(\mathbb{F}\langle z_1 \rangle \dots \langle z_s \rangle, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(z_i) = f_i\, z_i$, for $1 \leqslant i \leqslant s$. ♦

PROOF:
(1) $\iff$ (2) is established by Schneider (2010a, Theorem 9.1). (2) $\Longrightarrow$ (3) is obvious while (3) $\Longrightarrow$ (2) follows by iterative application of Schneider (2017, Corollary 2.6). ■

Throughout this chapter, let $(\mathbb{K}(x), \sigma)$ be the rational difference field over a constant field $\mathbb{K}$, where $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ is a rational function field over a field K. For algorithmic reasons we will assume in addition that K is strongly σ-computable (see Definition 4.1.1). In Section 5.1 we will treat Theorem 3.1.1 first for the special case $\mathrm{ProdE}(K)$. Next, we treat the case $\mathrm{ProdE}(\mathbb{K})$ in Section 5.2. In Section 5.3 we present simple criteria to check if a tower of Π-monomials t_i with $\frac{\sigma(t_i)}{t_i} \in \mathbb{K}[x]$ forms a Π-extension. Finally, in Section 5.4 we will utilize this extra knowledge to construct Π-extensions for the full case $\mathrm{ProdE}(\mathbb{K}(n))$.

5.1 Construction of $R\Pi$-extensions for nesting depth 1 expressions in $\mathrm{ProdE}(K)$

Our construction is based on the following theorem.

Theorem 5.1.1.
Let $\gamma_1, \dots, \gamma_s \in K^*$. Then there is an algebraic field extension K' of K together with a λ-th root of unity $\zeta \in K'$ and elements $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_w) \in K'^w$ with $\mathcal{M}(\boldsymbol{\alpha}, K') = \{0_w\}$ such that for all $i = 1, \dots, s$,

$$\gamma_i = \zeta^{\mu_i}\, \alpha_1^{u_{i,1}} \cdots \alpha_w^{u_{i,w}} \tag{5.2}$$

holds for some $1 \leqslant \mu_i \leqslant \lambda$ and $(u_{i,1}, \dots, u_{i,w}) \in \mathbb{Z}^w$.
If K is strongly σ-computable, then ζ, the α_i and the $\mu_i, u_{i,j}$ can be computed. ♢

PROOF:
We prove the Theorem by induction on s. The base case $s = 0$ obviously holds. Now assume that there are a λ-th root of unity ζ, elements $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_w) \in (K'^*)^w$ with $\mathcal{M}(\boldsymbol{\alpha}, K') = \{0_w\}$, $1 \leqslant \mu_i \leqslant \lambda$ and $(\nu_{i,1}, \dots \nu_{i,w}) \in \mathbb{Z}^w$ such that

$$\gamma_i = \zeta^{\mu_i}\, \alpha_1^{\nu_{i,1}} \cdots \alpha_w^{\nu_{i,w}}$$

holds for all $1 \leqslant i \leqslant s-1$.

Now consider in addition $\gamma_s \in K^*$. First suppose the case $\mathcal{M}((\alpha_1, \dots, \alpha_w, \gamma_s), K') = \{0_{w+1}\}$. With $\alpha_{w+1} := \gamma_s$, we can write γ_s as

$$\gamma_s = \zeta^{\lambda}\, \alpha_1^{\upsilon_1} \cdots \alpha_w^{\upsilon_w}\, \alpha_{w+1}$$

with $\lambda = \upsilon_1 = \cdots = \upsilon_w = 0$. Further, with $\nu_{i,w+1} = 0$, we can write

$$\gamma_i = \zeta^{\mu_i}\, \alpha_1^{\nu_{i,1}} \cdots \alpha_w^{\nu_{i,w}}\, \alpha_{w+1}^{\nu_{i,w+1}}$$

for all $1 \leqslant i \leqslant s-1$. This completes the proof for this case.
Otherwise, suppose that the $\mathbb{Z}$-module $\mathcal{M}((\alpha_1, \dots, \alpha_w, \gamma_s), K') \neq \{0_{w+1}\}$ and take a non-zero integer vector $(\upsilon_1, \dots, \upsilon_w, u_s) \in \mathcal{M}((\alpha_1, \dots, \alpha_w, \gamma_s), K') \setminus \{0_{w+1}\}$. Note that $u_s \neq 0$ since $\mathcal{M}(\boldsymbol{\alpha}, K') = \{0_w\}$. Then take all the non-zero integers in $(\upsilon_1, \dots, \upsilon_w, u_s)$ and define δ to be their least common multiple. Define $\tilde{\alpha}_j := \alpha_j^{1/|u_s|} \in K''$ for $1 \leqslant j \leqslant w$ where K'' is some algebraic field extension of K' and let

$\lambda' = \operatorname{lcm}(\delta, \lambda)$. Take a primitive λ'-th root of unity $\zeta' := e^{\frac{2\pi i}{\lambda'}}$. Then we can express γ_s in terms of $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_w$ by

$$\gamma_s = (\zeta')^{\nu_s} \prod_{j=1}^{w} \alpha_j^{-\frac{v_j}{u_s}} = (\zeta')^{\nu_s} \prod_{j=1}^{w} (\tilde{\alpha}_j)^{-v_j \cdot \operatorname{sign}(u_s)} \tag{5.3}$$

with $1 \leqslant \nu_s \leqslant \lambda'$. Note that for each $j, -v_j \cdot \operatorname{sign}(u_s) \in \mathbb{Z}$. Thus we have been able to represent γ_s as a power product of ζ' and elements $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_w) \in (K''^*)^w$ which are not roots of unity. Consequently, we can write

$$\gamma_i = (\zeta')^{\mu_i}\, \tilde{\alpha}_1^{u_{i,1}} \cdots \tilde{\alpha}_w^{u_{i,w}}$$

for $1 \leqslant i \leqslant s-1$, where $u_{i,j} = |u_s| v_{i,j}$ for $1 \leqslant j \leqslant w$ and $1 \leqslant \mu_i \leqslant \lambda'$. Now suppose that $\mathcal{M}(\tilde{\boldsymbol{\alpha}}, K'') \neq \{0_w\}$. Then there is a $(m_1, \ldots, m_w) \in \mathbb{Z}^w \setminus \{0_w\}$ such that

$$1 = \prod_{j=1}^{w} (\tilde{\alpha}_j)^{m_j} = \prod_{j=1}^{w} \left(\alpha_j^{\frac{1}{|u_s|}}\right)^{m_j} \Longrightarrow \prod_{j=1}^{w} \left(\alpha_j^{\frac{1}{|u_s|}}\right)^{|u_s|\, m_j} = 1^{|u_s|} \iff \prod_{j=1}^{w} \alpha_j^{m_j} = 1$$

with $(m_1, \ldots, m_w) \neq 0_w$; contradicting the assumption that $\mathcal{M}(\boldsymbol{\alpha}, K') = \{0_w\}$ holds. Consequently, $\mathcal{M}(\tilde{\boldsymbol{\alpha}}, K'') = \{0_w\}$ which completes the induction step.

Suppose that K is strongly σ-computable. Then one can decide if $\mathcal{M}(\boldsymbol{\alpha}, K')$ is the zero-module, and if not one can compute a non-zero integer vector. All other operations in the proof rely on basic operations that can be carried out. ■

Remark 5.1.2.
In the trivial case, ζ might simply be 1 in Theorem 5.1.1. In this case, it is redundant and we can even exclude it. ♠

Remark 5.1.3.
Let $\gamma_1, \ldots, \gamma_s \in K^*$ and suppose that the ingredients ζ, $\alpha_1, \ldots, \alpha_w$ and the μ_i and $u_{i,j}$ are given as stated in Theorem 5.1.1. Let $n \in \mathbb{N}$. Then by (5.2) we have that

$$\gamma_i^n = \prod_{k=1}^{n} \gamma_i = \prod_{k=1}^{n} \zeta^{\mu_i} \prod_{k=1}^{n} \alpha_1^{u_{i,1}} \cdots \prod_{k=1}^{n} \alpha_w^{u_{i,w}} = (\zeta^n)^{\mu_i} \left(\alpha_1^n\right)^{u_{i,1}} \cdots \left(\alpha_w^n\right)^{u_{i,w}}.$$

The following remarks are relevant.

(1) Since $\mathcal{M}(\boldsymbol{\alpha}, \tilde{K}) = \{0_w\}$, we know that there are no $g \in \tilde{K}^*$, and $(u_1, \ldots, u_w) \in \mathbb{Z}^w \setminus \{0_w\}$ with

$$1 = \frac{\sigma(g)}{g} = \alpha_1^{u_1} \cdots \alpha_w^{u_w}.$$

In short we say that $\alpha_1, \ldots, \alpha_w$ satisfy no integer relation. Thus it follows by Lemma 5.0.2 that there is a Π-extension $(\mathbb{E}, \sigma)$ of $(\tilde{K}, \sigma)$ with $\mathbb{E} = \tilde{K}[y_1, y_1^{-1}] \ldots [y_w, y_w^{-1}]$ and $\sigma(y_j) = \alpha_j\, y_j$ for $j = 1, \ldots, w$.

(2) Consider the A-extension $(\mathbb{E}[\vartheta], \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(\vartheta) = \zeta\,\vartheta$ of order λ. By Lemma 2.3.57 this is an R-extension. (Take the quotient field of $\mathbb{E}$, apply Lemma 2.3.57, and then take the corresponding subring.)

(3) Summarizing, the products $\gamma_1^n, \dots, \gamma_s^n$ can be rephrased in the RΠ-extension $(\tilde{K}\langle y_1\rangle \dots \langle y_w\rangle \langle \vartheta\rangle, \sigma)$ of $(\tilde{K}, \sigma)$. Namely, we can represent α_j^n by y_j and ζ^n by ϑ.

(4) If $K = \mathbb{Q}$ (or if K is the quotient field of a certain unique factorization domain like a rational function field over $\mathbb{Q}$), then this result can be obtained without any extension, i.e., $K = \tilde{K}$; see Schneider (2005). ♠

More precisely, we have the following Lemma.

Lemma 5.1.4.
Let K *be a number field and* (K, σ) *be the difference field with* $\sigma(c) = c$ *for all* $c \in K$. *Let* $(K\langle y_1\rangle \dots \langle y_s\rangle, \sigma)$ *be a* P*-extension of* (K, σ) *with* $\sigma(y_i) = \gamma_i\, y_i$ *where* $\gamma_i \in K^*$. *Let* $\mathrm{ev} : K\langle y_1\rangle \dots \langle y_s\rangle \times \mathbb{N} \to K$ *be the evaluation function defined as* $\mathrm{ev}(y_i, n) = \gamma_i^n$ *for* $1 \leqslant i \leqslant s$. *Then the following statements hold:*

(1) One can construct an RΠ*-extension* $(\tilde{K}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle, \sigma)$ *of* $(\tilde{K}, \sigma)$ *with*

$$\sigma(\vartheta) = \zeta\,\vartheta \qquad \textit{and} \qquad \sigma(\tilde{y}_k) = \alpha_k\, \tilde{y}_k \quad \textit{for} \quad 1 \leqslant k \leqslant w \tag{5.4}$$

where $\zeta \in \tilde{K}$ *is a primitive* λ*-th root of unity and an evaluation function* $\tilde{\mathrm{ev}} : \tilde{K}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle \times \mathbb{N} \to \tilde{K}$ *defined as*

$$\tilde{\mathrm{ev}}(\vartheta, n) = \zeta^n \qquad \textit{and} \qquad \tilde{\mathrm{ev}}(\tilde{y}_k, n) = \alpha_k^n. \tag{5.5}$$

(2) One can construct a difference ring homomorphism $\varphi : K\langle y_1\rangle \dots \langle y_s\rangle \to \tilde{K}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle$ *with*

$$\varphi(y_i) = \vartheta^{\mu_i}\, \tilde{y}_1^{\nu_{i,1}} \cdots \tilde{y}_w^{\nu_{i,w}} \tag{5.6}$$

where $\nu_{i,j} \in \mathbb{Z}$ *and* $0 \leqslant \mu_i < \lambda$ *such that for all* $f \in K\langle y_1\rangle \dots \langle y_s\rangle$ *and for all* $n \in \mathbb{N}$, $\mathrm{ev}(f, n) = \tilde{\mathrm{ev}}(\varphi(f), n)$ *holds.* ♦

PROOF:

(1) Given $\gamma_1, \dots, \gamma_s \in K^*$, it follows by Theorem 5.1.1 that there is a finite algebraic field extension $\tilde{K}$ of K together with a λ-th root of unity $\zeta \in \tilde{K}$ and $\alpha_1, \dots, \alpha_w \in \tilde{K}$ such that $\mathcal{M}((\alpha_1, \dots, \alpha_w), \tilde{K}) = \{0_w\}$ and (5.2) holds. Let $(\tilde{K}, \sigma)$ be a difference field with $\sigma(c) = c$ for all $c \in \tilde{K}$ and let $(\tilde{K}\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle, \sigma)$ be a P-extension of $(\tilde{K}, \sigma)$ with the automorphism (5.4) and evaluation function (5.5). Since $\mathcal{M}((\alpha_1, \dots, \alpha_w), \tilde{K}) = \{0_w\}$, it follows by Lemma 5.0.2 that $(\tilde{K}\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle, \sigma)$ is a Π-extension of $(\tilde{K}, \sigma)$. Consider the A-extension $(\mathbb{E}\langle\vartheta\rangle, \sigma)$ of $(\tilde{K}, \sigma)$ where $\mathbb{E} = \tilde{K}\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle$ with $\sigma(\vartheta) = \zeta\vartheta$ of order λ. By Lemma 2.3.57 this is an R-extension [1]. Note that, one can rearrange the generators in $\mathbb{E}\langle\vartheta\rangle = \tilde{K}\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle\langle\vartheta\rangle$ to get the RΠ-extension $(\tilde{K}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle, \sigma)$ of $(\tilde{K}, \sigma)$.

(2) Consider the uniquely determined ring homomorphism $\varphi : K\langle y_1\rangle \dots \langle y_s\rangle \to \tilde{K}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle$ subject to (5.6). Let $f \in K\langle y_1\rangle \dots \langle y_s\rangle$ with $f := a\, y_1^{u_1} \cdots y_s^{u_s}$. We show that φ is a difference ring homomorphism, i.e.,

$$\sigma(\varphi(f)) = \varphi(\sigma(f)) \tag{5.7}$$

[1] Note that if $\zeta = 1$, then we can consider only the Π-extension $(\mathbb{E}, \sigma)$ of $(\tilde{K}, \sigma)$. Otherwise, the construction of the single R-extension $(\mathbb{E}\langle\vartheta\rangle, \sigma)$ of $(\mathbb{E}, \sigma)$ is a redundant and causes no harm.

holds. For the left hand side of (5.7) we have:

$$\begin{aligned}\sigma(\varphi(a\,y_1^{u_1}\cdots y_s^{u_s})) &= \sigma\big(a\,(\vartheta^{\upsilon_1}\,\tilde{y}_1^{r_{1,1}}\cdots\tilde{y}_w^{r_{1,w}})\cdots(\vartheta^{\upsilon_s}\,\tilde{y}_1^{r_{s,1}}\cdots\tilde{y}_w^{r_{s,w}})\big)\\ &= a\,\big((\zeta\vartheta)^{\upsilon_1}\,(\alpha_1\,\tilde{y}_1)^{r_{1,1}}\cdots(\alpha_w\,\tilde{y}_w)^{r_{1,w}}\big)\cdots\big((\zeta\vartheta)^{\upsilon_s}\,(\alpha_1\,\tilde{y}_1)^{r_{s,1}}\cdots(\alpha_w\,\tilde{y}_w)^{r_{s,w}}\big).\end{aligned}$$

where $\upsilon_i = \mu_i\,u_i$ and $r_{i,j} = u_i\,v_{i,j}$ for $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant w$. Using (5.2), the right hand side of (5.7) is:

$$\begin{aligned}\varphi(\sigma(a\,y_1^{u_1}\cdots y_s^{u_s})) = \varphi\big(a\,(\gamma_1\,y_1)^{u_1}\cdots(\gamma_s\,y_s)^{u_s}\big) = a\,\big((\gamma_1\,\varphi(y_1))^{u_1}\cdots(\gamma_s\,\varphi(y_s))^{u_s}\big) =\\ a\,\big((\zeta^{\upsilon_1}\alpha_1^{r_{1,1}}\cdots\alpha_w^{r_{1,w}})(\vartheta^{\upsilon_1}\tilde{y}_1^{r_{1,1}}\cdots\tilde{y}_w^{r_{1,w}})\big)\cdots\big((\zeta^{\upsilon_s}\alpha_1^{r_{s,1}}\cdots\alpha_w^{r_{s,w}})(\vartheta^{\upsilon_s}\tilde{y}_1^{r_{s,1}}\cdots\tilde{y}_w^{r_{s,w}})\big) =\\ a\,\big((\zeta\vartheta)^{\upsilon_1}\,(\alpha_1\,\tilde{y}_1)^{r_{1,1}}\cdots(\alpha_w\,\tilde{y}_w)^{r_{1,w}}\big)\cdots\big((\zeta\vartheta)^{\upsilon_s}\,(\alpha_1\,\tilde{y}_1)^{r_{s,1}}\cdots(\alpha_w\,\tilde{y}_w)^{r_{s,w}}\big).\end{aligned}$$

Thus (5.7) holds. Suppose that the evaluation functions ev and $\tilde{\mathrm{ev}}$ satisfy the conditions (2.26), (2.27) and (2.28). Then since (5.2) holds, it follows that $\mathrm{ev}(f,n) = \tilde{\mathrm{ev}}(\varphi(f),n)$ holds for all $f \in \mathbb{K}\langle y_1\rangle\ldots\langle y_s\rangle$ and for all $n \in \mathbb{N}$. ∎

The Mathematica package `NestedProducts` contains the algorithmic part of Theorem 5.1.1 if $\mathbb{K}$ is an algebraic number field, i.e., a finite algebraic field extension of the field of rational numbers $\mathbb{Q}$. More precisely, the field is given in the form $\mathbb{K} = \mathbb{Q}(\theta)$ together with an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ with $f(\theta) = 0$ such that the degree $n := \deg f$ is minimal (f is also called the minimal polynomial of θ). Let $\theta_1, \ldots, \theta_n \in \mathbb{C}$ be the roots of the minimal polynomial $f(x)$. Then the mappings

$$\varphi_i : \mathbb{Q}(\theta) \to \mathbb{C}$$

defined as

$$\varphi_i\Big(\sum_{j=0}^{n-1}\gamma_j\,\theta^j\Big) = \sum_{j=0}^{n-1}\gamma_j\,\theta_i^j$$

with $\gamma_j \in \mathbb{Q}$ are the embeddings of $\mathbb{Q}(\theta)$ into the field of complex numbers $\mathbb{C}$ for all $i = 1, \ldots, n$. Note that any finite algebraic extension $\mathbb{K}'$ of $\mathbb{K}$ can be also represented in a similar manner and can be embedded into $\mathbb{C}$. Subsequently, we consider algebraic numbers as elements in the subfield $\varphi_i(\mathbb{Q}(\theta))$ of $\mathbb{C}$ for some i.

Now let $\mathbb{K}$ be such a number field. Applying the underlying algorithm of Theorem 5.1.1 to given $\gamma_1, \ldots, \gamma_s \in \mathbb{K}^*$ might lead to rather complicated algebraic field extensions in which the α_i are represented. It turned out that the following strategy improved this situation substantially. Namely, consider the map,

$$\begin{aligned}\|\ \| : \mathbb{K} &\to \mathbb{R}\\ \gamma &\mapsto \langle\gamma,\gamma\rangle^{\frac{1}{2}},\end{aligned}$$

where $\mathbb{R}$ is the set of real numbers and $\langle\gamma,\gamma\rangle$ denotes the product of γ with its complex conjugate. In this setting, one can solve the following problem.

Problem RU for $\gamma \in \mathbb{K}^*$.

Given $\gamma \in \mathbb{K}^*$. *Find*, if possible, a root of unity ζ such that $\gamma = \|\gamma\|\,\zeta$ holds.

Lemma 5.1.5.
If $\mathbb{K}$ is an algebraic number field, then Problem RU for $\gamma \in \mathbb{K}^*$ is solvable in $\mathbb{K}$ or some finite algebraic extension $\mathbb{K}'$ of $\mathbb{K}$. ♦

PROOF:
Let $\gamma \in K = \mathbb{Q}(\alpha)$ where $p(x)$ is the minimal polynomial of α. We consider two cases. Suppose that $\|\gamma\| \notin K$. Then using the Primitive Element Theorem (see, e.g., (Winkler, 2012, pp. 145)) we can construct a new minimal polynomial which represents the algebraic field extension K' of K with $\|\gamma\| \in K'$. Define $\zeta := \frac{\gamma}{\|\gamma\|} \in K'$. Note that $\|\zeta\| = 1$. It remains to check if ζ is a root of unity[2], i.e., if there is an $n \in \mathbb{N}$ with $\zeta^n = 1$. This is constructively decidable since K' is strongly σ-computable. In the second case we have $\|\gamma\| \in K$, and thus $\zeta := \frac{\gamma}{\|\gamma\|} \in K$. Since K is strongly σ-computable, one can decide again constructively if there is an $n \in \mathbb{N}$ with $\zeta^n = 1$. ∎

As preprocessing step (before we actually apply Theorem 5.1.1) we check algorithmically if we can solve Problem RU for each of the algebraic numbers $\gamma_1, \dots, \gamma_s$. Extracting their roots of unity and applying Proposition 5.1.6, we can compute a common λ-th root of unity that will represent all the other roots of unity.

Proposition 5.1.6.
Let a and b be distinct primitive roots of unity of order λ_a and λ_b, respectively. Then there is a primitive λ_c-th root of unity c such that for some $0 \leqslant m_a, m_b < \lambda_c$ we have $c^{m_a} = a$ and $c^{m_b} = b$. □

PROOF:
Take primitive roots of unity of orders λ_a and λ_b, say, $\alpha = e^{\frac{2\pi i}{\lambda_a}}$ and $\beta = e^{\frac{2\pi i}{\lambda_b}}$. Let $a = \alpha^u$ and $b = \beta^v$ for $0 \leqslant u < \lambda_a$ and $0 \leqslant v < \lambda_b$. Define $\lambda_c := \mathrm{lcm}(\lambda_a, \lambda_b)$ and take a primitive λ_c-th root of unity, $c = e^{\frac{2\pi i}{\lambda_c}}$. Then with $m_a = u\frac{\lambda_c}{\lambda_a} \mod \lambda_c$ and $m_b = v\frac{\lambda_c}{\lambda_b} \mod \lambda_c$ the Proposition is proven. ∎

Example 5.1.7.
With $K = \mathbb{Q}(i + \sqrt{3}, \sqrt{-13})$, we can extract the following products

$$\prod_{k=1}^{n} \underbrace{-13\sqrt{-13}}_{=:\gamma_1'}, \quad \prod_{k=1}^{n} \underbrace{\frac{-784}{13\sqrt{-13}\,(i+\sqrt{3})^4}}_{=:\gamma_2'}, \quad \prod_{k=1}^{n} \underbrace{\frac{-17210368}{13\sqrt{-13}\,(i+\sqrt{3})^{10}}}_{=:\gamma_3'} \tag{5.8}$$

from (3.4). Let $\gamma_1 = -13$, $\gamma_2 = \sqrt{-13}$, $\gamma_3 = -784$, $\gamma_4 = 13$, $\gamma_5 = (i + \sqrt{3})$ and $\gamma_6 = -17210368$. Applying Problem RU to each γ_i we get the roots of unity $1, -1, i, \frac{i+\sqrt{3}}{2}$ with orders $1, 2, 4, 12$, respectively. By Proposition 5.1.6, the order of the common root of unity is 12. Among all the possible 12-th root of unity, we take $\zeta := e^{\frac{\pi i}{6}} = (-1)^{1/6}$. Note that we can express the other roots of unity with less order in terms of our chosen root of unity, ζ. In particular, we can write $1, -1, i$ as $\zeta^{12}, \zeta^6, \zeta^3$, respectively. Consequently, (5.8) simplifies to

$$\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^9 \prod_{k=1}^{n} 13\sqrt{13}, \quad \left(\left((-1)^{\frac{1}{6}}\right)^n\right)^{11} \prod_{k=1}^{n} \frac{49}{13\sqrt{13}}, \quad \left(\left((-1)^{\frac{1}{6}}\right)^n\right)^5 \prod_{k=1}^{n} \frac{16807}{13\sqrt{13}}. \tag{5.9}$$

[2] ζ lies on the unity circle. However, not every algebraic number on the unit circle is a root of unity: Take for instance $\frac{1-\sqrt{3}}{2} + \frac{3^{\frac{1}{4}}}{\sqrt{2}}\, i$ and its complex conjugate; they are on the unit circle, but they are roots of the polynomial $x^4 - 2x^3 - 2x + 1$ which is irreducible in $\mathbb{Q}[x]$ and which is not a cyclotomic polynomial. For details on number fields containing such numbers see Parry (1975).

The pre-processing step yields the numbers $\gamma_1^* = \sqrt{13}$, $\gamma_2^* = 13$, $\gamma_3^* = 49$ and $\gamma_4^* = 16807$ which are not roots of unity. Now we carry out the steps worked out in the proof of Theorem 5.1.1. The package `NestedProducts` uses Ge's algorithm (Ge, 1993a); (Kauers, 2005, Algorithm 7.16, page 84) to given $\alpha_1 = \sqrt{13}$ and $\alpha_2' = 49$ and finds out that there is no integer relation, i.e, $\mathcal{M}((\alpha_1, \alpha_2'), \mathsf{K}') = \{0_2\}$ with $\mathsf{K}' = \mathbb{Q}\big((-1)^{\frac{1}{6}}, \sqrt{13}\big)$. For the purpose of working with primes whenever possible, we write $\alpha_2' = \alpha_2^2$ where $\alpha_2 = 7$. Note that, $\mathcal{M}((\alpha_1, \alpha_2), \mathsf{K}') = \{0_2\}$. Now take the AP-extension $(\mathsf{K}'\langle\vartheta\rangle\langle y_1\rangle\langle y_2\rangle, \sigma)$ of (K', σ) with

$$\sigma(\vartheta) = (-1)^{\frac{1}{6}}\,\vartheta, \qquad \sigma(y_1) = \sqrt{13}\,y_1, \qquad \sigma(y_2) = 7\,y_2.$$

By our construction and Remark 5.1.3 it follows that the AP-extension is an $R\Pi$-extension. Further, with α_1 and α_2 we can write

$$13 = \big(\sqrt{13}\big)^2 \cdot 7^0, \qquad 49 = \big(\sqrt{13}\big)^0 \cdot 7^2, \qquad 16807 = \big(\sqrt{13}\big)^0 \cdot 7^5.$$

Hence for

$$a_1' = \vartheta^9\, y_1^3, \qquad a_2' = \frac{\vartheta^{11}\, {y_2}^2}{y_1^3}, \qquad a_3' = \frac{\vartheta^5\, y_2^5}{y_1^3}$$

we get $\sigma(a_i') = \gamma_i'\, a_i'$ for $i = 1, 2, 3$. Thus the shift behavior of the products in (5.8) is modelled by a_1', a_2', a_3', respectively in $\mathsf{K}'[\vartheta][y_1, {y_1}^{-1}][y_2, y_2^{-1}]$. In particular, the products in (5.8) can be rewritten to

$$\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^9 \Big(\big(\sqrt{13}\big)^n\Big)^3, \quad \Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^{11} \frac{\big(7^n\big)^2}{\Big(\big(\sqrt{13}\big)^n\Big)^3}, \quad \Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^5 \frac{\big(7^n\big)^5}{\Big(\big(\sqrt{13}\big)^n\Big)^3}. \tag{5.10}$$

$\star$

5.2 Construction of $R\Pi$-extensions for nesting depth 1 expressions in $\mathrm{ProdE}(\mathbb{K})$

Next, we treat the case that $\mathbb{K} = \mathsf{K}(\kappa_1, \dots, \kappa_u)$ is a rational function field where we suppose that K is strongly σ-computable.

Theorem 5.2.1.
Let $\mathbb{K} = \mathsf{K}(\kappa_1, \dots, \kappa_u)$ be a rational function field over a field K and let $\gamma_1, \dots, \gamma_s \in \mathbb{K}^*$. Then there is an algebraic field extension K' of K together with a λ-th root of unity $\zeta \in \mathsf{K}'$ with $\lambda \geqslant 1$ and elements $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_w) \in \mathsf{K}'(\kappa_1, \dots, \kappa_u)^w$ with $\mathcal{M}(\boldsymbol{\alpha}, \mathsf{K}'(\kappa_1, \dots, \kappa_u)) = \{0_w\}$ such that for all $i = 1, \dots, s$ we have (5.2) for some $1 \leqslant \mu_i \leqslant \lambda$ and $(u_{i,1}, \dots, u_{i,w}) \in \mathbb{Z}^w$.
If K is strongly σ-computable, then ζ, the α_i and the $\mu_i, u_{i,j}$ can be computed. $\diamondsuit$

Proof:
There are monic irreducible[3] pairwise different polynomials $f_1, \dots, f_r$ from $\mathsf{K}[\kappa_1, \dots, \kappa_u]$ and elements $c_1, \dots, c_s \in \mathsf{K}^*$ such that for all i with $1 \leqslant i \leqslant s$ we have

$$\gamma_i = c_i\, f_1^{z_{i,1}}\, f_2^{z_{i,2}} \cdots f_r^{z_{i,r}} \tag{5.11}$$

[3] It would suffice to require that the $f_i \in \mathsf{K}[\kappa_1, \dots, \kappa_u] \setminus \mathsf{K}$ are monic and pairwise co-prime. For practical reasons we require in addition that the f_i are irreducible. For instance, suppose we have to deal with $(\kappa(\kappa+1))^n$. Then we could take $f_1 = \kappa(\kappa+1)$ and can adjoin the Π-monomial $\sigma(t) = f_1\, t$ to model the product. However, if in a later step also the unforeseen products κ^n and $(\kappa+1)^n$ arise, one has to split t into two monomials, say t_1, t_2, with $\sigma(t_1) = \kappa\, t_1$ and $\sigma(t_2) = (\kappa+1)\, t_2$. Requiring that the f_i are irreducible avoids such undesirable redesigns of an already constructed $R\Pi$-extension.

with $z_{i,j} \in \mathbb{Z}$. By Theorem 5.1.1 there exist $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_w) \in (\mathsf{K}'^*)^w$ in an algebraic field extension K' of K with $\boldsymbol{\mathcal{M}}(\boldsymbol{\alpha}, \mathsf{K}') = \{0_w\}$ and a root of unity $\zeta \in \mathsf{K}'$ such that

$$c_i = \zeta^{\mu_i}\, \alpha_1^{u_{i,1}} \cdots \alpha_w^{u_{i,w}} \tag{5.12}$$

holds for some $\mu_i, u_{i,j} \in \mathbb{N}$. Hence

$$\gamma_i = \zeta^{\mu_i}\, \alpha_1^{u_{i,1}} \cdots \alpha_w^{u_{i,w}}\, f_1^{z_{i,1}}\, f_2^{z_{i,2}} \cdots f_r^{z_{i,r}}. \tag{5.13}$$

Now let $(\nu_1, \dots, \nu_w, \lambda_1, \dots, \lambda_r) \in \mathbb{Z}^{w+r}$ with

$$1 = \alpha_1^{\nu_1}\, \alpha_2^{\nu_2} \cdots \alpha_w^{\nu_w}\, f_1^{\lambda_1}\, f_2^{\lambda_2} \cdots f_r^{\lambda_r}.$$

Since the f_i are all irreducible and the α_i are from $\mathsf{K}' \setminus \{0\}$, it follows that $\lambda_1 = \dots = \lambda_r = 0$. Note that

$$\alpha_1^{\nu_1}\, \alpha_2^{\nu_2} \cdots \alpha_w^{\nu_w} = 1$$

holds in K' if and only if it holds in $\mathsf{K}'(\kappa_1, \dots, \kappa_u)$. Thus by $\boldsymbol{\mathcal{M}}(\boldsymbol{\alpha}, \mathsf{K}') = \{0_w\}$ we conclude that $\nu_1 = \dots = \nu_w = 0$. Consequently, $\boldsymbol{\mathcal{M}}((\alpha_1, \dots, \alpha_w, f_1, \dots, f_r), \mathsf{K}'(\kappa_1, \dots, \kappa_u)) = \{0_{w+r}\}$.
Now suppose that the computational aspects hold. Since one can factorize polynomials in $\mathsf{K}[\kappa_1, \dots, \kappa_u]$, the representation (5.11) is computable. In particular, the representation (5.12) is computable by Theorem 5.1.1. This completes the proof. ■

Note that again Remark 5.1.3 is relevant where $\mathsf{K}'(\kappa_1, \dots, \kappa_u)$ takes over the role of K': using Theorem 5.2.1 in combination with Lemma 5.0.2 we can construct a Π-extension in which we can rephrase products defined over $\mathbb{K}$. Further, we remark that the package `NestedProducts` implements this machinery for the case that K is an algebraic number field. Summarizing, we allow products that depend on extra parameters. This will be used for the multibasic case with $\mathbb{K} = \mathsf{K}(q_1, \dots, q_e)$ for a field K (K might be again, e.g., a rational function field defined over an algebraic number field). We remark further that for the field $\mathbb{K} = \mathbb{Q}(\kappa_1, \dots, \kappa_u)$ this result can be accomplished without any field extension, i.e., $\mathbb{K}' = \mathbb{K}$; see Schneider (2005).

More precisely, Lemma 5.1.4 can be extended to the following Lemma.

Lemma 5.2.2.
Let $\mathbb{K} = \mathsf{K}(\kappa_1, \dots, \kappa_u)$ be a rational function field over a field K and $(\mathbb{K}, \sigma)$ be a difference field with $\sigma(c) = c$ for all $c \in \mathbb{K}$. Let $(\mathbb{K}\langle x_1\rangle \dots \langle x_s\rangle, \sigma)$ be a P-extension of $(\mathbb{K}, \sigma)$ with $\sigma(x_i) = \gamma_i\, x_i$ where $\gamma_i \in \mathbb{K}^*$. Let $\mathrm{ev} : \mathbb{K}\langle x_1\rangle \dots \langle x_s\rangle \times \mathbb{N} \to \mathbb{K}$ be an evaluation function defined as $\mathrm{ev}(x_i, n) = \gamma_i^n$ for $1 \leqslant i \leqslant s$. Then the following statements hold:

(1) One can construct an $\mathrm{R}\Pi$-extension $(\tilde{\mathbb{K}}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle, \sigma)$ [4] of $(\tilde{\mathbb{K}}, \sigma)$ with

$$\sigma(\vartheta) = \zeta\,\vartheta \qquad \textit{and} \qquad \sigma(\tilde{y}_k) = \alpha_k\, \tilde{y}_k \tag{5.14}$$

for $1 \leqslant k \leqslant w$ where $\tilde{\mathbb{K}} = \tilde{\mathsf{K}}(\kappa_1, \dots, \kappa_u)$ and $\tilde{\mathsf{K}}$ is a finite algebraic field extension of K with $\zeta \in \tilde{\mathsf{K}}$ being a λ-th root of unity and $\alpha_k \in \tilde{\mathbb{K}}^*$ together with an evaluation function $\tilde{\mathrm{ev}} : \tilde{\mathbb{K}}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle \times \mathbb{N} \to \tilde{\mathbb{K}}$ defined as

$$\tilde{\mathrm{ev}}(\vartheta, n) = \zeta^n \qquad \textit{and} \qquad \tilde{\mathrm{ev}}(\tilde{y}_k, n) = \alpha_k^n. \tag{5.15}$$

[4]Note that if $\zeta = 1$, then the construction of the single R-extension $(\tilde{\mathbb{K}}\langle\vartheta\rangle, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ is redundant.

(2) One can construct a difference ring homomorphism $\varphi : \mathbb{K}\langle x_1\rangle \dots \langle x_s\rangle \to \tilde{\mathbb{K}}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle$ *with*

$$\varphi(x_i) = \vartheta^{\mu_i}\,\tilde{\boldsymbol{y}}^{\boldsymbol{\nu}_i} = \vartheta^{\mu_i}\,\tilde{y}_1^{\nu_{i,1}} \cdots \tilde{y}_w^{\nu_{i,w}} \tag{5.16}$$

where $\nu_{i,k} \in \mathbb{Z}$ *and* $0 \leqslant \mu_i < \lambda$ *such that for all* $f \in \mathbb{K}\langle x_1\rangle \dots \langle x_s\rangle$ *and for all* $n \in \mathbb{N}$,

$$\mathrm{ev}(f, n) = \tilde{\mathrm{ev}}(\varphi(f), n)$$

holds. ♦

PROOF:

(1) Given $\gamma_1, \dots, \gamma_s \in \mathbb{K}^*$, it follows by Theorem 5.2.1 that there is a finite algebraic field extension $\tilde{K}$ of K together with a λ-th root of unity $\zeta \in \tilde{K}^*$ and elements $\alpha_1, \dots, \alpha_w \in \tilde{\mathbb{K}}^*$ such that $\mathcal{M}((\alpha_1, \dots, \alpha_w), \tilde{\mathbb{K}}) = \{0_w\}$ and (5.13) holds. Let $(\tilde{\mathbb{K}}, \sigma)$ be a difference field with $\sigma(c) = c$ for all $c \in \tilde{\mathbb{K}}$ and let $(\tilde{\mathbb{K}}\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle, \sigma)$ be a P-extension of $(\tilde{\mathbb{K}}, \sigma)$ with the automorphism (5.14) and evaluation function (5.15). Since $\mathcal{M}((\alpha_1, \dots, \alpha_w), \tilde{\mathbb{K}}) = \{0_w\}$, it follows by Lemma 5.0.2 that $(\tilde{\mathbb{K}}\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle, \sigma)$ is a Π-extension of $(\tilde{\mathbb{K}}, \sigma)$. In particular, we consider the A-extension [5] $(\mathbb{E}\langle\vartheta\rangle, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ where $\mathbb{E} = \tilde{\mathbb{K}}\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle$ with $\sigma(\vartheta) = \zeta\vartheta$ of order $\lambda \geqslant 1$. By Corollary 2.3.58 this is an R-extension. Note that, one can rearrange the generators in $\mathbb{E}\langle\vartheta\rangle = \tilde{\mathbb{K}}\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle\langle\vartheta\rangle$ to get the RΠ-extension $(\tilde{\mathbb{K}}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$.

(2) Consider the uniquely determined ring homomorphism $\varphi : \mathbb{K}\langle x_1\rangle \dots \langle x_s\rangle \to \tilde{\mathbb{K}}\langle\vartheta\rangle\langle\tilde{y}_1\rangle \dots \langle\tilde{y}_w\rangle$ subject to (5.16). Let $f \in \mathbb{K}\langle x_1\rangle \dots \langle x_s\rangle$ with $f := a\,\boldsymbol{x}^{\boldsymbol{m}} = a\,x_1^{m_1} \cdots x_s^{m_s}$. We show that φ is a difference ring homomorphism, i.e.,

$$\sigma(\varphi(f)) = \varphi(\sigma(f)) \tag{5.17}$$

holds. For the left hand side of (5.17) we have:

$$\begin{aligned}\sigma(\varphi(a\,\boldsymbol{x}^{\boldsymbol{m}})) = \sigma\big(a\,(\vartheta^{\mu_1}\,\tilde{\boldsymbol{y}}^{\boldsymbol{\nu}_1})^{m_1} \cdots (\vartheta^{\mu_s}\,\tilde{\boldsymbol{y}}^{\boldsymbol{\nu}_s})^{m_s}\big) = \sigma\big(a\,(\vartheta^{\upsilon_1}\,\tilde{\boldsymbol{y}}^{\boldsymbol{n}_1}) \cdots\, (\vartheta^{\upsilon_1}\,\tilde{\boldsymbol{y}}^{\boldsymbol{n}_s})\big) = \\ a\,\big((\zeta\vartheta)^{\upsilon_1}\,(\alpha_1\,\tilde{y}_1)^{n_{1,1}} \cdots (\alpha_w\,\tilde{y}_w)^{n_{1,w}}\big)\,\cdots\,\big((\zeta\vartheta)^{\upsilon_s}\,(\alpha_1\,\tilde{y}_1)^{n_{s,1}} \cdots (\alpha_w\,\tilde{y}_w)^{n_{s,w}}\big).\end{aligned}$$

where $\upsilon_i = \mu_i\,m_i$ and $n_{i,k} = m_i\,\nu_{i,k}$. Using (5.13), the right hand side of (5.17) is:

$$\begin{aligned}\varphi(\sigma(a\,x_1^{m_1} \cdots x_s^{m_s})) = \varphi\big(a\,(\gamma_1\,x_1)^{m_1} \cdots (\gamma_s\,x_s)^{m_s}\big) = a\,(\gamma_1\,\vartheta^{\mu_1}\,\tilde{\boldsymbol{y}}^{\boldsymbol{\nu}_1})^{m_1} \cdots (\gamma_s\,\vartheta^{\mu_s}\,\tilde{\boldsymbol{y}}^{\boldsymbol{\nu}_s})^{m_s} = \\ a\,\big((\zeta\vartheta)^{\upsilon_1}\,(\alpha_1\,\tilde{y}_1)^{n_{1,1}} \cdots (\alpha_w\,\tilde{y}_w)^{n_{1,w}}\big) \cdots \big((\zeta\vartheta)^{\upsilon_s}\,(\alpha_1\,\tilde{y}_1)^{n_{s,1}} \cdots (\alpha_w\,\tilde{y}_w)^{n_{s,w}}\big).\end{aligned}$$

Thus (5.17) holds. Since (5.13) holds and the evaluation functions ev and $\tilde{\mathrm{ev}}$ satisfy the conditions (2.26), (2.27) and (2.28) by Lemma 2.3.26, it follows that $\mathrm{ev}(f, n) = \tilde{\mathrm{ev}}(\varphi(f), n)$ holds for all $f \in \mathbb{K}\langle x_1\rangle \dots \langle x_s\rangle$ and for all $n \in \mathbb{N}$. ■

[5] By Remark 5.1.2, ζ can be 1 and thus the construction of the A-extension might be dropped, otherwise, it is redundant.

Example 5.2.3 (Cont. Example 5.1.7).
Let $\mathbb{K}' = \mathsf{K}'(\kappa)$ with $\mathsf{K}' = \mathbb{Q}\big((-1)^{\frac{1}{6}}, \sqrt{13}\big)$ and consider

$$\prod_{k=1}^{n} \underbrace{-13\sqrt{-13}\,\kappa}_{=:\gamma_1}, \quad \prod_{k=1}^{n} \underbrace{\frac{-784\,(\kappa+1)^2}{13\,\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^4\,\kappa}}_{=:\gamma_2}, \quad \prod_{k=1}^{n} \underbrace{\frac{-17210368\,(\kappa+1)^5}{13\,\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^{10}\,\kappa}}_{=:\gamma_3} \tag{5.18}$$

which are instances of the products from (3.4). By Example 5.1.7 the products in (5.8) can be modelled in the RΠ-extension $(\mathsf{K}'\langle\vartheta\rangle\langle y_1\rangle\langle y_2\rangle, \sigma)$ of (K', σ). Note that $\kappa, (\kappa+1) \in \mathsf{K}[\kappa] \setminus \mathsf{K}$ are both irreducible over K. Thus $\mathcal{M}\big((\sqrt{13}, 7, \kappa, \kappa+1), \mathbb{K}'\big) = \{0_4\}$ holds. Consequently by Remark 5.1.3, $(\mathbb{K}'\langle\vartheta\rangle\langle y_1\rangle\langle y_2\rangle\langle y_3\rangle\langle y_4\rangle, \sigma)$ is an RΠ-extension of $(\mathbb{K}', \sigma)$ with

$$\sigma(y_3) = \kappa\, y_3 \qquad \text{and} \qquad \sigma(y_4) = (\kappa+1)\, y_4.$$

Here the Π-monomials y_3 and y_4 model κ^n and $(\kappa+1)^n$, respectively. In particular, for $i = 1, 2, 3$ we get $\sigma(a_i) = \gamma_i\, a_i$ with

$$a_1 = \vartheta^9\, y_1^3\, y_3, \qquad a_2 = \frac{\vartheta^{11}\, y_2^2\, y_4^2}{y_1^3\, y_3}, \qquad a_3 = \frac{\vartheta^5\, y_2^5\, y_4^5}{y_1^3\, y_3}. \tag{5.19}$$

In short, a_1, a_2, a_3 model the shift behaviours of the products in (5.18), respectively. In particular, the products in (5.18) can be rewritten to

$$\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^9 \Big(\big(\sqrt{13}\big)^n\Big)^3 \kappa^n, \quad \frac{\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^{11} \big(7^n\big)^2 \big((\kappa+1)^n\big)^2}{\Big(\big(\sqrt{13}\big)^n\Big)^3\, \kappa^n}, \quad \frac{\Big(\big((-1)^{\frac{1}{6}}\big)^n\Big)^5 \big(7^n\big)^5 \big((\kappa+1)^n\big)^5}{\Big(\big(\sqrt{13}\big)^n\Big)^3\, \kappa^n}. \tag{5.20}$$

⋆

5.3 Structural results for single nested Π-extensions

Finally, we focus on products where non-constant polynomials are involved. Similar to Theorem 5.2.1 we will use irreducible factors as main building blocks to define our Π-extensions. The crucial refinement is that these factors are also shift co-prime; compare also Schneider (2005, 2014). Here the following two lemmas will be utilized.

Lemma 5.3.1.
Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ ($\alpha \in \mathbb{F}^*$ and $\beta = 0$ or $\alpha = 1$ and $\beta \in \mathbb{F}$). Let $\mathbf{f} = (f_1, \dots, f_s) \in (\mathbb{F}[t] \setminus \mathbb{F})^s$. Suppose that

$$\forall\, i, j\, (1 \leqslant i < j \leqslant s) : \gcd_\sigma(f_i, f_j) = 1 \tag{5.21}$$

holds and that for i with $1 \leqslant i \leqslant s$ we have that[6]

$$\frac{\sigma(f_i)}{f_i} \in \mathbb{F} \;\vee\; \forall\, k \in \mathbb{Z} \setminus \{0\} : \gcd(f_i, \sigma^k(f_i)) = 1. \tag{5.22}$$

Then for all $h \in \mathbb{F}^*$ there does not exist $(\nu_1, \dots, \nu_s) \in \mathbb{Z}^s \setminus \{0_s\}$ and $g \in \mathbb{F}(t)^*$ with

$$\frac{\sigma(g)}{g} = f_1^{\nu_1} \cdots f_s^{\nu_s}\, h. \tag{5.23}$$

In particular, $\mathcal{M}(\mathbf{f}, \mathbb{F}(t)) = \{0_s\}$. ♦

[6] We note that (5.22) could be also rephrased in terms of Abramov's dispersion Abramov (1971); Bronstein (2000).

PROOF:
Suppose that (5.21) and (5.22) hold. Now let $h \in \mathbb{F}^*$ and assume that there are a $g \in \mathbb{F}(t)^*$ and $(\nu_1, \dots, \nu_s) \in \mathbb{Z}^s \setminus \{0_s\}$ with (5.23). Suppose that $\beta = 0$ and $g = u\,t^m$ for some $m \in \mathbb{Z}$ and some $u \in \mathbb{F}^*$. Then

$$\frac{\sigma(g)}{g} \in \mathbb{F}. \tag{5.24}$$

Hence $\nu_i = 0$ for $1 \leqslant i \leqslant s$ since the f_i are pairwise co-prime by (5.21), a contradiction. Thus we can take a monic irreducible factor, say $p \in \mathbb{F}[t] \setminus \mathbb{F}$ of g where $p \neq t$ if $\beta = 0$. In addition, among all these possible factors we can choose one with the property that for $k > 0$, $\sigma^k(p)$ is not a factor in g. Note that this is possible by Lemma 4.1.5. Then $\sigma(p)$ does not cancel in $\frac{\sigma(g)}{g}$. Thus $\sigma(p) \mid f_i$ for some i with $1 \leqslant i \leqslant s$. On the other hand, let $r \leqslant 0$ be minimal such that $\sigma^r(p)$ is the irreducible factor in g with the property that $\sigma^r(p)$ does not occur in $\sigma(g)$. Note that this is again possible by Lemma 4.1.5. Then $\sigma^r(p)$ does not cancel in $\frac{\sigma(g)}{g}$. Therefore, $\sigma^r(p) \mid f_j$ for some j with $1 \leqslant j \leqslant s$. Consequently, $\gcd_\sigma(f_i, f_j) \neq 1$. By (5.21) it follows that $i = j$. In particular by (5.22) it follows that

$$\frac{\sigma(f_i)}{f_i} \in \mathbb{F}.$$

By Lemma 4.1.4 this implies $f_i = w\,t^m$ with $m \in \mathbb{Z}$, $w \in \mathbb{F}^*$ and $\beta = 0$. In particular, $p = t$, which we have already excluded. In any case, we arrive at a contradiction and conclude that $\nu_1 = \dots = \nu_e = 0$. ■

Note that condition (5.21) implies that the f_i are pairwise shift co-prime. In addition condition (5.22) implies that two different irreducible factors in f_i are shift-co-prime. The next lemma considers the other direction.

Lemma 5.3.2.
Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ with t transcendental over $\mathbb{F}$ and $\sigma(t) = \alpha\,t + \beta$ where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Let $\mathbf{f} = (f_1, \dots, f_s) \in (\mathbb{F}[t] \setminus \mathbb{F})^s$ be irreducible monic polynomials. If there are no $(\nu_1, \dots, \nu_s) \in \mathbb{Z}^s \setminus \{0_s\}$ and $g \in \mathbb{F}(t)^*$ with

$$\frac{\sigma(g)}{g} = f_1^{\nu_1} \cdots f_s^{\nu_s}, \tag{5.25}$$

i.e., if $\mathcal{M}(\mathbf{f}, \mathbb{F}(t)) = \{0_s\}$, then

$$\forall\, i, j\, (1 \leqslant i < j \leqslant s) : \gcd_\sigma(f_i, f_j) = 1$$

holds. ♦

PROOF:
Suppose there are i, j with $1 \leqslant i < j \leqslant s$ and $\gcd_\sigma(f_i, f_j) \neq 1$. Since f_i, f_j are irreducible, $f_i \sim f_j$. Thus by Lemma 4.1.7 there is a $g \in \mathbb{F}(t)^*$ with $f_i = \frac{\sigma(g)}{g} f_j$. Hence $\frac{\sigma(g)}{g} = f_i\, f_j^{-1}$ and thus we can find a $(\nu_1, \dots, \nu_s) \in \mathbb{Z}^s \setminus \{0_s\}$ with (5.23). ■

Summarizing, we arrive at the following result.

Theorem 5.3.3.
Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$. Let $\mathbf{f} = (f_1, \dots, f_s) \in (\mathbb{F}[t] \setminus \mathbb{F})^s$ be irreducible monic polynomials. Then the following statements are equivalent.

(1) $\forall\, i, j : 1 \leqslant i < j \leqslant s$, $\gcd_\sigma(f_i, f_j) = 1$.

(2) There does not exist $(\nu_1, \ldots, \nu_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}_s\}$ *and* $g \in \mathbb{F}(t)^*$ *with*

$$\frac{\sigma(g)}{g} = f_1^{\nu_1} \cdots f_s^{\nu_s},$$

i.e., $\mathcal{M}(\mathbf{f}, \mathbb{F}(t)) = \{\mathbf{0}_s\}$.

(3) One can construct a Π*-field extension* $(\mathbb{F}(t)(x_1)\ldots(x_s), \sigma)$ *of* $(\mathbb{F}(t), \sigma)$ *with* $\sigma(x_i) = f_i\, x_i$*, for* $1 \leqslant i \leqslant s$.

(4) One can construct a Π*-extension* $(\mathbb{F}(t)[x_1, x_1^{-1}]\ldots[x_s, x_s^{-1}], \sigma)$ *of* $(\mathbb{F}(t), \sigma)$ *with* $\sigma(x_i) = f_i\, x_i$*, for* $1 \leqslant i \leqslant s$. ◇

PROOF:
Since the f_i are irreducible, the condition (5.22) always holds. Therefore (1) $\Longrightarrow$ (2) follows from Lemma 5.3.1. Further, (2) $\Longrightarrow$ (1) follows from Lemma 5.3.2. The equivalences between (2), (3) and (4) follow by Lemma 5.0.2. ■

5.4 Construction of $R\Pi$-extensions for nesting depth 1 expressions in $\mathrm{ProdE}(\mathbb{K}(n))$

Finally, we combine Theorems 5.2.1 and 5.3.3 to get a Π-extension in which expressions from $\mathrm{ProdE}(\mathbb{K}(x))$ can be rephrased in general. In order to accomplish this task, we will show in Lemma 5.4.4 that the Π-monomials of the two constructions in the Sections 5.2 and 5.3 can be merged to one $R\Pi$-extension. Before we arrive at this result some preparation steps are needed.

Lemma 5.4.1.
Let $(\mathbb{F}(t), \sigma)$ *be a* Σ*-extension of* $(\mathbb{F}, \sigma)$ *with* $\sigma(t) = t + \beta$ *and let* $(\mathbb{E}, \sigma)$ *be a* Π*-extension of* $(\mathbb{F}, \sigma)$*. Then one can construct a* Σ*-extension* $(\mathbb{E}(t), \sigma)$ *of* $(\mathbb{E}, \sigma)$ *with* $\sigma(t) = t + \beta$. ♦

PROOF:
Let $(\mathbb{E}, \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1)\ldots(t_e)$ and suppose that there is a $g \in \mathbb{E}$ with $\sigma(g) = g + \beta$. Let i be minimal such that $g \in \mathbb{F}(t_1)\ldots(t_i)$. Since $\mathbb{F}(t)$ is a Σ-field extension of $\mathbb{F}$, it follows by Theorem 2.3.51 that there is no $g \in \mathbb{F}$ with $\sigma(g) = g + \beta$. Then (Karr, 1985, Lemma 4.1) implies that g cannot depend on t_i, a contradiction. Thus there is no $g \in \mathbb{E}$ with $\sigma(g) = g + \beta$ and by Theorem 2.3.51 we get the Σ-field extension $(\mathbb{E}(t), \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(t) = t + \beta$. ■

As a by-product of the above lemma it follows that the mixed $\mathbf{q}$-multibasic difference field is built by Π-monomials and one Σ-monomial.

Corollary 5.4.2.
The mixed $\mathbf{q}$*-multibasic difference ring* $(\mathbb{F}, \sigma)$ *with* $\mathbb{F} = \mathbb{K}(x)(t_1)\ldots(t_e)$ *from Example 2.3.10 is a* $\Pi\Sigma$*-extension of* $(\mathbb{K}, \sigma)$*. In particular,* $\mathrm{const}(\mathbb{F}, \sigma) = \mathbb{K}$. ◇

Proof:
Since the elements $q_1, \dots, q_e$ are algebraically independent among each other, there are no $g \in \mathbb{K}^*$ and $(\nu_1, \dots, \nu_e) \in \mathbb{Z}^e \setminus \{0_e\}$ with

$$1 = \frac{\sigma(g)}{g} = q_1^{\nu_1} \cdots q_e^{\nu_e}.$$

Therefore by Lemma 5.0.2, $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{K}(t_1)\dots(t_e)$ is a Π-extension of $(\mathbb{K}, \sigma)$ with $\sigma(t_i) = q_i\, t_i$ for $1 \leqslant i \leqslant e$. Since $(\mathbb{K}(x), \sigma)$ is a Σ-extension of $(\mathbb{K}, \sigma)$, we can activate Lemma 5.4.1 and can construct the Σ-extension $(\mathbb{E}(x), \sigma)$ of $(\mathbb{E}, \sigma)$. Note that $\text{const}(\mathbb{E}(x), \sigma) = \text{const}(\mathbb{E}, \sigma)$ also implies that $\text{const}(\mathbb{K}(x)(t_1)\dots(t_e), \sigma) = \text{const}(\mathbb{K}(x), \sigma) = \mathbb{K}$. In particular, the P-extension $(\mathbb{K}(x)(t_1)\dots(t_e), \sigma)$ of $(\mathbb{K}(x), \sigma)$ is a Π-extension. Consequently, $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{K}, \sigma)$. ■

Proposition 5.4.3.
Let $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t_i) = \alpha_i\, t_i + \beta_i$ where $\beta_i \neq 0$ or $\alpha_i = 1$. Let $f \in \mathbb{F}^*$. If there is a $g \in \mathbb{F}(t_1, \dots, t_e)^*$ with

$$\frac{\sigma(g)}{g} = f,$$

then $g = \omega\, t_1^{\nu_1} \cdots t_e^{\nu_e}$ where $\omega \in \mathbb{F}^*$. In particular, $\nu_i = 0$, if $\beta_i \neq 0$ (i.e., t_i is a Σ-monomial) or $\nu_i \in \mathbb{Z}$, if $\beta_i = 0$ (i.e., t_i is a Π-monomial). □

Proof:
See Schneider (2001, Corollary 2.2.6, page 76). ■

Lemma 5.4.4.
Let $(\mathbb{K}(x), \sigma)$ be the rational difference field with $\sigma(x) = x + 1$ and let $(\mathbb{K}(x)[\varkappa_1, \varkappa_1^{-1}]\dots[\varkappa_s, \varkappa_s^{-1}], \sigma)$ be a Π-extension of $(\mathbb{K}(x), \sigma)$ as given in Theorem 5.3.3 (item (4)). Further, let $\mathbb{K}'$ be an algebraic field extension of $\mathbb{K}$ and let $(\mathbb{K}'[y_1, y_1^{-1}]\dots[y_w, y_w^{-1}], \sigma)$ be a Π-extension of $(\mathbb{K}', \sigma)$ with $\frac{\sigma(y_i)}{y_i} \in \mathbb{K}' \setminus \{0\}$. Then the difference ring $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{K}'(x)[y_1, y_1^{-1}]\dots[y_w, y_w^{-1}][\varkappa_1, \varkappa_1^{-1}]\dots[\varkappa_s, \varkappa_s^{-1}]$ is a Π-extension of $(\mathbb{K}'(x), \sigma)$. Furthermore, the A-extension $(\mathbb{E}[\vartheta], \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(\vartheta) = \zeta\vartheta$ of order λ is an R-extension. ♦

Proof:
By iterative application of Schneider (2017, Corollary 2.6) it follows that $(\mathbb{F}, \sigma)$ is a Π-field extension of $(\mathbb{K}', \sigma)$ with $\mathbb{F} = \mathbb{K}'(y_1)\dots(y_w)$. Note that $(\mathbb{K}'(x), \sigma)$ is a Σ-extension of $(\mathbb{K}', \sigma)$. Thus by Lemma 5.4.1 $(\mathbb{F}(x), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$. We will show that $(\mathbb{H}, \sigma)$ with $\mathbb{H} = \mathbb{F}(x)(\varkappa_1)\dots(\varkappa_s)$ forms a Π-field extension of $(\mathbb{F}(x), \sigma)$. Since $(\mathbb{K}(x)[\varkappa_1, \varkappa_1^{-1}]\dots[\varkappa_s, \varkappa_s^{-1}], \sigma)$ is a Π-extension of $(\mathbb{K}(x), \sigma)$ as given in Theorem 5.3.3 (item (4)), we conclude that also (item 2) of the theorem, i.e., condition (5.21) holds. Now suppose that there is a $g \in \mathbb{F}(x)^*$ and $(l_1, \dots, l_s) \in \mathbb{Z}^s$ with

$$\frac{\sigma(g)}{g} = f_1^{l_1} \dots f_s^{l_s} \in \mathbb{K}(x).$$

By reordering of the generators in $(\mathbb{F}(x),\sigma)$ we get the Π-field extension $(\mathbb{K}'(x)(y_1)\dots(y_w),\sigma)$ of $(\mathbb{K}'(x),\sigma)$. By Proposition 5.4.3 we conclude that $g=\omega\, y_1^{n_1}\dots y_w^{n_w}$ with $(n_1,\dots,n_w)\in\mathbb{Z}^w$ and $\omega\in\mathbb{K}'(x)^*$. Thus $\frac{\sigma(g)}{g}=\frac{\sigma(\omega)}{\omega}\alpha_1^{n_1}\dots\alpha_w^{n_w}$ and hence

$$\frac{\sigma(\omega)}{\omega}=u\,f_1^{l_1}\dots f_s^{l_s} \tag{5.26}$$

for some $u\in\mathbb{K}'^*$. Now suppose that $f_i,f_j\in\mathbb{K}[x]\subset\mathbb{F}[x]$ with $i\neq j$ are not shift-co-prime in $\mathbb{F}[x]$. Then there are a $k\in\mathbb{Z}$ and $v,\tilde{f}_i,\tilde{f}_j\in\mathbb{F}[x]\setminus\mathbb{F}$ with $\sigma^k(f_j)=v\,\tilde{f}_j$ and $f_i=v\,\tilde{f}_i$. But this implies that $f_i\,\frac{\tilde{f}_j}{\tilde{f}_i}=\sigma^k(f_j)\in\mathbb{K}[x]$. Since $f_i,\sigma(f_j)\in\mathbb{K}[x]$, this implies that $\frac{\tilde{f}_j}{\tilde{f}_i}\in\mathbb{K}(x)$. Since $f_i,\sigma(f_j)$ are both irreducible in $\mathbb{K}[x]$ we conclude that $\frac{\tilde{f}_j}{\tilde{f}_i}\in\mathbb{K}$. Consequently, f_i and f_j are also not shift-co-prime in $\mathbb{K}[x]$, a contradiction. Thus the condition (5.21) holds not only in $\mathbb{K}[x]$ but also in $\mathbb{F}[x]$. Now suppose that $\gcd(f_i,\sigma^k(f_i))\neq 1$ holds in $\mathbb{F}[x]$ for some $k\in\mathbb{Z}\setminus\{0\}$. By the same arguments as above, it follows that $\sigma^k(f_i)=u\,f_i$ for some $u\in\mathbb{K}$. By Lemma 4.1.4 we conclude that $f_i=t$ and $\frac{\sigma(t)}{t}\in\mathbb{F}$. Therefore also condition (5.22) holds. Consequently, we can activate Lemma 5.3.1 and it follows from (5.26) that $l_1=\dots=l_m=0$. Consequently, we can apply Theorem 5.3.3 (equivalence (2) and (3)) and conclude that $(\mathbb{H},\sigma)$ is a Π-field extension of $(\mathbb{F}(x),\sigma)$. Finally, consider the A-extension $(\mathbb{H}[\vartheta],\sigma)$ of $(\mathbb{H},\sigma)$ with $\sigma(\vartheta)=\zeta\,\vartheta$ of order λ. By Lemma 2.3.57 it is an R-extension. Finally, consider the sub-difference ring $(\mathbb{H},\sigma)$ with $\mathbb{H}=\mathbb{K}'(x)[y_1,y_1^{-1}]\dots[y_w,y_w^{-1}][z_1,z_1^{-1}]\dots[z_s,z_s^{-1}][\vartheta]$ which is an AP-extension of $(\mathbb{K}'(x),\sigma)$. Since $\mathrm{const}(\mathbb{H},\sigma)=\mathrm{const}(\mathbb{K}'(x),\sigma)=\mathbb{K}'$, it is an R$\Pi$-extension. ■

Remark 5.4.5.
Take $(\mathbb{H},\sigma)$ with $\mathbb{H}=\mathbb{K}'(x)\langle y_1\rangle\dots\langle y_w\rangle\langle z_1\rangle\dots\langle z_s\rangle\langle\vartheta\rangle$ as constructed in Lemma 5.4.4. Then one can rearrange the generators in $\mathbb{H}$ to get the RΠ-extension $(\mathbb{K}'(x)\langle\vartheta\rangle\langle y_1\rangle\dots\langle y_w\rangle\langle z_1\rangle\dots\langle z_s\rangle,\sigma)$ of $(\mathbb{K}'(x),\sigma)$. ♠

With these considerations we can derive the following theorem that enables one to construct RΠ-extension to model product expressions in $\mathrm{ProdE}(\mathbb{K}(n))$ of nested depth-1.

Theorem 5.4.6.
Let $(\mathbb{K}(x),\sigma)$ be the rational difference field with $\sigma(x)=x+1$ where $\mathbb{K}=K(\kappa_1,\dots,\kappa_u)$ is a rational function field over a field K. Let $h_1,\dots,h_m\in\mathbb{K}(x)^*$. Then one can construct an RΠ-extension $(\mathbb{A},\sigma)$ of $(\mathbb{K}'(x),\sigma)$ with

$$\mathbb{A}=\mathbb{K}'(x)[\vartheta][y_1,y_1^{-1}]\dots[y_w,y_w^{-1}][z_1,z_1^{-1}]\dots[z_s,z_s^{-1}]$$

and $\mathbb{K}'=K'(\kappa_1,\dots,\kappa_u)$ where K' is an algebraic field extension of K such that

- *$\sigma(\vartheta)=\zeta\,\vartheta$ where $\zeta\in K'$ is a λ-th root of unity;*
- *$\dfrac{\sigma(y_j)}{y_j}=\alpha_j\in\mathbb{K}'\setminus\{0\}$ for $1\leqslant j\leqslant w$ where the α_j are not roots of unity;*
- *$\dfrac{\sigma(z_v)}{z_v}=f_v\in\mathbb{K}[x]\setminus\mathbb{K}$ are irreducible[7] and shift co-prime for $1\leqslant v\leqslant s$;*

[7] Instead of irreducibility it would suffice to require that the f_i satisfy property (5.22) for $1\leqslant i\leqslant s$. Restricting to irreducible factors simplifies the proof/construction below. In addition, it also turns the obtained difference ring to a rather robust version. E.g., suppose that one takes $f_1=x\,(2\,x+1)$ leading to the Π-monomial z with $\sigma(z)=x\,(2x+1)\,z$. Further, assume that one has to introduce unexpectedly also x and $2\,x+1$ in a later computation. Then one has to split z to the Π-monomials z_1,z_2 with $\sigma(z_1)=x\,z_1$ and $\sigma(z_2)=(2x+1)\,z_2$, i.e., one has to redesign the already constructed RΠ-extension. In short, irreducible polynomials provide an R$\Pi\Sigma$-extension which most probably need not be redesigned if other products have to be considered.

holds with the following property. For $1 \leqslant i \leqslant m$ one can define

$$g_i = r_i\,\vartheta^{\mu_i}\,y_1^{u_{i,1}} \cdots y_w^{u_{i,w}}\,z_1^{v_{i,1}} \cdots z_s^{v_{i,s}} \in \mathbb{A} \tag{5.27}$$

with $0 \leqslant \mu_i \leqslant \lambda - 1$, $u_{i,1}, \dots, u_{i,w}, v_{i,1}, \dots, v_{i,s} \in \mathbb{Z}$ and $r_i \in \mathbb{K}(x)^*$ such that

$$\sigma(g_i) = \sigma(h_i)\,g_i. \tag{5.28}$$

If K is strongly σ-computable, the components of the theorem can be computed. $\diamondsuit$

PROOF:
For $1 \leqslant i \leqslant m$ we can take pairwise different monic irreducible polynomials $p_1, \dots, p_n \in \mathbb{K}[x] \setminus \mathbb{K}$ $\gamma_1, \dots, \gamma_m \in \mathbb{K}^*$ and $d_{i,1}, \dots, d_{i,n} \in \mathbb{Z}$ such that

$$\sigma(h_i) = \gamma_i\,p_1^{d_{i,1}} \cdots p_n^{d_{i,n}}$$

holds. Note that this representation is computable if K is strongly σ-computable. By Theorem 5.2.1 it follows that there are a λ-th root of unity $\zeta \in K'$, elements $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_w) \in (\mathbb{K}'^*)^w$ with $\mathcal{M}(\boldsymbol{\alpha}, \mathbb{K}') = \{0_w\}$ and integer vectors $(u_{i,1}, \dots, u_{i,w}) \in \mathbb{Z}^w$ and $\mu_i \in \mathbb{N}$ with $0 \leqslant \mu_i < \lambda$ such that

$$\gamma_i = \zeta^{\mu_i}\,\alpha_1^{u_{i,1}} \cdots \alpha_w^{u_{i,w}}$$

holds for all $1 \leqslant i \leqslant m$. Obviously, the α_j with $1 \leqslant j \leqslant w$ are not roots of unity. By Lemma 5.0.2 we get the Π-extension $(\mathbb{K}'[y_1, y_1^{-1}] \dots [y_w, y_w^{-1}], \sigma)$ of $(\mathbb{K}', \sigma)$ with $\sigma(y_j) = \alpha_j\,y_j$ for $1 \leqslant j \leqslant w$ and we obtain

$$a_i = \vartheta^{\mu_i}\,y_1^{u_{i,1}} \cdots y_w^{u_{i,w}} \tag{5.29}$$

with

$$\sigma(a_i) = \gamma_i\,a_i \tag{5.30}$$

for $1 \leqslant i \leqslant m$. Next we proceed with the non-constant polynomials in $\mathbb{K}[x] \setminus \mathbb{K}$. Set $\mathcal{I} = \{p_1, \dots, p_n\}$. Then there is a partition $\mathcal{P} = \{\mathcal{E}_1, \dots, \mathcal{E}_s\}$ of $\mathcal{I}$ with respect to $\sim_\sigma$, i.e., each $\mathcal{E}_i$ contains precisely the shift equivalent elements of $\mathcal{P}$. Take a representative from each equivalence class $\mathcal{E}_i$ in $\mathcal{P}$ and collect them in $\mathcal{F} := \{f_1, \dots, f_s\}$. Since each f_i is shift equivalent with every element of $\mathcal{E}_i$, it follows by Lemma 4.1.7 that for all $h \in \mathcal{E}_i$, there is a rational function $r \in \mathbb{K}(x)^*$ with

$$h = \frac{\sigma(r)}{r}\,f_i$$

for $1 \leqslant i \leqslant s$. Consequently, we get $r_i \in \mathbb{K}(x)^*$ and $v_{i,j} \in \mathbb{Z}$ with

$$p_1^{d_{i,1}} \cdots p_n^{d_{i,n}} = \frac{\sigma(r_i)}{r_i}\,f_1^{v_{i,1}} \cdots f_s^{v_{i,s}}$$

for all $1 \leqslant i \leqslant s$. Further, by this construction, we know that $\gcd_\sigma(f_i, f_j) = 1$ for $1 \leqslant i < j \leqslant s$. Therefore, it follows by Theorem 5.3.3 that we can construct the Π-extension $(\mathbb{K}(x)[z_1, z_1^{-1}] \dots [z_s, z_s^{-1}], \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $\sigma(z_i) = f_i\,z_i$. Now define $b_i = r_i\,z_1^{v_{i,1}} \cdots z_s^{v_{i,s}}$. Then we get

$$\sigma(b_i) = p_1^{d_{i,1}} \cdots p_n^{d_{i,n}}\,b_i. \tag{5.31}$$

Finally, by Lemma 5.4.4 and Remark 5.4.5 we end up at the $R\Pi$-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}'(x), \sigma)$ with $\mathbb{A} = \mathbb{K}'(x)[\vartheta][y_1, y_1^{-1}] \dots [y_w, y_w^{-1}][z_1, z_1^{-1}] \dots [z_e, z_e^{-1}]$ with $\sigma(\vartheta) = \zeta\,\vartheta$, $\sigma(y_j) = \alpha_j\,y_j$ for $1 \leqslant j \leqslant w$ and $\sigma(z_i) = f_i\,z_i$ for $1 \leqslant i \leqslant s$.
Now let g_i be as defined in (5.27). Since $g_i = a_i\,b_i$ with (5.30) and (5.31), we conclude that (5.28) holds. If K is strongly σ-computable, all the ingredients delivered by Theorems 5.1.1 and 5.3.3 can be computed. This completes the proof. $\blacksquare$

Example 5.4.7.
Let $\mathbb{K} = \mathsf{K}(\kappa)$ be a rational function field over the algebraic number field $\mathsf{K} = \mathbb{Q}(\mathrm{i}+\sqrt{3}, \sqrt{-13})$ and take the rational difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x+1$. Given (3.5), we can write

$$\sigma(h_1) = \gamma_1\, p_1^{-1}, \qquad \sigma(h_2) = \gamma_2\, p_1\, p_2^{-2}, \qquad \sigma(h_3) = \gamma_3\, p_1\, p_2^{-5}$$

where the $\gamma_1, \gamma_2, \gamma_3$ are given in (5.18) and where we set $p_1 = x+1$, $p_2 = x+3$ as our monic irreducible polynomials. Note that p_1 and p_2 are shift equivalent:

$$\gcd(p_2, \sigma^2(p_1)) = p_2.$$

Consequently both factors fall into the same equivalence class

$$\mathscr{E} = \{\sigma^k(x+1) \mid k \in \mathbb{Z}\} = \{\sigma^k(x+3) \mid k \in \mathbb{Z}\}.$$

Take $p_1 = x+1$ as a representative of the equivalence class $\mathscr{E}$. Then by Lemma 4.1.7, it follows that there is a $g \in \mathbb{K}(x)^*$ that connects the representatives to all other elements in their respective equivalence classes. In particular with our example we have

$$x+3 = \frac{\sigma(g)}{g}\,(x+1) \quad \text{where} \quad g = (x+1)\,(x+2).$$

By Theorem 5.3.3, it follows that $(\mathbb{K}(x)[\varkappa, \varkappa^{-1}], \sigma)$ is a Π-extension of the difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(\varkappa) = (x+1)\,\varkappa$. In this ring, the Π-monomial $\varkappa$ models $n!$. By Lemma 5.4.4 the constructed difference rings $(\mathbb{K}'[\vartheta]\,\langle y_1\rangle\,\langle y_2\rangle\,\langle y_3\rangle\,\langle y_4\rangle, \sigma)$ and $(\mathbb{K}(x)\,\langle\varkappa\rangle, \sigma)$ from Example 5.2.3 with $\mathbb{K}' = \mathbb{Q}((-1)^{\frac{1}{6}}, \sqrt{13})(\kappa)$ can be merged into a single RΠ-extension $(\mathbb{A}, \sigma)$ where $\mathbb{A}$ is (3.6) with the automorphism defined accordingly. Further note that for

$$b_1 = \frac{1}{\varkappa}, \qquad b_2 = \frac{1}{(x+1)^2\,(x+2)^2\,\varkappa}, \qquad b_3 = \frac{1}{(x+1)^5\,(x+2)^5\,\varkappa^4}$$

we have that

$$\sigma(b_1) = p_1^{-1}\, b_1, \qquad \sigma(b_2) = p_1\, p_2^{-2}\, b_2, \qquad \sigma(b_3) = p_1\, p_2^{-5}\, b_3.$$

Taking a_1, a_2, a_3 in (5.19) with $\sigma(\gamma_i) = a_i\,\gamma_i$ for $i = 1,2,3$, we define $g_i = a_i\, b_i$ for $i = 1,2,3$ and obtain $\sigma(g_i) = \sigma(h_i)\, g_i$. Note that the g_i are precisely the elements given in (3.7). ⋆

Now we are ready to prove Theorem 3.1.1 for the special case $\mathrm{ProdE}(\mathbb{K}(n))$. Namely, consider the nested depth-1 product expressions

$$P_1(n) = \prod_{k=\ell_1}^{n} h_1(k), \ldots, P_m(n) = \prod_{k=\ell_m}^{n} h_m(k) \;\in \mathrm{Prod}(\mathbb{K}(n))$$

with $\ell_i \in \mathbb{N}$ where $\ell_i \geqslant Z(h_i)$. Further, suppose that we are given the components as claimed in Theorem 5.4.6.

• Now take the difference ring embedding $\tau(\frac{a}{b}) = \langle \mathrm{ev}\left(\frac{a}{b}, n\right)\rangle_{n\geqslant 0}$ for $a, b \in \mathbb{K}[x]$ where ev is defined in (2.1). Then by iterative application of part (1) of Lemma 2.4.3 we can construct the $\mathbb{K}'$-homomorphism $\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K}')$ determined by the homomorphic extension of

- $\tau(\vartheta) = \langle \zeta^n \rangle_{n\geqslant 0}$,

- $\tau(y_i) = \langle \alpha_i^n \rangle_{n\geqslant 0}$ for $1 \leqslant i \leqslant w$ and
- $\tau(z_i) = \langle \prod_{k=\ell_i'}^{n} f_i(k-1) \rangle_{n\geqslant 0}$ with $\ell_i' = Z(f_i) + 1$ for $1 \leqslant i \leqslant s$.

In particular, since $(\mathbb{A}, \sigma)$ is an $R\Pi$-extension of $(\mathbb{K}'(x), \sigma)$, it follows by part (3) of Lemma 2.4.3 that τ is a $\mathbb{K}'$-embedding.
• Finally, define for $1 \leqslant i \leqslant m$ the product expression

$$G_i(n) = r_i(n)\,(\zeta^n)^{\mu_i}\,(\alpha_1^n)^{u_{i,1}} \cdots (\alpha_w^n)^{u_{i,w}} \left(\prod_{k=\ell_1'}^{n} f_1(k-1)\right)^{\nu_{i,1}} \cdots \left(\prod_{k=\ell_s'}^{n} f_s(k-1)\right)^{\nu_{i,s}}$$

from $\mathrm{Prod}(\mathbb{K}'(n))$ and define $\delta_i = \max(\ell_i, \ell_1', \dots, \ell_s', Z(r_i))$. Observe that $\tau(g_i) = \langle G_i'(n) \rangle_{n\geqslant 0}$ with

$$G_i'(n) = \begin{cases} 0 & \text{if } 0 \leqslant n < \delta_i \\ G_i(n) & \text{if } n \geqslant \delta_i. \end{cases} \tag{5.32}$$

By (5.28) and the fact that τ is a $\mathbb{K}'$-embedding, it follows that

$$S(\tau(g_i)) = S(\tau(h_i))\,\tau(g_i).$$

In particular, for $n \geqslant \delta_i$ we have that

$$G_i(n+1) = h_i(n+1)\,G_i(n).$$

By definition, we have

$$P_i(n+1) = h_i(n+1)\,P_i(n)$$

for $n \geqslant \delta_i \geqslant \ell_i$. Since $G_i(n)$ and $P_i(n)$ satisfy the same first order recurrence relation, they differ only by a multiplicative constant. Namely, setting $Q_i(n) = c\,G_i(n)$ with

$$c = \frac{P_i(\delta_i)}{G_i(\delta_i)} \in (\mathbb{K}')^*$$

we have that $P_i(\delta_i) = Q_i(\delta_i)$ and thus $P_i(n) = Q_i(n)$ for all $n \geqslant \delta_i$. This proves part (1) of Theorem 3.1.1. Since τ is a $\mathbb{K}'$-embedding, the sequences

$$\langle \alpha_1^n \rangle_{n\geqslant 0}, \dots, \langle \alpha_w^n \rangle_{n\geqslant 0}, \Big\langle \prod_{k=\ell_1'}^{n} f_1(k-1) \Big\rangle_{n\geqslant 0}, \dots, \Big\langle \prod_{k=\ell_s'}^{n} f_s(k-1) \Big\rangle_{n\geqslant 0}$$

are among each other algebraically independent over $\tau\big(\mathbb{K}'(x)\big)\big[\langle \zeta^n \rangle_{n\geqslant 0}\big]$ which proves property (2) of Theorem 3.1.1.

Example 5.4.8 (Cont. Example 5.4.7).
We have

$$\sigma(g_i) = \sigma(h_i)\,g_i$$

for $i = 1, 2, 3$ where the h_i and g_i are given in (3.5) and (3.7), respectively. For the $\mathbb{K}'$-embedding defined in Example 3.1.2 we obtain

$$c_i\,\tau(g_i) = \langle P_i(n) \rangle_{n\geqslant 0} \text{ with } P_i(n) = \prod_{k=1}^{n} h_i(k)$$

and $c_1 = 1$, $c_2 = 4$ and $c_3 = 32$. Since there are no poles in the g_i we conclude that for

$$G_1(n) = \frac{\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^9 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n}{n!}, \qquad G_2(n) = \frac{4\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^{11} (7^n)^2 \left((\kappa+1)^n\right)^2}{(n+1)^2 (n+2)^2 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n\, n!},$$

$$G_3(n) = \frac{32\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^5 (7^n)^5 \left((\kappa+1)^n\right)^5}{(n+1)^5 (n+2)^5 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n (n!)^4}$$

we have

$$P_i(n) = G_i(n) \quad \text{for } n \geqslant 1.$$

With $P(n) = P_1(n) + P_2(n) + P_3(n)$ (see (3.4)) and $Q(n) = G_1(n) + G_2(n) + G_3(n)$ (see (3.8)) we get

$$P(n) = Q(n) \quad \text{for } n \geqslant 1. \tag{5.33}$$

⋆

Example 5.4.9.
The calculation steps illustrated in Examples 5.1.7, 5.2.3, 5.4.7, and 5.4.8 have all been implemented in my Mathematica package `NestedProducts`. The Mathematica Session 1 below demonstrate how one can use the Mathematica package `NestedProducts` to simplify the nested depth-1 hypergeometric product given in (3.4).

Mathematica Session 1

In[1]:= ``<<NestedProducts` ``

NestedProducts — A package by Evans Doe Ocansey — © RISC — Version 1.0 (October 1, 2019)

In[2]:= $P = {}^{a}\mathbf{FormalProduct}\left[\frac{-13\sqrt{-13}\,\kappa}{k}, \{k, 1, n\}\right] + \mathbf{FormalProduct}\left[\frac{-784\,(\kappa+1)^2\, k}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^4\,\kappa\,(k+2)^2}, \{k, 1, n\}\right] + \mathbf{FormalProduct}\left[\frac{-17210368\,(\kappa+1)^5\, k}{13\sqrt{-13}\,(\mathbb{i}+\sqrt{3})^{10}\,\kappa\,(k+2)^5}, \{k, 1, n\}\right];$

In[3]:= $Q = \mathbf{ProductReduce}[P]$

Out[3]= $$\frac{\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^9 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n}{n!} + \frac{4\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^{11} (7^n)^2 \left((\kappa+1)^n\right)^2}{(n+1)^2 (n+2)^2 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n\, n!} + \frac{32\left(\left((-1)^{\frac{1}{6}}\right)^n\right)^5 (7^n)^5 \left((\kappa+1)^n\right)^5}{(n+1)^5 (n+2)^5 \left(\left(\sqrt{13}\right)^n\right)^3 \kappa^n (n!)^4}$$

[a] Subsequently, we will use the shortcut FProduct which is an alias of FormalProduct.

Note that with Q given by the output Out[3] in Mathematica Session 1, the identity (5.33) holds. ⋆

Example 5.4.10.
Using the Mathematica package `NestedProducts`, we can simplify the nested depth-1 hypergeometric product

$$A(n) = \prod_{k=1}^{n-1} \frac{2\,(2\,k-1)}{k} \in \mathrm{Prod}(\mathbb{K}(n))$$

which evaluates to the same sequence as the central binomial coefficients $\binom{2\,(n-1)}{n-1}$ for all $n \geqslant 0$ as follows.

Mathematica Session 2

In[4]:= $\mathbf{A = FProduct\left[\frac{2\,(2\,k-1)}{k}, \{k, 1, n-1\}\right];}$

In[5]:= $\mathbf{B = ProductReduce[A]}$

Out[5]= $\frac{n}{2\,(2\,n-1)}\,\frac{(2^n)^2}{n!}\,\prod_{k=1}^{n}(k-\frac{1}{2})$

Internally, the package uses the function `SynchroniseProduct` to rewrite the hypergeometric product $A(n)$ by changing the upper bound from $n-1$ to n. This preprocessing step returns the nested depth-1 hypergeometric product

$$\tilde{A}(n) = \frac{n}{2\,(2\,n-1)} \prod_{k=1}^{n} \frac{2\,(2\,k-1)}{k}.$$

In particular,

$$A(n) = \tilde{A}(n) \quad \forall n \geqslant 1$$

holds. The package then reduces the hypergeometric product $\tilde{A}(n)$ to get the output Out[5] in Mathematica Session 2. ⋆

5.5 Construction of $R\Pi$-extensions for nesting depth 1 expressions in $\mathrm{ProdE}(\mathbb{K}(n, q^n))$

In this section we extend the results of Theorem 5.4.6 in the previous section to the case $\mathrm{ProdE}(\mathbb{K}(n, \mathbf{q}^n))$. As a consequence, we will also prove Theorem 3.1.1.

5.5.1 Structural results for nested Π-extensions

In the following let $(\mathbb{F}_e, \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}_0, \sigma)$ with $\mathbb{F}_e = \mathbb{F}_0(t_1)\dots(t_e)$ with $\sigma(t_i) = \alpha_i\,t_i + \beta_i$ and $\alpha_i \in \mathbb{F}_0^*$, $\beta_i \in \mathbb{F}_0$ for $1 \leqslant i \leqslant e$. We set $\mathbb{F}_i = \mathbb{F}_0(t_1)\dots(t_i)$ and thus $(\mathbb{F}_{i-1}(t_i), \sigma)$ is a $\Pi\Sigma$-extension of $(\mathbb{F}_{i-1}, \sigma)$ for $1 \leqslant i \leqslant e$.
We will use the following notations. For $\mathbf{f} = (f_1, \dots, f_s)$ and h we write $\mathbf{f} \wedge h = (f_1, \dots, f_s, h)$ for the concatenation of $\mathbf{f}$ and h. Moreover, the concatenation of $\mathbf{f}$ and $\mathbf{h} = (h_1, \dots, h_u)$ is denoted by $\mathbf{f} \wedge \mathbf{h} = (f_1, \dots, f_s, h_1, \dots, h_u)$.

Lemma 5.5.1.
Let $(\mathbb{F}_e, \sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{F}_0, \sigma)$ *as above. If the polynomials in* $\mathbf{f_i} \in (\mathbb{F}_{i-1}[t_i] \setminus \mathbb{F}_{i-1})^{s_i}$ *for* $1 \leqslant i \leqslant e$ *and* $s_i \in \mathbb{N} \setminus \{0\}$ *are such that*

$$\gcd_\sigma(f_{i,j}, f_{i,k}) = 1 \quad \text{for } 1 \leqslant j < k \leqslant s_i \tag{5.34}$$

holds and that for $1 \leqslant \ell \leqslant s_i$

$$\frac{\sigma(f_{i,\ell})}{f_{i,\ell}} \in \mathbb{F}_{i-1} \ \vee \ \forall k \in \mathbb{Z} \setminus \{0\} : \gcd(f_{i,\ell}, \sigma^k(f_{i,\ell})) = 1, \tag{5.35}$$

then $\mathcal{M}(\mathbf{f_1} \wedge \dots \wedge \mathbf{f_e}, \mathbb{F}_e) = \{\mathbf{0}_s\}$ *where* $s = s_1 + \dots + s_e$. ♦

Proof:
Let $\mathbf{v_1} \in \mathbb{Z}^{s_1}, \dots, \mathbf{v_e} \in \mathbb{Z}^{s_e}$ and $g \in \mathbb{F}_e^*$ with

$$\frac{\sigma(g)}{g} = \mathbf{f_1^{v_1}}\, \mathbf{f_2^{v_2}} \cdots \mathbf{f_e^{v_e}}. \tag{5.36}$$

Suppose that not all $\mathbf{v_i}$ with $1 \leqslant i \leqslant e$ are zero-vectors and let r be maximal such that $\mathbf{v_r} \neq \mathbf{0}_{s_r}$. Thus the right hand side of (5.36) is in $\mathbb{F}_r$ and it follows by Proposition 5.4.3 that $g = \gamma\, t_{r+1}^{u_{r+1}} \cdots t_e^{u_e}$ with $\gamma \in \mathbb{F}_r^*$ and $u_i \in \mathbb{Z}$; if t_i is a Σ-monomial, then $u_i = 0$. Hence

$$\frac{\sigma(\gamma)}{\gamma} = \alpha_{r+1}^{-u_{r+1}} \cdots \alpha_e^{-u_e}\, \mathbf{f_1^{v_1}} \cdots \mathbf{f_{r-1}^{v_{r-1}}}\, \mathbf{f_r^{v_r}} = h\, \mathbf{f_r^{v_r}}$$

with $h = \alpha_{r+1}^{-u_{r+1}} \cdots \alpha_e^{-u_e}\, \mathbf{f_1^{v_1}} \cdots \mathbf{f_{r-1}^{v_{r-1}}} \in \mathbb{F}_{r-1}^*$. Since conditions (5.34) and (5.35) hold for these entries, Lemma 5.3.1 is applicable and we get $\mathbf{v}_r = \mathbf{0}_{s_r}$, a contradiction. ■

We can now formulate a generalization of Theorem 5.3.3 for nested $\Pi\Sigma$-extensions.

Theorem 5.5.2.
Let $(\mathbb{F}_e, \sigma)$ *be the* $\Pi\Sigma$*-extension of* $(\mathbb{F}_0, \sigma)$ *from above. For* $1 \leqslant i \leqslant e$*, let* $\mathbf{f_i} = (f_{i,1}, \dots, f_{i,s_i}) \in (\mathbb{F}_{i-1}[t_i] \setminus \mathbb{F}_{i-1})^{s_i}$ *with* $s_i \in \mathbb{N} \setminus \{0\}$ *containing irreducible monic polynomials. Then the following statements are equivalent.*

(1) $\gcd_\sigma(f_{i,j}, f_{i,k}) = 1$ *for all* $1 \leqslant i \leqslant e$ *and* $1 \leqslant j < k \leqslant s_i$.

(2) There does not exist $\mathbf{v_1} \in \mathbb{Z}^{s_1}, \dots, \mathbf{v_e} \in \mathbb{Z}^{s_e}$ *with* $\mathbf{v_1} \wedge \dots \wedge \mathbf{v_e} \neq \mathbf{0}_s$ *and* $g \in \mathbb{F}_e^*$ *such that*

$$\frac{\sigma(g)}{g} = \mathbf{f_1^{v_1}} \cdots \mathbf{f_e^{v_e}}$$

holds. That is, $\mathcal{M}(\mathbf{f_1} \wedge \dots \wedge \mathbf{f_e}, \mathbb{F}_e) = \{\mathbf{0}_s\}$ *where* $s = s_1 + \dots + s_e$.

(3) One can construct a Π*-field extension* $(\mathbb{F}_e(\varkappa_{1,1}) \dots (\varkappa_{1,s_1}) \dots (\varkappa_{e,1}) \dots (\varkappa_{e,s_e}), \sigma)$ *of* $(\mathbb{F}_e, \sigma)$ *with* $\sigma(\varkappa_{i,k}) = f_{i,k}\, \varkappa_{i,k}$ *for* $1 \leqslant i \leqslant e$ *and* $1 \leqslant k \leqslant s_i$.

(4) One can construct a Π*-extension* $(\mathbb{E}, \sigma)$ *of* $(\mathbb{F}_e, \sigma)$ *with* $\mathbb{E} = \mathbb{F}_e\langle \varkappa_{1,1} \rangle \dots \langle \varkappa_{1,s_1} \rangle \dots \langle \varkappa_{e,1} \rangle \dots \langle \varkappa_{e,s_e} \rangle$ *and* $\sigma(\varkappa_{i,k}) = f_{i,k}\, \varkappa_{i,k}$ *for* $1 \leqslant i \leqslant e$ *and* $1 \leqslant k \leqslant s_i$. ♢

PROOF:
(1) $\Longrightarrow$ (2): Since the entries in $\mathbf{f}_i$ are shift co-prime and irreducible, conditions (5.34) and (5.35) hold and thus statement (2) follows by Lemma 5.5.1.
(2) $\Longrightarrow$ (3): We prove the statement by induction on the number of $\Pi\Sigma$-monomials $t_1,\dots,t_e$. For $e=0$ nothing has to be shown. Now suppose that the implication has been shown for $\mathbb{F}_{e-1}$, $e \geqslant 0$ and set $\mathbb{E} = \mathbb{F}_{e-1}(\varkappa_{1,1},\dots,\varkappa_{1,s_1})\dots(\varkappa_{e-1,1},\dots,\varkappa_{e-1,s_{e-1}})$. Suppose that $(\mathbb{E}(\varkappa_{e,1},\dots,\varkappa_{e,s_e}),\sigma)$ is not a Π-field extension of $(\mathbb{E},\sigma)$ and let ℓ be minimal with $s_\ell < s_e$ such that $(\mathbb{E}(\varkappa_{e,1},\dots,\varkappa_{e,s_\ell}),\sigma)$ is not a Π-field extension of $(\mathbb{E},\sigma)$. Then by Theorem 2.3.42 there are a $\nu_{e,s_\ell} \in \mathbb{Z}\setminus\{0\}$ and an $\omega \in \mathbb{E}(\varkappa_{e,1},\dots,\varkappa_{e,s_j})^*$ with $j = \ell - 1$ such that $\sigma(\omega) = f_{e_{s_\ell}}^{\nu_{e,s_\ell}}\,\omega$ holds. By Proposition 5.4.3, $\omega = g\,\varkappa_{e,1}^{\nu_{e,1}}\cdots\varkappa_{e,s_j}^{\nu_{e,s_j}}$ with $(\nu_{e,1},\dots,\nu_{e,s_j}) \in \mathbb{Z}^{s_j}$ and $g \in \mathbb{F}_{e-1}^*$. Thus

$$\frac{\sigma(g)}{g} = f_{e,1}^{-\nu_{e,1}}\cdots f_{e,s_j}^{-\nu_{e,s_j}}\, f_{e,s_\ell}^{\nu_{e,s_\ell}}.$$

(3) $\Longrightarrow$ (2). We prove the statement by induction on the number of $\Pi\Sigma$-monomials $t_1,\dots,t_e$. For the base case $e=0$ nothing has to be shown. Now suppose that the implication has been shown already for $e-1$ $\Pi\Sigma$-monomials and set $\mathbb{E} = \mathbb{F}_e(\varkappa_{1,1},\dots,\varkappa_{1,s_1})\dots(\varkappa_{e-1,1},\dots,\varkappa_{e-1,s_{e-1}})$. Suppose that $(\mathbb{E}(\varkappa_{e,1},\dots,\varkappa_{e,s_e}),\sigma)$ is a Π-field extension of $(\mathbb{E},\sigma)$ and assume on the contrary that there are a $g \in \mathbb{F}_e^*$ and $\boldsymbol{\nu_e} \in \mathbb{Z}^{s_e}\setminus\{\mathbf{0}_{s_e}\}$ such that

$$\frac{\sigma(g)}{g} = \mathbf{f}_1^{\boldsymbol{\nu}_1}\cdots\mathbf{f}_{e-1}^{\boldsymbol{\nu}_{e-1}}\,\mathbf{f}_e^{\boldsymbol{\nu}_e}$$

holds. Let j be maximal with $\nu_{e,j} \neq 0$ and define

$$\gamma := g\,\boldsymbol{\varkappa}_1^{-\boldsymbol{\nu}_1}\cdots\boldsymbol{\varkappa}_{e-1}^{-\boldsymbol{\nu}_{e-1}}\,\varkappa_{e,1}^{-\nu_{e,1}}\cdots\varkappa_{e,j-1}^{-\nu_{e,j-1}} \in \mathbb{E}(\varkappa_{e,1},\dots,\varkappa_{e,j-1})^*$$

where $\boldsymbol{\varkappa}_i^{-\boldsymbol{\nu}_i} = \varkappa_{i,1}^{-\nu_{i,1}}\cdots\varkappa_{i,s_i}^{-\nu_{i,s_i}}$ for $1 \leqslant i < e$ and $g \in \mathbb{F}_e^*$. Then

$$\frac{\sigma(\gamma)}{\gamma} = f_{e,j}^{\nu_{e,j}}$$

with $\nu_{e_j} \neq 0$; a contradiction since $(\mathbb{E}(\varkappa_{e,1},\dots,\varkappa_{e,j}),\sigma)$ is a Π-field extension of $(\mathbb{E}(\varkappa_{e,1},\dots,\varkappa_{e,j-1}),\sigma)$ by Theorem 2.3.42.
(2) $\Longrightarrow$ (1). We prove the statement by induction on the number of $\Pi\Sigma$-monomials $t_1,\dots,t_e$. For $e=0$ nothing has to be shown. Now assume that the implication holds for the first $e-1$ $\Pi\Sigma$-monomials. Now suppose that there are k,ℓ with $1 \leqslant k,\ell \leqslant s_e$ and $k \neq \ell$ such that $\gcd_\sigma(f_{e,k}, f_{e,\ell}) \neq 1$ holds. Since $\gcd_\sigma(f_{e,k}, f_{e,\ell}) \neq 1$ we know that they are shift equivalent and because $f_{e,k}$, $f_{e,\ell}$ are monic it follows by Lemma 4.1.7 that there is a $g \in \mathbb{F}_e^*$ with

$$\frac{\sigma(g)}{g} f_{e,k} = f_{e,\ell}$$

and thus

$$\frac{\sigma(g)}{g} = \mathbf{f}_1^{\boldsymbol{\nu}_1}\cdots\mathbf{f}_e^{\boldsymbol{\nu}_e}$$

holds with $\boldsymbol{\nu}_i = \mathbf{0}_{s_i}$ for $i \leqslant i \leqslant e-1$ and $\boldsymbol{\nu}_e = (0,\dots,0,\nu_{e,k},0,\dots,0,\nu_{e,\ell},0,\dots,0) \in \mathbb{Z}^{s_e}\setminus\{\mathbf{0}_{s_e}\}$ where $\nu_{e,k} = -1$ and $\nu_{e,\ell} = 1$.
(3) $\Longrightarrow$ (4) is obvious and (4) $\Longrightarrow$ (3) follows by Schneider (2017, Corollary 2.6). ∎

5.5.2 Proof of main result for nesting depth 1 products (Theorem 3.1.1)

Using the structural results for nested $\Pi\Sigma$-extensions from the previous chapter, we are now in the position to handle the mixed $\mathbf{q}$-multibasic case. More precisely, we will generalize Theorem 5.4.6 from the rational difference field to the mixed $\mathbf{q}$-multibasic difference field $(\mathbb{F}, \sigma)$ with $\mathbf{q} = (q_1, \dots, q_{e-1})$. Here we assume that $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)(q_1, \dots, q_{e-1})$ is a rational function field over a field K where K is strongly σ-computable. Following the notation from the previous subsection, we set $\mathbb{F}_0 := \mathbb{K}$ and $\mathbb{F}_i = \mathbb{F}_0(t_1)\dots(t_i)$ for $1 \leqslant i \leqslant e$. This means that $(\mathbb{F}_0(t_1), \sigma)$ is the Σ-extension of $(\mathbb{F}_0, \sigma)$ with $\sigma(t_1) = t_1 + 1$ and $(\mathbb{F}_{i-1}(t_i), \sigma)$ is the Π-field extension of $(\mathbb{F}_{i-1}, \sigma)$ with $\sigma(t_i) = q_{i-1}\, t_i$ for $2 \leqslant i \leqslant e$.

As for the rational case we have to merge difference rings coming from different constructions. Using Theorem 5.5.2 instead of Theorem 5.3.3, Lemma 5.4.4 generalizes straightforwardly to Lemma 5.5.3. Thus the proof is omitted here.

Lemma 5.5.3.
Let $(\mathbb{F}_e, \sigma)$ be the mixed $\mathbf{q}$-multi-basic difference field with $\mathbb{F}_0 = \mathbb{K}$ from above. Further, let $(\mathbb{K}\langle y_1\rangle \dots \langle y_w\rangle, \sigma)$ be a Π-extension of $(\mathbb{K}, \sigma)$ with $\alpha_i = \frac{\sigma(y_i)}{y_i} \in \mathbb{K}^*$ and $(\mathbb{F}_e\langle z_{1,1}\rangle \dots \langle z_{1,s_1}\rangle \dots \langle z_{e,1}\rangle \dots \langle z_{e,s_e}\rangle, \sigma)$ be a Π-extension of $(\mathbb{F}_0, \sigma)$ as given in item (4) of Theorem 5.5.2. Then the difference ring $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}_e\langle y_1\rangle \dots \langle y_w\rangle\langle z_{1,1}\rangle \dots \langle z_{1,s_1}\rangle \dots \langle z_{e,1}\rangle \dots \langle z_{e,s_e}\rangle$ is a Π-extension of $(\mathbb{F}_e, \sigma)$. Furthermore, the A-extension $(\mathbb{E}[\vartheta], \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(\vartheta) = \zeta\vartheta$ of order λ is an R-extension. ♦

Gluing everything together, we obtain a generalization of Theorem 5.4.6. Namely, one obtains an algorithmic construction of an $R\Pi$-extension in which one can represent a finite set of hypergeometric, q-hypergeometric, $\mathbf{q}$-multibasic hypergeometric and mixed $\mathbf{q}$-multibasic hypergeometric products.

Theorem 5.5.4.
Let $(\mathbb{F}_e, \sigma)$ be a mixed $\mathbf{q}$-multibasic difference field extension of $(\mathbb{F}_0, \sigma)$ with $\mathbb{F}_0 = \mathbb{K}$ where $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)(q_1, \dots, q_{e-1})$ is a rational function field, $\sigma(t_1) = t_1 + 1$ and $\sigma(t_\ell) = q_{\ell-1}\, t_\ell$ for $2 \leqslant \ell \leqslant e$. Let $h_1, \dots, h_m \in \mathbb{F}_e^*$. Then one can define an $R\Pi$-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}'(t_1), \sigma)$ with

$$\mathbb{A} = \mathbb{K}'(t_1)\dots(t_e)[\vartheta][y_1, y_1^{-1}]\dots[y_w, y_w^{-1}][z_{1,1}, z_{1,1}^{-1}]\dots[z_{1,s_1}, z_{1,s_1}^{-1}]\dots[z_{e,1}, z_{e,1}^{-1}]\dots[z_{e,s_e}, z_{e,s_e}^{-1}] \quad (5.37)$$

and $\mathbb{K}' = K'(\kappa_1, \dots, \kappa_u)(q_1, \dots, q_{e-1})$ where K' is an algebraic field extension of K such that

- *$\sigma(\vartheta) = \zeta\vartheta$ where $\zeta \in K'$ is a λ-th root of unity.*
- *$\dfrac{\sigma(y_j)}{y_j} = \alpha_j \in \mathbb{K}' \setminus \{0\}$ for $1 \leqslant j \leqslant w$ where the α_j are not roots of unity;*
- *$\dfrac{\sigma(z_{i,j})}{z_{i,j}} = f_{i,j} \in \mathbb{F}_{i-1}[t_i] \setminus \mathbb{F}_{i-1}$ are monic, irreducible and shift co-prime;*

holds with the following property. For $1 \leqslant k \leqslant m$ one can define[8]

$$g_k = r_k\, \vartheta^{\mu_k}\, y_1^{u_{k,1}} \cdots y_w^{u_{k,w}}\, z_{1,1}^{\nu_{k,1,1}} \cdots z_{1,s_1}^{\nu_{k,1,s_1}}\, z_{2,1}^{\nu_{k,2,1}} \cdots z_{2,s_2}^{\nu_{k,2,s_2}} \cdots z_{e,1}^{\nu_{k,e,1}} \cdots z_{e,s_e}^{\nu_{k,e,se}} \quad (5.38)$$

with $0 \leqslant \mu_k \leqslant \lambda - 1$, $u_{k,i} \in \mathbb{Z}$, $\nu_{k,i,j} \in \mathbb{Z}$ and $r_k \in \mathbb{F}_e^*$ such that

$$\sigma(g_k) = \sigma(h_k)\, g_k.$$

If K is strongly σ-computable, the components of the theorem can be computed. ♢

[8] We remark that this representation is related to the normal form given in Chen et al. (2011).

PROOF:
Take irreducible monic polynomials $\mathcal{B} = \{p_1, \dots, p_n\} \subseteq \mathbb{F}_0[t_1, t_2, \dots, t_e]$ and take $\gamma_1, \dots, \gamma_m \in \mathbb{F}_0^*$ such that for each k with $1 \leqslant k \leqslant m$ we get $d_{k,1}, \dots, d_{k,n} \in \mathbb{Z}$ with

$$\sigma(h_k) = \gamma_i\, p_1^{d_{k,1}} \cdots p_n^{d_{k,n}}.$$

Following the proof of Theorem 5.4.6, we can construct an RΠ-extension $\mathbb{F}'_0(x)\langle\vartheta\rangle\langle y_1\rangle \dots \langle y_w\rangle$ of $(\mathbb{F}'_0(x), \sigma)$ with constant field $\mathbb{F}'_0 = K'(\kappa_1, \dots, \kappa_u)(q_1, \dots, q_{e-1})$ where K' is an algebraic extension of K and the automorphism is defined as stated in Theorem 5.5.4 with the following property: we can define a_k of the form (5.29) in this ring with (5.30). Set $\mathcal{I}_i = \big\{\omega \in \mathcal{B} \,|\, \omega \in \mathbb{K}[t_1, t_2, \dots, t_i] \setminus \mathbb{K}[t_1, t_2, \dots, t_{i-1}]\big\}$ for $1 \leqslant i \leqslant e$ and define $I = \{1 \leqslant i \leqslant e \mid \mathcal{I}_i \neq \{\}\}$. Then for each $i \in I$ there is a partition $\mathcal{P}_i = \{\mathcal{E}_{i,1}, \dots, \mathcal{E}_{i,s_i}\}$ of $\mathcal{I}_i$ with respect to the shift-equivalence of the automorphism defined for each t_i, i.e., each $\mathcal{E}_{i,j}$ with $1 \leqslant j \leqslant s_i$ and $i \in I$ contains precisely the shift equivalent elements of $\mathcal{P}_i$. Take a representative from each equivalence class $\mathcal{E}_{i,j}$ in $\mathcal{P}_i$ and collect them in $\mathcal{F}_i := \{f_{i,1}, \dots, f_{i,s_i}\}$. By construction it follows that property (1) in Theorem 5.5.2 holds; here we put all t_i with $i \notin I$ in the ground field. Therefore by Theorem 5.5.2 we obtain the Π-field extension $(\mathbb{F}_e(\varkappa_{1,1}) \dots (\varkappa_{1,s_1}) \dots (\varkappa_{e,1}) \dots (\varkappa_{e,s_e}), \sigma)$ of $(\mathbb{F}_e, \sigma)$ with $\sigma(\varkappa_{i,k}) = f_{i,k}\, \varkappa_{i,k}$ for all $i \in I$ and $1 \leqslant k \leqslant s_i$ with $s_i \in \mathbb{N} \setminus \{0\}$; for $i \notin I$ we set $s_i = 0$. By Lemma 5.5.3 and Remark 5.4.5, $(\mathbb{A}, \sigma)$ with (5.37) is an RΠ-extension of $(\mathbb{F}'_0(t_1) \dots (t_e), \sigma)$. Let i, j with $i \in I$ and $1 \leqslant j \leqslant s_i$. Since each $f_{i,j}$ is shift equivalent with every element of $\mathcal{E}_{i,j}$, it follows by Lemma 4.1.7 that for all $h \in \mathcal{E}_{i,j}$, there is a rational function $0 \neq r \in \mathbb{F}_i \setminus \mathbb{F}_{i-1}$ with

$$h = \frac{\sigma(r)}{r}\, f_{i,j}.$$

Putting everything together we obtain for each k with $1 \leqslant k \leqslant m$, an $0 \neq r_k \in \mathbb{F}_e$ and $\boldsymbol{\nu}_{k,i} = (\nu_{k,i,1}, \dots, \nu_{k,i,s_i}) \in \mathbb{Z}^{s_i}$ with

$$p_1^{d_{k,1}} \cdots p_n^{d_{k,n}} = \frac{\sigma(r_k)}{r_k}\, \mathbf{f}_1^{\boldsymbol{\nu}_1} \cdots \mathbf{f}_e^{\boldsymbol{\nu}_e}.$$

Note that for

$$b_k := r_k\, \varkappa_{1,1}^{\nu_{k,1,1}} \cdots \varkappa_{1,s_1}^{\nu_{k,1,s_1}}\, \varkappa_{2,1}^{\nu_{k,2,1}} \cdots \varkappa_{2,s_2}^{\nu_{k,2,s_2}} \cdots \varkappa_{e,1}^{\nu_{k,e,1}} \cdots \varkappa_{e,s_e}^{\nu_{k,e,s_e}} \in \mathbb{A}$$

we have that

$$\sigma(b_k) = p_1^{d_{k,1}} \cdots p_n^{d_{k,n}}\, b_k.$$

Now let $g_k \in \mathbb{A}$ be as defined in (5.38). Since $g_k = a_k\, b_k$ where a_k equals (5.29) and has the property (5.30), we get

$$\sigma(g_k) = \sigma(h_k)\, g_k.$$

The proof of the computational part is the same as that of Theorem 5.4.6. ∎

We are now ready to complete the proof for Theorem 3.1.1. To link to the notations used there, we set $\mathbf{q} = (q_1, \dots, q_{e-1})$ and set further $(x, t_1, \dots, t_{e-1}) = (t_1, \dots, t_e)$, in particular we use the shortcut $\mathbf{t} = (t_2, \dots, t_{e-1})$.

PROOF (THEOREM 3.1.1):
Suppose we are given the products (3.1) and that we are given the components as stated in Theorem 5.5.4. Then we follow the strategy as in Section 5.4.

- Take the $\mathbb{K}'$-embedding

$$\tau : \mathbb{K}'(x, \mathbf{t}) \to \mathcal{S}(\mathbb{K}') \quad \text{where} \quad \tau(\tfrac{a}{b}) = \langle \text{ev}\left(\tfrac{a}{b}, n\right)\rangle_{n \geqslant 0}$$

for $a, b \in \mathbb{K}'[x, \mathbf{t}]$ is defined by (2.2). Then by iterative application of part (1) of Lemma 2.4.3 we can construct the $\mathbb{K}'$-homomorphism

$$\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K}')$$

determined by the homomorphic extension with

- $\tau(\vartheta) = \langle \zeta^n \rangle_{n \geqslant 0}$,
- $\tau(y_i) = \langle \alpha_i^n \rangle_{n \geqslant 0}$ for $1 \leqslant i \leqslant w$ and
- $\tau(z_{i,j}) = \langle \prod_{k=\ell'_{i,j}}^{n} f_{i,j}(k-1, \mathbf{q}^{k-1}) \rangle_{n \geqslant 0}$ with $\ell'_{i,j} = Z(f_{i,j}) + 1$ for $1 \leqslant i \leqslant e$, $1 \leqslant j \leqslant s_i$.

In particular, since $(\mathbb{A}, \sigma)$ is an $R\Pi$-extension of $(\mathbb{K}'(x, \mathbf{t}), \sigma)$, it follows by part (3) of Lemma 2.4.3 that τ is a $\mathbb{K}'$-embedding.

• Finally, define for $1 \leqslant i \leqslant m$ the product expressions

$$\begin{aligned} G_i(n) = r_i(n)\,(\zeta^n)^{\mu_i}\,(\alpha_1^n)^{u_{i,1}} \cdots (\alpha_w^n)^{u_{i,w}} \\ \Big(\prod_{k=\ell'_{1,1}}^{n} f_{1,1}(k-1, \mathbf{q}^{k-1})\Big)^{\nu_{i,1,1}} \cdots \Big(\prod_{k=\ell'_{1,s_1}}^{n} f_{1,s_1}(k-1, \mathbf{q}^{k-1})\Big)^{\nu_{i,1,s_1}} \cdots \\ \Big(\prod_{k=\ell'_{e,1}}^{n} f_{e,1}(k-1, \mathbf{q}^{k-1})\Big)^{\nu_{i,e,1}} \cdots \Big(\prod_{k=\ell'_{e,s_e}}^{n} f_{e,s_e}(k-1, \mathbf{q}^{k-1})\Big)^{\nu_{i,e,s_e}} \end{aligned}$$

and define $\delta_i = \max(\ell_i, \ell'_{1,1}, \ldots, \ell'_{1,s_1}, \ldots, \ell'_{e,1}, \ldots, \ell'_{e,s_e}, Z(r_i))$. Then observe that

$$\tau(g_i) = \langle G'_i(n) \rangle_{n \geqslant 0}$$

with (5.32). Now set $Q_i(n) := c\, G_i(n)$ with

$$c = \frac{P_i(\delta_i)}{G_i(\delta_i)} \in \mathbb{K}'.$$

Then as for the proof of the rational case we conclude that

$$P_i(n) = Q_i(n) \quad \text{for } n \geqslant \delta_i.$$

This proves part (1) of Theorem 3.1.1. Since τ is a $\mathbb{K}'$-embedding, the sequences

$$\langle \alpha_1^n \rangle_{n \geqslant 0}, \ldots, \langle \alpha_w^n \rangle_{n \geqslant 0}, \langle \prod_{k=\ell'_1}^{n} f_{1,1}(k-1, \mathbf{q}^{k-1}) \rangle_{n \geqslant 0}, \ldots, \langle \prod_{k=\ell'_{e,s_e}}^{n} f_{e,s_e}(k-1, \mathbf{q}^{k-1}) \rangle_{n \geqslant 0}$$

are among each other algebraically independent over $\tau\big(\mathbb{K}'(x)\big)[\langle \zeta^n \rangle_{n \geqslant 0}]$ which proves property (2) of Theorem 3.1.1. ∎

Example 5.5.5.
Let $\mathbb{K} = K(q_1, q_2)$ be the rational function field over the algebraic number field $K = \mathbb{Q}(\sqrt{-3}, \sqrt{-13})$, and consider the mixed $\mathbf{q} = (q_1, q_2)$-multibasic hypergeometric product expression

$$P(n) = \prod_{k=1}^{n} \frac{\sqrt{-13}\,(k\, q_1^k + 1)}{k^2\,(q_1^{k+1}\, q_2^{k+1} + k + 1)} + \prod_{k=1}^{n} \frac{k^2\,(k + q_1^k\, q_2^k)^2}{\sqrt{-3}\,(k+1)^2} + \prod_{k=1}^{n} \frac{169\,(k\, q_1^k\, q_2^k + q_2^k + k\, q_1^k + 1)}{(k\, q_1^{k+2} + 2\, q_1^{k+2} + 1)\, k^2}. \quad (5.39)$$

Now take the mixed $\mathbf{q}$-multibasic difference field extension $(\mathbb{K}(x)(t_1)(t_2), \sigma)$ of $(\mathbb{K}, \sigma)$ with

$$\sigma(x) = x + 1, \qquad \sigma(t_1) = q_1\, t_1, \qquad \sigma(t_2) = q_2\, t_2. \tag{5.40}$$

Note that $h_1(k, q_1^k, q_2^k)$, $h_2(k, q_1^k, q_2^k)$ and $h_3(k, q_1^k, q_2^k)$ with

$$h_1 = \frac{\sqrt{-13}\,(x\,t_1 + 1)}{x^2\,(q_1\,t_1\,q_2\,t_2 + x + 1)},\ h_2 = \frac{x^2\,(x + t_1\,t_2)^2}{\sqrt{-3}\,(x+1)^2},\ h_3 = \frac{169\,(x\,t_1\,t_2 + t_2 + x\,t_1 + 1}{(x\,q_1^2 t_1 + 2\,q_1^2 t_2 + 1)\,x^2} \in \mathbb{K}(x, t_1, t_2)$$

are the multiplicands of the above products, respectively. Applying Theorem 5.5.4 we construct the algebraic number field extension $\mathbb{K}' = \mathbb{Q}\big((-1)^{\frac{1}{2}}, \sqrt{3}, \sqrt{13}, q_1, q_2\big)$ of $\mathbb{K}$ and take the $\Pi\Sigma$-extension $(\mathbb{F}', \sigma)$ of $(\mathbb{K}', \sigma)$ with $\mathbb{F}' = \mathbb{K}'(x)(t_1)(t_2)$ with automorphisms defined as (5.40). On top of this mixed $\mathbf{q}$-multibasic difference field over $\mathbb{K}'$ we construct the $R\Pi$-extension $(\mathbb{A}, \sigma)$ with $\mathbb{A} = \mathbb{F}'\langle\vartheta\rangle\langle y_1\rangle\langle y_2\rangle\langle\varkappa_1\rangle\langle\varkappa_2\rangle\langle\varkappa_3\rangle\langle\varkappa_4\rangle$ where the R-monomial ϑ with $\sigma(\vartheta) = (-1)^{\frac{1}{2}}\,\vartheta$ and the Π-monomials y_1, y_2 with $\sigma(y_1) = \sqrt{3}\,y_1$ and $\sigma(y_2) = \sqrt{13}\,y_2$ are used to scope the content of the polynomials in h_1, h_2, h_3. Furthermore, the Π-monomials $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4$ with

$$\sigma(\varkappa_1) = (x{+}1)\,\varkappa_1,\ \sigma(\varkappa_2) = \big((x{+}1)\,q_1\,t_1{+}1\big)\,\varkappa_2\ \sigma(\varkappa_3) = (q_2\,t_2{+}1)\,\varkappa_3,\ \sigma(\varkappa_4) = (q_2\,q_1\,t_2\,t_1{+}x{+}1)\,\varkappa_4$$

are used to handle the monic polynomials in h_1, h_2, h_3. These Π-monomials are constructed in an iterative fashion as worked out in the proof of Theorem 5.5.4. In particular, within this construction we derive

$$Q = \underbrace{\frac{(q_2\,q_1 + 1)\,\vartheta\,y_2\,\varkappa_2}{\big(q_2\,q_1\,t_2\,t_1 + x + 1\big)\,\varkappa_1^2\,\varkappa_4}}_{=:g_1} + \underbrace{\frac{\vartheta^3\,\varkappa_4^2}{\big(x+1\big)^2\,y_1}}_{=:g_2} + \underbrace{\frac{\big(q_1+1\big)\,\big(2\,q_1^2+1\big)\,y_2^4\,\varkappa_3}{\big((x+1)\,q_1\,t_1 + 1\big)\,\big((x+2)\,q_1^2\,t_1 + 1\big)\,\varkappa_1^2}}_{=:g_3}$$

such that

$$\sigma(g_i) = \sigma(h_i)\,g_i$$

holds for $i = 1, 2, 3$. Now take the $\mathbb{K}'$-embedding

$$\tau : \mathbb{K}'(x, \mathbf{t}) \to \mathcal{S}(\mathbb{K}') \ \text{ where } \ \tau(\tfrac{a}{b}) = \langle \mathrm{ev}\left(\tfrac{a}{b}, n\right)\rangle_{n\geqslant 0}$$

for $a, b \in \mathbb{K}'[x, \mathbf{t}]$ is defined by (2.2). Then by iterative application of part (1) of Lemma 2.4.3 we can construct the $\mathbb{K}'$-embedding

$$\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K}')$$

determined by the homomorphic extension of

$$\tau(\vartheta) = \big\langle \big((-1)^{\frac{1}{2}}\big)^n \big\rangle_{n\geqslant 0}, \qquad \tau(y_1) = \langle (\sqrt{3})^n \rangle_{n\geqslant 0}, \qquad \tau(y_2) = \langle (\sqrt{13})^n \rangle_{n\geqslant 0},$$

$$\tau(\varkappa_1) = \langle n! \rangle_{n\geqslant 0}, \qquad \tau(\varkappa_2) = \langle \prod_{k=1}^{n} (k\,q_1^k + 1) \rangle_{n\geqslant 0}, \qquad \tau(\varkappa_3) = \langle \prod_{k=1}^{n} (q_2^k + 1) \rangle_{n\geqslant 0},$$

$$\tau(\varkappa_4) = \langle \prod_{k=1}^{n} (q_2^k\,q_1^k + k) \rangle_{n\geqslant 0}.$$

By our construction we can conclude that $\tau(g_1)$, $\tau(g_2)$ and $\tau(g_3)$ equal the sequences produced by the three products in (5.39), respectively. In particular, $\tau(Q) = \langle P(n)\rangle_{n\geqslant 0}$. Furthermore, if we define

$$\begin{aligned} Q(n) = {} & \frac{(q_2\,q_1 + 1)}{\big(q_2^{n+1}\,q_1^{n+1} + n + 1\big)} \big((-1)^{\frac{1}{2}}\big)^n \big(\sqrt{13}\big)^n \frac{1}{(n!)^2} \prod_{k=1}^{n} \big(k\,q_1^k + 1\big) \prod_{k=1}^{n} \frac{1}{\big(q_2^k\,q_1^k + k\big)} \\ & + \frac{1}{(n+1)^2} \Big(\big((-1)^{\frac{1}{2}}\big)^n\Big)^3 \Big(\big(\sqrt{3}\big)^n\Big)^{-1} \left(\prod_{k=1}^{n} \big(q_2^k\,q_1^k + k\big)\right)^2 \\ & + \frac{(q_1+1)\,(2\,q_1^2+1)}{\big((n+1)\,q_1^{n+1} + 1\big)\,\big((n+2)\,q_1^{n+2} + 1\big)} \Big(\big(\sqrt{13}\big)^n\Big)^4 \frac{1}{(n!)^2} \prod_{k=1}^{n} \big(q_2^k + 1\big), \end{aligned}$$

then we can guarantee that

$$P(n) = Q(n) \quad \text{for all } n \geqslant 1.$$

The sequences generated by

$$(\sqrt{3})^n,\ (\sqrt{13})^n,\ n!,\ \prod_{k=1}^{n}(k\,q_1^k+1),\ \prod_{k=1}^{n}(q_2^k+1),\ \prod_{k=1}^{n}(q_2^k\,q_1^k+k)$$

are algebraically independent among each other over $\tau(\mathbb{K}'(x,\mathbf{t}))[\langle((-1)^{\frac{1}{2}})^n\rangle_{n\geqslant 0}]$ by construction. $\star$

Example 5.5.6.
We use the Mathematica package `NestedProducts` to reduce the mixed $\mathbf{q} = (q_1, q_2)$-multibasic hypergeometric product (5.39) in Example 5.5.5 above.

Mathematica Session 3

In[6]:= $\mathbf{P = FProduct\Big[\dfrac{\sqrt{-13}\,(k\,q_1^k+1)}{k^2\,(q_1^{k+1}\,q_2^{k+1}+k+1)},\{k,1,n\}\Big] + FProduct\Big[\dfrac{k^2\,(k+q_1^k\,q_2^k)^2}{\sqrt{-3}\,(k+1)^2},\{k,1,n\}\Big] + FProduct\Big[\dfrac{169\,(k\,q_1^k\,q_2^k+q_2^k+k\,q_1^k+1)}{(k\,q_1^{k+2}+2\,q_1^{k+2}+1)\,k^2},\{k,1,n\}\Big];}$

In[7]:= $\mathbf{Q = ProductReduce[P]}$

Out[7]= $$\frac{(q_2\,q_1+1)}{(q_2^{n+1}\,q_1^{n+1}+n+1)}\,\frac{(\mathbb{i})^n\,(\sqrt{13})^n}{(n!)^2}\left(\prod_{k=1}^{n}(k\,q_1^k+1)\right)\left(\prod_{k=1}^{n}\frac{1}{(q_2^k\,q_1^k+k)}\right)+$$
$$\frac{1}{(n+1)^2}\,\frac{((\mathbb{i})^n)^3}{(\sqrt{3})^n}\left(\prod_{k=1} n(q_2^k\,q_1^k+k)\right)^2+$$
$$\frac{(q_1+1)\,(2\,q_1^2+1)}{((n+1)\,q_1^{n+1}+1)\,((n+2)\,q_1^{n+2}+1)}\,\frac{((\sqrt{13})^n)^4}{(n!)^2}\left(\prod_{k=1}^{n}(q_2^k+1)\right)$$

Note that the hypergeometric product, Q given by Out[7] in Mathematica Session 3 above is an element of $\mathrm{ProdE}(\tilde{\mathbb{K}}(n))$ where $\tilde{\mathbb{K}} = \mathbb{Q}(\mathbb{i}, \sqrt{3}, \sqrt{13})$. $\star$

Chapter 6

Algorithmic construction of RΠ-extensions for higher nesting depth expressions in $\mathrm{ProdE}(\mathbb{K}(n))$

In Chapter 5 we discussed how one can model product expressions of nesting depth 1 in an RΠ-extensions. In this chapter we will extend this to product expressions of higher nesting depths. That is, expression in $\mathrm{Prod}(\mathbb{S})$ with more than one product quantifier, where $\mathbb{S} \in \{\mathbb{K},\ \mathbb{K}(n)\}$.

6.1 Preprocessing hypergeometric products of finite nesting depth

In this section we will present a detailed discussion of how hypergeometric products of finite nesting depth in $\mathrm{Prod}(\mathbb{K}(n))$ are preprocessed. We will illustrate each preprocessing step with an example and then summarise the entire preprocessing step in Lemma 6.1.5 below.

Let $(\mathbb{K}(x), \sigma)$ be a rational difference field with $\sigma(x) = x + 1$ together with the evaluation function (2.1) and the Z-function (2.50). Let $P(n)$ be a hypergeometric product in n of nesting depth $m \in \mathbb{N}$ given by

$$P(n) = \prod_{k_1=l_1}^{n} f_1(k_1) \prod_{k_2=l_2}^{k_1} f_2(k_2) \cdots \prod_{k_m=l_m}^{k_{m-1}} f_m(k_m) \in \mathrm{Prod}(\mathbb{K}(n)) \tag{6.1}$$

where $f_i(x) \in \mathbb{K}(x)^*$ and $l_i \in \mathbb{N}$ for all $1 \leqslant i \leqslant m$. Then $P(n)$ in $\mathrm{Prod}(\mathbb{K}(n))$ is preprocessed as follows.

(1) Note that by Remark 2.1.11, (6.1) can be written in a product factored form

$$P(n) = \left(\prod_{k_1=l_1}^{n} f_1(k_1)\right)\left(\prod_{k_1=l_1}^{n}\prod_{k_2=l_2}^{k_1} f_2(k_2)\right) \cdots \left(\prod_{k_1=l_1}^{n}\prod_{k_2=l_2}^{k_1} \cdots \prod_{k_m=l_m}^{k_{m-1}} f_m(k_m)\right) \in \mathrm{ProdE}(\mathbb{K}(n)). \tag{6.2}$$

Write

$$f_i = u_i f_{i,1}^{e_{i,1}} \cdots f_{i,r_i}^{e_{i,r_i}} \tag{6.3}$$

where $u_i \in \mathbb{K}^*$ for $1 \leqslant i \leqslant m$, $f_{i,j} \in \mathbb{K}[x] \setminus \mathbb{K}$ are irreducible monic polynomials and $e_{i,j} \in \mathbb{Z}$ for all $1 \leqslant i \leqslant m$ and for $1 \leqslant j \leqslant r_i$ and for some $r_i \in \mathbb{N}$. Substituting (6.3) into (6.2) and expanding the product quantifiers over each factor in (6.3) we get

$$P(n) = A_1(n)\, A_2(n) \cdots A_m(n) \in \mathrm{ProdE}(\mathbb{K}(n))$$

where

$$A_i(n) = \left(\prod_{k_1=l_1}^{n} \cdots \prod_{k_i=l_i}^{k_{i-1}} u_i\right) \left(\prod_{k_1=l_1}^{n} \cdots \prod_{k_i=l_i}^{k_{i-1}} f_{i,1}(k_i)\right)^{e_{i,1}} \cdots \left(\prod_{k_1=l_1}^{n} \cdots \prod_{k_i=l_i}^{k_{i-1}} f_{i,r_i}(k_i)\right)^{e_{i,r_i}} \tag{6.4}$$

for all $1 \leqslant i \leqslant m$. In particular, the first product on the right hand side in (6.4) with innermost multiplicand $u_i \in \mathbb{K}^*$ is a geometric product of nesting depth i in $\mathrm{Prod}(\mathbb{K})$ while the rest are nesting depth i hypergeometric products in $\mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$.

Example 6.1.1.
Let $(\mathbb{K}(x), \sigma)$ be the rational difference field with $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$ and the automorphism $\sigma(x) = x + 1$ equipped with the evaluation function (2.1) and the Z-function (2.50). Suppose we are given the nesting depth 2 hypergeometric product

$$P(n) = \prod_{k=1}^{n} \frac{4\,k^2+1}{\sqrt{-3}\,k} \prod_{j=1}^{k} \frac{-2\,(j^3-7\,j+6)}{5\,(j^2-j-6)} \in \mathrm{Prod}(\mathbb{K}(n)). \tag{6.5}$$

Then with

$$A_1(n) = \left(\prod_{k=1}^{n} 4\right) \left(\prod_{k=1}^{n} k^2 + \tfrac{1}{4}\right) \left(\prod_{k=1}^{n} \frac{1}{\sqrt{-3}}\right) \left(\prod_{k=1}^{n} \frac{1}{k}\right) \tag{6.6}$$

$$\begin{aligned} A_2(n) = &\left(\prod_{k=1}^{n}\prod_{j=1}^{k} -2\right) \left(\prod_{k=1}^{n}\prod_{j=1}^{k} \frac{1}{5}\right) \left(\prod_{k=1}^{n}\prod_{j=1}^{k} \frac{1}{(j-3)}\right) \left(\prod_{k=1}^{n}\prod_{j=1}^{k} (j-2)\right) \\ &\left(\prod_{k=1}^{n}\prod_{j=1}^{k} (j-1)\right) \left(\prod_{k=1}^{n}\prod_{j=1}^{k} \frac{1}{(j+2)}\right) \left(\prod_{k=1}^{n}\prod_{j=1}^{k} (j+3)\right) \end{aligned} \tag{6.7}$$

equation (6.5) can be written in the form

$$P(n) = A_1(n)\,A_2(n) \in \mathrm{ProdE}(\mathbb{K}(n))$$

where the product factors of $A_1(n)$ and $A_2(n)$ which are elements of $\mathrm{ProdE}(\mathbb{K}(n)\setminus\mathbb{K})$ have innermost multiplicands being irreducible monic polynomials over $\mathbb{K}$. ⋆

(2) Next we choose a $\delta \in \mathbb{N}$ such that

$$\forall\, n \geqslant \delta,\; P(n) \neq 0.$$

More precisely, with $\mathcal{S} := \{f_{1,1}, \ldots, f_{1,r_1}, \ldots, f_{m,1}, \ldots, f_{m,r_m}\}$ and $\mathcal{L} = \{l_1, \ldots, l_m\}$ we take

$$\delta = \max\big(\{Z(f) \,|\, f \in \mathcal{S}\} \cup \mathcal{L}\big).$$

Example 6.1.2 (Cont. Example 6.1.1).
With $\mathcal{S} := \{4,\, x^2 + \frac{1}{4},\, -2,\, x-2,\, x-1,\, x+3,\, \sqrt{-3},\, k,\, 5,\, x-3,\, x+2\}$ and $\mathcal{L} := \{1\}$, we get

$$\delta = \max\big(\{Z(f) \,|\, f \in \mathcal{S}\} \cup \mathcal{L}\big) = 4.$$

⋆

(3) For all $1 \leqslant i \leqslant m$, rewrite each product factor in (6.4) such that the lower bounds are synchronised to the δ calculated in step (2). More precisely we apply the formula (6.9) to each of the products in (6.4). Note that in (6.9) all the products on the right hand side are from $\mathrm{Prod}(\mathbb{K})$ except the last one which is from $\mathrm{Prod}(\mathbb{K}(n) \setminus \mathbb{K})$. Summarising we obtain

$$\tilde{P}(n) = \tilde{A}_1(n)\,\tilde{A}_2(n) \cdots \tilde{A}_m(n)$$

with

$$\tilde{A}_i(n) = a_i \left(\prod_{k_1=\delta}^{n} \tilde{u}_{i,1}\right) \cdots \left(\prod_{k_1=\delta}^{n} \cdots \prod_{k_i=\delta}^{k_{i-1}} \tilde{u}_{i,i}\right) \left(\prod_{k_1=\delta}^{n} \cdots \prod_{k_i=\delta}^{k_{i-1}} f_{i,1}(k_i)\right)^{e_{i,1}} \cdots \left(\prod_{k_1=\delta}^{n} \cdots \prod_{k_i=\delta}^{k_{i-1}} f_{i,r_i}(k_i)\right)^{e_{i,r_i}} \tag{6.8}$$

where a_i, $\tilde{u}_{i,j} \in \mathbb{K}^*$ for some $j, \in \mathbb{N}$. Here we utilise the rewrite rule

$$\begin{aligned}\prod_{k_1=l_1}^{n} \prod_{k_2=l_2}^{k_1} \cdots \prod_{k_i=l_i}^{k_{i-1}} h(k_i) &= \left(\prod_{k_1=\lambda_1}^{\delta-1} \prod_{k_2=\lambda_2}^{k_1} \cdots \prod_{k_i=\lambda_i}^{k_{i-1}} h(k_i)\right) \left(\prod_{k_1=\delta}^{n} \prod_{k_2=\lambda_2}^{\delta-1} \cdots \prod_{k_i=\lambda_i}^{k_{i-1}} h(k_i)\right) \\ &\left(\prod_{k_1=\delta}^{n} \prod_{k_2=\delta}^{k_1} \prod_{k_3=\lambda_3}^{\delta-1} \cdots \prod_{k_i=\lambda_i}^{k_{i-1}} h(k_i)\right) \cdots \left(\prod_{k_1=\delta}^{n} \prod_{k_2=\delta}^{k_1} \cdots \prod_{k_{i-1}=\delta}^{k_{i-2}} \prod_{k_i=\lambda_i}^{\delta-1} h(k_i)\right) \left(\prod_{k_1=\delta}^{n} \prod_{k_2=\delta}^{k_1} \cdots \prod_{k_i=\delta}^{k_{i-1}} h(k_i)\right)\end{aligned} \tag{6.9}$$

where $\lambda_j = \max(Z(h), l_j) \leqslant \delta$ for all j with $1 \leqslant j \leqslant i$. In other words, the geometric products in (6.4) are updated accordingly. In particular we have that, $A_i(n) = \tilde{A}_i(n)$ for all $n \geqslant \delta$ and consequently,

$$P(n) = \tilde{P}(n)$$

holds for all $n \geqslant \delta$.

Example 6.1.3 (Cont. Example 6.1.2).
Synchronising the lower bounds of each product factor in (6.6) and (6.7) to 4 computed in Example 6.1.2 and rewriting each product factor in (6.6) and (6.7) we get the following:

$$\tilde{A}_1(n) = \frac{3145\,\mathbb{i}}{18\sqrt{3}} \left(\prod_{k=4}^{n} \frac{4}{\sqrt{-3}}\right) \left(\prod_{k=4}^{n} \frac{1}{k}\right) \left(\prod_{k=4}^{n} (k^2 + \tfrac{1}{4})\right) \tag{6.10}$$

$$\begin{aligned}\tilde{A}_2(n) = \frac{1024}{28125} &\left(\prod_{k=4}^{n} -\frac{32}{125}\right) \left(\prod_{k=4}^{n} \prod_{j=4}^{k} -\frac{2}{5}\right) \left(\prod_{k=4}^{n} \prod_{j=4}^{k} \frac{1}{(j-3)}\right) \left(\prod_{k=4}^{n} \prod_{j=4}^{k} (j-2)\right) \\ &\left(\prod_{k=4}^{n} \prod_{j=4}^{k} (j-1)\right) \left(\prod_{k=4}^{n} \prod_{j=4}^{k} \frac{1}{(j+2)}\right) \left(\prod_{k=4}^{n} \prod_{j=4}^{k} (j+3)\right).\end{aligned} \tag{6.11}$$

In particular, for $i = 1, 2$, and for all $n \geqslant \delta$ where $\delta = 4$,

$$A_i(n) = \tilde{A}_i(n)$$

holds. Consequently, with

$$\tilde{P}(n) = \tilde{A}_1(n)\,\tilde{A}_2(n)$$

we have that

$$P(n) = \tilde{P}(n)$$

holds for all $n \geqslant 4$. ⋆

(4) Finally for all $1 \leqslant i \leqslant m$, we rewrite each geometric product in (6.8) (i.e., the product factors in $\mathrm{Prod}(\mathbb{K})$) such that the lower bound δ are synchronised to 1. Here one can use a similar formula as given in (6.9). This yields

$$P'(n) = A'_1(n)\,A'_2(n)\cdots A'_m(n)$$

where

$$A'_i(n) = \tilde{a}_i \underbrace{\left(\prod_{k_1=1}^{n} \tilde{u}_{i,1}\right)}_{=:B_{i,1}(n)} \cdots \underbrace{\left(\prod_{k_1=1}^{n}\cdots\prod_{k_i=1}^{k_{i-1}} \tilde{u}_{i,i}\right)}_{=:B_{i,i}(n)} \underbrace{\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_i=\delta}^{k_{i-1}} f_{i,1}(k_i)\right)^{e_{i,1}} \cdots \left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_i=\delta}^{k_{i-1}} f_{i,r_i}(k_i)\right)^{e_{i,r_i}}}_{=:F_i(n)} \tag{6.12}$$

with $\tilde{a}_i,\ \tilde{u}_{i,j} \in \mathbb{K}^*$ for some $j \in \mathbb{N}$. In particular we have that, $A_i(n) = A'_i(n)$ holds for all $n \geqslant \delta$ and consequently,

$$P(n) = P'(n)$$

holds for all $n \geqslant \delta$. Further, $P'(n)$ can be rewritten as

$$P'(n) = c\,G(n)\,H(n)$$

where $c = \tilde{a}_1\,\tilde{a}_2\cdots\tilde{a}_m$ is a unit in $\mathbb{K}^*$, $G(n)$ is composed multiplicatively by geometric products in product factored form of nesting depth at most m. More precisely,

$$G(n) = E_1(n)\,E_2(n)\cdots E_m(n) \in \mathrm{ProdE}(\mathbb{K})$$

where

$$E_i(n) = B_{1,i}(n)\,\cdots\,B_{m,i}(n) \in \mathrm{ProdE}(\mathbb{K})$$

is a nesting depth i geometric product expression which is composed multiplicatively by the geometric products $B_{j,i}(n)$ in (6.12) for $1 \leqslant j \leqslant m$. The geometric product $B_{j,i}(n)$ is in product factored form and of nesting depth i. In particular, its lower bounds are all synchronised to 1. On the other hand, the product expression $H(n)$ is composed by multiplicatively by products over irreducible monic polynomials in $\mathbb{K}[x] \setminus \mathbb{K}$ of nesting depth at most m. That is,

$$H(n) = F_1(n)\,F_2(n)\cdots F_m(n) \in \mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$$

where $F_i(n)$ is the hypergeometric product expression in (6.12) which is composed multiplicatively by hypergeometric products in product factored form with their lower bounds all synchronised to δ.

Example 6.1.4 (Cont. Example 6.1.3).
Synchronising the lower bounds of each geometric product in (6.10) and (6.11) to 1 and rewriting these geometric products we get

$$A'_1(n) = \frac{3145}{384}\left(\prod_{k=1}^{n}\frac{4}{\sqrt{-3}}\right)\left(\prod_{k=4}^{n}\frac{1}{k}\right)\left(\prod_{k=4}^{n}(k^2+\tfrac{1}{4})\right)$$

$$A'_2(n) = \frac{5}{36}\left(\prod_{k=1}^{n}4\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}-\frac{2}{5}\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k}\frac{1}{(j-3)}\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k}(j-2)\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k}(j-1)\right)$$
$$\left(\prod_{k=4}^{n}\prod_{j=4}^{k}\frac{1}{(j+2)}\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k}(j+3)\right).$$

In particular, for $i = 1, 2$, and for all $n \geqslant 4$,

$$A_i(n) = A'_i(n)$$

holds. In total we obtain

$$P'(n) = A'_1(n)\, A'_2(n) = c\, G(n)\, H(n)$$

with

$$\begin{aligned}
c &= \frac{15725}{13824}\\
G(n) &= \left(\prod_{k=1}^{n} \frac{16}{\sqrt{-3}}\right) \left(\prod_{k=1}^{n}\prod_{j=1}^{k} -\frac{2}{5}\right)\\
H(n) &= \left(\prod_{k=4}^{n} k^2 + \tfrac{1}{4}\right) \left(\prod_{k=4}^{n} \frac{1}{k}\right) \left(\prod_{k=4}^{n}\prod_{j=4}^{k} \frac{1}{(j-3)}\right) \left(\prod_{k=4}^{n}\prod_{j=4}^{k} (j-2)\right) \left(\prod_{k=4}^{n}\prod_{j=4}^{k} (j-1)\right)\\
&\quad \left(\prod_{k=4}^{n}\prod_{j=4}^{k} \frac{1}{(j+2)}\right) \left(\prod_{k=4}^{n}\prod_{j=4}^{k} (j+3)\right)
\end{aligned} \tag{6.14}$$

such that

$$P(n) = P'(n)$$

holds for all $n \geqslant 4$. Note that c is a unit in $\mathbb{K}^*$, $G(n)$ is composed multiplicatively by geometric products in $\mathrm{Prod}(\mathbb{K})$ of nesting depth at most 2 where each geometric product is in product factored form and with all lower bounds synchronised to 1. On the other hand, the hypergeometric product expression $H(n)$ is composed multiplicatively by hypergeometric products over irreducible monic polynomials in $\mathbb{K}[x] \setminus \mathbb{K}$ and is of nesting depth at most m. In particular, $H(n)$ is an element of $\mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$ and the hypergeometric product factors in $H(n)$ are all in product factored form with all lower bounds synchronised to 4. $\star$

We summarise the preprocessing steps described in this section with the following lemma.

Lemma 6.1.5.
Let $(\mathbb{K}(x), \sigma)$ be a rational difference field with $\sigma(x) = x+1$ together with the evaluation function (2.1) and the Z-function (2.50). Let $\{P_1(n), \ldots, P_e(n)\}$ be a finite set of hypergeometric products of finite nesting depth in $\mathrm{Prod}(\mathbb{K}(n))$. Then one can compute a $\delta \in \mathbb{N}$ and can construct

(a) $c_1, \ldots, c_e \in \mathbb{K}^*$*;*

(b) geometric product expressions $G_i(n) \in \mathrm{ProdE}(\mathbb{K})$ for $1 \leqslant i \leqslant e$ with lower bounds all synchronised to 1 and composed multiplicatively by geometric products in product factored form;

(c) for $1 \leqslant i \leqslant e$ hypergeometric product expressions $H_i(n) \in \mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$ with lower bounds all synchronised to δ and composed multiplicatively by hypergeometic products in product factored form where the innermost multiplicand of each product factor in $H_i(n)$ is an irreducible monic polynomial in $\mathbb{K}[x] \setminus \mathbb{K}$;

such that for $1 \leqslant i \leqslant e$ and for all $n \geqslant \delta$:

$$P_i(n) = c_i\, G_i(n)\, H_i(n) \neq 0. \tag{6.15}$$

♦

Subsequently, we assume whenever necessary that hypergeometric products

$$P_1(n), \ldots, P_e(n) \in \mathrm{Prod}(\mathbb{K}(n))$$

have undergone the preprocessing step discussed in this section yielding (6.15). As a result, we will proceed as follows. For the geometric product expressions $G_i(n)$, we will begin in Section 6.2 by treating geometric product expressions over roots of unity. For this subclass of nested geometric products, we will show how they can be modelled in a single R-extension. The other subclass of nested geometric products will be treated in Section 6.5. In particular, we will show how they can be modelled in a Π-extension. For the hypergeometric product expressions $H_i(n)$, we refine its hypergeometric product factors in $\mathrm{Prod}(\mathbb{K}(n) \setminus \mathbb{K})$ in such a way that the refined version can be modelled automatically in a Π-extension. Combining this result with those result from Sections 6.2 and 6.5, we will show in Theorem 6.6.15 that, given any finite set of hypergeometric product expressions in $\mathrm{ProdE}(\mathbb{K}(n))$, one can always construct an $\mathrm{R}\Pi$-extension in which the given products can be modelled.

6.2 Construction of single R-extension for higher nesting depth expressions in $\mathrm{ProdE}(\mathbb{U})$

Throughout this section, K is a field containing $\mathbb{Q}$, $\mathbb{K}_m$ is a splitting field for the polynomial $x^m - 1$ over K (i.e., all roots of the polynomial $x^m - 1$ are in $\mathbb{K}_m$) for some $m \in \mathbb{N} \setminus \{0, 1\}$ and $\mathbb{U}_m$ is the set of all m-th roots of unity over K.

Proposition 6.2.1.
Let $\mathbb{U}_m$ be the set of all m-th roots of unity over K. Then $\mathbb{U}_m$ is a multiplicative cyclic group of order m. Moreover,

$$\mathbb{U}_{p\,q} = \big\{m\,n \,|\, m \in \mathbb{U}_p,\ n \in \mathbb{U}_q\big\} \subseteq \mathbb{K}_m.$$

Further $|\mathbb{U}_{p\,q}| = p\,q$, if $\gcd(p, q) = 1$. □

Subsequently, we write

$$\mathbb{U}_m = \langle \zeta_m \rangle$$

where $\zeta_m = e^{\frac{2\pi i}{m}} = (-1)^{\frac{2}{m}}$ is a primitive m-th root of unity that generates the multiplicative group $\mathbb{U}_m$. Let $m \in \mathbb{N} \setminus \{0\}$ with $\mathbb{K}_m$ and $\mathbb{U}_m$ as defined above. Then $\mathrm{Prod}(\mathbb{U}_m)$ is the set of all nested products over roots of unity in $\mathbb{U}_m$. Observe that we get the chain

$$\mathrm{Prod}(\mathbb{U}_m) \subseteq \mathrm{Prod}(\mathbb{K}_m) \subseteq \mathrm{Prod}(\mathbb{K}_m(n)).$$

We define $\mathrm{ProdE}_{\mathbb{K}_m(n)}(\mathbb{U}_m)$ as the set of all elements

$$\sum_{\nu=(\nu_1,\ldots,\nu_e)\in S} a_{\nu(n)}\, P_1(n)^{\nu_1} \cdots P_e(n)^{\nu_e}$$

with $e \in \mathbb{N}$, $S \subseteq \mathbb{Z}^e$ finite, $a_\nu(n) \in \mathbb{K}_m(n)$, $\nu_1, \ldots, \nu_e \in \mathbb{Z}$ and $P_1(n), \ldots, P_e(n) \in \mathrm{Prod}(\mathbb{U}_m)$.

The problem we would like to solve in this section is specified as follows.

Problem RGPEORU: Representation of Geometric Product Expressions Over Roots of Unity

Given

$$A(n) \in \mathrm{ProdE}_{\mathbb{K}_m(n)}(\mathbb{U}_m).$$

Find a natural number $\delta \in \mathbb{N}$ and

$$B(n) = \sum_{i=0}^{\lambda-1} f_i\, \zeta_\lambda^n \in \mathrm{ProdE}_{\mathbb{K}_\lambda(n)}(\mathbb{U}_\lambda)$$

with $f_i \in \mathbb{K}_\lambda(n)$ where $\mathbb{K}_\lambda$ is a finite algebraic extension of $\mathbb{K}_m$ such that for all $n \in \mathbb{N}$ with $n \geqslant \delta$,

$$A(n) = B(n).$$

We shall solve Problem RGPEORU by proving the following theorem.

Theorem 6.2.2.
Suppose we are given the nested geometric products $A_1(n), \dots, A_s(n) \in \mathrm{Prod}(\mathbb{U}_m)$ in n with

$$A_i(n) = \prod_{k_1=\ell_{i,1}}^{n} \zeta_{i,1} \prod_{k_2=\ell_{i,2}}^{k_1} \zeta_{i,2} \cdots \prod_{k_{r_i}=\ell_{i,r_i}}^{k_{r_i-1}} \zeta_{i,r_i} \tag{6.16}$$

for $1 \leqslant i \leqslant s$ where $\zeta_{i,j} \in \mathbb{U}_m$, $\ell_{i,j} \in \mathbb{N}$ for $1 \leqslant j \leqslant r_i$ and for some $r_i \in \mathbb{N}$. Then there exist a $\lambda \in \mathbb{N} \setminus \{0, 1\}$ with $m \,|\, \lambda$ and a primitive λ-th root of unity $\zeta_\lambda \in \mathbb{K}_\lambda$ satisfying the following property. For all $1 \leqslant i \leqslant s$ there exist a natural numbers $\delta \in \mathbb{N}$ and $f_{i,j} \in \mathbb{K}_\lambda$ for $0 \leqslant j < \lambda$ such that

$$A_i(n) = B_i(n)$$

holds for all $n \geqslant \delta$ where

$$B_i(n) = \sum_{j=0}^{\lambda-1} f_{i,j}\, (\zeta_\lambda^n)^j \in \mathrm{ProdE}_{\mathbb{K}_\lambda(n)}(\mathbb{U}_\lambda). \qquad \diamondsuit$$

6.2.1 Simple A-extensions

As in the previous chapters, (see for instance Chapter 3) we will prove Theorem 6.2.2 in the framework of difference ring theory, more precisely within the framework of simple A-extensions which are a subclass of nested A-extensions. This subsection only deals with the formulation of Problem RGPEORU in the setting of simple A-extensions. However, some preparation is required for this task.

Remark 6.2.3.
Observe that every nested APS-ring (resp. field) can be ordered with respect to the depth of its APS-monomials. ♠

Example 6.2.4.
Let $\mathbb{K}_{30} = \mathbb{Q}((-1)^{\frac{2}{3}}, (-1)^{\frac{2}{5}})$ and let $(\mathbb{K}_{30}[\vartheta_1][\vartheta_2][\vartheta_3][\vartheta_4][\vartheta_5][\vartheta_6], \sigma)$ be a nested A-extension of the difference field $(\mathbb{K}_{30}, \sigma)$ of depth $(1,2,1,3,2,3)$ with the automorphism $\sigma|_{\mathbb{K}_{30}} = \mathrm{id}_{\mathbb{K}_{30}}$,

$$\frac{\sigma(\vartheta_1)}{\vartheta_1} = (-1)^{\frac{2}{3}}, \qquad \frac{\sigma(\vartheta_2)}{\vartheta_2} = -\vartheta_1^2, \qquad \frac{\sigma(\vartheta_3)}{\vartheta_3} = (-1)^{\frac{2}{5}},$$
$$\frac{\sigma(\vartheta_4)}{\vartheta_4} = -\vartheta_2^3\,\vartheta_3^4, \qquad \frac{\sigma(\vartheta_5)}{\vartheta_5} = (-1)^{\frac{2}{3}}\,\vartheta_1^2\,\vartheta_3^3, \qquad \frac{\sigma(\vartheta_6)}{\vartheta_6} = -\vartheta_2^3\,\vartheta_5^5.$$

- Take all depth-1 A-monomials in $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6\}$. That is, $\{\vartheta_1, \vartheta_3\}$ and set
$$\vartheta_1 \mapsto \vartheta_{1,1} \quad \text{and} \quad \vartheta_3 \mapsto \vartheta_{2,1}.$$
- Take all depth-2 A-monomials in $\{\vartheta_2, \vartheta_4, \vartheta_5, \vartheta_6\}$. That is $\{\vartheta_2, \vartheta_5\}$ and set
$$\vartheta_2 \mapsto \vartheta_{1,2} \quad \text{and} \quad \vartheta_5 \mapsto \vartheta_{2,2}.$$
- The remaining set of A-monomials $\{\vartheta_4, \vartheta_6\}$ are of depth 3. Set
$$\vartheta_4 \mapsto \vartheta_{1,3} \quad \text{and} \quad \vartheta_6 \mapsto \vartheta_{2,3}.$$

In this way we get an ordered nested A-extension $(\mathbb{K}_{30}[\vartheta_{1,1}][\vartheta_{2,1}][\vartheta_{1,2}][\vartheta_{2,2}][\vartheta_{1,3}][\vartheta_{2,3}], \sigma)$ of $(\mathbb{K}_{30}, \sigma)$ with depth $(1,1,2,2,3,3)$. The difference rings

$$(\mathbb{K}_{30}[\vartheta_{1,1}][\vartheta_{2,1}][\vartheta_{1,2}][\vartheta_{2,2}][\vartheta_{1,3}][\vartheta_{2,3}], \sigma) \quad \text{and} \quad (\mathbb{K}_{30}[\vartheta_1][\vartheta_2][\vartheta_3][\vartheta_4][\vartheta_5][\vartheta_6], \sigma)$$

are isomorphic: the isomorphism simply takes care of the relabelling and rearrangement of the A-monomials by their depth. In particular, the set of constants remains unchanged for such an ordered extension. For further details on these notions we refer to Definition 6.2.8. $\star$

Definition 6.2.5.
Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle$ be an APS-extension of a difference ring $(\mathbb{A}, \sigma)$ and let G be a multiplicative subgroup of $\mathbb{A}^*$. We call

$$G_{\mathbb{A}}^{\mathbb{E}} := \{g\, t_1^{\nu_1} \cdots t_e^{\nu_e} \mid g \in G, \text{ and } \nu_i \in \mathbb{Z} \text{ where } \nu_i = 0 \text{ if } t_i \text{ is a S-monomial (or } \Sigma\text{-monomial)}\} \quad (6.17)$$

the *product group* over G with respect to AP-/AΠ-/RP-/RΠ-monomials for the APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$. If $G_{\mathbb{A}}^{\mathbb{E}}$ is free of any AS-monomial (or RΣ-monomial) i.e.,

$$\hat{G}_{\mathbb{A}}^{\mathbb{E}} := \{g\, t_1^{\nu_1} \cdots t_e^{\nu_e} \mid g \in G, \text{ and } \nu_i \in \mathbb{Z} \text{ where } \nu_i = 0 \text{ if } t_i \text{ is a AS-monomial (or R}\Sigma\text{-monomial)}\}, \quad (6.18)$$

then it is called the *product group* over G with respect to P-monomials (or Π-monomials) for the APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$. If $G_{\mathbb{A}}^{\mathbb{E}}$ is free of any PS-monomial (or $\Pi\Sigma$-monomial) i.e.,

$$\tilde{G}_{\mathbb{A}}^{\mathbb{E}} := \{g\, t_1^{\nu_1} \cdots t_e^{\nu_e} \mid g \in G, \text{ and } \nu_i \in \mathbb{Z} \text{ where } \nu_i = 0 \text{ if } t_i \text{ is a PS-monomial (or } \Pi\Sigma\text{-monomial)}\}, \quad (6.19)$$

then it is called the *product group* over G with respect to A-monomials (or R-monomials) for the APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$. $*$

Remark 6.2.6.

The product groups $G_{\mathbb{A}}^{\mathbb{E}}$ and $\hat{G}_{\mathbb{A}}^{\mathbb{E}}$ have already been introduced in Schneider (2016) and Schneider (2017) respectively. In addition for these three products groups introduced above we have that,

$$\tilde{G}_{\mathbb{A}}^{\mathbb{E}} \subseteq G_{\mathbb{A}}^{\mathbb{E}} \supseteq \hat{G}_{\mathbb{A}}^{\mathbb{E}}. \qquad \spadesuit$$

Proposition 6.2.7.

Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{A}\langle t_1 \rangle \ldots \langle t_e \rangle$ be an APS*-extension of a difference ring $(\mathbb{A}, \sigma)$ and let G be a multiplicative subgroup of $\mathbb{A}^*$. Then $G_{\mathbb{A}}^{\mathbb{E}}$ is a multiplicative subgroup of $\mathbb{E}^*$* □

Proof:

Let $f, h \in G_{\mathbb{A}}^{\mathbb{E}}$ be arbitrary but fixed with $f := u\, t_1^{m_1} \cdots t_e^{m_e}$ and $h := v\, t_1^{n_1} \cdots t_e^{n_e}$ where $u, v \in G$ and $m_i, n_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant e$. Then,

$$f\, h^{-1} = (u\, t_1^{m_1} \cdots t_e^{m_e})\, (v\, t_1^{n_1} \cdots t_e^{n_e})^{-1} = u\, v^{-1}\, t_1^{m_1-n_1}\, t_e^{m_e-n_e}.$$

Since G is a group, it follows that $u\, v^{-1} \in G$ and clearly $m_i - n_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant e$. Thus $f\, h^{-1} \in G_{\mathbb{A}}^{\mathbb{E}}$, which completes the proof. ■

Definition 6.2.8.

Let $(\mathbb{A}, \sigma)$ be a difference ring with $\text{const}(\mathbb{A}, \sigma) = \mathbb{K}_m$ for some $m \in \mathbb{N} \setminus \{0, 1\}$ with $\partial(f) = 0$ for all $f \in \mathbb{A}$. Let $\mathbb{U} := \langle \zeta_m \rangle$ be the multiplicative cyclic group of order m. An A-extension $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{A}[\vartheta_1] \ldots [\vartheta_e]$ of $(\mathbb{A}, \sigma)$ is called a $\mathbb{U}$-*simple* A-*extension* of depth-$(\nu_1, \ldots, \nu_e)$, if for all $1 \leqslant i \leqslant e$

- $\frac{\sigma(\vartheta_i)}{\vartheta_i} \in \mathbb{U}_{\mathbb{A}}^{\mathbb{A}[\vartheta_1]\ldots[\vartheta_{i-1}]}$ where $\mathbb{U}_{\mathbb{A}}^{\mathbb{A}[\vartheta_1]\ldots[\vartheta_{i-1}]}$ is the product group over $\mathbb{U}$ respect to A-monomials;
- $\partial(\vartheta_i) = \nu_i$;

If $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_e$, then $(\mathbb{E}, \sigma)$ is called an *ordered* $\mathbb{U}$-*simple* A-*extension* of $(\mathbb{A}, \sigma)$. Similarly, we call $(\mathbb{E}, \sigma)$ a $\mathbb{U}$-*simple* R-*extension* of $(\mathbb{A}, \sigma)$ of depth-$(\nu_1, \ldots, \nu_e)$ if it is a $\mathbb{U}$-simple A-extension of $(\mathbb{A}, \sigma)$ of depth-$(\nu_1, \ldots, \nu_e)$ and $\text{const}(\mathbb{E}, \sigma) = \text{const}(\mathbb{A}, \sigma)$. Furthermore, $(\mathbb{E}, \sigma)$ is an *ordered* $\mathbb{U}$-*simple* R-*extension* of $(\mathbb{A}, \sigma)$ if it is an ordered $\mathbb{U}$-simple A-extension of $(\mathbb{A}, \sigma)$ and $\text{const}(\mathbb{E}, \sigma) = \text{const}(\mathbb{A}, \sigma)$. Finally, we call $(\mathbb{E}, \sigma)$ a *simple* A-*extension* (resp. *simple* R-*extension*) of $(\mathbb{A}, \sigma)$, if it is a $\mathbb{A}^*$-simple A-extension (resp. $\mathbb{A}^*$-simple R-extension) of $(\mathbb{A}, \sigma)$. ∗

Remark 6.2.9.

From Definition 6.2.8, we observe that, if $\nu_1 = \cdots = \nu_e$, then $\partial(\vartheta_i) = 1$ for all $1 \leqslant i \leqslant e$. Thus, $(\mathbb{E}, \sigma)$ is a $\mathbb{U}$-simple A-extension/$\mathbb{U}$-simple R-extension/simple A-extension/simple R-extension of $(\mathbb{A}, \sigma)$ of monomial depth 1. ♠

Example 6.2.10 (Cont. Example 6.2.4).
The ordered nested A-extension $(\mathbb{K}_{30}[\vartheta_{1,1}][\vartheta_{2,1}][\vartheta_{1,2}][\vartheta_{2,2}][\vartheta_{1,3}][\vartheta_{2,3}],\sigma)$ of the difference field $(\mathbb{K}_{30},\sigma)$ with

$$\frac{\sigma(\vartheta_{1,1})}{\vartheta_{1,1}}=(-1)^{\frac{2}{3}},\qquad \frac{\sigma(\vartheta_{1,2})}{\vartheta_{1,2}}=-\vartheta_{1,1}^{2},\qquad \frac{\sigma(\vartheta_{1,3})}{\vartheta_{1,3}}=-\vartheta_{2,1}^{4}\,\vartheta_{1,2}^{3},$$
$$\frac{\sigma(\vartheta_{2,1})}{\vartheta_{2,1}}=(-1)^{\frac{2}{5}},\qquad \frac{\sigma(\vartheta_{2,2})}{\vartheta_{2,2}}=(-1)^{\frac{2}{3}}\,\vartheta_{1,1}^{2}\,\vartheta_{2,1}^{3},\qquad \frac{\sigma(\vartheta_{2,3})}{\vartheta_{2,3}}=-\vartheta_{1,2}^{3}\,\vartheta_{2,2}^{5},$$

in Example 6.2.4 is a $\mathbb{U}$-simple A-extension of the difference field $(\mathbb{K}_{30},\sigma)$. Here the multiplicative cyclic group $\mathbb{U}$ is

$$\mathbb{U}:=\langle -1,\,(-1)^{\frac{2}{3}},\,(-1)^{\frac{2}{5}}\rangle=\langle(-1)^{\frac{1}{15}}\rangle. \qquad \star$$

We are now ready to formulate the sub-problem to be tackled in the framework of difference rings that will aid in proving Theorem 6.2.2.

Problem SR-RC: Construction of Single R-Ring extension $(\mathbb{A},\sigma)$ over $\mathbb{K}$.

Given a simple A-extension $(\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e],\sigma)$ of a difference field $(\mathbb{K}_m,\sigma)$ with

$$\frac{\sigma(\vartheta_i)}{\vartheta_i}=\alpha_i=\zeta_m^{u_i}\,\vartheta_1^{\nu_{i,1}}\cdots\vartheta_{i-1}^{\nu_{i,i-1}}\in(\mathbb{K}_m^*)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\dots[\vartheta_{i-1}]}$$

for $1\leqslant i\leqslant e$ where ζ_m is a primitive m-th root of unity with $0\leqslant u_i<m$; given $\gamma_1,\dots,\gamma_s\in(\mathbb{K}_m^*)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]}$.
Find a single R-extension $(\mathbb{K}_\lambda[\vartheta],\sigma)$ of $(\mathbb{K}_\lambda,\sigma)$ with $\sigma(\vartheta)=\zeta_\lambda\,\vartheta$ where $\zeta_\lambda\in\mathbb{K}_\lambda$ is a primitive λ-th root of unity with $m\,|\,\lambda$ and $\mathbb{K}_\lambda$ is a finite algebraic field extension of $\mathbb{K}_m$ such that the following properties hold.

(1) The map $\varphi:\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]\to\mathbb{K}_\lambda[\vartheta]$ is a difference ring homomorphism.

(2) The diagram below commutes.

$$\begin{array}{ccc}\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e] & \xrightarrow{\tau_m} & \mathcal{S}(\mathbb{K}_m)\\ \varphi\downarrow & & \downarrow\varphi'\\ \mathbb{K}_\lambda[\vartheta] & \xrightarrow{\tau_\lambda} & \mathcal{S}(\mathbb{K}_\lambda).\end{array}$$

(3) For k with $1\leqslant k\leqslant s$ one can define $g_k:=\sum_{i=0}^{\lambda-1}g_{k,i}\,\vartheta^i\in\mathbb{K}_\lambda[\vartheta]$ with $g_{k,i}\in\mathbb{K}_\lambda$ such that

$$\mathrm{ev}(\sigma^j(\gamma_k),\ell)=\mathrm{ev}(\sigma^j(g_k),\ell)$$

holds for all $j,\,\ell\in\mathbb{N}$.

6.2.2 Solving Problem RGPEORU for nesting depth 1 products in ProdE($\mathbb{U}$)

In Schneider (2017) the necessary tools needed to prove Theorem 6.2.2 for nesting depth 1 products over primitive roots of unity have been discussed. In particular, it has been shown how a simple R-extension of monomial depth 1 (i.e., each R-monomial is of depth-1) can be merged into a single R-extension via a difference ring isomorphism. See Schneider (2017, Lemma 2.22). For the sake of completeness, we will discuss the following result and generalise it afterwards for simple A-extensions of any depth.

Lemma 6.2.11 (Schneider (2017), Lemma 2.22).
Let $(\mathbb{F},\sigma)$ be a constant-stable difference field. Let $(\mathbb{F}[\vartheta_1]\dots[\vartheta_e],\sigma)$ be a nested A-extension of $(\mathbb{F},\sigma)$ of monomial depth 1 with orders $\lambda_1,\dots,\lambda_e$ respectively. Further, suppose that $\zeta_i=\frac{\sigma(\vartheta_i)}{\vartheta_i}\in\mathrm{const}(\mathbb{F},\sigma)^*$ is a primitive λ_i-th root of unity for $1\leqslant i\leqslant e$ and that $\gcd(\lambda_i,\lambda_j)=1$ for pairwise distinct i and j. Then the following statements holds.

(1) There is an R-extension $(\mathbb{F}[\vartheta],\sigma)$ of order $\lambda:=\prod_{i=1}^{e}\lambda_i$ with $\sigma(\vartheta)=\zeta\vartheta$ where $\zeta:=\prod_{i=1}^{e}\zeta_i$.

(2) There is a difference ring isomorphism $\varphi:\mathbb{F}[\vartheta_1]\dots[\vartheta_e]\to\mathbb{F}[\vartheta]$ with $\varphi|_{\mathbb{F}}=\mathrm{id}_{\mathbb{F}}$ such that for $1\leqslant i\leqslant e$, $\varphi(\vartheta_i)=\vartheta^{\nu_i}$ for explicitly computable $\nu_i\in\mathbb{N}$. In particular, the inverse of φ is computable.

(3) $(\mathbb{F}[\vartheta_1]\dots[\vartheta_e],\sigma)$ is an R-extension of $(\mathbb{F},\sigma)$ of monomial depth 1. ♦

Proof:

(1) Observe that $\zeta=\zeta_1\cdots\zeta_e$ is a primitive λ-th root of unity. Thus it follows by Proposition 2.3.37 that $(\mathbb{F}[\vartheta],\sigma)$ is an R-extension of $(\mathbb{F},\sigma)$ with $\sigma(\vartheta)=\zeta\vartheta$.

(2) Let $\lambda=\lambda_1\cdots\lambda_e$. From $\gcd(\lambda_i,\lambda_j)=1$ for all pairwise distinct i and j, we know that $\gcd(\frac{\lambda}{\lambda_i},\lambda_i)=1$. By the extended Euclidean algorithm, we can compute $r_i,s_i\in\mathbb{Z}$ such that

$$r_i\,\frac{\lambda}{\lambda_i}+s_i\,\lambda_i=1 \tag{6.20}$$

and $0\leqslant r_i<\lambda_i$ holds. Set $\nu_i:=r_i\,\frac{\lambda}{\lambda_i}$. Observe that r_i and λ_i are relatively prime since otherwise, the left hand side of (6.20) is divisible by a non-negative integer while the right hand side is not; a contradiction. Consider the ring homomorphism defined by $\varphi(\vartheta_i)=\vartheta^{\nu_i}$. We show that φ is a difference ring homomorphism. Note that $\zeta_j^{\nu_i}=\zeta_j^{r_i\frac{\lambda}{\lambda_i}}=1$, if $j\neq i$. Hence, $\zeta^{\nu_i}=\zeta_i^{r_i\frac{\lambda}{\lambda_i}}=\zeta_i^{1-s_i\lambda_i}=\zeta_i$. Thus

$$\begin{aligned}
\varphi(\sigma(\vartheta_1^{m_1}\cdots\vartheta_e^{m_e}))&=\varphi(\sigma(\vartheta_1)^{m_1}\cdots\sigma(\vartheta_e)^{m_e})\\
&=\varphi\big((\zeta_1\,\vartheta_1)^{m_1}\big)\cdots\varphi\big((\zeta_e\,\vartheta_e)^{m_e}\big)\\
&=\zeta_1^{m_1}\cdots\zeta_e^{m_e}(\vartheta^{\nu_1})^{m_1}\cdots(\vartheta^{\nu_e})^{m_e}\\
&=(\zeta\,\vartheta)^{\nu_1\,m_1}\cdots(\zeta\,\vartheta)^{\nu_e\,m_e}\\
&=\sigma(\vartheta^{\nu_1\,m_1}\cdots\vartheta^{\nu_e\,m_e})\\
&=\sigma(\varphi(\vartheta_1)^{m_1}\cdots\varphi(\vartheta_e)^{m_e})\\
&=\sigma(\varphi(\vartheta_1^{m_1}\cdots\vartheta_e^{m_e}))
\end{aligned}$$

which shows that φ is an difference ring homomorphism by linearity. Finally, we show that φ is bijective. Note that since $\varphi|_{\mathbb{F}}=\mathrm{id}_{\mathbb{F}}$ is a bijection, it suffice to argue on the monomials. Suppose that φ is not injective. Then there is a $g\in\mathbb{F}[\vartheta_1]\dots[\vartheta_e]\setminus\{0\}$ with $\varphi(g)=0$. This also implies that there are at least two different monomials in g say $g_1=\vartheta_1^{a_1}\cdots\vartheta_e^{a_e}$ and $g_2=\vartheta_1^{b_1}\cdots\vartheta_e^{b_e}$ with $(a_1,\dots,a_e)\neq(b_1,\dots,b_e)$ and $0\leqslant a_i,\,b_i<\lambda_i$ such that $\varphi(g_1)=\varphi(g_2)$ holds. Multiplying $\varphi(\vartheta_1^{-b_1}\cdots\vartheta_e^{-b_e})$ on both sides of $\varphi(g_1)=\varphi(g_2)$ we obtain $\varphi(\vartheta_1^{a_1-b_1}\cdots\vartheta_1^{a_e-b_e})=1$. Consequently, we can assume that there is non-zero vector $(c_1, \ldots, c_e) \in \mathbb{N}^e$ with $0 \leqslant c_i < \lambda_i$ for $1 \leqslant i \leqslant e$ with $\varphi(\vartheta_1^{c_1} \cdots \vartheta_e^{c_e}) = 1$. By definition of φ,

$$\varphi(\vartheta_1^{c_1} \cdots \vartheta_e^{c_e}) = \varphi(\vartheta_1)^{c_1} \cdots \varphi(\vartheta_e)^{c_e} = \vartheta^{\nu_1 c_1 + \cdots + \nu_e c_e} = \vartheta^{r_1 \frac{\lambda}{\lambda_1} c_1 + \cdots + r_e \frac{\lambda}{\lambda_e} c_e} = 1.$$

Observe that λ divides $(r_1 \frac{\lambda}{\lambda_1} c_1 + \cdots + r_e \frac{\lambda}{\lambda_e} c_e) =: w$. Let j be arbitrary but fixed. Then λ_j divides w. In addition, λ_j divides $r_i \frac{\lambda}{\lambda_i} c_i$ for $1 \leqslant i \leqslant e$ with $i \neq j$. Hence it must also divide $r_j \frac{\lambda}{\lambda_j} c_j$. We know that λ_j is relatively prime with r_j and $\frac{\lambda}{\lambda_j}$. Therefore λ_j divides c_j. Since $c_j < \lambda_j$, it follows that $c_j = 0$. Consequently, $c_j = 0$ for all j, which is a contradiction to $0_e \neq (c_1, \ldots, c_e) \in \mathbb{N}^e$. Therefore, φ is injective.

We show surjectivity by inverting φ explicitly. Let $n \in \mathbb{N}$ with $0 \leqslant n < \lambda$ and define $d := \gcd(r_1 \frac{\lambda}{\lambda_1}, \ldots, r_e \frac{\lambda}{\lambda_e})$. By iterative application of the extended Euclidean algorithm, we can compute $(m_1, \ldots, m_e) \in \mathbb{N}^e$ with $d = m_1 r_1 \frac{\lambda}{\lambda_1} + \cdots + m_e r_e \frac{\lambda}{\lambda_e}$. Thus $\varphi(\vartheta_1^{m_1} \cdots \vartheta_e^{m_e}) = \vartheta^d$. Now we claim that d and λ are relatively prime: Let p be prime such that p divides $\lambda = \lambda_1 \cdots \lambda_e$. Then there is a j with $1 \leqslant j \leqslant e$ such that p divides λ_j. Since the λ_i with $1 \leqslant i \leqslant e$ are pairwise relatively prime, we have that p does not divide $\frac{\lambda}{\lambda_j}$. Also, since λ_j and r_j are relatively prime, p does not divide r_j. Thus p does not divide $r_j \frac{\lambda}{\lambda_j}$ and hence p does not divide d. Therefore, $\gcd(d, \lambda) = 1$. By the extended Euclidean algorithm, we can compute $u, k \in \mathbb{Z}$ with $u\,d + k\lambda = 1$ and get

$$\vartheta^{n\,u\,d} = \vartheta^{n - n\,k\,\lambda} = \vartheta^n$$

which implies that

$$\varphi(\vartheta_1^{u\,m_1} \cdots \vartheta_e^{u\,m_e}) = \vartheta^{n\,u\,d} = \vartheta^n.$$

By linearity, we obtain the inverse of φ.

(3) Suppose that $(\mathbb{F}[\vartheta_1] \ldots [\vartheta_e], \sigma)$ is not an R-extension of $(\mathbb{F}, \sigma)$. Then there is a $g \in \mathbb{F}[\vartheta_1] \ldots [\vartheta_e] \setminus \mathbb{F}$ such that $\sigma(g) = g$. Also, $\varphi(g) = \varphi(\sigma(g)) = \sigma(\varphi(g)) \in \mathrm{const}(\mathbb{F}[\vartheta], \sigma)$. Since $(\mathbb{F}[\vartheta], \sigma)$ is an R-extension of $(\mathbb{F}, \sigma)$, $\varphi(g) \in \mathrm{const}(\mathbb{F}, \sigma)$. In particular, $\varphi(g) \in \mathbb{F}$. Since $\varphi|_{\mathbb{F}} = \mathrm{id}_{\mathbb{F}}$, we have that $g = \varphi(g) \in \mathbb{F}$, a contradiction to the statement $g \in \mathbb{F}[\vartheta_1] \ldots [\vartheta_e] \setminus \mathbb{F}$. Hence $(\mathbb{F}[\vartheta_1] \ldots [\vartheta_e], \sigma)$ is a nested R-extension of $(\mathbb{F}, \sigma)$ of monomial depth 1. ■

Finally we obtain a criterion that allows one to determine whether a nested A-extension of monomial depth 1 is an R-extension of monomial depth 1; compare Schneider (2017, Proposition 2.23).

Proposition 6.2.12.
Let $(\mathbb{F}, \sigma)$ be a constant-stable difference field. Let $(\mathbb{F}[\vartheta_1] \ldots [\vartheta_e], \sigma)$ be a nested A-extension of $(\mathbb{F}, \sigma)$ monomial depth 1 of orders $\lambda_1, \ldots, \lambda_e$ respectively. Then $(\mathbb{F}[\vartheta_1] \ldots [\vartheta_e], \sigma)$ is a nested R-extension of $(\mathbb{F}, \sigma)$ of monomial depth 1 if and only if for $1 \leqslant i \leqslant e$, $\zeta_i := \frac{\sigma(\vartheta_i)}{\vartheta_i} \in \mathrm{const}(\mathbb{F}, \sigma)^*$ are primitive λ_i-th roots of unity and $\gcd(\lambda_i, \lambda_j) = 1$ for all pairwise distinct i and j. □

Proof:
" $\Longrightarrow$ " Suppose $(\mathbb{F}[\vartheta_1] \ldots [\vartheta_e], \sigma)$ is an R-extension of $(\mathbb{F}, \sigma)$ of monomial depth 1. Then by Proposition 2.3.37 $\zeta_i = \frac{\sigma(\vartheta_i)}{\vartheta_i} \in \mathrm{const}(\mathbb{F}, \sigma)^*$ for $1 \leqslant i \leqslant e$ are primitive λ_i-th roots of unity respectively. We show that the λ_i are pairwise relatively prime. Suppose that there are i and j with $i < j$ such that $d = \gcd(\lambda_i, \lambda_j) \neq 1$ holds. Define $\nu_i := \frac{\lambda_i}{d} \in \mathbb{N}$ and $\nu_j := \frac{\lambda_j}{d} \in \mathbb{N}$. Let $\zeta_i = \frac{\sigma(\vartheta_i)}{\vartheta_i}$ and

$\zeta_j = \frac{\sigma(\vartheta_j)}{\vartheta_j}$. Then $(\zeta_i^{\nu_i})^d = 1 = (\zeta_j^{\nu_j})^d$. Summarizing, given ν_i and ν_j with $0 < \nu_i < \lambda_i$ and $0 < \nu_j < \lambda_j$, $\zeta_i^{\nu_i}$ and $\zeta_j^{\nu_j}$ are d-th root of unity. Therefore, we can choose $0 < \nu_i' < \lambda_i$ and $0 < \nu_j' < \lambda_j$ with $\zeta_i^{\nu_i'} = \zeta_j^{\nu_j'}$ such that $\zeta_i^{\nu_i'}$ and $\zeta_j^{\nu_j'}$ are both d-th root of unity. Now let $\omega := \vartheta_j^{\nu_j'}/\vartheta_i^{\nu_i'}$. Then $\sigma(\omega) = \omega \in \mathrm{const}(\mathbb{F}[\vartheta_1]\dots[\vartheta_e], \sigma) = \mathrm{const}(\mathbb{F}, \sigma)$ and $\vartheta_j^{\nu_j'} = \omega\,\vartheta_i^{\nu_i'}$. Since $\nu_j' < \lambda_j$, $\vartheta_j^{\lambda_j} = 1$ is not the defining relation of ϑ_j; a contradiction.
" $\Longleftarrow$ " follows by statement (3) of Lemma 6.2.11. ∎

In the following example, we will demonstrate how Problem RGPEORU can be solved for the class of nesting depth 1 geometric products over roots of unity using the above tools.

Example 6.2.13.
Let $\mathbb{K}_{30} = \mathbb{Q}\big((-1)^{\frac{2}{3}}, (-1)^{\frac{2}{5}}\big)$ and let $(\mathbb{K}_{30}, \sigma)$ be a difference field. Consider the expression in terms of single products over roots of unity in $\mathbb{U}_{30} := \langle -1, (-1)^{\frac{2}{3}}, (-1)^{\frac{2}{5}} \rangle = \langle (-1)^{\frac{1}{15}} \rangle$:

$$P(n) = \prod_{k=1}^{n}(-1) + \prod_{k=1}^{n}(-1)^{\frac{2}{3}} + \prod_{k=1}^{n}(-1)^{\frac{2}{5}} \in \mathrm{ProdE}_{\mathbb{K}_{30}(n)}(\mathbb{U}_{30}) \tag{6.21}$$

Observe that $P(n)$ can be modelled in the simple A-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}_{30}, \sigma)$ of monomial depth 1 with $\mathbb{A} = \mathbb{K}_{30}[\vartheta_1][\vartheta_2][\vartheta_3]$ equipped with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ and the evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}_{30}$ defined by

$$\sigma(\vartheta_1) = -\vartheta_1, \qquad \sigma(\vartheta_2) = (-1)^{\frac{2}{3}}\,\vartheta_2, \qquad \sigma(\vartheta_3) = (-1)^{\frac{2}{5}}\,\vartheta_3,$$
$$\mathrm{ev}(\vartheta_1, n) = \prod_{k=1}^{n} -1, \qquad \mathrm{ev}(\vartheta_2, n) = \prod_{k=1}^{n}(-1)^{\frac{2}{3}}, \qquad \mathrm{ev}(\vartheta_3, n) = \prod_{k=1}^{n}(-1)^{\frac{2}{5}}$$

respectively. The orders of the A-monomials ϑ_1, ϑ_2 and ϑ_3 are 2, 3 and 5 respectively. Here, the A-monomials ϑ_1, ϑ_2 and ϑ_3 model the first, second and third product expression in (6.21) respectively. Since the order of the primitive roots of unity: (-1), $(-1)^{\frac{2}{3}}$ and $(-1)^{\frac{2}{5}}$ are pairwise relatively prime, it follows by Proposition 6.2.12 that $(\mathbb{K}_{30}[\vartheta_1][\vartheta_2][\vartheta_3], \sigma)$ is an R-extension of $(\mathbb{K}_{30}, \sigma)$ of monomial depth 1. Further, by statement (2) of Lemma 6.2.11, we know that the nested R-extension $(\mathbb{K}_{30}[\vartheta_1][\vartheta_2][\vartheta_3], \sigma)$ of $(\mathbb{K}_{30}, \sigma)$ of monomial depth 1, is isomorphic to the single R-extension $(\mathbb{K}_{30}[\vartheta], \sigma)$ of $(\mathbb{K}_{30}, \sigma)$ with $\sigma(\vartheta) = (-1)^{\frac{1}{15}}\,\vartheta$ and subject to the relation $\vartheta^{30} = 1$. In particular, the isomorphism

$$\varphi : \mathbb{K}_{30}[\vartheta_1][\vartheta_2][\vartheta_3] \to \mathbb{K}_{30}[\vartheta]$$

is defined by $\varphi(\vartheta_i) = \vartheta^{\nu_i}$ for $i = 1, 2, 3$ where $\nu_1 = 15$, $\nu_2 = 10$ and $\nu_3 = 6$. By Example 2.4.4 the map $\tau : \mathbb{K}_{30} \to \mathcal{S}(\mathbb{K}_{30})$ defined by $\tau(\frac{a}{b}) = \langle \mathrm{ev}\left(\frac{a}{b}, n\right) \rangle_{n \geqslant 0}$ with the evaluation function $\mathrm{ev} : \mathbb{K}_{30} \times \mathbb{N} \to \mathbb{K}_{30}$ given by $\mathrm{ev}(c, n) = c$ is a $\mathbb{K}_{30}$-embedding. Furthermore, since $(\mathbb{K}_{30}[\vartheta], \sigma)$ is an R-extension of $(\mathbb{K}_{30}, \sigma)$, it follows by statement (2) of Lemma 2.4.3 that the homomorphic extension

$$\tau : \mathbb{K}_{30}[\vartheta] \to \mathcal{S}(\mathbb{K}_{30})$$

with $\tau(\vartheta) = \big\langle \mathrm{ev}(\vartheta, n) \big\rangle_{n \geqslant 0}$ where $\mathrm{ev}(\vartheta, n) = \big((-1)^{\frac{1}{15}}\big)^n$ is also a $\mathbb{K}_{30}$-embedding. Now define,

$$Q' := \vartheta^{15} + \vartheta^{10} + \vartheta^{6} \in \mathbb{K}_{30}[\vartheta].$$

Note that we have defined Q' such that $\tau(Q') = \big\langle P(n) \big\rangle_{n \geqslant 0}$ holds. In particular, with

$$Q(n) = \mathrm{ev}(Q', n) = \Big(\big((-1)^{\frac{1}{15}}\big)^n\Big)^{15} + \Big(\big((-1)^{\frac{1}{15}}\big)^n\Big)^{10} + \Big(\big((-1)^{\frac{1}{15}}\big)^n\Big)^{6}$$

we have that

$$P(n) = Q(n)$$

holds for all $n \geqslant 0$. In other words, $P(n)$ and $Q(n)$ evaluate to the same sequence. $\star$

In the next two subsections, our goal is to generalise statements (1) and (2) of Lemma 6.2.11 for simple A-extension of any depth. We will require some algorithmic preparations.

6.2.3 An algorithmic machinery — order, σ-factorial, period and factorial order

An important ingredients that are necessary for generalising statements (1) and (2) of Lemma 6.2.11 for simple A-extension of any monomial depth are the order (see Definition 2.3.16), period and factorial order of elements in difference rings as discussed in Karr (1981); and Schneider (2016, Section 5).

Definition 6.2.14.
Let $(\mathbb{A}, \sigma)$ be a difference ring. The *rising factorial* or *σ-factorial* of $f \in \mathbb{A}^*$ to $k \in \mathbb{Z}$ is defined by

$$f_{(k,\sigma)} = \begin{cases} f\,\sigma(f)\,\cdots\sigma^{k-1}(f) & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ \sigma^{-1}(\frac{1}{f})\,\sigma^{-2}(\frac{1}{f})\,\cdots\sigma^{k}(\frac{1}{f}) & \text{if } k < 0. \end{cases}$$

If the automorphism is clear from the context, then we write $f_{(k)}$ instead of $f_{(k,\sigma)}$. $*$

Proposition 6.2.15.
Let $(\mathbb{A}, \sigma)$ be a difference ring, α, $g \in \mathbb{A}^*$ and $n \in \mathbb{Z}$. Then:

(1) $(\alpha\, g)_{(n)} = \alpha_{(n)}\, g_{(n)}$.

(2) If $\sigma(g) = \alpha\, g$, *then* $\sigma^n(g) = \alpha_{(n)}\, g$. $\square$

Proof:
(1) If $n = 0$, then $(\alpha\, g)_{(n)} = 1 = 1 \cdot 1 = \alpha_{(n)}\, g_{(n)}$.

If $n > 0$, then

$$\begin{aligned} (\alpha\, g)_{(n)} &= \alpha\, g\, \sigma(\alpha\, g) \cdots \sigma^{n-1}(\alpha\, g) \\ &= \alpha\, g\, \sigma(\alpha)\, \sigma(g) \cdots \sigma^{n-1}(\alpha)\, \sigma^{n-1}(g) \\ &= \big(\alpha\, \sigma(\alpha)\, \cdots \sigma^{n-1}(\alpha)\big)\, \big(g\, \sigma(g)\, \cdots \sigma^{n-1}(g)\big) \\ &= \alpha_{(n)}\, g_{(n)}. \end{aligned}$$

If $n < 0$, then

$$\begin{aligned} (\alpha\, g)_{(n)} &= \sigma^{-1}(\tfrac{1}{\alpha\, g})\, \sigma^{-2}(\tfrac{1}{\alpha\, g}) \cdots \sigma^{n}(\tfrac{1}{\alpha\, g}) \\ &= \sigma^{-1}(\tfrac{1}{\alpha})\, \sigma^{-1}(\tfrac{1}{g})\, \sigma^{-2}(\tfrac{1}{\alpha})\, \sigma^{-2}(\tfrac{1}{g}) \cdots \sigma^{n}(\tfrac{1}{\alpha})\, \sigma^{n}(\tfrac{1}{g}) \\ &= \big(\sigma^{-1}(\tfrac{1}{\alpha})\, \sigma^{-2}(\tfrac{1}{\alpha}) \cdots \sigma^{n}(\tfrac{1}{\alpha})\big)\, \big(\sigma^{-1}(\tfrac{1}{g})\, \sigma^{-2}(\tfrac{1}{g}) \cdots \sigma^{n}(\tfrac{1}{g})\big) \\ &= \alpha_{(n)}\, g_{(n)}. \end{aligned}$$

(2) If $n = 0$, then the statement holds. Suppose that $n > 0$ and $\sigma^k(g) = \alpha_{(k)}\, g$ holds for all $0 \leqslant k \leqslant n-1$. Then

$$\begin{aligned}
\sigma^n(g) = \sigma(\sigma^{n-1}(g)) &= \sigma(\alpha_{(n-1)}\, g) \\
&= \sigma(\alpha_{(n-1)})\, \sigma(g) \\
&= \sigma(\alpha\, \sigma(\alpha) \cdots \sigma^{n-1}(\alpha))\, \alpha\, g \\
&= \sigma(\alpha)\, \sigma^2(\alpha) \cdots \sigma^n(\alpha)\, \alpha\, g \\
&= \alpha_{(n)}\, g.
\end{aligned}$$

Now suppose that $n < 0$ and $\sigma^k(g) = \alpha_{(k)}\, g$ holds for all $n+1 \leqslant k \leqslant 0$. Then,

$$\begin{aligned}
\sigma^n(g) = \sigma^{-1}(\sigma^{n+1}(g)) &= \sigma^{-1}(\alpha_{(n+1)}\, g) \\
&= \sigma^{-1}(\alpha_{(n+1)})\, \sigma^{-1}(g) \\
&= \sigma^{-1}(\sigma^{-1}(\tfrac{1}{\alpha})\, \sigma^{-2}(\tfrac{1}{\alpha}) \cdots \sigma^{n+1}(\tfrac{1}{\alpha}))\, \tfrac{1}{\alpha}\, g \\
&= \sigma^{-2}(\tfrac{1}{\alpha})\, \sigma^{-3}(\tfrac{1}{\alpha}) \cdots \sigma^{n}(\tfrac{1}{\alpha})\, \tfrac{1}{\alpha}\, g \\
&= \sigma^{-1}(\tfrac{1}{\alpha})\, \sigma^{-2}(\tfrac{1}{\alpha})\, \cdots \sigma^{n}(\tfrac{1}{\alpha})\, g \\
&= \alpha_{(n)}\, g.
\end{aligned}$$

■

Furthermore we need the following definitions.

Definition 6.2.16.
Let $(\mathbb{A}, \sigma)$ be a difference ring. We define the period of $\alpha \in \mathbb{A}^*$ by

$$\mathrm{per}(\alpha) = \begin{cases} 0 & \text{if } \nexists\, n > 0 \text{ s.t. } \sigma^n(\alpha) = \alpha \\ \min\{n > 0 \,|\, \sigma^n(\alpha) = \alpha\} & \text{otherwise;} \end{cases}$$

and the factorial order of α by

$$\mathrm{ford}(\alpha) = \begin{cases} 0 & \text{if } \nexists\, n > 0 \text{ s.t. } \alpha_{(n)} = 1 \\ \min\{n > 0 \,|\, \alpha_{(n)} = 1\} & \text{otherwise.} \end{cases}$$

*

Lemma 6.2.17 (Schneider (2016), Lemma 5.1).
Let $(\mathbb{A}, \sigma)$ be a difference ring with α, $g \in \mathbb{A}$. Then the following holds:

(1) If $\alpha \in \mathrm{const}(\mathbb{A}, \sigma)^*$, then $\mathrm{per}(\alpha) = 1$ and $\mathrm{ford}(\alpha) = \mathrm{ord}(\alpha)$.

(2) If $\sigma(g) = \alpha\, g$, then $\mathrm{per}(g) = \mathrm{ford}(\alpha)$.

(3) If $\mathrm{per}(\alpha) > 0$ and $\mathrm{ord}(\alpha) > 0$, then $\mathrm{per}(\alpha) \,|\, \mathrm{ford}(\alpha) \,|\, \mathrm{ord}(\alpha)\, \mathrm{per}(\alpha)$ and

$$\mathrm{ford}(\alpha) = \min(i\, \mathrm{per}(\alpha) \,|\, 1 \leqslant i \leqslant \mathrm{ord}(\alpha) \text{ and } \alpha_{(i\, \mathrm{per}(\alpha))} = 1) > 0. \tag{6.22}$$

♦

PROOF:

(1) If $\alpha \in \mathrm{const}(\mathbb{A},\sigma)^*$, then $\sigma(\alpha) = \alpha$ and it follows that $\mathrm{per}(\alpha) = 1$. Also since $\alpha_{(n)} = \alpha^n$ for all $n \geqslant 0$, it follows that $\mathrm{ford}(\alpha) = \mathrm{ord}(\alpha)$.

(2) By Proposition 6.2.15, $\sigma^n(g) = g$ if and only if $\alpha_{(n)} = 1$. Therefore, $\mathrm{per}(g) = \mathrm{ford}(\alpha)$.

(3) Let $k = \mathrm{per}(\alpha) > 0$ and $\nu = \mathrm{ord}(\alpha) > 0$. Then we have that,

$$\alpha_{(k\,\nu)} = \alpha\,\sigma(\alpha)\cdots\sigma^{k\,\nu-1}(\alpha) = (\alpha\,\sigma(\alpha)\cdots\sigma^{k-1}(\alpha))^{\nu} = \alpha^{\nu}\,\sigma(\alpha^{\nu})\cdots\sigma^{k-1}(\alpha^{\nu}) = 1.$$

Therefore, we can choose $n = \mathrm{ord}(\alpha)\,\mathrm{per}(\alpha)$ to get $\alpha_{(n)} = 1$. In particular, for any $\ell \geqslant 0$ with $\alpha_{(\ell)} = 1$ we have that

$$1 = \frac{\sigma(1)}{1} = \frac{\sigma(\alpha_{(\ell)})}{\alpha_{(\ell)}} = \frac{\sigma^{\ell}(\alpha)}{\alpha}.$$

Hence $\mathrm{per}(\alpha) \mid \ell$. Thus the smallest natural number k with $\alpha_{(k)} = 1$ is given by (6.22). In particular, $\mathrm{per}(\alpha) \mid \mathrm{ford}(\alpha) \mid \mathrm{ord}(\alpha)\,\mathrm{per}(\alpha)$. ■

The next two lemmas below discuss methods for computing the order, factorial order and period of elements in the product group of a nested A-extension. In Schneider (2016, Lemma 5.2 and Lemma 5.3), these two lemmas have been proved for the class of nested R-extensions. In those proofs, the property that the set of constants remain unchanged was never used. Thus, the proofs hold for the class of nested A-extension. Nonetheless we will repeat these proofs again for the sake of completeness.

Lemma 6.2.18.
Let $(\mathbb{E},\sigma)$ with $\mathbb{E} = \mathbb{A}[\vartheta_1]\dots[\vartheta_e]$ be a nested A-extension of $(\mathbb{A},\sigma)$ and let $\alpha = \zeta\,\vartheta_1^{m_1}\cdots\vartheta_e^{m_e} \in (\mathbb{A}^*)_{\mathbb{A}}^{\mathbb{E}}$ with $\zeta \in \mathbb{A}^*$, $m_i \in \mathbb{N}$. Then $\mathrm{ord}(\alpha) > 0$ if and only if $\mathrm{ord}(\zeta) > 0$. If $\mathrm{ord}(\zeta) > 0$, then

$$\mathrm{ord}(\alpha) = \mathrm{lcm}\left(\mathrm{ord}(\zeta),\ \frac{\mathrm{ord}(\vartheta_1)}{\gcd(\mathrm{ord}(\vartheta_1),\ m_1)},\dots,\frac{\mathrm{ord}(\vartheta_e)}{\gcd(\mathrm{ord}(\vartheta_e),\ m_e)}\right). \tag{6.23}$$

♦

PROOF:

If $e = 0$, then $\alpha = \zeta$ and the lemma holds. Now let $n := \mathrm{ord}(\alpha) > 0$ and suppose that

$$1 \neq (\vartheta_1^{m_1}\cdots\vartheta_e^{m_e})^n = \vartheta_1^{m_1\,n}\cdots\vartheta_e^{m_e\,n}$$

holds. Let k be maximal such that $\mathrm{ord}(\vartheta_k) \nmid \nu_k\,n$. Then there is an ℓ with $0 < \ell < \mathrm{ord}(\vartheta_k)$ with

$$\vartheta_k^{\mathrm{ord}(\vartheta_k)-\ell} = \zeta^n\,\vartheta_1^{m_1}\cdots\vartheta_{k-1}^{m_{k-1}};$$

a contradiction to the assumption that $\vartheta_k^{\mathrm{ord}(\vartheta_k)} = 1$ is the defining relation for the A-monomial ϑ_k. Thus $(\vartheta_1^{m_1}\cdots\vartheta_e^{m_e})^n = 1$ and $\zeta^n = 1$, i.e., $\mathrm{ord}(\zeta) > 0$ and $\mathrm{ord}(\vartheta_1^{m_1}\cdots\vartheta_e^{m_e}) > 0$. In particular,

$$\mathrm{ord}(\alpha) = \mathrm{lcm}(\mathrm{ord}(\zeta), \mathrm{ord}(\vartheta_1^{m_1}\cdots\vartheta_e^{m_e})).$$

By similar arguments as above, we can show that

$$(\vartheta_1^{m_1})^n = \cdots = (\vartheta_e^{m_e})^n = 1$$

and consequently

$$\operatorname{ord}(\vartheta_1^{m_1}\cdots\vartheta_e^{m_e}) = \operatorname{lcm}(\operatorname{ord}(\vartheta_1^{m_1}),\dots,\operatorname{ord}(\vartheta_e^{m_e})).$$

Since also

$$\operatorname{ord}(\vartheta_i^{m_i}) = \frac{\operatorname{ord}(\vartheta_i)}{\gcd(\operatorname{ord}(\vartheta_i), m_i)}$$

holds, the identity (6.23) is proven.

Conversely, suppose $\operatorname{ord}(\zeta) > 0$, then the right hand side of (6.23) has a positive value say $n > 0$. Then one can check that $\alpha^n = 1$. Therefore, $\operatorname{ord}(\alpha) > 0$. ■

The next lemma sets the stage to compute the period and the factorial order of elements in the product group of a simple A-extension.

Lemma 6.2.19.
Let $(\mathbb{E},\sigma)$ *with* $\mathbb{E} = \mathbb{A}[\vartheta_1]\dots[\vartheta_e]$ *be a nested* A*-extension of* $(\mathbb{A},\sigma)$ *with* $\operatorname{per}(\vartheta_i) > 0$ *for* $1 \leqslant i \leqslant e$. *Let* $\alpha = \zeta\vartheta_1^{\nu_1}\cdots\vartheta_e^{\nu_e} \in (\mathbb{A}^*)_{\mathbb{A}}^{\mathbb{E}}$ *with* $\zeta \in \mathbb{A}^*$, $\nu_1,\dots,\nu_e \in \mathbb{N}$.

(1) Then $\operatorname{per}(\alpha) > 0$ *if and only if* $\operatorname{per}(\zeta) > 0$. *If* $\operatorname{per}(\zeta) > 0$, *then*

$$\operatorname{per}(\alpha) = \min(1 \leqslant j \leqslant u \,|\, \sigma^j(\alpha) = \alpha \textit{ and } j \,|\, u) \tag{6.24}$$

with $u = \operatorname{lcm}(\operatorname{per}(\zeta), \operatorname{per}(\vartheta_{i_1}),\dots,\operatorname{per}(\vartheta_{i_k}))$ *where* $\{i_1,\dots,i_k\} = \{i \,|\, \operatorname{ord}(\vartheta_i) \nmid \nu_i\}$.

(2) We have that $\operatorname{ford}(\alpha) > 0$ *if and only if* $\operatorname{ford}(\zeta) > 0$.

(3) If $\operatorname{per}(\zeta) > 0$ *and* $\operatorname{ord}(\zeta) > 0$, *then* $\operatorname{ford}(\alpha) > 0$ *and* $0 < \operatorname{per}(\alpha) \,|\, \operatorname{ford}(\alpha) \,|\, \operatorname{per}(\alpha)\operatorname{ord}(\alpha)$.

(4) If the values $\operatorname{ord}(\vartheta_i)$ *and* $\operatorname{per}(\vartheta_i)$ *for* $1 \leqslant i \leqslant e$ *and the values* $\operatorname{per}(\zeta) > 0$ *and* $\operatorname{ord}(\zeta) > 0$ *are given explicitly, then* $\operatorname{per}(\alpha)$ *and* $\operatorname{ford}(\alpha)$ *can be calculated.* ♦

Proof:

(1) If $\operatorname{per}(\zeta) > 0$, then $u > 0$. In particular, it follows that $\sigma^u(\alpha) = \alpha$. Therefore, $\operatorname{per}(\alpha) > 0$ with $\operatorname{per}(\alpha) \,|\, u$ and we have (6.24). Conversely, suppose that $\operatorname{per}(\alpha) > 0$. Then with $k := \operatorname{lcm}(\operatorname{per}(\alpha), \operatorname{per}(\vartheta_1),\dots,\operatorname{per}(\vartheta_e)) > 0$ we have

$$\zeta\vartheta_1^{\nu_1}\cdots\vartheta_e^{\nu_e} = \alpha = \sigma^k(\alpha) = \sigma^k(\zeta)\,\vartheta_1^{\nu_1}\cdots\vartheta_e^{\nu_e}.$$

Thus, it follows that $\sigma^k(\zeta) = \zeta$. Therefore, $\operatorname{per}(\zeta) > 0$.

(2) Since $\operatorname{per}(\vartheta_i) > 0$ and $\operatorname{ord}(\vartheta_i) > 0$, it follows by Lemma 6.2.17 that $\operatorname{ford}(\vartheta_i) > 0$ for all $1 \leqslant i \leqslant e$. If $\operatorname{ford}(\zeta) > 0$, then take $k = \operatorname{lcm}(\operatorname{ford}(\zeta), \operatorname{ford}(\vartheta_1),\dots,\operatorname{ford}(\vartheta_e)) > 0$. By statement (1) of Proposition 6.2.15,

$$\alpha_{(k)} = (\zeta\vartheta_1^{\nu_1}\cdots\vartheta_e^{\nu_e})_{(k)} = \zeta_{(k)}\,(\vartheta_1^{\nu_1})_{(k)}\cdots(\vartheta_e^{\nu_e})_{(k)} = 1.$$

Hence $\operatorname{ford}(\alpha) > 0$. Conversely, if $\operatorname{ford}(\alpha) > 0$, take $m = \operatorname{lcm}(\operatorname{ford}(\alpha), \operatorname{ford}(\vartheta_1),\dots,\operatorname{ford}(\vartheta_e)) > 0$. Then again by statement (1) of Proposition 6.2.15,

$$1 = \alpha_{(m)} = (\zeta\vartheta_1^{\nu_1}\cdots\vartheta_e^{\nu_e})_{(m)} = \zeta_{(m)}\,(\vartheta_1^{\nu_1})_{(m)}\cdots(\vartheta_1^{\nu_1})_{(m)} = \zeta_{(m)}.$$

Therefore, $\operatorname{ford}(\zeta) > 0$.

(3) If $\operatorname{per}(\zeta) > 0$, then by part (1) of the Lemma, $\operatorname{per}(\alpha) > 0$. Further, if $\operatorname{ord}(\zeta) > 0$, then by Lemma 6.2.18 $\operatorname{ord}(\alpha) > 0$. Therefore by part (3) of Lemma 6.2.17, $\operatorname{per}(\alpha) \mid \operatorname{ford}(\alpha) \mid \operatorname{per}(\alpha) \operatorname{ord}(\alpha)$. In particular, $\operatorname{ford}(\alpha) > 0$.

(4) If the values of $\operatorname{per}(\zeta)$ and $\operatorname{per}(\vartheta_i)$ for $1 \leqslant i \leqslant e$ are given, then u in statement (1) of the Lemma can be computed. Further, if the values of $\operatorname{ord}(\zeta)$ and $\operatorname{ord}(\vartheta_i)$ for $1 \leqslant i \leqslant e$ are given, then $\operatorname{ord}(\alpha)$ can be determined by Lemma 6.2.18. Thus, $\operatorname{per}(\alpha)$ can be calculated by (6.24) and then $\operatorname{ford}(\alpha)$ can be calculated by (6.22). ∎

Combining the above two lemmas, we arrive at the following result which have also been discussed for the class of simple R-extensions in Schneider (2016, Proposition 5.5). Moreover, since the property that the set of constants remain unchanged was not used in the proof for the class of simple R-extension, then the proof holds also for the class of simple A-extensions. For the sake of completeness, we repeat the proof here for the class of simple A-extensions.

Proposition 6.2.20.
Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{A}[\vartheta_1] \dots [\vartheta_e]$ be a simple A-extension of $(\mathbb{A}, \sigma)$ with $\frac{\sigma(\vartheta_i)}{\vartheta_i} = \zeta_i \, \vartheta_1^{\nu_{i,1}} \cdots \vartheta_{i-1}^{\nu_{i,i-1}}$ where $\zeta_i \in \mathbb{A}^*$ and $\nu_{i,j} \in \mathbb{N}$. Then the following statements holds:

(1) $\operatorname{ord}(\zeta_i) > 0$ for $1 \leqslant i \leqslant e$. In particular, if the values of $\operatorname{ord}(\zeta_i)$ are given (are computable), then the values $\operatorname{ord}(\vartheta_i)$ are also computable.

(2) If $\operatorname{per}(\zeta_i) > 0$ for $1 \leqslant i \leqslant e$, then $\operatorname{per}(\vartheta_i) > 0$ for $1 \leqslant i \leqslant e$. In particular, if the values of $\operatorname{ord}(\zeta_i)$ and $\operatorname{per}(\zeta_i)$ for $1 \leqslant i \leqslant e$ are given explicitly (are computable), then the values $\operatorname{per}(\vartheta_i)$ for all $1 \leqslant i \leqslant e$ are also computable. □

Proof:
(1) If $i = 1$, then by Definition 2.3.17, $\operatorname{ord}(\vartheta_1) = \operatorname{ord}(\zeta_i) > 0$. Now suppose that $\operatorname{ord}(\zeta_i)$ is given for $1 \leqslant i \leqslant e$ and assume that the values of $\operatorname{ord}(\vartheta_i)$ for $1 \leqslant i \leqslant e-1$ have already been determined. We prove the case $i = e$. Define $\alpha = \frac{\sigma(\vartheta_e)}{\vartheta_e}$. Then by (6.23), we obtain the value $\operatorname{ord}(\alpha)$ and by Definition 2.3.17, $\operatorname{ord}(\vartheta_e) = \operatorname{ord}(\alpha)$ which completes the inductive step.

(2) Suppose that $\operatorname{per}(\zeta_i) > 0$ for $1 \leqslant i \leqslant e$ and $\upsilon_i = \operatorname{per}(\vartheta_i)$ for all $1 \leqslant i \leqslant e-1$ have been shown already. We prove the inductive statement with $i = e$. Define $\alpha = \frac{\sigma(\vartheta_e)}{\vartheta_e}$. By statements (1) and (2) of Lemma 6.2.19 we have that $\operatorname{per}(\alpha) > 0$ and $\operatorname{ford}(\alpha) > 0$ respectively and by statement (2) Lemma 6.2.17 $\operatorname{per}(\vartheta_e) = \operatorname{ford}(\alpha)$. If the values of $\operatorname{ord}(\zeta_i)$ for $1 \leqslant i \leqslant e$ are given explicitly, then $\operatorname{ord}(\alpha)$ can be computed by statement (1) of the Proposition. If $\operatorname{per}(\zeta_e)$ is given explicitly and $\upsilon_1, \dots, \upsilon_{e-1}$ are given or have already been computed, then with (6.24), $\operatorname{per}(\alpha)$ can be computed. Hence $\operatorname{ford}(\alpha)$ can also be computed with (6.22). Thus by statement (2) of Lemma 6.2.17, $\operatorname{per}(\vartheta_e) = \operatorname{ford}(\alpha)$ which computes the inductive step. ∎

Finally we present some properties for the class of simple A-extensions over a strong constant-stable difference field. Again, these properties have already been discussed in Schneider (2016, Corollary 5.6) for the class of simple R-extensions over a strong constant-stable difference field. The proofs are basically the same to those presented for simple R-extensions.

Corollary 6.2.21.
Let $(\mathbb{E}, \sigma)$ *with* $\mathbb{E} = \mathbb{F}[\vartheta_1]\dots[\vartheta_e]$ *be a simple* A*-extension of a difference field* $(\mathbb{F}, \sigma)$ *with* $\mathrm{const}(\mathbb{F}, \sigma) = \mathbb{K}$ *such that all roots of unity in* $\mathbb{F}$ *are constants, i.e.,* $(\mathbb{F}, \sigma)$ *is strong constant-stable. Then the following statements holds.*

(1) For $1 \leqslant i \leqslant e$ *we have that*
$$\frac{\sigma(\vartheta_i)}{\vartheta_i} = \zeta_i\, \vartheta_1^{\nu_{i,1}} \cdots \vartheta_{i,i-1}^{\nu_{i,i-1}} \tag{6.25}$$
for some root of unity $\zeta_i \in \mathbb{K}^*$ *with* $\mathrm{ord}(\zeta_i)\,|\,\mathrm{ord}(\vartheta_i)$ *and* $\nu_{i,j} \in \mathbb{N}$ *for* $1 \leqslant j \leqslant i-1$.

(2) $(\mathbb{K}[\vartheta_1]\dots[\vartheta_e], \sigma)$ *is a simple* A*-extension of* $(\mathbb{K}, \sigma)$.

(3) Let $\alpha = \zeta\vartheta_1^{m_1}\cdots\vartheta_e^{m_e} \in (\mathbb{K}^*)_{\mathbb{K}}^{\mathbb{K}[\vartheta_1]\dots[\vartheta_e]}$ *with* $m_1, \dots, m_e \in \mathbb{N}$ *and* $\zeta \in \mathbb{K}^*$. *Then*
$$\mathrm{ord}(\zeta) > 0 \iff \mathrm{ord}(\alpha) > 0 \iff \mathrm{per}(\alpha) > 0 \iff \mathrm{ford}(\alpha) > 0.$$

(4) If $(\mathbb{K}, \sigma)$ *is computable and Problem O is solvable in* $\mathbb{K}^*$*, then for all* $\alpha \in (\mathbb{K}^*)_{\mathbb{K}}^{\mathbb{K}[\vartheta_1]\dots[\vartheta_e]}$*, the values of* $\mathrm{ord}(\alpha)$*,* $\mathrm{per}(\alpha)$ *and* $\mathrm{ford}(\alpha)$ *are computable.*

(5) If Problem O is solvable in $\mathbb{K}^*$ *and* $(\mathbb{F}, \sigma)$ *is computable, then Problem O is solvable in* $(\mathbb{F}^*)_{\mathbb{F}}^{\mathbb{E}}$. ◇

PROOF:

(1) Since $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}[\vartheta_1]\dots[\vartheta_e]$ is a simple A-extension of $(\mathbb{F}, \sigma)$, we have that (6.25) with $\nu_{i,j} \in \mathbb{N}$ and $\zeta_i \in \mathbb{F}^*$. By Lemma 6.2.18 it follows that $\mathrm{ord}(\zeta_i) > 0$ and $\mathrm{ord}(\zeta_i)\,|\,\mathrm{ord}(\vartheta_i)$. In particular, $\zeta_i \in \mathbb{K}^*$ since all roots of unity in $\mathbb{F}$ are constants by assumption.

(2) It is immediate that $\mathbb{K}[\vartheta_1]\dots[\vartheta_e]$ with $\sigma|_{\mathbb{K}} = \mathbb{K}$ and (6.25) forms the difference ring, $(\mathbb{K}[\vartheta_1]\dots[\vartheta_e], \sigma)$. In particular, it is a difference ring extension of $(\mathbb{K}, \sigma)$ and by Definition 6.2.8 it is a simple A-extension of $(\mathbb{K}, \sigma)$.

(3) By part (1) of the Corollary, we know that $\zeta_i \in \mathbb{K}^*$ with $\mathrm{ord}(\zeta_i) > 0$ for $1 \leqslant i \leqslant e$. In particular, $\mathrm{per}(\zeta_i) = 1 > 0$ for $1 \leqslant i \leqslant e$. By Proposition 6.2.20 we have that $\mathrm{ord}(\vartheta_i) > 0$ and $\mathrm{per}(\vartheta_i) > 0$ for $1 \leqslant i \leqslant e$. Since $\zeta \in \mathbb{K}^*$, it follows by statement (1) of Lemma 6.2.17 that $\mathrm{per}(\zeta) = 1 > 0$ and $\mathrm{ford}(\zeta) = \mathrm{ord}(\zeta) > 0$. Thus, the equivalences follows by Lemma 6.2.18 and statements (1) and (2) of Lemma 6.2.19.

(4) Since $\zeta_i \in \mathbb{K}^*$ for all $1 \leqslant i \leqslant e$, $\mathrm{ord}(\zeta_i)$ can be determined by solving Problem O for each ζ_i in $\mathbb{K}^*$. Further, by Proposition 6.2.20, $\mathrm{ord}(\vartheta_i)$ and $\mathrm{per}(\vartheta_i)$ for all $1 \leqslant i \leqslant e$ can be determined. Now let $\alpha = \zeta\vartheta_1^{m_1}\cdots\vartheta_e^{m_e} \in (\mathbb{K}^*)_{\mathbb{K}}^{\mathbb{K}[\vartheta_1]\dots[\vartheta_e]}$ with $\zeta \in \mathbb{K}^*$ and $m_1, \dots, m_e \in \mathbb{N}$. Note that $\mathrm{ord}(\zeta)$ can be determined by again solving Problem O for ζ in $\mathbb{K}^*$. Then with (6.23), $\mathrm{ord}(\alpha)$ can be computed. In addition, since $\mathrm{per}(\zeta) = 1$ and $\mathrm{ford}(\zeta) = \mathrm{ord}(\zeta)$ are given, we can utilise (6.24) and (6.22) to compute $\mathrm{per}(\alpha)$ and $\mathrm{ford}(\alpha)$ respectively.

(5) Let $\alpha = \zeta\vartheta_1^{m_1}\cdots\vartheta_e^{m_e} \in (\mathbb{F}^*)_{\mathbb{F}}^{\mathbb{E}}$ with $\zeta \in \mathbb{F}^*$ and $m_1, \dots, m_e \in \mathbb{N}$. By Lemma 6.2.18 $\mathrm{ord}(\alpha) > 0$ if and only if $\mathrm{ord}(\zeta) > 0$. By assumption, $\mathrm{ord}(\zeta) > 0$ implies that $\zeta \in \mathbb{K}^*$. Thus, if $\zeta \notin \mathbb{K}^*$, then $\mathrm{ord}(\alpha) = 0$. Otherwise, if $\zeta \in \mathbb{K}^*$, then by item (4) of the Corollary, we can compute $\mathrm{ord}(\alpha)$. ■

Example 6.2.22 (Cont. Example 6.2.10).
We compute the order, period and factorial order of the A-monomials in the ordered $\mathbb{U}_{30}$-simple A-extension $(\mathbb{K}_{30}[\vartheta_{1,1}][\vartheta_{2,1}][\vartheta_{1,2}][\vartheta_{2,2}][\vartheta_{1,3}][\vartheta_{2,3}], \sigma)$ of $(\mathbb{K}_{30}, \sigma)$ introduced in Example 6.2.10.

(1) Take the depth-1 A-monomials, i.e., $\vartheta_{1,1}$ and $\vartheta_{2,1}$, with

$$\sigma(\vartheta_{1,1}) = (-1)^{\frac{2}{3}}\,\vartheta_{1,1} \qquad \text{and} \qquad \sigma(\vartheta_{2,1}) = (-1)^{\frac{2}{5}}\,\vartheta_{2,1}.$$

Since $(-1)^{\frac{2}{3}}, (-1)^{\frac{2}{5}} \in \mathbb{K}^*$, it follows by statement (1) of Lemma 6.2.17 that

$\mathrm{per}((-1)^{\frac{2}{3}}) = \mathrm{per}((-1)^{\frac{2}{5}}) = 1$, $\mathrm{ford}((-1)^{\frac{2}{3}}) = \mathrm{ord}((-1)^{\frac{2}{3}}) = 3$ and $\mathrm{ford}((-1)^{\frac{2}{5}}) = \mathrm{ord}((-1)^{\frac{2}{5}}) = 5$.

By statement (2) of Lemma 6.2.17, the period of the depth-1 A-monomials are as follows:

$$\mathrm{per}(\vartheta_{1,1}) = \mathrm{ford}((-1)^{\frac{2}{3}}) = 3 \text{ and } \mathrm{per}(\vartheta_{2,1}) = \mathrm{ford}((-1)^{\frac{2}{5}}) = 5.$$

Note that the order of the depth-1 A-monomials are known a priori. That is,

$$\mathrm{ord}(\vartheta_{1,1}) = 3 \qquad \text{and} \qquad \mathrm{ord}(\vartheta_{2,1}) = 5.$$

Now with (6.22), we compute the factorial order of the depth-1 A-monomials:

$$\mathrm{ford}(\vartheta_{1,1}) = 3 \qquad \text{and} \qquad \mathrm{ford}(\vartheta_{2,1}) = 5.$$

(2) Take the depth-2 A-monomials, i.e., $\vartheta_{1,2}$ and $\vartheta_{2,2}$ with

$$\sigma(\vartheta_{1,2}) = -\vartheta_{1,1}^2\,\vartheta_{1,2} \qquad \text{and} \qquad \sigma(\vartheta_{2,2}) = (-1)^{\frac{2}{3}}\,\vartheta_{1,1}^2\,\vartheta_{2,1}^3\,\vartheta_{2,2}.$$

By Definition 2.3.17 we have that, $\mathrm{ord}(\vartheta_{1,2}) = \mathrm{ord}(-\vartheta_{1,1}^2)$ and $\mathrm{ord}(\vartheta_{2,2}) = \mathrm{ord}((-1)^{\frac{2}{3}}\,\vartheta_{1,1}^2\,\vartheta_{2,1}^3)$ and with (6.23) we compute

$$\mathrm{ord}(\vartheta_{1,2}) = \mathrm{ord}(-\vartheta_{1,1}^2) = 6 \qquad \text{and} \qquad \mathrm{ord}(\vartheta_{2,2}) = \mathrm{ord}((-1)^{\frac{2}{3}}\,\vartheta_{1,1}^2\,\vartheta_{2,1}^3) = 15.$$

By statement (2) of Lemma 6.2.17,

$$\mathrm{per}(\vartheta_{1,2}) = \mathrm{ford}(-\vartheta_{1,1}^2) = 6 \qquad \text{and} \qquad \mathrm{per}(\vartheta_{2,2}) = \mathrm{ford}((-1)^{\frac{2}{3}}\,\vartheta_{1,1}^2\,\vartheta_{2,1}^3) = 15.$$

The factorial order of $\vartheta_{1,2}$ and $\vartheta_{2,2}$ are computed with (6.22):

$$\mathrm{ford}(\vartheta_{1,2}) = 36 \qquad \text{and} \qquad \mathrm{ford}(\vartheta_{2,2}) = 45.$$

(3) Take the depth-3 A-monomials, i.e., $\vartheta_{1,3}$ and $\vartheta_{2,3}$ with

$$\sigma(\vartheta_{1,3}) = -\vartheta_{2,1}^4\,\vartheta_{1,2}^3\,\vartheta_{1,3} \qquad \text{and} \qquad \sigma(\vartheta_{2,3}) = -\vartheta_{1,2}^3\,\vartheta_{2,2}^5\,\vartheta_{2,3}.$$

Again by Definition 2.3.17, $\mathrm{ord}(\vartheta_{1,3}) = \mathrm{ord}(-\vartheta_{2,1}^4\,\vartheta_{1,2}^3)$ and $\mathrm{ord}(\vartheta_{2,3}) = \mathrm{ord}(-\vartheta_{1,2}^3\,\vartheta_{2,2}^5)$. Thus, with (6.23) we have that,

$$\mathrm{ord}(\vartheta_{1,3}) = \mathrm{ord}(-\vartheta_{2,1}^4\,\vartheta_{1,2}^3) = 10 \qquad \text{and} \qquad \mathrm{ord}(\vartheta_{2,3}) = \mathrm{ord}(-\vartheta_{1,2}^3\,\vartheta_{2,2}^5) = 6.$$

Again by statement (2) of Lemma 6.2.17,

$$\mathrm{per}(\vartheta_{1,3}) = \mathrm{ford}(-\vartheta_{2,1}^4\,\vartheta_{1,2}^3) = 20 \qquad \text{and} \qquad \mathrm{per}(\vartheta_{2,3}) = \mathrm{ford}(-\vartheta_{1,2}^3\,\vartheta_{2,2}^5) = 36.$$

Finally, with (6.22), we compute the factorial order of $\vartheta_{1,3}$ and $\vartheta_{2,3}$ as

$$\mathrm{ford}(\vartheta_{1,3}) = 20 \qquad \text{and} \qquad \mathrm{ford}(\vartheta_{2,3}) = 36.$$

$\star$

6.2.4 Idempotent representation of single RPS-extensions

Throughout this subsection, $(\mathbb{F}, \sigma)$ is a difference field with the constant field $\mathbb{K} = \mathrm{const}(\mathbb{F}, \sigma)$, $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}[\vartheta]\langle t_1\rangle \dots \langle t_e\rangle$ is an RPS-extension of $(\mathbb{F}, \sigma)$ where ϑ is an R-monomial of order λ with $\zeta = \frac{\sigma(\vartheta)}{\vartheta} \in \mathbb{K}^*$ and the t_i for $1 \leqslant i \leqslant e$ are PS-monomials with $\sigma(t_i) = \alpha_i\, t_i + \beta_i$, i.e., either $\alpha_i = 1$ which implies that t_i is an S-monomial or $\alpha_i \in (\mathbb{F}^*)_{\mathbb{F}}^{\mathbb{E}}$ with $\beta_i = 0$ implying that t_i is a P-monomial. Here, $(\mathbb{F}^*)_{\mathbb{F}}^{\mathbb{E}}$ is the product group over $\mathbb{F}^*$ with respect to P-monomials for the RPS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$, see (6.18). Most of the results presented in this subsection have already been discussed in Schneider (2017, Section 4). These ideas are also inspired by Put and Singer (1997).

Lemma 6.2.23.
Let $\mathbb{A}$ be an integral domain and let $\mathbb{A}[\vartheta]$ be a polynomial ring subject to the relation $\vartheta^\lambda = 1$ with $\lambda > 1$. Let $f = \sum_{i=0}^{\lambda-1} f_i\, \vartheta^i$ and $g = \sum_{i=0}^{\lambda-1} g_i\, \vartheta^i$ with $f_i,\, g_i \in \mathbb{A}$ and $f(\nu_i) = g(\nu_i)$ for distinct $\{\nu_1, \dots, \nu_\lambda\} \subseteq \mathbb{A}$. Then $f = g$. ♦

Proof:
Consider the polynomial ring $\mathbb{A}[y]$. Define $\tilde{f} := \sum_{i=0}^{\lambda-1} f_i\, y^i$, $\tilde{g} := \sum_{i=0}^{\lambda-1} g_i\, y^i \in \mathbb{A}[y]$. Let $h := \tilde{f} - \tilde{g} \in \mathbb{A}[y]$. Then

$$h(\nu_j) = \tilde{f}(\nu_j) - \tilde{g}(\nu_j) = \sum_{i=0}^{\lambda-1} f_i\, \nu_j^i - \sum_{i=0}^{\lambda-1} g_i\, \nu_j^i = 0$$

for all $1 \leqslant j \leqslant \lambda$. Since $\mathbb{A}$ is an integral domain, we have $h = 0$. Therefore, $\tilde{f} = \tilde{g}$ and hence $f = g$. ∎

The next Lemma is taken from Schneider (2017, Lemma 4.2); compare also Erocal (2011, Corollary 3.35).

Lemma 6.2.24.
Let $\mathbb{F}$ be a field and let ζ be a primitive λ-th root of unity. Let $\mathbb{F}[\vartheta]$ be a polynomial ring subject to the relation $\vartheta^\lambda = 1$. Then the following statements hold.

(1) There are idempotent elements $\boldsymbol{e}_0, \dots, \boldsymbol{e}_{\lambda-1} \in \mathbb{F}[\vartheta]$ where

$$\boldsymbol{e}_k = \boldsymbol{e}_k(\vartheta) := \prod_{\substack{i=0\\ i \neq \lambda-1-k}}^{\lambda-1} \frac{\vartheta - \zeta^i}{\zeta^{\lambda-1-k} - \zeta^i} \tag{6.26}$$

for all $0 \leqslant k < \lambda$ with

$$\boldsymbol{e}_k(\zeta^j) = \begin{cases} 1 & \text{if } j = \lambda - 1 - k \\ 0 & \text{if } j \neq \lambda - 1 - k \end{cases} \qquad \text{and} \qquad \boldsymbol{e}_k(\zeta\,\vartheta) = \boldsymbol{e}_{k+1 \bmod \lambda}. \tag{6.27}$$

(2) The idempotent elements defined in (6.26) are pairwise orthogonal and $\boldsymbol{e}_0 + \dots + \boldsymbol{e}_{\lambda-1} = 1$. ♦

PROOF:

(1) We will first prove identity (6.27) and use it to show that the $\boldsymbol{e}_k$ are idempotent elements. The first identity is obvious. For the second identity,

$$\boldsymbol{e}_k(\zeta\vartheta) = \prod_{\substack{i=0\\ i\neq\lambda-1-k}}^{\lambda-1} \frac{\zeta\vartheta - \zeta^i}{\zeta^{\lambda-1-k} - \zeta^i} = \prod_{\substack{i=0\\ i\neq\lambda-1-(k+1)}}^{\lambda-1} \frac{\vartheta - \zeta^{i-1}}{\zeta^{\lambda-1-(k+1)} - \zeta^{i-1}} = \boldsymbol{e}_{k+1 \bmod \lambda}.$$

Now let $0 \leqslant k < \lambda$ and define

$$w(\vartheta) := \boldsymbol{e}_k(\vartheta)^2 = \sum_{j=0}^{\lambda-1} f_j\,\vartheta^j$$

with $f_i \in \mathbb{F}$. By (6.27) we have that

$$w(\zeta^i) = \boldsymbol{e}_k(\zeta^i)^2 = \boldsymbol{e}_k(\zeta^i) \quad \text{for } 0 \leqslant i < \lambda.$$

Since ζ is a primitive λ-th root of unity, ζ^i for all $0 \leqslant i < \lambda$ are distinct, it follows by Lemma 6.2.23 that

$$\boldsymbol{e}_k(\vartheta) = w(\vartheta) = \boldsymbol{e}_k(\vartheta)^2.$$

(2) For $0 \leqslant i, j < \lambda$, the factors in $\boldsymbol{e}_i(\vartheta)\,\boldsymbol{e}_j(\vartheta)$ gives $\vartheta^\lambda - 1$ which shows that $\boldsymbol{e}_i(\vartheta)\,\boldsymbol{e}_j(\vartheta) = 0$. Hence the $\boldsymbol{e}_i$ are pairwise orthogonal. Now define $g(\vartheta) = \boldsymbol{e}_0(\vartheta) + \cdots + \boldsymbol{e}_{\lambda-1}(\vartheta) - 1$. Then it follows by (6.27) that $g(\zeta^i) = 0$ for all $0 \leqslant i < \lambda$. Thus by Lemma 6.2.23, $0 = g = \boldsymbol{e}_0 + \cdots + \boldsymbol{e}_{\lambda-1}$. ■

In the next theorem we discuss how one can decompose a simple RPS-extension $(\mathbb{E}, \sigma)$ of a difference field $(\mathbb{F}, \sigma)$ with $\mathbb{E} = \mathbb{F}[\vartheta]\langle t_1\rangle \dots \langle t_e\rangle$ whose product group $(\mathbb{F}^*)^{\mathbb{E}}_{\mathbb{F}}$ is free of any RS-/RΣ-monomial. For a more detailed result see Schneider (2017, Theorem 4.3); compare also Put and Singer (1997, Corollary 1.16), and Hardouin and Singer (2008, Lemma 6.8).

Theorem 6.2.25.
Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}[\vartheta]\langle t_1\rangle \dots \langle t_e\rangle$ be an RPS-extension of a difference field $(\mathbb{F}, \sigma)$ where ϑ is an R-monomial of order λ with $\zeta = \frac{\sigma(\vartheta)}{\vartheta} \in \mathrm{const}(\mathbb{F}, \sigma)^*$. Let $\boldsymbol{e}_0, \dots, \boldsymbol{e}_{\lambda-1}$ be the idempotent, pairwise orthogonal elements in (6.26) that sum up to one. Then the following statement holds:

(1) The ring $\mathbb{E}$ can be written as the direct sum

$$\mathbb{E} = \boldsymbol{e}_0\mathbb{E} \oplus \cdots \oplus \boldsymbol{e}_{\lambda-1}\mathbb{E} \tag{6.28}$$

where $\boldsymbol{e}_k\mathbb{E}$ is a ring with $\boldsymbol{e}_k$ as multiplicative identity for all $0 \leqslant k < \lambda$.

(2) We have that $\boldsymbol{e}_k\mathbb{E} = \boldsymbol{e}_k\tilde{\mathbb{E}}$ for $0 \leqslant k < \lambda$ where $\tilde{\mathbb{E}} = \mathbb{F}\langle t_1\rangle \dots \langle t_e\rangle$. ♢

PROOF:
Any $g \in \mathbb{E}$ can be written in the form $g = \sum_{i=0}^{\lambda-1} g_i\,\vartheta^i$ where $g_i \in \tilde{\mathbb{E}} = \mathbb{F}\langle t_1\rangle \dots \langle t_e\rangle$. With this representation, consider the map $\psi : \mathbb{E} \to \mathbb{E}$ defined by

$$\sum_{i=0}^{\lambda-1} g_i\,\vartheta^i \mapsto \sum_{i=0}^{\lambda-1} \boldsymbol{e}_i\,\tilde{g}_i$$

with $\tilde{g}_i = g(\zeta^{\lambda-1-i})$. Now define $\tilde{g}(\vartheta) := \psi(g)$. Then note that for all $0 \leqslant k < \lambda$,

$$\tilde{g}(\zeta^{\lambda-1-k}) = \sum_{i=0}^{\lambda-1} \tilde{g}_i\, \boldsymbol{e}_i(\zeta^{\lambda-1-k}) = \tilde{g}_k = g(\zeta^{\lambda-1-k}).$$

Since ζ is a primitive λ-th root of unity, ζ^k for $0 \leqslant k < \lambda$ are distinct. Thus it follows by Lemma 6.2.23 that $g = \tilde{g}$. In other words, if we consider $\mathbb{E}$ as an $\tilde{\mathbb{E}}$-module with basis $1, \vartheta, \vartheta^2, \ldots, \vartheta^{\lambda-1}$, then $\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{\lambda-1}$ is also a basis and ψ is a basis transformation. In particular, this gives the direct sum

$$\mathbb{E} = \boldsymbol{e}_0\tilde{\mathbb{E}} \oplus \cdots \oplus \boldsymbol{e}_{\lambda-1}\tilde{\mathbb{E}}$$

of modules. Note that $\boldsymbol{e}_k\tilde{\mathbb{E}}$ for $0 \leqslant k < \lambda$ is an integral domain with $\boldsymbol{e}_k$ as multiplicative identity. Thus we obtain a direct sum of rings $\boldsymbol{e}_k\tilde{\mathbb{E}}$. In particular, for $g \in \mathbb{E}$, we have

$$\boldsymbol{e}_k\, g = \boldsymbol{e}_k\, g(\zeta^{\lambda-1-k})$$

with $g(\zeta^{\lambda-1-k}) \in \tilde{\mathbb{E}}$ for all $0 \leqslant k < \lambda$. Thus $\boldsymbol{e}_k\mathbb{E} = \boldsymbol{e}_k\tilde{\mathbb{E}}$ for all $0 \leqslant k < \lambda$ which completes the proof. ∎

Proposition 6.2.26.
Let $(\mathbb{K}_m[\vartheta_1]\ldots[\vartheta_e], \sigma)$ be a simple A*-extension of a difference field $(\mathbb{K}_m, \sigma)$ with $\frac{\sigma(\vartheta_i)}{\vartheta_i} = \zeta_m^{u_i}\,\vartheta_1^{z_{i,1}} \cdots \vartheta_{i-1}^{z_{i,i-1}}$ for all $1 \leqslant i \leqslant e$ where $m \in \mathbb{N} \setminus \{0, 1\}$, $0 \leqslant u_i < m$ and $\zeta_m \in \mathbb{K}_m^*$ is a primitive m-th root of unity. Then there is a defining evaluation function $\mathrm{ev}_m : \mathbb{K}_m[\vartheta_1]\ldots[\vartheta_e] \times \mathbb{N} \to \mathbb{K}_m$ for the map $\tau : \mathbb{K}_m[\vartheta_1]\ldots[\vartheta_e] \to \mathcal{S}(\mathbb{K}_m)$ such that τ is a difference ring homomorphism.* □

PROOF:
Let $(\mathbb{A}, \sigma)$ with $\mathbb{A} = \mathbb{K}_m[\vartheta_1]\ldots[\vartheta_e]$ be a simple A-extension of $(\mathbb{K}_m, \sigma)$. Consider the map

$$\tau : \begin{cases} \mathbb{A} & \to \ \mathcal{S}(\mathbb{K}_m) \\ f & \mapsto \ \langle \mathrm{ev}_m(f, n) \rangle_{n \geqslant 0} \end{cases} \tag{6.29}$$

where $\mathrm{ev}_m : \mathbb{K}_m[\vartheta_1]\ldots[\vartheta_e] \times \mathbb{N} \to \mathbb{K}_m$ satisfies the properties in Lemma 2.4.2 and is defined recursively by

$$\mathrm{ev}_m\Big(\sum_{\nu_i \in \mathbb{N}^e} c_{\nu_i}\vartheta_1^{\nu_{i,1}} \cdots \vartheta_e^{\nu_{i,e}}, n\Big) \mapsto \sum_{\nu_i \in \mathbb{N}^e} \mathrm{ev}_m(c_{\nu_i}, n)\, \mathrm{ev}_m(\vartheta_1, n)^{\nu_{i,1}} \cdots \mathrm{ev}_m(\vartheta_e, n)^{\nu_{i,e}}$$

with $\mathrm{ev}_m(c_{\nu_i}, n) = c_{\nu_i}$ for all $c_{\nu_i} \in \mathbb{K}_m$ and

$$\mathrm{ev}_m(\vartheta_i, n) = \prod_{j=1}^{n} \mathrm{ev}_m(\alpha_i, j-1) \tag{6.30}$$

where $\alpha_i \in (\mathbb{K}_m^*)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\ldots[\vartheta_{i-1}]}$. Then by Schneider (2001, Lemma 2.5.1) τ is a difference ring homomorphism. ∎

Remark 6.2.27.
The evaluation function (6.30) introduced in the proof of Proposition 6.2.26 is also called the naturally induced evaluation function of $(\mathbb{K}_m[\vartheta_1]\ldots[\vartheta_e], \sigma)$. ♠

We are now ready to generalise items (1) and (2) of Lemma 6.2.11 for simple A-extension of any depth.

Theorem 6.2.28.
Let $(\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e],\sigma)$ be a simple A-extension of a constant-stable difference field $(\mathbb{K}_m,\sigma)$ with $\frac{\sigma(\vartheta_i)}{\vartheta_i} = \zeta_m^{u_i}\,\vartheta_1^{z_{i,1}}\cdots\vartheta_{i-1}^{z_{i,i-1}}$ for all $1\leqslant i\leqslant e$ where $m\in\mathbb{N}\setminus\{0,1\}$, $0\leqslant u_i<m$ and $\zeta_m\in\mathbb{K}_m^*$ is a primitive s-th root of unity. Furthermore, let $\mathrm{ev}_m:\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]\times\mathbb{N}\to\mathbb{K}_m$ be the naturally induced evaluation given in (6.30) and let $\tau_m:\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]\to\mathcal{S}(\mathbb{K})$ be the difference ring homomorphism defined by $\tau_m(f)=\langle\mathrm{ev}_m(f,n)\rangle_{n\geqslant0}$ Then the following statements hold.

(1) There is a difference ring $(\mathbb{K}_\lambda[\vartheta],\sigma)$ with a primitive λ-th root of unity, $\zeta_\lambda=\frac{\sigma(\vartheta)}{\vartheta}\in\mathbb{K}_\lambda^*$ with $\lambda:=\mathrm{lcm}(\mathrm{per}(\vartheta_1),\dots,\mathrm{per}(\vartheta_e))>0$ where $\mathbb{K}_\lambda$ is a finite algebraic extension of $\mathbb{K}_m$ with $m\,|\,\lambda$ such that

$$\varphi:\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]\to\mathbb{K}_\lambda[\vartheta]=\boldsymbol{e}_0\mathbb{K}_\lambda\oplus\cdots\oplus\boldsymbol{e}_{\lambda-1}\mathbb{K}_\lambda \tag{6.31}$$

defined with

$$\varphi(f)=f_0\boldsymbol{e}_0+\cdots+f_{\lambda-1}\boldsymbol{e}_{\lambda-1} \tag{6.32}$$

where $f_i=\mathrm{ev}_m(f,\lambda-1-i)\in\mathbb{K}_m\subseteq\mathbb{K}_\lambda$ for $0\leqslant i<\lambda$ is a difference ring homomorphism, where the $\boldsymbol{e}_k$ are the idempotent orthogonal elements defined in (6.26). In particular, $(\mathbb{K}_\lambda[\vartheta],\sigma)$ is an R-extension of $(\mathbb{K}_\lambda,\sigma)$.

(2) Take the evaluation function $\mathrm{ev}_\lambda:\mathbb{K}_\lambda\times\mathbb{N}\to\mathbb{K}_\lambda$ defined by $\mathrm{ev}_\lambda|_{\mathbb{K}_\lambda}=\mathrm{id}$ and $\mathrm{ev}_\lambda(\vartheta,n)=\zeta^n$ and consider the $\mathbb{K}_\lambda$-homomorphism $\tau_\lambda:\mathbb{K}_\lambda[\vartheta]\to\mathcal{S}(\mathbb{K}_\lambda)$ defined by $\tau_\lambda(f)=\langle\mathrm{ev}_\lambda(f,n)\rangle_{n\geqslant0}$. Then for the pairwise orthogonal idempotent elements $\boldsymbol{e}_k$ defined in (6.26) with $0\leqslant k<\lambda$, we have that

$$\mathrm{ev}_\lambda(\boldsymbol{e}_k,n)=\begin{cases}1 & \text{if } \lambda\,|\,n+k+1,\\ 0 & \text{if } \lambda\nmid n+k+1.\end{cases} \tag{6.33}$$

(3) The $\mathbb{K}_\lambda$-homomorphism $\tau_\lambda:\mathbb{K}_\lambda[\vartheta]\to\mathcal{S}(\mathbb{K}_\lambda)$ with the evaluation function defined in part (2) is injective.

(4) The diagram below commutes

$$\begin{array}{ccc}\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e] & \xrightarrow{\quad\tau_m\quad} & \mathcal{S}(\mathbb{K}_m)\\ \varphi\big\downarrow & & \big\downarrow\varphi'\\ \mathbb{K}_\lambda[\vartheta]\simeq\boldsymbol{e}_0\mathbb{K}_\lambda\oplus\cdots\oplus\boldsymbol{e}_{\lambda-1}\mathbb{K}_\lambda & \xrightarrow{\tau_\lambda} & \mathcal{S}(\mathbb{K}_\lambda)\end{array} \tag{6.34}$$

where $\varphi':\mathcal{S}(\mathbb{K}_m)\to\mathcal{S}(\mathbb{K}_\lambda)$ is the injective difference ring homomorphism defined by $\varphi'(a)=a$. $\diamondsuit$

Proof:
(1) Since $\zeta_m^{u_i}\in\mathbb{K}_m^*$, $\mathrm{per}(\zeta_m^{u_i})=1>0$ for all $1\leqslant u_i\leqslant e$. Hence it follows by statement (2) of Proposition 6.2.20 that $\mathrm{per}(\vartheta_i)>0$. Define $\lambda:=\mathrm{lcm}(\mathrm{per}(\vartheta_1),\dots,\mathrm{per}(\vartheta_e))>0$ and $\zeta_\lambda:=e^{\frac{2\pi i}{\lambda}}=(-1)^{\frac{2}{\lambda}}\in\mathbb{K}_\lambda^*$ where $\mathbb{K}_\lambda$ is some algebraic extension of $\mathbb{K}_m$. Note that m divides λ. Take the A-extension $(\mathbb{K}_\lambda[\vartheta],\sigma)$ of $(\mathbb{K}_\lambda,\sigma)$ with $\sigma(\vartheta)=\zeta_\lambda\,\vartheta$. By Proposition 2.3.37 it follows that $(\mathbb{K}_\lambda[\vartheta],\sigma)$ is an R-extension of $(\mathbb{K}_\lambda,\sigma)$. By Theorem 6.2.25 we have that

$$\mathbb{K}_\lambda[\vartheta]=\boldsymbol{e}_0\mathbb{K}_\lambda\oplus\cdots\oplus\boldsymbol{e}_{\lambda-1}\mathbb{K}_\lambda$$

where the $\boldsymbol{e}_k$ for $0 \leqslant k < \lambda$ are the orthogonal idempotent elements defined in (6.26). Now consider the map (6.31) defined by 6.32. We will now show that φ is a ring homomorphism. Observe that for any $c \in \mathbb{K}_m$, $\mathrm{ev}_m(c, i) = c$ for all $i \in \mathbb{N}$ and with statement (2) of Lemma 6.2.24 we have that

$$\varphi(c) = c\,\boldsymbol{e}_0 + \cdots + c\,\boldsymbol{e}_{\lambda-1} = c\,(\boldsymbol{e}_0 + \cdots + \boldsymbol{e}_{\lambda-1}) = c.$$

Thus, $\varphi(1) = 1$. Further, let $f, g \in \mathbb{K}_m[\vartheta_1]\ldots[\vartheta_e]$ with $f := a\,\vartheta_1^{\nu_1}\cdots\vartheta_e^{\nu_e}$ and $g := b\,\vartheta_1^{z_1}\cdots\vartheta_e^{z_e}$ where $a, b \in \mathbb{K}_m$ and $\nu_i, z_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant e$. Define $f_k := \mathrm{ev}_m(f, \lambda - 1 - k)$ and $g_k := \mathrm{ev}_m(g, \lambda - 1 - k)$ for $0 \leqslant k < \lambda$. Then,

$$\begin{aligned}
\varphi(f + g) &= \mathrm{ev}_m(f + g, \lambda - 1)\,\boldsymbol{e}_0 + \cdots + \mathrm{ev}_m(f + g, 0)\,\boldsymbol{e}_{\lambda-1}\\
&= \big(\mathrm{ev}_m(f, \lambda - 1) + \mathrm{ev}_m(g, \lambda - 1)\big)\,\boldsymbol{e}_0 + \cdots + \big(\mathrm{ev}_m(f, 0) + \mathrm{ev}_m(g, 0)\big)\,\boldsymbol{e}_{\lambda-1}\\
&= \big(\mathrm{ev}_m(f, \lambda - 1)\,\boldsymbol{e}_0 + \cdots + \mathrm{ev}_m(f, 0)\,\boldsymbol{e}_{\lambda-1}\big) + \big(\mathrm{ev}_m(g, \lambda - 1)\,\boldsymbol{e}_0 + \cdots + \mathrm{ev}_m(g, 0)\,\boldsymbol{e}_{\lambda-1}\big)\\
&= \big(f_0\,\boldsymbol{e}_0 + \cdots + f_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\big) + \big(g_0\,\boldsymbol{e}_0 + \cdots + g_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\big)\\
&= \varphi(f) + \varphi(g).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varphi(f\,g) &= \mathrm{ev}_m(f\,g, \lambda - 1)\,\boldsymbol{e}_0 + \cdots + \mathrm{ev}_m(f\,g, 0)\,\boldsymbol{e}_{\lambda-1}\\
&= \big(\mathrm{ev}_m(f, \lambda - 1)\ \mathrm{ev}_m(g, \lambda - 1)\big)\,\boldsymbol{e}_0 + \cdots + \big(\mathrm{ev}_m(f, 0)\ \mathrm{ev}_m(g, 0)\big)\,\boldsymbol{e}_{\lambda-1}\\
&= f_0\,g_0\,\boldsymbol{e}_0 + f_1\,g_1\,\boldsymbol{e}_1 + \cdots + f_{\lambda-1}\,g_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\\
&= \big(f_0\,\boldsymbol{e}_0 + \cdots + f_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\big)\,\big(g_0\,\boldsymbol{e}_0 + \cdots + g_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\big)\\
&= \varphi(f)\,\varphi(g).
\end{aligned}$$

The first equality follows since the $\boldsymbol{e}_i$ are idempotent. Thus, φ is a ring homomorphism. Next we show by induction on the number of A-monomials, $e \in \mathbb{N}$, that φ is a difference ring homomorphism. For the base case, i.e., $e = 0$, there are no A-monomials and we have that

$$\sigma(\varphi(c)) = \sigma(c) = c = \varphi(c) = \varphi(\sigma(c))$$

for all $c \in \mathbb{K}_m$. Thus, the statement holds for the base case. Now assume that the statement holds for all A-monomials ϑ_i with $0 \leqslant i < e$ and let $s_e \in \mathbb{N}$. We prove the statement for the A-monomial, ϑ_e, with $\sigma(\vartheta_e) = \tilde{\alpha}\,\vartheta_e$ with $\tilde{\alpha} \in (\mathbb{K}_m^*)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\cdots[\vartheta_{e-1}]}$. More precisely we show that

$$\sigma(\varphi(\vartheta_e^{s_e})) = \varphi(\sigma(\vartheta_e^{s_e})) \tag{6.35}$$

holds. For the left hand side of (6.35), we have that

$$\varphi(\vartheta_e^{s_e}) = \gamma_0\,\boldsymbol{e}_0 + \cdots + \gamma_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}$$

where $\gamma_i = \mathrm{ev}_m(\vartheta_e^{s_e}, \lambda - 1 - i) \in \mathbb{K}_m$ for $0 \leqslant i < \lambda$ are λ-th root of unity. Thus,

$$\sigma(\varphi(\vartheta_e^{s_e})) = \gamma_0\,\sigma(\boldsymbol{e}_0) + \cdots + \gamma_{\lambda-1}\,\sigma(\boldsymbol{e}_{\lambda-1}).$$

By (6.27) we have that $\sigma(\boldsymbol{e}_{\lambda-1}) = \boldsymbol{e}_0$ and $\sigma(\boldsymbol{e}_i) = \boldsymbol{e}_{i+1}$ for $0 \leqslant i < \lambda - 1$. Therefore,

$$\sigma(\varphi(\vartheta_e^{s_e})) = \tilde{\gamma}_0\,\boldsymbol{e}_0 + \cdots + \tilde{\gamma}_{\lambda-1}\,\boldsymbol{e}_{\lambda-1} \tag{6.36}$$

where $\tilde{\gamma}_0 = \gamma_{\lambda-1}$ and $\tilde{\gamma}_i = \gamma_{i-1}$ for $1 \leqslant i \leqslant \lambda - 1$.

For the right hand side of (6.35), we have that

$$\sigma(\vartheta_e^{s_e}) = \sigma(\vartheta_e)^{s_e} = (\tilde{\alpha}\,\vartheta_e)^{s_e} = \alpha\,\vartheta_e^{s_e}$$

where $\alpha = \tilde{\alpha}^{s_e} \in (\mathbb{K}_m^*)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\dots[\vartheta_{e-1}]}$. Thus,

$$\begin{aligned}\varphi(\sigma(\vartheta_e^{s_e})) = \varphi(\alpha\,\vartheta_e^{s_e}) &= \varphi(\alpha)\,\varphi(\vartheta_e^{s_e})\\ &= (\alpha_0\,\boldsymbol{e}_0 + \cdots + \alpha_{\lambda-1}\,\boldsymbol{e}_{\lambda-1})(\gamma_0\,\boldsymbol{e}_0 + \cdots + \gamma_{\lambda-1}\,\boldsymbol{e}_{\lambda-1})\\ &= \alpha_0\,\gamma_0\,\boldsymbol{e}_0 + \cdots + \alpha_{\lambda-1}\,\gamma_{\lambda-1}\,\boldsymbol{e}_{\lambda-1} \end{aligned} \tag{6.37}$$

where $\alpha_i = \mathrm{ev}_m(\alpha, \lambda-1-i)$ and $\gamma_i = \mathrm{ev}_m(\vartheta_e^{s_e}, \lambda-1-i)$ for $0 \leqslant i < \lambda$ are λ-th root of unity. Again (6.37) holds since the $\boldsymbol{e}_i$ are idempotent. By this construction,

$$\tilde{\gamma}_i = \alpha_i\,\gamma_i$$

for $0 \leqslant i < \lambda$. Together with (6.36), equation (6.35) holds. Thus, φ is a difference ring homomorphism.

(2) Evaluation of the idempotent elements $\boldsymbol{e}_k$ given in (6.33) follows by substituting j with n in (6.27).

(3) Since $(\mathbb{K}_\lambda[\vartheta], \sigma)$ is an R-extension of a difference field $(\mathbb{K}_\lambda, \sigma)$, it follows by Schneider (2017, Theorem 3.3) that it is simple, i.e., any ideal of $\mathbb{K}_\lambda[\vartheta]$ that is closed under σ is either $\mathbb{K}_\lambda[\vartheta]$ or $\{0\}$. Thus by Schneider (2017, Lemma 5.8) τ_λ is injective.

(4) Let $\alpha \in \mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]$ and let $\mathrm{ev}_m, \mathrm{ev}_\lambda$ be evaluation functions for $\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]$ and $\mathbb{K}_\lambda[\vartheta]$ defined by (6.30) and (6.33) respectively. We prove that

$$\varphi'(\tau_m(\alpha)) = \tau_\lambda(\varphi(\alpha)) \tag{6.38}$$

holds. For the left hand side of (6.38), we have

$$\varphi'(\tau_m(\alpha)) = \tau_m(\alpha) = \langle \mathrm{ev}_m(\alpha, n)\rangle_{n\geqslant 0} \in \mathcal{S}(\mathbb{K}_m) \subseteq \mathcal{S}(\mathbb{K}_\lambda).$$

For the right hand side of (6.38) we have the following. First of all, by (6.32),

$$\varphi(\alpha) = \alpha_0\,\boldsymbol{e}_0 + \cdots + \alpha_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}$$

where $\alpha_i = \mathrm{ev}_m(\alpha, \lambda-1-i) \in \mathbb{K}_m \subseteq \mathbb{K}_\lambda$ for $0 \leqslant i < \lambda$. Thus,

$$\begin{aligned}\tau_\lambda(\varphi(\alpha)) &= \langle \mathrm{ev}_\lambda(\alpha_0\,\boldsymbol{e}_0 + \cdots + \alpha_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}, n)\rangle_{n\geqslant 0}\\ &= \langle \mathrm{ev}_\lambda(\alpha_0\,\boldsymbol{e}_0, n)\rangle_{n\geqslant 0} + \cdots + \langle \mathrm{ev}_\lambda(\alpha_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}, n)\rangle_{n\geqslant 0}\\ &= \alpha_0\,\langle \mathrm{ev}_\lambda(\boldsymbol{e}_0, n)\rangle_{n\geqslant 0} + \cdots + \alpha_{\lambda-1}\,\langle \mathrm{ev}_\lambda(\boldsymbol{e}_{\lambda-1}, n)\rangle_{n\geqslant 0}\\ &= \langle \mathrm{ev}_m(\alpha, n)\rangle_{n\geqslant 0}.\end{aligned}$$

The last equality follows by (6.33). This proves the statement and consequently the diagram (6.34) commutes. ∎

Remark 6.2.29.
By statement (4) of Theorem 6.2.28 and (6.33) we observe that for a fixed $k \in \mathbb{N}$ and $\alpha \in \mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]$

$$\begin{aligned}\mathrm{ev}_m(\alpha, k) &= \mathrm{ev}_\lambda(\varphi(\alpha), k)\\ &= \alpha_0\ \mathrm{ev}_\lambda(\boldsymbol{e}_0, k) + \dots + \alpha_{\lambda-1}\ \mathrm{ev}_\lambda(\boldsymbol{e}_{\lambda-1}, k)\\ &= \alpha_j\ \mathrm{ev}_\lambda(\boldsymbol{e}_j, k)\\ &= \alpha_j\\ &= \mathrm{ev}_m(\alpha, j)\end{aligned} \tag{6.39}$$

holds for some $j \in \{0, 1, \dots, \lambda - 1\}$ with $\lambda \,|\, k - j$. ♠

Example 6.2.30 (Cont. Example 6.2.10).
From Example 6.2.10, we already know the period of the A-monomials in the $\mathbb{U}_{30}$-simple A-extension $(\mathbb{K}_{30}[\vartheta_{1,1}][\vartheta_{2,1}][\vartheta_{1,2}][\vartheta_{2,2}][\vartheta_{1,3}][\vartheta_{2,3}], \sigma)$ of $(\mathbb{K}_{30}, \sigma)$ where $\mathbb{K}_{30} = \mathbb{Q}((-1)^{\frac{2}{3}}, (-1)^{\frac{2}{5}})$. Set

$$\lambda = \mathrm{lcm}(\mathrm{per}(\vartheta_{1,1}), \mathrm{per}(\vartheta_{2,1}), \mathrm{per}(\vartheta_{1,2}), \mathrm{per}(\vartheta_{2,2}), \mathrm{per}(\vartheta_{1,3}), \mathrm{per}(\vartheta_{2,3})) = 180$$

and take a λ-th root of unity, say, $\zeta_\lambda := \mathrm{e}^{\frac{\pi i}{90}} = (-1)^{\frac{1}{90}}$. Then with $\mathbb{K}_\lambda = \mathbb{Q}(\zeta_\lambda)$ we can construct a difference ring homomorphism

$$\varphi : \mathbb{K}_{30}[\vartheta_{1,1}][\vartheta_{2,1}][\vartheta_{1,2}][\vartheta_{2,2}][\vartheta_{1,3}][\vartheta_{2,3}] \to \boldsymbol{e}_0\mathbb{K}_\lambda \oplus \dots \oplus \boldsymbol{e}_{\lambda-1}\mathbb{K}_\lambda = \mathbb{K}_\lambda[\vartheta].$$

where the $\boldsymbol{e}_k$ for $0 \leqslant k < \lambda$ are the pairwise orthogonal idempotent elements defined in (6.26). In particular, $(\mathbb{K}_\lambda[\vartheta], \sigma)$ with $\sigma(\vartheta) = \zeta_\lambda\,\vartheta$ is a single R-extension of $(\mathbb{K}_\lambda, \sigma)$. Moreover, we get the $\mathbb{K}_\lambda$-embedding

$$\tau_\lambda : \boldsymbol{e}_0\,\mathbb{K}_\lambda \oplus \dots \oplus \boldsymbol{e}_{\lambda-1}\,\mathbb{K}_\lambda \to \mathcal{S}(\mathbb{K}_\lambda)$$

with the evaluation function, $\mathrm{ev}_\lambda : \boldsymbol{e}_0\,\mathbb{K}_\lambda \oplus \dots \oplus \boldsymbol{e}_{\lambda-1}\,\mathbb{K}_\lambda \times \mathbb{N} \to \mathcal{S}(\mathbb{K}_\lambda)$ defined by

$$\mathrm{ev}_\lambda(\sum_{i=0}^{\lambda-1} f_i\,\boldsymbol{e}_i(\vartheta)^i, n) = \sum_{i=0}^{\lambda-1} \mathrm{ev}_\lambda(f_i, n)\ \mathrm{ev}_\lambda(\boldsymbol{e}_i(\vartheta), n)^i$$

where $\mathrm{ev}_\lambda(f_i, n)$ and $\mathrm{ev}_\lambda(\boldsymbol{e}_i(\vartheta), n)$ are the evaluation functions defined in (2.1) and (6.33) respectively. Consequently, with $\varphi'(a) = a$, the diagram

$$\begin{array}{ccc}\mathbb{K}_{30}[\vartheta_{1,1}][\vartheta_{2,1}][\vartheta_{1,2}][\vartheta_{2,2}][\vartheta_{1,3}][\vartheta_{2,3}] & \xrightarrow{\tau_{30}} & \mathcal{S}(\mathbb{K}_{30})\\ \varphi\big\downarrow & & \big\downarrow\varphi'\\ \mathbb{K}_\lambda[\vartheta] \simeq \boldsymbol{e}_0\mathbb{K}_\lambda \oplus \dots \oplus \boldsymbol{e}_{\lambda-1}\mathbb{K}_\lambda & \xrightarrow{\tau_\lambda} & \mathcal{S}(\mathbb{K}_\lambda)\end{array}$$

commutes. ⋆

Corollary 6.2.31.
Let $(\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e], \sigma)$ *be a simple* A*-extension of a difference field* $(\mathbb{K}_m, \sigma)$ *with*

$$\frac{\sigma(\vartheta_i)}{\vartheta_i} = \alpha_i = \zeta_m^{u_i}\,\vartheta_1^{\nu_{i,1}} \cdots \vartheta_{i-1}^{\nu_{i,i-1}} \in (\mathbb{K}_m^*)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\dots[\vartheta_{i-1}]}$$

for all $1 \leqslant i \leqslant e$ *where* $m \in \mathbb{N} \setminus \{0, 1\}$, $0 \leqslant u_i < m$ *and* $\zeta_m \in \mathbb{K}_m^*$ *is a primitive* m*-th root of unity. Let* $\mathrm{ev}_m : \mathbb{K}_m[\vartheta_1] \dots [\vartheta_e] \times \mathbb{N} \to \mathbb{K}_m$ *be the naturally induced evaluation function defined by* (6.30)*. Let* $\gamma_1, \dots, \gamma_s \in (\mathbb{K}_m^*)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]}$. *Then one can define a single* R*-extension* $(\mathbb{K}_\lambda[\vartheta], \sigma)$ *of* $(\mathbb{K}_\lambda, \sigma)$ *with* $\sigma(\vartheta) = \zeta_\lambda \vartheta$ *where* $\zeta_\lambda \in \mathbb{K}_\lambda^*$ *is primitive* λ*-th root of unity and* $\mathbb{K}_\lambda$ *is a finite algebraic field extension of* $\mathbb{K}_m$ *with* $m \,|\, \lambda$ *together with the evaluation function* $\mathrm{ev}_\lambda : \mathbb{K}_\lambda[\vartheta] \times \mathbb{N} \to \mathbb{K}_\lambda$ *defined by* $\mathrm{ev}_\lambda(\vartheta, n) = \zeta_\lambda^n$ *with* (6.33) *such that the following property hold. For all* k *with* $1 \leqslant k \leqslant s$ *one can define*

$$g_k := \sum_{i=0}^{\lambda-1} g_{k,i}\, \vartheta^i \in \mathbb{K}_\lambda[\vartheta] \tag{6.40}$$

with $g_{k,i} \in \mathbb{K}_\lambda$ *such that for all* $i, j \in \mathbb{N}$,

$$\mathrm{ev}_m(\sigma^i(\gamma_k), j) = \mathrm{ev}_\lambda(\sigma^i(g_k), j) \tag{6.41}$$

holds. In particular, $g_k \in \mathbb{K}_\lambda[\vartheta]$ *can be computed for all* $1 \leqslant k \leqslant s$. $\diamond$

Proof:
By Theorem 6.2.28, there is a single R-extension $(\mathbb{K}_\lambda[\vartheta], \sigma)$ of $(\mathbb{K}_\lambda, \sigma)$ subject to the relation $\vartheta^\lambda = 1$ with $\sigma(\vartheta) = \zeta_\lambda\, \vartheta$ where $\lambda := \mathrm{lcm}(\mathrm{per}(\gamma_1), \dots, \mathrm{per}(\gamma_s)) > 0$, $\zeta_\lambda := e^{\frac{2\pi i}{\lambda}} = (-1)^{\frac{2}{\lambda}} \in \mathbb{K}_\lambda$ and $\mathbb{K}_\lambda$ is some algebraic extension of $\mathbb{K}_m$ such that $m \,|\, \lambda$. Furthermore, there is a difference ring homomorphism

$$\varphi : \mathbb{K}_m[\vartheta_1] \dots [\vartheta_e] \to \boldsymbol{e}_0 \mathbb{K}_\lambda \oplus \dots \oplus \boldsymbol{e}_{\lambda-1} \mathbb{K}_\lambda = \mathbb{K}_\lambda[\vartheta]$$

such that the diagram (6.34). Define,

$$g_k := \varphi(\gamma_k) = g_{k,0}\, \boldsymbol{e}_0 + \dots + g_{k,\lambda-1}\, \boldsymbol{e}_{\lambda-1} \in \mathbb{K}_\lambda[\vartheta]$$

where $g_{k,\ell} = \mathrm{ev}_m(\gamma_k, \lambda - 1 - \ell) \in \mathbb{K}_\lambda$ for $0 \leqslant \ell < \lambda$. Now by (2.28), we have that $\mathrm{ev}_m(\sigma^i(\gamma_k), j) = \mathrm{ev}_m(\gamma_k, i + j)$. Similarly, $\mathrm{ev}_\lambda(\sigma^i(g_k), j) = \mathrm{ev}_\lambda(g_k, i + j) = \mathrm{ev}_\lambda(\varphi(\gamma_k), i + j)$. Finally by the first equality in (6.39) we have that

$$\mathrm{ev}_m(\gamma_k, i + j) = \mathrm{ev}_\lambda(\varphi(\gamma_k), i + j)$$

holds for all i, $j \in \mathbb{N}$ and for all k with $1 \leqslant k \leqslant s$. This completes the proof. Since Problem O is solvable in $\mathbb{K}_\lambda$, λ can be computed. Hence for all k with $1 \leqslant k \leqslant s$, g_k given by (6.40) is computable. This completes the proof. ■

Remark 6.2.32.
Note that with Theorem 6.2.28 and Corollary 6.2.31, Problem SR-RC is solved. More precisely, items (1) and (4) of Theorem 6.2.28 solves items (1) and (2) of Problem SR-RC respectively, while item (3) of Problem SR-RC is solved by Corollary 6.2.31. ♠

Example 6.2.33.
Let $\mathbb{K}_6 = \mathbb{Q}((-1)^{\frac{2}{3}})$ and let $(\mathbb{K}_6, \sigma)$ be a rational difference field. Consider the nested products over roots of unity in $\mathbb{U}_6 = \langle (-1)^{\frac{1}{3}} \rangle$:

$$P'(n) = \prod_{i=1}^{n} -1 + \prod_{i=1}^{n} -\left(\prod_{j=1}^{i} (-1)^{\frac{2}{3}} \prod_{k=1}^{j} -1 \right)^5 \in \mathrm{ProdE}_{\mathbb{K}_6}(\mathbb{U}_6). \tag{6.42}$$

We illustrate in detail with the help of the Mathematica package `NestedProducts` the various steps in order to construct an alternative representation Q of the sequence object $P(n)$ in a single R-extension of $(\mathbb{K}_6, \sigma)$ such that for all $n \in \mathbb{N}$, $P(n)$ and $Q(n)$ evaluate to the same sequence. Expanding the second product in (6.42) we have

$$P(n) = \prod_{i=1}^{n} -1 + \prod_{i=1}^{n} -1 \prod_{i=1}^{n} \prod_{j=1}^{i} (-1)^{\frac{4}{3}} \prod_{i=1}^{n} \prod_{j=1}^{i} \prod_{k=1}^{j} -1. \tag{6.43}$$

(1) Now, we construct a simple A-extension of $(\mathbb{K}_6, \sigma)$ that models $P(n)$ precisely. Here we have $(\mathbb{K}_6[\vartheta_{1,1}][\vartheta_{2,1}][\vartheta_{1,2}][\vartheta_{2,2}][\vartheta_{1,3}], \sigma)$ with

$$\frac{\sigma(\vartheta_{1,1})}{\vartheta_{1,1}} = -1, \qquad \frac{\sigma(\vartheta_{1,2})}{\vartheta_{1,2}} = -\vartheta_{1,1}, \qquad \frac{\sigma(\vartheta_{1,3})}{\vartheta_{1,3}} = -\vartheta_{1,1}\,\vartheta_{2,1},$$

$$\frac{\sigma(\vartheta_{2,1})}{\vartheta_{2,1}} = (-1)^{\frac{4}{3}}, \qquad \frac{\sigma(\vartheta_{2,2})}{\vartheta_{2,2}} = (-1)^{\frac{4}{3}}\,\vartheta_{2,1}$$

subject to the relations $\vartheta_{1,1}^2 = 1$, $\vartheta_{2,1}^3 = 1$, $\vartheta_{1,2}^2 = 1$, $\vartheta_{2,2}^3 = 1$ and $\vartheta_{1,3}^2 = 1$ together with the evaluation function

$$\mathrm{ev}(\vartheta_{1,1}, n) = \prod_{i=1}^{n} -1, \qquad \mathrm{ev}(\vartheta_{1,2}, n) = \prod_{i=1}^{n} \prod_{j=1}^{i} -1, \qquad \mathrm{ev}(\vartheta_{1,3}, n) = \prod_{i=1}^{n} \prod_{j=1}^{i} \prod_{k=1}^{j} -1,$$

$$\mathrm{ev}(\vartheta_{2,1}, n) = \prod_{i=1}^{n} (-1)^{\frac{4}{3}}, \qquad \mathrm{ev}(\vartheta_{2,2}, n) = \prod_{i=1}^{n} \prod_{j=1}^{i} (-1)^{\frac{4}{3}}.$$

In this ring, the expression

$$f := \vartheta_{1,1} + \vartheta_{1,1}\,\vartheta_{1,3}\,\vartheta_{2,2} \tag{6.44}$$

models $P(n)$ since $\mathrm{ev}(f, n) = P(n)$ for all $n \geqslant 1$.

(2) Next we use our package to compute the periods of the A-monomials. In particular, we compute:

$$\mathrm{per}(\vartheta_{1,1}) = 2,\ \mathrm{per}(\vartheta_{2,1}) = 3,\ \mathrm{per}(\vartheta_{2,1}) = 4,\ \mathrm{per}(\vartheta_{2,2}) = 3,\ \text{and}\ \mathrm{per}(\vartheta_{1,3}) = 4.$$

Let $\lambda := \mathrm{lcm}(\mathrm{per}(\vartheta_{1,1}), \mathrm{per}(\vartheta_{2,1}), \mathrm{per}(\vartheta_{2,1}), \mathrm{per}(\vartheta_{2,2}), \mathrm{per}(\vartheta_{1,3})) = 12$. Then the minimal single R-extension of $(\mathbb{K}_6(x), \sigma)$ that models (6.44) is $(\mathbb{K}_{12}(x)[\vartheta], \sigma)$ with $\sigma(\vartheta) = (-1)^{\frac{1}{6}}\,\vartheta$ subject to relation $\vartheta^{12} = 1$. Now consider the difference ring homomorphism

$$\varphi : \mathbb{K}_6[\vartheta_{1,1}][\vartheta_{2,1}][\vartheta_{1,2}][\vartheta_{2,2}][\vartheta_{1,3}] \to \mathbb{K}_{12}[\vartheta]$$

as defined in part (1) of Theorem 6.2.28. Then applying φ to (6.44) we get

$$\begin{aligned}
\varphi(\vartheta_{1,1} + \vartheta_{1,1}\,\vartheta_{1,3}\,\vartheta_{2,2}) = \frac{1}{6}\big(&(2 + 2\,\alpha^2 + 2\,\alpha^6 + \alpha^{10}) + (\alpha^3 + 2\,\alpha^7)\,\vartheta + (1 + \alpha^2)\,\vartheta^2 \\
&+ (-\alpha + \alpha^3 + \alpha^7 - 2\,\alpha^9 + \alpha^{11})\,\vartheta^3 + (4 - 2\,\alpha^2 + \alpha^4 + \alpha^6 + \alpha^8)\,\vartheta^4 \\
&+ (\alpha + \alpha^7 + \alpha^9 + \alpha^{11})\,\vartheta^5 + (2 + 6\,\alpha^2 - 4\,\alpha^4 + \alpha^{10})\,\vartheta^6 \\
&+ (2\,\alpha + \alpha^9)\,\vartheta^7 + (1 + \alpha^2)\,\vartheta^8 + (3\,\alpha + \alpha^5 + \alpha^7 + 3\,\alpha^9)\,\vartheta^9 \\
&+ (4 - 6\,\alpha^2 + 5\,\alpha^4 - 2\,\alpha^6 + \alpha^8)\,\vartheta^{10} + (\alpha + \alpha^3 + \alpha^5 + \alpha^7)\,\vartheta^{11}\big)
\end{aligned} \tag{6.46}$$

where $\alpha = e^{\frac{11\pi i}{6}} = (-1)^{\frac{11}{6}}$. Note that (6.46) models $P(n)$ in the single R-extension $(\mathbb{K}_{12}[\vartheta], \sigma)$ of $(\mathbb{K}_6, \sigma)$ with the automorphism $\sigma : \mathbb{K}_{12} \to \mathbb{K}_{12}$ and the evaluation function $\mathrm{ev}_{12} : \mathbb{K}_{12}[\vartheta] \times \mathbb{N} \to \mathbb{K}_{12}$ defined by

$$\sigma(\vartheta) = (-1)^{\frac{1}{6}}\,\vartheta \quad \text{and} \quad \mathrm{ev}_{12}(\vartheta, n) = \big((-1)^{\frac{1}{6}}\big)^n$$

respectively. Since $(\mathbb{K}_{12}[\vartheta],\sigma)$ is an R-extension of $(\mathbb{K}_{12},\sigma)$ it follows by part (2) of Lemma 2.4.3 that

$$\tau_{12}:\mathbb{K}_{12}[\vartheta]\to\mathcal{S}(\mathbb{K}_{12})$$

with

$$\tau(\vartheta)=\langle \mathrm{ev}(\vartheta,n)\rangle_{n\geqslant 0}$$

is a $\mathbb{K}_{12}$-embedding. Let $Q(n)=\tau(\varphi(\vartheta_{1,1}+\vartheta_{1,1}\,\vartheta_{1,3}\,\vartheta_{2,2}))$. Then,

$$\begin{aligned}Q(n)=\frac{1}{6}\Big(&(2+2\,\alpha^2+2\,\alpha^6+\alpha^{10})+(\alpha^3+2\,\alpha^7)\,((-1)^{\frac{1}{6}})^n+(1+\alpha^2)\,(((-1)^{\frac{1}{6}})^n)^2\\
&+(-\alpha+\alpha^3+\alpha^7-2\,\alpha^9+\alpha^{11})\,(((-1)^{\frac{1}{6}})^n)^3+(4-2\,\alpha^2+\alpha^4+\alpha^6+\alpha^8)\,(((-1)^{\frac{1}{6}})^n)^4\\
&+(\alpha+\alpha^7+\alpha^9+\alpha^{11})\,(((-1)^{\frac{1}{6}})^n)^5+(2+6\,\alpha^2-4\,\alpha^4+\alpha^{10})\,(((-1)^{\frac{1}{6}})^n)^6\\
&+(2\,\alpha+\alpha^9)\,(((-1)^{\frac{1}{6}})^n)^7+(1+\alpha^2)\,(((-1)^{\frac{1}{6}})^n)^8+(3\,\alpha+\alpha^5+\alpha^7+3\,\alpha^9)\,(((-1)^{\frac{1}{6}})^n)^9\\
&+(4-6\,\alpha^2+5\,\alpha^4-2\,\alpha^6+\alpha^8)\,(((-1)^{\frac{1}{6}})^n)^{10}+(\alpha+\alpha^3+\alpha^5+\alpha^7)\,(((-1)^{\frac{1}{6}})^n)^{11}\Big)\end{aligned}$$

with $\alpha=e^{\frac{11\pi i}{6}}=(-1)^{\frac{11}{6}}$ for all $n\in\mathbb{N}$. In particular, we have that

$$P(n)=Q(n)$$

holds for all $n\in\mathbb{N}$. ⋆

Remark 6.2.34.
Note that the simple A-extension of $(\mathbb{K}_6,\sigma)$ constructed in item (1) of Example 6.2.33 above to model (6.42) is not unique. In particular, with the simple A-extension $(\mathbb{K}_6[\vartheta_{1,1}][\vartheta_{1,2}][\vartheta_{1,3}],\sigma)$ of $(\mathbb{K}_6,\sigma)$ equipped with the automorphism

$$\sigma(\vartheta_{1,1})=-\vartheta_{1,1},\qquad \sigma(\vartheta_{1,2})=(-1)^{\frac{5}{3}}\,\vartheta_{2,1}\,\vartheta_{1,2},\qquad \sigma(\vartheta_{1,3})=(-1)^{\frac{4}{3}}\,\vartheta_{1,1}^5\,\vartheta_{1,2}^5\,\vartheta_{1,3}$$

and the naturally induced evaluation function $\mathrm{ev}:\mathbb{K}_6[\vartheta_{1,1}][\vartheta_{1,2}][\vartheta_{1,3}]\times\mathbb{N}\to\mathbb{K}_6$ defined by

$$\mathrm{ev}(\vartheta_{1,1},n)=\prod_{k=1}^{n}-1,\quad \mathrm{ev}(\vartheta_{1,2},n)=\prod_{k=1}^{n}-\left(\prod_{j=1}^{k}(-1)^{\frac{2}{3}}\right),\quad \mathrm{ev}(\vartheta_{1,3},n),=\prod_{i=1}^{n}-\left(\prod_{j=1}^{i}(-1)^{\frac{2}{3}}\prod_{k=1}^{j}-1\right)^5$$

and subject to the relations $\vartheta_{1,1}^2=1$, $\vartheta_{1,2}^6=1$ and $\vartheta_{1,3}^6=1$, the expression $\vartheta_{1,1}+\vartheta_{1,3}$ in the ring $\mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}][\vartheta_{1,3}]$ also models (6.42). ♠

Proof (Theorem 6.2.2):
Suppose we are given the geometric products over roots of unity $A_1(n),\dots,A_s(n)\in\mathrm{Prod}(\mathbb{U}_m)$ in n with (6.16). By Lemma 6.1.5 we can rewrite each $A_i(n)$ as

$$A_i(n)=u_i\,\tilde{A}_i(n)\quad\text{ where }\quad \tilde{A}_i(n)=\prod_{k_1=1}^{n}\zeta_{i,1}\prod_{k_2=1}^{k_1}\zeta_{i,2}\cdots\prod_{k_{r_i}=1}^{k_{r_i-1}}\zeta_{i,r_i}\tag{6.47}$$

and $u_i\in\mathbb{U}_m$. Observe that we can construct a simple A-extension $(\mathbb{K}_m[\vartheta_1]\dots[\vartheta_s],\sigma)$ of $(\mathbb{K}_m,\sigma)$ with

$$\frac{\sigma(\vartheta_i)}{\vartheta_i}=\alpha_i=\zeta_m^{u_i}\,\vartheta_1^{\nu_{i,1}}\cdots\vartheta_{i-1}^{\nu_{i,i-1}}\in(\mathbb{U}_m)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\dots[\vartheta_{i-1}]}$$

for $1 \leqslant i \leqslant s$ and the naturally induced evaluation function $\mathrm{ev}_m : \mathbb{K}_m[\vartheta_1]\dots[\vartheta_s] \times \mathbb{N} \to \mathbb{K}_m$ defined by

$$\mathrm{ev}_m(\vartheta_i, n) = \prod_{k=1}^{n} \mathrm{ev}_m(\alpha_i, k-1)$$

with the following property. For all i with $1 \leqslant i \leqslant s$, the geometric product $\tilde{A}_i(n)$ over roots of unity is modelled by some $\gamma_i \in (\mathbb{U}_m)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\dots[\vartheta_{i-1}]}$, i.e.,

$$\mathrm{ev}_m(\gamma_i, n) = \tilde{A}_i(n) \tag{6.48}$$

holds for all $n \geqslant \delta$ where δ is some natural number. By Corollary 6.2.31, we can construct the single R-extension $(\mathbb{K}_\lambda[\vartheta], \sigma)$ of $(\mathbb{K}_\lambda, \sigma)$ with the automorphism $\sigma(\vartheta) = \zeta_\lambda \vartheta$ together with the difference ring homomorphism (6.31) defined by (6.32) and for $1 \leqslant i \leqslant s$ we can define

$$g_i := \sum_{k=0}^{\lambda-1} g_{i,k}\, \vartheta^k \in \mathbb{K}_\lambda[\vartheta]$$

with $g_{i,k} \in \mathbb{K}_\lambda$ such that for all $k, j \in \mathbb{N}$,

$$\mathrm{ev}_m(\sigma^k(\gamma_i), j) = \mathrm{ev}_\lambda(\sigma^k(g_i), j) \tag{6.49}$$

holds. By item (1) of Lemma 2.4.3, we can construct the $\mathbb{K}_\lambda$-homomorphism

$$\tau_\lambda : \mathbb{K}_\lambda[\vartheta] \to \mathcal{S}(\mathbb{K}_\lambda) \quad \text{with} \quad \tau_\lambda(\vartheta) = \left\langle \zeta_\lambda^n \right\rangle_{n \geqslant 0}.$$

Since $(\mathbb{K}_\lambda[\vartheta], \sigma)$ is an R-extension, it follows by item (3) of Theorem 6.2.28 that τ_λ is a $\mathbb{K}_\lambda$-embedding. Finally, we define for $1 \leqslant i \leqslant s$

$$G_i(n) := \sum_{k=0}^{\lambda-1} g_{i,k}\, (\zeta_\lambda^n)^k$$

and get

$$\mathrm{ev}_\lambda(g_i,\, n) = G_i(n) \quad \forall\, n \geqslant \delta.$$

With (6.47), (6.49) and (6.48) we conclude that

$$\begin{aligned} A_i(n) &= u_i \tilde{A}_i(n) \\ &= u_i\, G_i(n) \end{aligned}$$

holds for all $n \geqslant \delta$ and this completes the proof. ∎

Example 6.2.35.
We represent the nesting depth 2 geometric product expression

$$A(n) = \left(\prod_{k=1}^{n} \left(- \prod_{i=1}^{k} (-1) \right)^3 - 1 \right) \left(\left(\prod_{k=1}^{n} -1 \right)^3 \left(\prod_{k=1}^{n} \prod_{i=1}^{k} (-1) \right)^5 + 1 \right) \in \mathrm{ProdE}_{\mathbb{K}_2}(\mathbb{U}_2)$$

in a single R-extension of $(\mathbb{K}_2, \sigma)$ where $\mathbb{K}_2 = \mathbb{Q}$. We will first represent $A(n)$ in a G-simple A-extension of $(\mathbb{K}_2, \sigma)$ where $G = \mathbb{U}_2 = \langle -1 \rangle$. More precisely, we take $\mathbb{A} = \mathbb{K}_2[x][y][z]$ with the automorphism and evaluation function defined by

$$\begin{aligned} &\sigma(x) = -x, && \sigma(y) = x^3\, y, && \sigma(z) = -x\, z, \\ &\mathrm{ev}(x, n) = (-1)^n, && \mathrm{ev}(y, n) = \prod_{k=1}^{n} \left(- \prod_{i=1}^{k} (-1) \right)^3, && \mathrm{ev}(z, n) = \prod_{k=1}^{n} \prod_{i=1}^{k} (-1) \end{aligned} \tag{6.50}$$

respectively. The constructed difference field $(\mathbb{A}, \sigma)$ is subject to the relations $x^2 = 1$, $y^2 = 1$ and $z^2 = 1$. In this ring, $A(n)$ is modelled by the expression $(y-1)(x\,z^5+1) \in \mathbb{A}$ which can be reduced to $A_1 = (y-1)(x\,z+1) \in \mathbb{A}$. In particular,

$$A(n) = \mathsf{ev}(A_1, n) \quad \forall n \geqslant 1$$

holds. Next we compute the periods of the A-monomials x, y and z. Thus we have:

$$\mathrm{per}(c) = 1 \quad \forall c \in \mathbb{K}_2, \qquad \mathrm{per}(x) = 2, \qquad \mathrm{per}(y) = 4, \qquad \mathrm{per}(z) = 4.$$

Let $\lambda := \mathrm{lcm}(\mathrm{per}(x), \mathrm{per}(y), \mathrm{per}(z)) = \mathrm{lcm}(2, 4, 4) = 4$ and let $\zeta := \mathbb{e}^{\frac{\pi \mathbb{i}}{2}} = \mathbb{i}$. Consider the A-extension $(\mathbb{K}_4[\vartheta], \sigma)$ of $(\mathbb{K}_4, \sigma)$ where $\mathbb{K}_4 = \mathbb{Q}(\mathbb{i})$ with the automorphism σ and the naturally induced evaluation function $\mathsf{ev} : \mathbb{K}_4[\vartheta] \times \mathbb{N} \to \mathbb{K}_4$ defined by

$$\sigma(\vartheta) = \mathbb{i}\,\vartheta \qquad \text{and} \qquad \mathsf{ev}(\vartheta, n) = (\mathbb{i})^n \tag{6.51}$$

subject to the relation $\vartheta^4 = 1$. By Proposition 2.3.37, $(\mathbb{K}_4[\vartheta], \sigma)$ is an R-extension of $(\mathbb{K}_4, \sigma)$ and by Theorem 6.2.25 the ring $\mathbb{K}_4[\vartheta]$ can be written as the direct sum $\boldsymbol{e}_0\mathbb{K}_4 \oplus \boldsymbol{e}_1\mathbb{K}_4 \oplus \boldsymbol{e}_2\mathbb{K}_4 \oplus \boldsymbol{e}_3\mathbb{K}_4$ where

$$\begin{aligned} \boldsymbol{e}_0 &= \frac{\mathbb{i}}{4}\left(\vartheta^3 + \mathbb{i}\,\vartheta^2 - \vartheta - \mathbb{i}\right); & \boldsymbol{e}_1 &= \frac{1}{4}\left(1 - \vartheta + \vartheta^2 - \vartheta^3\right); \\ \boldsymbol{e}_2 &= \frac{\mathbb{i}}{4}\left(-\vartheta^3 + \mathbb{i}\,\vartheta^2 + \vartheta - \mathbb{i}\right); & \boldsymbol{e}_3 &= \frac{1}{4}\left(1 + \vartheta + \vartheta^2 + \vartheta^3\right) \end{aligned} \tag{6.52}$$

with $\boldsymbol{e}_0 + \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 = 1$. Now consider the difference ring homomorphism

$$\varphi : \mathbb{K}_2[x][y][z] \to \mathbb{K}_4[\vartheta] \simeq \boldsymbol{e}_0\mathbb{K}_4 \oplus \boldsymbol{e}_1\mathbb{K}_4 \oplus \boldsymbol{e}_2\mathbb{K}_4 \oplus \boldsymbol{e}_3\mathbb{K}_4$$

defined by

$$\begin{aligned} \varphi(x) &= -\boldsymbol{e}_0 + \boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3 = \vartheta^2; \\ \varphi(y) &= \boldsymbol{e}_0 - \boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3 \;\; = \frac{1+\mathbb{i}}{2}\,\vartheta\,(\vartheta^2 - \mathbb{i}); \\ \varphi(z) &= -\boldsymbol{e}_0 - \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 = \frac{1-\mathbb{i}}{2}\,\vartheta\,(\vartheta^2 + \mathbb{i}). \end{aligned}$$

Then

$$\varphi(A_1) = \varphi\big((y-1)(x\,z+1)\big) = \left(\frac{(1-\mathbb{i})}{2}\,\vartheta^3\,(\vartheta^2+\mathbb{i}) - 1\right)\left(\frac{(1-\mathbb{i})}{2}\,\vartheta^3\,(\vartheta^2+\mathbb{i}) + 1\right) = 0.$$

Thus $A(n) = \mathsf{ev}(\varphi(A_1), n) = 0$ for all $n \in \mathbb{N}$, i.e., it evaluates to the zero sequence. ⋆

Example 6.2.36.
Using my Mathematica package `NestedProducts` the nested products

$$P_1(n) = \prod_{k=1}^{n}\prod_{j=1}^{k}\prod_{i=1}^{j} -1 \in \mathrm{Prod}(\mathbb{Q}) \quad \text{and} \quad P_2(n) = \prod_{k=1}^{n} \frac{-(\mathbb{i})^k}{k+1} \in \mathrm{Prod}(\mathbb{Q}(\mathbb{i}))$$

which appeared in Schneider (2016, Equations 4 and 7) respectively can be reduced. In Mathematica Session 4 below we reduce the nesting depth 3 geometric product

$$P_1(n) = \prod_{k=1}^{n}\prod_{j=1}^{k}\prod_{i=1}^{j} -1 \in \mathrm{Prod}(\mathbb{Q})$$

to a geometric product expression of nesting depth 1.

Mathematica Session 4

In[8]:= $P_1 = \text{FProduct}[\text{FProduct}[\text{FProduct}[-1, \{k, 1, j\}], \{j, 1, i\}], \{i, 1, n\}];$

In[9]:= $Q_1 = \text{ProductReduce}[P_1]$

Out[9]= $\frac{1}{2} + \frac{1}{2}\,\mathbb{i}\,(\mathbb{i})^n + \frac{1}{2}\left((\mathbb{i})^n\right)^2 - \frac{1}{2}\,\mathbb{i}\left((\mathbb{i})^n\right)^3$

Observe that while $P(n) \in \text{Prod}(\mathbb{Q})$, its reduced expression Q_1 which is given by Out[9] in the Mathematica Session 4 is an element of $\text{Prod}(\mathbb{Q}(\mathbb{i}))$.

In Mathematica Session 5 below, we reduce the nesting depth 2 hypergeometric product

$$P_2(n) = \prod_{k=1}^{n} \frac{-(\mathbb{i})^k}{k+1} \in \text{Prod}(\mathbb{Q}(\mathbb{i}))$$

to a nesting depth 1 hypergeometric product expression.

Mathematica Session 5

In[10]:= $P_2 = \text{FProduct}\left[-\frac{\text{FProduct}[\mathbb{i}, \{i, 1, k\}]}{k+1}, \{k, 1, n\}\right];$

In[11]:= $Q_2 = \text{ProductReduce}[P_2]$

Out[11]= $\frac{-(-1)^{\frac{1}{4}}\left((-1)^{\frac{1}{4}}\right)^n + \left(\left((-1)^{\frac{1}{4}}\right)^n\right)^3 + (-1)^{\frac{1}{4}}\left(\left((-1)^{\frac{1}{4}}\right)^n\right)^5 + \left(\left((-1)^{\frac{1}{4}}\right)^n\right)^7}{2\,(n+1)\,n!}$

Here, the 8-th root of unity, $(-1)^{\frac{1}{4}}$, is the complex number $\mathbb{e}^{\frac{\pi \mathbb{i}}{4}}$. While the simplification obtained for $P_2(n)$ in Schneider (2016, Equation 7), i.e.,

$$\left(-\frac{\mathbb{i}}{2} - \frac{1}{2}\right) \frac{-(-1)^n + \mathbb{i}}{n\,(n+1)} \left(\prod_{k=1}^{n-1} \frac{(\mathbb{i})^k}{k}\right)$$

is as the input expression an element of $\text{ProdE}(\mathbb{Q}(\mathbb{i}))$, the simplification Q_2 given by Out[11] in the Mathematica Session 5 is an element of $\text{ProdE}(\mathbb{Q}(\mathbb{e}^{\frac{\pi \mathbb{i}}{4}}))$. Furthermore, note that whiles the output in Schneider (2016, Equation 7) is a nesting depth 2 hypergeometric product expression, Q_2 given by Out[11] in the Mathematica Session 5 is a nesting depth 1 hypergeometric product expression. ⋆

6.3 Construction of multiple chain Π-extensions for higher nesting depth expressions in $\text{ProdE}(\mathbb{K})$

In this section we will discuss how one can model geometric products of higher nesting depth over a constant field $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)$ which is a rational function field over a σ-strongly computable field K.

Definition 6.3.1.
Let $(\mathbb{A},\sigma)$ be a difference ring with constant field $\mathbb{K}=\text{const}(\mathbb{A},\sigma)$ and set $\mathfrak{d}(f)=0$ for all $f\in\mathbb{A}$. Let $\mathbb{G}$ be a subgroup of $\mathbb{A}^*$. A P-extension $(\mathbb{E},\sigma)$ of $(\mathbb{A},\sigma)$ with $\mathbb{E}=\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle$ is called a $\mathbb{G}$-*simple* P-*extension* of depth-$(\nu_1,\dots,\nu_e)$, if for all $1\leqslant i\leqslant e$,

- $\sigma(t_i)=\alpha_i\,t_i$ where $\alpha_i\in\mathbb{G}_{\mathbb{A}}^{\mathbb{A}\langle t_1\rangle\dots\langle t_{i-1}\rangle}$ is the product group over $\mathbb{G}$ respect to P-monomials;
- $\mathfrak{d}(t_i)=\nu_i$.

If $\nu_1\leqslant\nu_2\leqslant\cdots\leqslant\nu_e$, then $(\mathbb{E},\sigma)$ is called an *ordered* $\mathbb{G}$-*simple* P-*extension* of $(\mathbb{A},\sigma)$. Similarly, we call $(\mathbb{E},\sigma)$ a $\mathbb{G}$-*simple* Π-*extension* of $(\mathbb{A},\sigma)$ of depth-$(\nu_1,\dots,\nu_e)$ if it is a $\mathbb{G}$-simple P-extension of $(\mathbb{A},\sigma)$ of depth-$(\nu_1,\dots,\nu_e)$ and $\text{const}(\mathbb{E},\sigma)=\text{const}(\mathbb{A},\sigma)$. Furthermore, $(\mathbb{E},\sigma)$ is an *ordered* $\mathbb{G}$-*simple* Π-*extension* of $(\mathbb{A},\sigma)$ if it is an ordered $\mathbb{G}$-simple P-extension of $(\mathbb{A},\sigma)$ and $\text{const}(\mathbb{E},\sigma)=\text{const}(\mathbb{A},\sigma)$. Finally, we call $(\mathbb{E},\sigma)$ a *simple* P-*extension* (resp. *simple* Π-*extension*) of $(\mathbb{A},\sigma)$, if it is an $\mathbb{A}^*$-simple P-extension (resp. $\mathbb{A}^*$-simple Π-extension) of $(\mathbb{A},\sigma)$. $*$

Example 6.3.2.
Consider the following geometric products

$$P_1(n)=\prod_{i=1}^{n}2,\quad P_2(n)=\prod_{i=1}^{n}5,\quad P_3(n)=\prod_{i=1}^{n}3\left(\prod_{j=1}^{i}7\right)^2,\quad P_4(n)=\prod_{i=1}^{n}\prod_{j=1}^{i}2\left(\prod_{k=1}^{j}3\right)^3 \tag{6.53}$$

in $\text{Prod}(\mathbb{Q})$. We construct a simple P-extension that models these products. Let $(\mathbb{Q}\langle y_1\rangle\dots\langle y_7\rangle,\sigma)$ be a simple P-extension of $(\mathbb{Q},\sigma)$ with the automorphism

$$\begin{array}{llll}\sigma(c)=c,\,\forall\,c\in\mathbb{Q}, & \sigma(y_1)=2\,y_1, & \sigma(y_2)=5\,y_2, & \sigma(y_3)=7\,y_3,\\ \sigma(y_4)=147\,y_3^2\,y_4, & \sigma(y_5)=3\,y_5, & \sigma(y_6)=54\,y_5^3\,y_6, & \sigma(y_7)=54\,y_5^3\,y_6\,y_7\end{array} \tag{6.54}$$

and the evaluation function $\text{ev}:\mathbb{Q}\langle y_1\rangle\dots\langle y_7\rangle\times\mathbb{N}\to\mathbb{Q}$ defined as follows: $\text{ev}(c,n)=c,\,\forall\,c\in\mathbb{Q}$,

$$\begin{array}{llll}\text{ev}(y_1,n)=\prod_{i=1}^{n}2, & \text{ev}(y_3,n)=\prod_{i=1}^{n}7, & \text{ev}(y_4,n)=\prod_{i=1}^{n}3\left(\prod_{j=1}^{i}7\right)^2, & \text{ev}(y_6,n)=\prod_{i=1}^{n}2\left(\prod_{j=1}^{i}3\right)^3,\\ \text{ev}(y_2,n)=\prod_{i=1}^{n}5, & \text{ev}(y_5,n)=\prod_{i=1}^{n}3, & \text{ev}(y_7,n)=\prod_{i=1}^{n}\prod_{j=1}^{i}2\left(\prod_{k=1}^{j}3\right)^3. & \end{array} \tag{6.55}$$

Then (6.53) can be modelled in the constructed simple P-extension $(\mathbb{Q}\langle y_1\rangle\dots\langle y_7\rangle,\sigma)$ of $(\mathbb{Q},\sigma)$. More precisely, the monomials y_1, y_2, y_4 and y_7 model $P_1(n)$, $P_2(n)$, $P_3(n)$ and $P_4(n)$ respectively. $\star$

We will now consider a special class of simple P-/Π-extensions called single chain and (ordered) multiple chain P-/Π-extensions that are closely related to the product factored form representation of nested products. Based on that we will demonstrate how any simple P-/Π-extension with an evaluation function can be transformed into a (ordered) multiple chain P-/Π-extension.

Definition 6.3.3.
Let $(\mathbb{E}, \sigma)$ be a difference ring extension of $(\mathbb{A}, \sigma)$ with $\mathfrak{d}(f) = 0$ for all $f \in \mathbb{E}$. We call $(\mathbb{E}\langle t_1 \rangle \ldots \langle t_e \rangle, \sigma)$ a *single chain* P-/Π-*extension* of $(\mathbb{E}, \sigma)$ over $\mathbb{A}$ if and only if for all $1 \leqslant k \leqslant e$,

$$\sigma(t_k) = c_k\, t_1 \cdots t_{k-1}\, t_k, \quad \text{with } c_k \in \mathbb{A}^*.$$

Note that $\mathfrak{d}(t_k) = k$. We call c_1 the *base* of the single chain P-/Π-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$. Further, we call $(\mathbb{E}, \sigma)$ a *multiple chain* P-/Π-*extension* of $(\mathbb{A}, \sigma)$ if it is a tower of single chain P-/Π-extensions over $\mathbb{A}$. $\ast$

Example 6.3.4.
We construct a multiple chain extension that models the geometric products in (6.53). First we write each geometric product in a product factored form; see Remark 2.1.11. Note that $P_1(n)$ and $P_2(n)$ are already in a product factored form since they are of nesting depth 1. For $P_3(n)$ and $P_4(n)$ we have

$$P_3'(n) = \left(\prod_{i=1}^{n} 3\right)\left(\prod_{i=1}^{n}\prod_{j=1}^{i} 7\right)^2 \qquad \text{and} \qquad P_4'(n) = \left(\prod_{i=1}^{n}\prod_{j=1}^{i} 2\right)\left(\prod_{i=1}^{n}\prod_{j=1}^{i}\prod_{k=1}^{j} 3\right)^3 \tag{6.56}$$

as their respective product factored form. Note that for all $n \in \mathbb{N}$

$$P_3(n) = P_3'(n) \qquad \text{and} \qquad P_4(n) = P_4'(n)$$

holds. Now consider the following single chain P-extensions of $(\mathbb{Q}, \sigma)$ over $\mathbb{Q}$.

(1) Define the single chain P-extension $(\mathbb{Q}\langle \varkappa_{1,1} \rangle, \sigma)$ of $(\mathbb{Q}, \sigma)$ over $\mathbb{Q}$ with 5 as its base and with the automorphism $\sigma : \mathbb{Q}\langle \varkappa_{1,1} \rangle \to \mathbb{Q}\langle \varkappa_{1,1} \rangle$ and the evaluation function $\tilde{\mathrm{ev}} : \mathbb{Q}\langle \varkappa_1 \rangle \times \mathbb{N} \to \mathbb{Q}$ defined as

$$\sigma(\varkappa_{1,1}) = 5\, \varkappa_{1,1} \qquad \text{and} \qquad \tilde{\mathrm{ev}}(c, n) = c,\ \forall\, c \in \mathbb{Q}, \qquad \tilde{\mathrm{ev}}(\varkappa_{1,1}, n) = \prod_{i=1}^{n} 5. \tag{6.57}$$

(2) Define the single chain P-extension $(\mathbb{Q}\langle \varkappa_{2,1} \rangle\langle \varkappa_{2,2} \rangle, \sigma)$ of $(\mathbb{Q}, \sigma)$ over $\mathbb{Q}$ with base 2 together with the automorphism σ and the evaluation function $\tilde{\mathrm{ev}} : \mathbb{Q}\langle \varkappa_{2,1} \rangle\langle \varkappa_{2,2} \rangle \times \mathbb{N} \to \mathbb{Q}$ defined as

$$\begin{aligned} &\sigma(\varkappa_{2,1}) = 2\, \varkappa_{2,1}, \qquad \sigma(\varkappa_{2,2}) = 2\, \varkappa_{2,1}\, \varkappa_{2,2} \qquad \text{and} \\ &\tilde{\mathrm{ev}}(c, n) = c,\ \forall\, c \in \mathbb{Q}, \quad \tilde{\mathrm{ev}}(\varkappa_{2,1}, n) = \prod_{i=1}^{n} 2, \qquad \tilde{\mathrm{ev}}(\varkappa_{2,2}, n) = \prod_{i=1}^{n}\prod_{j=1}^{i} 2. \end{aligned} \tag{6.58}$$

(3) Define the single chain P-extension $(\mathbb{Q}\langle \varkappa_{3,1} \rangle\langle \varkappa_{3,2} \rangle, \sigma)$ of $(\mathbb{Q}, \sigma)$ over $\mathbb{Q}$ with 7 as its base with the automorphism σ and the evaluation function $\tilde{\mathrm{ev}} : \mathbb{Q}\langle \varkappa_{3,1} \rangle\langle \varkappa_{3,2} \rangle \times \mathbb{N} \to \mathbb{Q}$ defined as

$$\begin{aligned} &\sigma(\varkappa_{3,1}) = 7\, \varkappa_{3,1}, \qquad \sigma(\varkappa_{3,2}) = 7\, \varkappa_{3,1}\, \varkappa_{3,2} \qquad \text{and} \\ &\tilde{\mathrm{ev}}(c, n) = c,\ \forall\, c \in \mathbb{Q}, \quad \tilde{\mathrm{ev}}(\varkappa_{3,1}, n) = \prod_{i=1}^{n} 7, \qquad \tilde{\mathrm{ev}}(\varkappa_{3,2}, n) = \prod_{i=1}^{n}\prod_{j=1}^{i} 7. \end{aligned} \tag{6.59}$$

(4) Define the single chain P-extension $(\mathbb{Q}\langle \varkappa_{4,1} \rangle\langle \varkappa_{4,2} \rangle\langle \varkappa_{4,3} \rangle, \sigma)$ of $(\mathbb{Q}, \sigma)$ over $\mathbb{Q}$ with base 3 together with the automorphism σ and the evaluation function $\tilde{\mathrm{ev}} : \mathbb{Q}\langle \varkappa_{4,1} \rangle\langle \varkappa_{4,2} \rangle\langle \varkappa_{4,3} \rangle \times \mathbb{N} \to \mathbb{Q}$ defined as

$$\begin{aligned} &\sigma(\varkappa_{4,1}) = 3\, \varkappa_{4,1}, \qquad \sigma(\varkappa_{4,2}) = 3\, \varkappa_{4,1}\, \varkappa_{4,2},\ \sigma(\varkappa_{4,3}) = 3\, \varkappa_{4,1}\, \varkappa_{4,2}\, \varkappa_{4,3}, \\ &\tilde{\mathrm{ev}}(c, n) = c,\ \forall\, c \in \mathbb{Q},\ \tilde{\mathrm{ev}}(\varkappa_{4,1}, n) = \prod_{i=1}^{n} 3,\ \ \tilde{\mathrm{ev}}(\varkappa_{4,2}, n) = \prod_{i=1}^{n}\prod_{j=1}^{i} 3,\ \tilde{\mathrm{ev}}(\varkappa_{4,3}, n) = \prod_{i=1}^{n}\prod_{j=1}^{i}\prod_{k=1}^{j} 3. \end{aligned} \tag{6.60}$$

Putting everything together, we have constructed $(\mathbb{Q}\langle\varkappa_{1,1}\rangle\langle\varkappa_{2,1}\rangle\langle\varkappa_{2,2}\rangle\langle\varkappa_{3,1}\rangle\langle\varkappa_{3,2}\rangle\langle\varkappa_{4,1}\rangle\langle\varkappa_{4,2}\rangle\langle\varkappa_{4,3}\rangle, \sigma)$ which is the multiple chain P-extension of $(\mathbb{Q}, \sigma)$ based at $5, 2, 7, 3$. In this ring together with a proper extension of the evaluation function, the sequences generated by $P_1(n)$, $P_2(n)$, $P_3(n)$ and $P_4(n)$ are modelled by $\varkappa_{2,1}$, $\varkappa_{1,1}$, $\varkappa_{4,1}\,\varkappa_{3,2}^2$ and $\varkappa_{2,2}\,\varkappa_{4,3}^3$ respectively. ⋆

Remark 6.3.5.
Given any simple P-extension $(\mathbb{E}, \sigma)$ of a difference ring $(\mathbb{A}, \sigma)$ with constant field $\mathbb{K} = \text{const}(\mathbb{A}, \sigma)$ and an evaluation function $\text{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$, one can always construct a multiple chain P-extension $(\tilde{\mathbb{E}}, \sigma)$ of $(\mathbb{A}, \sigma)$ together with a proper evaluation function by following the procedure outlined in Example 6.3.4. More precisely the following steps suffice.

(1) Write a product factored form of each product expression that is modelled by the simple P-monomials.

(2) Among the product factored forms with the same multiplicand, take one, say $P(n)$, with the highest nesting depth. Construct a single chain P-extension of $(\mathbb{A}, \sigma)$ over $\mathbb{A}$ and an evaluation function such that the outermost P-monomials models $P(n)$.

(3) Repeat step (2) for the remaining product factored forms.

(4) Combine the constructed single chain P-extensions of $(\mathbb{A}, \sigma)$ over $\mathbb{A}$ to obtain a multiple chain P-extension of $(\mathbb{A}, \sigma)$. In addition, compose the evaluation function accordingly.

As a consequence, we state the following Lemma without proof. ♠

Lemma 6.3.6.
Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\text{const}(\mathbb{A}, \sigma) = \mathbb{K}$ and let $(\mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle, \sigma)$ be a simple P-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t_i) = \alpha_i\, t_i$ for $1 \leqslant i \leqslant e$ where $\alpha_i = \gamma_i\, t_1^{\nu_{i,1}} \cdots t_{i-1}^{\nu_{i,i-1}} \in G_i = (\mathbb{A}^*)_{\mathbb{A}}^{\mathbb{A}\langle t_1\rangle\dots\langle t_{i-1}\rangle}$, $\nu_{i,j} \in \mathbb{Z}$ for $1 \leqslant j \leqslant i-1$. Let $\text{ev} : \mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle \times \mathbb{N} \to \mathbb{K}$ be an evaluation function defined as

$$\text{ev}(t_i, n) = \prod_{k=\delta}^{n} \text{ev}(\alpha_i, k-1).$$

In particular the P-monomials t_i model product expressions in $\text{ProdE}(\mathbb{K}(n))$ whose lower bounds are all synchronised to some $\delta \in \mathbb{N}$. Then one can construct

(i) a multiple chain P-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{A}, \sigma)$ where

$$\mathbb{E} = \mathbb{A}\langle z_{1,1}\rangle \dots \langle z_{1,s_1}\rangle\langle z_{2,1}\rangle \dots \langle z_{2,s_2}\rangle \dots \langle z_{m,1}\rangle \dots \langle z_{m,s_m}\rangle$$

with

$$\sigma(z_{\ell,k}) = \tilde{\alpha}_{\ell,k}\, z_{\ell,k} \quad \textit{where} \quad \tilde{\alpha}_{\ell,k} = \tilde{\gamma}_\ell\, z_{\ell,1} \cdots z_{\ell,k-1} \in (\mathbb{A}^*)_{\mathbb{A}}^{\mathbb{A}\langle z_{\ell,1}\rangle\dots\langle z_{\ell,k-1}\rangle}$$

for $1 \leqslant \ell \leqslant m$ and for $1 \leqslant k \leqslant s_\ell$;

(ii) and an evaluation function $\tilde{\text{ev}} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ defined as

$$\tilde{\text{ev}}(z_{\ell,k}, n) = \prod_{j=\delta}^{n} \tilde{\text{ev}}(\tilde{\alpha}_{\ell,k}, j-1)$$

for $1 \leqslant \ell \leqslant m$ and for $1 \leqslant k \leqslant s_\ell$;

such that the following holds. There are $\nu_{i,j,k} \in \mathbb{Z}$ *for* $1 \leqslant i \leqslant e$, $1 \leqslant j \leqslant m$ *and* $1 \leqslant k \leqslant s_j$ *such that*

(1) the map

$$\rho : \mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle \to \mathbb{A}\langle z_{1,1}\rangle \dots \langle z_{1,s_1}\rangle\langle z_{2,1}\rangle \dots \langle z_{2,s_2}\rangle \dots \langle z_{m,1}\rangle \dots \langle z_{m,s_m}\rangle$$

with

$$\rho(t_i) = z_{1,1}^{\nu_{i,1,1}} \cdots z_{1,s_1}^{\nu_{i,1,s_1}} z_{2,1}^{\nu_{i,2,1}} \cdots z_{2,s_2}^{\nu_{i,2,s_2}} \cdots z_{m,1}^{\nu_{i,m,1}} \cdots z_{m,s_m}^{\nu_{i,m,s_m}}$$

is a difference ring homomorphism,

(2) the diagram below commutes

$$\begin{tikzcd}
\mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle \arrow[r, "\psi"] \arrow[d, "\rho"'] & \mathcal{S}(\mathbb{K}) \arrow[d, "\mathrm{id}"] \\
\mathbb{A}\langle z_{1,1}\rangle \dots \langle z_{1,s_1}\rangle\langle z_{2,1}\rangle \dots \langle z_{2,s_2}\rangle \dots \langle z_{m,1}\rangle \dots \langle z_{m,s_m}\rangle \arrow[r, "\tilde{\psi}"] & \mathcal{S}(\mathbb{K})
\end{tikzcd}$$

where $\psi(f) = \langle \mathrm{ev}(f,n)\rangle_{n\geqslant 0}$ *and* $\tilde{\psi}(f) = \langle \tilde{\mathrm{ev}}(f,n)\rangle_{n\geqslant 0}$ *are both* $\mathbb{K}$*-homomorphism.*

Consequently, for all $g \in \mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle$,

$$\mathrm{ev}(\sigma^k(g), n) = \tilde{\mathrm{ev}}(\sigma^k(\rho(g)), n)$$

holds for all k, $n \in \mathbb{N}$. ♦

Example 6.3.7 (Cont. Examples 6.3.2 and 6.3.4).
Consider the simple P-extension $(\mathbb{Q}\langle y_1\rangle\langle y_2\rangle\langle y_3\rangle\langle y_4\rangle\langle y_5\rangle\langle y_6\rangle\langle y_7\rangle, \sigma)$ of $(\mathbb{Q}, \sigma)$ constructed in Example 6.3.2 with the automorphism and the evaluation function given by (6.54) and (6.55) respectively. Consider also the multiple chain P-extension $(\mathbb{Q}\langle x_{1,1}\rangle\langle x_{2,1}\rangle\langle x_{2,2}\rangle\langle x_{3,1}\rangle\langle x_{3,2}\rangle\langle x_{4,1}\rangle\langle x_{4,2}\rangle\langle x_{4,3}\rangle, \sigma)$ of $(\mathbb{Q}, \sigma)$ with the automorphism and evaluation function (6.57), (6.58), (6.59) and (6.60) constructed in Example 6.3.4. Then we can construct the difference ring homomorphism

$$\rho : \mathbb{Q}\langle y_1\rangle\langle y_2\rangle\langle y_3\rangle\langle y_4\rangle\langle y_5\rangle\langle y_6\rangle\langle y_7\rangle \to \mathbb{Q}\langle x_{1,1}\rangle\langle x_{2,1}\rangle\langle x_{2,2}\rangle\langle x_{3,1}\rangle\langle x_{3,2}\rangle\langle x_{4,1}\rangle\langle x_{4,2}\rangle\langle x_{4,3}\rangle$$

defined as follows:

$$\begin{aligned}
&\rho(y_1) = x_{2,1}, \quad \rho(y_2) = x_{1,1}, \qquad \rho(y_3) = x_{3,1}, \qquad \rho(y_4) = x_{4,1}\, x_{3,2}^2, \\
&\rho(y_5) = x_{4,1}, \quad \rho(y_6) = x_{2,1}\, x_{4,2}^3, \quad \rho(y_7) = x_{2,2}\, x_{4,3}^3.
\end{aligned}$$

In particular, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathrm{ev}(y_1, n) = \tilde{\mathrm{ev}}(x_{2,1}, n), \qquad \mathrm{ev}(y_2, n) = \tilde{\mathrm{ev}}(x_{1,1}, n), \quad \mathrm{ev}(y_3, n) = \tilde{\mathrm{ev}}(x_{3,1}, n), \\
&\mathrm{ev}(y_4, n) = \tilde{\mathrm{ev}}(x_{4,1}\, x_{3,2}^2, n), \quad \mathrm{ev}(y_5, n) = \tilde{\mathrm{ev}}(x_{4,1}, n), \quad \mathrm{ev}(y_6, n) = \tilde{\mathrm{ev}}(x_{2,1}\, x_{4,2}^3, n), \\
&\mathrm{ev}(y_7, n) = \tilde{\mathrm{ev}}(x_{2,2}\, x_{4,3}^3, n)
\end{aligned}$$

holds. ⋆

Proposition 6.3.8.
Let $(\mathbb{A}, \sigma)$ *be a difference field with* $\operatorname{const}(\mathbb{A}, \sigma) = \mathbb{K}$ *and let* $(\mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle, \sigma)$ *be a single chain* P*-extension of* $(\mathbb{A}, \sigma)$ *with* $\sigma(t_k) = c\, t_1 \cdots t_{k-1}\, t_k$ *for all* $1 \leqslant k \leqslant e$ *where* $c \in \mathbb{A}^*$ *and* c *is not a root of unity. Then* $(\mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle, \sigma)$ *is a* Π*-extension.* □

Proof:
Suppose that $(\mathbb{A}\langle t_1\rangle \dots \langle t_k\rangle, \sigma)$ is not a Π-extension of $(\mathbb{A}\langle t_1\rangle \dots \langle t_{k-1}\rangle, \sigma)$ for some k with $1 \leqslant k \leqslant e$. Then by Theorem 2.3.40 there is a $g \in \mathbb{A}\langle t_1\rangle \dots \langle t_{k-1}\rangle \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$ such that

$$\sigma(g) = (c\, t_1 \cdots t_{k-1})^m\, g.$$

Write

$$g = \sum_{i=\ell}^{r} g_i\, t_{k-1}^i$$

with $\deg(g) = r$ and $\operatorname{ldeg}(g) = \ell$. If $m > 0$, then $\deg(\sigma(g)) = r$ and $\deg\big((c\, t_1 \cdots t_{k-1})^m\, g\big) = m + r$. Thus $m = 0$, a contradiction. Conversely if $m < 0$, then $\operatorname{ldeg}(\sigma(g)) = \ell$ and $\operatorname{ldeg}\big((c\, t_1 \cdots t_{k-1})^m\, g\big) = m + \ell$. Again $m = 0$ which is a contradiction. Consequently, $(\mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle, \sigma)$ is a Π-extension of $(\mathbb{A}, \sigma)$. In particular, it is a single chain Π-extension of $(\mathbb{A}, \sigma)$ over $\mathbb{A}$. ■

Example 6.3.9.
Each of the single chain P-extension of $(\mathbb{Q}, \sigma)$ over $\mathbb{Q}$ constructed in items (1), (2), (3) and (4) of Example 6.3.4 is a single chain Π-extension of $(\mathbb{Q}, \sigma)$ over $\mathbb{Q}$. ⋆

Remark 6.3.10.
It is obvious that the building blocks of a multiple chain P-extension are "stand-alone" single chain Π-extensions. However, this does not mean that a multiple chain P-extension whose building blocks are "stand-alone" single chain Π-extensions will always be a multiple chain Π-extension. The criterion for a multiple chain P-extension to be a Π-extension is given in Theorem 6.4.14 below. ♠

In the following let $m \in \mathbb{N} \setminus \{0\}$, and for $1 \leqslant \ell \leqslant m$ let $(\mathbb{K}_\ell, \sigma)$ with $\mathbb{K}_\ell = \mathbb{K}\langle \boldsymbol{y}_\ell\rangle = \mathbb{K}\langle y_{\ell,1}\rangle \dots \langle y_{\ell,s_\ell}\rangle$ be a single chain Π-extension of $(\mathbb{K}, \sigma)$ with base $h_\ell \in \mathbb{K}^*$ where

$$\sigma(y_{\ell,k}) = \alpha_{\ell,k}\, y_{\ell,k} \quad \text{where} \quad \alpha_{\ell,k} = h_\ell\, y_{\ell,1} \cdots y_{\ell,k-1} \in (\mathbb{K}^*)_{\mathbb{K}}^{\mathbb{K}\langle y_{\ell,1}\rangle \dots \langle y_{\ell,k-1}\rangle} \tag{6.61}$$

and $\partial(y_{\ell,k}) = k$ for $1 \leqslant k \leqslant s_\ell$. Let $\operatorname{ev} : \mathbb{K}_\ell \times \mathbb{N} \to \mathbb{K}$ be an evaluation function defined as

$$\operatorname{ev}(y_{\ell,k}, n) = \prod_{j=1}^{n} \operatorname{ev}(\alpha_{\ell,k}, j-1) = \prod_{j=1}^{n} \alpha_{\ell,k}. \tag{6.62}$$

In particular, for all $c \in \mathbb{K}$, $\operatorname{ev}(c, n) = c$ for all $n \geqslant 0$. Let $(\mathbb{A}, \sigma)$ be the multiple chain P-extension of $(\mathbb{K}, \sigma)$ built by the single chain Π-extensions $(\mathbb{K}_\ell, \sigma)$ of $(\mathbb{K}, \sigma)$ over $\mathbb{K}$. That is,

$$\mathbb{A} = \mathbb{K}\langle \boldsymbol{y}_1\rangle\langle \boldsymbol{y}_2\rangle \dots \langle \boldsymbol{y}_m\rangle = \mathbb{K}\langle y_{1,1}\rangle \dots \langle y_{1,s_1}\rangle\langle y_{2,1}\rangle \dots \langle y_{2,s_2}\rangle \dots \langle y_{m,1}\rangle \dots \langle y_{m,s_m}\rangle.$$

Depending on the context, $\boldsymbol{y}_\ell$ denotes $(y_{\ell,1},\dots,y_{\ell,s_\ell})$ or $y_{\ell,1},\dots,y_{\ell,s_\ell}$ or $y_{\ell,1}\cdots y_{\ell,s_\ell}$. Note that the P-monomials $y_{\ell,k}$ can be ordered in increasing order of their depths. Let $d=\max(s_1,s_2,\dots,s_m)$ and $\mathbb{A}_0=\mathbb{K}$. Consider the tower of difference ring extensions $(\mathbb{A}_i,\sigma)$ of $(\mathbb{A}_{i-1},\sigma)$ with the ring

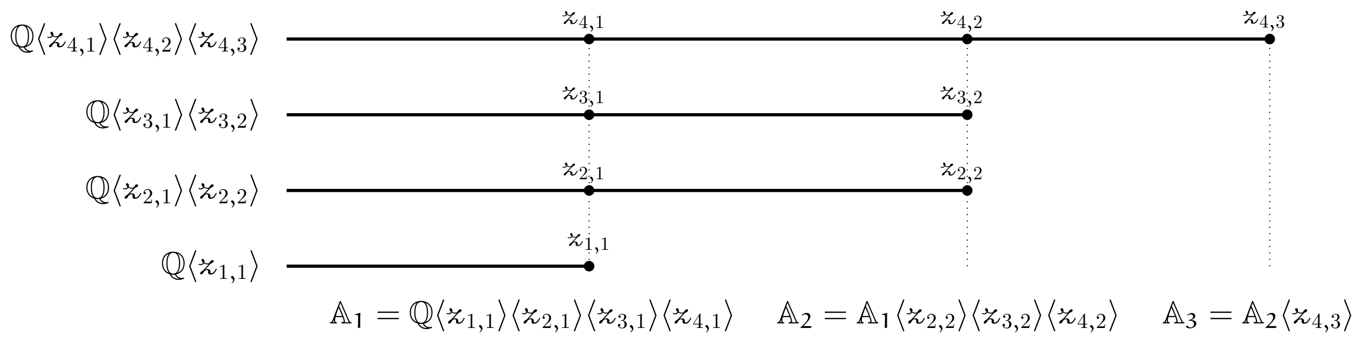
$$\mathbb{A}_i=\mathbb{A}_{i-1}\langle\boldsymbol{y}_i\rangle=\mathbb{A}_{i-1}\langle y_{1,i}\rangle\langle y_{2,i}\rangle\dots\langle y_{w_i,i}\rangle$$

for $1\leqslant i\leqslant d$ where $m=w_1\geqslant w_2\geqslant\cdots\geqslant w_d$ and with the automorphism

$$\sigma(y_{\ell,i})=\alpha_{\ell,i}\,y_{\ell,i}\quad\text{where}\quad\alpha_{\ell,i}=h_\ell\,y_{\ell,1}\cdots y_{\ell,i-1}\in(\mathbb{K}^*)_{\mathbb{K}}^{\mathbb{K}\langle y_{\ell,1}\rangle\dots\langle y_{\ell,i-1}\rangle}\tag{6.63}$$

for $1\leqslant\ell\leqslant w_i$. Note that the $\mathfrak{d}(y_{\ell,i})=i$. Further, the ring $\mathbb{A}_d$ is isomorphic to $\mathbb{A}$ up to reordering of the P-monomials. We call $(\mathbb{A}_d,\sigma)$ an *ordered multiple chain* P*-extension* of $(\mathbb{K},\sigma)$ of monomial depth at most d induced by the single chain Π-extensions $(\mathbb{K}_\ell,\sigma)$ of $(\mathbb{K},\sigma)$ for $1\leqslant\ell\leqslant m$ with (6.61) and (6.62). Observe that since $\mathbb{A}_d\simeq\mathbb{A}$, the evaluation function $\operatorname{ev}:\mathbb{A}_i\times\mathbb{N}\to\mathbb{K}$ for all i with $1\leqslant i\leqslant d$ is also defined by (6.62).

Example 6.3.11 (Cont. Example 6.3.4).
The difference ring $(\mathbb{Q}\langle z_{1,1}\rangle\langle z_{2,1}\rangle\langle z_{2,2}\rangle\langle z_{3,1}\rangle\langle z_{3,2}\rangle\langle z_{4,1}\rangle\langle z_{4,2}\rangle\langle z_{4,3}\rangle,\sigma)$ constructed in Example 6.3.4 is a multiple chain P-extension of $(\mathbb{Q},\sigma)$. In particular, the diagram below illustrates a graphical view of the ordered multiple chain P-extension $(\mathbb{Q}\langle z_{1,1}\rangle\langle z_{2,1}\rangle\langle z_{3,1}\rangle\langle z_{4,1}\rangle\langle z_{2,2}\rangle\langle z_{3,2}\rangle\langle z_{4,2}\rangle\langle z_{4,3}\rangle,\sigma)$ composed by the single chain Π-extensions $(\mathbb{Q}\langle z_{1,1}\rangle,\sigma)$, $(\mathbb{Q}\langle z_{2,1}\rangle\langle z_{2,2}\rangle,\sigma)$, $(\mathbb{Q}\langle z_{3,1}\rangle\langle z_{3,2}\rangle,\sigma)$ and $(\mathbb{Q}\langle z_{4,1}\rangle\langle z_{4,2}\rangle\langle z_{4,3}\rangle,\sigma)$ of $(\mathbb{Q},\sigma)$. In the diagram, each horizontal line represents a single chain Π-extension of $(\mathbb{Q},\sigma)$. On the other hand, if we adjoin stepwise the generators at the dotted lines from left to right, then we have an ordered multiple chain P-extension of $(\mathbb{Q},\sigma)$. Thus, in general, a row-wise view corresponds to the distinct single chain Π-extension of the ground difference ring, while a column-wise view corresponds to the ordered multiple chain P-extension of the ground difference ring at a particular depth.



⋆

6.4 Structural result for multiple chain Π-extensions

We begin with some general properties which will be essential in this section. Most of these results have already been discussed in Schneider (2016, Section 3 and 4). For the sake of completeness, the proofs are repeated.

Definition 6.4.1.
A ring $\mathbb{A}$ is said to be *reduced* if there are no non-zero nilpotent elements, i.e., for any $\alpha\in\mathbb{A}\setminus\{0\}$ and any $n>0$, $\alpha^n\neq 0$. $\mathbb{A}$ is said to be *connected* if 0 and 1 are the only idempotent elements, i.e., for any $\alpha\in\mathbb{A}\setminus\{0,1\}$, $\alpha^2\neq\alpha$. ∗

A polynomial $a_0 + a_1\,t + \cdots + a_n\,t^n \in \mathbb{A}[t]$ with a_i in the ring $\mathbb{A}$ is invertible if and only if $a_0 \in \mathbb{A}^*$ and a_i with $i \geqslant 1$ are nilpotent elements. Thus, if $\mathbb{A}[t]$ is a reduced ring, then $\mathbb{A}[t]^* = \mathbb{A}$. In Karpilovsky (1983, Theorem 1); see also Neher (2008), where this result has been extended to the ring of Laurent polynomials, $\mathbb{A}[t, \frac{1}{t}]$. In particular, a complete characterisation of the invertible elements in $\mathbb{A}[t, \frac{1}{t}]$ is presented. For our purpose, we extract the following result.

Lemma 6.4.2.
Let $\mathbb{A}$ be commutative ring with 1. If $\mathbb{A}$ is reduced, then $\mathbb{A}[t]^* = \mathbb{A}$. If $\mathbb{A}$ is reduced and connected, then $\mathbb{A}\langle t\rangle^* = \mathbb{A}[t, \frac{1}{t}]^* = \{h\,t^\nu \mid h \in \mathbb{A}^*, \nu \in \mathbb{Z}\}$. ♦

Definition 6.4.3.
Let $(\mathbb{A}, \sigma)$ be a difference ring. We define the set of semi-constants (also called semi-invariants in Bronstein (2000)) of $(\mathbb{A}, \sigma)$ by

$$\mathrm{sconst}(\mathbb{A}, \sigma) = \{c \in \mathbb{A} \mid \sigma(c) = u\,c \;\text{ for some } u \in \mathbb{A}^*\}. \qquad *$$

In Bronstein (2000, Lemma 3), it has been shown that for a general difference ring $(\mathbb{A}, \sigma)$, the set $\mathrm{sconst}(\mathbb{A}, \sigma) \setminus \{0\}$ is only a multiplicative monoid. However, for the purpose of constructing simple Π-extensions, we follow Schneider (2016) and introduce the following refinement in order to turn the set of semi-constants into a group.

Definition 6.4.4.
Let $(\mathbb{A}, \sigma)$ be a difference ring and let G be a multiplicative subgroup of $\mathbb{A}^*$ (in short $G \leqslant \mathbb{A}^*$). Then we define the set of semi-constants (semi-invariants) of $(\mathbb{A}, \sigma)$ over G as

$$\mathrm{sconst}_G(\mathbb{A}, \sigma) = \{c \in \mathbb{A} \mid \sigma(c) = u\,c \;\text{ for some } u \in G\}. \qquad *$$

Observe from Definition 6.4.4 that, if $G = \mathbb{A}^*$, then $\mathrm{sconst}_{(\mathbb{A}^*)}(\mathbb{A}, \sigma) = \mathrm{sconst}(\mathbb{A}, \sigma)$. On the other hand if $G = \{1\}$, then $\mathrm{sconst}_{\{1\}}(\mathbb{A}, \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$. For all our considerations, we will choose G such that $\mathrm{sconst}_G(\mathbb{A}, \sigma) \setminus \{0\}$ is a subgroup of $\mathbb{A}^*$. Then with such a group G, the $\mathbb{Z}$-submodule (5.1) can be refined to the set of semi-constants over a G.

Lemma 6.4.5 (Schneider (2016), Lemma 2.16).
Let $(\mathbb{A}, \sigma)$ be a difference ring and let $G \leqslant \mathbb{A}^*$ with $\mathrm{sconst}_G(\mathbb{A}, \sigma) \setminus \{0\} \leqslant \mathbb{A}^*$; let $\mathbf{f} = (f_1, \ldots, f_r) \in G^r$. Then $\boldsymbol{\mathcal{M}}(\mathbf{f}, \mathbb{A}) = \boldsymbol{\mathcal{M}}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A}, \sigma))$. Furthermore, $\boldsymbol{\mathcal{M}}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A}, \sigma))$ is a free submodule of $\mathbb{Z}^r$ over $\mathbb{Z}$ with $\mathrm{rank}(\boldsymbol{\mathcal{M}}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A}, \sigma))) \leqslant r$. ♦

Proof:
" $\subseteq$ ": If $\boldsymbol{v} = (v_1, \ldots, v_r) \in \boldsymbol{\mathcal{M}}(\mathbf{f}, \mathbb{A})$, then $\sigma(g) = \mathbf{f}^{\boldsymbol{v}}\,g$ for some $g \in \mathbb{A} \setminus \{0\}$. This implies that $g \in \mathrm{sconst}_G\mathbb{A}$ since $\mathbf{f}^{\boldsymbol{v}} \in G$. Therefore $\boldsymbol{v} \in \boldsymbol{\mathcal{M}}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A}, \sigma))$.

" $\supseteq$ ": If $\mathbf{u} = (u_1, \ldots, u_r) \in \boldsymbol{\mathcal{M}}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A}, \sigma))$, then $\sigma(g) = \mathbf{f}^{\mathbf{u}}\,g$ for some $g \in \mathrm{sconst}_G(\mathbb{A}, \sigma)$ with $\mathbf{f}^{\mathbf{u}} \in G$. Since G and $\mathrm{sconst}_G(\mathbb{A}, \sigma) \setminus \{0\}$ are both subgroups of $\mathbb{A}^*$, it follows that $\mathbf{f}^{\mathbf{u}} \in \mathbb{A}^*$ and $g \in \mathbb{A}^* \subseteq \mathbb{A} \setminus \{0\}$. Therefore $\mathbf{u} \in \boldsymbol{\mathcal{M}}(\mathbf{f}, \mathbb{A})$.

Next we prove that $\mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$ is an additive subgroup of $\mathbb{Z}^r$. Note that $\mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$ is not empty since $0_r \in \mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$. Let $\mathbf{u}, \mathbf{v} \in \mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$ with $\mathbf{u} = (u_1, \ldots, u_r)$ and $\mathbf{v} = (v_1, \ldots, v_r)$ where $u_i, v_i \in \mathbb{Z}$. We prove $\mathbf{u} - \mathbf{v} \in \mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$. Since the integer vectors $\mathbf{u}, \mathbf{v} \in \mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$, we know that there are $g_u, g_v \in \mathrm{sconst}_G(\mathbb{A},\sigma) \setminus \{0\}$ such that

$$\sigma(g_u) = \mathbf{f}^{\mathbf{u}}\, g_u \text{ and } \sigma(g_v) = \mathbf{f}^{\mathbf{v}}\, g_v.$$

Consider,

$$\sigma\left(g_u\, g_v^{-1}\right) = \sigma(g_u)\, \sigma(g_v)^{-1} = \mathbf{f}^{\mathbf{u}}\, g_u\, \mathbf{f}^{-\mathbf{v}}\, g_v^{-1} = \mathbf{f}^{\mathbf{u}-\mathbf{v}}\, g_u\, g_v^{-1}.$$

Since $\mathrm{sconst}_G(\mathbb{A},\sigma) \setminus \{0\}$ is a group, it follows that $g_u\, g_v^{-1} \in \mathrm{sconst}_G(\mathbb{A},\sigma) \setminus \{0\}$. Therefore, $\mathbf{u} - \mathbf{v} \in \mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$ which proves that $\mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$ is a subgroup of $\mathbb{Z}^r$. Let $m \in \mathbb{Z}$, we prove the scalar multiplication, $m\,\mathbf{u} \in \mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$. Since $\mathbf{u} \in \mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$, there is a $g \in \mathrm{sconst}_G(\mathbb{A},\sigma) \setminus \{0\}$ with $\sigma(g) = \mathbf{f}^{\mathbf{u}}\, g$. Consider,

$$\sigma(g^m) = \big(\sigma(g)\big)^m = \big(\mathbf{f}^{\mathbf{u}}\, g\big)^m = \mathbf{f}^{m\,\mathbf{u}}\, g^m.$$

Since $\mathrm{sconst}_G(\mathbb{A},\sigma) \setminus \{0\}$ is a group, it follows that $g^m \in \mathrm{sconst}_G\mathbb{A} \setminus \{0\}$ and $m\,\mathbf{u} \in \mathbb{Z}^r$. Therefore, $m\,\mathbf{u} \in \mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$ which completes the proof that $\mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$ is a submodule of $\mathbb{Z}^r$. With the standard basis $\mathcal{B} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r\}$ of $\mathbb{Z}^r$, we know that $\mathrm{rank}(\mathbb{Z}^r) = r$. Since $\mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))$ is a submodule of $\mathbb{Z}^r$, it follows that $\mathrm{rank}(\mathcal{M}(\mathbf{f}, \mathrm{sconst}_G(\mathbb{A},\sigma))) \leqslant r$. ■

Lemma 6.4.6.
Let $(\mathbb{A},\sigma)$ be a difference ring: $\mathrm{sconst}(\mathbb{A},\sigma) \setminus \{0\} \leqslant \mathbb{A}^*$ *if and only if* $\mathrm{sconst}(\mathbb{A},\sigma) \setminus \{0\} = \mathbb{A}^*$. ♦

Proof:
Suppose that $\mathrm{sconst}(\mathbb{A},\sigma) \setminus \{0\} \leqslant \mathbb{A}^*$. If $\alpha \in \mathbb{A}^*$, then $\sigma(\alpha) \in \mathbb{A}^*$. Thus $u := \frac{\sigma(\alpha)}{\alpha} \in \mathbb{A}^*$ and with $\sigma(\alpha) = u\,\alpha$ it follows that $\alpha \in \mathrm{sconst}(\mathbb{A},\sigma) \setminus \{0\}$. Therefore $\mathbb{A}^* \subseteq \mathrm{sconst}(\mathbb{A},\sigma) \setminus \{0\}$ and with $\mathrm{sconst}(\mathbb{A},\sigma) \setminus \{0\} \leqslant \mathbb{A}^*$ we have that $\mathrm{sconst}(\mathbb{A},\sigma) \setminus \{0\} \subseteq \mathbb{A}^*$. The other implication is immediate and obvious. ■

If the set of constants forms a group, then the result of Theorem 2.3.40 can be sharpened.

Theorem 6.4.7 (Schneider (2016), Theorem 3.17).
Let $(\mathbb{A},\sigma)$ be a difference ring and let $G \leqslant \mathbb{A}^*$ with $\mathrm{sconst}_G(\mathbb{A},\sigma) \setminus \{0\} \leqslant \mathbb{A}^*$. Let $(\mathbb{A}\langle t\rangle,\sigma)$ be a difference ring extension of $(\mathbb{A},\sigma)$ with $\sigma(t) = \alpha\, t$ for some $\alpha \in G$. Then $(\mathbb{A}\langle t\rangle,\sigma)$ is a Π-extension of $(\mathbb{A},\sigma)$ if and only if there does not exit $g \in \mathrm{sconst}_G(\mathbb{A},\sigma) \setminus \{0\}$ and $\nu \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = \alpha^\nu\, g$. ♢

Proof:
" $\Longrightarrow$ " Assume that t is not a Π-monomial. Then we can take a $g \in \mathbb{A} \setminus \{0\}$ and a $\nu \in \mathbb{Z} \setminus \{0\}$ such that $\sigma(g) = \alpha^\nu\, g$. Since G is a group and $\alpha \in G$, it follows that $\alpha^\nu \in G$. Therefore, $g \in \mathrm{sconst}_G(\mathbb{A},\sigma) \setminus \{0\} \leqslant \mathbb{A}^*$. Moreover, if $\nu < 0$ then let $\tilde{g} = \frac{1}{g} \in \mathbb{A} \setminus \{0\}$.

" $\Longleftarrow$ " This direction of the proof is immediate by the first part of the proof of Theorem 2.3.40. ■

We will now characterise the set of semi-constants for Π-extensions.

Proposition 6.4.8 (Schneider (2016), Proposition 3.19).
Let $(\mathbb{A}, \sigma)$ be a difference ring with $G \leqslant \mathbb{A}^*$ and $\mathrm{sconst}_G(\mathbb{A}, \sigma) \setminus \{0\} \leqslant \mathbb{A}^*$. Let $(\mathbb{A}\langle t\rangle, \sigma)$ be a Π-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ for some $\alpha \in G$. Then

$$\mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma) = \{h\, t^\nu \mid h \in \mathrm{sconst}_G(\mathbb{A}, \sigma) \text{ and } \nu \in \mathbb{Z}\} \quad \text{and} \quad \mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma) \setminus \{0\} \leqslant \mathbb{A}\langle t\rangle^*. \quad \square$$

PROOF:
" $\subseteq$ " : Let $f \in \mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma)$, i.e., $f = \sum_i f_i\, t^i \in \mathbb{A}\langle t\rangle$ with $\sigma(f) = u\, f$ for some $u \in G$. By Lemma 2.3.39 $\sigma(f_i) = u\, \alpha^{-i}\, f_i$. Suppose that there are $k, \ell \in \mathbb{Z}$ with $\ell > k$ and $f_k \neq 0 \neq f_\ell$. Since $u\, \alpha^{-\ell} \in G$, it follows that $f_\ell \in \mathrm{sconst}_G(\mathbb{A}, \sigma) \setminus \{0\} \leqslant \mathbb{A}^*$. Thus we have that

$$\sigma\left(\frac{f_k}{f_\ell}\right) = \alpha^{\ell-k}\, \frac{f_k}{f_\ell}$$

with $\ell - k > 0$; a contradiction to the assumption that $(\mathbb{A}\langle t\rangle, \sigma)$ is a Π-extension of $(\mathbb{A}, \sigma)$. Therefore, $f = h\, t^\nu$ for some $h \in \mathrm{sconst}_G(\mathbb{A}, \sigma)$, $\nu \in \mathbb{Z}$.

" $\supseteq$ " : Let $f = h\, t^\nu$ with $h \in \mathrm{sconst}_G(\mathbb{A}, \sigma)$ and $\nu \in \mathbb{Z}$. Then there is a $u \in G$ with $\sigma(h) = u\, h$. Hence

$$\sigma(f) = \sigma(h)\, \alpha^\nu\, t^\nu = u\, \alpha^\nu\, h\, t^\nu = u\, \alpha^\nu\, f$$

with $u\, \alpha^\nu \in G$. Therefore, $f \in \mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma)$ and obviously $\mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma) \setminus \{0\} \leqslant \mathbb{A}\langle t\rangle^*$. ■

Next, we lift the result above to the product group, $\hat{G} = G_{\mathbb{A}}^{\mathbb{A}\langle t\rangle} = \{h\, t^\nu \mid h \in G \text{ and } \nu \in \mathbb{Z}\}$.

Theorem 6.4.9 (Schneider (2016), Theorem 3.20).
Let $(\mathbb{A}, \sigma)$ be a difference ring with $G \leqslant \mathbb{A}^*$ and $\mathrm{sconst}_G(\mathbb{A}, \sigma) \setminus \{0\} \leqslant \mathbb{A}^*$. Let $(\mathbb{A}\langle t\rangle, \sigma)$ be a Π-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ for some $\alpha \in G$ and let $\hat{G} = G_{\mathbb{A}}^{\mathbb{A}\langle t\rangle}$. Then

$$\mathrm{sconst}_{\hat{G}}(\mathbb{A}\langle t\rangle, \sigma) = \mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma) = \{h\, t^\nu \mid h \in \mathrm{sconst}_G(\mathbb{A}, \sigma) \text{ and } \nu \in \mathbb{Z}\}. \quad \diamondsuit$$

PROOF:
By Proposition 6.4.8, we have that $\mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma) = \{g\, t^\nu \mid g \in \mathrm{sconst}_G(\mathbb{A}, \sigma) \text{ and } \nu \in \mathbb{Z}\}$. It remains to prove the equality, $\mathrm{sconst}_{\hat{G}}(\mathbb{A}\langle t\rangle, \sigma) = \mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma)$.

" $\supseteq$ ": Since $G \leqslant \hat{G}$, the inclusion $\mathrm{sconst}_{\hat{G}}(\mathbb{A}\langle t\rangle, \sigma) \supseteq \mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma)$ obviously follows.

" $\subseteq$ ": Suppose that $g = \sum_i g_i\, t^i \in \mathrm{sconst}_{\hat{G}}(\mathbb{A}\langle t\rangle, \sigma)$. Then there are a $h \in G$ and a $\nu \in \mathbb{Z}$ with $\sigma(g) = h\, t^\nu\, g$. By comparing coefficients, we have that $\sigma(g_i)\, \alpha^i = h\, g_{i-\nu}$. If $\nu \geqslant 1$, then take k minimal such that $g_k \neq 0$. Then $\sigma(g_k)\, \alpha^{-k} \neq 0$. But the choice of k, we get $h_{k-\nu} = 0$ and thus $h\, g_{k-\nu} = 0$, a contradiction. On the other hand, if $k < 0$, then take k maximal such that $g_{k-\nu} \neq 0$. Then $h\, g_{k-\nu} \neq 0$. However, by the choice of k, we have that $\sigma(g_k)\, \alpha^{-k} = 0$ which is also a contraction. Therefore, $m = 0$ and consequently, $g \in \mathrm{sconst}_G(\mathbb{A}\langle t\rangle, \sigma)$. ■

The next theorem gives a description of $\mathrm{sconst}(\mathbb{A}\langle t\rangle, \sigma)$ given that $\mathbb{A}$ is reduced and connected.

Theorem 6.4.10 (Schneider (2016), Theorem 3.21).
Let $(\mathbb{A}, \sigma)$ *be a difference ring which is reduced and connected with* $\mathrm{sconst}(\mathbb{A}, \sigma) \setminus \{0\} = \mathbb{A}^*$. *Let* $(\mathbb{A}\langle t\rangle, \sigma)$ *be a* Π*-extension of* $(\mathbb{A}, \sigma)$ *with* $\sigma(t) = \alpha\, t$ *for some* $\alpha \in \mathbb{A}^*$. *Then*

$$\mathrm{sconst}(\mathbb{A}\langle t\rangle, \sigma) = \{h\, t^{\nu} \mid h \in \mathrm{sconst}(\mathbb{A}, \sigma),\ \nu \in \mathbb{Z}\}. \qquad \diamond$$

PROOF:
Let $\hat{G} = (\mathbb{A}^*)_{\mathbb{A}}^{\mathbb{A}\langle t\rangle} = \{g\, t^{\nu} \mid g \in \mathbb{A},\ \nu \in \mathbb{Z}\}$. Then by Lemma 6.4.2 $G = \mathbb{A}\langle t\rangle^*$. Thus we conclude that $\mathrm{sconst}(\mathbb{A}\langle t\rangle, \sigma) = \mathrm{sconst}_{\mathbb{A}\langle t\rangle^*}(\mathbb{A}\langle t\rangle, \sigma) = \mathrm{sconst}_{\hat{G}}(\mathbb{A}\langle t\rangle, \sigma) = \{h\, t^{\nu} \mid h \in \mathrm{sconst}(\mathbb{A}, \sigma),\ \nu \in \mathbb{Z}\}$ by Theorem 6.4.9. ∎

Lemma 6.4.11 (Schneider (2016), Lemma 4.4).
Let $(\mathbb{A}\langle t\rangle, \sigma)$ *be a* $\Pi\Sigma$*-extension of* $(\mathbb{A}, \sigma)$. *If* $\mathbb{A}$ *is reduced, then* $(\mathbb{A}\langle t\rangle, \sigma)$ *is reduced. Furthermore, if* $\mathbb{A}$ *is reduced and connected, then* $\mathbb{A}\langle t\rangle$ *is also reduced and connected.* ♦

PROOF:
Let $\mathbb{A}$ be a reduced and t be a Π-monomial. Let $f = \sum_i f_i\, t^i \in \mathbb{A}\langle t\rangle$ with $f \neq 0$ and $f^n = 0$ for some $n > 0$. Since $\mathbb{A}$ is reduced, $f \notin \mathbb{A}$. Let $m \in \mathbb{Z}$ be maximal such that $f_m \neq 0$. Then the coefficient of $t^{m\,n}$ in f is $f_m^n = 0$ and thus f_m is a nilpotent element in $\mathbb{A}$ which contradicts the assumption that $\mathbb{A}$ is a reduced ring.

Now let $\mathbb{A}$ be reduced and connected and take $f = \sum_i f_i\, t^i \in \mathbb{A}\langle t\rangle$ with $f^2 = f$ and $f \notin \{0, 1\}$. Since $\mathbb{A}$ is connected, $f \in \mathbb{A}$. Let $m \in \mathbb{Z}$ be maximal such that $f_m \neq 0$. If $m > 0$, then the coefficient of $t^{2\,m}$ in f^2 is f_m^2 and thus with $f^2 = f$ we have that $f_m^2 = 0$; a contraction that $\mathbb{A}$ is reduced. Otherwise, if $m = 0$, take $\tilde{m}$ minimal with $f_{\tilde{m}}$. Note that $\tilde{m} < 0$ since $f \notin \mathbb{A}$. As above, it follows that $f_{\tilde{m}}^2 = 0$ which is again a contraction. Thus, if $\mathbb{A}$ is reduced (and connected), $\mathbb{A}\langle t\rangle$ is reduced (and connected). Note that for Σ-monomial t, the same implication holds since $\mathbb{A}[t] \leqslant \mathbb{A}\langle t\rangle$. ∎

We will now present the general structure of the semi-constants for nested Π-extensions. This result is a specialisation of Schneider (2016, Corollary 4.6).

Corollary 6.4.12.
Let $(\mathbb{E}, \sigma)$ *be a* Π*-extension of* $(\mathbb{A}, \sigma)$ *with* $\mathbb{E} = \mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle$.

(1) Let $G \leqslant \mathbb{A}^*$ *with* $\mathrm{sconst}_G(\mathbb{A}, \sigma) \setminus \{0\} \leqslant \mathbb{A}^*$ *and* $\tilde{G} = G_{\mathbb{A}}^{\mathbb{E}}$. *If* $(\mathbb{E}, \sigma)$ *is a* G*-simple* Π*-extension of* $(\mathbb{A}, \sigma)$*, then* $\mathrm{sconst}_{\tilde{G}}(\mathbb{E}, \sigma) \setminus \{0\} \leqslant \mathbb{E}^*$ *where*

$$\mathrm{sconst}_{\tilde{G}}(\mathbb{E}, \sigma) = \{h\, t_1^{\nu_1} \cdots t_e^{\nu_e} \mid h \in \mathrm{sconst}_G(\mathbb{A}, \sigma) \ \textit{and}\ \nu_i \in \mathbb{Z}\}.$$

(2) If $\mathbb{A}$ *is reduced and connected and* $\mathrm{sconst}(\mathbb{A}, \sigma) \setminus \{0\} = \mathbb{A}^*$*, then*

$$\mathrm{sconst}(\mathbb{E}, \sigma) \setminus \{0\} = \{h\, t_1^{\nu_1} \cdots t_e^{\nu_e} \mid h \in \mathbb{A}^* \ \textit{and}\ \nu_i \in \mathbb{Z}\} = \mathbb{E}^*. \tag{6.64}$$

(3) If $\mathbb{A}$ *is a field, then we have that* (6.64). ◇

PROOF:

(1) We prove the statement by induction on the number of Π-monomials, e. If $e = 0$, then there is nothing to prove. Suppose that the statement holds and consider an extra Π-monomial, t_{e+1}, with $\sigma(t_{e+1})/t_{e+1} \in \tilde{G}$. Define $\hat{G} = \tilde{G}_{\mathbb{E}}^{\mathbb{E}\langle t_{e+1}\rangle} = G_{\mathbb{A}}^{\mathbb{E}\langle t_{e+1}\rangle}$. By Theorem 6.4.9,

$$\mathrm{sconst}_{\hat{G}}(\mathbb{E}\langle t_{e+1}\rangle, \sigma) = \{h\, t_{e+1}^{\nu} \mid h \in \mathrm{sconst}_{\tilde{G}}(\mathbb{E}, \sigma),\ \nu \in \mathbb{Z}\}$$

and thus by the induction assumption we have that,

$$\mathrm{sconst}_{\hat{G}}(\mathbb{E}, \sigma) = \{h\, t_1^{\nu_1} \cdots t_{e+1}^{\nu_{e+1}} \mid h \in \mathrm{sconst}_{\tilde{G}}(\mathbb{A}, \sigma) \text{ and } \nu_i \in \mathbb{Z}\}$$

and thus $\mathrm{sconst}_{\hat{G}}(\mathbb{E}\langle t_{e+1}\rangle, \sigma) \setminus \{0\} \leqslant \mathbb{E}\langle t_{e+1}\rangle^*$. This completes the induction step.

(2) Again we prove the statement by induction on the number of Π-monomials, e. If $e = 0$, then there is nothing to prove. Suppose that the statement holds and consider an extra Π-monomial, t_{e+1}, with $\sigma(t_{e+1})/t_{e+1} \in \mathbb{E}^*$. By Theorem 6.4.10, $\mathrm{sconst}(\mathbb{E}\langle t_{e+1}\rangle, \sigma) = \{h\, t_{e+1}^{\nu} \mid h \in \mathrm{sconst}(\mathbb{E}, \sigma),\ \nu \in \mathbb{Z}\}$. Since $\mathrm{sconst}(\mathbb{E}, \sigma) \setminus \{0\} \leqslant \mathbb{E}^*$, it follows by Lemma 6.4.6 that

$$\mathrm{sconst}(\mathbb{E}\langle t_{e+1}\rangle, \sigma) \setminus \{0\} = \{h\, t_{e+1}^{\nu} \mid h \in \mathbb{E}^*,\ \nu \in \mathbb{Z}\} = \mathbb{E}\langle t_{e+1}\rangle^*.$$

The second equality follows by Lemma 6.4.2, because by recursive application of Lemma 6.4.11 $\mathbb{E}\langle t_{e+1}\rangle$ is reduced and connected. By the induction assumption we have that,

$$\mathrm{sconst}(\mathbb{E}\langle t_{e+1}\rangle, \sigma) \setminus \{0\} = \{h\, t_1^{\nu_1} \cdots t_{e+1}^{\nu_{e+1}} \mid h \in \mathbb{A}^*,\ \nu_i \in \mathbb{Z}\} = \mathbb{E}\langle t_{e+1}\rangle^*.$$

(3) Since any field is reduced and connected and $\mathrm{sconst}(\mathbb{A}, \sigma) \setminus \{0\} = \mathbb{A}^*$ by Lemma 6.4.6, statement (3) follows by statement (2). ■

The following remark is obvious.

Remark 6.4.13.
Let $(\mathbb{H}, \sigma)$ be a difference field and let $(\mathbb{H}_\ell, \sigma)$ with $\mathbb{H}_\ell = \mathbb{H}\langle t_{\ell,1}\rangle \ldots \langle t_{\ell,s_\ell}\rangle$ be the single chain Π-extension of $(\mathbb{H}, \sigma)$ with the automorphism σ defined by (6.65). Let $(\mathbb{A}, \sigma)$ be the ordered multiple chain P-extension of $(\mathbb{H}, \sigma)$ with $\mathbb{A} = \mathbb{H}\langle t_{1,1}\rangle \ldots \langle t_{w_1,1}\rangle \ldots \langle t_{1,d}\rangle \ldots \langle t_{w_d,d}\rangle$ of monomial depth $d := \max(s_1, \ldots, s_m)$ with $m = w_1 \geqslant w_2 \geqslant \cdots \geqslant w_d$ composed by the single chain Π-extensions $(\mathbb{H}_\ell, \sigma)$ of $(\mathbb{H}, \sigma)$ with the automorphism (6.65). By statement (3) of Corollary 6.4.12,

$$\mathrm{sconst}(\mathbb{H}_\ell, \sigma) = \{h\, t_{\ell,1}^{\nu_{\ell,1}} \cdots z_{\ell,s_\ell}^{\nu_{\ell,s_\ell}} \mid h \in \mathbb{H}^* \text{ and } \nu_{\ell,k} \in \mathbb{Z}\} = \mathbb{H}_\ell^*$$

for all $1 \leqslant \ell \leqslant m$ and also

$$\mathrm{sconst}(\mathbb{A}, \sigma) = \left\{h\, t_{1,1}^{\nu_{1,1}} \cdots t_{1,s_1}^{\nu_{1,s_1}}\, t_{2,1}^{\nu_{2,1}} \cdots t_{2,s_2}^{\nu_{2,s_2}} \cdots t_{m,1}^{\nu_{m,1}} \cdots t_{m,s_m}^{\nu_{m,s_m}} \mid h \in \mathbb{H}^* \text{ and } \nu_{i,j} \in \mathbb{Z}\right\} = \mathbb{A}^*.$$

In particular,

$$\mathrm{sconst}(\mathbb{H}_\ell, \sigma) \subseteq \mathrm{sconst}(\mathbb{A}, \sigma) \quad \text{for all} \quad 1 \leqslant \ell \leqslant m. \qquad ♠$$

We end this Section by giving the necessary and sufficient criteria for an ordered multiple chain P-extension to be a Π-extension.

Theorem 6.4.14.
Let $(\mathbb{H},\sigma)$ be a difference field and let $(\mathbb{H}_\ell,\sigma)$ with $\mathbb{H}_\ell=\mathbb{H}\langle t_{\ell,1}\rangle\dots\langle t_{\ell,s_\ell}\rangle$ for $1\leqslant\ell\leqslant m$ be the single chain Π-extensions of $(\mathbb{H},\sigma)$ over $\mathbb{H}$ with base $c_\ell\in\mathbb{H}$ together with the automorphism

$$\sigma(t_{\ell,k})=\alpha_{\ell,k}\,t_{\ell,k}\quad\textit{where}\quad\alpha_{\ell,k}=c_\ell\,t_{\ell,1}\cdots t_{\ell,k-1}\in(\mathbb{H}^*)_{\mathbb{H}}^{\mathbb{H}\langle t_{\ell,1}\rangle\dots\langle t_{\ell,k-1}\rangle}. \tag{6.65}$$

Let $(\mathbb{A},\sigma)$ be the ordered multiple chain P-extension of $(\mathbb{H},\sigma)$ with

$$\mathbb{A}=\mathbb{H}\langle t_{1,1}\rangle\dots\langle t_{w_1,1}\rangle\dots\langle t_{1,d}\rangle\dots\langle t_{w_d,d}\rangle$$

of monomial depth $d:=\max(s_1,\dots,s_m)$ with $m=w_1\geqslant w_2\geqslant\dots\geqslant w_d$ composed by the single chain Π-extensions $(\mathbb{H}_\ell,\sigma)$ of $(\mathbb{H},\sigma)$ with the automorphism (6.65). Then $(\mathbb{A},\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$ if and only if [1]

$$\boldsymbol{\mathcal{M}}((c_1,c_2,\dots,c_m),\,\mathbb{H})=\{0_m\}.\qquad\diamondsuit$$

Proof:
" $\Longrightarrow$ " Suppose that $(\mathbb{A},\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$. Then, it is a tower of Π-extensions $(\mathbb{A}_i,\sigma)$ of $(\mathbb{H},\sigma)$ where $\mathbb{A}_i=\mathbb{A}_{i-1}\langle t_{1,i}\rangle\dots\langle t_{w_i,i}\rangle$ for $1\leqslant i\leqslant d$ with $\mathbb{A}_0=\mathbb{H}$. Since $(\mathbb{A}_1,\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$, it follows by Lemma 5.0.2 that, $\boldsymbol{\mathcal{M}}((\alpha_{1,1},\dots,\alpha_{w_1,1}),\,\mathbb{H})=\boldsymbol{\mathcal{M}}((c_1,\dots,c_{w_1}),\,\mathbb{H})=\{0_{w_1}\}$ where $w_1=m$.
" $\Longleftarrow$ " Conversely, suppose that $\boldsymbol{\mathcal{M}}((\alpha_{1,1},\alpha_{2,1},\dots,\alpha_{w_1,1}),\,\mathbb{H})=\{0_{w_1}\}$ with $w_1=m$ and let $(\mathbb{A}_1,\sigma)$ with $\mathbb{A}_1=\mathbb{H}\langle t_{1,1}\rangle\dots\langle t_{w_1,1}\rangle$ be a P-extension of $(\mathbb{H},\sigma)$ with $\sigma(t_{j,1})=\alpha_{j,1}\,t_{j,1}$ for all $1\leqslant j\leqslant w_1$. By Lemma 5.0.2, $(\mathbb{A}_1,\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$. Let $(\mathbb{A}_i,\sigma)$ with $\mathbb{A}_i=\mathbb{A}_{i-1}\langle t_{1,i}\rangle\dots\langle t_{w_i,i}\rangle$ be the multiple chain P-extension of $(\mathbb{H},\sigma)$ of monomial depth i for all $1\leqslant i\leqslant d$ with the automorphism (6.65) and assume that $(\mathbb{A}_k,\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$ for all $1\leqslant k\leqslant d-1$. Suppose that $(\mathbb{A}_d,\sigma)$ is not a Π-extension of $(\mathbb{A}_{d-1},\sigma)$. Then by Lemma 5.0.2, we can take a $g\in\mathbb{A}_{d-1}\setminus\{0\}$ and $(\upsilon_1,\upsilon_2,\dots,\upsilon_{w_d})\in\mathbb{Z}^{w_d}\setminus\{0_{w_d}\}$ such that

$$\sigma(g)=\alpha_{1,d}^{\upsilon_1}\,\alpha_{2,d}^{\upsilon_2}\cdots\,\alpha_{w_d,d}^{\upsilon_{w_d}}\,g \tag{6.66}$$

holds. By Corollary 6.4.12 it follows that $g=h\,t_{1,1}^{\nu_{1,1}}\,t_{2,1}^{\nu_{2,1}}\cdots\,t_{w_1,1}^{\nu_{w_1,1}}\cdots\,t_{1,d-1}^{\nu_{1,d-1}}\,t_{2,d-1}^{\nu_{2,d-1}}\cdots\,t_{w_{d-1},d-1}^{\nu_{w_{d-1},d-1}}$ with $h\in\mathbb{H}^*$ and $\nu_{i,j}\in\mathbb{Z}$. For the left hand side of (6.66) we have that

$$\sigma(g)=\gamma\,t_{1,d-1}^{\nu_{1,d-1}}\,t_{2,d-1}^{\nu_{2,d-1}}\cdots\,t_{w_{d-1},d-1}^{\nu_{w_{d-1},d-1}}$$

where $\gamma\in\mathbb{A}_{d-2}$ and for the right hand side of (6.66) we have that

$$\alpha_{1,d}^{\upsilon_1}\,\alpha_{2,d}^{\upsilon_2}\cdots\,\alpha_{w_d,d}^{\upsilon_{w_d}}\,g=\omega\,t_{1,d-1}^{\nu_{1,d-1}+\upsilon_1}\,t_{2,d-1}^{\nu_{2,d-1}+\upsilon_2}\cdots\,t_{w_{d-1},d-1}^{\nu_{w_{d-1},d-1}+\upsilon_{w_d}}$$

where $\omega\in\mathbb{A}_{d-2}$. Consequently,

$$\nu_{k,d-1}=\nu_{k,d-1}+\upsilon_k$$

and thus $\upsilon_k=0$ for all $1\leqslant k\leqslant w_d$ which is a contradiction to the assumption that $(\upsilon_1,\dots,\upsilon_{w_d})\neq 0_{w_d}$ for all $1\leqslant k\leqslant w_d$. Thus $(\mathbb{A}_d,\sigma)$ is a Π-extension of $(\mathbb{A}_{d-1},\sigma)$ and consequently, a Π-extension of $(\mathbb{H},\sigma)$. ∎

Example 6.4.15 (Cont. Example 6.3.11).
The ordered multiple chain P-extension $(\mathbb{Q}\langle x_{1,1}\rangle\langle x_{2,1}\rangle\langle x_{3,1}\rangle\langle x_{4,1}\rangle\langle x_{2,2}\rangle\langle x_{3,2}\rangle\langle x_{4,2}\rangle\langle x_{4,3}\rangle,\sigma)$ of $(\mathbb{Q},\sigma)$ constructed in Example 6.3.11 is a Π-extension since $\boldsymbol{\mathcal{M}}((5,\,2,\,7,\,3),\,\mathbb{Q})=\{0_4\}$. That is, the difference ring $(\mathbb{Q}\langle x_{1,1}\rangle\langle x_{2,1}\rangle\langle x_{3,1}\rangle\langle x_{4,1}\rangle,\sigma)$ is a Π-extension of $(\mathbb{Q},\sigma)$. ⋆

[1] Note that $(c_1,\dots,c_m)=(\alpha_{1,1},\dots,\alpha_{w_1,1})$.

6.5 Construction of RΠ-extensions for higher nesting depth expressions in $\mathrm{ProdE}(\mathbb{K})$

In this section we discuss how one can construct a simple RΠ-extension to model geometric product expressions of higher nested depths in $\mathrm{ProdE}(\mathbb{K})$. More precisely, suppose we are given an ordered multiple chain P-extension $(\mathbb{A}_d, \sigma)$ of the difference field $(\mathbb{K}, \sigma)$ with $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ where K is some σ-strongly computable field. Then in Lemma 6.5.2 we will elaborate how this extension can be rephrased to a difference ring built by several nested Π-monomials and several nested A-monomials. Afterwards we will utilise the result from Section 6.2, more precisely Theorem 6.2.28 to rephrase these nested A-monomials in terms of one single R-monomial. Summarising we will succeed in Lemma 6.5.5 in rephrasing the multiple chain P-extension to a multiple chain Π-extension adjoined with one R-monomial.

Lemma 6.5.1.
Let $(\mathbb{A}\langle t\rangle, \sigma)$ be a Π-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ and let $(\mathbb{H}, \sigma)$ be a difference ring. Let $\tilde{\rho} : \mathbb{A} \to \mathbb{H}$ be a difference ring homomorphism and let $\rho : \mathbb{A}\langle t\rangle \to \mathbb{H}$ be a ring homomorphism defined by $\rho|_{\mathbb{A}} = \tilde{\rho}$ and $\rho(t) = g$ for some $g \in \mathbb{H}$. If $\sigma(g) = \rho(\alpha)\, g$, then ρ is a difference ring homomorphism. ♦

Proof:
Suppose that $\sigma(g) = \rho(\alpha)\, g$ holds. Then

$$\sigma(\rho(t)) = \sigma(g) = \rho(\alpha)\, g = \rho(\alpha\, t) = \rho(\sigma(t)).$$

Consequently, $\sigma(\rho(f)) = \rho(\sigma(f))$ for all $f \in \mathbb{A}\langle t\rangle$. ■

Lemma 6.5.2.
For $1 \leqslant \ell \leqslant m$, let $(\mathbb{K}_\ell, \sigma)$ with $\mathbb{K}_\ell = \mathbb{K}\langle y_{\ell,1}\rangle \dots \langle y_{\ell,s_\ell}\rangle$ be single chain Π-extensions of $(\mathbb{K}, \sigma)$ over $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ with base $h_\ell \in \mathbb{K}^*$, the automorphisms (6.61) and the naturally induced evaluation functions (6.62). Let $d := \max(s_1, \dots, s_m)$ and $\mathbb{A}_0 = \mathbb{K}$. Consider the tower of difference ring extensions $(\mathbb{A}_i, \sigma)$ of $(\mathbb{A}_{i-1}, \sigma)$ where $\mathbb{A}_i = \mathbb{A}_{i-1}\langle y_{1,i}\rangle\langle y_{2,i}\rangle \dots \langle y_{w_i,i}\rangle$ for $1 \leqslant i \leqslant d$ with $m = w_1 \geqslant \cdots \geqslant w_d$, the automorphism (6.63) and the evaluation function (6.62). In particular, one gets $(\mathbb{A}_d, \sigma)$ as the ordered multiple chain P-extension of $(\mathbb{K}, \sigma)$ of monomial depth at most d composed by the single chain Π-extensions $(\mathbb{K}_\ell, \sigma)$ of $(\mathbb{K}, \sigma)$ for $1 \leqslant \ell \leqslant m$ with (6.61) and (6.62). Then one can construct

(a) an ordered multiple chain AP-extension $(\mathbb{H}_d, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of monomial depth at most d with $\tilde{\mathbb{K}} = \tilde{K}(\kappa_1, \dots, \kappa_u)$ where $\tilde{K}$ is a finite algebraic field extension of K, with

$$\mathbb{H}_d = \tilde{\mathbb{K}}\langle \vartheta_{1,1}\rangle \dots \langle \vartheta_{\upsilon_1,1}\rangle\langle \tilde{y}_{1,1}\rangle \dots \langle \tilde{y}_{e_1,1}\rangle \dots \langle \vartheta_{1,d}\rangle \dots \langle \vartheta_{\upsilon_d,d}\rangle\langle \tilde{y}_{1,d}\rangle \dots \langle \tilde{y}_{e_d,d}\rangle \tag{6.67}$$

where[2] $\upsilon_i \geqslant 0$ and $e_d \geqslant 1$ and with the automorphism defined by

$$\sigma(\vartheta_{\ell,d}) = \gamma_{\ell,d}\, \vartheta_{\ell,d} \quad \text{where} \quad \gamma_{\ell,d} = \zeta^{\mu_\ell}\, \vartheta_{\ell,1} \cdots \vartheta_{\ell,d-1} \in \mathbb{U}_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}[\vartheta_{\ell,1}]\dots[\vartheta_{\ell,d-1}]} \tag{6.68}$$

where for $1 \leqslant \ell \leqslant \upsilon_d$ and $\mathbb{U} = \langle \zeta \rangle$ is a multiplicative cyclic subgroup of $\tilde{K}^*$ generated by a primitive λ-th root of unity $\zeta \in \tilde{K}^*$ and

$$\sigma(\tilde{y}_{\ell,d}) = \tilde{\alpha}_{\ell,d}\, \tilde{y}_{\ell,d} \quad \text{where} \quad \tilde{\alpha}_{\ell,d} = \tilde{h}_\ell\, \tilde{y}_{\ell,1} \cdots \tilde{y}_{\ell,d-1} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}\langle \tilde{y}_{\ell,1}\rangle \dots \langle \tilde{y}_{\ell,d-1}\rangle} \tag{6.69}$$

where $\tilde{\alpha}_{\ell,d} = \tilde{h}_\ell\, \tilde{y}_{\ell,1} \cdots \tilde{y}_{\ell,d-1} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}\langle \tilde{y}_{\ell,1}\rangle \dots \langle \tilde{y}_{\ell,d-1}\rangle}$ for $1 \leqslant \ell \leqslant e_d$;

[2]Note that if $\upsilon_i = 0$, then there is no depth-i A-monomial, $\vartheta_{\upsilon_i,i}$ for some i with $1 \leqslant i \leqslant d$.

(b) an evaluation function $\tilde{\text{ev}} : \mathbb{H}_d \times \mathbb{N} \to \tilde{\mathbb{K}}$ *defined as* [3]

$$\tilde{\text{ev}}(\vartheta_{\ell,d}, n) = \prod_{j=1}^{n} \tilde{\text{ev}}(\gamma_{\ell,d}, j-1) \quad \textit{and} \quad \tilde{\text{ev}}(\tilde{y}_{\ell,d}, n) = \prod_{j=1}^{n} \tilde{\text{ev}}(\tilde{\alpha}_{\ell,d}, j-1) \tag{6.70}$$

such that the following properties hold:

(1) The P*-extension* $(\tilde{\mathbb{A}}_d, \sigma)$ *of* $(\tilde{\mathbb{K}}, \sigma)$ *where*

$$\tilde{\mathbb{A}}_d = \tilde{\mathbb{K}}\langle \tilde{\mathbf{y}}_1\rangle\langle \tilde{\mathbf{y}}_2\rangle \dots \langle \tilde{\mathbf{y}}_d\rangle = \tilde{\mathbb{K}}\langle \tilde{y}_{1,1}\rangle \dots \langle \tilde{y}_{e_1,1}\rangle\langle \tilde{y}_{2,1}\rangle \dots \langle \tilde{y}_{e_2,2}\rangle \dots \langle \tilde{y}_{1,d}\rangle \dots \langle \tilde{y}_{e_d,d}\rangle \tag{6.71}$$

with the automorphism (6.69) is a Π*-extension. In particular, it is an ordered multiple chain* Π*-extension of monomial depth* d*.*

(2) One can construct a difference ring homomorphism

$$\begin{aligned} \rho_d : \mathbb{A}_d &\to \mathbb{H}_d \\ y_{\ell,d} &\mapsto \boldsymbol{\vartheta}_{\mathbf{d}}^{\boldsymbol{\mu}_{\ell,d}}\, \tilde{\mathbf{y}}_{\mathbf{d}}^{\boldsymbol{\nu}_{\ell,d}} = \vartheta_{1,d}^{\mu_{\ell,1,d}} \cdots \vartheta_{\upsilon_d,d}^{\mu_{\ell,\upsilon_d,d}}\, \tilde{y}_{1,d}^{\nu_{\ell,1,d}} \cdots \tilde{y}_{e_d,d}^{\nu_{\ell,e_d,d}} \end{aligned} \tag{6.72}$$

with $0 \leqslant \mu_{\ell,k,d} < \lambda$ *for* $1 \leqslant k \leqslant \upsilon_d$ *and* $\nu_{\ell,k,d} \in \mathbb{Z}$ *not all zero for* $1 \leqslant k \leqslant e_d$ *such that for all* $f \in \mathbb{A}_d$ *and for all* $n \in \mathbb{N}$*,*

$$\text{ev}(f, n) = \tilde{\text{ev}}(\rho_d(f), n)$$

holds. ♦

PROOF:
Let $(\mathbb{A}_d, \sigma)$ with $\mathbb{A}_d = \mathbb{A}_{d-1}\langle y_{1,d}\rangle\langle y_{2,d}\rangle \dots \langle y_{w_d,d}\rangle$ be the ordered multiple chain P-extension of $(\mathbb{K}, \sigma)$ of monomial depth $d \in \mathbb{N}$ as described above with the automorphism (6.63) and the evaluation function (6.62). We proof the Lemma by induction on the monomial depth d.

If $d = 1$, then the Lemma holds by Lemma 5.2.2. Let $d \geqslant 2$ and suppose that the Lemma holds for $d-1$. That is, we can construct $(\mathbb{H}_{d-1}, \sigma)$ with

$$\mathbb{H}_{d-1} = \tilde{\mathbb{K}}\langle \vartheta_{1,1}\rangle \dots \langle \vartheta_{\upsilon_1,1}\rangle\langle \tilde{y}_{1,1}\rangle \dots \langle \tilde{y}_{e_1,1}\rangle \dots \langle \vartheta_{1,d-1}\rangle \dots \langle \vartheta_{\upsilon_{d-1},d-1}\rangle\langle \tilde{y}_{1,d-1}\rangle \dots \langle \tilde{y}_{e_{d-1},d-1}\rangle$$

which is an ordered multiple chain AP-extension of $(\tilde{\mathbb{K}}, \sigma)$ of monomial depth at most $d-1$ with

$$\sigma(\vartheta_{\ell,k}) = \gamma_{\ell,k}\, \vartheta_{\ell,k} \qquad \text{and} \qquad \sigma(\tilde{y}_{\ell,k}) = \tilde{\alpha}_{\ell,k}\, \tilde{y}_{\ell,k} \tag{6.73}$$

where for $1 \leqslant k \leqslant d-1$,

$$\gamma_{\ell,k} = \zeta^{\mu_\ell}\, \vartheta_{\ell,1} \cdots \vartheta_{\ell,k-1} \in \mathbb{U}_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}[\vartheta_{\ell,1}]\dots[\vartheta_{\ell,k-1}]}$$

with $1 \leqslant \ell \leqslant \upsilon_k$ and

$$\tilde{\alpha}_{\ell,k} = \tilde{h}_\ell\, \tilde{y}_{\ell,1} \cdots \tilde{y}_{\ell,k-1} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}\langle \tilde{y}_{\ell,1}\rangle \dots \langle \tilde{y}_{\ell,k-1}\rangle}$$

with $1 \leqslant \ell \leqslant e_k$. In addition, we get the evaluation function $\tilde{\text{ev}} : \mathbb{H}_{d-1} \times \mathbb{N} \to \tilde{\mathbb{K}}$ defined as

$$\tilde{\text{ev}}(\vartheta_{\ell,k}, n) = \prod_{j=1}^{n} \tilde{\text{ev}}(\gamma_{\ell,k}, j-1) \quad \text{and} \quad \tilde{\text{ev}}(\tilde{y}_{\ell,k}, n) = \prod_{j=1}^{n} \tilde{\text{ev}}(\tilde{\alpha}_{\ell,k}, j-1) \tag{6.74}$$

[3] For all $c \in \mathbb{K}$, we set $\tilde{\text{ev}}(c, n) = c$ for all $n \geqslant 0$.

such that statements (1) and (2) of the Lemma hold. We prove the Lemma for the ordered multiple chain P-extension $(\mathbb{A}_d, \sigma)$ of $(\mathbb{K}, \sigma)$ with $\mathbb{A}_d = \mathbb{A}_{d-1}\langle y_{1,d}\rangle\langle y_{2,d}\rangle \dots \langle y_{w_d,d}\rangle$. Consider the P-monomials: $y_{1,d}, y_{2,d}, \dots, y_{w_d,d}$ in $\mathbb{A}_d$ of monomial depth d. Since the shift quotient of these P-monomials is contained in $\mathbb{A}_{d-1}$, i.e.,

$$\frac{\sigma(y_{\ell,d})}{y_{\ell,d}} = \alpha_{\ell,d} \in \mathbb{A}^*_{d-1},$$

we can iteratively apply the difference ring homomorphism $\rho_{d-1} : \mathbb{A}_{d-1} \to \mathbb{H}_{d-1}$ to rephrase each $\alpha_{\ell,d}$ in $\mathbb{H}_{d-1}$. That is,

$$\begin{aligned}\rho_{d-1}(\alpha_{\ell,d}) &= h_\ell\, \rho_{d-1}(y_{\ell,1}) \cdots \rho_{d-1}(y_{\ell,d-1})\\ &= h_\ell\, \rho_1(y_{\ell,1}) \cdots \rho_{d-1}(y_{\ell,d-1})\\ &= h_\ell\, (\boldsymbol{\vartheta}_1^{\mu_{\ell,1}}\, \tilde{\boldsymbol{y}}_1^{\nu_{\ell,1}}) \cdots (\boldsymbol{\vartheta}_{d-1}^{\mu_{\ell,d-1}}\, \tilde{\boldsymbol{y}}_{d-1}^{\nu_{\ell,d-1}})\end{aligned} \tag{6.75}$$

where

$$\boldsymbol{\vartheta}_i^{u_{\ell,i}} = \vartheta_{1,i}^{u_{\ell,1,i}} \cdots \vartheta_{\upsilon_i,i}^{u_{\ell,\upsilon_i,i}}$$

and

$$\tilde{\boldsymbol{y}}_i^{\nu_{\ell,i}} = \tilde{y}_{1,i}^{\nu_{\ell,1,i}} \cdots \tilde{y}_{e_i,i}^{\nu_{\ell,e_i,i}}$$

for $1 \leqslant i \leqslant d$ and $1 \leqslant \ell \leqslant w_d$ with $0 \leqslant \mu_{\ell,k,d-1} < \lambda$ for $1 \leqslant k \leqslant \upsilon_{d-1}$ and $\nu_{\ell,k,d-1} \in \mathbb{Z}$ not all zero for $1 \leqslant k \leqslant e_{d-1}$ such that for all $n \in \mathbb{N}$,

$$\operatorname{ev}(\alpha_{\ell,d}, n) = \tilde{\operatorname{ev}}(\rho_{d-1}(\alpha_{\ell,d}), n)$$

holds. Take all the depth-$(d-1)$ AP-single chain monomials in (6.75) with non-zero integer exponents. Then they belong to at least one of the single chain AP-extensions of $(\tilde{\mathbb{K}}, \sigma)$ in $(\mathbb{H}_{d-1}, \sigma)$. Suppose there are $e_d \geqslant 1$ of these single chains Π-extensions $(\tilde{\mathbb{K}}_r, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with

$$\tilde{\mathbb{K}}_r = \tilde{\mathbb{K}}\langle \tilde{y}_{r,1}\rangle\langle \tilde{y}_{r,2}\rangle \dots \langle \tilde{y}_{r,d-1}\rangle$$

for $1 \leqslant r \leqslant e_d$ and $\upsilon_d \geqslant 0$ of them that are single chain A-extensions $(\tilde{\mathbb{K}}_b, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with

$$\tilde{\mathbb{K}}_b = \tilde{\mathbb{K}}\langle \vartheta_{b,1}\rangle\langle \vartheta_{b,2}\rangle \dots \langle \vartheta_{b,d-1}\rangle$$

for $1 \leqslant b \leqslant \upsilon_d$. Consider the single chain P-extensions $(\mathbb{K}'_r, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of monomial depth d where

$$\mathbb{K}'_r = \tilde{\mathbb{K}}_r\langle \tilde{y}_{r,d}\rangle = \tilde{\mathbb{K}}\langle \tilde{y}_{r,1}\rangle\langle \tilde{y}_{r,2}\rangle \dots \langle \tilde{y}_{r,d-1}\rangle\langle \tilde{y}_{r,d}\rangle$$

and the single chain A-extensions $(\mathbb{K}'_b, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of monomial depth d where

$$\mathbb{K}'_b = \tilde{\mathbb{K}}_b\langle \vartheta_{b,d}\rangle = \tilde{\mathbb{K}}\langle \vartheta_{b,1}\rangle\langle \vartheta_{b,2}\rangle \dots \langle \vartheta_{b,d-1}\rangle\langle \vartheta_{b,d}\rangle$$

with the respective automorphisms

$$\sigma(\tilde{y}_{r,d}) = \tilde{\alpha}_{r,d}\, \tilde{y}_{r,d} \qquad \text{and} \qquad \sigma(\vartheta_{b,d}) = \gamma_{b,d}\, \vartheta_{b,d} \tag{6.76}$$

where

$$\tilde{\alpha}_{r,d} = \tilde{h}_r\, \tilde{y}_{r,1} \cdots \tilde{y}_{r,d-1} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}\langle \tilde{y}_{r,1}\rangle \dots \langle \tilde{y}_{r,d-1}\rangle}$$

and

$$\gamma_{b,d} = \zeta^{\mu_b}\, \vartheta_{b,1} \cdots \vartheta_{b,d-1} \in \mathbb{U}_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}\langle \vartheta_{b,1}\rangle \dots \langle \vartheta_{b,d-1}\rangle}$$

and the evaluation functions $\tilde{\operatorname{ev}} : \mathbb{K}'_r \times \mathbb{N} \to \tilde{\mathbb{K}}$ and $\tilde{\operatorname{ev}} : \mathbb{K}'_b \times \mathbb{N} \to \tilde{\mathbb{K}}$ defined by

$$\tilde{\operatorname{ev}}(\tilde{y}_{r,d}, n) = \prod_{j=1}^{n} \tilde{\operatorname{ev}}(\tilde{\alpha}_{r,d}, j-1) \qquad \text{and} \qquad \tilde{\operatorname{ev}}(\vartheta_{b,d}, n) = \prod_{j=1}^{n} \tilde{\operatorname{ev}}(\gamma_{b,d}, j-1) \tag{6.77}$$

respectively. By Proposition 6.3.8, each $(\mathbb{K}'_r, \sigma)$ is a Π-extension of $(\tilde{\mathbb{K}}, \sigma)$. Let the difference ring $(\tilde{\mathbb{A}}_d, \sigma)$ with

$$\tilde{\mathbb{A}}_d = \tilde{\mathbb{A}}_{d-1}\langle \tilde{y}_{1,d}\rangle\langle \tilde{y}_{2,d}\rangle \dots \langle \tilde{y}_{e_d,d}\rangle$$

be the ordered multiple chain P-extension of $(\tilde{\mathbb{K}}, \sigma)$ with the automorphism (6.73) and (6.76) and the evaluation function (6.74) and (6.77) where

$$\tilde{\mathbb{A}}_{d-1} = \tilde{\mathbb{K}}\langle \tilde{\boldsymbol{y}}_1\rangle \dots \langle \tilde{\boldsymbol{y}}_{d-1}\rangle = \tilde{\mathbb{K}}\langle \tilde{y}_{1,1}\rangle \dots \langle \tilde{y}_{e_1,1}\rangle \dots \langle \tilde{y}_{1,d}\rangle \dots \langle \tilde{y}_{e_{d-1},d-1}\rangle.$$

Since $\mathcal{M}((\tilde{h}_1, \dots, \tilde{h}_d), \tilde{\mathbb{K}}) = \{0_d\}$, it follows by Theorem 6.4.14 that $(\tilde{\mathbb{A}}_d, \sigma)$ is a Π-extension of $(\tilde{\mathbb{K}}, \sigma)$. In particular, it is an ordered multiple chain Π-extension of monomial depth d by construction. Furthermore, observe that for all $1 \leqslant \ell \leqslant w_d$ we can take

$$g_\ell := c_\ell\, \vartheta_{1,d}^{\mu_{\ell,1,d}} \cdots \vartheta_{v_d,d}^{\mu_{\ell,v_d,d}}\, \tilde{y}_{1,d}^{\nu_{\ell,1,d}} \cdots \tilde{y}_{e_d,d}^{\nu_{\ell,e_d,d}} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\mathbb{H}_d} \tag{6.78}$$

for some $c_\ell \in \tilde{\mathbb{K}}^*$ with

$$\frac{\sigma(g_\ell)}{g_\ell} = \rho_{d-1}(\alpha_{\ell,d}).$$

By iterative application of Lemma 6.5.1, the difference ring homomorphism $\rho_{d-1} : \mathbb{A}_{d-1} \to \mathbb{H}_{d-1}$ can be extended to

$$\rho_d : \mathbb{A}_{d-1}\langle y_{1,d}\rangle\langle y_{2,d}\rangle \cdots \langle y_{w_d,d}\rangle \to \mathbb{H}_{d-1}\langle \vartheta_{1,d}\rangle\langle \vartheta_{2,d}\rangle \cdots \langle \vartheta_{v_d,d}\rangle\langle \tilde{y}_{1,d}\rangle\langle \tilde{y}_{2,d}\rangle \cdots \langle \tilde{y}_{e_d,d}\rangle$$

with

$$\rho_d|_{\mathbb{A}_{d-1}} = \rho_{d-1} \qquad \text{and} \qquad \rho_d(y_{\ell,d}) = g_\ell$$

for $1 \leqslant \ell \leqslant w_d$. Now we show that, for all $f \in \mathbb{A}_d$ and for all $n \in \mathbb{N}$ we can choose $c \in \tilde{\mathbb{K}}^*$ such that $\mathrm{ev}(f, n) = \tilde{\mathrm{ev}}(\rho_d(f), n)$ holds. Note that for all $n \geqslant 1$

$$\mathrm{ev}(y_{\ell,d}, n+1) = \mathrm{ev}(\sigma(y_{\ell,d}), n) = \mathrm{ev}(\alpha_{\ell,d}, n)\ \mathrm{ev}(y_{\ell,d}, n). \tag{6.79}$$

On the other hand, since ρ_d is a difference ring homomorphism, we have that

$$\sigma(\rho_d(y_{\ell,d})) = \rho_d(\sigma(y_{\ell,d})) = \rho_d(\alpha_{\ell,d})\, \rho_d(y_{\ell,d}) = \rho_{d-1}(\alpha_{\ell,d})\, \rho_d(y_{\ell,d})$$

for all $n \geqslant 1$. Applying the evaluation function $\tilde{\mathrm{ev}}$ we get

$$\tilde{\mathrm{ev}}(\rho_d(y_{\ell,d}), n+1) = \tilde{\mathrm{ev}}(\sigma(\rho_d(y_{\ell,d})), n) = \tilde{\mathrm{ev}}(\rho_{d-1}(\alpha_{\ell,d}),\ n)\, \tilde{\mathrm{ev}}(\rho_d(y_{\ell,d}), n). \tag{6.80}$$

By the induction hypothesis, $\mathrm{ev}(\alpha_{\ell,d}, n) = \tilde{\mathrm{ev}}(\rho_{d-1}(\alpha_{\ell,d}),\ n)$ holds for all $n \in \mathbb{N}$. Since $\mathrm{ev}(y_{\ell,d}, n)$ and $\tilde{\mathrm{ev}}(\rho(y_{\ell,d}), n)$ satisfy the same first order recurrence relation, they differ only by a multiplicative constant. Namely, choosing

$$c_\ell = \frac{\mathrm{ev}(y_{\ell,d}, 1)}{\tilde{\mathrm{ev}}(\rho(y_{\ell,d}), 1)}$$

in (6.78) we have that $\mathrm{ev}(y_{\ell,d}, 1) = \tilde{\mathrm{ev}}(g_\ell, 1)$ and thus $\mathrm{ev}(y_{\ell,d}, n) = \tilde{\mathrm{ev}}(g_\ell, n)$ holds for all $n \geqslant 1$. Together with the induction hypothesis

$$\mathrm{ev}(f, n) = \tilde{\mathrm{ev}}(\rho_d(f), n)$$

for all $f \in \mathbb{A}$ and for all $n \geqslant 1$. This completes the proof. ∎

Remark 6.5.3.

(1) The evaluation function (6.74) introduced in the proof of Lemma 6.5.2 is also called the naturally induced evaluation function of $(\mathbb{H}_d, \sigma)$ with (6.67).

(2) The generators of $\mathbb{H}_d$ with (6.67) constructed in Lemma 6.5.2 can be rearranged to get the AΠ-extension $(\tilde{\mathbb{K}}[\vartheta_{1,1}] \dots [\vartheta_{\upsilon_1,1}] \dots [\vartheta_{1,d}] \dots [\vartheta_{\upsilon_d,d}]\langle\tilde{y}_{1,1}\rangle \dots \langle\tilde{y}_{e_1,1}\rangle \dots \langle\tilde{y}_{1,d}\rangle \dots \langle\tilde{y}_{e_d,d}\rangle, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$. Furthermore, a consequence of statement (2) of Lemma 6.5.2 is that the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\psi"] \arrow[d, "\rho"'] & \mathcal{S}(\mathbb{K}) \arrow[d, hook, "\rho'"] \\ \tilde{\mathbb{H}} \arrow[r, "\tilde{\tau}"] & \mathcal{S}(\tilde{\mathbb{K}}) \end{tikzcd}$$

commutes where $\mathbb{A} = \mathbb{A}_d$, $\rho = \rho_d$, $\rho' = \mathrm{id}$ and the difference ring homomorphism $\tilde{\tau}$ and ψ are defined by $\tilde{\tau}(f) = \left\langle \tilde{\mathrm{ev}}(f, n)\right\rangle_{n\geqslant 0}$ and $\psi(g) = \left\langle \mathrm{ev}(g, n)\right\rangle_{n\geqslant 0}$ respectively. ♠

Example 6.5.4.
We will represent the geometric product expression

$$H(n) = \prod_{k=1}^{n} -\kappa^{\frac{\prod_{i=1}^{k} \frac{-(2\,\kappa+3)^2}{\sqrt{2}\,\kappa^3}}{(2\,\kappa+3)^3}} \tag{6.81}$$

of nesting depth 2 in $\mathrm{ProdE}(\mathbb{K})$ with $\mathbb{K} = K(\kappa)$ where $K = \mathbb{Q}(\sqrt{2})$ in a simple RΠ-extension of $(\mathbb{K}, \sigma)$. We first write (6.81) in a product factored form:

$$H_1(n) = \left(\prod_{k=1}^{n} -\kappa\right) \left(\prod_{k=1}^{n} \frac{1}{(2\,\kappa+3)^3}\right) \left(\prod_{k=1}^{n}\prod_{i=1}^{k} \frac{1}{\sqrt{2}\,\kappa^3}\right) \left(\prod_{k=1}^{n}\prod_{i=1}^{k} -(2\,\kappa+3)^2\right) \in \mathrm{Prod}(\mathbb{K}). \tag{6.82}$$

Note that

$$H(n) = H_1(n) \quad \forall n \geqslant 1$$

holds. Let $(\mathbb{A}, \sigma)$ with $\mathbb{A} = \mathbb{K}\langle w_{1,1}\rangle\langle w_{2,1}\rangle\langle w_{3,1}\rangle\langle w_{4,1}\rangle\langle w_{2,2}\rangle\langle w_{4,2}\rangle$ be the ordered multiple chain P-extension of $(\mathbb{K}, \sigma)$ composed by the following single chain Π-extensions.

(1) The single chain Π-extension $(\mathbb{K}_1, \sigma)$ of $(\mathbb{K}, \sigma)$ over $\mathbb{K}$ with base $-\kappa$ where $\mathbb{K}_1 = \mathbb{K}\langle w_{1,1}\rangle$ with the automorphism σ and the evaluation function $\mathrm{ev} : \mathbb{K}_1 \times \mathbb{N} \to \mathbb{K}$ is defined as:

$$\begin{aligned} \sigma(c) &= c,\ \forall\, c \in \mathbb{K}, & \sigma(w_{1,1}) &= -\kappa\, w_{1,1}, \\ \mathrm{ev}(c, n) &= c,\ \forall\, c \in \mathbb{K}, & \mathrm{ev}(w_{1,1}, n) &= \prod_{k=1}^{n} -\kappa. \end{aligned} \tag{6.83}$$

(2) The single chain Π-extension $(\mathbb{K}_2, \sigma)$ of $(\mathbb{K}, \sigma)$ over $\mathbb{K}$ with base $\sqrt{2}\,\kappa^3$ where $\mathbb{K}_2 = \mathbb{K}\langle w_{2,1}\rangle\langle w_{2,2}\rangle$ with the automorphism σ and the evaluation function $\mathrm{ev} : \mathbb{K}_2 \times \mathbb{N} \to \mathbb{K}$ is defined as:

$$\begin{aligned} \sigma(c) &= c,\ \forall\, c \in \mathbb{K}, & \sigma(w_{2,1}) &= \sqrt{2}\,\kappa^3\, w_{2,1}, & \sigma(w_{2,2}) &= \sqrt{2}\,\kappa^3\, w_{2,1}\, w_{2,2}, \\ \mathrm{ev}(c, n) &= c,\ \forall\, c \in \mathbb{K}, & \mathrm{ev}(w_{2,1}, n) &= \prod_{k=1}^{n} \sqrt{2}\,\kappa^3, & \mathrm{ev}(w_{2,2}, n) &= \prod_{k=1}^{n}\prod_{i=1}^{k} \sqrt{2}\,\kappa^3. \end{aligned} \tag{6.84}$$

(3) The single chain Π-extension $(\mathbb{K}_3, \sigma)$ of $(\mathbb{K}, \sigma)$ over $\mathbb{K}$ with base $(2\kappa+3)^3$ where $\mathbb{K}_3 = \mathbb{K}\langle w_{3,1}\rangle$ with the automorphism σ and the evaluation function $\mathrm{ev} : \mathbb{K}_3 \times \mathbb{N} \to \mathbb{K}$ is defined as:

$$\begin{aligned} \sigma(c) &= c, \ \forall c \in \mathbb{K}, & \sigma(w_{3,1}) &= (2\kappa+3)^3\, w_{3,1}, \\ \mathrm{ev}(c, n) &= c, \ \forall c \in \mathbb{K}, & \mathrm{ev}(w_{3,1}, n) &= \prod_{k=1}^{n} (2\kappa+3)^3. \end{aligned} \tag{6.85}$$

(4) The single chain Π-extension $(\mathbb{K}_4, \sigma)$ of $(\mathbb{K}, \sigma)$ over $\mathbb{K}$ with $-(2\kappa+3)^2$ as its base where $\mathbb{K}_4 = \mathbb{K}\langle w_{4,1}\rangle\langle w_{4,2}\rangle$ with the automorphism σ and the evaluation function $\mathrm{ev} : \mathbb{K}_4 \times \mathbb{N} \to \mathbb{K}$ is defined as:

$$\begin{aligned} \sigma(c) &= c, \ \forall c \in \mathbb{K}, & \sigma(w_{4,1}) &= -(2\kappa+3)^2\, w_{4,1}, & \sigma(w_{4,2}) &= -(2\kappa+3)^2\, w_{4,1}\, w_{4,2}, \\ \mathrm{ev}(c, n) &= c, \ \forall c \in \mathbb{K}, & \mathrm{ev}(w_{4,1}, n) &= \prod_{k=1}^{n} -(2\kappa+3)^2, & \mathrm{ev}(w_{4,2}, n) &= \prod_{k=1}^{n}\prod_{i=1}^{k} -(2\kappa+3)^2. \end{aligned} \tag{6.86}$$

Now we merge the single chain Π-extensions to an ordered multiple chain P-extension yielding the tower of ring extensions $\mathbb{K} \leqslant \mathbb{A}_1 \leqslant \mathbb{A}_2$ where $\mathbb{A}_1 = \mathbb{K}\langle w_{1,1}\rangle\langle w_{2,1}\rangle\langle w_{3,1}\rangle\langle w_{4,1}\rangle$ and $\mathbb{A}_2 = \mathbb{A}_1\langle w_{1,2}\rangle\langle w_{4,2}\rangle$. Then the product expression (6.82) is modelled by

$$H_1 = \frac{w_{1,1}\, w_{4,2}}{w_{3,1} w_{2,2}} \in \mathbb{A}_2, \tag{6.87}$$

that is,

$$\mathrm{ev}(H_1, n) = H_1(n) \quad \forall n \geqslant 1$$

holds. We follow the proof of Lemma 6.5.2 to construct an ordered multiple chain AΠ-extension in which (6.81) can be modelled.

(1) Take the shift quotient of all depth-1 P-monomials $w_{1,1}$, $w_{2,1}$, $w_{3,1}$ and $w_{4,1}$ in the ordered multiple chain P-extension $(\mathbb{A}_1, \sigma)$ of $(\mathbb{K}, \sigma)$ of monomial depth 1 where $\mathbb{A}_1 = \mathbb{K}\langle w_{1,1}\rangle\langle w_{2,1}\rangle\langle w_{3,1}\rangle\langle w_{4,1}\rangle$, with

$$\frac{\sigma(w_{1,1})}{w_{1,1}} = -\kappa, \qquad \frac{\sigma(w_{2,1})}{w_{2,1}} = \sqrt{2}\,\kappa^3, \qquad \frac{\sigma(w_{3,1})}{w_{3,1}} = (2\kappa+3)^3, \qquad \frac{\sigma(w_{4,1})}{w_{4,1}} = -(2\kappa+3)^2$$

in $\mathbb{K}^*$. Observe that there are monic irreducible pairwise distinct polynomials κ, $(\kappa+\frac{3}{2}) \in \mathbb{K}\setminus K$ and the algebraic numbers (-1), $\sqrt{2} \in K$ such that the elements in $\mathbb{K}^*$ can be written as a power product of (-1), $\sqrt{2}$, κ and $(\kappa+\frac{3}{2})$; see (5.13) of Theorem 5.2.1. Solving Problem GO with $(\sqrt{2}, \kappa, \kappa+\frac{3}{2})$ as input we have that $\boldsymbol{\mathcal{M}}((\sqrt{2}, \kappa, \kappa+\frac{3}{2}), \mathbb{K}) = \{0_3\}$. Let $(\tilde{\mathbb{A}}_1, \sigma)$ with

$$\tilde{\mathbb{A}}_1 = \mathbb{K}\langle \boldsymbol{y}_1\rangle = \mathbb{K}\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle$$

be a P-extension of $(\mathbb{K}, \sigma)$ with

$$\begin{aligned} \sigma(c) &= c, \ \forall c \in \mathbb{K}, & \sigma(y_{1,1}) &= \sqrt{2}\, y_{1,1}, & \sigma(y_{2,1}) &= \kappa\, y_{2,1}, & \sigma(y_{3,1}) &= (\kappa+\tfrac{3}{2})\, y_{3,1}, \\ \mathrm{ev}(c, n) &= c, \ \forall c \in \mathbb{K}, & \tilde{\mathrm{ev}}(y_{1,1}, n) &= \prod_{k=1}^{n} \sqrt{2}, & \tilde{\mathrm{ev}}(y_{2,1}, n) &= \prod_{k=1}^{n} \kappa, & \tilde{\mathrm{ev}}(y_{3,1}, n) &= \prod_{k=1}^{n} (\kappa+\tfrac{3}{2}). \end{aligned} \tag{6.88}$$

Since $\boldsymbol{\mathcal{M}}((\sqrt{2}, \kappa, \kappa+\frac{3}{2}), \mathbb{K}) = \{0_3\}$ it follows by Lemma 5.0.2 that $(\tilde{\mathbb{A}}_1, \sigma)$ is a Π-extension of $(\mathbb{K}, \sigma)$. In particular it is an ordered multiple chain of monomial depth 1 composed by the single chains Π-extensions $(\mathbb{K}\langle y_{1,1}\rangle, \sigma)$, $(\mathbb{K}\langle y_{2,1}\rangle, \sigma)$ and $(\mathbb{K}\langle y_{3,1}\rangle, \sigma)$ of monomial depth 1. Furthermore, the difference ring extension $(\tilde{\mathbb{A}}_1[\vartheta_1], \sigma)$ of $(\tilde{\mathbb{A}}_1, \sigma)$ with

$$\sigma(\vartheta_1) = -\vartheta_1 \qquad \text{and} \qquad \tilde{\mathrm{ev}}(\vartheta_1, n) = \prod_{k=1}^{n} (-1) \tag{6.89}$$

is an A-extension of order 2. In particular, by Lemma 2.3.57 it is also an R-extension. The depth-1 P-monomials $w_{1,1}$, $w_{2,1}$, $w_{3,1}$ and $w_{4,1}$ can be written in terms of the depth-1 AΠ-monomials ϑ_1, $y_{1,1}$, $y_{2,1}$ and $y_{3,1}$ via the difference ring homomorphism

$$\rho_1 : \mathbb{A}_1 \to \mathbb{H}_1$$

with

$$\begin{aligned} \rho_1(w_{1,1}) &= \vartheta_1\, y_{2,1}, & \rho_1(w_{2,1}) &= y_{1,1}\, y_{2,1}^3, \\ \rho_1(w_{3,1}) &= y_{1,1}^6\, y_{3,1}^3, & \rho_1(w_{4,1}) &= \vartheta_1\, y_{1,1}^4\, y_{3,1}^2 \end{aligned} \tag{6.90}$$

where $\mathbb{H}_1 = \mathbb{K}\langle\vartheta_1\rangle\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle$.

(2) Take the shift quotient of all depth-2 simple P-monomials: $w_{2,2}$, $w_{4,2}$, in the ordered multiple chain P-extension $(\mathbb{A}_2, \sigma)$ of $(\mathbb{K}, \sigma)$ of monomial depth at most 2 with $\mathbb{A}_2 = \mathbb{A}_1\langle w_{2,2}\rangle\langle w_{4,2}\rangle$, that is,

$$\frac{\sigma(w_{2,2})}{w_{2,2}} = \sqrt{2}\,\kappa^3\, w_{2,1} \qquad \text{and} \qquad \frac{\sigma(w_{4,2})}{w_{4,2}} = -(2\,\kappa+3)^2 w_{4,1}$$

in the product group $(\mathbb{K}^*)_{\mathbb{K}}^{\mathbb{A}_1}$. Applying ρ_1 to these elements we get

$$\sqrt{2}\,\kappa^3\, y_{1,1}\, y_{2,1}^3,\ -(2\,\kappa+3)^2\, \vartheta_1^2\, y_{1,1}^4\, y_{3,1}^2 \in (\mathbb{K}^*)_{\mathbb{K}}^{\mathbb{H}_1}. \tag{6.91}$$

Since the depth-1 single chain AP-monomials ϑ_1, $y_{1,1}$, $y_{2,1}$, $y_{3,1}$ have non-zero integer exponents in (6.91), we extend the single chain AP-extension of $(\mathbb{K}, \sigma)$ that each AP-monomial belongs to. More precisely, consider the following single chain A-/P-extensions: $(\mathbb{K}[\vartheta_1][\vartheta_2], \sigma)$, $(\mathbb{K}\langle y_{1,1}\rangle\langle y_{1,2}\rangle, \sigma)$, $(\mathbb{K}\langle y_{2,1}\rangle\langle y_{2,2}\rangle, \sigma)$ and $(\mathbb{K}\langle y_{3,1}\rangle\langle y_{3,2}\rangle, \sigma)$ of $(\mathbb{K}, \sigma)$ with the automorphism and evaluation function

$$\sigma(\vartheta_2) = -\vartheta_1\, \vartheta_2, \qquad \mathrm{ev}(\vartheta_2, n) = \prod_{k=1}^{n}\prod_{i=1}^{k}(-1) \tag{6.92}$$

for the A-extension and

$$\begin{aligned} \sigma(y_{1,2}) &= \sqrt{2}\, y_{1,1}\, y_{1,2}, & \sigma(y_{2,2}) &= \kappa\, y_{2,1}\, y_{2,2}, & \sigma(y_{3,2}) &= (\kappa + \tfrac{3}{2})\, y_{3,1}\, y_{3,2}, \\ \tilde{\mathrm{ev}}(y_{1,2}, n) &= \prod_{k=1}^{n}\prod_{i=1}^{k}\sqrt{2}, & \tilde{\mathrm{ev}}(y_{2,2}, n) &= \prod_{k=1}^{n}\prod_{i=1}^{k}\kappa, & \tilde{\mathrm{ev}}(y_{3,2}, n) &= \prod_{k=1}^{n}\prod_{i=1}^{k}(\kappa + \tfrac{3}{2}) \end{aligned} \tag{6.93}$$

for the single chain P-extensions respectively. Consider the ordered multiple chain P-extension $(\tilde{\mathbb{A}}_2, \sigma)$ of $(\mathbb{K}, \sigma)$ with (6.88) and (6.93) where

$$\tilde{\mathbb{A}}_2 = \tilde{\mathbb{A}}_1\langle \boldsymbol{y}_2\rangle = \mathbb{K}\langle \boldsymbol{y}_1\rangle\langle \boldsymbol{y}_2\rangle = \mathbb{K}\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle.$$

Since $\boldsymbol{\mathcal{M}}((\sqrt{2}, \kappa, \kappa + \frac{3}{2}), \mathbb{K}) = \{0_3\}$, it follows by Theorem 6.4.14 that $(\tilde{\mathbb{A}}_2, \sigma)$ is a Π-extension of $(\mathbb{K}, \sigma)$ and consequently, a Π-extension of $(\tilde{\mathbb{A}}_1, \sigma)$. By construction, it is an ordered multiple chain Π-extension of $(\mathbb{K}, \sigma)$ of monomial depth 2. Observe that the difference ring extension $(\mathbb{H}_2, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ where $\mathbb{H}_2 = \mathbb{H}_1\langle\vartheta_2\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle$ with the automorphism (6.88), (6.89), (6.92) and (6.93) is an AP-extension of $(\mathbb{K}, \sigma)$. Now we claim that for the product group elements in (6.91), there are g_1, $g_2 \in (\mathbb{K}^*)_{\mathbb{K}}^{\mathbb{H}_2}$ such that

$$\frac{\sigma(g_1)}{g_1} = \sqrt{2}\,\kappa^3\, y_{1,1}\, y_{2,1}^3 \qquad\qquad \frac{\sigma(g_2)}{g_2} = -(2\,\kappa+3)^3\, \vartheta_1\, y_{1,1}^4\, y_{3,1}^2. \tag{6.94}$$

Looking at $\rho_1(w_{2,1}) = y_{1,1}\, y_{2,1}^3$ and $\rho_1(w_{4,1}) = \vartheta_1\, y_{1,1}^4\, y_{3,1}^2$ in (6.90), we can choose

$$g_1 = y_{1,2}\, y_{2,2}^3 \qquad \text{and} \qquad g_2 = \vartheta_2\, y_{1,2}^4\, y_{3,2}^2$$

and (6.94) holds. Thus we can define the difference ring homomorphism

$$\rho_2 : \mathbb{A}_2 \to \mathbb{H}_2$$

with $\mathbb{H}_2 = \mathbb{H}_1\langle\vartheta_2\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle$ by:

$$\rho_2|_{\mathbb{A}_1} = \rho_1, \qquad \rho_2(w_{2,2}) = y_{1,2}\, y_{2,2}^3, \qquad \rho_2(w_{4,2}) = \vartheta_2\, y_{1,2}^4\, y_{3,2}^2.$$

By Remark 6.5.3 the generators in $\mathbb{H}_2$ can be rearranged to get the $\mathrm{A}\Pi$-extension $(\tilde{\mathbb{H}}, \sigma)$ with

$$\tilde{\mathbb{H}} = \mathbb{K}[\vartheta_1][\vartheta_2]\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle$$

of $(\mathbb{K}, \sigma)$. Applying ρ_2 to H_1 in (6.87) we get

$$\tilde{H} := \rho_2(H_1) = \frac{\rho_2(w_{1,1})\, \rho_2(w_{4,2})}{\rho_2(w_{3,1})\, \rho_2(w_{2,2})} = \frac{\vartheta_1\, \vartheta_2\, y_{2,1}\, y_{1,2}^3\, y_{3,2}^2}{y_{1,1}^6\, y_{3,1}^3\, y_{2,2}^3} \in \tilde{\mathbb{H}}. \tag{6.95}$$

In particular, by part (2) of Lemma 6.5.2,

$$\mathrm{ev}(f, n) = \tilde{\mathrm{ev}}(\rho_2(f), n)$$

for all $f \in \mathbb{A}$ and for all $n \geqslant 1$. Thus (6.95) models (6.81). That is, with

$$\tilde{H}(n) := \tilde{\mathrm{ev}}(\tilde{H}, n) = \frac{(-1)^n\, (-1)^{\binom{n+1}{2}}\, \kappa^n \left(\left(\sqrt{2}\right)^{\binom{n+1}{2}}\right)^3 \left(\left(\kappa + \frac{3}{2}\right)^{\binom{n+1}{2}}\right)^2}{\left(\left(\sqrt{2}\right)^n\right)^6 \left(\left(\kappa + \frac{3}{2}\right)^n\right)^3 \left(\kappa^{\binom{n+1}{2}}\right)^3} \tag{6.96}$$

we have that

$$H(n) = \tilde{H}(n) \quad \forall\, n \in \mathbb{N}$$

holds.

By Remark 6.5.3 the diagram:

$$\begin{array}{ccc}
\mathbb{A} := \mathbb{K}\langle w_{1,1}\rangle\langle w_{2,1}\rangle\langle w_{3,1}\rangle\langle w_{4,1}\rangle\langle w_{2,2}\rangle\langle w_{4,2}\rangle & \xrightarrow{\ \psi\ } & \mathcal{S}(\mathbb{K}) \\
{\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\rho'} \\
\tilde{\mathbb{H}} := \mathbb{K}[\vartheta_1][\vartheta_2]\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle & \xrightarrow{\ \tilde{\tau}\ } & \mathcal{S}(\mathbb{K}).
\end{array}$$

commutes where $\rho = \rho_2$, $\tilde{\tau}(f) = \left\langle \tilde{\mathrm{ev}}(f, n)\right\rangle_{n \geqslant 0}$, $\psi(g) = \left\langle \mathrm{ev}(g, n)\right\rangle_{n \geqslant 0}$ and $\rho' = \mathrm{id}$. $\star$

Lemma 6.5.5.
Let $(\mathbb{H}_d, \sigma)$ with $\mathbb{H}_d = \tilde{\mathbb{K}}\langle\vartheta_{1,1}\rangle \dots \langle\vartheta_{\upsilon_1,1}\rangle\langle\tilde{y}_{1,1}\rangle \dots \langle\tilde{y}_{e_1,1}\rangle \dots \langle\vartheta_{1,d}\rangle \dots \langle\vartheta_{\upsilon_d,d}\rangle\langle\tilde{y}_{1,d}\rangle \dots \langle\tilde{y}_{e_d,d}\rangle$ be the ordered $\mathrm{A}\Pi$-extension of $(\tilde{\mathbb{K}}, \sigma)$ with the automorphism defined in (6.68) and (6.69) and the evaluation function defined in (6.70), that satisfy properties (1) and (2) of Lemma 6.5.2 where $\tilde{\mathbb{K}} = \tilde{K}(\kappa_1, \dots, \kappa_u)$. Then there is an R-extension $(\tilde{\mathbb{D}}[\vartheta], \sigma)$ of $(\tilde{\mathbb{D}}, \sigma)$ where

$$\tilde{\mathbb{D}} = \mathbb{K}'\langle\tilde{y}_{1,1}\rangle \dots \langle\tilde{y}_{e_1,1}\rangle \dots \langle\tilde{y}_{1,d}\rangle \dots \langle\tilde{y}_{e_d,d}\rangle \tag{6.97}$$

with the automorphism $\sigma(\vartheta)=\zeta'\,\vartheta$, of order λ'. Here, $\mathbb{K}'=\mathrm{K}'(\kappa_1,\dots,\kappa_u)$, ζ' is a primitive λ'-th root of unity in K' and K' is a finite algebraic field extension of $\tilde{\mathrm{K}}$. Furthermore, the difference ring $(\mathbb{D},\sigma)$ with

$$\mathbb{D}=\mathbb{K}'[\vartheta]\langle\tilde{y}_{1,1}\rangle\dots\langle\tilde{y}_{e_1,1}\rangle\dots\langle\tilde{y}_{1,d}\rangle\dots\langle\tilde{y}_{e_d,d}\rangle \tag{6.98}$$

is an $\mathrm{R}\Pi$-extension of $(\mathbb{K}',\sigma)$ and the ring $\mathbb{D}$ can be written as the direct sum

$$\mathbb{D}=\boldsymbol{e}_0\tilde{\mathbb{D}}\oplus\boldsymbol{e}_1\tilde{\mathbb{D}}\oplus\dots\oplus\boldsymbol{e}_{\lambda'-1}\tilde{\mathbb{D}} \tag{6.99}$$

where $\boldsymbol{e}_0,\dots,\boldsymbol{e}_{\lambda'-1}$ are the idempotent, pairwise orthogonal elements in (6.26) that sum up to one. ♦

PROOF:
Note that by Remark 6.5.3 the generators in $\mathbb{H}_d$ can be rearranged to get the $\mathrm{A}\Pi$-extension $(\tilde{\mathbb{H}},\sigma)$ of $(\tilde{\mathbb{K}},\sigma)$ where

$$\tilde{\mathbb{H}}=\tilde{\mathbb{K}}[\vartheta_{1,1}]\dots[\vartheta_{\upsilon_1,1}]\dots[\vartheta_{1,d}]\dots[\vartheta_{\upsilon_d,d}]\langle\tilde{y}_{1,1}\rangle\dots\langle\tilde{y}_{e_1,1}\rangle\dots\langle\tilde{y}_{1,d}\rangle\dots\langle\tilde{y}_{e_d,d}\rangle \tag{6.100}$$

with the automorphism (6.68) and (6.69) and the evaluation function and (6.70) satisfying properties (1) and (2) of Lemma 6.5.2. Consider the sub-difference ring $(\tilde{\mathbb{K}}[\vartheta_{1,1}]\dots[\vartheta_{\upsilon_1,1}]\dots[\vartheta_{1,d}]\dots[\vartheta_{\upsilon_d,d}],\sigma)$ of $(\tilde{\mathbb{H}},\sigma)$ which is a difference ring extension of $(\tilde{\mathbb{K}},\sigma)$, with the automorphism defined by

$$\sigma(\vartheta_{\ell,k})=\gamma_{\ell,k}\,\vartheta_{\ell,k}\quad\text{ where }\quad\gamma_{\ell,k}=\zeta^{\mu_\ell}\,\vartheta_{\ell,1}\cdots\vartheta_{\ell,k-1}\in\mathbb{U}_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}[\vartheta_{\ell,1}]\dots[\vartheta_{\ell,k-1}]} \tag{6.101}$$

for $1\leqslant k\leqslant d$ and for $1\leqslant\ell\leqslant\upsilon_k$ where $\mathbb{U}=\langle\zeta\rangle$ is the multiplicative cyclic subgroup of $\tilde{\mathrm{K}}$ generated by a primitive λ-th root of unity, $\zeta\in\tilde{\mathrm{K}}^*$. Observe that the difference ring extension $(\mathbb{G},\sigma)$ of $(\tilde{\mathbb{K}},\sigma)$ where $\mathbb{G}=\tilde{\mathbb{K}}[\vartheta_{1,1}]\dots[\vartheta_{\upsilon_1,1}]\dots[\vartheta_{1,d}]\dots[\vartheta_{\upsilon_d,d}]$ with (6.101) is a simple A-extension to which statement (1) of Theorem 6.2.28 can be applied. Thus there is an R-extension $(\mathbb{K}'[\vartheta],\sigma)$ of $(\mathbb{K}',\sigma)$ with

$$\sigma(\vartheta)=\zeta'\,\vartheta \tag{6.102}$$

of order λ' where $\mathbb{K}'=\mathrm{K}'(\kappa_1,\dots,\kappa_u)$, ζ' is a primitive λ'-th root of unity in K' and K' is a finite algebraic field extension of $\tilde{\mathrm{K}}$. Note that the difference ring $(\tilde{\mathbb{D}},\sigma)$ where $\tilde{\mathbb{D}}$ is given by (6.97) with the automorphism defined by (6.69) is a Π-extension of $(\mathbb{K}',\sigma)$. Thus by Corollary 2.3.58 it follows that the A-extension $(\tilde{\mathbb{D}}[\vartheta],\sigma)$ of $(\tilde{\mathbb{D}},\sigma)$ with (6.102) of order λ' is an R-extension. By Remark 5.4.5, the difference ring $(\mathbb{D},\sigma)$ where $\mathbb{D}=\mathbb{K}'[\vartheta]\langle\tilde{y}_{1,1}\rangle\dots\langle\tilde{y}_{e_1,1}\rangle\dots\langle\tilde{y}_{1,d}\rangle\dots\langle\tilde{y}_{e_d,d}\rangle$ with the automorphism (6.102) and (6.69) is an $\mathrm{R}\Pi$-extension of $(\mathbb{K}',\sigma)$. By statement (1) of Theorem 6.2.25

$$\mathbb{D}=\boldsymbol{e}_0\mathbb{D}\oplus\dots\oplus\boldsymbol{e}_{\lambda'-1}\mathbb{D}$$

and by stetement (2) of the same Theorem, $\boldsymbol{e}_k\mathbb{D}=\boldsymbol{e}_k\tilde{\mathbb{D}}$ for $0\leqslant k<\lambda'$. Thus (6.99) holds. ■

Example 6.5.6 (Cont. Example 6.5.4).
Take the $\mathrm{A}\Pi$-extension $(\tilde{\mathbb{H}},\sigma)$ of $(\mathbb{K},\sigma)$ where $\tilde{\mathbb{H}}=\mathbb{K}[\vartheta_1][\vartheta_2]\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle$ constructed in Example 6.5.4 with the automorphism (6.88), (6.89), (6.92), (6.93) and consider the sub-difference ring $(\mathbb{K}[\vartheta_1][\vartheta_2],\sigma)$ of $(\tilde{\mathbb{H}},\sigma)$ with

$$\sigma(\vartheta_1)=-\vartheta_1\quad\text{ and }\quad\sigma(\vartheta_2)=-\vartheta_1\,\vartheta_2$$

which is a simple A-extension of $(\mathbb{K}, \sigma)$ where $\mathbb{K} = \mathbb{Q}(\sqrt{2})$. Then by statement (1) of Theorem 6.2.28 there is an R-extension $(\mathbb{K}'[\vartheta], \sigma)$ of $(\mathbb{K}', \sigma)$ of order 4 with

$$\sigma(\vartheta) = \mathbb{i}\,\vartheta \tag{6.103}$$

where $\mathbb{K}' = \mathbb{Q}(\mathbb{i}, \sqrt{2})$. Furthermore, $(\mathbb{D}, \sigma)$ where $\mathbb{D} = \mathbb{K}'[\vartheta]\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle$ with the automorphism (6.103) and

$$\begin{aligned} &\sigma(c) = c,\ \forall\, c \in \mathbb{K}, && \sigma(y_{1,1}) = \sqrt{2}\, y_{1,1}, && \sigma(y_{2,1}) = \kappa\, y_{2,1},\\ &\sigma(y_{3,1}) = (\kappa + \tfrac{3}{2})\, y_{3,1}, && \sigma(y_{1,2}) = \sqrt{2}\, y_{1,1}\, y_{1,2}, && \sigma(y_{2,2}) = \kappa\, y_{2,1}\, y_{2,2},\\ &\sigma(y_{3,2}) = (\kappa + \tfrac{3}{2})\, y_{3,1}\, y_{3,2}. \end{aligned} \tag{6.104}$$

is an $\mathrm{R\Pi}$-extension of $(\mathbb{K}', \sigma)$. With $\tilde{\mathbb{D}} = \mathbb{K}'\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle$, the ring $\mathbb{D}$ can be written as the direct sum

$$\mathbb{D} = \boldsymbol{e}_0\tilde{\mathbb{D}} \oplus \boldsymbol{e}_1\tilde{\mathbb{D}} \oplus \boldsymbol{e}_2\tilde{\mathbb{D}} \oplus \boldsymbol{e}_3\tilde{\mathbb{D}}$$

where the idempotent elements $\boldsymbol{e}_k$ for $0 \leqslant k \leqslant 3$ are defined by (6.52). ⋆

Lemma 6.5.7.
Let $(\tilde{\mathbb{H}}, \sigma)$ with (6.100) be the ordered AP*-extension of $(\tilde{\mathbb{K}}, \sigma)$ equipped with the automorphism defined by (6.68) and (6.69) and the evaluation function (6.70) satisfying properties (1) and (2) of Lemma 6.5.2 and let $(\mathbb{D}, \sigma)$ with (6.98) be the single $\mathrm{R\Pi}$-extension of $(\mathbb{K}', \sigma)$ with the automorphism (6.102) and (6.69) and the evaluation function $\tilde{\mathrm{ev}} : \mathbb{D} \times \mathbb{N} \to \mathbb{K}'$ defined by* [4]

$$\tilde{\mathrm{ev}}(\vartheta, n) = \prod_{k=1}^{n} \zeta' \qquad \textit{and} \qquad \tilde{\mathrm{ev}}(\tilde{y}_{\ell,d}, n) = \prod_{j=1}^{n} \tilde{\mathrm{ev}}(\tilde{\alpha}_{\ell,d}, j-1) \tag{6.105}$$

that satisfies Lemma 6.5.5. Then the map $\phi : \tilde{\mathbb{H}} \to \mathbb{D}$ defined as

$$\phi(\tilde{y}_{\ell,k}) = \tilde{y}_{\ell,k} \tag{6.106}$$

$$\phi(\vartheta_{\ell,k}) = \beta_{\ell,k,0}\boldsymbol{e}_0 + \cdots + \beta_{\ell,k,\lambda'-1}\boldsymbol{e}_{\lambda'-1} \tag{6.107}$$

where $\beta_{\ell,k,i} = \mathrm{ev}(\vartheta_{\ell,k}, \lambda' - 1 - i)$ for $0 \leqslant i < \lambda'$ is a difference ring homomorphism. Furthermore, for all $f \in \tilde{\mathbb{H}}$ and for all $n \in \mathbb{N}$, $\mathrm{ev}(f, n) = \mathrm{ev}(\phi(f), n)$ holds. ♦

Proof:
We show that $\phi : \tilde{\mathbb{H}} \to \mathbb{D}$ defined as (6.106) and (6.107) is a difference ring homomorphism. By statement (1) of Theorem 6.2.28, $\phi|_{\mathbb{G}}$ where $\mathbb{G} = \tilde{\mathbb{K}}[\vartheta_{1,1}]\dots[\vartheta_{\upsilon_1,1}]\dots[\vartheta_{1,d}]\dots[\vartheta_{\upsilon_d,d}]$ which is defined by (6.107) is a difference ring homomorphism. Since ϕ maps $\tilde{y}_{\ell,k}$ to itself, ϕ itself is a difference ring homomorphism. Furthermore, for all $f \in \tilde{\mathbb{H}}$ and for all $n \in \mathbb{N}$, we have that $\mathrm{ev}(f, n) = \mathrm{ev}(\phi(f), n)$ holds. ■

[4]Note that for all $c \in \mathbb{K}'$, $\tilde{\mathrm{ev}}(c, n) = c$ for all $n \geqslant 0$.

Remark 6.5.8.
As a consequence of Lemma 6.5.7, the diagram

$$\begin{tikzcd} \tilde{\mathbb{H}} \arrow[rr, "\tilde{\tau}"] \arrow[d, "\phi"'] & & \mathcal{S}(\tilde{\mathbb{K}}) \arrow[d, "\phi'"] \\ \mathbb{D} \simeq \boldsymbol{e}_0\tilde{\mathbb{D}} \oplus \cdots \oplus \boldsymbol{e}_{\lambda-1}\tilde{\mathbb{D}} \arrow[rr, "\tau"] & & \mathcal{S}(\mathbb{K}') \end{tikzcd}$$

with $\tau(f) = \langle \mathsf{ev}(\phi(f), n)\rangle_{n\geqslant 0}$, ϕ defined by (6.106) and (6.107), $\tilde{\tau}(f) = \langle \mathsf{ev}(f, n)\rangle_{n\geqslant 0}$ and $\phi' = \mathrm{id}$, commutes. ♠

Example 6.5.9 (Cont. Example 6.5.4 and 6.5.6).
Consider the AΠ-extension $(\tilde{\mathbb{H}}, \sigma)$ of $(\mathbb{K}, \sigma)$ constructed in Example 6.5.4 and the RΠ-extension $(\mathbb{D}, \sigma)$ of $(\mathbb{K}', \sigma)$ constructed in Example 6.5.6 with the automorphism σ and the evaluation function $\mathsf{ev} : \mathbb{D}\times\mathbb{N} \to \mathbb{K}'$ defined by (6.88), (6.93), (6.103) and

$$\tilde{\mathsf{ev}}(\vartheta, n) = \prod_{k=1}^{n} \mathbb{i}. \tag{6.108}$$

Then the map

$$\begin{aligned} \phi : \tilde{\mathbb{H}} &\to \mathbb{D} \\ y_{\ell,k} &\mapsto y_{\ell,k} \\ \vartheta_i &\mapsto \beta_{i,0}\boldsymbol{e}_0 + \beta_{i,1}\boldsymbol{e}_1 + \beta_{i,2}\boldsymbol{e}_2 + \beta_{i,3}\boldsymbol{e}_3 \end{aligned}$$

with (6.52) where $\beta_{i,j} = \mathsf{ev}(\vartheta_i, 3-j)$ for $i \in \{1, 2\}$ and $0 \leqslant j \leqslant 3$ is a difference ring homomorphism. More precisely, for the A-monomials we have that

$$\begin{aligned} \phi(\vartheta_1) &= -\boldsymbol{e}_0 + \boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3 = \vartheta^2; \\ \phi(\vartheta_2) &= -\boldsymbol{e}_0 - \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 = \frac{(1-\mathbb{i})}{2}\,\vartheta\,(\vartheta^2 + \mathbb{i}). \end{aligned}$$

Applying ϕ to the expression (6.95) in $\tilde{\mathbb{H}}$ we have that

$$\hat{H} = \frac{\phi(\vartheta_1)\,\phi(\vartheta_2)\,\phi(y_{2,1})\,\phi(y_{1,2}^3)\,\phi(y_{3,2}^2)}{\phi(y_{1,1}^6)\,\phi(y_{3,1}^3)\,\phi(y_{2,2}^3)} = \frac{(1-\mathbb{i})\,\vartheta\,(\vartheta^2+\mathbb{i})\,y_{2,1}\,y_{1,2}^3\,y_{3,2}^2}{2\,y_{1,1}^6\,y_{3,1}^3\,y_{2,2}^3} \in \mathbb{E}$$

which models (6.96). That is, with

$$\hat{H}(n) := \mathsf{ev}(\hat{H}, n) = \frac{(1-\mathbb{i})\,(\mathbb{i})^n\,((\mathbb{i}^n)^2+\mathbb{i})\;\kappa^n\left(\left(\sqrt{2}\right)^{\binom{n+1}{2}}\right)^3\left(\left(\kappa+\frac{3}{2}\right)^{\binom{n+1}{2}}\right)^2}{2\left(\left(\sqrt{2}\right)^n\right)^6\left(\left(\kappa+\frac{3}{2}\right)^n\right)^3\left(\kappa^{\binom{n+1}{2}}\right)^3}$$

we have that

$$\tilde{H}(n) = \hat{H}(n) \quad \forall\, n \in \mathbb{N}$$

holds. In addition, by Remark 6.5.8 the diagram

$$\begin{tikzcd} \tilde{\mathbb{H}} := \mathbb{K}[\vartheta_1][\vartheta_2]\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle \arrow[rr, "\tilde{\tau}"] \arrow[d, "\phi"'] & & \mathcal{S}(\mathbb{K}) \arrow[d, "\phi'"] \\ \mathbb{D} := \mathbb{K}'[\vartheta]\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle \simeq \boldsymbol{e}_0\tilde{\mathbb{D}} \oplus \boldsymbol{e}_1\tilde{\mathbb{D}} \oplus \boldsymbol{e}_2\tilde{\mathbb{D}} \oplus \boldsymbol{e}_3\tilde{\mathbb{D}} \arrow[rr, "\tau"] & & \mathcal{S}(\mathbb{K}') \end{tikzcd}$$

commutes where $\tilde{\mathbb{D}} = \mathbb{K}'\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{3,2}\rangle$. Putting Example 6.5.4 and Example 6.5.6 together we have that

$$H(n) = \tilde{H}(n) = \hat{H}(n) \quad \forall n \in \mathbb{N}$$

holds and the following diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\rho"] \arrow[d, "\psi"'] & \tilde{\mathbb{H}} \arrow[r, "\phi"] \arrow[d, "\tilde{\tau}"] & \mathbb{D} \arrow[d, "\tau"] \\
\mathcal{S}(\mathbb{K}) \arrow[r, hook, "\rho'"] & \mathcal{S}(\mathbb{K}) \arrow[r, hook, "\phi'"] & \mathcal{S}(\mathbb{K}').
\end{tikzcd}$$

commutes. ⋆

Summarising we have the following theorem.

Theorem 6.5.10.
For $1 \leqslant \ell \leqslant m$, let $(\mathbb{K}_\ell, \sigma)$ with $\mathbb{K}_\ell = \mathbb{K}\langle y_{\ell,1}\rangle \dots \langle y_{\ell,s_\ell}\rangle$ be the single chain Π-extensions of $(\mathbb{K}, \sigma)$ over $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ with base $h_\ell \in \mathbb{K}^*$ for $1 \leqslant \ell \leqslant m$, the automorphisms (6.61) and the naturally induced evaluation functions (6.62). Let $d := \max(s_1, \dots, s_m)$ and $\mathbb{A}_0 = \mathbb{K}$. Consider the tower of difference ring extensions $(\mathbb{A}_i, \sigma)$ of $(\mathbb{A}_{i-1}, \sigma)$ where

$$\mathbb{A}_i = \mathbb{A}_{i-1}\langle y_{1,i}\rangle\langle y_{2,i}\rangle \dots \langle y_{w_i,i}\rangle$$

for $1 \leqslant i \leqslant d$ with $m = w_1 \geqslant w_2 \geqslant \dots \geqslant w_d$ and the automorphism (6.63) and the evaluation function (6.62). This yields $(\mathbb{A}_d, \sigma)$ as an ordered multiple chain P-extension of $(\mathbb{K}, \sigma)$ of monomial depth at most d composed by the single chain Π-extensions $(\mathbb{K}_\ell, \sigma)$ of $(\mathbb{K}, \sigma)$ for $1 \leqslant \ell \leqslant m$ with (6.61) and (6.62). Then one can construct

(1) an $R\Pi$-extension $(\mathbb{D}, \sigma)$ of $(\mathbb{K}', \sigma)$ with

$$\mathbb{D} = \mathbb{K}'[\vartheta]\langle \tilde{y}_{1,1}\rangle \dots \langle \tilde{y}_{e_1,1}\rangle \dots \langle \tilde{y}_{1,d}\rangle \dots \langle \tilde{y}_{e_d,d}\rangle,$$ [5]

where $\mathbb{K}' = K'(\kappa_1, \dots, \kappa_u)$ and K' is a finite algebraic field extension of K and with the automorphism

$$\sigma(\vartheta) = \zeta'\,\vartheta \qquad \text{and} \qquad \sigma(\tilde{y}_{\ell,d}) = \tilde{\alpha}_{\ell,d}\,\tilde{y}_{\ell,d}$$

where $\zeta' \in K'$ is a λ'-th root of unity and

$$\tilde{\alpha}_{\ell,d} = \tilde{h}_\ell\,\tilde{y}_{\ell,1} \cdots \tilde{y}_{\ell,k-1} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}\langle \tilde{y}_{\ell,1}\rangle \dots \langle \tilde{y}_{\ell,d-1}\rangle}$$

for $1 \leqslant \ell \leqslant e_d$;

(2) a naturally induced evaluation function $\tilde{\mathrm{ev}} : \mathbb{D} \times \mathbb{N} \to \mathbb{K}'$ defined as [6]

$$\tilde{\mathrm{ev}}(\vartheta, n) = \prod_{j=1}^{n} \zeta' \quad \text{and} \quad \tilde{\mathrm{ev}}(\tilde{y}_{\ell,d}, n) = \prod_{j=1}^{n} \tilde{\mathrm{ev}}(\tilde{\alpha}_{\ell,d}, j-1)$$

[5]If $\zeta = 1$, then we can consider only the Π-extension $(\tilde{\mathbb{D}}, \sigma)$ of $(\mathbb{K}', \sigma)$ with $\tilde{\mathbb{D}} = \mathbb{K}'\langle \tilde{y}_{1,1}\rangle \dots \langle \tilde{y}_{e_1,1}\rangle \dots \langle \tilde{y}_{1,d}\rangle \dots \langle \tilde{y}_{e_d,d}\rangle$. Otherwise, the construction of the single R-extension $(\tilde{\mathbb{D}}[\vartheta], \sigma)$ of $(\tilde{\mathbb{D}}, \sigma)$ is a redundant and causes no harm.

[6]Note that for all $c \in \mathbb{K}'$, $\tilde{\mathrm{ev}}(c, n) = c$ for all $n \geqslant 0$.

such that the map $\varphi : \mathbb{A}_d \to \mathbb{D}$ *defined as*

$$\begin{aligned}\varphi : \mathbb{A}_d &\to \mathbb{D}\\ y_{\ell,d} &\mapsto (\beta_{\ell,d,0}\boldsymbol{e}_0 + \cdots + \beta_{\ell,d,\lambda'-1}\boldsymbol{e}_{\lambda'-1})\,\tilde{y}_{1,d}^{\nu_{\ell,1,d}} \cdots \tilde{y}_{e_d,d}^{\nu_{\ell,e_d,d}}\end{aligned}$$

is a difference ring homomorphism where for all $f \in \mathbb{A}_d$ *and for all* $n \in \mathbb{N}$,

$$\text{ev}(f, n) = \text{ev}(\varphi(f), n)$$

holds. Here, $\beta_{\ell,k,i} = \text{ev}(\vartheta_{\ell,d}, \lambda' - 1 - i)$ *for* $0 \leqslant i < \lambda'$ *and* $\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{\lambda'-1}$ *are the idempotent, pairwise orthogonal elements in* (6.26) *that sum up to one.* ♢

PROOF:
Given the ordered multiple chain P-extension $(\mathbb{A}_d, \sigma)$ of $(\mathbb{K}, \sigma)$ of monomial depth at most d with the automorphism $\sigma : \mathbb{A}_d \to \mathbb{A}_d$ defined by (6.63) and the evaluation function $\text{ev} : \mathbb{A}_d \times \mathbb{N} \to \mathbb{K}$ defined by (6.62), it follows by Lemma 6.5.2 that we can construct an ordered multiple chain AP-extension $(\mathbb{H}_d, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of monomial depth at most d where $\mathbb{H}_d$ is given by (6.67) with the automorphism (6.68) and (6.69) and the evaluation function (6.70) such that the sub-difference ring $(\tilde{\mathbb{A}}_d, \sigma)$ of $(\mathbb{H}_d, \sigma)$ where $\tilde{\mathbb{A}}_d$ is given by (6.71) is a Π-extension of $(\tilde{\mathbb{K}}, \sigma)$. Furthermore, we can also construct a difference ring homomorphism, $\rho_d : \mathbb{A}_d \to \mathbb{H}_d$ with (6.72) such that for all $f \in \mathbb{A}_d$ and for all $n \in \mathbb{N}$,

$$\text{ev}(f, n) = \tilde{\text{ev}}(\rho_d(f), n)$$

holds. Given the AΠ-extension $(\mathbb{H}_d, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ it follows by Lemma 6.5.5 that we can construct the single RΠ-extension $(\mathbb{D}, \sigma)$ of $(\mathbb{K}', \sigma)$ where $\mathbb{D}$ is given by (6.98) with (6.102) of order λ' that can be written as the direct sum (6.99). Observe that, by Remark 6.5.3 the generators in $\mathbb{H}_d$ can be rearranged to get the AΠ-extension $(\tilde{\mathbb{H}}, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with (6.100). By Lemma 6.5.7, the map $\phi : \tilde{\mathbb{H}} \to \mathbb{D}$ defined by (6.106) and (6.107) is a difference ring homomorphism such that $f \in \tilde{\mathbb{H}}$ and for all $n \in \mathbb{N}$,

$$\text{ev}(f, n) = \text{ev}(\phi(f), n)$$

holds. Putting everything together, the map $\varphi : \mathbb{A}_d \to \mathbb{D}$ with

$$\varphi(y_{\ell,d}) = \phi(\rho(y_{\ell,d})) = (\beta_{\ell,d,0}\boldsymbol{e}_0 + \cdots + \beta_{\ell,d,\lambda'-1}\boldsymbol{e}_{\lambda'-1})\,\tilde{y}_{1,d}^{\nu_{\ell,1,d}} \cdots \tilde{y}_{e_d,d}^{\nu_{\ell,e_d,d}}$$

is a difference ring homomorphism. Furthermore, for all $f \in \mathbb{A}_d$ and for all $n \in \mathbb{N}$,

$$\text{ev}(f, n) = \tilde{\text{ev}}(\varphi(f), n) = \tilde{\text{ev}}(\phi(\rho(f)), n)$$

holds. ∎

Example 6.5.11.
We will represent the nesting depth 2 geometric product expression

$$G(n) = \prod_{k=1}^{n} \frac{3}{2} \left(\prod_{i=1}^{k} \frac{-1}{\sqrt{6}} \right) \in \text{ProdE}(\mathbb{K}) \tag{6.109}$$

where $\mathbb{K} = \mathbb{Q}(\sqrt{6})$ in an ordered multiple chain Π-extension of $(\mathbb{K}, \sigma)$. We first write (6.109) in a product factored form:

$$G_1(n) = \left(\prod_{k=1}^{n} \frac{1}{2}\right) \left(\prod_{k=1}^{n} 3\right) \left(\prod_{k=1}^{n}\prod_{i=1}^{k} -1\right) \left(\prod_{k=1}^{n}\prod_{i=1}^{k} \frac{1}{\sqrt{6}}\right) \in \operatorname{Prod}(\mathbb{K}). \tag{6.110}$$

Note that

$$G(n) = G_1(n) \quad \forall n \geqslant 1.$$

Let $(\mathbb{A}, \sigma)$ with $\mathbb{A} = \mathbb{K}\langle\vartheta_1\rangle\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle\vartheta_2\rangle\langle y_{3,2}\rangle$ be an ordered multiple chain AP-extension of $(\mathbb{K}, \sigma)$ whose building blocks are the following single chain A-/Π-extensions.

(1) The single chain A-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{K}, \sigma)$ over $\mathbb{K}$ of order 2 where $\mathbb{G} = \mathbb{K}\langle\vartheta_1\rangle\langle\vartheta_2\rangle$ with the automorphism $\sigma : \mathbb{G} \to \mathbb{G}$ and evaluation function $\mathrm{ev} : \mathbb{G} \times \mathbb{N} \to \mathbb{K}$ is defined as:

$$\begin{aligned} &\sigma(x) = x + 1, && \sigma(\vartheta_1) = -\vartheta_1, && \sigma(\vartheta_2) = -\vartheta_1\,\vartheta_2, \\ &\mathrm{ev}(x, n) = n, && \mathrm{ev}(\vartheta_1, n) = \prod_{k=1}^{n} -1, && \mathrm{ev}(\vartheta_2, n) = \prod_{k=1}^{n}\prod_{i=1}^{k} -1. \end{aligned} \tag{6.111}$$

(2) The single chain Π-extension $(\mathbb{K}_1, \sigma)$ of $(\mathbb{K}, \sigma)$ over $\mathbb{K}$ where $\mathbb{K}_1 = \mathbb{K}\langle y_{1,1}\rangle$ with the automorphism $\sigma : \mathbb{K}_1 \to \mathbb{K}_1$ and the naturally induced evaluation function $\mathrm{ev} : \mathbb{K}_1 \times \mathbb{N} \to \mathbb{K}$ is defined as

$$\begin{aligned} &\sigma(c) = c,\ \forall\, c \in \mathbb{K}, && \sigma(y_{1,1}) = 2\, y_{1,1}, \\ &\mathrm{ev}(c, n) = c,\ \forall\, c \in \mathbb{K}, && \mathrm{ev}(y_{1,1}, n) = \prod_{k=1}^{n} 2. \end{aligned} \tag{6.112}$$

(3) The single chain Π-extension $(\mathbb{K}_2, \sigma)$ of $(\mathbb{K}, \sigma)$ over $\mathbb{K}$ where $\mathbb{K}_2 = \mathbb{K}\langle y_{2,1}\rangle$ with the automorphism $\sigma : \mathbb{K}_2 \to \mathbb{K}_2$ and the naturally induced evaluation function $\mathrm{ev} : \mathbb{K}_2 \times \mathbb{N} \to \mathbb{K}$ is defined as

$$\begin{aligned} &\sigma(c) = c,\ \forall\, c \in \mathbb{K}, && \sigma(y_{2,1}) = 3\, y_{2,1}, \\ &\mathrm{ev}(c, n) = c,\ \forall\, c \in \mathbb{K}, && \mathrm{ev}(y_{2,1}, n) = \prod_{k=1}^{n} 3. \end{aligned} \tag{6.113}$$

(4) The single chain Π-extension $(\mathbb{K}_3, \sigma)$ of $(\mathbb{K}, \sigma)$ over $\mathbb{K}$ where $\mathbb{K}_3 = \mathbb{K}\langle y_{3,1}\rangle\langle y_{3,2}\rangle$ with the automorphism $\sigma : \mathbb{K}_3 \to \mathbb{K}_3$ and the naturally induced evaluation function $\mathrm{ev} : \mathbb{K}_3 \times \mathbb{N} \to \mathbb{K}$ is defined as:

$$\begin{aligned} &\sigma(c) = c,\ \forall\, c \in \mathbb{K}, && \sigma(y_{3,1}) = \sqrt{6}\, y_{3,1}, && \sigma(y_{3,2}) = \sqrt{6}\, y_{3,1}\, y_{3,2}, \\ &\mathrm{ev}(c, n) = c,\ \forall\, c \in \mathbb{K}, && \mathrm{ev}(y_{3,1}, n) = \prod_{k=1}^{n} \sqrt{6}, && \mathrm{ev}(y_{3,2}, n) = \prod_{k=1}^{n}\prod_{i=1}^{k} \sqrt{6}. \end{aligned} \tag{6.114}$$

Now we merge the single chain A-/Π-extensions to an ordered multiple chain AP-extension yielding the tower of ring extensions $\mathbb{K} \leqslant \mathbb{A}_1 \leqslant \mathbb{A}_2$ where $\mathbb{A}_1 = \mathbb{K}\langle\vartheta_1\rangle\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle$ and $\mathbb{A}_2 = \mathbb{A}_1\langle\vartheta_2\rangle\langle y_{3,2}\rangle$. Then the product expression (6.109) is modelled by

$$G_1 = \frac{\vartheta_2\, y_{2,1}}{y_{1,1}\, y_{3,2}} \in \mathbb{A}_2, \tag{6.115}$$

that is,

$$\mathrm{ev}(G_1, n) = G_1(n) \quad \forall n \geqslant 1$$

holds. Let $(\mathbb{H}, \sigma)$ with $\mathbb{H} = \mathbb{K}\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{3,2}\rangle$ be the ordered multiple chain P-extension of $(\mathbb{K}, \sigma)$ composed by the single chain Π-extensions $(\mathbb{K}_i, \sigma)$ for $i = 1, 2, 3$ based at the algebraic numbers 2, 3, $\sqrt{6} \in \mathbb{K}^*$ respectively which were constructed in items (2), (3) and (4) above. In particular, we have

$$\frac{y_{2,1}}{y_{1,1}\, y_{3,2}} \in \mathbb{H}. \tag{6.116}$$

We follow the proof of Lemma 6.5.2 to construct an ordered multiple chain AΠ-ring in which (6.109) can be modelled.

(1) Take the shift quotient of all depth-1 P-monomials $y_{1,1}$, $y_{2,1}$ and $y_{3,1}$ in the ordered multiple chain P-extension $(\mathbb{H}_1, \sigma)$ of $(\mathbb{K}, \sigma)$ of monomial depth 1 with $\mathbb{H}_1 = \mathbb{K}\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle$, i.e.,

$$\frac{\sigma(y_{1,1})}{y_{1,1}} = 2, \qquad \frac{\sigma(y_{2,1})}{y_{2,1}} = 3, \qquad \frac{\sigma(y_{3,1})}{y_{3,1}} = \sqrt{6}$$

in $\mathbb{K}^*$. By item (1) of Lemma 5.1.4[7] we can construct the Π-extension $(\tilde{\mathbb{H}}_1, \sigma)$ of $(\mathbb{K}', \sigma)$ where $\tilde{\mathbb{H}}_1 = \mathbb{K}'\langle \tilde{y}_{1,1}\rangle\langle \tilde{y}_{2,1}\rangle$ with the automorphism $\sigma : \mathbb{H}_1 \to \mathbb{H}_1$ and the evaluation function $\tilde{\mathrm{ev}} : \tilde{\mathbb{H}}_1 \times \mathbb{N} \to \mathbb{K}'$

$$\begin{aligned} \sigma(\tilde{y}_{1,1}) &= \sqrt{2}\, \tilde{y}_{1,1}, & \sigma(\tilde{y}_{2,1}) &= \sqrt{3}\, \tilde{y}_{2,1}, \\ \tilde{\mathrm{ev}}(\tilde{y}_{1,1}, n) &= \prod_{k=1}^{n} \sqrt{2}, & \tilde{\mathrm{ev}}(\tilde{y}_{2,1}, n) &= \prod_{k=1}^{n} \sqrt{3} \end{aligned} \tag{6.117}$$

together with the difference ring homomorphism $\rho_1 : \mathbb{H}_1 \to \tilde{\mathbb{H}}_1$ defined by

$$\rho_1(y_{1,1}) = \tilde{y}_{1,1}^2, \qquad \rho_1(y_{2,1}) = \tilde{y}_{2,1}^2, \qquad \rho_1(y_{3,1}) = \tilde{y}_{1,1}\, \tilde{y}_{2,1}. \tag{6.118}$$

Here, $\mathbb{K}' = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Further, note that $(\tilde{\mathbb{H}}_1, \sigma)$ is an ordered multiple chain Π-extension of $(\mathbb{K}', \sigma)$ of monomial depth 1 which is composed by the single chain Π-extensions $(\mathbb{K}'_i, \sigma)$ of monomial depth 1 where $\mathbb{K}'_i = \mathbb{K}'\langle \tilde{y}_{i,1}\rangle$ for $i = 1, 2$ with (6.117).

(2) Take the shift quotient of the depth-2 P-monomial, $y_{3,2}$ in the ordered multiple chain P-extension $(\mathbb{H}_2, \sigma)$ of $(\mathbb{K}, \sigma)$ of monomial depth 2 with $\mathbb{H}_2 = \mathbb{H}_1\langle y_{3,2}\rangle$, i.e.,

$$\frac{\sigma(y_{3,2})}{y_{3,2}} = \sqrt{6}\, y_{3,1} \in (\mathbb{K}^*)^{\mathbb{H}_1}_{\mathbb{K}}. \tag{6.119}$$

Applying the difference ring homomorphism $\rho_1 : \mathbb{H}_1 \to \tilde{\mathbb{H}}_1$ to the right hand side of (6.119) we get

$$\rho_1(\sqrt{6}\, y_{3,1}) = \sqrt{6}\, \tilde{y}_{1,1}\, \tilde{y}_{2,1} \in (\mathbb{K}'^*)^{\tilde{\mathbb{H}}_1}_{\mathbb{K}'}.$$

Since the depth-1 single chain Π-monomials $\tilde{y}_{1,1}, \tilde{y}_{2,1}$ have non-zero integer exponents in the expression above, we extend the single chain Π-extension that each Π-monomial belongs to. More precisely, consider the following single chain Π-extensions $(\mathbb{K}''_i, \sigma)$ of $(\mathbb{K}', \sigma)$ for $i = 1, 2$ where $\mathbb{K}''_i = \mathbb{K}'_i\langle \tilde{y}_{i,2}\rangle$ with

$$\begin{aligned} \sigma(\tilde{y}_{1,2}) &= \sqrt{2}\, \tilde{y}_{1,1}\, \tilde{y}_{1,2}, & \sigma(\tilde{y}_{2,2}) &= \sqrt{3}\, \tilde{y}_{2,1}\, \tilde{y}_{2,2}, \\ \tilde{\mathrm{ev}}(\tilde{y}_{1,2}, n) &= \prod_{k=1}^{n}\prod_{i=1}^{k} \sqrt{2}, & \tilde{\mathrm{ev}}(\tilde{y}_{2,2}, n) &= \prod_{k=1}^{n}\prod_{i=1}^{k} \sqrt{3}. \end{aligned} \tag{6.120}$$

Now consider the ordered multiple chain Π-extension $(\tilde{\mathbb{H}}_2, \sigma)$ of $(\mathbb{K}', \sigma)$ of monomial depth 2 where $\tilde{\mathbb{H}}_2 = \tilde{\mathbb{H}}_1\langle \tilde{y}_{1,2}\rangle\langle \tilde{y}_{2,2}\rangle$ with (6.120) and composed by the single chain Π-extensions $(\mathbb{K}''_i, \sigma)$ of $(\mathbb{K}', \sigma)$

[7] Since $\zeta = 1$, we only consider the Π-extension $(\mathbb{K}'\langle \tilde{y}_{1,1}\rangle\langle \tilde{y}_{2,1}\rangle, \sigma)$ of $(\mathbb{K}', \sigma)$.

for $i = 1,2$. Since $\boldsymbol{\mathcal{M}}\big((\sqrt{2},\sqrt{3}),\,\mathbb{K}'\big) = \{0_2\}$, it follows by Theorem 6.4.14 that $(\tilde{\mathbb{H}}_2,\sigma)$ is a Π-extension of $(\mathbb{K}',\sigma)$. In particular, $(\tilde{\mathbb{H}}_2,\sigma)$ is an ordered multiple chain Π-extension of $(\mathbb{K}',\sigma)$ of monomial depth 2. It remains to show that there is an element $g \in (\mathbb{K}')^{\tilde{\mathbb{H}}_2}_{\mathbb{K}'}$ with

$$\sigma(g) = \sqrt{6}\,\tilde{y}_{1,1}\,\tilde{y}_{2,1}\,g.$$

Looking at $\rho_1(y_{3,1}) = \tilde{y}_{1,1}\,\tilde{y}_{2,1}$ we can choose $g = \tilde{y}_{1,2}\,\tilde{y}_{2,2}$ which satisfies the relation above. Thus we define the difference ring homomorphism

$$\rho_2 : \mathbb{H}_2 \to \tilde{\mathbb{H}}_2$$

with

$$\rho_2|_{\mathbb{H}_1} = \rho_1, \qquad\qquad \rho_2(y_{3,2}) = g.$$

Merging $(\tilde{\mathbb{H}}_2,\sigma)$ and the single chain A-extension $(\mathbb{G},\sigma)$ of $(\mathbb{K},\sigma)$ with (6.111) we get the ordered multiple chain $A\Pi$-extension $(\tilde{\mathbb{H}},\sigma)$ of $(\mathbb{K}',\sigma)$ where

$$\tilde{\mathbb{H}} = \mathbb{K}'\langle\vartheta_1\rangle\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\vartheta_2\rangle\langle\tilde{y}_{1,2}\rangle\langle\tilde{y}_{2,2}\rangle$$

with (6.111), (6.117) and (6.120). We extend the difference ring homomorphism $\rho_2 : \mathbb{H}_2 \to \tilde{\mathbb{H}}_2$ to $\rho : \mathbb{A} \to \tilde{\mathbb{H}}$ by defining

$$\rho|_{\mathbb{H}_2} = \rho_2, \qquad \text{and} \qquad \rho(\vartheta_i) = \vartheta_i \quad \text{for } i = 1,2.$$

Applying ρ to (6.115) we have

$$\rho(F_1) = \frac{\rho(\vartheta_2)\,\rho(y_{2,1})}{\rho(y_{1,1})\,\rho(y_{3,2})} = \frac{\vartheta_2\,\tilde{y}_{2,1}^2}{\tilde{y}_{1,1}^2\,\tilde{y}_{1,2}\,\tilde{y}_{2,2}} \in \tilde{\mathbb{H}}.$$

Given the difference ring $(\mathbb{A},\sigma)$, it follows by Lemma 6.5.5 that we can construct the simple $R\Pi$-extension $(\mathbb{D},\sigma)$ of $(\tilde{\mathbb{K}},\sigma)$ where $\tilde{\mathbb{K}} = \mathbb{K}'(\mathbb{i})$ and $\mathbb{D} = \tilde{\mathbb{K}}[\vartheta]\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{1,2}\rangle\langle\tilde{y}_{2,2}\rangle$ which can be written as the direct sum

$$\mathbb{D} = \boldsymbol{e}_0\tilde{\mathbb{D}} \oplus \boldsymbol{e}_1\tilde{\mathbb{D}} \oplus \boldsymbol{e}_2\tilde{\mathbb{D}} \oplus \boldsymbol{e}_3\tilde{\mathbb{D}}$$

where $\tilde{\mathbb{D}} = \tilde{\mathbb{K}}\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{1,2}\rangle\langle\tilde{y}_{2,2}\rangle$ and $\boldsymbol{e}_k$ for $0 \leqslant k \leqslant 3$ are idempotent elements given by (6.52), with the automorphism, $\sigma : \mathbb{D} \to \mathbb{D}$ and the evaluation function, $\tilde{\mathrm{ev}} : \mathbb{D} \times \mathbb{N} \to \tilde{\mathbb{K}}$ defined by (6.103), (6.108), (6.117), and (6.120). Further, the map $\phi : \tilde{\mathbb{H}} \to \mathbb{D}$ defined by

$$\begin{aligned}
\phi(\vartheta_1) &= -\boldsymbol{e}_0 + \boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3 = \vartheta^2;\\
\phi(\vartheta_2) &= -\boldsymbol{e}_0 - \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 = \frac{(1-\mathbb{i})}{2}\,\vartheta\,(\vartheta^2 + \mathbb{i});\\
\phi(\tilde{y}_{\ell,k}) &= \tilde{y}_{\ell,k}
\end{aligned}$$

is a difference ring homomorphism. Applying ϕ to $\rho(F_1) \in \tilde{\mathbb{H}}$ we get

$$\tilde{G}_1 = \phi(\rho(G_1)) = \frac{(1-\mathbb{i})\,\vartheta\,(\vartheta^2+\mathbb{i})\,\tilde{y}_{2,1}^2}{2\,\tilde{y}_{1,1}^2\,\tilde{y}_{1,2}\,\tilde{y}_{2,2}} \in \mathbb{D}$$

and

$$\tilde{G}(n) = \tilde{\mathrm{ev}}(\tilde{G}_1,n) = \frac{(1-\mathbb{i})\,(\mathbb{i})^n\,((\mathbb{i}^n)^2+\mathbb{i})\left(\left(\sqrt{3}\right)^n\right)^2}{2\left(\left(\sqrt{2}\right)^n\right)^2\,\left(\sqrt{2}\right)^{\binom{n+1}{2}}\,\left(\sqrt{3}\right)^{\binom{n+1}{2}}} \tag{6.121}$$

with

$$G(n) = \tilde{G}(n) \quad \forall\, n \geqslant 1.$$

$\star$

6.6 Construction of RΠ-extensions for higher nesting depth expressions in $\mathrm{ProdE}(\mathbb{K}(n))$

In this section, we will extend the result in Section 6.5 to the class of hypergeometric product expressions in $\mathrm{ProdE}(\mathbb{K}(n))$. More precisely, suppose we are given the rational function field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x+1$ and the evaluation function (2.1) where $\mathbb{K}$ is the rational function field $K(\kappa_1, \dots, \kappa_u)$ over some σ-strongly computable field K. Suppose further that we are give a finite set of hypergeometric products $\{P_1(n), \dots, P_e(n)\} \subseteq \mathrm{Prod}(\mathbb{K}(n))$ of finite nesting depth. Then in Lemma 6.6.3, we will refine these given hypergeometric products to obtain other hypergeometric products in product factored form whose distinct innermost multiplicands are irreducible monic polynomials in $\mathbb{K}[x] \setminus \mathbb{K}$ and are shift co-prime among each other. For these refined hypergeometric products, we will succeed in Lemma 6.6.10 by constructing a multiple chain Π-extension of $(\mathbb{K}(x), \sigma)$ where they can be modelled. Furthermore, the refinement procedure in Lemma 6.6.3 also yields other geometric products which can be modelled in a simple RΠ-extension. Combining these two difference rings, we will succeed in Theorem 6.6.13 by merging them to get a RΠ-extension in which the given hypergeometric products $\{P_1(n), \dots, P_e(n)\}$ of finite nesting depth came be modelled.

We begin with the Theorem below which is a generalisation of Theorem 5.3.3 for multiple chain Π-extensions.

Theorem 6.6.1.
Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ ($\alpha \in \mathbb{F}^*$ and $\beta = 0$ or $\alpha = 1$ and $\beta \in \mathbb{F}$). Let $\mathbf{f} = (f_1, \dots, f_m) \in (\mathbb{F}[t] \setminus \mathbb{F})^m$ be irreducible monic polynomials. For all $1 \leqslant \ell \leqslant m$, let $(\mathbb{F}_\ell, \sigma)$ with

$$\mathbb{F}_\ell := \mathbb{F}(t)\langle z_{\ell,1}\rangle \dots \langle z_{\ell,s_\ell}\rangle$$

be a single chain Π-extension of $(\mathbb{F}(t), \sigma)$ with base $f_\ell \in \mathbb{F}[t] \setminus \mathbb{F}$ with the automorphism (6.122). Let $(\mathbb{H}_b, \sigma)$ with

$$\mathbb{H}_b = \mathbb{F}(t)\langle \boldsymbol{z_1}\rangle \dots \langle \boldsymbol{z_b}\rangle = \mathbb{F}(t)\langle z_{1,1}\rangle \dots \langle z_{w_1,1}\rangle \dots \langle z_{1,b}\rangle \dots \langle z_{w_b,b}\rangle$$

be an ordered multiple chain P-extension of $(\mathbb{F}(t), \sigma)$ of monomial depth $b = \max(s_1, \dots, s_m)$ with bases $f_1, \dots, f_m$ where $m = w_1 \geqslant w_2 \geqslant \dots \geqslant w_b$ which is composed by the single chain Π-extensions $(\mathbb{F}_\ell, \sigma)$ of $(\mathbb{F}(t), \sigma)$. Then $(\mathbb{H}_b, \sigma)$ is a Π-extension of $(\mathbb{F}(t), \sigma)$ if and only if $\gcd_\sigma(f_i, f_j) = 1$ for all i, j with $1 \leqslant i < j \leqslant m$. ♢

Proof:
" $\Longrightarrow$ " If $(\mathbb{H}_b, \sigma)$ is a Π-extension of $(\mathbb{F}(t), \sigma)$, then by Theorem 6.4.14 $\mathcal{M}(\mathbf{f}, \mathbb{F}(t)) = \{0_m\}$ and by Theorem 5.3.3 $\gcd_\sigma(f_i, f_j) = 1$ for all i, j with $1 \leqslant i < j \leqslant m$.
" $\Longleftarrow$ " Conversely, if $\gcd_\sigma(f_i, f_j) = 1$ for all i, j with $1 \leqslant i < j \leqslant m$, then by Theorem 5.3.3 $\mathcal{M}(\mathbf{f}, \mathbb{F}(t)) = \{0_m\}$ and by Theorem 6.4.14 $(\mathbb{H}_b, \sigma)$ is a Π-extension of $(\mathbb{F}(t), \sigma)$. ■

Throughout this section, $(\mathbb{K}(x), \sigma)$ is a rational difference field with $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ where K is some σ-strongly computable field. As in the previous section, we suppose the following. Let $(\mathbb{F}, \sigma)$ with $\mathbb{F} = \mathbb{K}(x)$ be the rational difference field over $\mathbb{K}$ as defined in Example 2.3.6 that satisfies Lemma 4.1.6. In addition, we will use the evaluation function defined in (2.1). For $1 \leqslant \ell \leqslant m$ with $m \in \mathbb{N} \setminus \{0\}$, let $(\mathbb{F}_\ell, \sigma)$ with

$$\mathbb{F}_\ell = \mathbb{F}\langle \boldsymbol{z_\ell}\rangle = \mathbb{F}\langle z_{\ell,1}\rangle \dots \langle z_{\ell,s_\ell}\rangle$$

be a single chain Π-extension of $(\mathbb{F},\sigma)$ with base $f_\ell \in \mathbb{F}^*$ together with the automorphism

$$\sigma(z_{\ell,k}) = \alpha_{\ell,k}\, z_{\ell,k} \quad \text{where} \quad \alpha_{\ell,k} = f_\ell\, z_{\ell,1} \cdots z_{\ell,k-1} \in (\mathbb{F}^*)_{\mathbb{F}}^{\mathbb{F}\langle z_{\ell,1}\rangle \ldots \langle z_{\ell,k-1}\rangle} \tag{6.122}$$

and $\partial(z_{\ell,k}) = k$ for $1 \leqslant k \leqslant s_\ell$. Let $\mathrm{ev} : \mathbb{F}_\ell \times \mathbb{N} \to \mathbb{K}$ be an evaluation function for $(\mathbb{F}_\ell,\sigma)$ defined as

$$\mathrm{ev}(z_{\ell,k}, n) = \prod_{j=\delta}^{n} \mathrm{ev}(\alpha_{\ell,k}, j-1) \tag{6.123}$$

where $\delta \in \mathbb{N}$ is chosen big enough such that $\mathrm{ev}(\alpha_{\ell,k},\delta) \neq 0$ for $1 \leqslant \ell \leqslant m$ and $1 \leqslant k \leqslant s_\ell$. Note that these input assumptions are justified by Lemma 6.1.5. We will follow the construction on page 124. Let $(\mathbb{H},\sigma)$ be the multiple chain P-extension of $(\mathbb{F},\sigma)$ built by the single chain Π-extensions $(\mathbb{F}_\ell,\sigma)$ of $(\mathbb{F},\sigma)$ over $\mathbb{F}$ based at $f_\ell \in \mathbb{F}^*$. That is,

$$\mathbb{H} = \mathbb{F}\langle \boldsymbol{z_1}\rangle\langle \boldsymbol{z_2}\rangle \ldots \langle \boldsymbol{z_m}\rangle = \mathbb{F}\langle z_{1,1}\rangle \ldots \langle z_{1,s_1}\rangle\langle z_{2,1}\rangle \ldots \langle z_{2,s_2}\rangle \ldots \langle z_{m,1}\rangle \ldots \langle z_{m,s_m}\rangle.$$

Depending on the context, $\boldsymbol{z_\ell}$ denotes $(z_{\ell,1},\ldots,z_{\ell,s_\ell})$ or $z_{\ell,1},\ldots,z_{\ell,s_\ell}$ or $z_{\ell,1}\cdots z_{\ell,s_\ell}$. Again observe that the P-monomials $z_{\ell,k}$ can be ordered in increasing order of their depths. Let $b = \max(s_1, s_2, \ldots, s_m)$ and $\mathbb{H}_0 = \mathbb{F}$. Consider the tower of difference ring extensions $(\mathbb{H}_i,\sigma)$ of $(\mathbb{H}_{i-1},\sigma)$ where

$$\mathbb{H}_i = \mathbb{H}_{i-1}\langle \boldsymbol{z_i}\rangle = \mathbb{H}_{i-1}\langle z_{1,i}\rangle\langle z_{2,i}\rangle \ldots \langle z_{w_i,i}\rangle$$

for $1 \leqslant i \leqslant b$ with $m = w_1 \geqslant w_2 \geqslant \cdots \geqslant w_b$ and the automorphism

$$\sigma(z_{\ell,i}) = \alpha_{\ell,i}\, z_{\ell,i} \quad \text{where} \quad \alpha_{\ell,i} = f_\ell\, z_{\ell,1} \cdots z_{\ell,i-1} \in (\mathbb{F}^*)_{\mathbb{F}}^{\mathbb{F}\langle z_{\ell,1}\rangle \ldots \langle z_{\ell,i-1}\rangle} \tag{6.124}$$

for $1 \leqslant \ell \leqslant w_i$. Note that the depth of each $z_{\ell,i}$ is i. Further, the ring $\mathbb{H}_b$ is a isomorphic to $\mathbb{H}$ up to reordering of the P-monomials. Recall that $(\mathbb{H}_b,\sigma)$ is an *ordered multiple chain* P*-extension* of $(\mathbb{F},\sigma)$ of monomial depth at most b induced by single chain Π-extensions $(\mathbb{F}_\ell,\sigma)$ of $(\mathbb{F},\sigma)$ for $1 \leqslant \ell \leqslant m$ with the automorphism (6.122) and the evaluation function (6.123). Observe that since $\mathbb{H}_b \simeq \mathbb{H}$, the evaluation function $\mathrm{ev} : \mathbb{H}_i \times \mathbb{N} \to \mathbb{K}$ for all i with $1 \leqslant i \leqslant b$ is also defined by (6.123).

Remark 6.6.2.
Subsequently, we will call the evaluation function (6.123) the naturally induced evaluation function of the single chain Π-extension $(\mathbb{F}\langle z_{\ell,1}\rangle \ldots \langle z_{\ell,s_\ell}\rangle,\sigma)$ with respect to $\delta \in \mathbb{N}$. We will use the same terminology for the (ordered) multiple chain P-/Π-extensions as well. ♠

Lemma 6.6.3.
Let $(\mathbb{K}(x),\sigma)$ with $\sigma(x) = x+1$ be the rational difference field satisfying Lemma 4.1.6 together with the evaluation function $\mathrm{ev} : \mathbb{K}(x)\times\mathbb{N} \to \mathbb{K}$ defined by (2.1) and the Z-function defined by (2.50). Suppose we are given a finite set of hypergeometric product expressions $\{P_1(n),\ldots,P_e(n)\} \subseteq \mathrm{ProdE}(\mathbb{K}(n))$ of nesting depth at most d. Then one can choose a $\delta \in \mathbb{N}$ and can construct

(1) $\tilde{c}_1,\ldots,\tilde{c}_e \in \mathbb{K}^*$;

(2) for all $1 \leqslant \ell \leqslant e$ *rational functions* $r_\ell(n) \in \mathbb{K}(n)^*$;

(3) geometric product expressions $\tilde{G}_\ell(n) \in \mathrm{ProdE}(\mathbb{K})$ *for* $1 \leqslant \ell \leqslant e$ *with lower bounds all synchronised to* 1 *and composed multiplicatively by geometric products in product factored form;*

(4) for $1 \leqslant \ell \leqslant e$ hypergeometric product expressions $\tilde{H}_\ell(n) \in \mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$ with lower bounds all synchronised to δ and composed multiplicatively by hypergeometric products in product factored form where the innermost multiplicands are given by irreducible monic polynomials in $\mathbb{K}[x] \setminus \mathbb{K}$ that are shift co-prime among each other;

such that for $1 \leqslant \ell \leqslant e$ and for all $n \geqslant \delta$:

$$P_\ell(n) = \tilde{c}_\ell \, \tilde{r}_\ell(n) \, \tilde{G}_\ell(n) \, \tilde{H}_\ell(n) \neq 0. \tag{6.125}$$

♦

PROOF:
By Lemma 6.1.5 we can compute a $\delta \in \mathbb{N}$ and construct $c_1, \ldots, c_e \in \mathbb{K}$, geometric product expressions $G_\ell(n) \in \mathrm{ProdE}(\mathbb{K})$ for $1 \leqslant \ell \leqslant e$ and hypergeometric product expressions $H_\ell(n) \in \mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$ for $1 \leqslant \ell \leqslant e$ satisfying properties (a), (b) and (c) respectively such that

$$P_\ell(n) = c_\ell \, G_\ell(n) \, H_\ell(n) \neq 0$$

holds. We will process the hypergeometric products in the $H_\ell(n)$ further and update the geometric product expressions $G_\ell(n)$ and the units c_ℓ accordingly.

(1) Take all the irreducible monic polynomials which are the innermost multiplicands of each product factor $H_i(n)$ say $f_1, \ldots, f_s \in \mathbb{K}[x] \setminus \mathbb{K}$ and set $\mathcal{F} := \{f_1, \ldots, f_s\} \subseteq \mathbb{K}[x] \setminus \mathbb{K}$. With part (2) of Lemma 4.1.6, we can construct a partition $\mathcal{P} = \{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ with respect to $\sim_\sigma$. That is, each $\mathcal{E}_i$ for $1 \leqslant i \leqslant m$, contains precisely the shift equivalent elements of $\mathcal{F}$. For each i with $1 \leqslant i \leqslant m$ take the leftmost element in $\mathcal{E}_i$ say $\tilde{f}_i$, as the representative of the equivalence class $\mathcal{E}_i$ in $\mathcal{P}$ and collect them in $\mathcal{R} = \{\tilde{f}_1, \ldots, \tilde{f}_m\}$. Since each $\tilde{f}_i$ is shift equivalent with every element of $\mathcal{E}_i$, we can take $k \geqslant 0$ with $\frac{\sigma^k(\tilde{f}_i)}{h} \in \mathbb{K}^*$ and it follows by Lemma 4.1.7 that for each $h \in \mathcal{E}_i$ one can construct an $r \in \mathbb{K}(x)^*$ with

$$r = \tilde{f}_i \, \sigma(\tilde{f}_i) \, \sigma^2(\tilde{f}_i) \cdots \sigma^{k-1}(\tilde{f}_i)$$

such that

$$h = \frac{\sigma(r)}{r} \, \tilde{f}_i \tag{6.126}$$

holds. Note that since $\tilde{f}_i$ is monic, $\sigma^j(\tilde{f}_i)$ is monic for all $1 \leqslant j \leqslant k-1$ and thus r is also monic. Further, since $\mathrm{ev}(\sigma^i(\tilde{f}_i), n) \neq 0$ for all $i \in \mathbb{N}$ and for all $n \geqslant \delta$, we have that,

$$r(n) = \mathrm{ev}(r, n) = \mathrm{ev}(\tilde{f}_i,\, n)\ \mathrm{ev}(\sigma(\tilde{f}_i),\, n)\ \mathrm{ev}(\sigma^2(\tilde{f}_i),\, n) \cdots \mathrm{ev}(\sigma^{k-1}(\tilde{f}_i),\, n) \neq 0.$$

(2) Now take all nesting depth d hypergeometric product factors in each product expression $H_\ell(n)$. Each of them can be treated as follows. Let h be its irreducible monic multiplicand and choose i such that $h \in \mathcal{E}_i$. Using the relation (6.126) in the preprocessing step (1), the hypergeometric product under construction can be reduced as follows:

$$\left(\prod_{k_1=\delta}^{n} \cdots \prod_{k_d=\delta}^{k_{d-1}} h(k_d)\right) = \left(\prod_{k_1=\delta}^{n} \cdots \prod_{k_d=\delta}^{k_{d-1}} \frac{r(k_d+1)}{r(k_d)}\right) \underbrace{\left(\prod_{k_1=\delta}^{n} \cdots \prod_{k_d=\delta}^{k_{d-1}} \tilde{f}_i(k_d)\right)}_{=: A_{\ell,d}(n)}. \tag{6.127}$$

Note that the product

$$\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_d=\delta}^{k_{d-1}}\frac{r(k_d+1)}{r(k_d)}\right)$$

telescopes multiplicatively. That is,

$$\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_d=\delta}^{k_{d-1}}\frac{r(k_d+1)}{r(k_d)}\right)=\underbrace{\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_{d-1}=\delta}^{k_{d-2}}\frac{1}{r(\delta)}\right)}_{=:F_{\ell,d-1}(n)}\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_{d-1}=\delta}^{k_{d-2}}r(k_{d-1}+1)\right). \tag{6.128}$$

Note further that the nesting depth of the product on the right hand side of (6.128) reduces to $d-1$. Further, $r(\delta)\in\mathbb{K}^*$. Thus the product $F_{d-1}(n)$ is a geometric product with all lower bounds synchronised to δ. By item (4) of Section 6.1, we can rewrite the geometric product $F_{d-1}(n)$ of nesting depth $d-1$ into product factored forms where all lower bounds are synchronised to 1. That is

$$\tilde{F}_{\ell,d-1}(n)=\tilde{a}_{\ell,d-1}\,B_{\ell,d-1}(n)$$

where $\tilde{a}_{\ell,d-1}\in\mathbb{K}^*$ and

$$B_{\ell,d-1}(n)=\left(\prod_{k_1=1}^{n}\tilde{u}_{\ell,d-1,1}\right)\left(\prod_{k_1=1}^{n}\prod_{k_2=1}^{k_1}\tilde{u}_{\ell,d-1,2}\right)\cdots\left(\prod_{k_1=1}^{n}\cdots\prod_{k_{d-1}=1}^{k_{d-2}}\tilde{u}_{d-1,d-1}\right)$$

with $\tilde{u}_{\ell,d-1,i}\in\mathbb{K}^*$ for all $1\leqslant i\leqslant d-1$. In particular, for all $n\geqslant\delta$

$$F_{\ell,d-1}(n)=\tilde{F}_{\ell,d-1}(n)$$

holds. Thus (6.128) becomes

$$\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_d=\delta}^{k_{d-1}}\frac{r(k_d+1)}{r(k_d)}\right)=\tilde{F}_{\ell,d-1}(n)\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_{d-1}=\delta}^{k_{d-2}}r(k_{d-1}+1)\right). \tag{6.129}$$

We observe that the inner most multiplicand $r(k_{d-1}+1)$ of the second product on the right hand side of (6.129) is monic and it corresponds to $\sigma(r)\in\mathbb{K}(x)^*$. Let $r_1,\dots,r_s\in\mathbb{K}[x]\setminus\mathbb{K}$ be irreducible monic polynomials such that $\sigma(r)=r_1^{\nu_1}\cdots r_s^{\nu_s}$. Then (6.129) becomes

$$\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_d=\delta}^{k_{d-1}}\frac{r(k_d+1)}{r(k_d)}\right)=\tilde{F}_{\ell,d-1}(n)\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_{d-1}=\delta}^{k_{d-2}}r_1(k_{d-1})\right)^{\nu_1}\cdots\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_{d-1}=\delta}^{k_{d-2}}r_s(k_{d-1})\right)^{\nu_s}. \tag{6.130}$$

Substituting (6.130) into (6.127) we get

$$\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_d=\delta}^{k_{d-1}}h(k_d)\right)=\tilde{a}_{\ell,d-1}\,B_{\ell,d-1}(n)\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_{d-1}=\delta}^{k_{d-2}}r_1(k_{d-1})\right)^{\nu_1}\cdots\left(\prod_{k_1=\delta}^{n}\cdots\prod_{k_{d-1}=\delta}^{k_{d-2}}r_s(k_{d-1})\right)^{\nu_s}A_{\ell,d}(n). \tag{6.131}$$

Finally replace the hypergeometric product on the left hand side of (6.131) in $H_\ell(n)$ for $1\leqslant\ell\leqslant e$ by the hypergeometric product $A_{\ell,d}(n)$ on the right hand side of (6.131). In addition multiply the geometric product expression $B_{\ell,d-1}(n)$ and the units $\tilde{a}_{\ell,d-1}$ to $G_\ell(n)$ and c_ℓ respectively. After the treatment of all products with nesting depth d, we proceed to step (3).

(3) Set $\mathscr{F} = \mathscr{F} \cup \{r_1, \dots, r_s\}$. Note that all irreducible monic polynomials r_i for $i = 1, \dots, s$ are shift-equivalent to $\tilde{f}_i$ by (6.126). In particular, $\sigma^{m_i}(r_i) = \tilde{f}_i$ with $m_i \geqslant 0$. Thus the elements $r_1, \dots, r_s$ can be stored accordingly in the existing equivalence classes $\mathscr{E}_1, \dots, \mathscr{E}_m$ and $\tilde{f}_1, \dots, \tilde{f}_m$ are still the leftmost elements of the updated equivalence classes $\mathscr{E}_1, \dots, \mathscr{E}_m$ respectively. Now repeat the above process in step (2) for all product factors in $H_\ell(n)$ for $1 \leqslant \ell \leqslant e$ with nesting depth, $d = d - 1$ until $d = 0$. For each of these steps, the representative $\tilde{f}_i$ of the equivalence class $\mathscr{E}_i$, used in the previous step, i.e., d, is used to reduce each product factor at step $d - 1$. Furthermore, at depth $d = 1$, we get a rational $r(n) \in \mathbb{K}(n)^*$.

Summarising, we obtain

(1) $\tilde{c}_1, \dots, \tilde{c}_e \in \mathbb{K}^*$;

(2) rational functions $\tilde{r}_1(n), \dots, \tilde{r}_e(n) \in \mathbb{K}(n)^*$;

(3) geometric product expressions $\tilde{G}_1(n), \dots, \tilde{G}_e(n) \in \mathrm{ProdE}(\mathbb{K})$ with lower bounds all synchronised to 1 and composed multiplicatively by geometric products in product factored form;

(4) hypergeometric product expressions $\tilde{H}_1(n), \dots, \tilde{H}_e(n) \in \mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$ with lower bounds all synchronised to δ and composed multiplicatively by hypergeometric products in product factored form where the innermost multiplicands are irreducible monic polynomials in $\mathscr{R}$ which are all shift co-prime among each other.

In particular, (6.125) holds for all ℓ with $1 \leqslant \ell \leqslant e$ and for all $n \geqslant \delta$. ■

Remark 6.6.4.
Recall that by earlier discussions in Sections 6.2 and 6.5, the geometric product expressions

$$\tilde{G}_\ell(n) \in \mathrm{ProdE}(\mathbb{K})$$

for all $1 \leqslant \ell \leqslant e$ in (6.125) can be modelled in an $\mathrm{R}\Pi$-extension; see Theorem 6.5.10. Subsequently, we focus on the hypergeometric product expressions

$$\tilde{H}_\ell(n) \in \mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$$

for all $1 \leqslant \ell \leqslant e$ in (6.125) and show how they can be modelled in a Π-extension. ♠

Example 6.6.5.
Let $(\mathbb{K}(x), \sigma)$ be the rational difference field with $\sigma(x) = x + 1$ with $\mathbb{K} = \mathbb{Q}$. Given the product expression

$$P(n) = \prod_{k=1}^{n} k \, \frac{\left(\prod_{i=1}^{k} \frac{1}{(i+3)}\right)\left(\prod_{i=1}^{k} \frac{(i+2)}{(i+\frac{1}{2})}\right)}{(k+3)} \in \mathrm{ProdE}(\mathbb{K}(n)), \tag{6.132}$$

we follow the procedure in Lemma 6.6.3 as follows. By Lemma 6.1.5 we compute $\delta = 1$, such that for all $n \geqslant 1$

$$P(n) = c\, G(n)\, H(n)$$

holds. Here $c = 1$, $G(n) = 1$ and $H(n)$ is given by

$$H(n) = \left(\prod_{k=1}^{n} k\right)\left(\prod_{k=1}^{n} \frac{1}{(k+3)}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k} \frac{1}{(i+3)}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k} (i+2)\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k} \frac{1}{(i+\frac{1}{2})}\right). \tag{6.133}$$

In particular, $H(n)$ with (6.133) is composed multiplicatively by hypergeometric products of nesting depth at most 2 which are all in product factored form. Let

$$\mathcal{F} = \{h_1, h_2, h_3, h_4\}$$

where $h_1 = x$, $h_2 = x + \frac{1}{2}$, $h_3 = x + 2$ and $h_4 = x + 3$ are monic and irreducible polynomials. Since h_1 is shift equivalent with h_3 and h_4, i.e.,

$$\gcd(h_1, \sigma^2(h_3)) = h_3 \quad \text{and} \quad \gcd(h_1, \sigma^3(h_4)) = h_4,$$

they fall into the same equivalence class

$$\mathcal{E}_1 = \{h_1,\ h_3,\ h_4\}.$$

The other equivalence class is

$$\mathcal{E}_2 = \{h_4\}$$

and we get the set partition

$$\boldsymbol{\mathcal{P}} = \{\mathcal{E}_1,\ \mathcal{E}_2\}.$$

Take the leftmost elements, h_1 and h_2 as the representatives of the equivalence classes $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. Thus

$$\mathcal{R} = \{h_1,\ h_2\}.$$

By Lemma 4.1.7 with $g_1 = x\,(x+1)$ and $g_2 = x\,(x+1)\,(x+2)$ in $\mathbb{K}(x)^*$ we have that

$$h_3 = \frac{\sigma(g_1)}{g_1}\,h_1 \qquad \text{and} \qquad h_4 = \frac{\sigma(g_2)}{g_2}\,h_1$$

holds. Thus for the hypergeometric products of nesting depth 2 over h_3 and h_4 in (6.133) we have that

$$\prod_{k=1}^{n}\prod_{i=1}^{k} h_3(i) = \left(\prod_{k=1}^{n} \frac{g_1(k+1)}{g_1(k)}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k} i\right) = \left(\prod_{k=1}^{n} \frac{1}{2}\right)\left(\prod_{k=1}^{n} (k+1)\right)\left(\prod_{k=1}^{n} (k+2)\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k} i\right)$$

and

$$\prod_{k=1}^{n}\prod_{i=1}^{k} \frac{1}{h_4(i)} = \left(\prod_{k=1}^{n} \frac{g_2(k)}{g_2(k+1)}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k} \frac{1}{i}\right) =$$
$$\left(\prod_{k=1}^{n} 6\right)\left(\prod_{k=1}^{n} \frac{1}{(k+1)}\right)\left(\prod_{k=1}^{n} \frac{1}{(k+2)}\right)\left(\prod_{k=1}^{n} \frac{1}{(k+3)}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k} \frac{1}{i}\right).$$

After reducing all nesting depth 2 hypergeometric products in (6.133) the new monic and irreducible polynomial that emerges is $h_5 = x + 1$. Thus

$$\mathcal{F} = \mathcal{F} \cup \{h_5\}.$$

Note that since h_1 is shift equivalent with h_5, i.e., $\gcd(\sigma(h_1), h_5) = h_5$, we insert h_5 into $\mathscr{E}_1$. In particular, the leftmost factor of $\mathscr{E}_1$, h_1 remains unchanged. By Lemma 4.1.7 with $g_3 = x$, we have that

$$h_5 = \frac{\sigma(g_3)}{g_3}\, h_1$$

holds. Thus the nesting depth 1 hypergemtric products over h_3, h_4 and h_5 reduce as follows:

$$\begin{aligned}
\prod_{k=1}^{n} \frac{1}{h_5(k)} &= \frac{1}{(n+1)} \left(\prod_{k=1}^{n} \frac{1}{k}\right),\\
\prod_{k=1}^{n} \frac{1}{h_3(k)} &= \frac{2}{(n+1)\,(n+2)} \left(\prod_{k=1}^{n} \frac{1}{k}\right),\\
\prod_{k=1}^{n} \frac{1}{h_4(k)} &= \frac{6}{(n+1)\,(n+2)\,(n+3)} \left(\prod_{k=1}^{n} \frac{1}{k}\right).
\end{aligned}$$

Substituting the above into (6.133), and setting

$$\begin{aligned}
\tilde{c} &= 36, \qquad \tilde{r}(n) = \frac{1}{(n+1)^2\,(n+2)^2\,(n+3)^2}\\
\tilde{G}(n) &= \prod_{k=1}^{n} 3, \quad \tilde{H}(n) = \left(\prod_{k=1}^{n} \frac{1}{k}\right) \left(\prod_{k=1}^{n} \prod_{i=1}^{k} \frac{1}{(i+\frac{1}{2})}\right)
\end{aligned} \tag{6.134}$$

we have that for all $n \geqslant 1$,

$$P(n) = \tilde{c}\,\tilde{r}(n)\,\tilde{G}(n)\,\tilde{H}(n)$$

holds. ⋆

Example 6.6.6 (Cont. Examples 6.1.1, 6.1.2, 6.1.3, 6.1.4).
Given the nesting depth 2 hypergeometric product

$$P(n) = \prod_{k=1}^{n} \frac{4\,k^2+1}{\sqrt{-3}\,k} \prod_{j=1}^{k} \frac{-2\,(j^3-7\,j+6)}{5\,(j^2-j-6)} \in \mathrm{Prod}(\mathbb{K}(n)). \tag{6.135}$$

in Example 6.1.1 we follow the procedure in Lemma 6.6.3 as follows. From Example 6.1.2, $\delta = 4$ and from Example 6.1.4 with c, $G(n)$ and $H(n)$ given by

$$\begin{aligned}
c &= \frac{15725}{13824}\\
G(n) &= \left(\prod_{k=1}^{n} \frac{16}{\sqrt{-3}}\right) \left(\prod_{k=1}^{n} \prod_{j=1}^{k} -\frac{2}{5}\right)\\
H(n) &= \left(\prod_{k=4}^{n} k^2+\tfrac{1}{4}\right) \left(\prod_{k=4}^{n} \frac{1}{k}\right) \left(\prod_{k=4}^{n} \prod_{j=4}^{k} \frac{1}{(j-3)}\right) \left(\prod_{k=4}^{n} \prod_{j=4}^{k} (j-2)\right) \left(\prod_{k=4}^{n} \prod_{j=4}^{k} (j-1)\right)\\
&\quad \left(\prod_{k=4}^{n} \prod_{j=4}^{k} \frac{1}{(j+2)}\right) \left(\prod_{k=4}^{n} \prod_{j=4}^{k} (j+3)\right)
\end{aligned} \tag{6.137}$$

we have that

$$P(n) = c\, G(n)\, H(n)$$

holds for all $n \geqslant 4$. We will now focus on the hypergeometric expression $H(n)$ and update c and $G(n)$ accordingly. Let

$$\mathscr{F} = \{h_1,\, h_2,\, h_3,\, h_4,\, h_5,\, h_6, h_7\}$$

where

$$h_1 = x-3,\ h_2 = x-2,\ h_3 = x-1,\ h_4 = x,\ h_5 = x+2,\ h_6 = x+3,\ h_7 = x^2 + \tfrac{1}{4}$$

are irreducible monic polynomials. Since h_7 is shift co-prime to all polynomials in $\mathscr{F} \setminus \{h_7\}$, we have the equivalence class

$$\mathscr{E}_1 = \{h_7\}.$$

The other equivalence class is

$$\mathscr{E}_2 = \{h_1,\, h_2,\, h_3,\, h_4,\, h_5,\, h_6\}$$

since

$$\gcd_\sigma(h_1,\, \sigma(h_2)) = h_2, \qquad \gcd_\sigma(h_1,\, \sigma^2(h_3)) = h_3, \qquad \gcd_\sigma(h_1,\, \sigma^3(h_4)) = h_4,$$
$$\gcd_\sigma(h_1,\, \sigma^5(h_5)) = h_5, \qquad \gcd_\sigma(h_1,\, \sigma^6(h_6)) = h_6.$$

Thus we get the set partition

$$\boldsymbol{\mathscr{P}} = \{\mathscr{E}_1,\, \mathscr{E}_2\}.$$

Take the left most elements, h_7 and h_1 as the representatives of the equivalence classes $\mathscr{E}_1$ and $\mathscr{E}_2$ respectively. Then

$$\mathscr{R} = \{h_7,\, h_1\}.$$

By Lemma 4.1.7 with $g_1,\, g_2,\, g_3,\, g_4,\, g_5 \in \mathbb{K}(x)^*$ where

$$g_1 = (x-3), \qquad g_2 = (x-3)\,(x-2),$$
$$g_3 = (x-3)\,(x-2)\,(x-1), \qquad g_4 = (x-3)\,(x-2)\,(x-1)\,x\,(x+1),$$
$$g_5 = (x-3)\,(x-2)\,(x-1)\,x\,(x+1)\,(x+2),$$

we have that

$$h_2 = \frac{\sigma(g_1)}{g_1}\, h_1, \qquad h_3 = \frac{\sigma(g_2)}{g_2}\, h_1, \qquad h_4 = \frac{\sigma(g_3)}{g_3}\, h_1,$$
$$h_5 = \frac{\sigma(g_4)}{g_4}\, h_1, \qquad h_6 = \frac{\sigma(g_5)}{g_5}\, h_1.$$

Thus the nesting depth 2 hypergeometric products over h_2, h_3, h_5 and h_6 reduces as follows:

$$\begin{aligned}\prod_{k=4}^{n}\prod_{j=4}^{k} h_2(j) &= \left(\prod_{k=4}^{n}\prod_{j=4}^{k} \frac{g_1(j+1)}{g_1(j)}\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k} h_1(j)\right)\\ &= \left(\prod_{k=4}^{n} (k-2)\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k} h_1(j)\right).\end{aligned}$$

$$
\begin{aligned}
\prod_{k=4}^{n}\prod_{j=4}^{k} h_3(j) &= \left(\prod_{k=4}^{n}\prod_{j=4}^{k}\frac{g_2(j+1)}{g_2(j)}\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k} h_1(j)\right) \\
&= \left(\prod_{k=4}^{n}\frac{1}{2}\right)\left(\prod_{k=4}^{n}(k-2)\right)\left(\prod_{k=4}^{n}(k-1)\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k} h_1(j)\right) \\
&= 8\left(\prod_{k=1}^{n}\frac{1}{2}\right)\left(\prod_{k=4}^{n}(k-2)\right)\left(\prod_{k=4}^{n}(k-1)\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k} h_1(j)\right).
\end{aligned}
$$

$$
\begin{aligned}
\prod_{k=4}^{n}\prod_{j=4}^{k}\frac{1}{h_5(j)} &= \left(\prod_{k=4}^{n}\prod_{j=4}^{k}\frac{g_4(j)}{g_4(j+1)}\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k}\frac{1}{h_1(j)}\right) \\
&= \left(\prod_{k=4}^{n}120\right)\left(\prod_{k=4}^{n}\frac{1}{(k-2)}\right)\left(\prod_{k=4}^{n}\frac{1}{(k-1)}\right)\left(\prod_{k=4}^{n}\frac{1}{k}\right)\left(\prod_{k=4}^{n}\frac{1}{(k+1)}\right)\left(\prod_{k=4}^{n}\frac{1}{(k+2)}\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k}\frac{1}{h_1(j)}\right) \\
&= \frac{1}{1728000}\left(\prod_{k=1}^{n}120\right)\left(\prod_{k=4}^{n}\frac{1}{(k-2)}\right)\left(\prod_{k=4}^{n}\frac{1}{(k-1)}\right)\left(\prod_{k=4}^{n}\frac{1}{k}\right)\left(\prod_{k=4}^{n}\frac{1}{(k+1)}\right)\left(\prod_{k=4}^{n}\frac{1}{(k+2)}\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k}\frac{1}{h_1(j)}\right).
\end{aligned}
$$

$$
\begin{aligned}
\prod_{k=4}^{n}\prod_{j=4}^{k} h_6(j) &= \left(\prod_{k=4}^{n}\prod_{j=4}^{k}\frac{g_5(j+1)}{g_5(j)}\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k} h_1(j)\right) \\
&= \left(\prod_{k=4}^{n}\frac{1}{720}\right)\left(\prod_{k=4}^{n}k-2\right)\left(\prod_{k=4}^{n}(k-1)\right)\left(\prod_{k=4}^{n}k\right)\left(\prod_{k=4}^{n}(k+1)\right)\left(\prod_{k=4}^{n}(k+2)\right)\left(\prod_{k=4}^{n}(k+3)\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k} h_1(j)\right) \\
&= 373248000\left(\prod_{k=1}^{n}\frac{1}{720}\right)\left(\prod_{k=4}^{n}(k-2)\right)\left(\prod_{k=4}^{n}(k-1)\right)\left(\prod_{k=4}^{n}k\right)\left(\prod_{k=4}^{n}(k+1)\right)\left(\prod_{k=4}^{n}(k+2)\right)\left(\prod_{k=4}^{n}(k+3)\right)\left(\prod_{k=4}^{n}\prod_{j=4}^{k} h_1(j)\right).
\end{aligned}
$$

After reducing reducing all nesting depth 2 hypergeometric products in the hypergeometric product expression $H(n)$ given in (6.137), the new irreducible monic polynomial that emerges is $h_8 = x + 1$. Thus

$$\mathcal{F} = \mathcal{F} \cup \{h_8\}.$$

Note that since h_1 is shift equivalent with h_8, i.e., $\gcd_\sigma(\sigma(h_1), h_8) = h_8$, we insert h_8 into $\mathcal{E}_1$. In particular, the leftmost factor of $\mathcal{E}_1$, h_1 remains unchanged. By Lemma 4.1.7 with $g_6 = (x-3)(x-2)(x-1)\,x$ we have that

$$h_8 = \frac{\sigma(g_6)}{g_6}\, h_1$$

holds. Thus the nesting depth 1 hypergeometric products over h_2, h_3, h_4, h_5, h_6 and h_8 are reduced as follows:

$$\prod_{k=4}^{n} h_2(k) = \left(\prod_{k=4}^{n} \frac{g_1(k+1)}{g_1(k)}\right)\left(\prod_{k=4}^{n} h_1(k)\right) = (n-2)\left(\prod_{k=4}^{n} h_1(k)\right).$$

$$\prod_{k=4}^{n} h_3(k) = \left(\prod_{k=4}^{n} \frac{g_2(k+1)}{g_2(k)}\right)\left(\prod_{k=4}^{n} h_1(k)\right) = \frac{(n-2)\,(n-1)}{2}\left(\prod_{k=4}^{n} h_1(k)\right).$$

$$\prod_{k=4}^{n} h_4(k) = \left(\prod_{k=4}^{n} \frac{g_3(k+1)}{g_3(k)}\right)\left(\prod_{k=4}^{n} h_1(k)\right) = \frac{(n-2)\,(n-1)\,n}{6}\left(\prod_{k=4}^{n} h_1(k)\right).$$

$$\prod_{k=4}^{n} h_8(k) = \left(\prod_{k=4}^{n} \frac{g_6(k+1)}{g_6(k)}\right)\left(\prod_{k=4}^{n} h_1(k)\right) = \frac{(n-2)\,(n-1)\,n\,(n+1)}{24}\left(\prod_{k=4}^{n} h_1(k)\right).$$

$$\prod_{k=4}^{n} h_5(k) = \left(\prod_{k=4}^{n} \frac{g_4(k+1)}{g_4(k)}\right)\left(\prod_{k=4}^{n} h_1(k)\right) = \frac{(n-2)\,(n-1)\,n\,(n+1)\,(n+2)}{120}\left(\prod_{k=4}^{n} h_1(k)\right).$$

$$\prod_{k=4}^{n} h_6(k) = \left(\prod_{k=4}^{n} \frac{g_5(k+1)}{g_5(k)}\right)\left(\prod_{k=4}^{n} h_1(k)\right) = \frac{(n-2)\,(n-1)\,n\,(n+1)\,(n+2)\,(n+3)}{720}\left(\prod_{k=4}^{n} h_1(k)\right).$$

Substituting the reduced nesting depth 2 and nesting depth 1 hypergeometric product factors into the hypergeometric product expression $H(n)$ given in (6.137) and updating the geometric product expression $G(n)$ and units $c \in \mathbb{K}^*$ in (6.137) accordingly, we get the following:

$$\tilde{c} = \frac{3145}{384}, \qquad r(n) = (n-2)^3\,(n-1)\,(n+1)\,(n+2)\,(n+3),$$
$$\tilde{G}(n) = \left(\prod_{k=1}^{n} \frac{4}{3\sqrt{-3}}\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k} -\frac{2}{5}\right), \quad \tilde{H}(n) = \left(\prod_{k=4}^{n} (k^2+\tfrac{1}{4})\right)\left(\prod_{k=4}^{n} (k-3)\right)^3\left(\prod_{k=4}^{n}\prod_{j=4}^{k} (j-3)\right). \tag{6.138}$$

In particular, all innermost multiplicands of the hypergeometric products in $\tilde{H}(n)$ given in (6.138) are shift co-prime among each other. Furthermore,

$$P(n) = \tilde{c}\, r(n)\, \tilde{G}(n)\, \tilde{H}(n)$$

holds for all $n \geqslant 4$. ⋆

Definition 6.6.7.
Let $(\mathbb{K}(x), \sigma)$ be a rational difference field with $\sigma(x) = x+1$ and let $H_1(n), \dots, H_e(n)$ be hypergeometric product expressions in $\mathrm{ProdE}(\mathbb{K}(n))$ which are composed multiplicatively by hypergeometric products in product factored form. We say that $H_1(n), \dots, H_e(n)$ are in *reduced normal form* if

(1) the innermost multiplicand of each hypergeometric product factor in $H_\ell(n)$ for $1 \leqslant \ell \leqslant e$ is an irreducible monic polynomial in $\mathbb{K}[x]$;

(2) all the distinct innermost multiplicands of $H_1(n), \dots, H_e(n)$ are shift co-prime among each other.

Furthermore, we say that $H_1(n), \dots, H_e(n)$ are δ-*refined* if they are in reduced normal form and all lower bounds in each $H_\ell(n)$ for $1 \leqslant \ell \leqslant e$ are synchronised to $\delta \in \mathbb{N}$ such that for all $n \geqslant \delta$, $H_\ell(n) \neq 0$. ∗

Example 6.6.8 (Cont. Example 6.6.5).
The product expression $\tilde{H}(n)$ in (6.134) is in reduced normal form and furthermore, it is 1-refined. $\star$

Example 6.6.9 (Cont. Example 6.6.6).
The product expression $\tilde{H}(n)$ given in (6.138) above is in reduced normal form. In addition, since its lower bounds are all synchronised to 4 and $\tilde{H}(n) \neq 0$ for all $n \geqslant 4$, it is 4-refined. $\star$

Lemma 6.6.10.
Let $(\mathbb{K}(x), \sigma)$ *be a rational difference field with the automorphism* $\sigma(x) = x+1$ *and the evaluation function* $\mathrm{ev} : \mathbb{K}(x) \times \mathbb{N} \to \mathbb{K}$ *given by* (2.1)*. Let* $\tilde{H}_1(n), \dots, \tilde{H}_e(n)$ *be hypergeometric product expressions in* $\mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$ *of nesting depth at most* b *which are all in reduced normal form and* δ*-refined for some* $\delta \in \mathbb{N}$*. Then one can construct an ordered multiple chain* Π*-extension* $(\tilde{\mathbb{H}}_b, \sigma)$ *of* $(\mathbb{K}(x), \sigma)$ *with*

$$\tilde{\mathbb{H}}_b = \mathbb{K}(x)\langle \tilde{\boldsymbol{z}}_1 \rangle \dots \langle \tilde{\boldsymbol{z}}_b \rangle = \mathbb{K}(x)\langle \tilde{z}_{1,1} \rangle \dots \langle \tilde{z}_{p_1,1} \rangle \dots \langle \tilde{z}_{1,b} \rangle \dots \langle \tilde{z}_{p_b,b} \rangle \tag{6.139}$$

which is composed by the single chain Π*-extensions* $(\tilde{\mathbb{F}}_\ell, \sigma)$ *of* $(\mathbb{K}(x), \sigma)$ *where* $\tilde{\mathbb{F}}_\ell = \mathbb{K}(x)\langle \tilde{z}_{\ell,1} \rangle \dots \langle \tilde{z}_{\ell,s_\ell} \rangle$ *with*

(1) the automorphism $\sigma : \tilde{\mathbb{F}}_\ell \to \tilde{\mathbb{F}}_\ell$ *defined by*

$$\sigma(\tilde{z}_{\ell,k}) = \tilde{\alpha}_{\ell,k}\, \tilde{z}_{\ell,k} \quad \text{where} \quad \tilde{\alpha}_{\ell,k} = \tilde{f}_\ell\, \tilde{z}_{\ell,1} \cdots \tilde{z}_{\ell,k-1} \in (\mathbb{K}(x)^*)^{\mathbb{K}(x)\langle \tilde{z}_{\ell,1} \rangle \dots \langle \tilde{z}_{\ell,k-1} \rangle}_{\mathbb{K}(x)} \tag{6.140}$$

where $\tilde{f}_\ell \in \mathbb{K}[x] \setminus \mathbb{K}$ *is an irreducible monic polynomial for* $1 \leqslant \ell \leqslant p_1$ *and* $1 \leqslant k \leqslant s_\ell$*, and*

(2) the naturally induced evaluation function $\tilde{\mathrm{ev}} : \tilde{\mathbb{F}}_\ell \times \mathbb{N} \to \mathbb{K}$ *with respect to* δ *given by* $\tilde{\mathrm{ev}}|_{\mathbb{K}(x)} = \mathrm{ev}$ *with* (2.1) *and*

$$\tilde{\mathrm{ev}}(\tilde{z}_{\ell,k}, n) = \prod_{j=\delta}^{n} \tilde{\mathrm{ev}}(\tilde{\alpha}_{\ell,k}, j-1) \tag{6.141}$$

for $1 \leqslant \ell \leqslant p_1$ *and* $1 \leqslant k \leqslant s_\ell$*.*

Furthermore, for all $g \in \tilde{\mathbb{H}}_b$*, the map* $\tilde{\tau} : \tilde{\mathbb{H}}_b \to \mathcal{S}(\mathbb{K})$ *defined by*

$$\tilde{\tau}(g) = \left\langle \tilde{\mathrm{ev}}(g, n) \right\rangle_{n \geqslant 0} \tag{6.142}$$

is a $\mathbb{K}$*-embedding.* ♦

Proof:
The construction of the ordered multiple chain P-extension $(\tilde{\mathbb{H}}_b, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with (6.139) follows by the procedure outlined in Remark 6.3.5. Suppose we have constructed such an ordered multiple chain P-extension $(\tilde{\mathbb{H}}_b, \sigma)$ of $(\mathbb{K}(x), \sigma)$. Then since the bases $\tilde{f}_1, \dots, \tilde{f}_{p_1}$ of the single chain Π-extensions $(\tilde{\mathbb{F}}_1, \sigma), \dots, (\tilde{\mathbb{F}}_{p_1}, \sigma)$ that composes $(\tilde{\mathbb{H}}_b, \sigma)$ are shift co-prime, i.e.,

$$\gcd_\sigma(\tilde{f}_i, \tilde{f}_j) = 1$$

for all i, j with $1 \leqslant i < j \leqslant p_1$, by Theorem 6.6.1 it follows that $(\tilde{\mathbb{H}}_b, \sigma)$ is a Π-extension of $(\mathbb{K}(x), \sigma)$. In particular, it is an ordered multiple chain Π-extension of monomial depth at most b. Since $(\tilde{\mathbb{H}}_b, \sigma)$ is a Π-extension of a rational difference field $(\mathbb{K}(x), \sigma)$, it follows by statement (2) of Lemma 2.4.3 that

$$\tilde{\tau} : \tilde{\mathbb{H}}_b \to \mathcal{S}(\mathbb{K})$$

defined by (6.142) is a $\mathbb{K}$-embedding. ■

Example 6.6.11 (Cont. Example 6.6.8).
Since the nesting depth 2 hypergeometric product expression $\tilde{H}(n)$ in (6.134) is in reduced normal form and furthermore 1-refined, it follows by Lemma 6.6.10 that there is an ordered multiple chain Π-extension $(\tilde{\mathbb{H}}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ of monomial depth 2 with

$$\tilde{\mathbb{H}} = \mathbb{K}(x)\langle \tilde{z}_{1,1}\rangle\langle \tilde{z}_{2,1}\rangle\langle \tilde{z}_{2,2}\rangle$$

where $(\tilde{\mathbb{H}}, \sigma)$ is composed by the following single chain Π-extensions:

(1) $(\tilde{\mathbb{F}}_1, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}(x)$ where $\tilde{\mathbb{F}}_1 = \mathbb{K}(x)\langle \tilde{z}_{1,1}\rangle$ and the automorphism $\sigma : \tilde{\mathbb{F}}_1 \to \tilde{\mathbb{F}}_1$ and the naturally induced evaluation function $\tilde{ev} : \tilde{\mathbb{F}}_1 \times \mathbb{N} \to \mathbb{K}$ with respect to 1 are defined as:

$$\begin{aligned} \sigma(x) &= x + 1, & \sigma(\tilde{z}_{1,1}) &= (x + 1)\, \tilde{z}_{1,1}, \\ \tilde{ev}(x, n) &= n, & \tilde{ev}(\tilde{z}_{1,1}, n) &= \prod_{k=1}^{n} k. \end{aligned} \tag{6.143}$$

(2) $(\tilde{\mathbb{F}}_2, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}(x)$ where $\tilde{\mathbb{F}}_2 = \mathbb{K}(x)\langle \tilde{z}_{2,1}\rangle\langle \tilde{z}_{2,2}\rangle$ and the automorphism $\sigma : \tilde{\mathbb{F}}_2 \to \tilde{\mathbb{F}}_2$ and the naturally induced evaluation function $\tilde{ev} : \tilde{\mathbb{F}}_2 \times \mathbb{N} \to \mathbb{K}$ with respect to 1 are defined as:

$$\begin{aligned} \sigma(x) &= x + 1, & \sigma(\tilde{z}_{2,1}) &= (x + \tfrac{3}{2})\, \tilde{z}_{2,1}, & \sigma(\tilde{z}_{2,2}) &= (x + \tfrac{3}{2})\, \tilde{z}_{2,1}\, \tilde{z}_{2,2}, \\ \tilde{ev}(x, n) &= n, & \tilde{ev}(\tilde{z}_{2,1}, n) &= \prod_{k=1}^{n} (k + \tfrac{1}{2}), & \tilde{ev}(\tilde{z}_{2,2}, n) &= \prod_{k=1}^{n}\prod_{i=1}^{k} (i + \tfrac{1}{2}). \end{aligned} \tag{6.144}$$

Furthermore, the map

$$\tilde{\tau} : \tilde{\mathbb{H}} \to \mathcal{S}(\mathbb{K})$$

defined by

$$\tilde{\tau}(f) = \big\langle \tilde{ev}(f, n) \big\rangle_{n \geqslant 0}$$

for all $f \in \tilde{\mathbb{H}}_b$ is a $\mathbb{K}$-embedding. ⋆

Example 6.6.12 (Cont. 6.6.9).
From Example 6.6.9, we know that the nesting depth 2 hypergeometric expression $\tilde{H}(n)$ given in (6.138) is 4-refined. Thus by Lemma 6.6.10 we can construct the ordered multiple chain Π-extension $(\tilde{\mathbb{H}}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ of monomial depth 2 with

$$\tilde{\mathbb{H}} = \mathbb{K}(x)\langle \tilde{z}_{1,1}\rangle\langle \tilde{z}_{2,1}\rangle\langle \tilde{z}_{1,2}\rangle$$

where $(\tilde{\mathbb{H}}, \sigma)$ is composed by the following single chain Π-extensions:

(1) $(\tilde{\mathbb{F}}_1, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}(x)$ where $\tilde{\mathbb{F}}_1 = \mathbb{K}(x)\langle \tilde{z}_{1,1}\rangle\langle \tilde{z}_{1,2}\rangle$ and the automorphism $\sigma : \tilde{\mathbb{F}}_1 \to \tilde{\mathbb{F}}_1$ and the naturally induced evaluation function $\tilde{ev} : \tilde{\mathbb{F}}_1 \times \mathbb{N} \to \mathbb{K}$ with respect to 4 are defined as:

$$\begin{aligned} \sigma(x) &= x + 1, & \sigma(\tilde{z}_{1,1}) &= (x - 2)\, \tilde{z}_{1,1}, & \sigma(\tilde{z}_{1,2}) &= (x - 2)\, \tilde{z}_{1,1}\, \tilde{z}_{1,2}, \\ \tilde{ev}(x, n) &= n, & \tilde{ev}(\tilde{z}_{1,1}, n) &= \prod_{k=4}^{n} (k - 3), & \tilde{ev}(\tilde{z}_{1,2}, n) &= \prod_{k=4}^{n}\prod_{j=4}^{k} (j - 3). \end{aligned} \tag{6.145}$$

(2) $(\tilde{\mathbb{F}}_2, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}(x)$ where $\tilde{\mathbb{F}}_2 = \mathbb{K}(x)\langle \tilde{z}_{2,1}\rangle$ and the automorphism $\sigma : \tilde{\mathbb{F}}_2 \to \tilde{\mathbb{F}}_2$ and the naturally induced evaluation function $\tilde{\mathrm{ev}} : \tilde{\mathbb{F}}_2 \times \mathbb{N} \to \mathbb{K}$ with respect to 4 are defined as:

$$\begin{aligned} \sigma(x) &= x+1, & \sigma(\tilde{z}_{2,1}) &= ((x+1)^2 + \tfrac{1}{4})\, \tilde{z}_{2,1}, \\ \tilde{\mathrm{ev}}(x, n) &= n, & \tilde{\mathrm{ev}}(\tilde{z}_{2,1}, n) &= \prod_{k=1}^{n} k^2 + \tfrac{1}{4}. \end{aligned} \tag{6.146}$$

Furthermore, by statement (2) of Lemma 2.4.3 it follows that the $\mathbb{K}$-homomorphism

$$\tilde{\tau} : \tilde{\mathbb{H}} \to \mathcal{S}(\mathbb{K})$$

defined by

$$\tilde{\tau}(f) = \left\langle \tilde{\mathrm{ev}}(f, n) \right\rangle_{n \geqslant 0}$$

for all $f \in \tilde{\mathbb{H}}_b$ is a $\mathbb{K}$-embedding. ⋆

So far we have treated hypergeometric products over monic and irreducible polynomials of finite nesting depth say b that are δ-refined for some $\delta \in \mathbb{N}$; see Definition 6.6.7. Given such hypergeometric products, it follows by Lemma 6.6.10 that we can construct an ordered multiple chain Π-extension $(\tilde{\mathbb{H}}_b, \sigma)$ of $(\mathbb{K}(x), \sigma)$ where $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ and

$$\tilde{\mathbb{H}}_b = \mathbb{K}(x)\langle \tilde{\boldsymbol{z}}_1\rangle \dots \langle \tilde{\boldsymbol{z}}_b\rangle = \mathbb{K}(x)\langle \tilde{z}_{1,1}\rangle \dots \langle \tilde{z}_{p_1,1}\rangle \dots \langle \tilde{z}_{1,b}\rangle \dots \langle \tilde{z}_{p_b,b}\rangle. \tag{6.147}$$

In particular, $(\tilde{\mathbb{H}}_b, \sigma)$ is composed by the single chain Π-extensions $(\tilde{\mathbb{F}}_\ell, \sigma)$ of $(\mathbb{K}(x), \sigma)$ where

$$\tilde{\mathbb{F}}_\ell = \mathbb{K}(x)\langle \tilde{z}_{\ell,1}\rangle\langle \tilde{z}_{\ell,2}\rangle \dots \langle \tilde{z}_{\ell,s_\ell}\rangle$$

for $1 \leqslant \ell \leqslant p_1$ and $1 \leqslant k \leqslant s_\ell$ together with the automorphism $\sigma : \tilde{\mathbb{F}}_\ell \to \tilde{\mathbb{F}}_\ell$ defined by

$$\sigma(\tilde{z}_{\ell,k}) = \tilde{\alpha}_{\ell,k}\, \tilde{z}_{\ell,k} \quad \text{where} \quad \tilde{\alpha}_{\ell,k} = \tilde{f}_\ell\, \tilde{z}_{\ell,1} \cdots \tilde{z}_{\ell,k-1} \in (\mathbb{K}(x)^*)_{\mathbb{K}(x)}^{\mathbb{K}(x)\langle \tilde{z}_{\ell,1}\rangle \dots \langle \tilde{z}_{\ell,k-1}\rangle} \tag{6.148}$$

and the naturally induced evaluation function $\tilde{\mathrm{ev}} : \tilde{\mathbb{F}}_\ell \times \mathbb{N} \to \mathbb{K}$ with respect to $\delta \in \mathbb{N}$ defined by

$$\tilde{\mathrm{ev}}(\tilde{z}_{\ell,k}, n) = \prod_{j=\delta}^{n} \tilde{\mathrm{ev}}(\tilde{\alpha}_{\ell,k}, j-1). \tag{6.149}$$

On the other hand, geometric products over the contents was also treated in Section 6.5. In particular these products appeared additionally by the approach described in Lemmas 6.1.5 and 6.6.3. All these geometric products can be modelled in an $R\Pi$-extension. More precisely, in Theorem 6.5.10 we constructed a simple $R\Pi$-extension $(\mathbb{D}, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ where $\tilde{\mathbb{K}} = \tilde{K}(\kappa_1, \dots, \kappa_u)$, $\tilde{K}$ is a finite algebraic field extension of K and

$$\mathbb{D} = \tilde{\mathbb{K}}[\vartheta]\langle \tilde{y}_{1,1}\rangle \dots \langle \tilde{y}_{e_1,1}\rangle \dots \langle \tilde{y}_{1,d}\rangle \dots \langle \tilde{y}_{e_d,d}\rangle \tag{6.150}$$

with

(a) the automorphism $\sigma : \mathbb{D} \to \mathbb{D}$ defined by

$$\sigma(\vartheta) = \zeta\, \vartheta, \tag{6.151}$$

$$\sigma(\tilde{y}_{\ell,k}) = \tilde{\gamma}_{\ell,k}\, \tilde{y}_{\ell,k} \tag{6.152}$$

where $\zeta \in \tilde{K}$ is a λ-th root of unity and

$$\tilde{\gamma}_{\ell,k} = \tilde{h}_\ell\, \tilde{y}_{\ell,1} \cdots \tilde{y}_{\ell,k-1} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}\langle \tilde{y}_{\ell,1}\rangle \dots \langle \tilde{y}_{\ell,k-1}\rangle}$$

for $1 \leqslant k \leqslant d$ and $1 \leqslant \ell \leqslant e_k$ and

(b) the naturally evaluation function $\tilde{\mathrm{ev}}: \mathbb{D} \times \mathbb{N} \to \tilde{\mathbb{K}}$ defined by

$$\tilde{\mathrm{ev}}(\vartheta, n) = \prod_{j=1}^{n} \zeta, \tag{6.153}$$

$$\tilde{\mathrm{ev}}(\tilde{y}_{\ell,k}, n) = \prod_{j=1}^{n} \tilde{\mathrm{ev}}(\tilde{\gamma}_{\ell,k}, j-1). \tag{6.154}$$

In particular, by reordering we obtain the difference ring extension $(\tilde{\mathbb{A}}_d, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with

$$\tilde{\mathbb{A}}_d = \tilde{\mathbb{K}}\langle \tilde{\mathbf{y}}_1 \rangle \dots \langle \tilde{\mathbf{y}}_d \rangle = \tilde{\mathbb{K}}\langle \tilde{y}_{1,1} \rangle \dots \langle \tilde{y}_{e_1,1} \rangle \dots \langle \tilde{y}_{1,d} \rangle \dots \langle \tilde{y}_{e_d,d} \rangle, \tag{6.155}$$

the automorphism (6.152) and the naturally induced evaluation function (6.154) which is a sub-difference ring of $(\mathbb{D}, \sigma)$. Furthermore, it is an ordered multiple chain Π-extension of $(\tilde{\mathbb{K}}, \sigma)$ and is composed by the single chain Π-extensions $(\tilde{\mathbb{K}}_\ell, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ where

$$\tilde{\mathbb{K}}_\ell = \tilde{\mathbb{K}}\langle \tilde{y}_{\ell,1} \rangle \langle \tilde{y}_{\ell,2} \rangle \dots \langle \tilde{y}_{\ell,\tilde{e}_\ell} \rangle$$

for $1 \leqslant \ell \leqslant e_1$.

Putting the two difference rings $(\tilde{\mathbb{H}}_b, \sigma)$ with (6.147) and $(\mathbb{D}, \sigma)$ with (6.150) together, we are able to completely model any finite set of hypergeometric product expressions of finite nesting depth coming from $\mathrm{ProdE}(\mathbb{K}(n))$ in a simple RΠ-extension. More precisely, we have the following theorem.

Theorem 6.6.13.
Let $(\mathbb{K}(x), \sigma)$ *be a rational difference field over* $\mathbb{K}$ *with* $\sigma(x) = x + 1$ *and let the difference ring* $(\tilde{\mathbb{H}}_b, \sigma)$ *with* (6.147) *be an ordered multiple chain* Π*-extension of* $(\mathbb{K}(x), \sigma)$ *with the automorphism* (6.148)*. Further, let* $\tilde{\mathbb{K}}$ *be an algebraic field extension of* $\mathbb{K}$ *and let the difference ring* $(\tilde{\mathbb{A}}_d, \sigma)$ *with* (6.155) *be the ordered multiple chain* Π*-extension of* $(\tilde{\mathbb{K}}, \sigma)$ *with the automorphism* (6.152)*. Then the difference ring* $(\tilde{\mathbb{E}}, \sigma)$ *with*

$$\tilde{\mathbb{E}} = \tilde{\mathbb{K}}(x)\langle \tilde{\mathbf{y}}_1 \rangle \langle \tilde{\mathbf{z}}_1 \rangle \dots \langle \tilde{\mathbf{y}}_d \rangle \langle \tilde{\mathbf{z}}_b \rangle \tag{6.156}$$

where $\langle \tilde{\mathbf{y}}_i \rangle = \langle \tilde{y}_{1,i} \rangle \dots \langle \tilde{y}_{e_i,i} \rangle$ *for* $1 \leqslant i \leqslant d$ *and* $\langle \tilde{\mathbf{z}}_k \rangle = \langle \tilde{z}_{1,k} \rangle \dots \langle \tilde{z}_{p_k,k} \rangle$ *for* $1 \leqslant k \leqslant b$ *is an ordered multiple chain* Π*-extension of* $(\tilde{\mathbb{K}}(x), \sigma)$*. Furthermore, the* A*-extension* $(\mathbb{E}, \sigma)$ *of* $(\tilde{\mathbb{E}}, \sigma)$ *where* $\mathbb{E} = \tilde{\mathbb{E}}[\vartheta]$ *with* (6.151) *of order* λ *is an* R*-extension.* ♢

Proof:
Take the Π-extensions $(\tilde{\mathbb{H}}_1, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $\tilde{\mathbb{H}}_1 = \mathbb{K}(x)\langle z_{1,1} \rangle \dots \langle z_{p_1,1} \rangle$ and $(\tilde{\mathbb{A}}_1, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with $\tilde{\mathbb{A}}_1 = \tilde{\mathbb{K}}\langle \tilde{y}_{1,1} \rangle \dots \langle \tilde{y}_{e_1,1} \rangle$ which are both of monomial depth 1. By Lemma 5.4.4 the difference ring $(\mathbb{E}_1, \sigma)$ with

$$\tilde{\mathbb{E}}_1 = \tilde{\mathbb{K}}(x)\langle \tilde{y}_{1,1} \rangle \dots \langle \tilde{y}_{e_1,1} \rangle \langle \tilde{z}_{1,1} \rangle \dots \langle \tilde{z}_{p_1,1} \rangle$$

is a Π-extension of $(\tilde{\mathbb{K}}(x), \sigma)$ of monomial depth 1. Consider the ordered multiple chain P-extension $(\tilde{\mathbb{E}}, \sigma)$ of $(\tilde{\mathbb{K}}(x), \sigma)$ with (6.156) which is composed by the single chain Π-extensions in the ordered multiple chains $(\tilde{\mathbb{H}}_b, \sigma)$ and $(\tilde{\mathbb{A}}_d, \sigma)$. By Theorem 6.4.14 it follows that $(\tilde{\mathbb{E}}, \sigma)$ is a Π-extension of $(\tilde{\mathbb{K}}(x), \sigma)$. By Corollary 2.3.58 the A-extension $(\mathbb{E}, \sigma)$ of $(\tilde{\mathbb{E}}, \sigma)$ where $\mathbb{E} = \tilde{\mathbb{E}}[\vartheta]$ with the automorphism (6.151) is an R-extension. ■

Example 6.6.14 (Cont. Examples 6.6.5, 6.6.8, 6.6.11 and 6.5.11).
Let $(\mathbb{K}(x), \sigma)$ be the rational difference field with $\mathbb{K} = \mathbb{Q}(\sqrt{6})$ equipped with the automorphism $\sigma(x) = x + 1$ and the evaluation function (2.1). We will represent the hypergeometric product expression

$$P(n) = \prod_{k=1}^{n} k \, \frac{\left(\prod_{i=1}^{k} \frac{-1}{\sqrt{6}\,(i+3)}\right) \left(\prod_{i=1}^{k} \frac{(i+2)}{(i+\frac{1}{2})}\right)}{2\,(k+3)} \in \mathrm{ProdE}(\mathbb{K}(n)) \tag{6.157}$$

of nesting depth 2 in an RΠ-extension $(\mathbb{E}, \sigma)$ of the rational difference field $(\tilde{\mathbb{K}}(x), \sigma)$ where $\tilde{\mathbb{K}}$ is some finite algebraic field extension of $\mathbb{K}$.

(1) By Lemma 6.6.3 we can compute a $\delta \in \mathbb{N}$ and construct a $\tilde{c} \in \mathbb{K}^*$, an $\tilde{r}(n) \in \mathbb{K}(n)^*$, a geometric product expression $\tilde{G}(n) \in \mathrm{ProdE}(\mathbb{K})$ and a hypergeometric product expression $\tilde{H}(n) \in \mathrm{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$ with

$$\begin{aligned}
\tilde{c} &= 36, \\
\tilde{r}(n) &= \frac{1}{(n+1)^2\,(n+2)^2\,(n+3)^2}, \\
\tilde{G}(n) &= \left(\prod_{k=1}^{n} \frac{1}{2}\right) \left(\prod_{k=1}^{n} 3\right) \left(\prod_{k=1}^{n} \prod_{i=1}^{k} -1\right) \left(\prod_{k=1}^{n} \prod_{i=1}^{k} \frac{1}{\sqrt{6}}\right), \\
\tilde{H}(n) &= \left(\prod_{k=1}^{n} \frac{1}{k}\right) \left(\prod_{k=1}^{n} \prod_{i=1}^{k} \frac{1}{(i+\frac{1}{2})}\right)
\end{aligned}$$

such that for all $n \geqslant 1$

$$P(n) = \tilde{c}\,\tilde{r}(n)\,\tilde{G}(n)\,\tilde{H}(n)$$

holds.

(2) Since the nesting depth 2 hypergeometric expression $\tilde{H}(n)$ is in reduced normal form and also 1-refined, by Lemma 6.6.10 we can construct the ordered multiple chain Π-extension $(\tilde{\mathbb{H}}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with

$$\tilde{\mathbb{H}} = \mathbb{K}(x)\langle \tilde{z}_{1,1} \rangle \langle \tilde{z}_{2,1} \rangle \langle \tilde{z}_{2,2} \rangle$$

of monomial depth 2. In particular, $(\tilde{\mathbb{H}}, \sigma)$ is composed by the single chain Π-extensions $(\tilde{\mathbb{F}}_1, \sigma)$ and $(\tilde{\mathbb{F}}_2, \sigma)$ given by items (1) and (2) respectively in Example 6.6.11. Further $\tilde{H}(n)$ is modelled by

$$\tilde{H}_1 = \frac{1}{\tilde{z}_{1,1}\,\tilde{z}_{2,2}} \in \tilde{\mathbb{H}}.$$

That is,

$$\tilde{H}(n) = \tilde{\mathrm{ev}}(H_1, n)$$

holds for all $n \geqslant 1$.

(3) We continue with the geometric product expression $\tilde{G}(n)$. By Theorem 6.5.10 we can construct an RΠ-extension $(\mathbb{D}, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with

$$\mathbb{D} = \tilde{\mathbb{K}}[\vartheta]\langle \tilde{y}_{1,1} \rangle \langle \tilde{y}_{2,1} \rangle \langle \tilde{y}_{1,2} \rangle \langle \tilde{y}_{2,2} \rangle = \boldsymbol{e}_0 \tilde{\mathbb{D}} \oplus \boldsymbol{e}_1 \tilde{\mathbb{D}} \oplus \boldsymbol{e}_2 \tilde{\mathbb{D}} \oplus \boldsymbol{e}_3 \tilde{\mathbb{D}}$$

where $\tilde{\mathbb{D}} = \tilde{\mathbb{K}}\langle \tilde{y}_{1,1} \rangle \langle \tilde{y}_{2,1} \rangle \langle \tilde{y}_{1,2} \rangle \langle \tilde{y}_{2,2} \rangle$, $\tilde{\mathbb{K}} = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $\boldsymbol{e}_k$ for $0 \leqslant k \leqslant 3$ are the idempotent elements given by (6.52). The automorphism $\sigma : \mathbb{D} \to \mathbb{D}$ and the naturally induced evaluation function $\tilde{\text{ev}} : \mathbb{D} \times \mathbb{N} \to \tilde{\mathbb{K}}$ are defined by (6.103), (6.108), (6.117) and (6.120). In particular $\tilde{G}(n)$ is modelled by

$$\tilde{G}_1 = \frac{(1 - \mathbb{i})\,\vartheta\,(\vartheta^2 + \mathbb{i})\,\tilde{y}_{2,1}^2}{2\,\tilde{y}_{1,1}^2\,\tilde{y}_{1,2}\,\tilde{y}_{2,2}} \in \mathbb{D}$$

and with

$$\tilde{F}(n) = \tilde{\text{ev}}(\tilde{G}_1, n) = \frac{(1 - \mathbb{i})\,(\mathbb{i})^n\,((\mathbb{i}^n)^2 + \mathbb{i})\left((\sqrt{3})^n\right)^2}{2\left((\sqrt{2})^n\right)^2 (\sqrt{2})^{\binom{n+1}{2}}\,(\sqrt{3})^{\binom{n+1}{2}}}$$

we have that

$$\tilde{G}(n) = \tilde{F}(n)$$

holds for all $n \geqslant 1$.

(4) Putting everything together, it follows by Theorem 6.6.13 that $(\mathbb{E}, \sigma)$ is an $R\Pi$-extension of $(\tilde{\mathbb{K}}(x), \sigma)$ where

$$\mathbb{E} = \tilde{\mathbb{K}}(x)[\vartheta]\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{z}_{1,1}\rangle\langle\tilde{z}_{2,1}\rangle\langle\tilde{y}_{1,2}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{z}_{2,2}\rangle = \boldsymbol{e}_0\tilde{\mathbb{E}} \oplus \boldsymbol{e}_1\tilde{\mathbb{E}} \oplus \boldsymbol{e}_2\tilde{\mathbb{E}} \oplus \boldsymbol{e}_3\tilde{\mathbb{E}}$$

with $\tilde{\mathbb{E}} = \tilde{\mathbb{K}}(x)\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{z}_{1,1}\rangle\langle\tilde{z}_{2,1}\rangle\langle\tilde{y}_{1,2}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{z}_{2,2}\rangle$. The automorphism $\sigma : \mathbb{E} \to \mathbb{E}$ and the evaluation function $\tilde{\text{ev}} : \mathbb{E} \times \mathbb{N} \to \tilde{\mathbb{K}}$ are given by (6.103), (6.108), (6.117), (6.120), (6.143) and (6.144). In this difference ring, the nesting depth 2 hypergeometric product expression $P(n)$ given by (6.157) is modelled by

$$\tilde{P} = \frac{36\,\tilde{G}_1\,\tilde{H}_1}{(x+3)^2\,(x+2)^2\,(x+1)^2} = \frac{18\,(1 - \mathbb{i})\,\vartheta\,(\vartheta^2 + \mathbb{i})\,\tilde{y}_{2,1}^2}{(x+3)^2\,(x+2)^2\,(x+1)^2\,\tilde{y}_{1,1}^2\,\tilde{y}_{1,2}\,\tilde{y}_{2,2}\,\tilde{z}_{1,1}\,\tilde{z}_{2,2}} \in \mathbb{E}$$

and with

$$\tilde{P}(n) = \tilde{\text{ev}}(\tilde{P}, n) = \frac{18\,(1 - \mathbb{i})\,(\mathbb{i})^n\,((\mathbb{i}^n)^2 + \mathbb{i})\left((\sqrt{3})^n\right)^2}{(n+3)^2\,(n+2)^2\,(n+1)^2\left((\sqrt{2})^n\right)^2 (\sqrt{2})^{\binom{n+1}{2}}\,(\sqrt{3})^{\binom{n+1}{2}}\,n!}\left(\prod_{k=1}^{n}\prod_{i=1}^{k}\frac{1}{(i+\frac{1}{2})}\right)$$

we have that

$$P(n) = \tilde{P}(n)$$

holds for all $n \geqslant 1$. ⋆

Summarising we have the following theorem.

Theorem 6.6.15.
Let $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x + 1$ be the rational difference field satisfying Lemma 4.1.6 together with the evaluation function $\text{ev} : \mathbb{K}(x) \times \mathbb{N} \to \mathbb{K}$ defined by (2.1) and the Z-function defined by (2.50). In particular, $\mathbb{K}$ is the rational function field $K(\kappa_1, \dots, \kappa_u)$ over a field K. Suppose we are given a finite set of hypergeometric product expressions

$$\{P_1(n), \dots, P_e(n)\} \subseteq \mathrm{ProdE}(\mathbb{K}(n)) \tag{6.158}$$

of nesting depth at most d for some $d \in \mathbb{N}$. Then there is a $\delta \in \mathbb{N}$ and an $R\Pi$-extension $(\mathbb{E}, \sigma)$ of $(\tilde{\mathbb{K}}(x), \sigma)$ of monomial depth at most d where $\tilde{\mathbb{K}}$ is a finite algebraic field extension of $\mathbb{K}$ equipped with a naturally induced evaluation function $\tilde{\text{ev}} : \mathbb{E} \times \mathbb{N} \to \tilde{\mathbb{K}}$ with respect to δ with the following properties:

(1) The map $\tau : \mathbb{E} \to \mathcal{S}(\tilde{\mathbb{K}})$ *with* $\tau(f) = \langle \tilde{\text{ev}}(f, n) \rangle_{n \geqslant 0}$ *is a* $\tilde{\mathbb{K}}$*-embedding.*

(2) There are elements $g_1, \dots, g_e \in \mathbb{E}^*$ *such that for* j *with* $1 \leqslant j \leqslant e$ *and for all* $n \geqslant \delta$

$$P_j(n) = \tilde{\text{ev}}(g_j, n)$$

holds.

If $\mathbb{K}$ *is a strongly* σ*-computable, the components of the theorem can be computed.* ◇

PROOF:

(a) Given the hypergeometric product expressions in (6.158), it follows by Lemma 6.6.3 that we can compute a $\delta \in \mathbb{N}$ and construct for all $1 \leqslant j \leqslant e$, $c_j \in \mathbb{K}^*$, rational functions $r_j \in \mathbb{K}(n)^*$, geometric product expressions $\tilde{G}_j(n) \in \text{ProdE}(\mathbb{K})$ and hypergeometric product expressions $\tilde{H}_j(n) \in \text{ProdE}(\mathbb{K}(n) \setminus \mathbb{K})$ such that

$$P_j(n) = \tilde{c}_j \, \tilde{r}_j(n) \, \tilde{G}_j(n) \, \tilde{H}_j(n) \neq 0 \tag{6.159}$$

holds for all $n \geqslant \delta$.

(b) Take the hypergeometric product expressions $\tilde{H}_1(n), \dots, \tilde{H}_e(n)$ in (6.159). Note that since each $\tilde{H}_\ell(n)$ is in reduced normal form with all lower bounds are synchronised to $\delta \in \mathbb{N}$ and they are non-zero for all $n \geqslant \delta$, it follows by Definition 6.6.7 that they are δ-refined. By Lemma 6.6.10 we can construct an ordered multiple chain Π-extension $(\tilde{\mathbb{H}}_b, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with (6.147) which is composed by the single chain Π-extensions $(\tilde{\mathbb{F}}_\ell, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $1 \leqslant \ell \leqslant p_1$ for some $p_1 \in \mathbb{N}$ where

$$\tilde{\mathbb{F}}_\ell = \mathbb{K}(x)\langle \tilde{z}_{\ell,1} \rangle \langle \tilde{z}_{\ell,2} \rangle \dots \langle \tilde{z}_{\ell,s_\ell} \rangle$$

with the automorphism $\sigma : \tilde{\mathbb{F}}_\ell \to \tilde{\mathbb{F}}_\ell$ and the naturally induced evaluation function $\tilde{\text{ev}} : \tilde{\mathbb{F}}_\ell \times \mathbb{N} \to \mathbb{K}$ with respect to δ defined by (6.148) and (6.149) respectively. In particular, there is an $\tilde{h}_j \in \tilde{\mathbb{H}}_b$ that models $\tilde{H}_j(n)$ for all j with $1 \leqslant j \leqslant e$. That is,

$$\tilde{\text{ev}}(\tilde{h}_j, n) = \tilde{H}_j(n) \quad \forall n \geqslant \delta. \tag{6.160}$$

(c) Next we take the geometric product expressions $\tilde{G}_1(n), \dots, \tilde{G}_e(n)$ in (6.159). Then by Theorem 6.5.10 we can construct an RΠ-extension $(\mathbb{D}, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with (6.150) together with the automorphism $\sigma : \mathbb{D} \to \mathbb{D}$ and the naturally induced evaluation function $\tilde{\text{ev}} : \mathbb{D} \times \mathbb{N} \to \tilde{\mathbb{K}}$ defined by (6.151), (6.152) and (6.105), (6.154) respectively. In particular, for each $\tilde{G}_j(n)$ there is a $\tilde{g}_j \in \mathbb{D}$ that models it. That is,

$$\tilde{\text{ev}}(\tilde{g}_j, n) = \tilde{G}_j(n) \quad \forall n \geqslant \delta. \tag{6.161}$$

(d) By Theorem 6.6.13 we can merge these two difference rings to obtain an RΠ-extension $(\mathbb{E}, \sigma)$ of $(\tilde{\mathbb{K}}(x), \sigma)$ with (6.156) and the automorphism $\sigma : \mathbb{E} \to \mathbb{E}$ and the evaluation function $\tilde{\text{ev}} : \mathbb{E} \times \mathbb{N} \to \tilde{\mathbb{K}}$ defined accordingly.

(e) In particular, since $(\mathbb{E}, \sigma)$ is an RΠ-extension of a rational difference field $(\tilde{\mathbb{K}}(x), \sigma)$, it follows by part (2) of Lemma 2.4.3 that $\tilde{\tau} : \mathbb{E} \to \mathcal{S}(\tilde{\mathbb{K}})$ defined by (6.142) is a $\tilde{\mathbb{K}}$-embedding. For $1 \leqslant j \leqslant e$, define

$$g_j := \tilde{c}_j\, \tilde{r}_j\, \tilde{g}_j\, \tilde{h}_j$$

where $\tilde{\text{ev}}(\tilde{r}_j, n) = \tilde{r}_j(n)$ and the evaluation of $\tilde{h}_j$ and $\tilde{g}_j$ are given by (6.160) and (6.161) respectively. Then with (6.159) it follows that for all j with $1 \leqslant j \leqslant e$ and for all $n \geqslant \delta$ we get

$$P_j(n) = \tilde{\text{ev}}(g_j, n).$$

Finally observe that if K is strongly σ-computable, all the ingredients delivered by Theorems 5.1.1 and 5.3.3 can be computed and consequently, $(\mathbb{E}, \sigma)$ can be constructed and τ and $g_1, \ldots, g_e$ can be computed explicitly. ■

Example 6.6.16 (Cont. Examples 6.6.6, 6.6.9, 6.6.12).
Let $(\mathbb{K}(x), \sigma)$ be the rational difference field with $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$ equippped with the field automorphism $\sigma : \mathbb{K}(x) \to \mathbb{K}(x)$ and the evaluation function $\text{ev} : \mathbb{K}(x) \times \mathbb{N} \to \mathbb{K}$ defined by $\sigma(x) = x + 1$ and (2.1) respectively. Given the nesting depth 2 hypergeometric product $P(n)$ with (6.135) in Example 6.6.6, we compueted $\delta = 4$ and refined the given hypergeometric product $P(n)$ to get

$$P(n) = \tilde{c}\, r(n)\, \tilde{G}(n)\, \tilde{H}(n) \tag{6.162}$$

where $\tilde{c}$, $r(n)$, $\tilde{G}(n)$ and $\tilde{H}(n)$ are given in (6.138). In particular, (6.162) holds for all $n \in \mathbb{N}$ with $n \geqslant 4$. Furthermore, from Example 6.6.9 we know that the hypergeometric product expression $\tilde{H}(n)$, is 4-refined and in Example 6.6.12 we constructed the ordered multiple chain Π-extension $(\tilde{\mathbb{H}}, \sigma)$ of $(\mathbb{K}(x), \sigma)$. There, $(\tilde{\mathbb{H}}, \sigma)$ was composed by the single chain Π-extensions $(\tilde{\mathbb{F}}_1, \sigma)$ and $(\tilde{\mathbb{F}}_2, \sigma)$ of $(\mathbb{K}(x), \sigma)$ defined in items (1) and (2) respectively. The automorphism and the naturally induced evaluation function with respect to δ were defined as (6.145) and (6.146) respectively. In particular, the hypergeometric product expression $\tilde{H}(n)$ is modelled by the expression

$$\tilde{h} = \tilde{z}_{2,1}\, \tilde{z}_{1,1}^2\, \tilde{z}_{1,2} \in \tilde{\mathbb{H}}.$$

It only remains to construct an RΠ-extension to model the geometric product expression $\tilde{G}(n)$. Using my Mathematica package `NestedProducts`, we are able to construct the RΠ-extension $(\mathbb{D}, \sigma)$ of $(\mathbb{K}, \sigma)$ with

$$\mathbb{D} = \mathbb{K}[\vartheta]\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{3,1}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{y}_{3,2}\rangle = \boldsymbol{e}_0\tilde{\mathbb{D}} \oplus \boldsymbol{e}_1\tilde{\mathbb{D}} \oplus \boldsymbol{e}_2\tilde{\mathbb{D}} \oplus \boldsymbol{e}_3\tilde{\mathbb{D}}$$

where $\tilde{\mathbb{D}} = \mathbb{K}\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{3,1}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{y}_{3,2}\rangle$ and $\boldsymbol{e}_k$ for $0 \leqslant k \leqslant 3$ are the idempotent elements given by (6.52). In particular, $(\mathbb{D}, \sigma)$ is composed by the following difference rings.

(1) The single R-extension $(\mathbb{K}[\vartheta], \sigma)$ of $(\mathbb{K}, \sigma)$ with the automorphism $\sigma : \mathbb{K}[\vartheta] \to \mathbb{K}[\vartheta]$ and the naturally induced evaluation function $\tilde{\text{ev}} : \mathbb{K}[\vartheta] \times \mathbb{N} \to \mathbb{K}$ defined by (6.103) and (6.108) respectively.

(2) The ordered multiple chain Π-extension $(\tilde{\mathbb{D}}, \sigma)$ of $(\mathbb{K}, \sigma)$ which is composed by the following single chain Π-extensions over $\mathbb{K}$.

   (i) The single chain Π-extension $(\mathbb{K}_1, \sigma)$ of $(\mathbb{K}, \sigma)$ where $\mathbb{K}_1 = \mathbb{K}\langle\tilde{y}_{1,1}\rangle$ with the automorphism $\sigma : \mathbb{K}_1 \to \mathbb{K}_1$ and the naturally induced evaluation function $\tilde{\text{ev}} : \mathbb{K}_1 \times \mathbb{N} \to \mathbb{K}$ defined as

$$\begin{gathered} \sigma(c) = c,\, \forall\, c \in \mathbb{K}, \qquad \sigma(\tilde{y}_{1,1}) = \sqrt{3}\, \tilde{y}_{1,1}, \\ \tilde{\text{ev}}(c, n) = c,\, \forall\, c \in \mathbb{K}, \quad \tilde{\text{ev}}(\tilde{y}_{1,1}, n) = \prod_{k=1}^{n} \sqrt{3}. \end{gathered} \tag{6.163}$$

(ii) The single chain Π-extension $(\mathbb{K}_2,\sigma)$ of $(\mathbb{K},\sigma)$ where $\mathbb{K}_2=\mathbb{K}\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{2,2}\rangle$ with the automorphism $\sigma:\mathbb{K}_2\to\mathbb{K}_2$ and the naturally induced evaluation function $\tilde{\mathrm{ev}}:\mathbb{K}_2\times\mathbb{N}\to\mathbb{K}$ defined as

$$\begin{aligned}\sigma(c)&=c,\;\forall\,c\in\mathbb{K}, &\sigma(\tilde{y}_{2,1})&=2\,\tilde{y}_{2,1}, &\sigma(\tilde{y}_{2,2})&=2\,\tilde{y}_{2,1}\,\tilde{y}_{2,2},\\ \tilde{\mathrm{ev}}(c,n)&=c,\;\forall\,c\in\mathbb{K}, &\tilde{\mathrm{ev}}(\tilde{y}_{2,1},n)&=\prod_{k=1}^{n}2, &\tilde{\mathrm{ev}}(\tilde{y}_{2,2},n)&=\prod_{k=1}^{n}\prod_{i=1}^{k}2.\end{aligned}\tag{6.164}$$

(iii) The single chain Π-extension $(\mathbb{K}_3,\sigma)$ of $(\mathbb{K},\sigma)$ where $\mathbb{K}_3=\mathbb{K}\langle\tilde{y}_{3,1}\rangle\langle\tilde{y}_{3,2}\rangle$ with the automorphism $\sigma:\mathbb{K}_3\to\mathbb{K}_3$ and the naturally induced evaluation function $\tilde{\mathrm{ev}}:\mathbb{K}_3\times\mathbb{N}\to\mathbb{K}$ defined as:

$$\begin{aligned}\sigma(c)&=c,\;\forall\,c\in\mathbb{K}, &\sigma(\tilde{y}_{3,1})&=5\,\tilde{y}_{3,1}, &\sigma(\tilde{y}_{3,2})&=5\,\tilde{y}_{3,1}\,\tilde{y}_{3,2},\\ \tilde{\mathrm{ev}}(c,n)&=c,\;\forall\,c\in\mathbb{K}, &\tilde{\mathrm{ev}}(\tilde{y}_{3,1},n)&=\prod_{k=1}^{n}5, &\tilde{\mathrm{ev}}(\tilde{y}_{3,2},n)&=\prod_{k=1}^{n}\prod_{i=1}^{k}5.\end{aligned}\tag{6.165}$$

In the constructed difference ring $(\mathbb{D},\sigma)$, $\tilde{G}(n)$ is modelled by

$$\tilde{g}=\frac{(1-\mathbb{i})\,(\mathbb{i}+\vartheta^2)\,\tilde{y}_{2,1}^2\,\tilde{y}_{2,2}}{2\,\tilde{y}_{1,1}^3\,\tilde{y}_{3,2}}\in\mathbb{D}.$$

Merging the two difference rings $(\tilde{\mathbb{H}},\sigma)$ and $(\mathbb{D},\sigma)$, we get the RΠ-extension $(\mathbb{E},\sigma)$ of $(\mathbb{K}(x),\sigma)$ where

$$\mathbb{E}=\mathbb{K}(x)[\vartheta]\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{3,1}\rangle\langle\tilde{z}_{1,1}\rangle\langle\tilde{z}_{2,1}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{y}_{3,2}\rangle\langle\tilde{z}_{1,2}\rangle=\boldsymbol{e}_0\tilde{\mathbb{E}}\oplus\boldsymbol{e}_1\tilde{\mathbb{E}}\oplus\boldsymbol{e}_2\tilde{\mathbb{E}}\oplus\boldsymbol{e}_3\tilde{\mathbb{E}}$$

with

$$\tilde{\mathbb{E}}=\mathbb{K}(x)\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{3,1}\rangle\langle\tilde{z}_{1,1}\rangle\langle\tilde{z}_{2,1}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{y}_{3,2}\rangle\langle\tilde{z}_{1,2}\rangle.$$

The automorphism $\sigma:\mathbb{E}\to\mathbb{E}$ and the evaluation function $\tilde{\mathrm{ev}}:\mathbb{E}\times\mathbb{N}\to\mathbb{K}$ are defined by (6.103), (6.108), (6.163), (6.164) and (6.165). By statement (2) of Theorem 6.6.15, there is an element $p\in\mathbb{E}$ such that for all $n\geqslant 4$,

$$P(n)=\tilde{\mathrm{ev}}(p,\,n)\tag{6.166}$$

holds. Looking at the rational function $r(n)$ in (6.138), we can take $r=(x-2)^3\,(x-1)\,(x+1)\,(x+2)\,(x+3)$ and set

$$p:=\frac{3145}{384}\,r\,\tilde{g}\,\tilde{h}\in\mathbb{E}.$$

Then (6.166) holds. In particular,

$$\begin{aligned}\tilde{\mathrm{ev}}(p,n)=\frac{3145}{768}&(1-\mathbb{i})\,(n-2)^3\,(n-1)\,(n+1)\,(n+2)\,(n+3)\,\left(\mathbb{i}+\left((\mathbb{i})^n\right)^2\right)\\ &\left((\sqrt{3})^n\right)^{-3}(2^n)^2\,2^{\binom{n+1}{2}}\,\left(5^{\binom{n+1}{2}}\right)^{-1}\left(\prod_{k=4}^{n}(k^2+\tfrac{1}{4})\right)\left(\prod_{k=4}^{n}(k-3)\right)^3\left(\prod_{k=4}^{n}\prod_{j=4}^{k}(j-3)\right).\end{aligned}$$

Since $\tau:\mathbb{E}\to\mathcal{S}(\mathbb{K})$ is a $\mathbb{K}$-embedding, the sequences generated by

$$(\sqrt{3})^n,\,2^n,\,2^{\binom{n+1}{2}},\,5^{\binom{n+1}{2}},\,\left(\prod_{k=4}^{n}(k^2+\tfrac{1}{4})\right),\,\left(\prod_{k=4}^{n}(k-3)\right),\,\left(\prod_{k=4}^{n}\prod_{j=4}^{k}(j-3)\right)$$

for all $n\in\mathbb{N}$ with $n\geqslant 4$, are algebraically independent among each other over the ring $\tau(\mathbb{K}(x))[\langle(\mathbb{i})^n\rangle_{n\geqslant 0}]$ by construction. ⋆

In the following examples below, I will demonstrate how one can use my Mathematica package, `NestedProduct` in the Mathematica computer algebra system to reduce some hypergeometric products that appeared in some scientific literature.

Example 6.6.17.
We demonstrate how one can use my Mathematica package `NestedProducts` to simplify the hypergeometric product

$$P(n) = \frac{1}{2}\prod_{k=1}^{n-1}\frac{1}{36}\left(\prod_{i=1}^{k-1}\frac{(i+1)(i+2)}{4(2i+3)^2}\right) \in \mathrm{ProdE}(\mathbb{Q}(n)) \tag{6.167}$$

which appeared in Kauers (2018, Example 3). The product was guessed using the Mathematica package RATE which was written by Christian Krattenthaler; see Krattenthaler (1997). More generally, formulae for determinants that tend to involve nested products can be guessed using Krattenthaler (1997) on Mathematica or more recently, the algorithms described in Hebisch and Rubey (2011, Section 3.3) implemented on the `FriCAS`[8] system.

Mathematica Session 6

In[12]:= $\mathbf{P} = \frac{1}{2}\mathbf{FProduct}\left[\frac{1}{36}\left(\mathbf{FProduct}\left[\frac{(i+1)(i+2)}{4(2i+3)^2}, \{i, 1, k-1\}\right]\right), \{k, 1, n-1\}\right];$

In[13]:= $\mathbf{Q} = \mathbf{ProductReduce}[\mathbf{P}]$

Out[13]= $\frac{9}{(2n+3)^2}\,\frac{(2^n)^5}{(3^n)^2\, n!\,\left(2^{\binom{n+1}{2}}\right)^4}\left(\prod_{k=1}^{n}\left(k+\tfrac{3}{2}\right)\right)^4\left(\prod_{k=1}^{n}\prod_{i=1}^{k} i\right)^2\left(\prod_{k=1}^{n}\prod_{i=1}^{k}\frac{1}{\left(i+\frac{3}{2}\right)}\right)^2$

Internally, the package uses the function `SynchroniseProduct` to rewrite the hypergeometric product (6.167) by changing the upper bounds from $n-1$ and $k-1$ to n and k respectively. This preprocessing step returns the hypergeometric product expression

$$\frac{9(n+1)(n+2)}{2(2n+3)}\left(\prod_{j=1}^{n}\frac{4(2j+3)^2}{(j+1)(j+2)}\right)\left(\prod_{k=1}^{n}\frac{(2k+3)^2}{9(k+1)(k+2)}\left(\prod_{j=1}^{k}\frac{(j+1)(j+2)}{4(2j+3)^2}\right)\right) \tag{6.168}$$

which evaluates to the same sequence as (6.167) for all $n \geqslant 1$. The package then reduces (6.168) to get the hypergeometric product expression Q given by Out[13] in the Mathematica Session 6 above. $\star$

Example 6.6.18.
The product expression

$$A(n) = \frac{1!\,4!\,7!\cdots(3n-2)!}{n!\,(n+1)!\cdots(2n-1)!} = \prod_{k=0}^{n-1}\frac{(3k+1)!}{(n+k)!} \tag{6.169}$$

[8] `FriCAS` is freely available at `http://fricas.sourceforge.net`

was conjectured in the 1980s by Mills et al. (1983) as the number of $n \times n$ alternating sign matrices. The conjecture was later proved by Zeilberger (1996b). For more information on alternating sign matrices see (Zeilberger, 1996a; Robbins and Rumsey Jr., 1986; Robbins, 1991; Kuperberg, 1996; Bressoud, 1999; Bressoud and Propp, 1999; Fischer, 2011). The product expression (6.169) can be transformed to

$$H(n) = \prod_{k=1}^{n-1} 2 \prod_{j=1}^{k-1} \frac{3\,(3\,j+2)\,(3\,j+4)}{4\,(2\,j+1)\,(2\,j+3)}. \tag{6.170}$$

See Krattenthaler (2001, page 409) or Hebisch and Rubey (2011, Section 3.3). We demonstrate how the hypergeometric product (6.169) can be reduced further using the `NestedProducts` package.

Mathematica Session 7

In[14]:= $H = \mathrm{FProduct}\left[2\,\mathrm{FProduct}\left[\frac{3\,(3\,j+2)\,(3\,j+4)}{4\,(2\,j+1)\,(2\,j+3)}, \{j, 1, k-1\}\right] \{k, 1, n-1\}\right];$

In[15]:= $G = \mathrm{ProductReduce}[H]$

Out[15]= $$\frac{(3\,n+2)\,(3\,n+4)\,(2^n)^8\,\left(3^{\binom{n+1}{2}}\right)^3}{8\,(2\,n+1)\,(3^n)^5\left(2^{\binom{n+1}{2}}\right)^4}\left(\prod_{k=1}^{n}(k+\tfrac{1}{2})\right)^3\left(\prod_{k=1}^{n}\frac{1}{(k+\frac{2}{3})}\right)^2\left(\prod_{k=1}^{n}\frac{1}{(k+\frac{4}{3})}\right)^2 \left(\prod_{k=1}^{n}\prod_{j=1}^{k}(j+\tfrac{2}{3})\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}(j+\tfrac{4}{3})\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j+\frac{1}{2})}\right)^2$$

In particular, the hypergeometric product expression G, given by Out[15] in the Mathematica Session 7, satisfies the identity

$$H(n) = G(n) \quad \forall\, n \geqslant 1.$$

In Zeilberger (1996a, Main Theorem) and Fischer (2007, Thoerem 1) it has been shown that the product expression

$$A(n, r) = \frac{(r)_{n-1}\,(n-r+1)_{n-1}}{(n-1)!}\prod_{k=1}^{n-1}\frac{(3\,k-2)!}{(n+k-1)!} \tag{6.171}$$

where $(r)_n = r\,(r+1)\cdots(r+n-1)$, counts the number of $n \times n$ alternating sign matrices for which the unique 1 of the first row is at the r-th column. Combining the Mathematica package RATE or the alternative by Hebisch and Rubey (2011, Section 3.3) with the `NestedProducts` package we can reduce (6.171) further for any positive integer r. Let's consider the case $r = 2$.

$$A_2(n) = A(n, 2) = \frac{1}{2}\prod_{k=1}^{n-1} 2 \prod_{j=1}^{k-1}\frac{3\,j\,(j+2)\,(3\,j-1)\,(3\,j+1)}{4\,(j+1)^2\,(2\,j-1)\,(2\,j+1)}.$$

Mathematica Session 8

In[16]:= $A_2 = \frac{1}{2}\text{FProduct}\Big[2\,\text{FProduct}\Big[\frac{3j(j+2)(3j-1)(3j+1)}{4(j+1)^2(2j-1)(2j+1)}, \{j, 1, k-1\}\Big]\{k, 1, n-1\}\Big];$

In[17]:= $G_2 = \text{ProductReduce}[A_2]$

Out[17]= $$\frac{3n(9n^2-1)(2^n)^7\left(3^{\binom{n+1}{2}}\right)^3}{8(2n-1)(3^n)^6\left(2^{\binom{n+1}{2}}\right)^4}\left(\prod_{k=1}^{n}(k-\tfrac{1}{2})\right)^3\left(\prod_{k=1}^{n}\frac{1}{(k-\frac{1}{3})}\right)^2\left(\prod_{k=1}^{n}\frac{1}{(k+\frac{1}{3})}\right)^2\left(\prod_{k=1}^{n}\prod_{j=1}^{k}(j-\tfrac{1}{3})\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}(j+\tfrac{1}{3})\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j-\frac{1}{2})}\right)^2$$

This means that for the hypergeometric product expression G_2 defined by Out[17] in the Mathematica Session 8, the identity

$$A_2(n) = G_2(n)$$

holds for all $n \geqslant 1$. ⋆

Example 6.6.19.
We use our Mathematica package, `NestedProducts` to reduce the following hypergeometric products

$$H_1(n) := \prod_{k=0}^{n-1} k! \prod_{k=2n}^{3n-1} k! \prod_{k=n}^{2n-1}\frac{1}{k!} = 2\prod_{k=1}^{n-1}120\prod_{j=1}^{k-1}\frac{27(j+1)(3j+1)(3j+2)^2(3j+4)^2(3j+5)}{16(2j+1)^2(2j+3)^2} \tag{6.172}$$

$$H_2(n) := \frac{(-1)^{\binom{n}{2}}}{2}\prod_{k=1}^{n-1}\frac{(k!)^6}{(2k)!(2k+1)!} = \frac{1}{2}\prod_{k=1}^{n-1}\frac{-1}{12}\prod_{j=1}^{k-1}\frac{-(j+1)^4}{4(2j+1)(2j+3)} \tag{6.173}$$

$$H_3(n) := \frac{(-1)^{\binom{n+1}{2}}}{2}\prod_{k=1}^{n-1}\frac{(k!)^3\left((k+1)!\right)^3}{(2k+1)!(2k+2)!} = -\frac{1}{2}\prod_{k=1}^{n-1}\frac{1}{18}\prod_{j=1}^{k-1}\frac{-(j+1)^2(2+j)^2}{4(3+2j)} \tag{6.174}$$

$$H_4(n) := \frac{(-1)^{\binom{n}{2}}}{6}\prod_{k=1}^{n-1}\frac{k!\left((k+1)!\right)^4(k+2)!}{(2k+2)!(2i+3)!} = \frac{1}{6}\prod_{k=1}^{n-1}\frac{-1}{30}\prod_{j=1}^{k-1}\frac{-(j+1)(j+2)^2(j+3)}{4(2j+3)(2j+5)} \tag{6.175}$$

$$H_5(n) := \prod_{k=0}^{n-1}\frac{(2k)!\left((2k+1)!\right)^4(2k+2)!}{(4k+2)!(4k+3)!} = \frac{1}{6}\prod_{k=1}^{n-1}\frac{3}{175}\prod_{j=1}^{k-1}\frac{(j+1)^3(j+2)(2j+1)(2j+3)^3}{(4j+3)(4j+5)^2(4j+7)} \tag{6.176}$$

$$H_6(n) := (-1)^n\prod_{k=1}^{n-1}\frac{(2k-1)!\left((2k)!\right)^4(2k+1)!}{(4k)!(4k+1)!} = (-1)^n\prod_{k=1}^{n-1}\frac{1}{30}\prod_{j=1}^{k-1}\frac{j(j+1)^3(2j+1)^3(2j+3)}{(4j+1)(4j+3)^2(4j+5)} \tag{6.177}$$

which appeared in Krattenthaler (2001, Theorem 49, Equations (3.56), (3.57), (3.58), (3.59), (3.60)) respectively. The right most expression of (6.172), (6.173), (6.174), (6.175) were obtained using the Mathematica package RATE by Christian Krattenthaler; see Krattenthaler (1997), whiles the right most expression of (6.176) and (6.177) were obtained using the algorithms in Hebisch and Rubey (2011, Section 3.3) which is implemented on the `FriCAS` system.

For the hypergeometric product $H_1(n)$ given by (6.172) we obtain the following simplification.

Mathematica Session 9

In[18]:= $H_1 = 2\,\text{FProduct}\Big[120\,\text{FProduct}\Big[\frac{27(j+1)(3j+1)(3j+2)^2(3j+4)^2(3j+5)}{16(2j+1)^2(2j+3)^2},\{j,1,k-1\}\Big],\{k,1,n-1\}\Big];$

In[19]:= $F_1 = \text{ProductReduce}[H_1]$

Out[19]= $$\frac{(3n+1)(3n+2)^2\left(2^n\right)^{13}\left(3^{\binom{n+1}{2}}\right)^9}{4(2n+1)^2\left(3^n\right)^{12}\left(2^{\binom{n+1}{2}}\right)^8 n!}\left(\prod_{k=1}^{n}(k+\tfrac{1}{2})\right)^6\left(\prod_{k=1}^{n}\frac{1}{(k+\frac{1}{3})}\right)^4\left(\prod_{k=1}^{n}\frac{1}{(k+\frac{2}{3})}\right)^5 \left(\prod_{k=1}^{n}\prod_{j=1}^{k}j\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}(j+\tfrac{1}{3})\right)^3\left(\prod_{k=1}^{n}\prod_{j=1}^{k}(j+\tfrac{2}{3})\right)^3\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j+\frac{1}{2})}\right)^4$$

Note that the identity
$$H_1(n) = F_1(n)$$
where F_1 is given by Out[19] in the Mathematica Session 9 holds for all $n \geqslant 1$.

For the hypergeometric product $H_2(n)$ given by (6.173) we obtain the following result.

Mathematica Session 10

In[20]:= $H_2 = \frac{1}{2}\,\text{FProduct}\Big[\frac{-1}{12}\,\text{FProduct}\Big[\frac{-(j+1)^4}{4(2j+1)(2j+3)},\{j,1,k-1\}\Big],\{k,1,n-1\}\Big];$

In[21]:= $F_2 = \text{ProductReduce}[H_2]$

Out[21]= $$\frac{(1-i)\left((i)^n\right)^3\left(i+\left((i)^n\right)^2\right)\left(2^n\right)^5}{4(2n+1)\left(2^{\binom{n+1}{2}}\right)^4(n!)^4}\left(\prod_{k=1}^{n}(k+\tfrac{1}{2})\right)^3\left(\prod_{k=1}^{n}\prod_{j=1}^{k}j\right)^4\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j+\frac{1}{2})}\right)^2$$

Again the identity,
$$H_2(n) = F_2(n) \qquad \star$$
where the hypergeometric product expression F_2 is given by Out[21] in Mathematica Session 10 holds for all $n \geqslant 1$.

Similarly, all the remaining identities (6.174), (6.175), (6.176) and (6.177) can be derived.

The product expressions we have covered in this thesis are all indefinite. However, there seems to be at least some classes of definite products objects that can be transformed into indefinite products. For example the product expression (6.178) below as well as the expressions (6.169) and (6.171) from above.

Example 6.6.20.
The product expression,

$$H(n) = \prod_{k=1}^{n}\prod_{j=1}^{k}\prod_{i=1}^{j}\frac{i+j+k}{i+j+k+2} \tag{6.178}$$

is definite since the multiplicand

$$\frac{i+j+k}{i+j+k+2}$$

depends on the symbolic variables k and j which are upper bound symbolic variables of some product quantifiers in (6.178). Using "heuristic" methods implemented in the Mathematica package `Sigma`, one can transform (6.178) to the indefinite hypergeometric product

$$M_1 = \frac{2}{n+2}2^n\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{4j(j+2)(2j+1)^2}{3(j+1)^2(3j+1)(3j+2)} \in \mathrm{ProdE}(\mathbb{Q}(n)).$$

On the other hand, using the Mathematica package RATE by Krattenthaler (1997), the definite product expression (6.178) can be transformed to the indefinite hypergeometric products

$$M_2 = \frac{3}{5}\prod_{k=1}^{n-1}\frac{5}{14}\prod_{j=1}^{k-1}\frac{4(2j+5)^2}{3(3j+7)(3j+8)} \in \mathrm{ProdE}(\mathbb{Q}(n))$$

or

$$M_3 = \frac{3}{5}\prod_{k=1}^{n-1}\frac{5}{14}\prod_{j=1}^{k-1}\frac{98}{165}\prod_{i=1}^{j-1}\frac{(2i+7)^2(3i+7)(3i+8)}{(2i+5)^2(3i+10)(3i+11)} \in \mathrm{ProdE}(\mathbb{Q}(n)).$$

In particular, for all $n \geqslant 1$,

$$H(n) = M_1(n) = M_2(n) = M_3(n)$$

holds. The Mathematica sessions below illustrates how the hypergeometric products, M_1, M_2 and M_3 are reduced further using the Mathematica package `NestedProducts`.

Mathematica Session 11

In[22]:= $M_1 = \frac{2}{n+2}\mathrm{FProduct}\left[2\,\mathrm{FProduct}\left[\frac{4j(j+2)(2j+1)^2}{3(j+1)^2(3j+1)(3j+2)}, \{j, 1, k\}\right], \{k, 1, n\}\right];$

In[23]:= $Q_1 = \mathrm{ProductReduce}[M_1]$

Out[23]= $$\frac{\left(2^{\binom{n+1}{2}}\right)^4}{\left(3^{\binom{n+1}{2}}\right)^3}\left(\prod_{k=1}^{n}\prod_{j=1}^{k}(j+\tfrac{1}{2})\right)^2\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j+\frac{1}{3})}\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j+\frac{2}{3})}\right)$$

With the calculation in the Mathematica Session 11, we conclude that for the hypergeometric product expression Q_1 given by Out[23], we have that for all $n \geqslant 1$ the identity

$$M_1(n) = Q_1(n)$$

holds.

Mathematica Session 12

In[24]:= $M_2 = \frac{3}{5}\text{FProduct}\left[\frac{5}{14}\text{FProduct}\left[\frac{4(2j+5)^2}{3(3j+7)(3j+8)}, \{j, 1, k-1\}\right], \{k, 1, n-1\}\right];$

In[25]:= $Q_2 = \text{ProductReduce}[M_2]$

Out[25]= $$\frac{56(2n+5)^2 5^n (3^n)^6 \left(2^{\binom{n+1}{2}}\right)^4}{25(3n+7)(3n+8)7^n (2^n)^9 \left(3^{\binom{n+1}{2}}\right)^3}\left(\prod_{k=1}^{n}(k+\tfrac{7}{3})\right)^2\left(\prod_{k=1}^{n}(k+\tfrac{8}{3})\right)^2\left(\prod_{k=1}^{n}\frac{1}{(k+\frac{5}{2})}\right)^4$$
$$\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j+\frac{7}{3})}\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j+\frac{8}{3})}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k}(j+\tfrac{5}{2})\right)^2$$

With the output Q_2 given by Out[25] in the Mathematica Session 12 we obtain the identity

$$M_2(n) = Q_2(n)$$

which holds for all $n \geqslant 1$.

Mathematica Session 13

In[26]:= $M_3 = \frac{3}{5}\text{FProduct}\left[\frac{5}{14}\text{FProduct}\left[\frac{98}{165}\text{FProduct}\left[\frac{(2i+7)^2(3i+7)(3i+8)}{(2i+5)^2(3i+10)(3i+11)},\right.\right.\right.$
$\left.\left.\left.\{i, 1, j-1\}\right], \{j, 1, k-1\}\right], \{k, 1, n-1\}\right];$

In[27]:= $Q_3 = \text{ProductReduce}[M_3]$

Out[27]= $$\frac{56(2n+5)^2 5^n (3^n)^6 \left(2^{\binom{n+1}{2}}\right)^4}{25(3n+7)(3n+8)7^n (2^n)^9 \left(3^{\binom{n+1}{2}}\right)^3}\left(\prod_{k=1}^{n}(k+\tfrac{7}{3})\right)^2\left(\prod_{k=1}^{n}(k+\tfrac{8}{3})\right)^2\left(\prod_{k=1}^{n}\frac{1}{(k+\frac{5}{2})}\right)^4$$
$$\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j+\frac{7}{3})}\right)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{(j+\frac{8}{3})}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k}(j+\tfrac{5}{2})\right)^2$$

With Q_3 given by Out[27] in the Mathematica Session 13, we derive the identity

$$M_3(n) = Q_3(n)$$

which holds for all $n \geqslant 1$. Observe that, the hypergeometric product expressions Q_2 and Q_3 are the same. We observe that while (6.178) is a nesting depth 3 product expression, the factors of its reduced expressions, namely Q_1, Q_2 and Q_3 given by Out[23], Out[25] and Out[27] respectively are all at most nesting depth 2 products. In particular, the algorithms described in this thesis enable the user to reduced the nesting depth 3 hypergeometric product M_3 to the nesting depth 2 hypergeometric product Q_2. In addition we want to emphasis once more that we obtain an extra bonus by the underlying difference ring theory. Namely, the sequences of the output product expressions in each of the Mathematica Sessions above are algebraically independent among each other over the field of rational sequences. ⋆

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