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Power Series Solutions of AODEs – Existence, Uniqueness, Convergence and Computation

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Kurzfassung

Differentialgleichungen werden seit langer Zeit intensiv studiert. Für diverse Spezialfälle wurden Methoden zur Beschreibung von exakten Lösungen entwickelt, jedoch gibt es im allgemeinen Fall keinen Algorithmus zur Berechnung von expliziten Lösungen.

In dieser Arbeit werden algebraische gewöhnliche Differentialgleichungen studiert und neue Methoden zur Berechnung aller formalen Potenzreihenlösungen mit nicht- negativen ganzzahligen Exponenten oder gebrochen rationalen Exponenten, sogenannte Puiseux-Reihen, vorgestellt. Mit Berechnung aller Lösungen ist die Beschreibung der Lösungsmenge bestehend aus Potenz- und Puiseux-Reihen durch eine Menge von abgebrochenen Reihen mit endlich vielen Summanden gemeint, sodass sich die Elemente dieser beiden Mengen eindeutig miteinander identifizieren lassen. Zusätzlich zu den Potenz- und Puiseux-Reihen betrachten wir auch algebraische Lösungen, welche implizit durch ihr definierendes Minimalpolynom dargestellt werden können. Zu diesem Zweck verwenden wir drei verschiedene Ansätze: den direkten Ansatz mittels Koeffizientenvergleich, die Newton-Polygon Methode für Differentialgleichungen und den algebraisch-geometrischen Ansatz.

Die ersten beiden Methoden sind relativ gut Beschrieben in der bereits existierenden Literatur, aber ihre Anwendung garantiert im Allgemeinen weder Existenz, Eindeutigkeit noch Konvergenz der Lösungen. Für bestimmte Familien von Differentialgleichungen jedoch sind wir in der Lage diese Eigenschaften zu beweisen.
Der algebraisch-geometrische Ansatz transformiert das gegebene differentielle Problem in ein algebraisches. Algebraische Differentialgleichungen definieren implizit algebraische Mengen, an denen Methoden aus der algebraischen Geometrie angewandt werden können. Das Hauptresultat der Dissertation ist die exakte Formulierung des Zusammenhangs zwischen dem differentiellen und algebraischen Problem und die Anwendung der algebraischen Geometrie, um die oben genannten Eigenschaften wie Existenz, Eindeutigkeit und Konvergenz von Lösungen zu zeigen. Insbesondere können für algebraische Kurven die entsprechende gut durchleuchtete Theorie von formalen Parametrisierungen und deren Äquivalenzklassen verwendet werden. Zum Beispiel algebraische gewöhnliche Differentialgleichungen erster Ordnung mit konstanten Koeffizienten entsprechen ebenen Kurven. Von einer bereits bestimmten lokalen Parametrisierung kann nun eine assozierte Differentialgleichung eines bestimmten Types aufgestellt werden, die sich durch die Newton-Polygon Methode für Differentialgleichungen vollständig analysieren und lösen lassen. Die Hintereinanderausführung der lokalen Parametrisierung und der Lösungen der assozierten Differentialgleichung sind Lösungen der ursprünglichen Differentialgleichung.
Für Differentialgleichungen, welche die unbekannte Funktion und die zweite Ableitung davon enthalten, lassen sich durch diese Herangehensweise ähnliche Resultate erzielen. Wir behandeln auch Systeme von gewöhnlichen Differentialgleichungen welche algebraische Mengen in Form von Raumkurven entsprechen. Diese Systeme lassen sich durch die Anwendung von sogenannten regulären Ketten auf eine einzelne Differentialgleichung zurückführen, welche von obigem Typ ist, wodurch sich die Eigenschaften der Lösungen auf solche Systeme verallgemeinern lassen.

Abstract

Differential equations have been intensively studied for a long time. There exist several methods for describing exact solutions for specific cases. Nevertheless, there is no general algorithm for computing explicit exact solutions.

In this thesis we consider algebraic ordinary differential equations (shortly AODEs) and investigate new methods for computing all formal power series solutions with non-negative integer exponents or fractional exponents, so-called formal Puiseux series. By "computing all" solutions we mean to describe the set of formal power series or Puiseux series solutions by a set of truncations such that these two sets are in one-to-one correspondence. Additionally to these solutions we also consider algebraic solutions, which can be represented implicitly by its defining minimal polynomial. For this purpose, we study three different approaches: the direct approach by comparison of coefficients, the Newton polygon method for differential equations and the algebro-geometric approach.

The first two approaches are well described in the literature, but both neither ensure existence, uniqueness nor convergence in the general situation. For certain families of differential equations, however, we are able to prove these properties.
The algebro-geometric approach transforms the differential problem into an algebraic one by considering the given differential equations as algebraic equations. Algebraic equations implicitly define algebraic sets where tools from algebraic geometry can be applied. The main result of the thesis is to precisely state the relation between the differential and algebraic problem and use the results from algebraic geometry in order to show properties of the solutions of the algebraic problem such as the existence, uniqueness and convergence. In particular, for algebraic curves we can use the well-developed theory on formal parametrizations and its equivalence classes under substitution with formal power series of order one, namely places. Plane algebraic curves on the algebraic side get derived for example from first order autonomous AODEs on the differential side. From a given local parametrization we can then derive an associated differential equation which is exactly of the type we can fully analyze by the Newton polygon method for differential equations; in particular it is of order one and degree one. Then the composition of the parametrization and a solution of the associated differential equation yields a solution of the original differential equation.
For differential equations involving the differential indeterminate and the second derivative of it we obtain similar properties of the solutions.
We also deal with systems of autonomous AODEs whose corresponding algebraic set is of dimension one, namely a space curve. For those systems, by using regular chains, we are able to derive a single first order autonomous AODE which enables us to generalize the main properties of formal Puiseux series and algebraic solutions to such systems.

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Contents

Chapter 1

Introduction

Exact solution methods of ordinary differential equations have been extensively studied in the literature. An overview of the most known techniques and introduction to some of these methods is presented in [Zwi98][Section II]. The main aim of the present work is to contribute new results on two well-known solution methods, which are called "direct approach" and "Newton polygon method for differential equations" herein, and to provide an alternative exact method for solving in particular first order autonomous algebraic ordinary differential equations (AODEs).

In this chapter we introduce the topic and the main ideas. We will focus but not limit ourselves to the three different solution methods presented in this thesis and the methods used for them. In Section 1.1 we give an overview of similar and more specific methods and their historical background. Our contributions and new results are described in Section 1.2. In Section 1.3 we recall some basic notions and fix the main notations used throughout the thesis.

1.1 Historical Background

The problem of finding exact solutions of ordinary differential equations has been extensively studied in the literature. The huge majority of these methods, however, make implicit assumptions on the structure of the equations such as that the system is in normal form where the well known Cauchy-Kovalevskaya Theorem can be applied. We focus here on methods where these assumptions are dropped and note that there are a lot of famous examples which are not of normal form such as Navier-Stokes equations, Maxwell equations and many others.

A naive approach for finding solutions is to plug a function of the desired type with unknown coefficients, for example a formal power series, into the given differential equation and try to find relations on the coefficients. Recently there have been several theories developed in order to find a description of these relations and give them a geometrical meaning such as the theory of arc-spaces and jets, see e.g. [Sei09]. In order to find explicit solutions, the goal is to obtain a solvable recursion formula.
For linear differential equations this turned out to be very successful (see [Zwi98][Section II, 90] or [Inc26][Chapter 16]). For non linear AODEs this method works in general as

well. Already Hurwitz (in [Hur89]) and later Denef and Lipshitz (in [DL84]) proved some recursion formulas which can be used to decide, in almost all cases, existence and uniqueness of formal power series solutions with some given initial values.

A method to compute generalized formal power series solutions, i.e. power series with real exponents, and describe their properties is the Newton polygon method for differential equations. A description of this method is given in [Fin89, Fin90] and more recently in [GS91, DDRJ97, Aro00b].
A variation of this method, using power transformations, can be found in [Bru00]. Using this approach, one can derive necessary conditions on computing more general types of solutions such as power series with complex exponents and power-logarithmic series. For details, see [Bru04].
In the general case, the description of the solutions is again not completely algorithmic. In particular, there is no bound on the number of computational steps in order to guarantee existence and uniqueness of a series solution.

For non-linear first order AODEs an implicit description of the solutions can be given by using Gröbner bases, see [Hub96]. In order to find explicit solutions, algebraic-geometric solution methods have been developed recently. The main idea is to disregard the differential aspect of the problem and consider the differential indeterminate and its derivatives as independent variables. This implicitly defines an algebraic set which might be parametrized, by rational, algebraic or local parametrizations, depending on the solutions one is looking for. This approach was first introduced by Feng and Gao in [FG04, FG06] for autonomous first order AODEs and rational general solutions. A generalization to non-autonomous first order AODEs can be found in [NW10, NW11, GVW16], to systems of algebro-geometric dimension equal to one in [LSNW15] and to AODEs of higher order in [HNW13]. Algebraic general solutions of first order autonomous AODEs are computed in [ACFG05]. For an overview of these results explained by examples we refer to [SW19].

1.2 Structure of the Thesis

In this thesis we consider three different approaches: the direct approach by using coefficient comparison, the Newton polygon method for differential equations and a local version of the algebro-geometric approach. This is also the basic structure of the thesis.

In Chapter 2 we develop the method of undetermined coefficients for finding formal power series solutions of (higher order) AODEs with polynomial coefficients. For many initial values a differential version of the implicit function theorem can be used in order to find a recursion formula for the coefficients of the solution. In the cases where this is not possible, derivatives of the underlying differential polynomial can be studied in order to find a similar recursion formula. This idea was already introduced by Hurwitz and rediscovered recently. Our contribution in this thesis is to develop this idea by using matrix representation, which simplifies some reasonings and relations. Moreover, for a given differential equation we present a classification of initial values with respect to the number of derivatives of the differential polynomials which have to be considered before existence and uniqueness of the solution can be ensured. In the last part of this chapter we show necessary and sufficient conditions on the differential polynomial such that the number of derivatives for all suitable initial values is bounded by a particular number.

Next we present the Newton polygon method for differential equations in Chapter 3 for finding formal Puiseux series solutions of (higher order) AODEs with Puiseux series coefficients. This method consists mainly in the study of the Newton polygon, a convex set in the plane, obtained from the given differential equation. In this way one constructs candidates for solutions term-by-term. In contrast to the Newton polygon method for algebraic equations, in this dynamical procedure there is no bound on the number of terms known from which on existence and uniqueness of the solutions are guaranteed. In some particular situations, however, this bound can be derived. An important example where we are able to show such a bound and also convergence of the solutions, provided that the original differential equation itself is convergent, are the associated differential equations we obtain from the algebro-geometric approach. Another example are some important families of differential equations fulfilling the sufficient conditions in Chapter 2. This allows us to generalize the results on existence and uniqueness obtained in Chapter 2 from formal power series to the case of formal Puiseux series. Up to our knowledge none of these bounds were known before.

In Chapter 4 we develop a local version of the algebro-geometric approach by using local parametrizations and their equivalence classes, namely places. In the case of a first order autonomous AODE we derive an associated differential equation of a specific type where all solutions can be found by the Newton polygon method for differential equations. The well-known theory of local parametrizations and places and the results from Chapter 3 allow us to prove bounds on the number of computation steps from which on existence and uniqueness are guaranteed. Additionally we show convergence of all formal Puiseux series solutions provided that coefficients of the given differential equation are convergent. These results can be seen as the major contribution of this thesis. Additionally, for a given point in the complex plane and

any first order autonomous AODE, we give a new proof of the fact that there exists an analytic curve passing through this point.
Algebraic functions are a special instance of formal Puiseux series which can be expressed in a closed form, namely by its minimal polynomial. This allows us to achieve also a closed form description of the solutions, see Section 4.1. Here we prove order bounds for the minimal polynomials of the solutions of a first order autonomous AODE and show that all solutions have the same minimal polynomials up to a shift in the independent variable.

Later in Section 4.2 we consider systems of autonomous AODEs whose corresponding algebraic set is of dimension one. In the case of one differential indeterminate, by using regular chains, a concept introduced by Kalkbrenner [Kal93] and studied recently by many others, we are able to derive a single first order autonomous differential equation with the same non-constant formal Puisuex series solutions. This allows us to generalize the results on existence, uniqueness and convergence obtained in the beginning of this chapter to such systems. In the case of several differential indeterminates, studied in Section 4.2, this leads to solution candidates and in general only convergence and uniqueness is ensured. For algebraic solutions we are able to additionally present a method for checking whether the solution candidates are indeed solutions. This allows us to compute all algebraic solutions of such systems.

Finally, in Section 4.3, we consider autonomous AODEs involving a differential indeterminate and an arbitrary derivative of it. Interestingly some new problems arise in the study of the associated differential equations. In the case that the derivative is the second derivative, we are able to prove existence, uniqueness and convergence of the formal Puiseux series solutions. For higher derivatives our proves can most probably be adapted, but it remains as an open problem.

Additional information on the algebraic structure of formal power series and formal Puiseux series including the Newton polygon method for algebraic equations differential algebra can be found in the appendix. Moreover, we provide additional information on differential algebra and algebraic geometry and in particular on space curves and regular chains.

The content of this thesis is based on but not limited to the papers [FZTV19, FS19, CFS19, CFS20], co-authored by the writer of the thesis.

1.3 Preliminaries

In this section we recall some basics of Differential Algebra and Algebraic Geometry and fix some notations that will be used throughout this thesis. Further details are given in the appendix chapters A, B and C.

Let us fix some further notations used throughout this thesis, when not specified differently: For every set A containing a zero-element, i.e. a neutral element with respect to addition, we use the notation $A^* = A \setminus \{0\}$. In particular, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$. By $\mathbb{K}$ we denote an algebraically closed field of characteristic zero. We will use the following notation for the basic algebraic structures with coefficients in $\mathbb{K}$

$\mathbb{K}[x]$	ring of polynomials in x
$\mathbb{K}(x)$	field of rational functions in x
$\mathbb{K}[[x - x_0]]$	ring of formal power series in x expanded around $x_0 \in \mathbb{K}$
$\mathbb{K}[[x^{-1}]]$	ring of formal power series in x expanded around infinity
$\mathbb{K}((x - x_0))$	field of formal Laurent series in x expanded around $x_0 \in \mathbb{K}$
$\mathbb{K}\langle\langle x - x_0 \rangle\rangle$	field of formal Puiseux series in x expanded around $x_0 \in \mathbb{K}$
$\mathbb{K}\langle\langle x^{-1} \rangle\rangle$	field of formal Puiseux series in x expanded around infinity

Table 1.1: Notation for basic algebraic structures.

From time to time we will also write

$$\mathbb{K}[[x - \infty]] = \mathbb{K}[[x^{-1}]] \quad \text{and} \quad \mathbb{K}\langle\langle x - \infty \rangle\rangle = \mathbb{K}\langle\langle x^{-1} \rangle\rangle.$$

By $\mathbb{K}_\infty = \mathbb{K} \cup \{\infty\}$ we denote the one-point compactification of $\mathbb{K}$. For $x_0 \in \mathbb{K}_\infty$ let us recall the relation

$$\mathbb{K}[[x - x_0]] \subset \mathbb{K}((x - x_0)) \subset \mathbb{K}\langle\langle x - x_0 \rangle\rangle.$$

For each $f(x) \in \mathbb{K}\langle\langle x - x_0 \rangle\rangle$ and a given $k \in \mathbb{Q}$, we use the notation $[(x - x_0)^k]f$ to refer to the coefficient of $(x - x_0)^k$ in f. For $x_0 = \infty$ we set $[(x - \infty)^k]f = [(x^{-1})^k]f$ and refer to the coefficient of $(x^{-1})^k$ in f.

For a formal power series we define its encoding map $\boldsymbol{\pi}$ as

$$\begin{array}{cccc} \boldsymbol{\pi}: & \mathbb{K}[[x - x_0]] & \longrightarrow & \mathbb{K}^{\mathbb{N}} \\ & y(x) = \sum_{i \geq 0} \frac{c_i}{i!} (x - x_0)^i & \longmapsto & (c_0, c_1, \ldots). \end{array}$$

Moreover, for $n, m \in \mathbb{N}$ and $m \geq n$ we define the projection map π_n as

$$\begin{array}{cccc} \pi_n: & \mathbb{K}^{m+1} & \longrightarrow & \mathbb{K}^{n+1} \\ & (c_0, \ldots, c_m) & \longmapsto & (c_0, \ldots, c_n). \end{array}$$

We also allow $m = \infty$ here and note that the composition $\pi_n \circ \boldsymbol{\pi}$ is the extraction of the first $n + 1$ coefficients of a formal power series.

With abuse of notation we define for formal Puiseux series its encoding map as

$$\begin{array}{rccc} \boldsymbol{\pi}: & \mathbb{K}\langle\langle x-x_0\rangle\rangle & \longrightarrow & \mathbb{K}^{\mathbb{N}} \\ & y(x)=\sum_{i\geq 0} c_i\,(x-x_0)^{p+i/m} & \longmapsto & (c_0,c_1,\ldots), \end{array}$$

where $p=\mathrm{ord}_x(y(x))$ and $m\in\mathbb{N}^*$ is minimal. Observe that in the definition of $\boldsymbol{\pi}$ on formal power series we always set $p=0$. Additionally, let us remark that the definition of $\boldsymbol{\pi}$ for formal power series, introduced above, and the current definition of $\boldsymbol{\pi}$ for formal Puiseux series are slightly different: in the first one the series is encoded by taking the coefficients multiplied by the corresponding factorial, while in the second one the codification is done through the coefficients. The reason for this difference is essentially in the simplification of notation, as it will be clear in the subsequent chapters.
For simplification of notation we might omit the encoding map $\boldsymbol{\pi}$ in calculations and define π_n also as a mapping on formal Puiseux series for truncating the series. More precisely, for a formal Puiseux series $y(x)=\sum_{i\geq 0} c_i\,(x-x_0)^{p+i/m}$ we also write

$$\pi_n(y(x))=\sum_{i=0}^{n} c_i\,(x-x_0)^{p+i/m}.$$

For vectors or sets of formal power series we define the projection map π_n as the mapping on every component and for every element, respectively.

Let $\mathcal{R}\{y\}$ be a ring of differential polynomials in the differential indeterminate y with coefficients in the ring $\mathcal{R}\supseteq\mathbb{K}$. Usually $\mathcal{R}$ will be taken as an element of $\{\mathbb{K},\mathbb{K}[x],\mathbb{K}\langle\langle x\rangle\rangle\}$. For the derivative with respect to the independent variable x we use the notation $\frac{d}{dx}$ or $'$ and for higher derivatives $y'=y^{(1)}$ and recursively $y^{(n)}=(y^{(n-1)})'$. A differential polynomial is of *order* $n\in\mathbb{N}$ if the n-th derivative $y^{(n)}$ is the highest derivative appearing in it.
Consider the algebraic ordinary differential equation (AODE) of the form

$$F(x,y,y',\ldots,y^{(n)})=0, \tag{1.1}$$

where F is a differential polynomial in $\mathcal{R}\{y\}$ of order n. For simplicity, we may also write (1.1) as $F(y)=0$, call n the *order* of the AODE (1.1) and shortly write $\mathrm{ord}(F)=n$. If $F(y)$ is independent of x, i.e. $\mathcal{R}=\mathbb{K}$ and F can be chosen as element in $\mathbb{K}\{y\}$, then we say that the differential equation is *autonomous*.

We are looking for solutions of (1.1) in the ring of formal power series or in the field of formal Puiseux series expanded around some x_0, respectively. Fixing an algebraic structure $\mathcal{R}$ with coefficients in $\mathbb{K}$, where the solutions are sought, the set of solutions will be denoted by $\mathbf{Sol}_{\mathcal{R}}(F)$. Since constant solutions of differential equations play a similar role as the neutral element with respect to the addition in algebraic structures, we also use the notation

$$\mathbf{Sol}^*_{\mathcal{R}}(F)=\mathbf{Sol}_{\mathcal{R}}(F)\setminus\mathbf{Sol}_{\mathbb{K}}(F).$$

Let us note that if $F(y)=0$ is an autonomous differential equation, we can perform the change of variables $\bar{x}=x+x_0$ for any $x_0\in\mathbb{K}$: Since F is independent of x, and from the chain rule it directly follows that

$$\frac{d\,y(\bar{x})}{dx} = \frac{d\,y(x+x_0)}{dx} = y'(\bar{x}),$$

solutions of $F(y(x)) = 0$ are solutions of $F(y(x+x_0)) = 0$ and vice versa. In other words, the solution set is invariant under this transformation and it holds that

$$\mathbf{Sol}_{\mathcal{R}}(F(y(x)) = \mathbf{Sol}_{\mathcal{R}}(F(y(x+x_0))).$$

Therefore, for autonomous differential equations it is sufficient to study solutions expanded around zero or around infinity.
For non-autonomous differential equations we can still apply the above described change of variables, but the solution set has to be changed the same way. More precisely,

$$\mathbf{Sol}_{\mathbb{K}\langle\langle x-x_0\rangle\rangle}(F(y(x))) = \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F(y(x+x_0)))$$

and

$$\mathbf{Sol}_{\mathbb{K}[[x-x_0]]}(F(y(x))) = \mathbf{Sol}_{\mathbb{K}[[x]]}(F(y(x+x_0))).$$

But for $F \in \mathbb{K}[x]\{y\}$ of order n we can define a new differential polynomial $\bar{F}$, again of order n, such that

$$F(y(x+x_0)) = F(x+x_0, y(x+x_0), \ldots, y^{(n)}(x+x_0)) = \bar{F}(x, y(x+x_0), \ldots, y^{(n)}(x+x_0)).$$

Therefore, for non-autonomous differential equations $F = 0$ with $F \in \mathbb{K}[x]\{y\}$ it is sufficient to study solutions expanded around zero or at infinity and the corresponding transformed differential equation.
Note that for the more general case of $F \in \mathbb{K}\langle\langle x\rangle\rangle\{y\}$, such an $\bar{F}$ cannot be found as it is described more detailed in the Appendix A.

For a solution $y(x) \in \mathbf{Sol}_{\mathcal{R}}(F)$ of (1.1) expanded around zero we may additionally ask the solution to fulfill some given initial values $y(0) = y_0, y'(0) = y_1, \ldots$. Let $\mathbf{p}_0 = (y_0, y_1, \ldots)$ be such a finite tuple of initial values containing $y_0, y_1, \ldots \in \mathbb{K}_\infty$. Then we use the notation $\mathbf{Sol}_{\mathcal{R}}(F; \mathbf{p}_0)$ for solutions additionally fulfilling the initial values. Often we will neglect the subscript $\mathcal{R}$ or F in this notation if they are clear from to context and simply write $\mathbf{Sol}$ for the set of solutions and $\mathbf{Sol}(\mathbf{p}_0)$ if some additional initial values $\mathbf{p}_0$ are given. In the following chapter we devote a big part to the question on what a suitable number of given initial values for finding formal power series solutions might be, see Section 2.2.

Often we will require the given differential polynomial $F \in \mathcal{R}\{y\}$ to be square-free or even irreducible as polynomial in y and its derivatives. If F can be factored into several factors $F_j \in \mathcal{R}\{y\}$, then it holds that

$$\mathbf{Sol}_{\mathcal{R}}(F) = \bigcup_j \mathbf{Sol}_{\mathcal{R}}(F_j). \tag{1.2}$$

Since factorization is unique, up to constant factors and reordering, it is enough to study the factors of F.

For systems of differential equations $\Sigma \subset \mathcal{R}\{y_1, \ldots, y_p\}$ which will be of particular interest in Section 4.2, we denote by $\mathbf{Sol}_{\mathcal{R}}(\Sigma)$ the set of solution tuples of Σ and again write $\mathbf{Sol}_{\mathcal{R}}(\Sigma, \mathbf{p}_0)$ if there are additionally initial tuples $\mathbf{p}_0$ given. Let us note that

$$\mathbf{Sol}_{\mathcal{R}}(\Sigma) = \bigcap_{F \in \Sigma} \mathbf{Sol}_{\mathcal{R}}(F). \tag{1.3}$$

The similar relations to (1.2) and (1.3) hold in the case of non-constant solutions and if there are additional initial tuples given as well.

Chapter 2

Direct Approach

A common strategy to find formal power series solutions of AODEs is by making an ansatz of unknown coefficients, plugging the power series formally into the differential equation and comparing coefficients. This will be formalized by using so-called jet ideals in Section 2.1. In the generic case this is indeed a good strategy and if a particular initial value is given, the method will lead to a unique solution as Proposition 2.1.7 shows. However, it might be the case that after some steps there is no solution for the next coefficient. In other words, the computed truncation cannot be continued to a solution of the differential equation. Example 2.1.4 illustrates this phenomenon.

In this chapter we present the ansatz-based general method for determining formal power series solutions of AODEs. In order to overcome the difficulty described above, we follow the method which is inherited from the work by Hurwitz [Hur89], Limonov in [Lim15] and Denef and Lipshitz in [DL84]. There the authors give an expression of the derivatives of a differential polynomial with respect to the independent variable in terms of lower order differential polynomials (see [Hur89, page 328–329], [Lim15, Corollary 1] and [DL84, Lemma 2.2]). Our first contribution is to enlarge the class of differential equations where all formal power series solutions with a given initial value can be computed algorithmically. This class is given by a sufficient condition on the given differential equation and initial value which is described by the local vanishing order. Moreover, we give a necessary and sufficient condition on the given differential equation such that for every initial value all formal power series solutions can be computed in this way. For differential equations satisfying this condition, we give an algorithm to compute all formal power series solutions up to an arbitrary order and illustrate it by some examples.

The chapter is organized as follows. Section 2.1 is devoted to present a question (phrased as Conjecture 2.1.5) which occurs by simply performing coefficient comparison to compute formal power series solutions of AODEs and we show how the well-known formula of Ritt (Lemma 2.1.6) can be used in partially solving this problem, see Proposition 2.1.7. Since in general not all formal power series solutions can be found in this way, one may use a refinement of Ritt's formula presented in [Hur89, Lim15]. We summarize it by Theorem 2.2.2 in Section 2.2. In order to simplify some of the subsequent reasonings, we also use a slightly different notation and define *separant matrices*. Moreover, we give some sufficient conditions on the given differential equa-

tion, which is called the *vanishing order*, such that the refined formula can be used in an algorithmic way for computing all formal power series solutions and present new results in this direction, see Theorems 2.2.6, 2.2.18 and Algorithm 1. In Section 2.2 we focus on solutions with given initial values and study the vanishing order locally, whereas in Section 2.2 we generalize the vanishing order to arbitrary initial values. For the global situation, Proposition 2.2.13 and 2.2.14 show that a large class of AODEs indeed satisfy our sufficient conditions.
An interesting application of our method is to give an explicit statement of a result by Hurwitz in [Hur89]. Hurwitz proved that for every formal power series solution of an AODE there exists a large enough positive integer N such that coefficients of order greater than N are determined by a recursion formula. Our result can be used to determine a sharp upper bound for such an N and the corresponding recursion formula (compare with [vdH19]).

2.1 Implicit Function Theorem for AODEs

For $k \in \mathbb{N}$, the coefficient of x^k in a formal power series can be expressed by means of its k-th derivative, as stated in the following lemma (see [KP10, Theorem 2.3, page 20]).

Lemma 2.1.1. *Let $f \in \mathbb{K}[[x]]$ and $k \in \mathbb{N}$. Then $[x^k]f = [x^0]\left(\frac{1}{k!} f^{(k)}\right)$.*

By Lemma 2.1.1, we know that for a given $F \in \mathbb{K}[x]\{y\}$ with $\operatorname{ord}(F) = n$ and $y(x) \in \mathbb{K}[[x]]$ it holds that $F(y(x)) = 0$ if and only if

$$[x^0](F^{(k)}(y(x))) = F^{(k)}(0, \pi_{n+k}(y(x))) = 0 \quad \text{for each } k \in \mathbb{N},$$

where π_{n+k} projects the coefficients of the formal power series $y(x) = \sum_{i\geq 0} \frac{c_i}{i!} x^i$ to the first $n+k+1$ coefficients $(c_0, \ldots, c_{n+k})$. This fact motivates the following definition.

Definition 2.1.2. Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}^*$. Assume that $\mathbf{c} = (c_0, c_1, \ldots)$ is a sequence of indeterminates and $m \in \mathbb{N}$. We call the ideal

$$\mathcal{J}_m(F) = \langle F(0, \pi_n(\mathbf{c})), \ldots, F^{(m)}(0, \pi_{n+m}(\mathbf{c}))\rangle \subseteq \mathbb{K}[c_0, \ldots, c_{n+m}]$$

the m-th *jet ideal* of F. Assume that $\tilde{\mathbf{c}} = (c_0, c_1, \ldots, c_k) \in \mathbb{K}^{k+1}$ for some $k \in \mathbb{N}$ and $\tilde{y}(x) = c_0 + c_1 x + \cdots + \frac{c_k}{k!} x^k$. We say that $\tilde{\mathbf{c}}$, or $\tilde{y}(x)$, can be extended to a formal power series solutions of $F(y) = 0$, if there exists $y(x) \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$ such that

$$y(x) \equiv \tilde{y}(x) \mod x^{k+1}.$$

With this definition we know that $y(x) \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$ implies that every projection of its coefficient set is a zero of the corresponding jet ideal, i.e. for every $k \in \mathbb{N}$ it holds that

$$\pi_{n+k}(y(x)) \in \mathbb{V}_{\mathbb{K}}(\mathcal{J}_k(F)).$$

The converse direction is called "Strong approximation Theorem" and is proven for a more general situation in [DL84][Theorem 2.10]. Because it is such an important result, we state it here as its own theorem. Let us note that the assumption on $\mathbb{K}$ to be algebraically closed is essential here, see Remark 2.12 in the same reference.

Theorem 2.1.3 (Strong Approximation Theorem)**.** *Let $\Sigma \subset \mathbb{K}[x]\{y\}$ be a set of differential polynomials of order at most n such that for every $k \in \mathbb{N}$ there exists*

$$(c_0, \ldots, c_{n+k}) \in \mathbb{V}_{\mathbb{K}}\left(\bigcup_{F\in\Sigma} \mathcal{J}_k(F)\right).$$

Then there exists a solution $y(x) = \sum_{i\geq 0} \frac{c_i}{i!}\, x^i \in \mathbf{Sol}_{\mathbb{K}[[x]]}(\Sigma)$.

Since it is impossible to compute for every $k \in \mathbb{N}$ the algebraic set $\mathbb{V}_{\mathbb{K}}(\mathcal{J}_k(F))$ and check whether this set is empty or not, we need an upper bound thereof. The following example shows that this number, if it exists, can in general be arbitrarily big.

Example 2.1.4. For each $m \in \mathbb{N}^*$, consider the AODE

$$F = x\, y' - m\, y + x^m = 0$$

with the initial tuple $(c_0, c_1) = (0, 0) \in \mathbb{V}(\mathcal{J}_0(F))$. For every $0 \leq k < m$, we have

$$F^{(k)} = x\, y^{(k+1)} + (k - m)\, y^{(k)} + m(m-1)\cdots(m-k+1)\, x^{m-k}.$$

Therefore, we have that $(c_0, \ldots, c_{k+1}) \in \mathbb{V}(\mathcal{J}_k)$ for all $0 \leq k < m$ if and only if $c_0 = \cdots = c_k = 0$. However,

$$F^{(m)}(0, \ldots, 0, c_m, c_{m+1}) = m! \neq 0$$

and $(c_0, c_1) = (0, 0)$ cannot be extended to a formal power series solution of $F(y) = 0$.

The following conjecture is formulated in a classical way and would allow to at least decide the existence of a formal power series solution extending a given initial tuple.

Conjecture 2.1.5. Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. The descending chain

$$\mathbb{V}_{\mathbb{K}}(\mathcal{J}_0(F)) \supseteq \pi_n(\mathbb{V}_{\mathbb{K}}(\mathcal{J}_1(F))) \supseteq \pi_n(\mathbb{V}_{\mathbb{K}}(\mathcal{J}_2(F))) \supseteq \cdots \tag{2.1}$$

stabilizes, i.e. there exists an $N \in \mathbb{N}$ such that $\pi_n(\mathbb{V}_{\mathbb{K}}(\mathcal{J}_N(F))) = \pi_n(\mathbb{V}_{\mathbb{K}}(\mathcal{J}_{N+k}(F)))$ for every $k \geq 0$.

We note that in Conjecture 2.1.5 the sets $\pi_n(\mathbb{V}_{\mathbb{K}}(\mathcal{J}_j(F)))$ are in general not algebraic sets. Otherwise the statement would follow from Hilbert's Basis Theorem.
Let us assume that the chain (2.1) indeed stabilizes with an $N \in \mathbb{N}$. Therefore,

$$\pi_n(\mathbb{V}_{\mathbb{K}}(\mathcal{J}_N(F))) = \pi_n(\boldsymbol{\pi}(\mathbf{Sol}_{\mathbb{K}[[x]]}(F))).$$

This means that every $(c_0, \ldots, c_n) \in \mathbb{K}^{n+1}$ can be extended to a formal power series solution if and only if there exist some $c_{n+1}, \ldots, c_{n+N} \in \mathbb{K}$ such that

$$F^{(k)}(0, c_0, \ldots, c_{n+k}) = 0$$

for every $0 \leq k \leq N$. Provided that $N \in \mathbb{N}$ is known, this can be checked algorithmically.
However, since up to our knowledge there is no proof or counterexample of Conjecture 2.1.5, we have to come up with other ideas.

First let us recall a lemma showing that for $k \in \mathbb{N}^*$ the highest occurring derivative appears linearly in the k-th derivative of F with respect to x (see [Rit50, page 30]).

Lemma 2.1.6. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Then for each $k \in \mathbb{N}^*$, there exists a differential polynomial $R_k \in \mathbb{K}[x]\{y\}$ of order at most $n+k-1$ such that*

$$F^{(k)} = S_F \cdot y^{(n+k)} + R_k, \tag{2.2}$$

where $S_F = \frac{\partial F}{\partial y^{(n)}}$ is the separant of F.

Based on Lemma 2.1.6 and the reasonings in the beginning of the section, we have the following proposition.

Proposition 2.1.7. [1] Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Assume that $\mathbf{p}_0 = (c_0, c_1, \ldots, c_n) \in \mathbb{V}_{\mathbb{K}}(\mathcal{J}_0(F))$ and $S_F(\mathbf{p}_0) \neq 0$. For $k \in \mathbb{N}^*$, set

$$c_{n+k} = -\frac{R_k(0, c_0, \ldots, c_{n+k-1})}{S_F(\mathbf{p}_0)},$$

where R_k is specified in Lemma 2.1.6. Then $y(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$.

In the above proposition, if the separant of F vanishes at the initial value $\mathbf{p}_0$, we may expand R_k in Lemma 2.1.6 further in order to find formal power series solutions, as the following example illustrates.

Example 2.1.8. Consider the AODE

$$F = x\,y' + y^2 - y - x^2 = 0.$$

Since $S_F = x$, we cannot apply Proposition 2.1.7. Instead, we observe that for $k \in \mathbb{N}^*$,

$$F^{(k)} = x\,y^{(k+1)} + (2y + k - 1)\,y^{(k)} + \tilde{R}_{k-1}, \tag{2.3}$$

where $\tilde{R}_{k-1} \in \mathbb{K}[x]\{y\}$ is of order $k-1$.
Assume that $y(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$, where $c_i \in \mathbb{K}$ are to be determined. From $[x^0]F(y(x)) = 0$, we have that $c_0^2 - c_0 = 0$.
If we take $c_0 = 1$, then we can deduce from (2.3) that for each $k \in \mathbb{N}^*$,

$$[x^0]F^{(k)}(y(x)) = (k+1)\,c_k + \tilde{R}_{k-1}(0, 1, c_1, \ldots, c_{k-1}) = 0.$$

Thus,

$$c_k = -\frac{\tilde{R}_{k-1}(0, 1, c_1, \ldots, c_{k-1})}{k+1}.$$

Therefore, we derive uniquely a formal power series solution of $F(y) = 0$ with $c_0 = 1$.
If we take $c_0 = 0$, then we observe that

$$[x^0]F'(y(x)) = 2c_0\,c_1 = 0.$$

It implies that there is no constraint for c_1 in the equation $[x^0]F'(y(x)) = 0$. For $k \geq 2$, it follows from (2.3) that

$$[x^0]F^{(k)}(y(x)) = (k-1)\,c_k + \tilde{R}_{k-1}(0, 0, c_1, \ldots, c_{k-1}) = 0.$$

[1] This proposition is sometimes called Implicit Function Theorem for AODEs as a folklore.

Thus,

$$c_k = -\frac{\tilde{R}_{k-1}(0, 0, c_1, \ldots, c_{k-1})}{k-1},$$

where $k \geq 2$. Therefore, we derive uniquely a formal power series solutions of $F(y) = 0$ with $c_0 = 0$ for every $c_1 \in \mathbb{K}$. In other words, we can compute a general solution $y(x) \in \mathbb{K}(c_1)[[x]] \setminus \mathbb{K}[[x]]$, where c_1 is a new variable.

In the above example, we have expanded $F^{(k)}$ to the second highest derivative $y^{(k)}$. Evaluated at the given initial tuple, the coefficients of this term are non-zero and all formal power series solutions of $F(y) = 0$ can be constructed recursively up to an arbitrary order. In the next sections we will develop this idea in a systematical way.

2.2 Generalized Separants

In [Hur89, Lim15] the author present an expansion formula for derivatives of F with respect to x showing that not only the highest occurring derivative appears linearly, also the second-highest one, third-highest one, and so on, after taking sufficiently many derivatives. This is a refinement of Lemma 2.1.6 and [DL84][Lemma 2.2].
Throughout the section we will use the notation $F(\mathbf{c}) = F(0, \pi_n(\mathbf{c}))$, where $\mathbf{c} = (c_0, c_1, \ldots)$ is a sequence of indeterminates or elements in $\mathbb{K}$.

Definition 2.2.1. For a differential polynomial $F \in \mathbb{K}[x]\{y\}$ of order $n \in \mathbb{N}$ and $k, i \in \mathbb{N}$ let us define

$$f_i = \begin{cases} \frac{\partial F}{\partial y^{(i)}}, & i = 0, \ldots, n; \\ 0, & \text{otherwise}; \end{cases}$$

and

$$S_{F,k,i} = \sum_{j=0}^{i} \binom{k}{j} f_{n-i+j}^{(j)}.$$

We call $S_{F,k,i}$ the *generalized separants* of F.

Note that for any $k \in \mathbb{N}$ the generalized separant $S_{F,k,0}$ coincides with the usual separant $\frac{\partial F}{\partial y^{(n)}}$ of F. Moreover, the order of $S_{F,k,i}$ is less or equal to $n + i$.

Theorem 2.2.2. *Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Then for each $m \in \mathbb{N}$ and $k > 2m$ there exists a differential polynomial $r_{n+k-m-1}$ with order less than or equal to $n + k - m - 1$ such that*

$$F^{(k)} = \sum_{i=0}^{m} S_{F,k,i}\, y^{(n+k-i)} + r_{n+k-m-1}. \tag{2.4}$$

Proof. See [Lim15][Corollary 1]. □

Let us remark that the remaining term $r_{n+k-m-1}$, introduced in Theorem 2.2.2, depends on both m and k. Although the index might be the same, for distinct pairs $(m, k) \neq (\ell, r)$ in general also $r_{n+k-m-1} \neq r_{n+r-\ell-1}$. Nevertheless, here we simplify the notation without explicitly referring to this dependency in the notation.

For $F \in \mathbb{K}[x]\{y\}$ of order n and $m, k \in \mathbb{N}$, we define

$$\mathcal{B}_m(k) = \left[\binom{k}{0} \quad \binom{k}{1} \quad \cdots \quad \binom{k}{m} \right],$$

and

$$\mathcal{S}_{F,m} = \begin{bmatrix} f_n & f_{n-1} & f_{n-2} & \cdots & f_{n-m} \\ 0 & f_n^{(1)} & f_{n-1}^{(1)} & \cdots & f_{n-m+1}^{(1)} \\ 0 & 0 & f_n^{(2)} & \cdots & f_{n-m+2}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & f_n^{(m)} \end{bmatrix},$$

and

$$Y_m = \begin{bmatrix} y^{(m)} \\ y^{(m-1)} \\ \vdots \\ y \end{bmatrix}.$$

Then we can represent formula (2.4) of Theorem 2.2.2 as

$$F^{(k)} = \mathcal{B}_m(k) \cdot \mathcal{S}_{F,m} \cdot Y_m^{(n+k-m)} + r_{n+k-m-1}, \tag{2.5}$$

where $Y_m^{(j)}$ is the matrix obtained by considering the $j-$th derivatives of its entries.

In the following we will refer to $\mathcal{S}_{F,m}$ by the *m-th separant matrix of F*. An easy but important relation between the generalized separants and the separant matrix is the following one.

Corollary 2.2.3. *Let $F \in \mathbb{K}[x]\{y\}$ with $\operatorname{ord}(F) = n$, $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathbb{K}^{\mathbb{N}}$. Then the separant matrix $\mathcal{S}_{F,m}(\boldsymbol{c}) = 0$ if and only if for all $0 \le i \le m$ and $k \in \mathbb{N}^*$ it holds that $S_{F,k,i}(\boldsymbol{c}) = 0$.*

Proof. If all entries of the separant matrix are zero, then obviously the generalized separants are zero.
Vice versa, let for all $0 \le i \le m$ and $k \in \mathbb{N}^*$ be

$$S_{F,k,i}(\mathbf{c}) = \sum_{j=0}^{i} \binom{k}{j} f_{n-i+j}^{(j)} = 0.$$

In particular, if we set $i = 0$, we obtain $f_n(\mathbf{c}) = 0$, which is the left top entry of $\mathcal{S}_{F,m}(\mathbf{c})$. For $i = 1$ we obtain $f_{n-1}(\mathbf{c}) + k\, f_n^{(1)}(\mathbf{c}) = 0$, which can be seen as polynomial in k. Since this holds for every $k \in \mathbb{N}^*$, we obtain $f_{n-1}(\mathbf{c}) = f_n^{(1)}(\mathbf{c}) = 0$, which are the two entries of the second column of the separant matrix. Now continuing this process iteratively for all i up to m, the statement follows. □

Local Vanishing Order

In this part of the section we consider the problem of deciding when a solution modulo a certain power of x of a given AODE can be extended to a full formal power series solution. As a nice application of Theorem 2.2.2, we present a partial answer for this problem. In particular, given a certain number of coefficients satisfying some additional assumptions, we propose an algorithm to check whether there is a formal power series solution whose first coefficients are the given ones, and in the affirmative case, compute all of them (see Theorem 2.2.6 and Algorithm 1).
Let us start with a technical lemma which we will later often implicitly use.

Lemma 2.2.4. *Let $m, n \in \mathbb{N}$ and $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n and $\boldsymbol{c} = (c_0, c_1, \ldots)$ be a sequence of indeterminates. Assume that the generalized separants $S_{F,k,i}(\boldsymbol{c}) = 0$ for all $0 \le i < m$. Then the polynomial $F^{(k)}(\boldsymbol{c})$ involves only*

1. $c_0, \dots, c_{n+\lfloor k/2 \rfloor}$ *for* $0 \le k \le 2m$*;*

2. $c_0, \dots, c_{n+k-m}$ *for* $k > 2m$*.*

Proof. Let $0 \le k \le 2m$ and let us set $\tilde{m} = \lceil k/2 \rceil = k - \lfloor k/2 \rfloor$. Then $k > 2\tilde{m} - 1$ and by assumption and Theorem 2.2.2,

$$F^{(k)}(\mathbf{c}) = r_{n+k-\tilde{m}}(\mathbf{c}),$$

where $r_{n+k-\tilde{m}}$ is a differential polynomial of order at most $n + k - \tilde{m} = n + \lfloor k/2 \rfloor$, and item 1 follows.
Let $k > 2m$. Then, again by (2.5) and by the assumption,

$$F^{(k)}(\mathbf{c}) = S_{F,k,m}(\mathbf{c})\, c_{n+k-m} + r_{n+k-m-1}(\mathbf{c}),$$

and item 2 holds. □

Definition 2.2.5. Let $m, n \in \mathbb{N}$ and $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Let $\mathbf{p}_0 = (c_0, \dots, c_{n+m}) \in \mathbb{K}^{n+m+1}$. We say that F has *(local) vanishing order* m *at* $\mathbf{p}_0$ if

$$\mathcal{S}_{F,i}(\mathbf{p}_0) = 0 \quad \text{for all } 0 \le i < m, \quad \text{and} \quad \mathcal{S}_{F,m}(\mathbf{p}_0) \ne 0,$$

and

$$\mathbf{p}_0 \in \mathbb{V}_{\mathbb{K}}(\mathcal{J}_{2m}(F)).$$

As a consequence of Lemma 2.2.4 and the first part of the above definition, $\mathbb{V}_{\mathbb{K}}(\mathcal{J}_{2m}(F))$ can be seen as a subset of $\mathbb{K}^{n+m+1}$ and the second part of the definition is well-defined.

In the remaining part of the section we want to use the notion of local vanishing orders to ensure existence and uniqueness of solutions. Therefore let us first introduce some notations.
Let $m, n \in \mathbb{N}$ and $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Assume that $\mathbf{p}_0 = (c_0, \dots, c_{n+m}) \in \mathbb{K}^{n+m+1}$ and F has vanishing order m at $\mathbf{p}_0$. We consider $S_{F,t,m}$ as polynomial in t and denote

$$\mathbf{r}_{F,\mathbf{p}_0} = \text{the number of integer roots of } S_{F,t,m} \text{ greater than } 2m,$$

$$\mathbf{q}_{F,\mathbf{p}_0} = \begin{cases} \text{the largest integer root of } S_{F,t,m}, & \text{if } \mathbf{r}_{F,\mathbf{p}_0} \ge 1, \\ 2m, & \text{otherwise.} \end{cases}$$

Theorem 2.2.6. *Let* $F \in \mathbb{K}[x]\{y\}$ *with* $\operatorname{ord}(F) = n$ *be of vanishing order* $m \in \mathbb{N}$ *at* $\mathbf{p}_0 \in \mathbb{K}^{n+m+1}$.

1. *Then* $\mathbf{p}_0$ *can be extended to a formal power series solution of* $F(y) = 0$ *if and only if it can be extended to an element of* $\mathbb{V}_{\mathbb{K}}(\mathcal{J}_{\mathbf{q}_{F,\mathbf{p}_0}}(F))$.

2. Let

$$\mathcal{V}_{\mathbf{p}_0}(F) = \pi_{n+\boldsymbol{q}_{F,\mathbf{p}_0}-m}(\{\boldsymbol{c} \in \mathbb{V}_{\mathbb{K}}(\mathcal{J}_{\boldsymbol{q}_{F,\mathbf{p}_0}}(F)) \mid \pi_{n+m}(\boldsymbol{c}) = \mathbf{p}_0\}).$$

Then $\mathcal{V}_{\mathbf{p}_0}(F)$ is an affine variety of dimension at most $\mathbf{r}_{F,\mathbf{p}_0}$. Moreover, each point of it can be extended uniquely to a formal power series solution of $F(y) = 0$.

Proof. 1. Let $\mathbf{c} = (c_0, c_1, \ldots) \in \mathbb{K}^{\mathbb{N}}$ be such that $\pi_{n+m}(\mathbf{c}) = \mathbf{p}_0$, where c_k is to be determined for $k > n + m$. We recall that $\mathbf{c}$ defines a solution of $F(y) = 0$ if and only if $F^{(k)}(\mathbf{c}) = 0$ for each $k > 2m$. Since F has vanishing order m at $\mathbf{p}_0$, and by Theorem 2.2.2, there is a differential polynomial $r_{n+k-m-1}$ of order at most $n + k - m - 1$ such that

$$F^{(k)}(\mathbf{c}) = S_{F,k,m}(\mathbf{p}_0)\, c_{n+k-m} + r_{n+k-m-1}(\mathbf{c}) = 0. \tag{2.6}$$

If $\mathbf{p}_0$ can be extended to a solution, then also to a zero of $\mathcal{J}_{\mathbf{q}_{F,\mathbf{p}_0}}(F)$. Conversely, if $\mathbf{p}_0$ can be extended to a zero of $\mathcal{J}_{\mathbf{q}_{F,\mathbf{p}_0}}(F)$, there exist $c_{n+m+1}, \ldots, c_{n+\mathbf{q}_{F,\mathbf{p}_0}-m} \in \mathbb{K}$ such that equation (2.6) holds for $k = 2m, \ldots, \mathbf{q}_{F,\mathbf{p}_0}$. For $k > \mathbf{q}_{F,\mathbf{p}_0}$, we set

$$c_{n+k-m} = -\frac{r_{n+k-m-1}(0, c_0, \ldots, c_{n+k-m-1})}{S_{F,k,m}(\mathbf{p}_0)} \tag{2.7}$$

and $y(x) = \sum_{i\geq 0} \frac{c_i}{i!}\, x^i \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$.
2. By item 1, $\mathbb{V}_{\mathbf{p}_0}(F)$ is the set of points fulfilling

1. $\tilde{c}_k = c_k \quad$ for $0 \leq k \leq n + m$;
2. $S_{F,k,m}(\mathbf{c})\, \tilde{c}_{n+k-m} + r_{n+k-m-1}(\tilde{c}_0, \ldots, \tilde{c}_{n+k-m-1}) = 0 \quad$ for $2m < k \leq \mathbf{q}_{F,\mathbf{p}_0}$,

and therefore it is an affine variety.
If $\mathbf{r}_{F,\mathbf{p}_0} = 0$, then $\mathbf{q}_{F,\mathbf{p}_0} = 2m$ and thus, $\mathbb{V}_{\mathbf{p}_0}(F) = \{\mathbf{p}_0\}$ contains one point.
Assume that $\mathbf{r}_{F,\mathbf{p}_0} \geq 1$. Let $k_1 < \cdots < k_{\mathbf{r}_{F,\mathbf{p}_0}} = \mathbf{q}_{F,\mathbf{p}_0}$ be integer roots of $S_{F,t,m}(\mathbf{p}_0)$ which are greater than $2m$. If $k \notin \{k_1, \ldots, k_{\mathbf{r}_{F,\mathbf{p}_0}}\}$, then it follows from (2.6) that $\tilde{c}_{n+k-m}$ is uniquely determined from the previous coefficients and

$$\begin{array}{rccc} \phi: & \mathcal{V}_{\mathbf{p}_0}(F) & \longrightarrow & \mathbb{K}^{\mathbf{r}_{F,\mathbf{p}_0}} \\ & \tilde{\mathbf{c}} & \longmapsto & (\tilde{c}_{n+k_1-m}, \ldots, \tilde{c}_{n+k_{\mathbf{r}_{F,\mathbf{p}_0}}-m}) \end{array}$$

defines an injective map. Therefore, we conclude that $\mathcal{V}_{\mathbf{p}_0}(F)$ is of dimension at most $\mathbf{r}_{F,\mathbf{p}_0}$. Moreover, it follows from item 1 that each point of $\mathcal{V}_{\mathbf{p}_0}(F)$ can be uniquely extended to a formal power series solution of $F(y) = 0$. □

The proof of the above theorem is constructive. More precisely, if a tuple $\mathbf{p}_0 \in \mathbb{K}^{n+m+1}$ satisfies the condition that F has vanishing order m at $\mathbf{p}_0$, then the proof gives an algorithm to decide whether $\mathbf{p}_0$ can be extended to a formal power series solution of $F(y) = 0$ or not, and in the affirmative case determine all of them.

For $F \in \mathbb{K}[x]\{y\}$ let $y(x) \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$ and let $\tilde{y}(x) \in \mathbb{K}[x]$ with $d = \deg(\tilde{y}(x))$ and

$$y(x) \equiv \tilde{y}(x) \mod x^d.$$

Then we call $\tilde{y}(x)$ a *determined solution truncation* of $F = 0$ if there is no other solution $z(x) \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$ different from $y(x)$ such that $z(x) \equiv \tilde{y}(x) \mod x^d$. In other words, the truncation $\tilde{y}(x)$ extends uniquely to the solution $y(x)$.
Observe that a determined solution truncations is not unique, since by adding further consecutive terms of the solution one again obtains a determined solution truncation of the same solution.

Let us restate the conclusion of Theorem 2.2.6 in the notation introduced here.

Corollary 2.2.7. *Let $F \in \mathbb{K}[x]\{y\}$ with $\operatorname{ord}(F) = n$ be of vanishing order $m \in \mathbb{N}$ at $\mathbf{p}_0 \in \mathbb{K}^{n+m+1}$. Then the mapping*

$$\begin{array}{rccc} \pi_{n+\mathbf{q}_{F,\mathbf{p}_0}-m} \circ \boldsymbol{\pi} : & \mathbf{Sol}_{\mathbb{K}[[x]]}(F, \mathbf{p}_0) & \longrightarrow & \mathcal{V}_{\mathbf{p}_0}(F) \\ & y(x) = \sum_{i \geq 0} \frac{c_i}{i!} x^i & \longmapsto & (\mathbf{p}_0, c_{n+m+1}, \ldots, c_{n+\mathbf{q}_{F,\mathbf{p}_0}-m}) \end{array}$$

is bijective and the elements in the image are determined solution truncations of $F = 0$.

We summarize the results obtained above as the following algorithm.

Algorithm 1 DirectMethodLocal

Input: $\ell \in \mathbb{N}$, $\mathbf{p}_0 = (c_0, \ldots, c_{n+m}) \in \mathbb{K}^{n+m+1}$, and a differential polynomial F of order n which has vanishing order m at $\mathbf{p}_0$.

Output: The set $\mathbf{Sol}_{\mathbb{K}[[x]]}(\mathbf{p}_0)$ described by determined solution truncations with $\mathbf{p}_0$ as initial values up to an order of at least ℓ represented by a finite number of indeterminates and algebraic conditions on them.

Compute $S_{F,k,m}(\mathbf{p}_0), \mathbf{r}_{F,\mathbf{p}_0}, \mathbf{q}_{F,\mathbf{p}_0}$ and the defining equations of $\mathcal{V}_{\mathbf{p}_0}(F)$:

1. $\tilde{c}_k = c_k \quad \text{for } 0 \leq k \leq n+m;$

2. $S_{F,k,m}(\mathbf{p}_0)\, \tilde{c}_{n+k-m} + r_{n+k-m-1}(\tilde{c}_0, \ldots, \tilde{c}_{n+k-m-1}) = 0 \quad \text{for } 2m < k \leq \mathbf{q}_{F,\mathbf{p}_0},$

where $\tilde{c}_k$'s are indeterminates.

Check whether $\mathcal{V}_{\mathbf{p}_0}(F)$ is empty or not by using Gröbner bases.

if $\mathcal{V}_{\mathbf{p}_0}(F) = \emptyset$ **then**

  Output the string "$\mathbf{p}_0$ can not be extended to a formal power series solution of $F(y) = 0$".

else

  Compute $\tilde{c}_{n+\mathbf{q}_{F,\mathbf{p}_0}-m+1}, \ldots, \tilde{c}_\ell$ by using (2.7).

  return $\sum_{i=0}^{\max(\mathbf{q}_{F,\mathbf{p}_0},\ell)} \frac{\tilde{c}_i}{i!} x^i$ and $\mathcal{V}_{\mathbf{p}_0}(F)$.

end if

The termination of the above algorithm is evident. The correctness follows from Theorem 2.2.6 and Corollary 2.2.7.

Example 2.2.8. Consider the second order AODE

$$F = x\, y'' - 3y' + x^2\, y^2 = 0.$$

Let $\mathbf{p}_0 = (c_0, 0, 0, 2c_0^2) \in \pi_3(\mathbb{V}_{\mathbb{C}}(\mathcal{J}_2(F)))$, where c_0 is an arbitrary constant in $\mathbb{C}$. One can verify that each point of $\pi_3(\mathbb{V}_{\mathbb{C}}(\mathcal{J}_2(F)))$ is of the form of $\mathbf{p}_0$. A direct calculation shows that F has vanishing order 1 at $\mathbf{p}_0$. Moreover, we have that $S_{F,t,1}(\mathbf{p}_0) = t - 3$ and $\mathbf{q}_{F,\mathbf{p}_0} = 3$. Thus, it follows that

$$\mathcal{V}_{\mathbf{p}_0}(F) = \{\mathbf{c} = (c_0, 0, 0, 2c_0^2, c_4) \in \mathbb{C}^5 \mid c_4 \in \mathbb{C}\}.$$

So, the dimension of $\mathcal{V}_{\mathbf{p}_0}(F)$ is equal to one and the corresponding formal power series solutions computed up to the order $\ell = 10$ are

$$y(x) \equiv c_0 + \frac{c_0^2}{3}\, x^3 + \frac{c_4}{24}\, x^4 - \frac{c_0^3}{18}\, x^6 - \frac{c_0\, c_4}{252}\, x^7 - \frac{c_0^2\, c_4}{3024}\, x^{10} \mod x^{11}.$$

Above all, the sets $\mathbf{Sol}_{\mathbb{C}[[x]]}(F, \mathbf{p}_0)$ and $\mathcal{V}_{\mathbf{p}_0}(F)$ and $\mathbb{C}^2$ are in bijection.

In the following remark we explain that every solution of a given AODE, regular or singular, is a solution of an AODE with a finite vanishing order.

Remark 2.2.9. Recall that for $F \in \mathbb{K}[x]\{y\}$ with $\operatorname{ord}(F) = n$ a solution $y(x) \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$ is called *non-singular* if it does not vanish at the separant $S_F = \frac{\partial F}{\partial y^{(n)}}$ identically. If $y(x) = \sum_{i \geq 0} \frac{c_i}{i!}\, x^i$ is a non-singular solution of $F(y) = 0$, then there exists an $m \in \mathbb{N}$ such that

$$S_F^{(m)}(0, c_0, \ldots, c_{n+m}) \neq 0.$$

Therefore,

$$\mathcal{S}_{F,m}(0, c_0, \ldots, c_{n+m}) \neq 0$$

and F has a vanishing order of at most m at $(c_0, \ldots, c_{n+m})$.
If $y(x)$ is a singular solution of $F_0(x, y, \ldots, y^{(n)}) = F(y) = 0$, then it is also a solution of the resultant (with respect to $y^{(n)}$)

$$F_1(x, y, \ldots, y^{(n-1)}) = \operatorname{Res}_{y^{(n)}}(F(y), S_F(y)) = 0.$$

The differential polynomial F_1 is of order less than n and non-trivial by choosing F to be square-free. If $y(x)$ is a non-singular solution of $F_1(y) = 0$, there exists $m \in \mathbb{N}$ such that F_1 has vanishing order at most equal to m at $(c_0, \ldots, c_{n+m-1})$. Otherwise we compute the resultant of F_1 and its separant with respect to $y^{(n-1)}$, which is again of order less than F_1. Continuing this procedure we derive up to $n+1$ differential polynomials

$$F_0(x, y, \ldots, y^{(n)}), F_1(x, y, \ldots, y^{(n-1)}), \ldots, F_n(x, y)$$

such that for every solution of $F(y) = 0$ there exists $i \in \{0, \ldots, n\}$ and an $m \in \mathbb{N}$ such that F_i has vanishing order at most m at $(c_0, \ldots, c_{n+m-i})$.

Let us emphasize that in Remark 2.2.9 there is no bound on the vanishing order provided, just that there exists a finite one. Algorithmically this is really problematic and we devote a big part of the following section to solve this problem.

Global Vanishing Order

In the first part of this section, in particular Theorem 2.2.6 and Algorithm 1, we have seen that for a given differential polynomial F of order n we require initial values $\mathbf{p}_0 \in \mathbb{K}^{n+m+1}$ such that F has vanishing order m at $\mathbf{p}_0$. In general, the natural number m can be arbitrarily large. In this section, we give a sufficient condition for differential polynomials for which the existence for an upper bound of m is guaranteed. It the sufficient condition is fulfilled, Algorithm 1 can be applied to every initial tuple of appropriate length.

Definition 2.2.10. Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order $n \in \mathbb{N}$. Assume that $\mathbf{c} = (c_0, c_1, \ldots)$ is a sequence of indeterminates and $m \in \mathbb{N}$. We define $\mathcal{I}_m(F) \subseteq \mathbb{K}[c_0, \ldots, c_{n+m}]$ as the ideal generated by the entries of the separant matrix $\mathcal{S}_{F,m}(\mathbf{c})$. We say that F has *(global) vanishing order m* if m is the smallest natural number such that

$$1 \in \mathcal{I}_m(F) + \mathcal{J}_{2m}(F) \subseteq \mathbb{K}[c_0, \ldots, c_{n+2m}],$$

where $\mathcal{J}_{2m}(F)$ is the $2m$-th jet ideal of F. If there exists no such $m \in \mathbb{N}$, we define the vanishing order as ∞.

Example 2.2.11. Consider the AODE from Example 2.1.8

$$F = x\,y' + y^2 - y - x^2 = 0.$$

By computation, we find that $1 \notin \langle c_0^2 - c_0 \rangle_{\mathbb{C}[c_0]} = \mathcal{I}_0 + \mathcal{J}_0$. Furthermore, we have $1 \in \mathcal{I}_1 + \mathcal{J}_2$, because $(S_F)' = 1$. Therefore, the differential polynomial F has vanishing order 1.

In the next proposition we consider a very particular case, namely those of irreducible bivariate polynomials, and show that they have finite vanishing order. Note that bivariate polynomials can be seen as differential polynomials of order zero.

Proposition 2.2.12. Let $A \in \mathbb{K}[x, y]$ with total degree $\deg(A)$ be irreducible. Then A has a vanishing order of at most $2\deg(A)\,(\deg(A)/2 + 1)$.

Proof. Assume that $A(x, y)$ is irreducible in $\mathbb{K}[x, y]$ and set

$$S_A = \frac{\partial A}{\partial y} \in \mathbb{K}[x, y].$$

By Gauss's Lemma, $A(x, y)$ is also irreducible in $\mathbb{K}(x)[y]$. Therefore, $\gcd(A, S_A) = 1$ in $\mathbb{K}(x)[y]$. By Bézout's identity, there exist $U, V \in \mathbb{K}(x)[y]$ such that

$$U\,A + V\,S_A = 1.$$

By clearing the denominators of the above equation, we know that there exist $\tilde{U}, \tilde{V} \in \mathbb{K}[x, y]$ and $D \in \mathbb{K}[x]^*$ such that

$$\tilde{U}\,A + \tilde{V}\,S_A = D. \tag{2.8}$$

Let $d = \deg_x(D)$. Differentiating both sides of the above equation for d times, we have that

$$\sum_{i=0}^{d} \binom{d}{i} \tilde{U}^{(i)} A^{(d-i)} + \sum_{i=0}^{d} \binom{d}{i} \tilde{V}^{(i)} S_A^{(d-i)} = c,$$

where $c \in \mathbb{K}^*$. It implies that

$$1 \in \mathcal{I}_d(A) + \mathcal{J}_{2d}(A).$$

Therefore, we conclude from Definition 2.2.10 that A has a vanishing order of at most d.
Let us now bound the degree of D: By choosing the lexicographical ordering $y > x$ we could compute a reduced Gröbner basis G of the ideal

$$I = \langle A, S_A \rangle \subseteq \mathbb{K}[x, y].$$

Since $D \in \mathcal{I} \cap \mathbb{K}[x]$, there is a divisor of D in G. Thus, by [Dub90] and the fact that $\deg(S_A) \leq \deg(A)$,

$$\deg_x(D) \leq 2\deg(A) \left(\frac{\deg(A)}{2} + 1 \right)$$

follows and the Proposition is shown. □

Alternatively to the proof of Proposition 2.2.12 given here, one could say that an irreducible polynomial $A \in \mathbb{K}[x, y]$ does not have a singular solution. Then, as it is in Remark 2.2.9 described for specific initial values, it follows that a A has a finite vanishing number.
Now let us consider differential polynomials where one of the derivatives is separated and of degree one.

Proposition 2.2.13. Every differential polynomial of the type

$$A(x)\, y^{(m)} + B(x, y, \ldots, y^{(m-1)}, y^{(m+1)}, \ldots, y^{(n)})$$

has a vanishing order of at most $\deg_x(A) + n - m$.

Proof. In the separant matrix $\mathcal{S}_{F,\deg_x(A)+n-m}$ appears the non-zero entry

$$f_m^{(\deg_x(A))} = \deg_x(A)!\ \mathrm{lc}(A).$$

Hence, $1 \in \mathcal{I}_{\deg_x(A)} \subseteq \mathcal{I}_{\deg_x(A)} + \mathcal{J}_{2\deg_x(A)}$. □

Proposition 2.2.13 is a generalization of [Lim15][Corollary 2] where only the case $A(x) \in \mathbb{K}$ is treated. The following proposition gives a characterization of differential polynomials with finite vanishing order.

Proposition 2.2.14. Let $F \in \mathbb{K}[x]\{y\}$ be a differential polynomial of order n. Then F has a finite vanishing order if and only if the differential system

$$F = \frac{\partial F}{\partial y} = \cdots = \frac{\partial F}{\partial y^{(n)}} = 0 \tag{2.9}$$

has no solution in $\mathbb{K}[[x]]$. In particular, if the differential ideal $\left[F, \frac{\partial F}{\partial y}, \ldots, \frac{\partial F}{\partial y^{(n)}}\right]$ in $\mathbb{K}(x)\{y\}$ contains 1, then F has finite vanishing order.

Proof. Assume that the system (2.9) has a solution $y(x) = \sum_{i\geq 0} \frac{c_i}{i!} x^i \in \mathbb{K}[[x]]$. Then for every $m \in \mathbb{N}$, $(c_0, \ldots, c_{n+m})$ is in $\mathbb{V}(\mathcal{I}_m(F) + \mathcal{J}_{2m}(F))$. Therefore, by Hilbert's weak Nullstellensatz, $1 \notin \mathcal{I}_m(F) + \mathcal{J}_{2m}(F)$ and F does not have finite vanishing order.

Conversely, assume that F does not have finite vanishing order, i.e. for each $k \in \mathbb{N}$ we have $1 \notin \mathcal{I}_{k+n}(F) + \mathcal{J}_{2(k+n)}(F)$ and the ideals have a common root $\mathbf{p}_0 = (c_0, \ldots, c_{3n+2k}) \in \mathbb{K}^{3n+2k+1}$. In particular, we have for all $i = 0, \ldots, k$

$$F^{(i)}(\mathbf{p}_0) = \left(\frac{\partial F}{\partial y}\right)^{(i)}(\mathbf{p}_0) = \ldots = \left(\frac{\partial F}{\partial y^{(n)}}\right)^{(i)}(\mathbf{p}_0) = 0, \tag{2.10}$$

and therefore, by Lemma 2.1.1, for $\tilde{y}(x) = \sum_{i=0}^{3n+2k} \frac{c_i}{i!} x^i$ and every $i = 0, \ldots, k$ also

$$[x^i]F(\tilde{y}(x)) = [x^i]\frac{\partial F}{\partial y}(\tilde{y}(x)) = \cdots = [x^i]\frac{\partial F}{\partial y^{(n)}}(\tilde{y}(x)) = 0.$$

Thus, $\tilde{y}(x)$ is a solution of $F = \frac{\partial F}{\partial y} = \cdots = \frac{\partial F}{\partial y^{(n)}} = 0$ modulo x^k. Due to the Strong Approximation Theorem, we conclude that the system (2.9) admits a solution in $\mathbb{K}[[x]]$ and the equivalence is proven.

Now let $1 \in \left[F, \frac{\partial F}{\partial y}, \ldots, \frac{\partial F}{\partial y^{(n)}}\right]$. Then system (2.9) does not have a solution in $\mathbb{K}[[x]]$ and F has a finite vanishing order. □

Remark 2.2.15. In order to test whether $1 \in \left[F, \frac{\partial F}{\partial y}, \ldots, \frac{\partial F}{\partial y^{(n)}}\right] \subseteq \mathbb{K}(x)\{y\}$, one can use the *RosenfeldGroebner* command in the Maple package *DifferentialAlgebra*.

Below is an example of a differential polynomial with vanishing order ∞.

Example 2.2.16. Consider the AODE

$$F = (y')^2 + y^3 = 0.$$

A direct computation implies that for each $m \in \mathbb{N}$, we have

$$\mathcal{I}_m(F) + \mathcal{J}_{2m}(F) \subseteq \langle c_0, c_1, \ldots, c_{2m+1}\rangle.$$

Therefore, F has vanishing order ∞. Note that the system

$$F = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial y'} = 0$$

from Proposition 2.2.14 has a common solution $y(x) = 0$.

Next, we show that for each $m \in \mathbb{N}$, there exists a differential polynomial with vanishing order m.

Example 2.2.17. Assume that $m \in \mathbb{N}$. Consider the AODE

$$F = \frac{(y' + y)^2}{2} + x^{2m} = 0.$$

For $m = 0$, it is straightforward to see that F has vanishing order 0.
Let $m > 0$. By computation, we find that $\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y} = y' + y$. Therefore, we have that

$$\mathcal{I}_m(F) = \langle c_1 + c_0, \ldots, c_{m+1} + c_m \rangle. \tag{2.11}$$

For each $k \geq 0$, it is straightforward to see that $((y' + y)^2)^{(k)}$ is a $\mathbb{C}$-linear combination of terms of the form $(y^{(i)} + y^{(i+1)})\,(y^{(j)} + y^{(j+1)})$ with $i + j = k$, and $i, j \geq 0$. Therefore, we conclude that for each $0 \leq k \leq m-1$, the jet ideal $\mathcal{J}_{2k}(F)$ is contained in $\mathcal{I}_m(F)$. This implies that

$$\mathcal{I}_k(F) + \mathcal{J}_{2k}(F) \subseteq \mathcal{I}_m(F) \quad \text{for } 0 \leq k \leq m - 1.$$

By (2.11) and the above formula, we have

$$1 \notin \mathcal{J}_{2k}(F) + \mathcal{I}_k(F) \quad \text{for } 0 \leq k \leq m - 1.$$

Furthermore, we have that

$$F^{(2m)}(0, c_0, \ldots, c_{2m+1}) \equiv (2m)! \mod \mathcal{I}_m(F).$$

Thus,

$$1 \in \mathcal{I}_m(F) + \mathcal{J}_{2m}(F)$$

and by definition, F has vanishing order m.

The following theorem is a generalization of the Implicit Function Theorem for AODEs (Proposition 2.1.7).

Theorem 2.2.18. *Let $F \in \mathbb{K}[x]\{y\}$ be of order n with vanishing order $m \in \mathbb{N}$ and let $\mathbf{p}_0 \in \mathbb{V}_{\mathbb{K}}(\mathcal{J}_{2m}(F))$.*

1. *Then there exists $i \in \{0, \ldots, m\}$ such that F has vanishing order i at $\pi_{n+i}(\mathbf{p}_0)$.*
2. *Let $M = \max\{2m + i, \boldsymbol{q}_{F,\mathbf{p}_0}\}$. Then $\mathbf{p}_0$ can be extended to a formal power series solution of $F(y) = 0$ if and only if it can be extended to a zero point of $\mathcal{J}_M(F)$.*
3. *Let*
$$\mathcal{V}_{\mathbf{p}_0}(F) = \pi_{n+M-i}(\{\boldsymbol{c} \in \mathbb{V}_{\mathbb{K}}(\mathcal{J}_M(F)) \mid \pi_{n+2m}(\boldsymbol{c}) = \mathbf{p}_0\}).$$
Then $\mathcal{V}_{\mathbf{p}_0}(F)$ is an affine variety of dimension at most $\boldsymbol{r}_{F,\mathbf{p}_0}$. Moreover, each point of it can be uniquely extended to a formal power series solution of $F(y) = 0$.

Proof. 1. Since $\mathbf{p}_0 \in \mathbb{V}_{\mathbb{K}}(\mathcal{J}_{2m}(F))$ and F has vanishing order m, it follows that there exists a minimal $i \in \{0, \ldots, m\}$ such that

$$\mathcal{S}_{F,i}(\mathbf{p}_0) \neq 0.$$

By item 2 of Lemma 2.2.4, we have $F^{(k)}(\mathbf{p}_0) = 0$ for $k = 0, \ldots, 2i$. Taking into account Lemma 2.2.4, only the first $n + i + 1$ coefficients of $\mathbf{p}_0$ are relevant and therefore, F has vanishing order i at $\pi_{n+i}(\mathbf{p}_0)$.
2. and 3. The proofs are literally the same as those in Theorem 2.2.6. □

As a consequence of the above theorem, the solution set $\mathbf{Sol}_{\mathbb{K}[[x]]}(F)$ is in bijection with the set

$$\bigcup_{\mathbf{c}\in\mathbb{V}(\mathcal{J}_{2m}(F))} \mathcal{V}_{\mathbf{p}_0}(F).$$

Moreover, Algorithm 1 can be applied to every given initial tuple $\mathbf{p}_0 \in \mathbb{K}^{n+m+1}$. So it can be decided whether there exists a formal power series solution of $F(y) = 0$ extending $\mathbf{p}_0$ or not and in the affirmative case, all formal power series solutions can be described in finite terms. Let us note that, in contrast to the local case (see Corollary 2.2.7), the length of the initial values is in general not the same. Of course initial values leading to solutions could always be extended uniquely, but algorithmically we want to avoid this.
Theorem 2.2.18 gives also a positive answer to Conjecture 2.1.5 under the additional assumption that F has a finite vanishing order. The upper bound can then be given by M as defined in item 2.

In [DL84], Lemma 2.3 only concerns non-singular formal power series solutions of a given AODE. Our method also can be used to find singular solutions of AODEs defined by differential polynomials with finite vanishing order, as the following example illustrates.

Example 2.2.19. Consider the AODE

$$F = y'^2 + y' - 2y - x = 0.$$

From Proposition 2.2.13 we know that F has a vanishing order of at most 1.
Let $\mathbf{p}_0 = (-\frac{1}{8}, -\frac{1}{2}, 0, c_3)$, where c_3 is an arbitrary constant in $\mathbb{C}$. It is straightforward to verify that $\mathbf{p}_0$ is a zero point of $\mathcal{J}_2(F)$. Furthermore, we have that F has vanishing order 1 at $\pi_2(\mathbf{c}) = (-\frac{1}{8}, -\frac{1}{2}, 0)$. Therefore, we also know that F has indeed vanishing order equal to 1. We find that $S_{F,k,1}(\mathbf{p}_0) = -2$ and $M = 3$. From item 2 of Theorem 2.2.18, we know that $\mathbf{p}_0$ can be extended into a formal power series solution of $F(y) = 0$ if and only if it can be extended to a zero point of $\mathcal{J}_3(F)$. By calculation, we see that $\mathbf{p}_0 \in \mathbb{V}_{\mathbb{C}}(\mathcal{J}_3(F))$ if and only if $c_3 = 0$. Hence, $\mathcal{V}_{\mathbf{p}_0}(F) = \{(-\frac{1}{8}, -\frac{1}{2}, 0, 0)\}$ and $\mathbf{p}_0$ extends uniquely to

$$y_1(x) = -\frac{1}{8} - \frac{x}{2} \; \in \mathbf{Sol}_{\mathbb{C}[[x]]}(F, \mathbf{p}_0).$$

It is straightforward to verify that $y_1(x)$ is a singular solution of $F(y) = 0$.
Similarly, let $\mathbf{p}_1 = (-\frac{1}{8}, -\frac{1}{2}, 1, c_3)$, where c_3 is an arbitrary constant in $\mathbb{C}$. Using item 2 of Theorem 2.2.18, we deduce that $\mathbf{p}_1$ can be extended into a formal power solution of $F(y) = 0$ if and only if $c_3 = 0$. In the affirmative case, we find

$$y_2(x) = -\frac{1}{8} - \frac{x}{2} + \frac{x^2}{2} \; \in \mathbf{Sol}_{\mathbb{C}[[x]]}(F, \mathbf{p}_1).$$

Actually, one can verify that $y_1(x), y_2(x)$ are all the formal power series solutions of $F(y) = 0$ where the Implicit Function Theorem for AODEs cannot be applied. Therefore, the set of formal power series solutions can be decomposed into

$$\mathbf{Sol}_{\mathbb{C}[[x]]}(F) = \{y_1(x), y_2(x)\} \cup \{y(x) \in \mathbf{Sol}_{\mathbb{C}[[x]]}(F) \mid [x^0]S_F(y) \neq 0\}.$$

Chapter 3

Newton Polygon Method for AODEs

In this chapter we give a short description of the Newton polygon method for AODEs following the works [Fin89], [Can05] and [Can93a]. In the general case of AODEs, by this or any other method known to us, it is not possible to describe all formal Puiseux series solutions such that existence and uniqueness is guaranteed algorithmically. However, later we will consider equations of a very specific type where we are able to provide exactly such algorithms.

The present chapter is structured into two sections. Section 3.1 gives a description of the Newton polygon method for differential equations. It can be seen as an adjustment of the Newton polygon method for algebraic equations which is described in the appendix (see Section A.1) and gives necessary conditions on solution truncations term by term. In contrast to the Newton polygon method for algebraic equations, however, the solution truncations cannot always be extended. This can already be seen by the cases explained in Section 3.1 and Example 3.1.8. Since the Newton polygon changes in each computation step, it might happen after an arbitrary high number of computational steps that the truncation cannot be extended anymore. The same problem arises for the uniqueness of the solutions, see Example 3.1.4.

Section 3.2 is devoted to give some criteria on the differential equations such that the properties of existence and uniqueness of the solutions can be guaranteed. For some specific families of differential equations, similar to those where the direct approach can be used algorithmically as well, we can prove a sufficient condition on existence and uniqueness after a particular number of computational steps. These results generalize the results obtained by the direct approach and are not presented in this framework anywhere else.

The remaining Section 3.3 is devoted to a specific family of differential equations, see (3.11). For equations of this type we give a finite description of all solutions and show how many steps have to be computed such that existence and uniqueness are guaranteed. We present results on convergence and the dependence of the coefficients, see Lemma 3.3.1. These results are new and essential for our results in later sections as in Section 4.1, where we will relate first order autonomous AODEs to equations of the type (3.11).

3.1 Description

In this section we describe the term by term construction of possible formal Puiseux series solutions. For more details on formal Puiseux series see Section A. Therefor are two steps necessary, the construction of the Newton polygon and the change of variables.
First, for a given differential equation of arbitrary order $F(x, y, \ldots, y^{(n)}) = 0$, we make the ansatz

$$y(x) = c\,x^{\mu} + y_1(x) \in \mathbb{K}\langle\langle x\rangle\rangle \tag{3.1}$$

with $\mathrm{ord}_x(y_1(x)) > \mu$ and $\mu \in \mathbb{Q}$. The lowest order terms in the differential equation have to cancel out or $y(x)$ cannot be a solution. This leads to the necessary condition that (c, μ) has to be a root of (3.6) or (3.7) and to the construction of the Newton polygon.
Secondly, after determining (c, μ), simplify $F(x, c\,x^{\mu} + y_1, \ldots, (c\,x^{\mu} + y_1)^{(n)}) = 0$ and continue with the new differential equation in y_1.
Let us note that since we start with the lowest order terms, $\mathrm{ord}(y(x)) = \mu$ and the denominator of μ is a divisor of the ramification index of the potential solution $y(x)$.

The Newton polygon

A differential polynomial $F \in \mathbb{K}\langle\langle x\rangle\rangle\{y\}$ of order n can uniquely be written as

$$F = \sum a_{\alpha,\rho_0,\ldots,\rho_n} x^{\alpha}\, y^{\rho_0} \cdots (y^{(n)})^{\rho_n} \quad \text{with } a_{\alpha,\boldsymbol{\rho}} \in \mathbb{K}.$$

For the subsequent reasonings we may assume that the order of F with respect to x is non-negative, because otherwise we multiply the differential equation $F = 0$ with $x^{-\alpha}$, where α is the minimal exponent. In order to find formal Puiseux series solutions let $y(x) = c\,x^{\mu} + y_1(x) \in \mathbb{K}\langle\langle x\rangle\rangle$ with $c \in \mathbb{K},\ \mu \in \mathbb{Q},\ y_1(x) \in \mathbb{K}\langle\langle x\rangle\rangle,\ \mathrm{ord}(y_1(x)) > \mu$ be a solution of $F = 0$. For $k \in \mathbb{N}^*$, let us denote by $(\mu)_k$ the falling factorials, i.e. $(\mu)_k = \mu(\mu-1)\cdots(\mu-k+1)$. Then

$$\begin{aligned} &F(c\,x^{\mu} + y_1(x)) = \\ &\sum a_{\alpha,\boldsymbol{\rho}}\, x^{\alpha}\, (c\,x^{\mu} + y_1(x))^{\rho_0}\, (\mu\, c\,x^{\mu-1} + y_1'(x))^{\rho_1} \cdots ((\mu)_n\, c\,x^{\mu-n} + y_1^{(n)}(x))^{\rho_n} = \\ &\sum a_{\alpha,\boldsymbol{\rho}}\, c^{\sum_{i=0}^{n}\rho_i}\, (\mu)_1^{\rho_1} \cdots (\mu)_n^{\rho_n}\, x^{\alpha+\mu\sum_{i=0}^{n}\rho_i - \sum_{i=1}^{n} i\,\rho_i} + \quad \text{higher order terms} \\ &\qquad = 0. \end{aligned} \tag{3.2}$$

The coefficients of terms with lowest order in x, i.e. where

$$\alpha + \mu \sum_{i=0}^{n} \rho_i - \sum_{i=1}^{n} i\,\rho_i \tag{3.3}$$

is minimal, have to cancel out. In particular, there are at least two distinct terms with coefficients $a_{\alpha,\boldsymbol{\rho}}, a_{\tilde{\alpha},\tilde{\boldsymbol{\rho}}} \neq 0$ such that the orders, expressed in (3.3), are equal. If

$\sum_{i=0}^{n} \rho_i \neq \sum_{i=0}^{n} \tilde{\rho}_i$, then we can solve the resulting equation for μ and obtain

$$\mu = \frac{-(\alpha - \sum_{i=1}^{n} i\,\rho_i) + (\tilde{\alpha} - \sum_{i=1}^{n} i\,\tilde{\rho}_i)}{\sum_{i=0}^{n} \rho_i - \sum_{i=0}^{n} \tilde{\rho}_i}. \tag{3.4}$$

In the case of $\sum_{i=0}^{n} \rho_i = \sum_{i=0}^{n} \tilde{\rho}_i$, also

$$\alpha - \sum_{i=1}^{n} i\,\rho_i = \tilde{\alpha} - \sum_{i=1}^{n} i\,\tilde{\rho}_i$$

and we cannot solve for μ. In fact, the value of μ in (3.3) is arbitrary and μ is only bounded by $\mathrm{ord}_x(y_1(x))$.

The following definitions are based on the previous observations: Associated to F we define the point set

$$\mathcal{P}(F) = \left\{ P_{\alpha,\boldsymbol{\rho}} = \left(\alpha - \sum_{i=1}^{n} i\,\rho_i, \sum_{i=0}^{n} \rho_i \right) \;\middle|\; a_{\alpha,\boldsymbol{\rho}} \neq 0 \right\} \subseteq \frac{1}{m} \cdot \mathbb{Z} \times \mathbb{N}. \tag{3.5}$$

The second component of a $P \in \mathcal{P}(F)$ is called the *height* of P. For every monomial in F with non-zero coefficient there exists a corresponding point in $\mathcal{P}(F)$. In the case of differential polynomials F of order zero, i.e. when $F \in \mathbb{K}\langle\langle x \rangle\rangle[y]$ defines an algebraic equation, the correspondence is one-to-one. If the order of F is positive, this is not necessarily the case. We will distinguish between the cases where a point $P \in \mathcal{P}(F)$ corresponds to a single monomial and where it corresponds to several monomials in F.

The *Newton polygon* $\mathcal{N}(F)$ of F is defined as the convex hull of the set

$$\bigcup_{P \in \mathcal{P}(F)} (P + \{(a, 0) \mid a \geq 0\}),$$

where "$+$" denotes the Minkowski sum. We remark that $\mathcal{N}(F)$ is a convex set having a finite number of vertices, all of them in the upper closed half plane. Moreover, applied to algebraic equations, the definition of the Newton polygon given here coincides with that one in the Appendix A.1. Hence, it can be seen as a generalization thereof.

Example 3.1.1. Let

$$F = x^4 y^2 y' y''^2 - x y'^3 + x y y' y'' + y y'^2 + x^2 y''^2 + x y' y'' - x y' + 4y - x^4 y y' y'' - x^4.$$

Then we obtain the point set

$$\mathcal{P}(F) = \{(-1, 5), (-2, 3), (-2, 2), (0, 1), (1, 3), (4, 0)\},$$

where $(-1, 5), (1, 3)$ and $(0, 4)$ are derived from one monomial and the other points from at least two terms in F. The Newton polygon $\mathcal{N}(F)$ looks as follows.

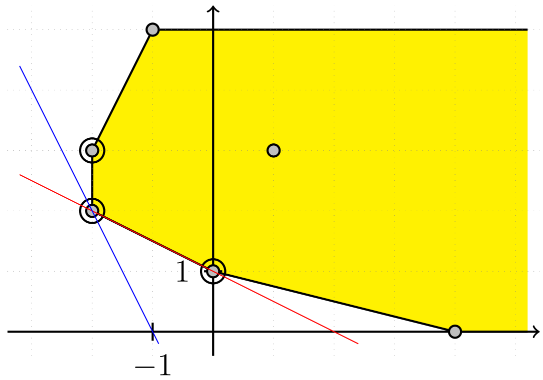

Figure 3.1: The Newton polygon $\mathcal{N}(F)$ in yellow. The red line indicates case (I) and the blue case (II). Multiple vertices are marked with double circles.

Let us denote by $L(F;\mu) \subset \mathbb{R}^2$ a line with slope $-1/\mu$ (with respect to the x-axis) such that the Newton polygon $\mathcal{N}(F)$ is contained in the right closed half plane defined by $L(F;\mu) \subset \mathbb{R}^2$ and such that

$$L(F;\mu) \cap \mathcal{N}(F) \neq \emptyset.$$

We say that $L(F;\mu)$ has *inclination* μ and corresponding to every $L(F,\mu)$ we define the polynomial

$$\Phi_{(F;\mu)}(C) = \sum_{P_{\alpha,\boldsymbol{\rho}} \in L(F;\mu) \cap \mathcal{P}(F)} a_{\alpha,\boldsymbol{\rho}} \, C^{\sum_{i=0}^{n} \rho_i} \prod_{i=1}^{n} (\mu)_i^{\rho_i}, \tag{3.6}$$

where $\mathcal{P}(F)$ is defined as in (3.5). Note that $\Phi_{(F;\mu)}(C)$ is a summand from equation (3.2) of the same order in x without terms involving $y_1(x)$ or any derivative of it.
There are two possibilities for $L(F;\mu)$ such that $L(F;\mu) \cap \mathcal{N}(F) \neq \emptyset$ holds, namely that the line intersects with a side (case I) or with a vertex (case II) of $\mathcal{N}(F)$. Note that there are only finitely many different choices of sides or vertices. Moreover, for $F \neq 0$, $\mathcal{N}(F)$ contains at least one vertex and a side.
In case (I), the inclination μ is given as in equation (3.4) and we call $\Phi_{(F;\mu)}(C)$ the *characteristic polynomial* of the side $L(F;\mu)$. If the characteristic polynomial is non-constant and not of the form $\lambda\, C^k$ for some $\lambda \in \mathbb{K}^*, k \in \mathbb{N}$ (case Ia), there are finitely many roots $c \in \mathbb{K}^*$ of (3.6). However, if the characteristic polynomial is constantly zero (case Ib), then the parameter c can be chosen arbitrarily. If $\Phi_{(F;\mu)}(C)$ is constant but non-zero or of the form $\lambda\, C^k$ (case Ic), then there is no valid root.
In the latter case (II), $L(F;\mu)$ and $\mathcal{N}(F)$ intersect only at a single point $P \in \mathcal{P}(F)$. In particular, the height of every $P_{\alpha,\boldsymbol{\rho}} = P$ coincides. Let us denote by h this height, i.e. $\sum_{i=0}^{n} \rho_i = h$. Then the characteristic polynomial simplifies to $\Phi_{(F;\mu)}(C) = C^h \Psi_{(F;P)}(\mu)$, where

$$\Psi_{(F;P)}(\mu) = \sum_{P_{\alpha,\boldsymbol{\rho}} = P} a_{\alpha,\boldsymbol{\rho}} \prod_{i=1}^{n} (\mu)_i^{\rho_i}, \tag{3.7}$$

which can be seen as polynomial in μ. We call $\Psi_{(F;P)}(\mu)$ the *indicial polynomial* of P. Similar as before, $\Psi_{(F;P)}(\mu)$ can be of positive degree in μ, be constantly zero or constant but non-zero. In the first case, there are finitely many roots of the indicial polynomial. Nevertheless, not all of them might be suitable. The Newton polygon can have sides with inclination μ_1 and μ_2, respectively, such that $\mu_1 < \mu < \mu_2$. Then we additionally have to look for the roots μ of $\Psi_{(F;P)}(\mu)$ which fulfill $\mu_1 < \mu < \mu_2$ and are rational numbers. If there exists a valid choice, we speak of case (IIa). In this case the coefficient c, such that (c, μ) is a root of (3.6), is a free parameter. If $\Psi_{(F;P)}(\mu)$ is constantly zero, both values c and μ are free parameters up to the restriction $\mu_1 < \mu < \mu_2$ (case IIb). If $\Psi_{(F;P)}(\mu)$ is constantly non-zero or has no suitable root, we speak of case (IIc) and there exists no valid root (c, μ) of (3.6).

Let us summarize all the possibilities described above in the following table.

A side with inclination $\mu \in \mathbb{Q}$ in $\mathcal{N}(F)$ is chosen		
case	description of $\Phi_{(F;\mu)}(C)$	choices for (c, μ): fixed $\mu \in \mathbb{Q}$
(Ia)	not equals $\lambda\, C^k$ with $\lambda \in \mathbb{K}^*, k \in \mathbb{N}$	finitely many choices $c \in \mathbb{K}^*$
(Ib)	equals zero	free parameter $c \in \mathbb{K}$
(Ic)	equals $\lambda\, C^k$ with $\lambda \in \mathbb{K}^*, k \in \mathbb{N}$	no valid $c \in \mathbb{K}^*$

A vertex P in $\mathcal{N}(F)$ is chosen		
case	description of $\Psi_{(F;P)}(\mu)$	choices for (c, μ): free $c \in \mathbb{K}$
(IIa)	having roots in (μ_1, μ_2)	finitely many $\mu \in (\mu_1, \mu_2) \cap \mathbb{Q}$
(IIb)	equals zero	free parameter $\mu \in (\mu_1, \mu_2) \cap \mathbb{Q}$
(IIc)	no roots in (μ_1, μ_2)	no valid $\mu \in \mathbb{Q}$

Table 3.1: List of possible cases in the Newton polygon.

In the following we will be particularly interested in predicting when a case different from (Ia) is possible by only studying $\mathcal{N}(F)$.

Let us give some relations which will be important in the proceeding reasonings.

Remark 3.1.2.

1. If the Newton polygon $\mathcal{N}(F)$ consists only of points with positive height, then every monomial in F has y or derivatives of y as a factor and $y(x) = 0$ is a solution of $F = 0$. Note that the constant zero solution is not detected by our ansatz (3.1).

2. The characteristic polynomial (3.6) can always be written in terms of indicial polynomials. Let $L(F;\mu)$ be a side in the Newton polygon including points $P_h \in L(F;\mu) \cap \mathcal{P}(F)$ of height h between h_0 and h_1. Then
$$\Phi_{(F,\mu)}(C) = \sum_{h=h_0}^{h_1} \Psi_{(F;P_h)}(\mu)\, C^h. \tag{3.8}$$

3. Indicial polynomials $\Psi_{(F;P)}(\mu)$ of a vertex $P \in \mathcal{N}(F)$ corresponding to a single monomial in F have a special behavior. First, they have exactly the roots

$0, 1, \ldots, \ell - 1 \in \mathbb{N}$, where ℓ is the highest order of the monomial corresponding to P. Second, if P is of height $h = 0$, i.e. it corresponds to a monomial depending only on x, $\Psi_{(F;P)}(\mu)$ is constantly non-zero and case (IIc) necessarily occurs if we choose $L(F;\mu) \cap \mathcal{N}(F) = \{P\}$.

Example 3.1.3. Let us continue Example 3.1.1 from above. A complete table of the characteristic polynomials and indicial polynomials, respectively, is listed here.

μ	$\Phi_{(F;\mu)}(C)$	case
$\mu = -1/2$	$-9/32C^5$	(Ic)
$\mu = 0$	0	(Ib)
$\mu = 2$	$2C(4C+1)$	(Ia)
$\mu = 4$	-1	(Ic)

μ	$\Psi_{(F;P)}(\mu)$	case
$-\infty < \mu < -1/2$	$\mu^3(\mu-1)^2$	(IIc)
$-1/2 < \mu < 0$	0	(IIb)
$0 < \mu < 2$	$\mu^3(\mu-1)$	(IIa)
$2 < \mu < 4$	$-\mu+4$	(IIc)
$4 < \mu < \infty$	-1	(IIc)

Table 3.2: The characteristic (left) and indicial polynomials (right) of F.

We note that in the cases $-\infty < \mu \leq -1/2$ and $2 < \mu < 4$ there are no suitable choices for (c, μ), whereas in the case $0 < \mu < 2$ the corresponding indicial polynomial has the valid root $\mu = 1$.

Let us now consider some examples which show the limitations of the Newton polygon method for differential equations in the general situation. The first example shows that the cases (IIa) and (IIb) are indeed possible and lead to a family of solutions.

Example 3.1.4. Let us consider the differential equation

$$F_1(x, y, y', y'') = xyy'' - xy'^2 + yy' = 0.$$

All the monomials in F define the vertex $(-1, 2)$, see the Newton polygon $\mathcal{N}(F_1)$ in Figure 3.2. Therefore, we can choose the side with inclination 0 and obtain the constant zero solution or we choose the vertex $(-1, 2)$ and consider its indicial polynomial

$$\Psi_{(F_1;(-1,2))}(\mu) = \mu(\mu - 1) - \mu^2 + \mu,$$

which is constantly zero (case IIb). In fact, $y(x) = c\,x^\mu$ is a solution of $F_1 = 0$ for any $(c, \mu) \in \mathbb{C} \times \mathbb{Q}$.
Let us now consider

$$F_2(x, y, y') = xy' - d\,y = 0$$

for $d \in \mathbb{C}$. Similar to the situation before, $\mathcal{N}(F_2)$ contains only the vertex $(0, 1)$. The corresponding indicial polynomial is

$$\Psi_{(F_2;(0,1))}(\mu) = \mu - d,$$

which has the unique root $\mu = d$. If $d \in \mathbb{Q}$, then this is a valid value (case IIa) and $y(x) = c\,x^d$ is already a solution of $F_2 = 0$, where $c \in \mathbb{C}$ is arbitrary. If $d \in \mathbb{C} \setminus \mathbb{Q}$, then $\mu = d$ is not a valid choice and case (IIc) occurs.
Let us remark that in both examples there are solutions of the described type but with real or even complex exponents. Since they do not define formal Puiseux series, we do not handle them here.

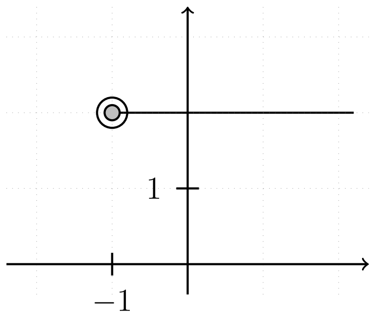

Figure 3.2: The Newton polygon $\mathcal{N}(F_1)$.

In Example 3.1.4 we have seen a second order differential equation with a family of solutions where the exponent is a free parameter. In the following proposition, taken from [Can05], we show that for a big class of differential equations this cannot happen.

Proposition 3.1.5. Let $F \in \mathbb{K}\langle\langle x \rangle\rangle[y, y']$ define the first order differential equation $F = 0$. Then case (IIb) from above cannot happen.

Proof. Let $P = (\alpha, h) \in \mathcal{P}(F)$ and let F_P be the sum of all monomials of F whose corresponding point is P. We have that

$$F_P = \sum_{i=0}^{h} a_i\, x^\alpha y^i (xy')^{h-i}.$$

The indicial polynomial associated to P is $\Psi_{(F;P)}(\mu) = \sum_{i=0}^{h} a_i\, \mu^{h-i}$. Let us assume that case (IIb) happens, i.e. $\Psi_{(F;P)}(\mu) = 0$ has an infinite number of solutions. Then $a_0 = \cdots = a_h = 0$ and F_P is identically zero which is in contradiction to $P \in \mathcal{P}(F)$. $\square$

Changing the Variable

In the remaining part of this section we describe how to continue after considering the Newton polygon of the original differential equation. For a given $F \in \mathbb{K}((x^{1/m}))\{y\}$ of order n, if there exists a pair $(c_0, \mu_0) \in \mathbb{K}^* \times \mathbb{Q}$ fulfilling the necessary conditions arising from the Newton polygon, we apply the change of variable $y = c_0\, x^{\mu_0} + y_1$ and plug it into F to obtain the new differential equation

$$F(x, c_0\, x^{\mu_0} + y_1, c_0\, \mu_0\, x^{\mu_0 - 1} + y_1', \ldots, c_0\, (\mu_0)_n\, x^{\mu_0 - n} + y_1^{(n)}) = 0 \tag{3.9}$$

where the defining differential polynomial is an element in $\mathbb{K}((x^{1/\ell}))\{y\}$ of order n. The exponent ℓ can be bounded by the product of m and the denominator of μ_0. We repeat the process by computing the Newton polygon of equation (3.9) and look for a pair $(c_1, \mu_1) \in \mathbb{K}^* \times \mathbb{Q}_{>\mu_0}$, where $\mathbb{Q}_{>\mu_0}$ denotes the set of rational numbers strictly bigger than μ_0. If we find such a pair, we plug $y_1 = c_1\, x^{\mu_1} + y_2$ into equation (3.9) and continue. In this way we obtain a sequence of differential polynomials $F = F_0, F_1, \ldots$ which might be finite or infinite. We refer to a sequence of the form $F_0, F_1, \ldots$ by a *Newton polygon sequence of* F and define its *length* as the number of variable changes in the Newton polygon sequence which might be equal to infinity.

Let us remark that for a given differential polynomial F there can be infinitely many Newton polygon sequences of F. This happens if case (Ib), case (IIa) or case (IIb) occurs, which is indeed possible as we have seen in Example 3.1.4. However, by not choosing particular values for the free parameters and continue computations symbolically plus the additional conditions on them, there can be constructed a finite tree describing the truncations of possible solutions. For more details on this we refer to [Can05].

Remark 3.1.6. Let us give a collection of important remarks on the Newton polygon sequences $F_0, F_1, \ldots$ of $F \in \mathbb{K}\langle\langle x \rangle\rangle\{y\}$ where (c_i, μ_i) is a suitable root of the characteristic polynomial of F_i and $y_i = c_i\, x^{\mu_i} + y_{i+1}$.

1. For every $k \in \mathbb{N}$ it holds that $\sum_{i \geq 0} c_i\, x^{\mu_i}$ is a solution of $F(y) = 0$ if and only if $\sum_{i \geq k} c_i\, x^{\mu_i}$ is a solution of $F_k(y_k) = 0$.
2. The Newton polygon sequence has finite length $N \in \mathbb{N}$ if and only if the characteristic polynomial (3.6) of F_N does not have a valid root $(c_N, \mu_N) \in \mathbb{K}^* \times \mathbb{Q}_{>\mu_{N-1}}$. There are two possibilities in this situation:
    (a) The Newton polygon $\mathcal{N}(F_N)$ contains only points of positive height (compare item (1) in Remark 3.1.2). Then

    $$y(x) = \sum_{i \geq 0}^{N-1} c_i\, x^{\mu_i} \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F).$$

    (b) In the other case, when the characteristic polynomial neither has a valid nor the trivial root, then $\sum_{i \geq 0}^{N-1} c_i\, x^{\mu_i}$ cannot be extended to a formal Puiseux series solution of $F(y) = 0$.

3. By construction of (3.6), the intersection point of $L(F_i;\mu_i)$ with the x-axis is not an element of $\mathcal{N}(F_{i+1})$. Thus, in $\mathcal{N}(F_{i+1})$ there always exists a side with inclination bigger than μ_i or $y_{i+1}(x)=0$ is already a solution.

4. Let $P \in L(F_i;\mu_i) \cap \mathcal{N}(F_i)$ corresponding to a monomial $f(x,y_i)$ in F_i. Then, after the change of variables, the monomials of $f(x, c_i\, x^{\mu_i} + y_{i+1})$ lead to points in $\mathcal{N}(F_{i+1})$ with height less or equal to the height of P lying on $L(F_i;\mu_i)$. Moreover, the term $f(x,y_{i+1})$ is the only term producing a vertex of the same height as P (in fact it produces exactly P itself) and therefore, P appears in $\mathcal{N}(F_{i+1})$. In other words, the boarder of $\mathcal{N}(F_{i+1})$ coincides with $\mathcal{N}(F_i)$ on the region left and above of the point in $L(F_i;\mu_i) \cap \mathcal{N}(F_i)$ where the height is maximal.

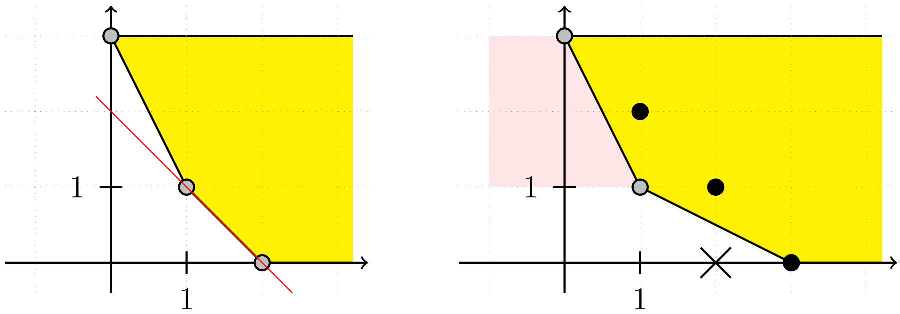

Figure 3.3: After choosing the side with inclination 1 (marked red in the left Newton polygon), the Newton polygon obtained after changing the variables (right) has the same boarder above and left of the vertex $(1,1)$. The vertex $(2,0)$ disappears and $(0,3)$ may contribute new vertices.

Based on item (1) and (2) in Remark 3.1.6 and the notation therein, we will speak about Newton polygon sequences $F_0, F_1, \ldots$ *corresponding to the solution* $y(x) = \sum_{i\geq 0} c_i\, x^{\mu_i}$ if either the sequence has infinite length or the sequence has finite length $N \in \mathbb{N}$ such that $y_N = 0$ is a solution of $F_N(y_N)=0$. In the other case we will say that $F_0, F_1, \ldots$ *does not correspond to a solution*.

The following lemma directly follows from the construction of the Newton polygon method for differential equations, which derives necessary conditions on every formal Puiseux series solution.

Lemma 3.1.7. *Let $F \in \mathbb{K}\{y\}$ and let $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$. Then there exists a Newton polygon sequence $F_0, F_1, \ldots$ of F corresponding to $y(x)$.*

In the next example we show that it is indeed possible that the process might stop after some steps and the first terms cannot be continued to a solution of the original differential equation. In other words, there are Newton polygon sequences with positive length that do not correspond to a solution.

Example 3.1.8. Let us consider

$$F(x,y,y') = -x^2\, y\, y' + y^2 - x^2\, y' = 0.$$

Then

$$\mathcal{P}(F) = \{(1,2),(0,2),(1,1)\}$$

and the Newton polygon $\mathcal{N}(F)$ has the vertices $(0,2)$ and $(1,1)$, see the left picture of Figure 3.4. Both vertices correspond to a single monomial, so the only possible inclinations of a solution are $\mu_0 = 0$ corresponding the the bottom line and $\mu_1 = 1$ corresponding to the line defined by $(0,2)$ and $(1,1)$. For μ_0 we directly obtain the solution $y(x) = 0$ of $F = 0$. For μ_1, the corresponding characteristic polynomial

$$\Phi_{(F,\mu_1)}(C) = C^2 - C$$

has the non-zero root $c = 1$ (case Ia). Let us perform the change of variables $y(x) = x + y_1(x)$ in order to obtain the differential equation

$$F_1(x, y_1, y_1') = F(x, x+y_1, 1+y_1') = -x^3 - x^3 y_1' - x^2 y_1 - x^2 y_1 y_1' + 2x y_1 + y_1^2 - x^2 y_1' = 0,$$

where $\operatorname{ord}_x(y_1(x)) > 1$. In the Newton polygon $\mathcal{N}(F_1)$, see the right picture in Figure 3.4, we obtain two possibilities: First, we can choose the side defined by $P_1 = (1,1)$ (corresponding to the monomials $2\,x\,y_1$ and $-x^2\,y_1'$) and $P_0 = (3,0)$ (corresponding to $-x^3$). Then the corresponding characteristic polynomial is

$$\Phi_{(F_1;2)}(C) = -1$$

with no root (case Ic). We could also choose P_1 and consider its indicial polynomial

$$\Psi_{(F_1;P_1)}(\mu) = 2 - \mu$$

having only $\mu = 2$ as root. However, this corresponds to the side defined by P_1 and P_0 which we already excluded and case (IIc) occurs.

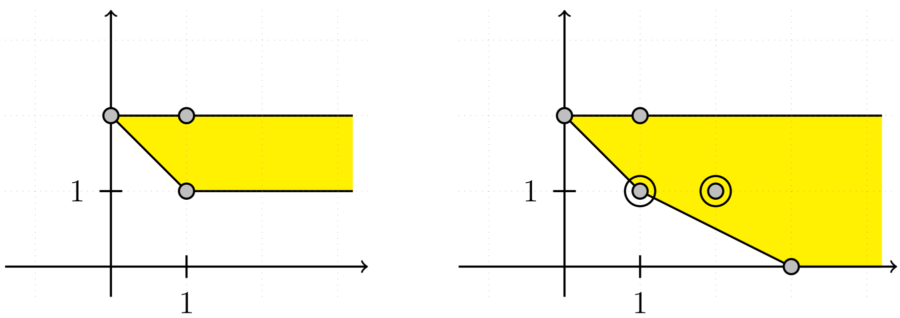

Figure 3.4: The Newton polygons $\mathcal{N}(F)$ (left) and $\mathcal{N}(F_1)$ (right).

3.2 Some Results on Existence and Uniqueness

In this section we present some results on existence and uniqueness of solutions. The first result, Lemma 3.2.1, is known in the field, but we want to restate and proof it here independently. The subsequent Proposition 3.2.2 and 3.2.8 are applications of this lemma. They are also directly related to results in [Can93b] and [Can05], but we want to present it here in a slightly different way. This enables us to state them as generalizations of Proposition 2.2.13 and 2.2.14, respectively, from formal power series solutions to formal Puiseux series solutions.

Let $F_0, F_1, \ldots$ be a Newton polygon sequence of F with corresponding $L(F_i; \mu_i)$ chosen in the respective Newton polygons $\mathcal{N}(F_i)$ and let

$$P_i \in L(F_i; \mu_i) \cap \mathcal{N}(F_i)$$

be the point of biggest height in $L(F_i; \mu_i)$. Then, by Remark 3.1.6, the height of $P_0, P_1, \ldots$ can only decrease finitely many times and remains the same after a certain number of steps $N \in \mathbb{N}$, i.e.

$$P_N = P_{N+1} = \cdots .$$

We refer to this minimal number N by the *stabilization number* and call P_N the *pivot point* of the sequence.
Let us remark that if a Newton polygon sequence $F_0, F_1, \ldots$ of F has stabilization number N and a pivot point P corresponding to a solution

$$y(x) = \sum_{i \geq 0} c_i \, x^{\mu_i} \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F),$$

then for any $M \in \mathbb{N}$ such that F_M is in this sequence, $F_M, F_{M+1}, \ldots$ is a Newton polygon sequence of F_M with stabilization number $\max(N - M, 0)$, pivot point P and corresponding to a solution

$$y_M(x) = \sum_{i \geq M} c_i \, x^{\mu_i} \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F_M).$$

First let us give a dynamical criterion which determines the continuation of an initial part of a Newton polygon sequence uniquely. In other words, if the Newton polygon has reached a particular shape, then existence of a formal Puiseux series solution extending the initial part follows and the remaining terms are uniquely determined by the truncation.

Lemma 3.2.1. *Let $F \in \mathbb{K}\langle\langle x \rangle\rangle\{y\}$ and let $F_0, \ldots, F_N$ be the initial part of a Newton polygon sequence of F with corresponding $L(F_i; \mu_i)$ such that $P \in L(F_N; \mu_N)$ is a vertex of height one and no point of bigger height is in $L(F_N; \mu_N) \cap \mathcal{N}(F_N)$. Then stabilization is already reached, i.e. the stabilization number of the full Newton polygon sequence is at most N, and P is the pivot point of $F_0, F_1, \ldots$.*
Moreover, the indicial polynomial of P remains the same, i.e.

$$\Psi_{(F_N;P)}(\mu) = \Psi_{(F_{N+1};P)}(\mu) = \cdots ,$$

and if $\Psi_{(F_N;P)}(\mu)$ does not have a root $\mu \in \mathbb{Q}_{>\mu_N}$, then existence and uniqueness of a solution $y(x) = \sum_{i \geq 1} c_i \, x^{\mu_i} \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$ follows.

Proof. Let us first show that the initial part of the Newton polygon sequence uniquely extends. The equation $F_{N+1}(y_{N+1}) = 0$ is obtained after the change of variables

$$y_N = c_N\, x^{\mu_N} + y_{N+1}$$

in $F_N(y_N) = 0$. Now we have to find a line $L(F_{N+1}; \mu_{N+1})$ with inclination $\mu_{N+1} \in \mathbb{Q}_{>\mu_N}$ such that

$$L(F_{N+1}; \mu_{N+1}) \cap \mathcal{N}(F_{N+1}) \neq \emptyset$$

and $\mathcal{N}(F_{N+1})$ is contained in the right closed half plane defined by $L(F; \mu_{N+1})$. From Remark 3.1.6 we know several important properties about P and the Newton polygon of F_{N+1} which we recall here:

- $P \in \mathcal{N}(F_{N+1})$ and the Newton polygon $\mathcal{N}(F_{N+1})$ has the same structure as $\mathcal{N}(F_N)$ above and left of P.
- The terms in F_{N+1} corresponding to P are the same as those in F_N.
- For any valid $\mu_{N+1} \in \mathbb{Q}_{>\mu_N}$, there is no point in $L(F_{N+1}; \mu_{N+1}) \cap \mathcal{N}(F_{N+1})$ with height bigger than the height of P.
- The intersection point of $L(F_N; \mu_N)$ and the x-axis is not an element of $\mathcal{N}(F_{N+1})$.

Since P has height one and all the sides and vertices above and left of P remained unchanged, P has to be an element of $L(F_{N+1}; \mu_{N+1})$. From the second item from above we obtain for the indicial polynomials corresponding to P

$$\Psi_{(F_N;P)}(\mu) = \Psi_{(F_{N+1};P)}(\mu).$$

By assumption, $\Psi_{(F_{N+1};P)}(\mu)$ does not have a suitable root μ_{N+1} and we have to choose a side containing P and not only P itself. If there is no vertex of $\mathcal{N}(F_{N+1})$ lying on the x-axis, then $y_{N+1} = 0$ is already a solution. Now let $(v, 0) \in \mathcal{N}(F_{N+1})$. Since P has height one, the side $L(F_{N+1}; \mu_{N+1})$ containing P and $(v, 0)$ does not contain any other point of $\mathcal{N}(F_{N+1})$ and $\mu_{N+1} > \mu_N$ holds. By equation (3.8), we can write

$$\Phi_{(F_{N+1},\mu_{N+1})}(C) = \Psi_{(F_{N+1};P)}(\mu_{N+1})\, C + \Psi_{(F_{N+1};(v,0))}(\mu_{N+1}),$$

which has a unique root $c \in \mathbb{K}^*$, since, by assumption, $\Psi_{(F_{N+1};P)}(\mu_{N+1}) \neq 0$ and by Remark 3.1.2, $\Psi_{(F_{N+1};(v,0))}$ is a non-zero constant. Hence, $(c_{N+1}, \mu_{N+1}) \in \mathbb{K}^* \times \mathbb{Q}$ is uniquely determined.
Now the proof of the lemma can be continued iteratively obtaining in every step $i > N$ a unique pair $(c_i, \mu_i) \in \mathbb{K}^* \times \mathbb{Q}_{>\mu_i}$ or $y_i = 0$. □

In the general case we cannot give a bound on the number of steps such that the particular situation in Lemma 3.2.1 is reached. Even worse, this situation may not even occur at all. In the following propositions we consider the families of differential polynomials already known from Section 2.2, where we are able to show that the assumptions of Lemma 3.2.1 are fulfilled after a bounded number of steps.

Proposition 3.2.2. Every Newton polygon sequence of a differential equation of the type

$$F(y) = A(x)\, y^{(k)} + B(x, y, \dots, y^{(k-1)}, y^{(k+1)}, y^{(n)}) = 0,$$

where $F \in \mathbb{K}\langle\langle x \rangle\rangle\{y\}$, is finite or has a pivot point of height one.
Moreover, for a solution $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$ with ramification index $m \in \mathbb{N}^*$ and order p, the stabilization number is at most $m\,(\deg_x(A) + n - k - p)$.

Proof. The term $A(x)\, y^{(k)}$ provides a point

$$P = (\mathrm{ord}_x(A) - k, 1) \in \mathcal{P}(F).$$

After the changes of variables in a Newton polygon sequence $F_0, F_1, \dots$ of F, the point P remains unchanged and therefore, it is for every $i \geq 0$ an element in the point set $\mathcal{P}(F_i) \subset \mathcal{N}(F_i)$.
Let us now assume that $F_0, F_1, \dots$ has infinite length. Then there is a finite number $N_0 \in \mathbb{N}$ such that P or another point with height equal to one is an element of $\mathcal{N}(F_{N_0})$ and of biggest height. This can be seen by the following observations:
First, since $\mathrm{ord}(F) = n$, the points in $\mathcal{P}(F)$ have $-n$ or bigger values as first coordinate and the most left point with minimal height bigger than one possibly in $\mathcal{N}(F)$ is $(-n, 2)$. Consequently, a line $L(F; \mu)$ going through P and additionally a point with height bigger than P has inclination less or equal to $\mathrm{ord}_x(A) + n - k$. Every other possible side of $\mathcal{N}(F)$, containing P or not, has a smaller inclination or does not contain a point of height strictly bigger than one.

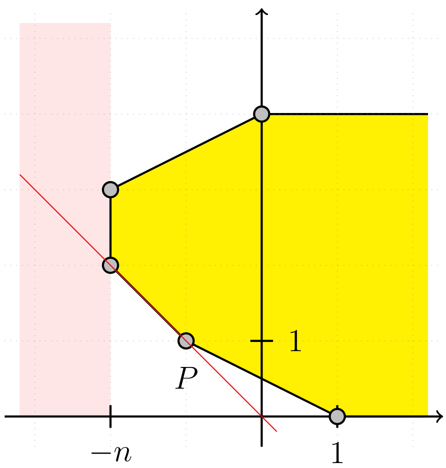

Figure 3.5: The Newton polygon $\mathcal{N}(F)$ with $(-n, 2) \in \mathcal{P}(F)$. The red line indicates the side with maximal inclination including a point of height bigger than one. The red region cannot contain any point in $\mathcal{P}(F)$.

Second, for a solution $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$ with ramification index equal to $m \in \mathbb{N}^*$, the difference between μ_i and μ_{i+1} is at least $1/m$. If the order of $y(x)$ is equal to $\mu_1 = p/m \in \mathbb{Q}$, then there are at most $m\,(\mathrm{ord}_x(A) + n - k - p)$ many steps until

$$\mu_{m(\mathrm{ord}_x(A)+n-k)} \geq n + \mathrm{ord}_x(A) - k.$$

Let us set

$$N_0 = m\,(\mathrm{ord}_x(A) + n - k - p).$$

Then the inclination μ_{N_0+1} is strictly bigger than $\mathrm{ord}_x(A) + n - k$. Recall that P is for every $i \in \mathbb{N}^*$ an element of $\mathcal{P}(F_i) \subset \mathcal{N}(F_i)$. Since $L(F_{N_0}; \mu_{N_0}) \cap \mathcal{N}(F_{N_0}) \neq \emptyset$ and by the reasoning above, $L(F_{N_0}; \mu_{N_0})$ contains P and no point of bigger height, i.e.

$$L(F_{N_0}; \mu_{N_0}) \cap \mathcal{N}(F_{N_0}) \in \{\{P\}, \{P, (v, 0)\}\}$$

for some $v \in \mathbb{Q}$.

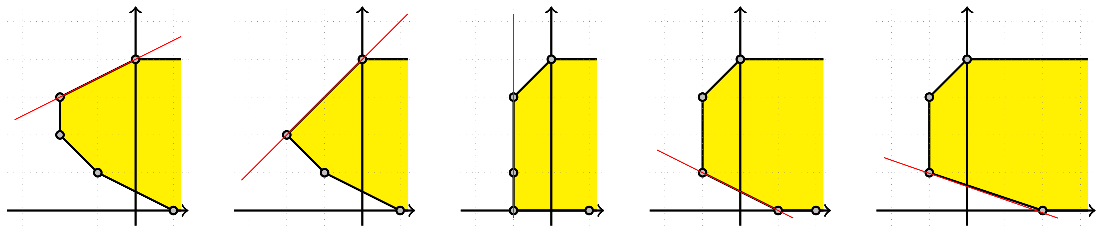

Figure 3.6: The Newton polygons $\mathcal{N}(F_0), \ldots, \mathcal{N}(F_3)$ of a possible Newton polygon sequence of F starting with the side of minimal inclination and ending with P as pivot point. The chosen sides are indicated in red.

By item (3) and (4) in Remark 3.1.6, in the next steps $i \geq N_0$ there will always be a side including P, no point with height bigger than one and possibly a point with height zero. The indicial polynomial corresponding to P is equal to $\Psi_{(F_{N_0};P)}(\mu)$ and has only a finite number of valid roots

$$\mu_{N_1} < \cdots < \mu_{N_\ell} \in \mathbb{Q}_{>\mu_{N_0}}.$$

After at most $m\,(\mu_{N_\ell} - \mu_{N_0}) + 1$ many additional steps, the biggest valid root is surpassed and the assumptions of Lemma 3.2.1 are fulfilled. If there is no valid root of $\Psi_{(F_{N_0};P)}(\mu)$, then no additional steps have to be performed until the existence and uniqueness of a solution of

$$y(x) = \sum_{i \geq 0} c_i\, x^{\mu_i} \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$$

corresponding to the Newton polygon sequence $F_0, F_1, \ldots$ follows.
In the case that the Newton polygon sequence $F_0, F_1, \ldots$ has finite length $N \in \mathbb{N}^*$ such that $N < N_0$ or $N < N_\ell$ in the above steps, there is no valid root $(c_N, \mu_N) \in \mathbb{K}^* \times \mathbb{Q}_{>\mu_{N-1}}$. Then there are two possibilities (see item (2) in Remark 3.1.6): either

$$y(x) = \sum_{i=0}^{N} c_i\, x^{\mu_i} \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$$

or the Newton polygon sequence does not correspond to a solution. In both cases the statement follows. □

In Proposition 3.2.2, if we are looking for formal power series solutions $y(x) \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$, then after at most $N_0 = \deg_x(A) + n - k$ many steps, the pivot point P is determined, and the roots of its indicial polynomial $\Psi_{(F;P)}(\mu)$ can be computed. Let μ_{N_ℓ} be its biggest root. Then it has to be checked whether the Newton polygon sequence can be continued up to $F_{\max(N_0,N_\ell)}$ and in the affirmative case, existence and uniqueness of the remaining exponents and coefficients follows by Lemma 3.2.1. This procedure corresponds to checking the existence and computation the zeros of the jet ideals $\mathcal{J}_{N_0+1}, \ldots, \mathcal{J}_{\max(N_0,N_\ell)}$ as it is done in the direct approach.

Example 3.2.3. Let us consider

$$F = x\,y' + y^2 - y - x^2 = 0$$

from Example 2.1.8, which is of the type described in Proposition 3.2.2 with $A(x) = x$ and $k = 1$. Then we obtain for $c \notin \{0, \pm 1\}$ the Newton polygon sequence starting with

$$\begin{aligned} F_0 &= F(x, y, y'),\\ F_1 &= x\,y_1' + y_1^2 + 2c\,x\,y_1 - y_1 - (1-c^2)\,x,\\ F_2 &= x\,y_2' + y_2^2 + 2(1-c^2)\,x^2\,y_2 + 2c\,x\,y_2 - y_2 + (1-c^2)^2\,x^4 + 2(1-c^2)\,c\,x^3, \end{aligned}$$

depicted in Figure 3.7, corresponding to the solution

$$y(x) = c\,x + (1-c^2)\,x^2 - c\,(1-c^2)\,x^3 + \mathcal{O}(x^4) \in \mathbf{Sol}_{\mathbb{C}[[x]]}(F).$$

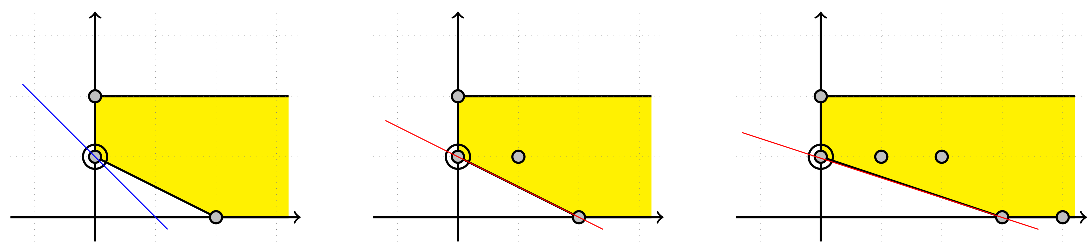

Figure 3.7: The Newton polygons $\mathcal{N}(F_0), \mathcal{N}(F_1), \mathcal{N}(F_2)$ from left to right corresponding to $y(x)$.

The pivot point is $P = (1, 0)$ which is the vertex corresponding to the term $x\,y'$. Since $\Psi_{(F;P)}(\mu)$ has only the root $\mu_0 = 1$, we know from Lemma 3.2.1 that already after the first transformation $y(x) = c\,x + y_1(x)$ that existence and uniqueness of the solution $y_1(x) \in \mathbb{C}[[x]]$ with $\operatorname{ord}_x(y_1(x)) > 1$ follows. This is also what we have seen in Example 2.1.8 by using the direct approach and what we expected from Proposition 3.2.2.

Let us now consider an equation of the type of Proposition 3.2.2 where the pivot point is different from the vertex corresponding to the special term $A(x)\,y^{(k)}$ and where the Newton polygon sequence is finite.

Example 3.2.4. Consider the AODE

$$F = y'^2 + y' - 2y - x = 0$$

from Example 2.2.19. Then we obtain the finite Newton polygon sequence depicted in Figure 3.8 corresponding to the solution

$$y_1(x) = -\frac{1}{8} - \frac{x}{2} \in \mathbf{Sol}_{\mathbb{C}[[x]]}(F).$$

After two steps we obtain the Newton polygon $\mathcal{N}(F_2)$, where every vertex has positive height and corresponds to a single monomial. Since there is no side with inclination bigger than one, the Newton polygon sequence has to stop with F_2 (see item 2 in 3.1.6).

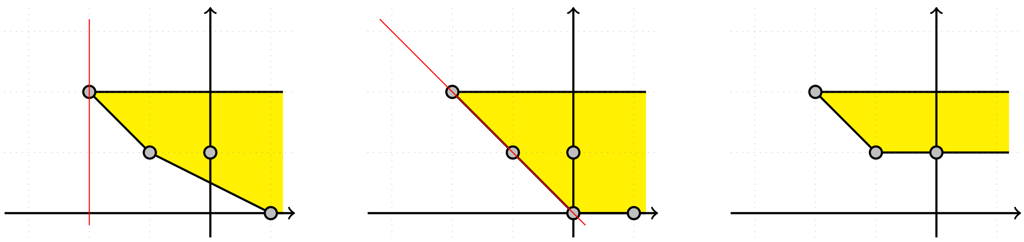

Figure 3.8: The Newton polygons $\mathcal{N}(F_0), \mathcal{N}(F_1), \mathcal{N}(F_2)$ from left to right corresponding to $y_1(x)$.

For the following proposition we need to generalize the concept of constructing the Newton polygon for differential equations, see also [Fin89, Fin90, Can93b].
Let $F \in \mathbb{K}\langle\langle x \rangle\rangle\{y\}$ be of order n and $\boldsymbol{\nu} = (\nu_0, \ldots, \nu_n) \in \mathbb{N}^{n+1}$. Then, using the notation from (3.5), we define the point set

$$\mathcal{P}_{\boldsymbol{\nu}}(F) = \{P \in \mathcal{P}(F) \mid \nu_i \geq \rho_i\}$$

and the Newton polygon $\mathcal{N}_{\boldsymbol{\nu}}(F)$ as the convex hull of

$$\bigcup_{P \in \mathcal{P}_{\boldsymbol{\nu}}(F)} (P + \{(a, 0) \mid a \geq 0\}).$$

Obviously for $\boldsymbol{\nu}, \boldsymbol{\gamma} \in \mathbb{N}^{n+1}$ with $\nu_i \geq \gamma_i$ for every $i \in \{0, \ldots, n\}$ it follows that

$$\mathcal{P}_{\boldsymbol{\gamma}}(F) \subseteq \mathcal{P}_{\boldsymbol{\nu}}(F) \subseteq \mathcal{P}(F)$$

and therefore,

$$\mathcal{N}_{\boldsymbol{\gamma}}(F) \subseteq \mathcal{N}_{\boldsymbol{\nu}}(F) \subseteq \mathcal{N}(F).$$

Note that for every $\boldsymbol{\nu} \in \mathbb{N}^{n+1}$ with $|\boldsymbol{\nu}| = h$ and $P \in \mathcal{P}_{\boldsymbol{\nu}}$ the height of P is less or equal to h.

Example 3.2.5. For F from Example 3.1.1 we obtain the point sets

$\mathcal{P}_{(0,0,1)} = \{(-1,5),(-2,3),(-2,2),(1,3)\}$, $\mathcal{P}_{(0,1,1)} = \{(-1,5),(-2,3),(-2,2),(1,3)\}$, $\mathcal{P}_{(1,1,1)} = \{(-1,5),(-2,3),(1,3)\}$,

and the corresponding Newton polygons looking as follows.

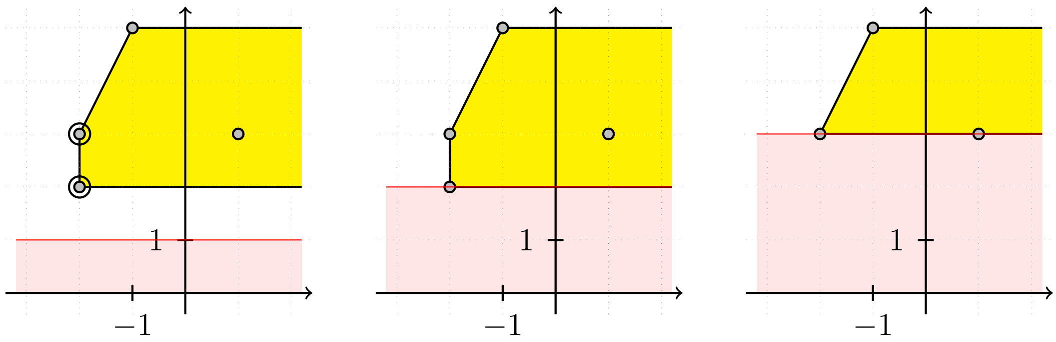

Figure 3.9: The Newton polygons $\mathcal{N}_{(0,0,1)}(F)$ (left), $\mathcal{N}_{(0,1,1)}(F)$ (middle) and $\mathcal{N}_{(1,1,1)}(F)$ (right). The red part indicates which vertices necessarily disappear.

For $F \in \mathbb{K}\langle\langle x\rangle\rangle\{y\}$ with $\operatorname{ord}(F) = n$ and a fixed $\boldsymbol{\nu} \in \mathbb{N}^{n+1}$ we can define correspondingly for a Newton polygon sequence $F_0, F_1, \ldots$ of F, the *stabilization number* and the *pivot point* by simply replacing every Newton polygon $\mathcal{N}(F_i)$ with $\mathcal{N}_{\boldsymbol{\nu}}(F_i)$.
Since $\mathcal{P}_{\boldsymbol{\nu}} \subseteq \mathcal{P}(F)$, the pivot point $P_{\boldsymbol{\nu}}$ corresponding to $\boldsymbol{\nu}$ has height at least of the height of the pivot point P of $F_0, F_1, \ldots$. The following Lemma shows that there is a (non-zero) multiindex with equality where equality of the height holds. In fact, the pivot point is exactly the same in this case.

Lemma 3.2.6. *Let $F_0, F_1, \ldots$ be a Newton polygon sequence of $F \in \mathbb{K}\langle\langle x\rangle\rangle\{y\}$ and P its pivot point. Then there exists $\boldsymbol{\nu} \in \mathbb{N}^{n+1}$ with $|\boldsymbol{\nu}| = 1$ such that the pivot point $P_{\boldsymbol{\nu}}$ corresponding to $\boldsymbol{\nu}$ is equal to P.*

Proof. Let $N \in \mathbb{N}^*$ be the stabilization number of the Newton polygon sequence $F_0, F_1, \ldots$. Since P has positive height, there is $k \in \{0, \ldots, n\}$ such that a monomial in F_N associated to P has $y_N^{(k)}$ as factor. After changing the variable in the Newton polygon sequence, $y_i^{(k)}$ remains a factor of a monomial associated to $P \in \mathcal{N}(F_i)$ for every $i \geq N$. Hence, by choosing $\boldsymbol{\nu}$ as the unit vector with all components equal zero except on the k-th position equal to one,

$$P \in \mathcal{N}_{\boldsymbol{\nu}}(F_i).$$

Since there cannot be a point in $\mathcal{P}_{\boldsymbol{\nu}}(F) \setminus \mathcal{P}(F)$, P is the pivot point of $F_0, F_1, \ldots$ corresponding to $\boldsymbol{\nu}$ of F. □

From [Can05] we have a very useful relation between the pivot point corresponding to a multiindex and solutions of the original differential equation. Note that [Can93b][Proposition 1] the statement is written in almost the same way as we present it here, but in this reference are more general differential equations considered which makes it necessary to weaken the statement in there.

Proposition 3.2.7. Let $F \in \mathbb{K}\langle\langle x \rangle\rangle\{y\}$ be of order n, let $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$ and let $P_{\boldsymbol{\nu}}$ be the pivot point corresponding to $\boldsymbol{\nu} \in \mathbb{N}^{n+1}$ of height h. Then $h > |\boldsymbol{\nu}|$ if and only if $y(x)$ is a solution of

$$\frac{\partial^{|\boldsymbol{\nu}|} F}{\partial y^{\nu_0} \cdots \partial (y^{(n)})^{\nu_n}} = 0.$$

Proof. The first direction is exactly Lemma 4 plus Remark 1 in [Can05]. The converse direction follows by the chain rule as it can be seen in [Can93b][Lemma 3, item (ii)]. □

Combining Lemma 3.2.6 and Proposition 3.2.7 we obtain the generalization of Proposition 2.2.14. Note that we do not provide a bound for the stabilization number and the algorithmic usage of this result remains as an open problem.

Proposition 3.2.8. Let $F \in \mathbb{K}\langle\langle x \rangle\rangle\{y\}$ be of order n. Then

$$F = \frac{\partial F}{\partial y} = \cdots = \frac{\partial F}{\partial y^{(n)}} = 0 \tag{3.10}$$

does not have a solution in $\mathbb{K}\langle\langle x \rangle\rangle$ if and only if every Newton polygon sequence of F stabilizes with a pivot point of height one.

Proof. Let $F_0, F_1, \ldots$ be a Newton polygon sequence of F stabilizing with a pivot point P of height one. Then, by Proposition 3.2.7, there is no $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$ which is also a solution of $\frac{\partial F}{\partial y^{(k)}} = 0$ for any $k = 0, \ldots, n$.
Conversely, let us assume that there exists a solution $y(x) \in \mathbb{K}\langle\langle x \rangle\rangle$ of system (3.10) such that the corresponding Newton polygon sequence $F_0, F_1, \ldots$ stabilizes with a pivot point P of height one. By Lemma 3.2.6, there is a unit vector e_k such that $F_0, F_1, \ldots$ corresponding to e_k has a pivot point P_{e_k} of height one. Again by Proposition 3.2.7, $y(x)$ is not a solution of $\frac{\partial F}{\partial y^{(k)}} = 0$ in contradiction to the assumption. □

3.3 The Associated Differential Equation

In Proposition 3.1.5 we have already seen that AODEs of order one have some special properties. In this section we use the Newton polygon method for AODEs to further analyze a specific type of differential equations of order one, namely

$$F(x, y, y') = f(y)\, y' - x^k\, g(y) = 0, \tag{3.11}$$

where $f, g \in \mathbb{K}[[y]]$ and $k \in \mathbb{Z}$. Note that in Section 3.1 we have considered equations where the coefficient x appeared as formal power series, whereas here we consider F where y may appear as formal power series. This can only be done because for equations of the type (3.11) we can still draw the full Newton polygon where we know all sides and inclinations.
In addition, we will slightly change the initialization of the numbering for the inclination, the Newton polygon sequence, etc. In Sections 3.1 and 3.2 this initial value was 0. However, in the current section we initialize it at 1, because the conclusions of the section are easier to follow later in Chapter 3, where we will be working with formal power series solutions of equation (3.11) of order one.

There are several alternative possibilities to the Newton polygon method for AODEs for proving at least some of the properties of solutions of equation (3.11). In particular, if one is interested in only formal power series solutions, then equation (3.11) can be transformed into Briot-Bouquet type and the results on them can be used, see Appendix D.

Lemma 3.3.1. *Let F be as in (3.11) and $p = \operatorname{ord}_y(f(y))$, $q = \operatorname{ord}_y(g(y))$. Then the following statement holds for solutions $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$ with $\operatorname{ord}_x(y(x)) > 0$:*

1. *If $1 + p - q \neq 0$, then*

$$\operatorname{ord}_x(y(x)) = \frac{k+1}{1+p-q} =: \mu_1$$

and for $\mu_1 \in \mathbb{N}^*$ it follows that $y(x) \in \mathbb{K}[[x]]$.

2. *Let $\mu = q - \mu_1 p$. If $k \geq 0,\ 1 + p > q,\ \mu \leq \mu_1$, there exist exactly $1 + p - q$ distinct solutions.*
If $k \geq 0,\ 1 + p \leq q$ or $k = -1,\ 1 + p \neq q$ or $k < -1,\ 1 + p \geq q$, then there is no solution.
If $k = -1,\ 1 + p = q$, then there is either no solution or a family of solutions involving one free parameter c in the lowest order term.
In the other cases there is either no solution or a family of solutions involving one free parameter c in the term of order μ.

3. *If $\mathbb{K} = \mathbb{C}$ and f, g are convergent, then $y(x)$ is convergent.*

4. *Let the coefficients of $f(y)$ and $g(y)$ belong to a subfield $\mathbb{L}$ of $\mathbb{K}$, c_1 be the first coefficient of $y(x)$ and c be an arbitrary constant. If $k \geq 0,\ 1+p > q,\ \mu \leq \mu_1$, then $y(x) \in \mathbb{L}(c_1)$ with $c_1^{1+p-q} \in \mathbb{L}$. In other words, in the first case the field*

extension is simple radical.
If $k = -1,\ 1 + p = q$, *then* $y(x) \in \mathbb{L}(c)$.
If $k < -1,\ 1 + p < q$, *then* $y(x) \in \mathbb{L}(c_1, c)$ *with* $c_1^{q-1-p} \in \mathbb{L}$.

5. *For any* $m \in \mathbb{N}^*$, *the first* m *coefficients of* $y(x)$ *depend on the first* m *coefficients of* $f(y)$ *and* $g(y)$, *namely* $[y^p]f, \dots, [y^{p+m-1}]f, [y^q]g, \dots, [y^{q+m-1}]g$, *and possibly on* c.

Proof. Let us write $y(x) = c_1\, x^{\mu_1} + y_2$, where $c_1 \neq 0$ and y_2 is of order greater than μ_1. From Section 3.1 we know that there are several possibilities in the Newton polygon $\mathcal{N}(F)$ for possible solutions. We divide the proof into several cases, where the Newton polygon has different behavior.

Case 1: $\mathbf{k \geq 0}$. There is one vertex at $P_1 = (-1, 1 + p) \in \mathcal{N}(F)$ and one at $P_2 = (k, q) \in \mathcal{P}$ possibly in $\mathcal{N}(F)$. The other vertices in $\mathcal{P} \subset \mathcal{N}(F)$ have the same first coordinate as P_1 and P_2, respectively, but a bigger height. For a sketch of $\mathcal{N}(F)$ see Figure 3.10.

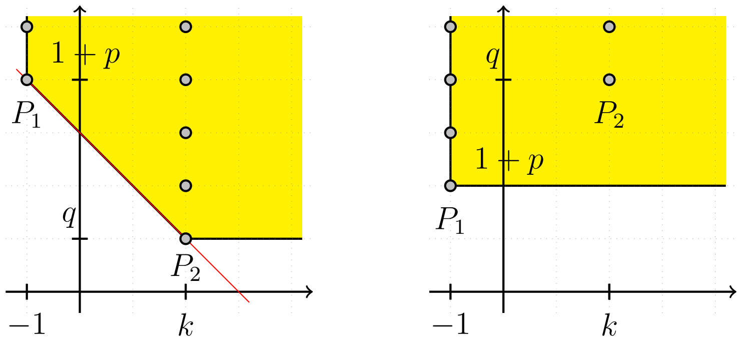

Figure 3.10: The Newton polygon $\mathcal{N}(F)$ with $k \geq 0, 1 + p > q$ (left) and $k \geq 0, 1 + p \leq q$ (right).

In the situation of $k \geq 0$ the points P_1 and P_2 are defined by only one monomial and choosing one of these vertices would result in case (IIc) as it is explained in item (3) in Remark 3.1.2. Note that the bottom line might produce the constant zero solution $y(x) = 0$ (see item (1) in Remark 3.1.2), where the statements of the lemma trivially hold. We do not consider them here anymore. Hence, choosing a side is the only possibility. In order that P_1 and P_2 define a line with inclination $\mu_1 > 0$, it has to be the case that $1 + p > q$. In the case of $1 + p \leq q$, no non-trivial solution exists.
Assume that $\mathbf{1 + p > q}$. Let $L(F; \mu_1)$ be the side defined by P_1 and P_2 with

$$\mu_1 = \frac{k + 1}{1 + p - q} > 0.$$

Correspondingly we obtain the characteristic polynomial

$$\Phi_{(F;\mu_1)}(C) = f_p\, C^{1+p} - g_q\, C^q. \tag{3.12}$$

Since $f_p, g_q \neq 0$, there are exactly $1 + p - q$ possibilities to choose c_1 as a root of $\Phi_{(F;\mu_1)}(C)$, namely as the distinct roots

$$c_1 = \sqrt[1+p-q]{\frac{g_q}{f_p}}.$$

We perform the change of variable $y(x) = c_1\, x^{\mu_1} + y_2$ to obtain

$$F_2(x, y_2, y_2') = F(x, c_1\, x^{\mu_1} + y_2, c_1\mu_1\, x^{\mu_1 - 1} + y_2') = 0. \tag{3.13}$$

The Newton polygon $\mathcal{N}(F_2)$ is sketched in the right picture of Figure 3.11.

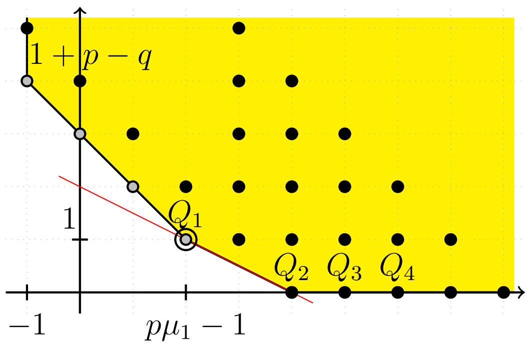

Figure 3.11: The Newton polygon $\mathcal{N}(F_2)$.

By expanding (3.13), it can be seen that the vertex

$$Q_1 = (k + (q-1)\,\mu_1, 1) = (p\,\mu_1 - 1, 1) \in \mathcal{N}(F_2)$$

is guaranteed. The monomials corresponding to Q_1 are $A\, x^{p\mu_1 - 1}\, y_2 + B\, x^{p\mu_1}\, y_2'$, where

$$A = f_p\, p\, \mu_1\, c_1^p - g_q\, q\, c_1^{q-1} \quad \text{and} \quad B = f_p\, c_1^p.$$

The indicial polynomial corresponding to Q_1 is

$$\Psi_{(F_2;Q_1)}(\mu) = A + \mu\, B$$

with only one root being equal to

$$\mu = -A/B = \frac{g_p\, q\, c^{q-1-p}}{f_p} - \mu_1\, p = q - \mu_1\, p.$$

In particular, if $\mu \leq \mu_1$, then there is no valid root of $\Psi_{(F_2;Q_1)}(\mu)$. In this case, by Lemma 3.2.1, existence and uniqueness of $y_2(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F_2)$ follows.
In the case that $\mu > \mu_1$, the critical value $\mu = q - \mu_1\, p$ might be a possible exponent of a term in $y_2(x)$ and Lemma 3.2.1 cannot directly be applied. Note that if $\mu_1 \geq 1$, then $\mu < q - p < 1 \leq \mu_1$ always holds.
Case 1.1: $\boldsymbol{\mu \leq \mu_1}$. Since f and g have integer exponents, the possible points of height zero $Q_2, Q_3, \ldots$ are of the form

$$Q_i = (k + (q+i)\,\mu_1, 0) = ((p + i - 1)\,\mu_1 - 1, 0),$$

see also Figure 3.11. In a Newton polygon sequence $F_2, F_3, \dots$ corresponding to a solution $y_2(x)$ the vertices Q_j, with $j \geq i \geq 2$ are also the possible points of height zero in $\mathcal{N}(F_i)$ (see item (3) and (4) in Remark 3.1.6). In other words, the exponents in the terms of a solution $y_2(x)$ grow by multiples of μ_1 and in the particular case of $\mu_1 \in \mathbb{N}^*$, it follows that $y(x) \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$.
Since the pivot point is Q_1 and the coefficient of the highest derivative $B \neq 0$, by Theorem 2 in [Can93b], $y(x)$ is convergent provided that f and g are convergent[1]. This proves items (1)-(3).
In order to prove the remaining statements (4) and (5), we explicitly describe how to compute the further coefficients of

$$y(x) = \sum_{i \geq 1} c_i \, x^{i\mu_1} \quad \text{with} \quad c_i \in \mathbb{K}.$$

Here, in contrast to the usual case in the Newton polygon method for differential equations, we allow coefficients equal to zero. The reason is that we do not ensure that $\mathcal{N}(F_i)$ indeed has a vertex Q_i. If there is no such vertex we will obtain $c_i = 0$. The characteristic polynomial associated to $\mu_2 = 2\mu_1$ is

$$\Phi_{(F_2;\mu_2)}(C) = \Psi_{(F_2;Q_1)}(\mu_2) + \Psi_{(F_2;Q_2)}(\mu_2) = (A + \mu_2 \, B) \, C + \Psi_{(F_2;Q_2)}(\mu_2),$$

where the latter summand is the coefficient of $x^{(p+2)\mu_1 - 1}$ in F_2, namely

$$\Psi_{(F_2;Q_2)}(\mu_2) = f_{p+1} \, \mu_1 \, c_1^{p+2} - g_{q+1} \, c_1^{q+1}.$$

Since $A + \mu_2 \, B \neq 0$, case (Ia) occurs and

$$c_2 = \frac{-f_{p+1} \, \mu_1 \, c_1^{p+2} + g_{q+1} \, c_1^{q+1}}{A + 2\mu_1 \, B} \in \mathbb{L}(c_1)$$

is uniquely determined.
In the following differential polynomials F_i for $i \geq 2$ we obtain the characteristic polynomial

$$\Phi_{(F_i,\mu_i)}(C) = (A + \mu_i \, B) \, C + \Psi_{(F_i;Q_i)}(\mu_i),$$

where $\mu_i = i \, \mu_1$ and the indicial polynomial $\Psi_{(F_i;Q_i}(\mu_i)$ is the coefficient of $x^{(p+i)\mu_1 - 1}$ in F_i. Because $A + \mu_i \, B \neq 0$, then

$$c_i = \frac{-\Psi_{(F_i;Q_i)}(\mu_i)}{A + \mu_i \, B} \tag{3.14}$$

is uniquely determined and an element of $\mathbb{L}(c_1)$. This shows item (4). Since A and B only depend on c_1, in order to prove item (5), it is enough to show that for every $i \geq 2$ the indicial polynomial $\Psi_{(F_i;Q_i)}(i\mu_1)$ depends only on $c_1, \dots, c_{i-1}, f_p, \dots, f_{p+i-1}$

[1]Alternatively one could argue that the linearized operator along $y(x)$ has a regular singularity and by the main result from [Mal89], $y(x)$ is convergent.

and $g_q, \dots, g_{q+i-1}$, where $f_\ell = [x^\ell]f, g_\ell = [x^\ell]g$. The vertex Q_i is corresponding to the coefficient of $x^{(p+i)\mu_1 - 1}$ in F_i, or equivalently, the same coefficient in

$$F(x, \tilde{y}(x), \tilde{y}'(x)) = \left(\sum_{\ell \geq p} f_\ell \, \tilde{y}(x)^\ell \right) \tilde{y}'(x) - x^k \sum_{\ell \geq q} g_\ell \, \tilde{y}(x)^\ell,$$

where $\tilde{y}(x) = \sum_{j=1}^{i} c_j \, x^{j\mu_1}$. Hence, $\operatorname{ord}_x(\tilde{y}(x)^\ell) = \ell \, \mu_1$ and the statement follows.

Case 1.2: $\boldsymbol{\mu > \mu_1}$. Then the indicial polynomial corresponding to Q_1 has the valid root

$$\mu = q - \mu_1 \, p \in \mathbb{Q}_{>\mu_1}.$$

Let $\ell \in \mathbb{N}^*$ be the biggest number such that $\ell \, \mu_1 < q - \mu_1 \, p$. For $1 \leq i \leq \ell$ it holds that

$$\mu_1 < \mu_i = i \, \mu_1 < q - \mu_1 \, p$$

and case (Ia) occurs. The characteristic polynomials $\Phi_{(F_i, \mu_i)}$ have unique roots given by $\mu_i = i \, \mu$ and by the recursion formula (3.14) as for the Case 1.1.
If the critical value $\mu \in \mu_1 \cdot \mathbb{N}^*$, for $\mu_\ell = q - \mu_1 \, p$ case (Ib) or (Ic) happens, i.e. either the coefficient $[x^{q - \mu_1 p}]y(x)$ can be chosen arbitrarily or there exist no solution. In the case that a solutions exists, for $i > \ell + 1$ the vertices of height zero in $\mathcal{N}(F_i)$ are still the points Q_i. Then $\mu_i = i \, \mu_1$ and since $\mu_i > q - \mu_1 \, p$, again case (Ia) happens and the coefficients are uniquely determined by (3.14).
If the critical value $\mu \notin \mu_1 \times \mathbb{N}^*$, then after the ℓ-th step there can be chosen the line $L(\mu_{\ell+1}; F_{\ell+1})$ intersecting with $\mathcal{N}(F_{\ell+1})$ only in the vertex Q_1 with $\ell \, \mu_1 < \mu_{\ell+1} = \mu < (\ell + 1) \, \mu_1$ as valid exponent and case (IIa) occurs. Afterwards we can apply Lemma 3.2.1 and existence and uniqueness of $y_{\ell+2}(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F_{\ell+2})$ follows. Although the following exponents are not necessarily equal to $(\mu_i = i\mu_1, c_i)$, we can determine the ramification index by the least common multiple of the denominators of μ_1 and μ, denoted by $\operatorname{den}(\mu_1)$ and $\operatorname{den}(\mu)$, respectively, and set for $i > \ell + 1$

$$\mu_i = \mu_{\ell+1} + (i - \ell - 1) \operatorname{lcm}(\operatorname{den}(\mu_1), \operatorname{den}(\mu)).$$

This is because the points of height zero in $\mathcal{N}(F_i)$ are of the form

$$\begin{aligned} Q_i &= Q_\ell + (i - \ell - 1) \cdot (\gcd(\operatorname{den}(\mu_1), \operatorname{den}(\mu)), 0) \\ &= ((p + \ell - 1) \, \mu_1 - 1 + (i - \ell - 1)(\gcd(\operatorname{den}(\mu_1), \operatorname{den}(\mu)), 0). \end{aligned}$$

The corresponding coefficients $c_{\ell+1+i}$ are now given by the recursion formula (3.14) as well, but with a different μ_i.
The remaining statements follow as in Case 1.1.

Case 2: $\mathbf{k < 0}$. Let us consider

$$F(x, y, y') = x^{-k} \, f(y) \, y' - g(y) = 0.$$

Similarly as before, there are vertices

$$P_1 = (0, q) \in \mathcal{N}(F) \ \text{ and } \ P_2 = (-k-1, 1+p) \in \mathcal{P}(F).$$

The vertices P_1 and P_2 define a line $L(F; \mu_1)$ on $\mathcal{N}(F)$ with

$$\mu_1 = \frac{-k-1}{q-1-p} > 0$$

if and only if $k < -1$ and $q > 1+p$ as it can be seen left in Figure 3.12.

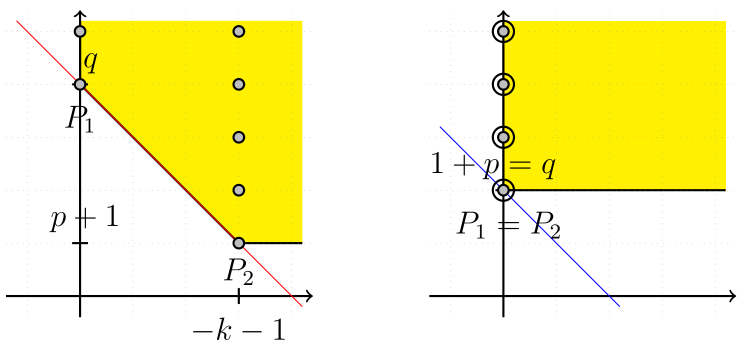

Figure 3.12: The Newton polygon $\mathcal{N}(F)$ with $k < 0,\ q > 1+p$ (left) and $k = -1,\ q = 1+p$ (right).

Assume that $\mathbf{k < -1}$ and $\mathbf{q > 1+p}$. The corresponding characteristic polynomial $\Phi_{(F,\mu_1)}$ is again as in (3.12), but having the roots

$$c_1 = \sqrt[q-1-p]{\frac{f_p}{g_q}}.$$

After the change of variables $y(x) = c_1\, x^{\mu_1} + y_2(x)$, the Newton polygon $\mathcal{N}(F_2)$ has a vertex

$$Q_1 = (-k + p\,\mu_1 - 1, 1) = ((q-1)\,\mu_1, 1)$$

corresponding to the monomials $A\,x^{-k+p\mu_1-1}\,y_2 + B\,x^{-k+p\mu_1}\,y_2'$, where again

$$A = f_p\,p\,\mu_1\,c_1^p - g_q\,q\,c_1^{q-1} \ \text{ and } \ B = f_p\,c_1^p.$$

The indicial polynomial corresponding to Q_1 is $\Psi_{(F_2;Q_1)}(\mu) = A + \mu\,B$ with the root $\mu = -A/B = q - \mu_1\,p$ possibly in $\mathbb{Q}_{\geq \mu_1}$. As in Case 1.2 above, for $1 < \mu_i < q - \mu_1\,p$ case (Ia) occurs and the characteristic polynomials are uniquely solvable. If $\mu \in \mu_1 \cdot \mathbb{N}^*$ case (Ib) or (Ic) happens for this value, and if $\mu \notin \mu_1 \cdot \mathbb{N}^*$ case (IIa) occurs. In the case that a solutions exists, for $\mu_i > \mu = q - \mu_1\,p$ again case (Ia) appears and the coefficients are uniquely determined.
Now, the statements of the lemma follow as in Case 1.1.

In the case of $k < 0$ there is also the possibility that P_1 and P_2 coincide and the corresponding indicial polynomial might lead to a non-zero solution. This is the case if and only if $k = -1$ and $q = 1+p$ as it can be seen right in Figure 3.12.

Assume that $\mathbf{k = -1}$ and $\mathbf{q = 1 + p}$. The indicial polynomial then is

$$\Psi_{(F;P_1)}(\mu_1) = f_p\,\mu_1 - g_q$$

with the unique root $\mu_1 = \frac{g_q}{f_q}$. If $\mu_1 \in \mathbb{Q}_{>0}$, then case (IIa) occurs and we perform for an arbitrary constant c the change of variables $y(x) = c\,x^{\mu_1} + y_2(x)$ and continue as in Case 1.1. Otherwise there is no solution with positive order.
In the other possibilities of Case 2 there is no (non-zero) formal Puiseux series solution of $F = 0$, which concludes the lemma. □

For proving Lemma 3.3.1 we have used the Newton polygon method for differential equations, since it provides the theoretical tools we need. However, by using the results of the lemma, algorithmically we are not limited to the Newton polygon method for differential equations. One alternative option would be for example to simply make an ansatz and compute the unknown coefficients as we do in the following algorithm.

Algorithm 2 AssocSolve

Input: A first-order AODE $F(x, y, y') = 0$ of the type (3.11) and $N \in \mathbb{N}^*$.
Output: The first terms of the elements in $\mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$ with positive order up to the degree N.

Set $p = \operatorname{ord}_y(f(y)),\ q = \operatorname{ord}_y(g(y))$ and for $1 + p - q \neq 0$ set $\mu_1 = \frac{k+1}{1+p-q},\ \mu = q - \mu_1\,p$.
if $k \geq 0,\ 1 + p > q,\ \mu \leq \mu_1$ or $k < -1,\ 1 + p < q$ **then**
  Set $\mu_i = i\,\mu_1$ for $i \in \{1, \ldots, M\}$ such that $\mu_M \leq N$ is maximal.
else if $k \geq 0,\ 1 + p > q,\ \mu > \mu_1$ **then**
  Set

$$\mu_i = \begin{cases} i\,\mu_1, & i \in \{1, \ldots, \ell\} \\ \mu, & i = \ell + 1 \\ \mu_1 + (i - \ell - 1)\operatorname{lcm}(\operatorname{den}(\mu_1), \operatorname{den}(\mu)), & i \in \{\ell + 2, \ldots, M\} \end{cases}$$

  such that $\mu_\ell < \mu$ and $\mu_M \leq N$ are maximal.
else if $k = -1,\ 1 + p = q$ **then**
  Set $\mu_1 = [x^q]g/[x^p]f$.
  if $\mu_1 \in \mathbb{Q}_{>0}$ **then**
    Set $\mu_i = i\,\mu_1$ for $i \in \{1, \ldots, M\}$ such that $\mu_M \leq N$ is maximal.
  end if
else
  return "No solution".
end if
Plug $\tilde{y}(x) = \sum_{i=1}^{N} c_i\,x^{\mu_i}$ into $F = 0$ and solve the resulting algebraic equations for c_i with $i \in \{1, \ldots, M\}$.
return $\tilde{y}(x)$.

Correctness of Algorithm 2 follows from Lemma 3.3.1. Termination is guaranteed if the possible test $\mu_1 \in \mathbb{Q}_{>0}$ terminates.

Example 3.3.2. Let us consider the differential equation

$$F = y' - x^{-2}\, y^2$$

of the type (3.11) with $k = -2,\ p = 0,\ q = 2$. Then $\mu_1 = 1$ and the critical value is equal to $\mu = 2$. Following Algorithm 2 with the input $N = 3$, we plug

$$\tilde{y}(x) = c_1\, x + c_2\, x^2 + c_3\, x^3 + \mathcal{O}(x^4)$$

into $F = 0$ to obtain

$$c_1 - c_1^2 + (2c_2 - 2c_1c_2)\, x + (3c_3 - c_2^2 - 2c_1c_3)\, x^2 + \mathcal{O}(x^3) = 0.$$

By doing coefficient comparison we obtain the non-zero solution given by $c_1 = 1,\ c_3 = c_2^2$ and arbitrary $c_2 \in \mathbb{C}$. In fact, the set of solutions $\mathbf{Sol}_{\mathbb{C}\langle\langle x\rangle\rangle}$ of positive order is given by

$$y(x) = \frac{x}{1 - c_2\, x} = x + c_2\, x^2 + c_2^2\, x^3 + \mathcal{O}(x^4).$$

Following the proof of Lemma 3.3.1 and using the Newton polygon method for differential equations we obtain the same result. The first Newton polygons of the Newton polygon sequence corresponding to $y(x)$,

$$F_1 = F(x, y, y'),\ F_2 = y_2' - 2x^{-1}\, y_2 - x^{-2}\, y_2^2,$$
$$F_3 = y_3' - 2c_2\, y_3 - x^{-2}\, y_3^2 - 2x^{-1} y_3 - c_2^2\, x^2,$$

are depicted in Figure 3.13.

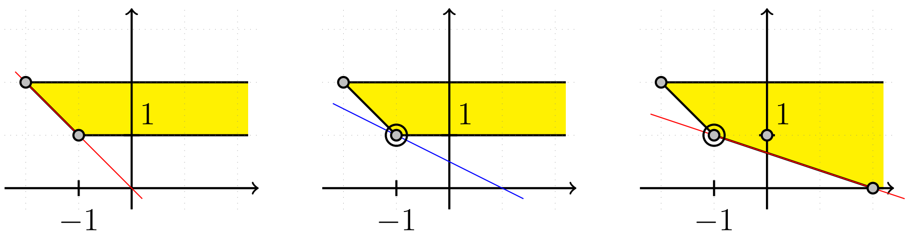

Figure 3.13: The Newton polygons $\mathcal{N}(F_1), \mathcal{N}(F_2), \mathcal{N}(F_3)$ from left to right of the Newton polygon sequence corresponding to $y(x)$.

In the following example we show that by only adding a term in x in equations of the form (3.11), we might loose the results obtained above such as convergence.

Example 3.3.3. Let us consider $F(x, y, y') = y' - x^{-2}\, y + x^{-1}$. As one can verify easily, there is a non-convergent formal power series

$$y(x) = \sum_{i \geq 1} (i-1)!\, x^i \in \mathbf{Sol}_{\mathbb{C}[[x]]}(F).$$

Chapter 4

Algebro-geometric Approach

In this chapter we study the existence, uniqueness and convergence of formal Puiseux series solutions and describe algorithmically all solutions. In Section 4.1 we study first order autonomous AODEs. By using the Newton polygon method for differential equations, see Chapter 3, we have faced some algorithmic problems and we were able to prove convergence only for particular cases. The direct application of the Newton polygon method to autonomous AODEs does not provide any advantage with respect to the non-autonomous case, because after the first change of variables the characteristic of being autonomous gets lost.

In [VGW18] they derive an associated differential system to find rational general solutions of non-autonomous first order differential equations by considering rational parametrizations of the implicitly defined curve. We instead consider its places and obtain an associated differential equation of first order and first degree which is treated by the Newton polygon method in Lemma 3.3.1. Using the known bounds for computing places of algebraic curves (see e.g. [Duv89]), existence and uniqueness of the solutions and the termination of our computations can be ensured.

In Section 4.1 we handle algebraic solutions such as in [ACFG05] and show that all solutions have the same minimal polynomial up to a shift in the independent variable.

In Section 4.2, we generalize the main results to systems of higher order autonomous AODEs in one differential indeterminate and in more differential indeterminates, respectively, which associated algebraic set is of dimension one. Note that in [DL84] it is shown that for general systems of algebraic ordinary differential equations the existence of non-constant formal power series solutions can not be decided algorithmically. Nevertheless, in the case of systems as above, this undecidability property does not hold.

We conclude the chapter by dealing with differential equations involving one differential indeterminate and a higher derivative of it. In the case that this derivative is the second derivative, we again show convergence of solutions expanded around a finite point.

Most of the results shown in this chapter are new and based on the papers [FS19, CFS19, CFS20], co-authored by the writer of the thesis. The main results are certainly the results on existence, uniqueness and convergence shown in Theorem 4.1.8 and Theorem 4.1.9, respectively, for the case of one autonomous first-order AODE. The result on convergence gets generalized in Theorem 4.2.11 to systems of dimension one. In the case of existence and uniqueness we are limited to one differential indeterminate (Theorem 4.2.2) or algebraic solutions (Proposition 4.2.14). The convergence of solutions expanded around a finite point in the case where the second derivative instead of the first derivative appears is shown in Theorem 4.3.6.

4.1 First Order Autonomous AODEs

In Section 4.1 we show that every non-constant formal Puiseux series solution defines a place of the associated curve. We give a necessary condition on a place of the curve to contain in its equivalence class formal Puiseux series solutions of the original differential equation, and show the analyticity of them. In the case where the solutions are expanded around a finite point, the necessary condition turns out to be sufficient as well. As a byproduct, we obtain a new proof of the fact that there is an analytic solution curve of $F(y, y') = 0$ passing through any given point in the plane. This result is a consequence of Section 6.10 in [Aro00a]. In Section 4.1 algorithms for computing all Puiseux series solutions are presented and illustrated by examples. In the case of solutions expanded around zero we give a precise bound on the number of terms such that the solutions are in bijection with the corresponding truncations. For solutions expanded at infinity we are able to compute for every solution a corresponding truncation, but in general uniqueness of the extension is not guaranteed.

Let us consider the autonomous first-order AODE

$$F(y, y') = 0 \tag{4.1}$$

with $F \in \mathbb{K}[y, y']$. Without loss of generality we can consider F to be square-free and have no factor in y or y'. Otherwise we first compute the linear solutions of equation (4.1) by making the ansatz $y(x) = c_1 x + c_0$ and computing all values $c_0, c_1 \in \mathbb{K}$ such that $F(c_1 x + c_0, c_1) = 0$ is fulfilled. Then we take the square-free part of F with no factor in y or y' and continue.
Formal Puiseux series can either be expanded around a finite point or at infinity. In the first case, since equation (4.1) is invariant under translations of the independent variable, without loss of generality we can assume that the formal Puiseux series is expanded around zero. In the case of infinity we can use the transformation $x = 1/z$ obtaining the (non-autonomous) differential equation $F(y(z), -z^2 y'(z)) = 0$. In order to deal with both cases in a unified way, we will study equations of the type

$$F(y(x), x^h y'(x)) = 0, \tag{4.2}$$

with $h \in \mathbb{Z} \setminus \{1\}$ and its formal Puiseux series solutions expanded around zero. We note that for $h = 0$ equation (4.2) is equal to (4.1) and for $h = 2$ the case of formal Puiseux series solutions expanded at infinity is treated. For the case $h = 1$ some different phenomena occur, since y and xy' provide the same vertices in the Newton polygon, and for example convergence cannot be proven anymore as we see in Example 3.3.3. In the sequel, we assume that h is fixed.
Additionally to (4.1), we may require that a formal Puiseux series solution $y(x)$ fulfills the initial conditions $y(0) = y_0, (x^h y'(x))(0) = p_0$ for some fixed $\mathbf{p}_0 = (y_0, p_0) \in \mathbb{K} \times \mathbb{K}_\infty$, where $\mathbb{K}_\infty = \mathbb{K} \cup \{\infty\}$ is the projective compactification of $\mathbb{K}$. In the case

where $y(0) = \infty$, $\tilde{y}(x) = 1/y(x)$ is a Puiseux series solution of a new first order differential equation of the same type, namely the equation given by the numerator of the rational function $F(1/y, -x^h p/y^2)$, and $\tilde{y}(0) \in \mathbb{K}$.
For more informations on formal Puiseux series, algebraic geometry or our algebro-geometric approach used in this chapter, see the appendix.

Solution Places

Let us consider the following mappings for the set of non-constant formal Puiseux series solutions denoted by

$$\mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0) = \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F;\mathbf{p}_0) \setminus \mathbf{Sol}_{\mathbb{K}}(F;\mathbf{p}_0).$$

For use the star notation here for excluding constant solution, since these solutions play for first order AODEs a similar same role as the zero in rings and fields. Let the set $\mathbf{IFP}(\mathbf{p}_0)$ containing all irreducible formal parametrizations of $\mathscr{C}(F)$ centered at $\mathbf{p}_0$ and the set $\mathbf{Places}(\mathbf{p}_0)$ containing the places of $\mathscr{C}(F)$ centered at $\mathbf{p}_0$. Then we define the mappings

$$\begin{array}{rccc} \Delta: & \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0) & \longrightarrow & \mathbf{IFP}(\mathbf{p}_0) \\ & y(x) & \longmapsto & \left(y(t^m), t^{hm}\frac{dy}{dx}(t^m)\right), \end{array}$$

where m is the ramification index of $y(x)$, and

$$\begin{array}{rccc} \delta: & \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0) & \longrightarrow & \mathbf{Places}(\mathbf{p}_0) \\ & y(x) & \longmapsto & [\Delta(y(x))]. \end{array}$$

The map Δ is well defined because on the one hand, $\Delta(y(x))$ is a formal parametrization of $\mathscr{C}(F)$ centered at $\mathbf{p}_0$ and on the other hand, by the definition of the ramification index, one deduces that $\Delta(y(x))$ is irreducible. We remark that, since Δ is well defined, a necessary condition for $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ is that $\mathbf{p}_0 \in \mathscr{C}(F)$.

Definition 4.1.1. A place $\mathcal{P} \in \mathbf{Places}(\mathbf{p}_0)$ is a *solution place* of (4.2) if there exists $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ such that $\delta(y(x)) = \mathcal{P}$. Moreover, we say that $y(x)$ is a *generating Puiseux series solution* of the place $\mathcal{P}$. An irreducible formal parametrization $A(t) \in \mathbf{IFP}(\mathbf{p}_0)$ is called a *solution parametrization* if $A(t) \in \Delta(\mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0))$.

Now we give a characterization for an irreducible formal parametrization to be a solution parametrization. Later we will show how to decide whether a given place contains a solution parametrization, i.e. whether it is a solution place.

Lemma 4.1.2. *Let $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ be of ramification index m, and let $(a(t), b(t)) = \Delta(y(x))$. It holds that*

$$a'(t) = m\, t^{m(1-h)-1}\, b(t). \tag{4.3}$$

$$m(1-h) = \operatorname{ord}(a(t) - y_0) - \operatorname{ord}(b(t)). \tag{4.4}$$

Proof. Since $a(t) = y(t^m)$ and $b(t) = t^{hm}\, y'(t^m)$, by the chain rule

$$a'(t) = m\, t^{m-1}\, y'(t^m) = m\, t^{m(1-h)-1}\, b(t).$$

Equation (4.4) is obtained by taking the order in t on both sides of equation (4.3). $\square$

Proposition 4.1.3. Let $(a(t), b(t)) \in \mathbf{IFP}(\mathbf{p}_0)$. Then $(a(t), b(t))$ is a solution parametrization if and only if there exists $m \in \mathbb{N}^*$ such that equation (4.3) holds. In the affirmative case, m is the ramification index of the generating Puiseux series solution. As a consequence, the map Δ is injective.

Proof. The first implication follows from Lemma 4.1.2. Let us now assume that (4.3) holds for an $m \in \mathbb{N}^*$ and write $a(t) = y_0 + \sum_{j=k}^{\infty} a_j\, t^j$ with $k > 0$, $a_k \neq 0$, and $b(t) = \sum_{j=k-m(1-h)}^{\infty} b_j\, t^j$. Let us consider $y(x) = y_0 + \sum_{j=k}^{\infty} a_j\, x^{j/m}$. By assumption, $y'(x) = x^{-h}\, b(x^{1/m})$ and

$$F(y(x), x^h y'(x)) = F(a(x^{1/m}), b(x^{1/m})) = 0.$$

Thus, $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$. It remains to show that m is the ramification index of $y(x)$. Otherwise, there exists a natural number $n \geq 2$, such that n divides m and if $a_i \neq 0$ then n divides i. By assumption, we have that $a_{j+m(1-h)} \neq 0$ if and only if $b_j \neq 0$. Hence, if $b_j \neq 0$, then n divides j. This implies that $(a(t), b(t))$ is reducible in contradiction to our assumption. Therefore, m is the ramification index of $y(x)$ and $\Delta(y(x)) = (a(t), b(t))$.
As a consequence of the first part, two solutions $y_1(x), y_2(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$ with

$$\Delta(y_1(x)) = \Delta(y_2(x)) = (a(t), b(t))$$

have the same ramification index $m \in \mathbb{N}^*$. Since the coefficients $[t^k]a(t)$ are equal to the coefficients of $[x^{k/m}]y_1(x)$ and $[x^{k/m}]y_2(x)$ for every $k \geq \operatorname{ord}(a(t))$, also the solutions $y_1(x)$ and $y_2(x)$ are equal, which proves the injectivity of Δ. $\square$

Lemma 4.1.4. *All Puiseux series solutions in* $\mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$*, generating the same solution place in* $\mathbf{Places}(\mathbf{p}_0)$*, have the same ramification index.*

Proof. Let $y_1(x), y_2(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ be such that $\delta(y_1(x)) = \delta(y_2(x))$. Then there exists an order one formal power series $s(t)$ such that

$$\Delta(y_1(x))(s(t)) = \Delta(y_2(x))(t).$$

Let m_i be the ramification index of $y_i(x)$ and let $\Delta(y_i(x)) = (a_i(t), b_i(t))$ for $i = 1, 2$. By equation (4.3),

$$a_2'(t) = m_2\, t^{m_2(1-h)-1}\, b_2(t) = m_1\, t^{m_1(1-h)-1}\, b_1(s(t))$$

and

$$a_2'(t) = (a_1(s(t)))' = a_1'(s(t))\, s'(t) = m_1\, s(t)^{m_1(1-h)-1}\, b_1(s(t))\, s'(t).$$

Since $y_1(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ is non-constant, $b_1(s(t))$ is non-zero. Therefore,

$$m_1\, s(t)^{m_1(1-h)-1}\, s'(t) = m_2\, t^{m_2(1-h)-1}. \tag{4.5}$$

Finally, comparing orders and taking into account that $h \neq 1$ by assumption, we get that $m_1 = m_2$. □

Definition 4.1.5. Since Δ is injective, we can define the *ramification index of a solution parametrization* $A(t)$ as the ramification index of $\Delta^{-1}(A(t))$. As a consequence of Lemma 4.1.4, we additionally may speak about the *ramification index of the solution place* $[A(t)]$.

In the following we analyze the number of solution parametrizations in a solution place. We start with a technical lemma.

Lemma 4.1.6. *Let $a(t) \in \mathbb{K}((t))$ be non-constant and let $\alpha_1, \alpha_2 \in \mathbb{K}$ be two different k-th roots of unity. If $a(\alpha_1 t) = a(\alpha_2 t)$, then there exists a minimal $n \in \mathbb{N}^*$, with $1 < n \leq k$, such that $a(t)$ can be written as $a(t) = \sum_{j \geq p/n} a_{jn}\, t^{jn}$.*

Proof. Let $a(t) = \sum_{j \geq j_0} a_j\, t^j$. Since $a(\alpha_1 t) = a(\alpha_2 t)$, then $a_j\, \alpha_1^j = a_j\, \alpha_2^j$. So, if $a_j \neq 0$ then $(\alpha_1/\alpha_2)^j = 1$. Let $n \in \mathbb{N}^*$ be minimal such that α_1/α_2 is an n-th primitive root of unity. Then $(\alpha_1/\alpha_2)^j = 1$ if and only if j is a multiple of n and this implies that $a(t) = \sum_{j \geq p/n} a_{jn}\, t^{jn}$. □

Lemma 4.1.7. *Let $[A(t)]$ be a solution place with ramification index $m \in \mathbb{N}^*$.*

1. *If $h \leq 0$, then there are exactly $m\,(1-h)$ solution parametrizations in $[A(t)]$, $A(t)$ is a solution parametrization, and all solution parametrizations in the place are of the form $A(\alpha\, t)$ where $\alpha^{m(1-h)} = 1$.*

2. *If $h \geq 2$, then there are infinitely many solution parametrizations in $[A(t)]$.*

Proof. Let, for $i = 1, 2$, $(a_i(t), b_i(t)) \in [A(t)]$ be two different solution parametrizations. As a consequence of equation (4.5), we get that the order one formal power series $s(t)$ relating $(a_1(t), b_1(t))$ and $(a_2(t), b_2(t))$ satisfies

$$s(t)^{m(1-h)-1}\, s'(t) = t^{m(1-h)-1}, \tag{4.6}$$

where m is the ramification index of the place. Conversely, let $s(t)$ be a solution of (4.6) with $\operatorname{ord}(s(t)) = 1$ and $(a_3(t), b_3(t)) = (a_1(s(t)), b_1(s(t))$. Then

$$a_3'(t) = (a_1(s(t)))' = a_3'(s(t))\, s'(t) = m\, s(t)^{m(1-h)-1}\, b_1(s(t))\, s'(t),$$

and by using equation (4.6),

$$a_3'(t) = m\, t^{m(1-h)-1}\, b_3(t).$$

Then, by Proposition 4.1.3, $(a_3(t), b_3(t)) = (a_1(s(t)), b_1(s(t))$ is a solution parametrization. Let us compute the solutions of (4.6) by separation of variables (see [Zwi98][Section IIA, 89]). If $h \leq 0$, then the solutions are

$$s(t) = \sqrt[m(1-h)]{t^{m(h-1)} + c},$$

where c is in the field of constants, which is equal to $\mathbb{K}$. Since $s(0) = c = 0$, we obtain $s(t) = \alpha\, t$, where α is an $m\,(1-h)$-th root of unity and the set of all solution parametrizations in $[A(t)]$ is

$$\mathcal{A} := \{(a_1(\alpha\, t), b_1(\alpha\, t)) \mid \alpha^{m(1-h)} = 1\}.$$

Let us verify that

$$\#\mathcal{A} = m\,(1-h).$$

If $m\,(1-h) = 1$, the result is trivial. Let $m\,(1-h) > 1$, and let us assume that $\#\mathcal{A} < m(1-h)$. Then, there exist two different $m\,(1-h)$-th roots of unity, α_1, α_2, such that

$$(a_1(\alpha_1\, t), b_1(\alpha_1\, t)) = (a_1(\alpha_2\, t), b_1(\alpha_2\, t)).$$

By Lemma 4.1.6 there exists $n \in \mathbb{N}$ with $1 < n \leq m\,(1-h)$, such that $a_1(t), b_1(t)$ can be written as $a_1(t) = \sum_{j \geq p/n} c_{jn}\, t^{jn}$ and $b_1(t) = \sum_{j \geq q/n} d_{jn}\, t^{jn}$. This implies that (a_1, b_1) is reducible, which is a contradiction.
If $h \geq 2$, the solutions of (4.6) are of the form

$$s(t) = \frac{\alpha\, t}{\sqrt[m(h-1)]{1 + t^{m(h-1)}\, c}},$$

where c is an arbitrary element in the field of constants and $\alpha^{m(h-1)} = 1$. Note that $s(t)$ can indeed be written as a formal power series of first order and for every choice $c \in \mathbb{K}$ the solution parametrization is distinct. □

For a given parametrization $(a(t), b(t)) \in \mathbf{IFP}(\mathbf{p}_0)$ satisfying (4.4), our strategy for finding the solutions will be to find local parametrizations in the place $[(a(t), b(t))]$ fulfilling (4.3). More precisely, we want to determine reparametrizations $s(t) \in \mathbb{K}[[t]]$ with $\operatorname{ord}(s(t)) = 1$ such that $(a(s(t)), b(s(t))$ satisfies

$$\frac{d\,(a(s(t)))}{dt} = m\, t^{m(1-h)-1}\, b(s(t)).$$

By applying the chain rule, this is equivalent to

$$a'(s(t)) \cdot s'(t) = m\, t^{m(1-h)-1}\, b(s(t)). \tag{4.7}$$

Equation (4.7) will be called the *associated differential equation*.
In Section 3.3 we have studied equations of the type (4.7), its solvability and properties of possible solutions.
Now, in the case of non-positive h, we are in the position to decide whether a given place $\mathcal{P} \in \mathbf{Places}(\mathbf{p}_0)$ is a solution place by a simple order comparison.

Theorem 4.1.8. *Let $\mathcal{P} = [(a(t), b(t))] \in$ $\mathbf{Places}(\mathbf{p}_0)$ and $h \leq 0$. Then $\mathcal{P}$ is a solution place if and only if equation* (4.4) *holds for an $m \in \mathbb{N}^*$. In the affirmative case the ramification index of $\mathcal{P}$ is equal to m.*

Proof. The first direction is Lemma 4.1.2. For the other direction let $(a(t), b(t))$ and $m \in \mathbb{N}^*$ be such that equation (4.4) holds. Moreover, let $s(t) \in \mathbb{K}[[t]]$ with $\operatorname{ord}(s(t)) = 1$ be such that the associated differential equation (4.7) is fulfilled. By setting $f(s) = a'(s),\ g(s) = m\, b(s)$ and $k = m\,(1-h) - 1 \geq 0$ in Lemma 3.3.1, such a solution exists. Then

$$(\bar{a}(t), \bar{b}(t)) = (a(s(t)), b(s(t))$$

fulfills equation (4.3) and by Proposition 4.1.3, $(\bar{a}(t), \bar{b}(t))$ is a solution parametrization with ramification index equal to m. □

Theorem 4.1.9. *Let $\mathbb{K} = \mathbb{C}$. Then every formal Puiseux series solution of* (4.1), *expanded around a finite point or at infinity, is convergent.*

Proof. In order to prove the statement we show that every formal Puiseux series solution expanded around zero of equation (4.2), in particular for $h \in \{0, 2\}$, is convergent.
Let $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$. Performing the change of variable $\tilde{y}(x) = 1/y(x)$ if necessary, we can assume that $y_0 \in \mathbb{C}$. Let $m \geq 1$ be the ramification index of $y(x)$ and

$$\Delta(y(x)) = (a(t), b(t)).$$

By Lemma 4.1.2, equations (4.3) and (4.4) hold. Let $k = \operatorname{ord}(a(t) - y_0) \geq 1$. There exists a reparametrization $s(t) \in \mathbb{C}[[t]]$, with $\operatorname{ord}(s(t)) = 1$, to bring $a(t)$ into the form of a classical Puiseux parametrization, i.e.

$$a(s(t)) - y_0 = t^k.$$

Let $\bar{a}(t) = a(s(t))$ and $\bar{b}(t) = b(s(t))$. Then

$$(\bar{a}(t) - y_0, \bar{b}(t)) = (t^k, \bar{b}(t))$$

is a local parametrization of the non-trivial algebraic curve defined by $F(y - y_0, p)$. Hence, by Puiseux's theorem, $\bar{b}(t)$ is convergent. Let $r(t)$ be the compositional inverse of $s(t)$, i.e.

$$r(s(t)) = t = s(r(t)).$$

Then $r(t)$ is a formal power series of order one and

$$a(t) = \bar{a}(r(t)), b(t) = \bar{b}(r(t)).$$

Since equation (4.7) holds for $(\bar{a}(t), \bar{b}(t))$ and $r(t)$, by Lemma 3.3.1, $r(t)$ is convergent. This implies that $a(t)$ is convergent and therefore, $y(x) = a(x^{1/m})$ is convergent as a Puiseux series. □

Theorem 4.1.10. *Let $F(y,p)$ be a non-constant polynomial with no factor in $\mathbb{K}[y]$ or $\mathbb{K}[p]$. For any point in the plane $(x_0, y_0) \in \mathbb{K}^2$, there exists a solution $y(x)$ of $F(y, y') = 0$ such that $y(x_0) = y_0$. If $\mathbb{K} = \mathbb{C}$, as a consequence of Theorem 4.1.9, $y(x)$ is additionally convergent.*

Proof. It is sufficient to prove the existence of a formal Puiseux series solution

$$y(x) = y_0 + \sum_{i=1}^{\infty} c_i \, (x - x_0)^{i/m}.$$

Performing the change of variable $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$, we may assume that $x_0 = 0$ and $y_0 = 0$. Let us write

$$F(y,p) = \sum F_{i,j} \, y^i \, p^j.$$

If $F(0,0) = F_{0,0} = 0$, then we have that $y(x) = 0$ is a solution of $F(y, y') = 0$ and $\Delta(y(x))$ passes through $(0,0)$. We may assume that $F_{0,0} \neq 0$. Consider $\mathcal{N}(F)$, the algebraic Newton polygon of the algebraic curve $F(y,p) = 0$ in the variables y and p, depicted in Figure 4.1. The point $(0,0)$ is a vertex of $\mathcal{N}(F)$, because $F_{0,0} \neq 0$. This implies that all the sides of $\mathcal{N}(F)$ have slope greater or equal to zero.

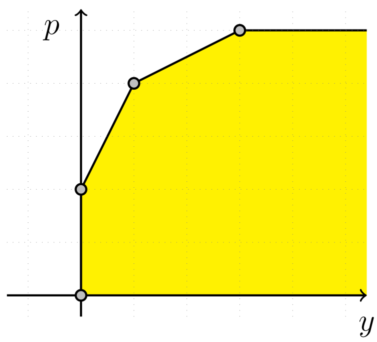

Figure 4.1: The Newton polygon of the algebraic curve $F(y,p) = 0$. All its sides have non-negative slope, because the point $(0,0) \in \mathcal{N}(F)$.

Since the degree of $F(y,p)$ with respect to p is positive, $\mathcal{N}(F)$ has at least one side. Therefore, by the Newton polygon method for algebraic equations as it is explained in Appendix C.1, there exists a classical Puiseux parametrization

$$(a(t), b(t)) = \left(t^m, \sum_{i=k}^{\infty} c_i \, t^i \right) \in \mathbf{IFP}(\mathbf{p}_0)$$

satisfying

$$n = \operatorname{ord}(a(t) - a(0)) - \operatorname{ord}(b(t)) = m - k \geq m \geq 1.$$

Then, by Theorem 4.1.8, there exists a $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$ with $\operatorname{ord}(y(x)) > 0$, which proves the theorem. □

Notice that in Theorem 4.1.10 we can give a lower and an upper bound for the number of solution parametrizations passing through a given point $(x_0, y_0) \in \mathbb{K}^2$. First, every side with slope greater or equal to zero defines a different solution parametrization. Thus, a lower bound can easily be derived after computing the Newton polygon $\mathcal{N}(F)$. Second, let $\Sigma_{(x_0,y_0)}$ denote the set of solution parametrizations passing through (x_0, y_0). The set of corresponding solution places is denoted by

$$\mathcal{P}(y_0) = \{[(a(t), b(t))] \mid (a(t), b(t)) \in \Sigma_{(x_0,y_0)}\}.$$

Since every solution parametrization passing through (x_0, y_0) is a solution parametrization centered at (y_0, p_0) for some $p_0 \in \mathbb{K}_\infty$, by Lemma 4.1.7,

$$\#\Sigma_{(x_0,y_0)} = \sum_{P \in \mathcal{P}(y_0)} \text{ramification index of } P \leq \deg_p(F).$$

The last inequality is a well known result for algebraic curves and follows from Theorem C.1.1 or can be found for example in [Duv89][Theorem 1].
As a consequence for example the family of functions

$$y(x) = x + c\,x^2,$$

where c is an arbitrary constant, cannot be a solution of any first order autonomous ordinary differential equation. Otherwise, there are infinitely many distinct formal parametrizations $(y(x), y'(x))$ with $y_0 = 0$ as initial value and the sum of the ramification indexes of $P \in \mathcal{P}(y_0)$ is infinite in contradiction to the bound above.
We note that there might be families of formal Puiseux series solutions at infinity for an autonomous first order ordinary differential equation as we will see in Example 4.1.18.

Algorithms and Examples

In this part of the section we outline an algorithm that is derived from the results on solution places, in particular, for $h \in \{0, 2\}$. We can describe algorithmically all formal Puiseux series solutions of the differential equation (4.1). For each formal Puiseux series solution we will provide what we call a *determined solution truncation*. A determined solution truncation is an element of $\mathbb{K}[x^{1/m}][x^{-1}]$, for some $m \in \mathbb{N}^*$, that can be extended uniquely to a formal Puiseux series solution.

If F is reducible, one could factor it and consider its irreducible components and the solutions of the corresponding differential equations. However, from a computational point of view, this is not optimal, and we compute the square-free part of F instead. So let us assume $F \in \mathbb{K}[y, p]$ to be square-free and have no factor in $\mathbb{K}[y]$ or $\mathbb{K}[p]$ in the remaining of the section. Since each formal Puiseux series solution $y(x)$ gives rise to an initial tuple

$$\mathbf{p}_0 = (y(0), (x^h y'(x))(0)) \in \mathscr{C}(F),$$

we will describe for each curve point $\mathbf{p}_0$ the set $\mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$. We note that if $\operatorname{ord}(y(x)) \geq 0$ and $h \geq 2$, then $\mathbf{p}_0$ will necessarily be of the type $(y_0, 0)$ for some $y_0 \in \mathbb{K}$.

Solutions expanded around Zero

In this subsection we consider formal Puiseux series solutions of (4.1), or equivalently, solutions of (4.2) with $h = 0$ expanded around zero. A point $\mathbf{p}_0 = (y_0, p_0) \in \mathscr{C}(F)$ is called a *critical curve point* if

$$p_0 \in \{0, \infty\} \quad \text{or} \quad \frac{\partial F}{\partial p}(\mathbf{p}_0) = S_F(\mathbf{p}_0) = 0.$$

Let us remark that there cannot be a solution with an initial tuple of the form (∞, p_0) with $p_0 \in \mathbb{K}$, because if $\operatorname{ord}(y(x)) < 0$ then $\operatorname{ord}(y'(x)) < 0$ as well. This motivates why we do not consider those points as critical curve points.

Lemma 4.1.11. *Let $F \in \mathbb{K}[y, p]$ be as in (4.1). Then the set of critical curve points $\mathcal{B}(F)$ is finite.*

Proof. Let F be of total degree d_F. Since F is square-free and F does not have a factor in $\mathbb{K}[p]$, by Bézout's Theorem (see e.g. [Wal50]), it holds that

$$\#\mathbb{V}_{\mathbb{K}}(\{F(y, p), p\}) \leq d_F.$$

Because F depends on p, the separant S_F is non-zero and it holds that the total degree of S_F, denoted by d_S is less than d_F. Since F is square-free, F and its separant S_F do not have a common factor. Thus, again by Bézout's Theorem, $\mathbb{V}_{\mathbb{K}}(\{F(y, z), S_F(y, z)\})$ is finite too and

$$\#\mathbb{V}_{\mathbb{K}}(\{F(y, z), S_F(y, z)\}) \leq d_F\, d_S \leq d_F\, (d_F - 1).$$

It remains to consider the curve points $\mathbf{p}_0 = (y_0, p_0) \in \mathscr{C}(F)$ with $p_0 = \infty$. For $y_0 \in \mathbb{K}$, set

$$G(y,p) = \mathrm{num}(F(y, 1/p)).$$

Then $\mathbf{p}_0 \in \mathscr{C}(F)$ corresponds to $(y_0, 0) \in \mathscr{C}(G)$. For $y_0 = \infty$ set

$$H(y,p) = \mathrm{num}(F(1/y, 1/p))$$

and $\mathbf{p}_0 \in \mathscr{C}(F)$ corresponds to $(0,0) \in \mathscr{C}(H)$. The defining polynomials G and H are again square-free and of total degree d_F. As before, there are only finitely many such points. More precisely,

$$\#\mathbb{V}_{\mathbb{K}}(\{G(y,p), p\}),\ \#\mathbb{V}_{\mathbb{K}}(\{H(y,p), p, y\}) \leq d_F$$

and the statement follows. □

Let us also give a geometrical interpretation of the results obtained in Theorem 4.1.8: Let $(a(t), b(t)) \in \mathbf{IFP}(\mathbf{p}_0)$ with $\mathbf{p}_0 = (y_0, p_0) \in \mathscr{C}(F)$ and $y_0 \in \mathbb{K}$. Then, by (4.4), $m = \mathrm{ord}(a(t) - y_0) - \mathrm{ord}(b(t))$ has to be a positive integer or $[(a(t), b(t))]$ is not a solution place. Moreover, as it is explained in the Appendix C,

- If $p_0 = 0$, then $\mathrm{ord}(a(t) - y_0) > \mathrm{ord}(b(t)) = \mathrm{ord}(b(t) - p_0)$ and $\mathbf{p}_0$ is of p-ramification. Note that the constant $y(x) = y_0$ is always a solution in this case.

- If $p_0 \neq 0$, then $m = \mathrm{ord}(a(t) - y_0)$ and the ramification index of the solutions is equal to $\mathrm{ord}(a(t) - y_0)$. In particular, if $\mathbf{p}_0 \in \mathscr{C}(F) \setminus \mathcal{B}(F)$, i.e. $\mathbf{p}_0$ is regular and not of y-ramification, the solution with $\mathbf{p}_0$ as initial tuple is unique and a formal power series. In this case we can apply the direct approach and the determined solution truncation is given by $y_0 + p_0\,x$.

The proof of Lemma 4.1.11 also tells us how to compute the special cases where the initial tuple is a critical curve point, i.e. $\mathbf{p}_0 \in \mathcal{B}(F)$, and the direct approach is not applicable.

Example 4.1.12. Let us consider

$$F(y, y') = ((y'-1)^2 + y^2)^3 - 4(y'-1)^2\,y^2 = 0.$$

The corresponding curve $\mathscr{C}(F)$ is a rational degree 6 curve with a non-ordinary singularity at $(0,1)$ (see Figure 4.1.12).

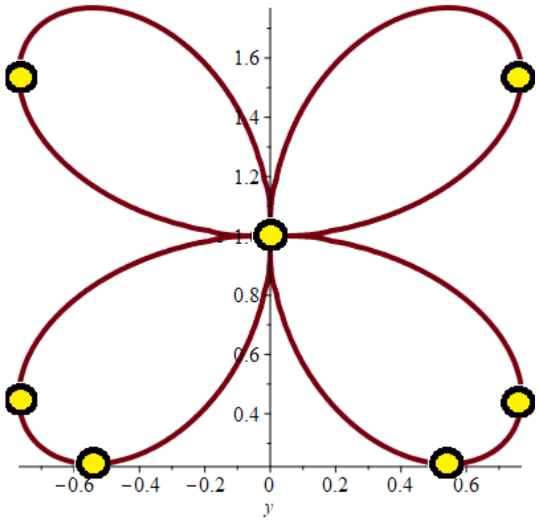

Figure 4.2: Plot of $\mathscr{C}(F)$.

The set of critical curve points is

$$\mathcal{B}(F) = \left\{(0,1), (\alpha, 0), \left(\frac{4\beta}{9}, \gamma\right), (\infty, \infty)\right\},$$

where $\alpha^6 + 3\alpha^4 - \alpha^2 + 1 = 0$, $\beta^2 = 3$, and $27\gamma^2 - 54\gamma + 19 = 0$. Observe that, since the leading coefficient of F with respect to y is 1, there is no curve point of the form (y_0, ∞) with $y_0 \in \mathbb{C}$.

Let $\mathbf{CPP}(\mathbf{p}_0) \subset \mathbf{IFP}(\mathbf{p}_0)$ denote the set of non-equivalent classical Puiseux parametrizations at $\mathbf{p}_0 \in \mathscr{C}(F)$. Recall that the elements in $\mathbf{CCP}(\mathbf{p}_0)$ are of the form

$$(a(t), b(t)) = (y_0 + t^k, b(t))$$

and for every $\lambda \in \mathbb{K}$ with $\lambda^k = 1$ the formal parametrization

$$(a(\lambda\, t), b(\lambda\, t)) = (y_0 + t^k, b(\lambda\, t))$$

is an equivalent classical Puiseux parametrization.

Let $\mathbf{RTrunc}(\mathbf{p}_0) \subseteq \mathbb{K}[t][t^{-1}]$ denote a set of truncations of elements in $\mathbf{CPP}(\mathbf{p}_0)$ such that $\mathbf{RTrunc}(\mathbf{p}_0)$ is in one-to-one correspondence to $\mathbf{Places}(\mathbf{p}_0)$. We additionally require the elements of $\mathbf{RTrunc}(\mathbf{p}_0)$ to satisfy that the ramification orders of the approximated places are determined such that we can check whether equation (4.4) holds and no further extensions of the ground field for computing the coefficients are necessary. In [Duv89], for the monic case, a bound for the number of steps to compute $\mathbf{RTrunc}(\mathbf{p}_0)$, namely

$$2(\deg_p(F) - 1)\deg_y(F) + 1,$$

is presented. For the general case, a similar bound based on the degree of the polynomial can be derived. Alternatively in [Sta00] a bound is derived from the Milnor number. From now on we will denote by N the difference between the maximal degree and the minimal order of the elements in $\mathbf{RTrunc}(\mathbf{p}_0)$.

If $\mathbf{p}_0 \in \mathscr{C}(F)$ is not a critical curve point, there is exactly one place at $\mathbf{p}_0$ and the classical Puiseux parametrization is of the form $(y_0 + t, b(t))$.

Following the proof of Lemma 3.3.1, we can describe all formal Puiseux series solutions with $\mathbf{p}_0$ as initial tuple as Algorithm PuiseuxSolve shows.

Algorithm 3 PuiseuxSolve

Input: A first-order AODE $F(y, y') = 0$, where $F \in \mathbb{K}[y, p]$ is square-free with no factor in $\mathbb{K}[y]$ or $\mathbb{K}[p]$.

Output: A set consisting of all determined solution truncations of $F(y, y') = 0$ expanded around zero.

if $(\infty, \infty) \in \mathscr{C}(F)$ **then**
  Perform the transformation $\tilde{y} = 1/y$ and apply the following steps additionally to $\mathrm{num}(F(1/y, -p/y^2))$ and $\mathbf{p}_0 = (0, 0)$.
end if
for every point $(y_0, p_0) \in \mathscr{C}(F) \setminus \mathcal{B}(F)$ **do**
  The generic determined solution truncation is $y_0 + p_0\, x$.
end for
Compute the set of critical curve points $\mathcal{B}(F)$.
for every critical curve point $\mathbf{p}_0 = (y_0, p_0) \in \mathcal{B}(F)$ with $y_0 \in \mathbb{K}$ **do**
  Compute the finite set $\mathbf{RTrunc}(\mathbf{p}_0)$.
  if $p_0 = 0$ **then**
    add to the output the constant solution $y(x) = y_0$.
  end if
  for every truncation $\pi_N(a(t), b(t)) \in \mathbf{RTrunc}(\mathbf{p}_0)$ **do**
    if equation (4.4) is fulfilled with an $m \in \mathbb{N}^*$ **then**
      Apply Algorithm AssocSolve to compute the first N terms of the solutions $s_1(t), \ldots, s_m(t)$ of (4.7).
      Add $\pi_N(a(s_i(x^{1/m})))$ to the set of determined solution truncations.
    end if
  end for
end for
return all determined solution truncations.

Let us remark that in Algorithm PuiseuxSolve we compute the first N terms of the classical Puiseux parametrizations and the first N terms of the reparametrization. That this is exactly the number of elements to compute the first N terms of their composition, which is the determined solution truncation, is a consequence of item 5 in Lemma 3.3.1.

Moreover, observe that in the algorithm it does not matter which representation of the truncated classical Puiseux parametrizations are chosen. More precisely, let $A(t) \in \mathbf{CPP}(\mathbf{p}_0)$ and $B(t) = A(\lambda t)$, with $\lambda^k = 1$. Then we obtain the corresponding reparametrizations $s_i(t)$ and $s_{i;\lambda}(t)$, $1 \leq i \leq n$, respectively. By Lemma 4.1.7, the sets of solution parametrizations coincide, i.e.

$$\{A(s_1(t)), \dots, A(s_n(t))\} = \{B(s_{1;\lambda}(t)), \dots, B(s_{n;\lambda}(t))\}.$$

Therefore also the sets of solution truncations are the same.

Lemma 4.1.13. *Let $F \in \mathbb{K}[y,p]$ define a plane curve and let $\mathbf{RTrunc}(\mathbf{p}_0)$ and N be as before. Let $A_1(t), A_2(t) \in \mathbf{IFP}(\mathbf{p}_0)$ with $\pi_N(A_1(t)) = \pi_N(A_2(t))$. Then*

$$A_1(t) = A_2(t).$$

Proof. Let us write

$$A_i(t) = (a_i(t), b_i(t)) = \left(\sum_{j \geq 0} a_{i,j}\, t^{k+j}, \sum_{j \geq 0} b_{i,j}\, t^{r+j} \right)$$

with $a_{i,0}, b_{i,0} \neq 0$. For $i = 1, 2$, there exist formal power series

$$s_i(t) = \sum_{j \geq 1} \sigma_{i,j}\, t^j$$

of order one such that $a_i(s_i(t)) = y_0 + t^k$. The first coefficient $\sigma_{i,1}$ should satisfy $a_{i,0}\, \sigma_{i,1}^k = 1$. Since $a_{1,0} = a_{2,0}$, we can choose $\sigma_{1,1} = \sigma_{2,1}$. The other coefficients $\sigma_{i,2}, \dots, \sigma_{i,N+1}$ are completely determined and depend only on $a_{i,0}, \dots, a_{i,N}$, see Theorem 2.2 in [Wal50]. Hence, by assumption, they are equal. Now the first N coefficients of $b_i(s_i(t))$ depend only on $\sigma_{i,1}, \dots, \sigma_{i,N+1}, b_{i,0}, \dots, b_{i,N}$ and therefore coincide as well. Now, both

$$\tilde{A}_1(t) = A_1(s_1(t)),\ \tilde{A}_2(t) = A_2(s_2(t)) \in \mathbf{CPP}(\mathbf{p}_0)$$

are classical Puiseux parametrizations with ramification index equal to k. Hence there exist two k-roots of unity α_1, α_2 such that we obtain for the truncations that $\pi_N(\tilde{A}_i(\alpha_i\, t)) \in \mathbf{RTrunc}(\mathbf{p}_0)$ and from the coefficient dependency described above,

$$\pi_N(\tilde{A}_1(t)) = \pi_N(\tilde{A}_2(t)).$$

Therefore,

$$\pi_N(A_1(t)) = \pi_N(A_2(\alpha\, t)),$$

where $\alpha = \alpha_2/\alpha_1$ is also a k-root of unity. In particular, $A_1(t)$ and $A_2(t)$ represent the same place. By the properties of the set $\mathbf{RTrunc}(\mathbf{p}_0)$, it follows that $A_1(t) = A_2(t)$ and $\alpha = 1$. Hence $\alpha_1 = \alpha_2$ and $A_1(s_1(t)) = A_2(s_2(t))$. Since $s_i(t)$ has a compositional inverse, then also $A_1(t) = A_2(t)$. □

In the following theorem we show that the output truncations are indeed determined solution truncations, which also proves the correctness of Algorithm 3.

Theorem 4.1.14. *Let $F \in \mathbb{K}[y,p]$ be square-free with no factor in $\mathbb{K}[y]$ or $\mathbb{K}[p]$. Then the set of truncated solutions obtained by the Algorithm* PuiseuxSolve *with* $\mathbf{p}_0$ *as initial tuple, denoted by* $\mathbf{STrunc}(\mathbf{p}_0)$*, and* $\mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F;\mathbf{p}_0)$ *are in one-to-one correspondence.*

Proof. Let $\mathbf{p}_0 \in \mathscr{C}(F)$. As we explained in the paragraph before Algorithm Puiseux-Solve, the elements in $\mathbf{RTrunc}(\mathbf{p}_0)$ and $\mathbf{Places}(\mathbf{p}_0)$ are in one-to-one correspondence and ramification orders are determined. By Proposition 4.1.3,

$$\#\mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F;\mathbf{p}_0) = \sum_{i=1}^{\#\mathbf{Places}(\mathbf{p}_0)} n_i,$$

where each summand n_i is equal to the ramification order of the corresponding place (or 0, if (4.4) is not fulfilled). Hence,

$$\#\mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F;\mathbf{p}_0) = \sum_{i=1}^{\#\,\mathbf{RTrunc}(\mathbf{p}_0)(\mathbf{p}_0)} n_i.$$

The number of output elements of Algorithm PuiseuxSolve is the same. Therefore,

$$\#\,\mathbf{STrunc}(\mathbf{p}_0) \le \#\mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F;\mathbf{p}_0)$$

with equality if and only if all output elements are distinct. Let us assume that two outputs $\hat{y}_1(x), \hat{y}_2(x) \in \mathbf{STrunc}(\mathbf{p}_0)$ coincide. Let us denote their corresponding expansions by $y_1(x)$ and $y_2(x)$, respectively. Necessarily the ramification orders of $y_1(x)$ and $y_2(x)$ coincide and we denote it by n. Moreover, the corresponding truncated solution parametrizations coincide, i.e.

$$\pi_N(y_1(t^n), y_1'(t^n)) = (\hat{y}_1(t^n), \hat{y}_1'(t^n)) = (\hat{y}_2(t^n), \hat{y}_2'(t^n)) = \pi_N(y_2(t^n), y_2'(t^n)).$$

Then, by Lemma 4.1.13, also the solution parametrizations $(y_1(t^n), y_1'(t^n))$ and $(y_2(t^n), y_2'(t^n))$ coincide and in particular, $y_1(x) = y_2(x)$. □

Example 4.1.15. Let us consider $F = ((y'-1)^2 + y^2)^3 - 4(y'-1)^2 y^2 = 0$ from Example 4.1.12. The generic solution is given by $y(x; y_0) = y_0 + p_0 x + \mathcal{O}(x^2) \in \mathbb{C}[[x]]$ with $p_0 \in \mathbb{C}$ such that $F(y_0,p_0) = 0$ and $\frac{\partial F}{\partial y'}(y_0,p_0) \neq 0$.
We now analyze the critical curve points. Let $\mathbf{c}_\alpha = (\alpha, 0)$ where $\alpha^6 + 3\alpha^4 - \alpha^2 + 1 = 0$. We get the place

$$\left(\alpha + t, \frac{\alpha}{19}\,(11\alpha^4 + 36\alpha^2 + 4)\,t + \mathcal{O}(t^2)\right),$$

which does not provide any solution (see equation (4.4)). Thus, the constant α is the only solution with the initial tuple $\mathbf{c}_\alpha$.
Let $\mathbf{c}_1 = (0,1)$. The truncated classical Puiseux parametrizations at $\mathbf{c}_1$ are

$$\begin{aligned}
(a_1(t), b_1(t)) &= (t^2, 1 + \sqrt{2}t - \tfrac{3t^3}{4\sqrt{2}} - \tfrac{15t^5}{64\sqrt{2}} + \mathcal{O}(t^6)),\\
(a_2(t), b_2(t)) &= (t^2, 1 - \sqrt{2}it - \tfrac{3it^3}{4\sqrt{2}} + \tfrac{15it^5}{64\sqrt{2}} + \mathcal{O}(t^6)),\\
(a_3(t), b_3(t)) &= (t, 1 + \tfrac{it^2}{2} + \tfrac{3it^4}{8} + \mathcal{O}(t^6)),\\
(a_4(t), b_4(t)) &= (t, 1 - \tfrac{it^2}{2} - \tfrac{3it^4}{8} + \mathcal{O}(t^6)).
\end{aligned}$$

So we have $m = 2$ for $(a_1(t), b_1(t))$ and $(a_2(t), b_2(t))$ and $m = 1$ for $(a_3(t), b_3(t))$ and $(a_4(t), b_4(t))$. Then equation (4.7) corresponding to $(a_1(t), b_1(t))$ is

$$s(t)\, s'(t) = t \left(1 + \sqrt{2}s(t) - \frac{3s(t)^3}{4\sqrt{2}} - \frac{15s(t)^5}{64\sqrt{2}}\right).$$

By Algorithm **AssocSolve** we obtain the solutions

$$s_1(t) = t + \tfrac{\sqrt{2}t^2}{3} + \tfrac{t^3}{18} - \tfrac{89t^4}{540\sqrt{2}} + \mathcal{O}(t^5),$$
$$s_2(t) = -t + \tfrac{\sqrt{2}t^2}{3} - \tfrac{t^3}{18} - \tfrac{89t^4}{540\sqrt{2}} + \mathcal{O}(t^5).$$

Then $a_1(s_1(x^{1/2})),\ a_1(s_2(x^{1/2}))$ are determined solution truncations of $F(y, y') = 0$. Similarly we can find two determined solution truncations coming from $(a_2(t), b_2(t))$ and one for each $(a_3(t), b_3(t))$ and $(a_4(t), b_4(t))$. We note that the solutions corresponding to $(a_3(t), b_3(t))$ and $(a_4(t), b_4(t))$ are formal power series and already detected by the direct approach. Thus,

$$\mathcal{U}_{\mathbf{c}_1} = \left\{\begin{array}{l} a_1(s_1(x^{1/2})) = x + \frac{2\sqrt{2}x^{3/2}}{3} + \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ a_1(s_2(x^{1/2})) = x - \frac{2\sqrt{2}x^{3/2}}{3} + \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ a_2(\tilde{s}_1(x^{1/2})) = x + \frac{2\sqrt{2}ix^{3/2}}{3} - \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ a_2(\tilde{s}_2(x^{1/2})) = x - \frac{2\sqrt{2}ix^{3/2}}{3} - \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\ a_3(s(x)) = x + \frac{x^3}{6} + \frac{17x^5}{240} + \mathcal{O}(x^6), \\ a_4(s(x)) = x - \frac{x^3}{6} + \frac{17x^5}{240} + \mathcal{O}(x^6) \end{array}\right\}$$

is the set of all determined solution truncations with $\mathbf{c}_1$ as initial tuple.
Let $\mathbf{c}_{\beta,\gamma} = \left(\frac{4\beta}{9}, \gamma\right)$, where $\beta^2 = 3$, and $27\gamma^2 - 54\gamma + 19 = 0$. We get the place

$$\left(\frac{4\beta}{9} + t^2, \gamma + \frac{\beta\, i}{\sqrt{3}}t + \mathcal{O}(t^2)\right).$$

Thus, (4.4) is fulfilled with $m = 2$. Similarly as before, we obtain at $\mathbf{c}_{\beta,\gamma}$ the set of solutions

$$\mathcal{U}_{\mathbf{c}_{\beta,\gamma}} = \left\{\begin{array}{l} \frac{4\beta}{9} + \gamma x + \frac{2\sqrt{-\gamma\beta}}{3\sqrt{3}}x^{3/2} + \left(\frac{5\gamma}{32} - \frac{143}{864}\right)\beta x^2 + \mathcal{O}(x^{5/2}), \\ \frac{4\beta}{9} + \gamma x - \frac{2\sqrt{-\gamma\beta}}{3\sqrt{3}}x^{3/2} + \left(\frac{5\gamma}{32} - \frac{143}{864}\right)\beta x^2 + \mathcal{O}(x^{5/2}) \end{array}\right\}.$$

Let us analyze $\mathbf{c}_\infty = (\infty, \infty)$. The numerator of $F(1/y, -y'/y^2)$ is equal to

$$\begin{aligned} G =& y^{12} + (6y' - 1)y^{10} + (15y'^2 + 4y' + 3)y^8 + (20y'^3 + 14y'^2 + 6y' + 1)y^6 + \\ & (15y'^4 + 12y'^3 + 3y'^2)y^4 + (6y'^5 + 3y'^4)y^2 + y'^6. \end{aligned}$$

The places at the origin of $\mathscr{C}(G)$ are given by

$$(t^3, \pm i\, t^3 + \mathcal{O}(t^4)),$$

which do not define a solution place.
Now the set $\{y(x; y_0)\} \cup \{\alpha\} \cup \mathcal{U}_{\mathbf{c}_1} \cup \mathcal{U}_{\mathbf{c}_{\beta,\gamma}}$ is in one-to-one correspondence to $\mathbf{Sol}_{\mathbb{C}[[x]]}(F)$.

Solutions expanded around Infinity

In this subsection we describe the formal Puiseux series solutions of (4.1) expanded around infinity, or equivalently up to the sign, formal Puiseux series solutions of (4.2) with $h = 2$ expanded around zero. The different sign in the second component requires to use, in all reasonings and results, a change in the sign of the second component of the formal parametrizations as well. This implies, in the same notation as in equation (4.7), the slightly different *associated differential equation for solutions expanded around infinity*, namely

$$a'(s(t))\, s'(t) = -m\, t^{-m-1}\, b(s(t)). \tag{4.8}$$

As we have already remarked, condition (4.4) is only a necessary and not a sufficient condition. That there are some cases which fulfill (4.4) for some $m \in \mathbb{N}^*$ but do not lead to solutions is shown in the following example.

Example 4.1.16. We are looking for formal Puiseux series solutions expanded around infinity of $F(y, y') = (1 + y)\, y' + y^2$. Instead we consider

$$F(y, -x^2 y') = -x^2\, (1 + y)\, y' + y^2 = 0.$$

In Example 3.1.8 we have seen that there is no formal Puiseux series solution with $y(0) = 0$ except the constant zero. On the other hand,

$$(t, -t^2 + \mathcal{O}(t^3))$$

is a formal parametrization of $\mathscr{C}(F)$ fulfilling (4.4) with $m = 1$.

Nevertheless, it can still be checked whether there exists a solution fulfilling the necessary condition (4.4) or not and therefore, we can algorithmically compute all solutions as in the previous subsection. For this purpose let $\mathbf{RTrunc}(\mathbf{p}_0)$ and N be as in Section 4.1, but additionally require that $N \geq \deg_p(F) + 1$.

Algorithm 4 PuiseuxSolveInfinity

Input: A first-order AODE $F(y,y')=0$, where $F \in \mathbb{K}[y,p]$ is square-free with no factor in $\mathbb{K}[y]$ or $\mathbb{K}[p]$.

Output: A set consisting of all solution truncations of $F(y,y')=0$ expanded around infinity.

```
Compute the algebraic set 𝕍_𝕂(F(y,0)).
for every y_0 ∈ 𝕍_𝕂(F(y,0)) do
  Set p_0 = (y_0,0) and compute the finite set RTrunc(p_0).
  Add to the output the constant solutions y(x) = y_0.
  for every truncation π_N(a(t),b(t)) ∈ RTrunc(p_0) do
    if equation (4.4) is fulfilled with an m ∈ ℕ* then
      Check by Algorithm AssocSolve whether the associated differential equa-
      tion (4.8) is solvable.[1]
      In the affirmative case, apply the algorithm to compute the first N terms
      of the solutions s_1(t),...,s_m(t), which contain a free parameter.
      Add π_N(a(s_i(x^{-1/m}))) to the set of solution truncations.
    end if
  end for
end for
Apply the previous steps additionally to the numerator of F(1/y, p/y^2) in order
to obtain the outputs corresponding to solutions of negative order.
return the set of solution truncations.
```

Let $y_0 \in \mathbb{K}$ be such that $\mathbf{p}_0 = (y_0,0) \in \mathscr{C}(F)$ and let us denote the output of Algorithm PuiseuxSolveInfinity by $\mathbf{STrunc}(\mathbf{p}_0)$.
Since $\mathbb{V}_{\mathbb{K}}(F(y,0))$ is a finite set and termination of Algorithm AssocSolve is ensured, also termination of Algorithm 4 follows. Correctness of the algorithm follows from Section 4.1 and the following corollary.

Corollary 4.1.17. *Let $F \in \mathbb{K}[y,p]$ be square-free with no factor in $\mathbb{K}[y]$ or $\mathbb{K}[p]$. Then every solution truncation $\tilde{y}(x) \in \mathbf{STrunc}(\mathbf{p}_0)$ can be extended to a solution $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x^{-1}\rangle\rangle}(F)$. Conversely, for every $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x^{-1}\rangle\rangle}(F)$ there exists a truncation $\tilde{y}(x) \in \mathbf{STrunc}(\mathbf{p}_0)$.*

Proof. In Algorithm 4 all places fulfilling the necessary condition (4.4) are treated. By Lemma 3.3.1, all solutions of (4.8) are found and by Proposition 4.1.3 the statement holds. □

In Theorem 4.1.14 we were able to additionally show that the output elements in Algorithm PuiseuxSolve, namely the elements of $\mathbf{RTrunc}(\mathbf{p}_0)$, are distinct. However, in Algorithm PuiseuxSolveInfinity we cannot guarantee this. The problem for adapting the proof of Theorem 4.1.14 lies in the free parameter of the reparametrizations.

Example 4.1.18. Let us consider

$$F(y,y') = y' + y^2 = 0$$

and its formal Puiseux series solutions expanded around infinity. We obtain that $\mathbb{V}_{\mathbb{C}}(F(y,0)) = \{0\}$. For $\mathbf{p}_0 = (0,0)$ let us compute the formal parametrization $(a,b) = (t, -t^2)$, which fulfills (4.4) with $m = 1$. Equation (4.8) simplifies to

$$s'(t) = t^{-2}\, s(t)^2,$$

having the solutions

$$s(t) = \frac{t}{1 - c\,t} = t + c\,t^2 + c^2\,t^3 + \mathcal{O}(t^4)$$

for an arbitrary constant c as we have seen in Example 3.3.2. Hence,

$$a(s(x^{-1})) = \frac{1}{x-c} = \frac{1}{x} + \frac{c}{x^2} + \frac{c^2}{x^3} + \mathcal{O}(x^{-4})$$

describes all formal Puiseux series solutions expanded around infinity.

Algebraic Puiseux Series Solutions

In this section we consider a subclass of formal Puiseux series, namely algebraic series. These are $y(x) \in \mathbb{K}\langle\langle x\rangle\rangle$ such that there exists a non-zero $G \in \mathbb{K}[x,y]$ with $G(x,y(x)) = 0$. Note that since the field of formal Puiseux series is algebraically closed, all algebraic solutions can be represented as (formal) Puiseux series.
In [ACFG05] the authors give a necessary and sufficient condition for a first order autonomous AODE to have an algebraic general solution. Moreover, they give an upper bound for the degree of the algebraic general solution and show how to compute them. They indicate how to use these results in order to compute all algebraic solutions of such a given differential equation. Here we follow their ideas and present a detailed proof of this fact. Moreover, we present all results for all algebraic solutions.
It is worth mentioning that rational functions are particular instances of algebraic functions. Therefore, the results obtained here generalize the results in [FG04, FG06].

In this section we show in Theorem 4.1.20 that if there exists one (non-constant) formal Puiseux series solution, algebraic over $\mathbb{K}(x)$, of a first order autonomous AODE, then all of them are algebraic over $\mathbb{K}(x)$. Moreover, the minimal polynomials are equal up to a shift and the multiplication with a constant (Theorem 4.1.22). Later in Section 4.2 we will use the results from this section for computing all algebraic solutions of systems of autonomous AODEs of dimension one.

Let us recall that since $\mathbb{K}\langle\langle x\rangle\rangle$ is algebraically closed, every algebraic function can be represented as a formal Puiseux series through its Puiseux expansion. For $d_x, d_y \in \mathbb{N}$, we say that a formal Puiseux series $y(x) \in \mathbb{K}\langle\langle x\rangle\rangle$ is (d_x, d_y)-*algebraic* if and only if $y(x)$ is algebraic over $\mathbb{K}(x)$ with a minimal polynomial $G(x,y) \in \mathbb{K}[x,y]$ such that

$$\deg_x(G) \leq d_x, \ \ \deg_y(G) \leq d_y.$$

Rational functions $y(x) = \frac{f(x)}{g(x)} \in \mathbb{K}(x)^*$ are $(d_x, 1)$-algebraic, where $\gcd(f,g) = 1$ and d_x is the maximum of the degrees of $f(x)$ and $g(x)$, since they have the minimal polynomial

$$G(x,y) = g(x)\,y - f(x).$$

Lemma 4.1.19. *Let $d_x, d_y \in \mathbb{N}^*$ and $y(x) \in \mathbb{K}\langle\langle x\rangle\rangle$. Then $y(x)$ is (d_x, d_y)-algebraic over $\mathbb{K}(x)$ if and only if there exists $G \in \mathbb{K}\{y\}$ with $\operatorname{ord}(G) \leq (d_x+1)(d_y+1)-1$ such that $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(G)$.*

Proof. A formal Puiseux series $y(x) \in \mathbb{K}\langle\langle x\rangle\rangle$ is (d_x, d_y)-algebraic if and only if the set

$$D = \{1, x, \ldots, x^{d_x}, y(x), \ldots, x^{d_x}y(x), \ldots, y(x)^{d_y}, \ldots, x^{d_x}y(x)^{d_y}\}$$

is linearly independent over $\mathbb{K}$. It is well known that D is linearly dependent if and only if its Wronskian determinant

$$W(x,y(x)) = \det \mathcal{D}(x,y(x))$$

is non-zero, where $\mathcal{D}$ is the matrix generated by the entries of D and their derivatives up to the order $d = (d_x+1)(d_y+1)-1$ (see for example [Rit50][Chapter II]). The matrix $\mathcal{D}$ can also be written as

$$\mathcal{D} = \begin{bmatrix} \mathcal{B} & * \\ \mathbf{0} & \mathcal{A} \end{bmatrix},$$

where $\mathbf{0}$ denotes the zero matrix, "$*$" denotes a matrix with entries which will be irrelevant in the later computations,

$$\mathcal{B}(x) = \begin{bmatrix} 1 & x & \dots & x^{d_x} \\ 0 & 1 & \dots & d_x\, x^{d_x-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_x! \end{bmatrix},$$

$$\mathcal{A}(x,y) = \left[\mathcal{A}_y(y) \,\middle|\, \mathcal{A}_y(y^2) \,\middle|\, \cdots \,\middle|\, \mathcal{A}_y(y^{d_y}) \right] \cdot \begin{bmatrix} \mathcal{B} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathcal{B} \end{bmatrix},$$

where

$$\mathcal{A}_y(y) = \begin{bmatrix} \binom{d_x+1}{0} y^{(d_x+1)} & \binom{d_x+1}{1} y^{(d_x)} & \dots & \binom{d_x+1}{d_y} y^{(d_x+1-d_y)} \\ \binom{d_x+2}{0} y^{(d_x+2)} & \binom{d_x+2}{1} y^{(d_x)} & \dots & \binom{d_x+2}{d_y} y^{(d_x+2-d_y)} \\ \vdots & \vdots & & \vdots \\ \binom{d}{0} y^{(d)} & \binom{d}{1} y^{(d-1)} & \dots & \binom{d}{d_y} y^{(d-d_y)} \end{bmatrix}.$$

Hence,

$$\begin{aligned} W(x,y(x)) &= \det(\mathcal{B}(x))^{d_y+1} \cdot \det \left[\mathcal{A}_y(y(x)) \,\middle|\, \mathcal{A}_y(y(x)^2) \,\middle|\, \cdots \,\middle|\, \mathcal{A}_y(y(x)^{d_y}) \right] \\ &= (d_x!)^{d_y+1} \cdot G(y(x)) = 0 \end{aligned}$$

holds if and only if $G(y(x)) = 0$, which proves the lemma. □

In the following we show that if there exists one algebraic solution, then all solutions are algebraic and they have the same minimal polynomial up to a shift in the independent variable. Moreover, by finding an arbitrary algebraic solutions and its minimal polynomial, all of them can be found by shifting the independent variable.

Theorem 4.1.20. *Let $F \in \mathbb{K}[y, y']$ be an irreducible polynomial with a solution $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$ algebraic over $\mathbb{K}(x)$. Then all elements in $\mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$ are algebraic over $\mathbb{K}(x)$.*

Proof. Let $Q(x,y) \in \mathbb{K}[x,y]$ be the minimal polynomial of $y(x)$ with ramification index equal to $m \in \mathbb{N}^*$. By Lemma 4.1.19, there exists $G \in \mathbb{K}\{y\}$ with

$$\operatorname{ord}(G) \le (\deg_x(Q)+1)(\deg_y(Q)+1) - 1 = d$$

such that $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x \rangle\rangle}(G)$. Then we can compute the differential pseudo remainder of G with respect to F and the ordering $y < y' < \cdots$ (see Appendix B)

$$\operatorname{prem}(G, F)(y, y') = \operatorname{init}(F)\, S_F\, G - \sum_{0 \le i \le d-1} G_i\, F^{(i)}, \tag{4.9}$$

where $\operatorname{init}(F)$ denotes the initial of F, S_F its separant and $G_i \in \mathbb{K}\{y\}$. By plugging $y(x)$ into equation (4.9), we obtain that

$$\operatorname{prem}(G, F)(y(x), y'(x)) = 0.$$

The pair $(y(x^m), y'(x^m))$ is a formal parametrization of $\mathscr{C}(F)$ and $\mathscr{C}(\operatorname{prem}(G, F))$. Let us consider the resultant

$$H(y) = \operatorname{Res}_{y'}(F, \operatorname{prem}(G, F)) = R_1(y, y')\, F(y, y') + R_2(y, y')\, \operatorname{prem}(G, F)(y, y')$$

for some $R_1, R_2 \in \mathbb{K}[y, y']$. Evaluated at $(y(x^m), y'(x^m))$, we get that $H(y(x)) = 0$ and since $y(x)$ is non-constant, H is constantly zero. The well known theory on resultants tells us that F and $\operatorname{prem}(G, F)$ have a common component. From the pseudo remainder reduction we additionally have

$$\deg_{y'}(\operatorname{prem}(G, F)) < \deg_{y'}(F)$$

and because of the irreducibility of F,

$$\operatorname{prem}(G, F)(y, y') = 0$$

follows. Now let $z(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$. Then $F^{(i)}(z(x)) = 0$ and from equation (4.9),

$$\operatorname{init}(F)(z(x))\, S_F(z(x), z'(x))\, G(z(x), \ldots, z^{(d)}(x)) = 0.$$

Similar as before, since $z(x)$ is non-constant and F is irreducible, it follows that the first two factors are non-zero and $z(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(G)$. Then the statement follows from Lemma 4.1.19. □

Lemma 4.1.21. *Let $F \in \mathbb{K}[y, y']$ be irreducible and let $y_1(x), y_2(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x \rangle\rangle}(F)$ be algebraic over $\mathbb{K}(x)$ with minimal polynomials $G_1, G_2 \in \mathbb{K}[x, y]$. Then there are $\lambda, c \in \mathbb{K}$ such that*

$$G_1(x, y) = \lambda\, G_2(x + c, y).$$

Proof. For the proof we need $y_0 \in \mathbb{K}$ such that $\mathbf{p}_0 = (y_0, p_0) \in \mathscr{C}(F)$ and $(x_0, y_0) \in \mathscr{C}(G_i)$, for $i \in \{1, 2\}$, are somehow generic points. To be precise, let $y_0 \in \mathbb{K}$ be such that

- $S_F(y_0, p) \in \mathbb{K}[p]$ does not have multiple roots (i.e. there is no p_0 such that $\mathbf{p}_0$ is of p-ramification);
- $\frac{\partial G_i}{\partial x}(x, y_0)$, $\frac{\partial G_i}{\partial y}(x, y_0) \in \mathbb{K}[x]$ do not have multiple roots;

- $\operatorname{init}(F)(y_0),\ \operatorname{init}(G_i)(y_0) \neq 0$.

Note that, similar to Lemma 4.1.11, there are only finitely many exceptional values for $y_0 \in \mathbb{K}$.
Since $\mathbb{K}$ is algebraically closed, the first item tells us that $S_F(y_0, p) \in \mathbb{K}[p]$ has exactly $d = \deg_{y'}(F)$ many distinct roots $p_1, \ldots, p_d \in \mathbb{K}$. From Theorem 3.4 in [ACFG05] we know that

$$\deg_x(G_i) = \deg_{y'}(F).$$

Hence, by the second item above, the polynomials $G_i(x, y_0) \in \mathbb{K}[x]$ have d distinct roots $x_1, \ldots, x_d \in \mathbb{K}$ and $x'_1, \ldots, x'_d \in \mathbb{K}$, respectively. Centered at $(x_j, y_0) \in \mathscr{C}(G_1)$, for $j \in \{1, \ldots, d\}$, we can compute by the Newton polygon method for algebraic equations (see Appendix C.1) the Puiseux expansions

$$\varphi_j(x) = y_0 + \sum_{k \geq 1} a_{j,k}(x - x_j)^k.$$

Since $(x_j, y_0) \in (G_1)$ are regular curve points and by Theorem C.1.1, φ_j are indeed formal power series. Moreover, from [ACFG05][Lemma 2.4] it follows that $\varphi_j \in \mathbf{Sol}_{\mathbb{K}[[x]]}(F)$. In particular,

$$F(\varphi_j(x_j), \varphi'_j(x_j)) = F(y_0, a_{j,1}) = 0.$$

Again by Theorem C.1.1, there is no other place of $\mathscr{C}(F)$ centered at $(y_0, a_{j,1})$ and therefore,

$$\{a_{1,1}, \ldots, a_{d,1}\} = \{p_1, \ldots, p_d\}.$$

Similarly, for G_2 and its Puiseux expansions

$$\psi_j(x) = y_0 + \sum_{k \geq 1} a'_{j,k}(x - x'_j)^k$$

we obtain

$$\{a'_{1,1}, \ldots, a'_{d,1}\} = \{p_1, \ldots, p_d\}.$$

For $a'_{1,1}$ there exists $a_{j,1}$ such that $a'_{1,1} = a_{j,1}$. Since the Puiseux expansion at $(y_0, a'_{1,1}) \in \mathscr{C}(F)$ is unique, for all coefficients $a'_{k,1} = a_{k,j}$ and

$$\psi_1(x + (x_1 - x_j)) = \varphi_1(x),$$

where $c = x_1 - x_j \neq 0$. Now the irreducible polynomials $G_1(x+c, y),\ G_2(x, y)$ share the common non-trivial solution $\psi_1(x)$, which is only possible if and only if they divide each other and the statement follows. $\square$

Theorem 4.1.22. *Let $F \in \mathbb{K}[y, y']$ be irreducible and let $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$ be algebraic over $\mathbb{K}(x)$ with minimal polynomial $G \in \mathbb{K}[x, y]$. Then all formal Puiseux series solutions $\mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$ are algebraic and given by $G(x + c, y)$, where $c \in \mathbb{K}$.*

Proof. Since F is independent of x, for every $c \in \mathbb{K}$ it follows that $G(x+c, y)$ defines an algebraic solution of $F = 0$. This leads to the set relation

$$\{y_c(x) \in \mathbb{K}\langle\langle x\rangle\rangle \mid c \in \mathbb{K},\ G(x+c, y_c(x)) = 0\} \subseteq \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(F). \tag{4.10}$$

For the converse direction let $z(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$. From Lemma 4.1.21 we know that for every solution there exist $\lambda, c \in \mathbb{K}$ such that $\lambda\, G(x+c, y)$ is the minimal polynomial of $z(x)$. Hence, also $G(x+c, y)$ is a minimal polynomial of $z(x)$ and $z(x)$ is an element of the left hand side in (4.10), which proves the set equality. □

In Section 4 of [ACFG05] there is a description of an algorithm that decides whether a given irreducible autonomous AODE $F(y, y') = 0$ has algebraic solutions and compute them in the affirmative case. This is done by first using the direct approach in order to compute a formal power series solution at an initial tuple which does not vanish at the separant up to a specific order. Note that in the case of autonomous first order AODEs Proposition 2.1.7 can be applied to almost every curve point. Then check whether this formal power series is algebraic over $\mathbb{K}(x)$. Let us call this algorithm AlgebraicSolve and let the output be equal to the minimal polynomial of an algebraic solution, if it exists, or empty otherwise.

We illustrate Algorithm AlgebraicSolve in the following example.

Example 4.1.23. Let us consider the differential equation

$$F(y, y') = y\, y' - 1 = 0.$$

For the initial tuple $\mathbf{p}_0 = (1, 1) \in \mathscr{C}(F)$ the separant is $S_F(\mathbf{p}_0) = 1 \neq 0$. By using the direct approach and Proposition 2.1.7 we obtain the formal power series solution

$$y(x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{5x^4}{8} + \mathcal{O}(x^5).$$

Let $G(x, y) = \sum_{0 \le i,j \le 2} g_{i,j}\, x^i\, y^j$. Note that from [ACFG05][Theorem 3.8] we have the order bound $\operatorname{ord}_x(G), \operatorname{ord}_y(G) \le 2$. Then $G(x, y(x)) = 0$ leads to the possible choice of coefficients $g_{0,0} = -1,\ g_{1,0},\ g_{0,2} = 1$ and all other coefficients equal to zero. Thus, the roots of

$$G(x, y) = y^2 - 2x - 1,$$

namely $y(x) = \pm\sqrt{2x+1}$, are an algebraic solution of $F = 0$. The (non-constant) solutions are then given by the zeros of $G(x+c, y)$. There are no constant solutions and, by setting $\tilde{c} = c + 1/2$, all formal Puiseux series solutions are given by

$$\mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F) = \{y_{\tilde{c}}(x) = \sqrt{2(x+\tilde{c})} \mid \tilde{c} \in \mathbb{C}\}.$$

4.2 Systems of Dimension One

In the previous section we have studied first order AODEs. In this section, we generalize the results obtained there to systems of higher order autonomous ordinary differential equations in one differential indeterminate which associated algebraic set is a finite union of curves and, maybe, points.

Finding rational general solutions of systems in one indeterminate has been studied in [LSNW15]. There, a necessary condition on the degree of the associated algebraic curve is provided. If the condition is fulfilled, the solutions are constructed from a rational parametrization of a birational planar projection of the associated space curve. Here, we provide an algorithm which decides the existence of not only rational but also algebraic Puiseux series solutions of such systems. Alternatively to the method described in [LSNW15], in the current approach we do not need to consider a rational parametrization of the associated curve. We instead triangularize the given system and we derive from there a single autonomous ordinary differential equation of first order with the same non-constant formal Puiseux series solutions. We call it the reduced differential equation of the system. Since rational or algebraic functions are determined by their Puiseux series expansion, the reduced differential equation has also the same algebraic solutions as the original system.
In the literature there are several methods to triangularize differential systems and to obtain resolvent representations of them, see for instance [CH03] and references therein. The description of these methods are quite involved because they apply to general differential systems. Since we work with systems of a particular type, a straightforward differential elimination process can be used. For the algebraic part we use regular chains, which is explained in Appendix C.3 more detailed. This simple description of the process allow us to have a precise relation between the formal Puiseux series solutions of the original system and those of the reduced differential equation.

We split this section into two parts. In the first part we consider systems of dimension one in one differential indeterminate. Here the reduced differential equation has exactly the same (non-constant) formal Puiseux series solutions. In the second part we consider systems of dimension one in several differential indeterminates, where the first component of a solution vector consisting of (non-constant) formal Puiseux series is also a solution of the reduced differential equation. Checking whether this candidates of solution vectors are indeed solutions can be done for algebraic solutions as it is shown in Theorem 4.2.14. Therefor we need to show that if the first component of the solution vector is algebraic, then all of them are algebraic, see Theorem 4.1.20.
The relation to the reduced differential equation allows us to prove convergence for both types of systems as it is shown in Theorem 4.1.9 and Theorem 4.2.11, respectively.

Systems in One Indeterminate

Let us consider systems of differential equations of the form

$$\widetilde{\mathcal{S}} = \{F_j(y, y', \dots, y^{(m)}) = 0\}_{1 \le j \le M}, \tag{4.11}$$

where $F_1, \dots, F_M \in \mathbb{K}[y, y', \dots, y^{(m)}]$ with $m > 0$. We assume the dimension of $\mathbb{V}_{\mathbb{K}}(\widetilde{\mathcal{S}})$ to be one, i.e. $\mathbb{V}_{\mathbb{K}}(\widetilde{\mathcal{S}})$ is a finite union of curves and, maybe, a finite union of points. Note that a single AODE of order one can be seen as a system of the type (4.11) with $M = m = 1$ and is of dimension one.

Lemma 4.2.1. *For every $\widetilde{\mathcal{S}}$ as in* (4.11) *we can compute a finite union of regular chains with the same non-constant formal Puiseux series solutions.*

Proof. Let us choose the ordering $y < y' < \cdots < y^{(m)}$. By Theorem C.3.3 there is a regular chain decomposition

$$\mathbb{V}_{\mathbb{K}\langle\langle x\rangle\rangle}(\widetilde{\mathcal{S}}) = \bigcup \mathbb{V}_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathcal{S}) \setminus \mathbb{V}_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathrm{pinit}(\mathcal{S})),$$

where $\mathrm{pinit}(\mathcal{S})$ denotes the product of the initials, such that every regular chain $\mathcal{S}$ has a zeroset of dimension zero or one. We omit systems of regular chains starting with an algebraic equation in y, since they only lead to constant solutions. Thus, the remaining systems are of dimension one and of the form

$$\mathcal{S} = \begin{cases} G_1(y, y') = \sum_{j=0}^{r_1} G_{1,j}(y) \cdot (y')^j = 0 \\ G_2(y, y', y'') = \sum_{j=0}^{r_2} G_{2,j}(y, y') \cdot (y'')^j = 0 \\ \quad \vdots \\ G_m(y, \ldots, y^{(m)}) = \sum_{j=0}^{r_m} G_{m,j}(y, \ldots, y^{(m-1)}) \cdot (y^{(m)})^j = 0 \end{cases} \tag{4.12}$$

with $r_j \geq 1$ and $\mathrm{init}(G_j) = G_{j,r_j} \neq 0$ for every $1 \leq j \leq m$.
Now we want to study in the regular chain decomposition which kind of solutions might be a solution of a regular chain but not of the original system, i.e. the common solutions of $\mathcal{S}$ and

$$\mathrm{pinit}(\mathcal{S}) = \mathrm{init}(G_1) \cdots \mathrm{init}(G_m) = 0.$$

If $y(x)$ is a non-constant formal Puiseux series solution of a $\mathcal{S}$, then $y(x)$ is transcendental over $\mathbb{K}$ and $(y(x), y'(x), \ldots, y^{(m)}(x))$ is a regular zero of $\mathcal{S}$. For every $1 \leq k \leq m$ we have

$$(\mathcal{S} \cap \mathbb{K}[y, \ldots, y^{(k)}]) \setminus \mathbb{K}[y, \ldots, y^{(k-1)}] = G_k \neq \emptyset.$$

Then, by Theorem C.3.2,

$$\mathrm{init}(G_2)(y(x), y'(x)), \ldots, \mathrm{init}(G_m)(y(x), \ldots, y^{(m-1)}(x)) \neq 0.$$

Since $\mathrm{init}(G_1)(y) = 0$ is an algebraic equation in y, there can only be constant common zeros of $\mathcal{S}$ and $\mathrm{pinit}(\mathcal{S})$. □

System (4.12) could be further decomposed into systems with the factors of G_1 as initial equations. However, for our purposes it is sufficient that G_1 and its separant, namely $S_{G_1} = \frac{\partial G_1}{\partial u_1}$, have no common differential solutions, i.e. if $G_1(y(x), y'(x)) = 0$ for $y(x) \in \mathbb{K}\langle\langle x\rangle\rangle$ then $S_{G_1}(y(x), y'(x)) \neq 0$. To ensure this we consider $G_1 \in \mathbb{C}[u_0, u_1]$ to be square-free and with no factor in $\mathbb{K}[u_0]$ or $\mathbb{K}[u_1]$; compare with the hypotheses in Section 4.1.
Moreover, it is enough to consider solutions expanded around zero or infinity. Additionally, we can assume without loss of generality for every solution $y(x) \in \mathbb{K}\langle\langle x\rangle\rangle$

of a system $\mathcal{S}$ as in (4.12) that $y(0) = y_0 \in \mathbb{K}$. Otherwise, if $y_0 = \infty$, consider the change of variable $\tilde{y} = 1/y$. Let $G_j^*(\tilde{y}, \tilde{y}', \dots, \tilde{y}^{(j)})$ be the numerator of $G_j(1/\tilde{y}, (1/\tilde{y})', \dots, (1/\tilde{y})^{(j)})$, and let $\mathcal{S}^*$ be the system $\{G_j^* = 0\}_{1 \le j \le m}$. In this situation, if $y(x) \in \mathbb{K}((x))^*$ is a solution of $\mathcal{S}$ such that $y(0) = \infty$, then $\tilde{y}(x) = 1/y(x)$ is a formal Puiseux series solution of $\mathcal{S}^*$ with $\tilde{y}(x_0) \in \mathbb{K}$. Moreover, for $j > 0$, the j-th derivative of $\tilde{y}$ can be written as

$$\tilde{y}^{(j)} = \frac{-y^{j-1}\, y^{(j)} + P_j(y, \dots, y^{(j-1)})}{y^{j+1}},$$

where $P_j \in \mathbb{K}[u_0, \dots, u_{j-1}]$. In this situation, we consider the rational map

$$\Phi : \mathbb{K}^{m+1} \setminus \mathbb{V}(u_0) \to \mathbb{K}^{m+1} \setminus \mathbb{V}(w_0); (u_0, \dots, u_m) \mapsto (w_0, \dots, w_m),$$

where $w_0 = 1/u_0$ and

$$w_j = \frac{-u_0^{j-1}\, u_j + P_j(u_0, \dots, u_{j-1})}{u_0^{j+1}}.$$

Since the equality above is linear in u_j, Φ is birational. In addition, taking into account that u_0 is not a factor of $G_1(u_0, u_1)$, one has that the Zariski closure of $\Phi(\mathbb{V}_{\mathbb{K}}(\mathcal{S}))$ is $\mathbb{V}_{\mathbb{K}}(\mathcal{S}^*)$. Since $\dim(\mathbb{V}_{\mathbb{K}}(\mathcal{S})) = 1$, also $\dim(\mathbb{V}_{\mathbb{K}}(\mathcal{S}^*)) = 1$ and one may proceed with $\mathcal{S}^*$ instead of $\mathcal{S}$.

For a given system $\mathcal{S}$ as in (4.12) we now associate a finite set of bivariate polynomials $\mathcal{H}(\mathcal{S}) = \{H_1, \dots, H_m\} \subset \mathbb{K}[u_0, u_1]$. According to 2.1.6, for every $j \ge 2$ there exists a differential polynomial R_{j-1} of order $j-1$ such that

$$G_1^{(j-1)}(y, \dots, y^{(j)}) = S_{G_1}(y, y') \cdot y^{(j)} + R_{j-1}(y, \dots, y^{(j-1)}). \tag{4.13}$$

Then, for $2 \le j \le m$, we introduce the rational functions

$$A_j(u_0, \dots, u_{j-1}) = \frac{-R_{j-1}(u_0, \dots, u_{j-1})}{S_{G_1}(u_0, u_1)}. \tag{4.14}$$

We recursively substitute the variables $u_2, \dots, u_m$ by $A_2(u_0, u_1), \dots, A_m(u_0, \dots, u_{m-1})$ to obtain the new rational functions $B_2, \dots, B_m \in \mathbb{K}(u_0, u_1)$:

$$\begin{cases} B_2(u_0, u_1) = A_2(u_0, u_1) \\ B_3(u_0, u_1) = A_3(u_0, u_1, B_2(u_0, u_1)) \\ \quad \vdots \\ B_m(u_0, u_1) = A_m(u_0, u_1, B_2(u_0, u_1), \dots, B_{m-1}(u_0, u_1)). \end{cases} \tag{4.15}$$

Observe that the denominators of the rational functions A_j are powers of the separant and depend only on u_0 and u_1. Finally we set

$$\begin{cases} H_1(u_0, u_1) = \operatorname{num}(G_1(u_0, u_1)) \\ H_2(u_0, u_1) = \operatorname{num}(G_2(u_0, u_1, B_2(u_0, u_1))) \\ \vdots \\ H_m(u_0, u_1) = \operatorname{num}(G_m(u_0, u_1, B_2(u_0, u_1), \dots, B_m(u_0, u_1))), \end{cases} \tag{4.16}$$

where $\operatorname{num}(f)$ denotes the numerator of the rational function f.
In this situation, we introduce a new autonomous first order algebraic differential equation, namely

$$H(y, y') = \gcd(H_1, \ldots, H_m)(y, y') = 0, \tag{4.17}$$

and call it the *reduced differential equation* (of $\mathcal{S}$). Note that by construction, H divides G_1. Moreover, if $\mathcal{S}$ contains only one single equation G_1, then the reduced differential equation of $\mathcal{S}$ is equal to G_1.

Let us note that the equations in (4.16) could also be found by using differential pseudo remainder computation. To be more precise,

$$H_j(u_0, u_1) = \operatorname{prem}(G_j; G_1).$$

The advantage of our more cumbersome computation is that we directly see the dependency only on the two variables u_0 and u_1 and which computational steps have to be done. In the reduction it is sufficient to use G_1, its separant S_{G_1} and the derivatives of G_1 with respect to x up to the order j.

Theorem 4.2.2. *Let $\mathcal{S}$ be as in* (4.12)*. Then $\mathcal{S}$ and its reduced differential equation have the same non-constant formal Puiseux series solutions.*

Proof. Let G_1 be the square-free starting equation of $\mathcal{S}$. Then $\gcd(G_1, S_{G_1}) = 1$ and if $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(G_1)$ is non-constant, then $S_{G_1}(y(x), y'(x)) \neq 0$.
Let $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathcal{S})$ be non-constant. Then $G_1(y(x), y'(x)) = 0$ and hence,

$$G_1^{(j-1)}(y(x), \ldots, y^{(j)}(x)) = 0.$$

Applying formula (4.13) and since $S_{G_1}(y(x), y'(x)) \neq 0$, we get that

$$A_j(y(x), \ldots, y^{(j-1)}(x)) = y^{(j)}(x)$$

and we obtain that

$$B_j(y(x), y'(x)) = y^{(j)}(x).$$

Therefore,

$$\begin{aligned} H_j(y(x), y'(x)) &= \operatorname{num}(G_j(y(x), y'(x), B_2(y(x), y'(x)), \ldots, B_j(y(x), y'(x)))) \\ &= G_j(y(x), \ldots, y^{(r)}(x)) = 0 \end{aligned}$$

for every $2 \leq j \leq m$. By Bézout's identity, there exists $Q \in \mathbb{K}[y]$ such that the polynomial $Q \cdot \gcd(H_1, \ldots, H_m)$ is an algebraic combination of the H_j. Since the equation $Q(y) = 0$ has only constant solutions, $y(x)$ is a solution of $H(y, y') = 0$.

Conversely, let $H(y, y') = \gcd(H_1, \ldots, H_m)(y, y') = 0$ be the reduced differential equation of $\mathcal{S}$ and $y(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(H)$ be a non-constant. Then,

$$G_1(y(x), y'(x)) = H_1(y(x), y'(x)) = 0$$

and as observed above, $S_{G_1}(y(x), y'(x)) \neq 0$. Thus, for every $1 \leq j \leq m$ the denominator of $G_j(u_0, u_1, B_2(u_0, u_1), \ldots, B_j(u_0, u_1))$ does not vanish at $(y(x), y'(x))$. Taking into account (4.16), it follows that

$$G_j(y(x), y'(x), B_2(y(x), y'(x)), \ldots, B_j(y(x), y'(x))) = 0.$$

Now, let us recursively show $B_j(y(x), y'(x)) = y^{(j)}(x)$ for $2 \leq j \leq m$, which proves the theorem: Since $B_2(y(x), y'(x)) = A_2(y(x), y'(x))$ and by (4.14),

$$R_1(y(x), y'(x)) = -B_2(y(x), y'(x))\, S_{G_1}(y(x), y'(x)).$$

Then, by (4.13),

$$0 = G_1^{(1)}(y(x), y'(x), y''(x)) = S_{G_1}(y(x), y'(x))(y''(x) - B_2(y(x), y'(x))).$$

Using that $S_{G_1}(y(x), y'(x)) \neq 0$, we obtain $B_2(y(x), y'(x)) = y''(x)$. Now, let us assume that $B_i(y(x), y'(x)) = y^{(i)}(x)$ for $2 \leq i \leq j$. By equation (4.15),

$$B_{j+1}(y(x), y'(x)) = A_{j+1}(y(x), \ldots, y^{(j)}(x)).$$

Then, reasoning as above, we obtain

$$\begin{aligned} 0 &= G_1^{(j)}(y(x), \ldots, y^{(j)}(x)) \\ &= S_{G_1}(y(x), y'(x))(y^{(j+1)}(x) - B_{j+1}(y(x), y'(x))), \end{aligned}$$

and hence, $B_{j+1}(y(x), y'(x)) = y^{(j+1)}(x)$. □

Corollary 4.2.3. *Let $\mathcal{S}$ be as in* (4.12) *and let its reduced differential equation H be a product of factors in $\mathbb{K}[y]$ and $\mathbb{K}[y']$. Then $\mathcal{S}$ has only linear formal Puiseux series solutions, i.e. solutions of the form $\alpha\, x + \beta$ for some $\alpha, \beta \in \mathbb{K}$.*

Proof. From the construction of the reduced differential equations, and the assumption that $G_1, \ldots, G_m \neq 0$, we know that $H \neq 0$. For every factor in $\mathbb{K}[y]$ of H, there are only constant solutions, and for every factor in $\mathbb{K}[y']$, there are only linear solutions of $H(y, y') = 0$. Then from Theorem 4.2.2 the statement follows. □

Let $\widetilde{\mathcal{S}}$ be as in (4.11). Then, by Lemma 4.2.1, it can be written as the union of systems $\mathcal{S}_1, \ldots, \mathcal{S}_K$ of the form (4.12). Let $H_1, \ldots, H_K$ be the reduced differential equations of these systems $\mathcal{S}_1, \ldots, \mathcal{S}_K$. Then, as a consequence of Theorem 4.2.2, $\widetilde{\mathcal{S}}$ and

$$H(y, y') = \operatorname{lcm}(H_1, \ldots, H_K)(y, y') = 0 \tag{4.18}$$

have the same non-constant formal Puiseux series solutions. Equation (4.18) is called a reduced differential equation of $\widetilde{\mathcal{S}}$. Now we are in the position to generalize the theoretical results obtained in Section 4.1 to systems of dimension one.

Theorem 4.2.4. *Let $\mathbb{K} = \mathbb{C}$. Then all formal Puiseux series solutions of the system of differential equations* (4.11)*, expanded around a finite point or at infinity, are convergent.*

Proof. By Theorem 4.2.2, the system (4.11) and its reduced differential equation $H(y, y') = 0$ have the same non-constant formal Puiseux series solutions. Then by 4.2.4, and the fact that constant solutions are convergent, the statement follows. □

Theorem 4.2.5. *Let $\widetilde{\mathcal{S}}$, as in (4.11), have a non-linear solution. Then for any point $(x_0, y_0) \in \mathbb{K}^2$ there exists a solution $y(x)$ of (4.11) such that $y(x_0) = y_0$. If $\mathbb{K} = \mathbb{C}$, then as a consequence of Theorem 4.2.4, $y(x)$ is additionally convergent.*

Proof. By Corollary 4.2.3, the reduced differential equation of $\widetilde{\mathcal{S}}$ has at least one irreducible factor depending on y and y'. Then by Theorem 4.1.10 the statement follows. □

Theorem 4.2.6. *Let $\widetilde{\mathcal{S}}$ as in (4.11) have the reduced differential equation $H(y, y') = 0$ and let $y(x)$ be a non-constant formal Puiseux series solution of $\widetilde{\mathcal{S}}$ algebraic over $\mathbb{K}(x)$. Then the minimal polynomial $G(x, y)$ of $y(x)$ fulfills the degree bounds*

$$\deg_x(G) \leq \deg_{y'}(H), \quad \text{and} \quad \deg_y(G) \leq \deg_y(H) + \deg_{y'}(H). \tag{4.19}$$

In particular if $y(x)$ is a rational solution of $\widetilde{\mathcal{S}}$, then its degree, the maximum of the degree of the numerator and denominator, is less than or equal to $\deg_{y'}(H)$.

Proof. By Theorem 4.2.2, $y(x)$ is a solution of the autonomous first order differential equation $H(y, y') = 0$. Then by Theorem 3.4 and 3.8 in [ACFG05] the degree bounds (4.19) follow. □

In the remaining part of this section we outline an algorithm based on the results in Section 4.2 to derive the reduced differential equation from a given system (4.11). By **Triangularize**($\widetilde{\mathcal{S}}$) we refer to the computation of a regular chain decomposition of a given system $\widetilde{\mathcal{S}}$ as in Theorem C.3.3. Then, using the algorithms in Section 4.1, it is possible to algorithmically describe all formal Puiseux series solutions of the given system. We illustrate this in the subsequent examples.

Algorithm 5 ReduceSystem

Input: A finite system of autonomous algebraic ordinary differential equations $\widetilde{\mathcal{S}} \subset \mathbb{K}[y, \ldots, y^{(m)}]$ which associated algebraic set is of dimension one.

Output: The reduced differential equation of $\widetilde{\mathcal{S}}$.

1: Set $\mathfrak{S} =$**Triangularize**($\widetilde{\mathcal{S}}$) and $H = 1$.
2: **for** every $\mathcal{S} \in \mathfrak{S}$ of dimension one let G_1 be the polynomial in $\mathbb{K}[u_0, u_1]$ associated to the equation of $\mathcal{S}$ depending on y, y' **do**
3: Take the square-free part of $G_1(u_0, u_1)$ and divide by its factors in $\mathbb{K}[u_0]$ and $\mathbb{K}[u_1]$; call it $G_1^*(u_0, u_1)$.
4: Replace in $\mathcal{S}$ the equation $G_1(y, y') = 0$ by $G_1^*(y, y')$; call it $\mathcal{S}^*$.
5: Compute the associated set $\mathcal{H}(\mathcal{S}^*) = \{H_1, \ldots, H_m\}$ as in (4.13)-(4.16) and set $H = \operatorname{lcm}(H, \gcd(H_1, H_2, \ldots, H_m))$.
6: **end for**
7: **return** H.

Example 4.2.7. Let us consider the system of differential equations given by

$$\widetilde{\mathcal{S}} = \begin{cases} yy'y'' + y'^3 - yy'' - y'^2 = 0 \\ yy' - 1 - y'^2 - yy'' = 0. \end{cases} \tag{4.20}$$

The system $\widetilde{\mathcal{S}}$ can be decomposed into the system of regular chains

$$\mathcal{S}_1 = \begin{cases} G_1 = yy' - 1 = 0 \\ G_2 = y'^2 + yy'' = 0 \end{cases} \quad \text{and} \quad \mathcal{S}_2 = \begin{cases} y' - 1 = 0 \\ 2 - y + yy'' = 0 \end{cases}$$

For the system $\mathcal{S}_1$ the starting equation G_1 is already square-free with no factor in $\mathbb{C}[y]$ or $\mathbb{C}[y']$ and we set $H_1 = G_1$. By computing $G_1^{(1)}(y, y', y'')$ and setting it to zero we obtain $y'' = \frac{-y'^2}{y}$. Hence,

$$H_2(y, y') = \operatorname{num}\left(G_2\left(y, y', \frac{-y'^2}{y}\right)\right) = H_1(y, y').$$

Then the reduced differential equation of $\mathcal{S}$ is

$$H(y, y') = \gcd(H_1, H_2)(y, y') = yy' - 1 = 0.$$

For the system $\mathcal{S}_2$ we obtain $H_1(y, y') = y' - 1$ and $H_2(y, y') = 2 - y'$, which are coprime. Hence, the reduced differential equation of $\mathcal{S}_2$ is equal to one and therefore, the reduced differential equation of $\widetilde{\mathcal{S}}$ is $H(y, y') = yy' - 1$.
We remark that by using the `RosenfeldGroebner`-command from `Maple`, which uses regular differential chains as described in [CH03], the reduced differential equation $H(y, y') = 0$ of $\widetilde{\mathcal{S}}$ can be found as well.

The next algorithm describes all formal Puiseux series solutions of a system $\widetilde{\mathcal{S}}$ which associated algebraic set is of dimension one. We use Algorithm PuiseuxSolve described in Section 4.1 whose input is an autonomous AODE of order one and Algorithm ReduceSystem from above. The output is a finite set of truncations in one-to-one correspondence to all Puiseux series solutions.

Algorithm 6 PuiseuxSolveSystem

Input: A finite system of autonomous algebraic ordinary differential equations $\widetilde{\mathcal{S}} \subset \mathbb{K}[y, \ldots, y^{(m)}]$ which associated algebraic set is of dimension one.
Output: A set Σ of all solution truncations of $\widetilde{\mathcal{S}}$ such that the truncation can be uniquely extended. Σ has non constant solutions if and only if $\Sigma \neq \emptyset$.
Set $\Sigma = \emptyset$ and $H(y, y') =$ReduceSystem$(\widetilde{\mathcal{S}})$.
Let $H^*(y, y')$ be the polynomial obtained after factoring out factors in $\mathbb{K}[y]$ and $\mathbb{K}[y']$ and taking the square-free part of $H(y, y')$.
if H^* is not a constant **then**
  Set $\Sigma =$PuiseuxSolve(H^*).
end if
Add to Σ the non constant linear solutions of $\widetilde{\mathcal{S}}$. This can be done by making the ansatz $y(x) = \alpha\, x + \beta$ with unknown α and β and plug it into the equations and solving the algebraic system obtained in α and β.
return Σ.

We can devise a similar algorithm to compute a set of truncations of Puiseux solutions expanded at the infinity point replacing in the above algorithm the Algorithm PuiseuxSolve by the Algorithm PuiseuxSolveInfinity from Section 4.1. However, here we do not ensure the uniqueness of the extension.

The next algorithm decides if a system $\widetilde{\mathcal{S}}$ as (4.11) has an algebraic solution and compute some of them in the affirmative case. Its correctness is based on the proof of Theorem 4.2.6 where it is shown that the non constant algebraic solutions of the system $\widetilde{\mathcal{S}}$ are the non constant algebraic solutions of the reduced equation of $\widetilde{\mathcal{S}}$. Moreover, we use Algorithm AlgebraicSolve described in Section 4.1 for computing all algebraic solutions. Recall, if one algebraic solution with minimal polynomial $G \in \mathbb{K}[x, y]$ exists, then all of them are determined as the zeros of $G(x+c, y)$ with $c \in \mathbb{K}$. This algorithm needs that the polynomials $H_i(y, y')$ are irreducible. Hence, the next algorithm is not factorization free.

Algorithm 7 AlgSolutionSystem

Input: A finite system of autonomous algebraic ordinary differential equations $\widetilde{\mathcal{S}} \subset \mathbb{C}[y, \ldots, y^{(m)}]$ which associated algebraic set is of dimension one.

Output: A set Σ of algebraic solutions of $\widetilde{\mathcal{S}}$ or the emptyset such that system $\widetilde{\mathcal{S}}$ has an algebraic solution if and only if $\Sigma \neq \emptyset$.

1: Set $\Sigma = \emptyset$ and $H(y, y')$ =ReduceSystem($\widetilde{\mathcal{S}}$).
2: Let $H^*(y, y')$ be the polynomial obtained after factoring out factors in $\mathbb{K}[y]$ and $\mathbb{K}[y']$.
3: **for** each irreducible factor $H_i(y, y')$ of $H^*(y, y')$ **do**
4: Add to Σ the output of Algorithm AlgebraicSolve(H_i).
5: **end for**
6: Add to Σ the non constant linear solutions of $\widetilde{\mathcal{S}}$.
7: **return** Σ.

Example 4.2.8. Let us consider system (4.20) of Example 4.2.7. By Theorem 4.2.2, the solutions are those of the reduced differential equation

$$H(y, y') = y\, y' - 1 = 0.$$

We obtain all the formal Puiseux series solutions, expanded around zero, by the one-parameter family of solutions

$$y(x) = y_0 + \frac{x}{y_0} - \frac{x^2}{2y_0^3} + \frac{x^3}{2y_0^5} + \mathcal{O}(x^4)$$

with $y_0 \in \mathbb{C} \setminus \{0\}$, and the particular solutions

$$y(x) = \pm\sqrt{2}\, x^{1/2}.$$

There is no formal Puiseux series solution with the initial value $y(0) = \infty$. The only linear solutions of $\widetilde{\mathcal{S}}$ are $y(x) = \pm 1$. The possible algebraic solutions $y(x)$ have a minimal polynomial $G(x, y)$ with degree bound of $\deg_x G \leq \deg_{y'}(H) = 1$ and $\deg_y G \leq \deg_{y'}(H) + \deg_y(H) = 2$. They are given by the zeros of

$$G_{y_0}(x, y) = y^2 - 2\left(x + \frac{y_0^2}{2}\right).$$

The assumption on the dimension of the given system is crucial in our work. Otherwise for instance Theorem 4.2.4 does not hold anymore as the following example shows.

Example 4.2.9. Let us consider $F(x, y, y') = x^2\, y' - y + x$ with the non-convergent formal power series

$$y(x) = \sum_{j \geq 0} j!\, x^{j+1} \in \mathbf{Sol}_{\mathbb{C}[[x]]}(F)$$

from Example 3.3.3. The solution $y(x)$ is also a zero of $F^{(1)}(x, y, y', y'')$ and consequently, of the resultant

$$\mathrm{Res}_x(F, F^{(1)}) = y'' + y''^2 y^2 - y'' y' + 4y'' y' y - y'^2 - 2y'' y'^2 y - 4y'^3 y + y'^4.$$

Note that $\{\mathrm{Res}_x(F, F^{(1)})(y, y', y'') = 0\}$ defines a system of autonomous ordinary algebraic differential equations of algebro-geometric dimension two.

Systems in Several Indeterminates

In this section let us consider systems of differential equations in several differential indeterminates, i.e. systems of the type

$$\widetilde{\mathcal{S}} = \{F_j(y_1, \ldots, y_1^{(m_1-1)}, \ldots, y_p, \ldots, y_p^{(m_p-1)}) = 0\}_{1 \leq j \leq M}, \tag{4.21}$$

where $F_1, \ldots, F_M \in \mathbb{K}[y_1, \ldots, y_p^{(m_p-1)}]$ with $p \geq 1$ and $m_1 > 1, m_2, \ldots, m_p \geq 1$. Let $m = m_1 + \cdots + m_p$ and $\mathbb{V}_{\mathbb{K}}(\widetilde{\mathcal{S}})$ be of dimension one.

Lemma 4.2.10. *For every $\widetilde{\mathcal{S}}$ as in (4.21) we can compute a finite union of regular chains with the same solution vector $(y_1(x), \ldots, y_p(x)) \in (\mathbb{K}\langle\langle x \rangle\rangle \setminus \mathbb{K})^p$.*

Proof. Let us choose the term ordering

$$y_1 < \cdots < y_1^{(m_1-1)} < \cdots < y_p < \cdots < y_p^{(m_p-1)}.$$

Then, by Theorem C.3.3, there exists a regular chain decomposition of $\widetilde{\mathcal{S}}$ into finitely many systems, i.e.

$$\mathbb{V}_{\mathbb{K}\langle\langle x \rangle\rangle}(\widetilde{\mathcal{S}}) = \bigcup \mathbb{V}_{\mathbb{K}\langle\langle x \rangle\rangle}(\mathcal{S}) \setminus \mathbb{V}_{\mathbb{K}\langle\langle x \rangle\rangle}(\mathrm{pinit}(\mathcal{S})),$$

where the $\mathcal{S}$ are of the form

$$\begin{cases} G_1(y_1, y_1') = \sum_{j=0}^{r_1} G_{1,j}(y_1) \cdot (y_1')^j = 0 \\ G_2(y_1, y_1', y_1'') = \sum_{j=0}^{r_2} G_{2,j}(y_1, y_1') \cdot (y_1'')^j = 0 \\ \quad \vdots \\ G_{m_1}(y_1, \ldots, y_1^{(m_1-1)}, y_2) = \sum_{j=0}^{r_{m_1}} G_{m_1,j}(y_1, \ldots, y_1^{(m_1-1)}) \cdot y_2^j = 0 \\ \quad \vdots \\ G_{m-1}(y_1, \ldots, y_p^{(m_p-1)}) = \sum_{j=0}^{r_{m-1}} G_{m-1,j}(y_1, \ldots, y_p^{(m_p-2)}) \cdot (y_p^{(m_p-1)})^j = 0 \end{cases} \tag{4.22}$$

with $r_j \geq 1$ and $\mathrm{init}(G_j) = G_{j,r_j} \neq 0$ for every $1 \leq j \leq m$. If one component of a solution vector is constant, we can plug this value into $\mathcal{S}$ and we obtain a new system $\mathcal{S}^*$ involving one variable less. Since $\mathbb{V}_{\mathbb{K}}(\mathcal{S}^*)$ is a projection of $\mathbb{V}_{\mathbb{K}}(\mathcal{S})$ on an affine space of smaller dimension, we get that

$$\dim(\mathbb{V}_{\mathbb{K}}(\mathcal{S}^*)) \leq \dim(\mathbb{V}_{\mathbb{K}}(\mathcal{S})).$$

So, the system $\mathcal{S}^*$ is of dimension one or zero. Since there are only finitely many variables, this splitting into new systems terminates and if there is only one variable remaining, we can apply Lemma 4.2.1. Hence, by applying these splittings, we can make the following assumptions for finding solutions where not all components are constant.

- The system $\mathcal{S}$ does not start with an algebraic equation in y_1 as it is already written in (4.22).

- The differential polynomials $G_1, \ldots, G_m$ do not have a factor $F_i \in \mathbb{K}[y_i]$ for any $1 \leq i \leq p$. Otherwise we replace G_j by G_j/F_i for any $1 \leq j \leq m$ with F_i as factor and additionally consider the system obtained by the constant solution of $F_i(y_i) = 0$.

Now we show that under these constraints on the regular chain decomposition there are no common solution vectors of $\mathcal{S}$ and $\operatorname{pinit}(\mathcal{S}) = 0$. If $y_1(x) \in \mathbb{K}\langle\langle x \rangle\rangle \setminus \mathbb{K}$ is the first component of a solution vector $(y_1(x), \ldots, y_p(x)) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x \rangle\rangle}(\mathcal{S})$, then

$$Y(x) = (y_1(x), \ldots, y_1^{(m_1-1)}(x), \ldots, y_p(x), \ldots, y_p^{(m_p-1)}(x))$$

is a regular zero of $\mathcal{S}$. Hence, by Theorem C.3.2,

$$\operatorname{init}(G_2)(Y(x)), \ldots, \operatorname{init}(G_m)(Y(x)) \neq 0.$$

Since $\operatorname{init}(G_1)(y_1) = 0$ is an algebraic equation in y_1, there can only be common solutions of $\mathcal{S}$ and $\operatorname{pinit}(\mathcal{S})$ where $y_1(x) \in \mathbb{K}$ which we excluded. □

Similarly to the case of one differential indeterminate, we will impose in the further reasoning that in (4.22) the starting equation $G_1 = 0$ and its separant have no common (non-constant) differential solutions by considering $G_1 \in \mathbb{K}[y_1, y_1']$ to be square-free and with no factor in $\mathbb{K}[y_1]$ or $\mathbb{K}[y_1']$.

Let us set $m_0 = 1$. We will consider the subsystems

$$\mathcal{S}_j = \{G_{m_1+\cdots+m_{j-1}} = 0, \ldots, G_{m_1+\cdots+m_j-1} = 0\}$$

involving only $y_1, \ldots, y_j$ and its derivatives. Then we have the disjoint union

$$\mathcal{S} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_p.$$

For system (4.22) we look for formal Puiseux series solutions expanded around zero or around infinity and where each component is of non-negative order. Otherwise we perform for the change of variable $\tilde{y}_i = 1/y_i$ for every $1 \leq i \leq p$ where the order is negative. Let us call the resulting system $\mathcal{S}^*$ and denote the subsystems by $\mathcal{S}_1^*, \ldots, \mathcal{S}_p^*$. Let $\mathbf{\Phi}$ denote the birational mapping defined as Φ, from the subsection devoted to systems in one differential indeterminate, on the components where we want to perform the change of variables, and defined as the identity map on the other components. For the subsystem $\mathcal{S}_1$ we have shown in the previous part of the section that such a change of variable preserves the structure and the algebraic-geometric dimension of $\mathcal{S}_1^*$ is again one. As explained in the proof of Lemma 4.2.10, it can be assumed that $\mathcal{S}_2, \ldots, \mathcal{S}_p$ do not have any $y_1, \ldots, y_p$ as factor. Then $\mathbb{V}_{\mathbb{K}}(\mathcal{S}_i^*)$ is the Zariski closure of $\mathbf{\Phi}(\mathbb{V}_{\mathbb{K}}(\mathcal{S}_i))$ and using that $\mathbf{\Phi}$ is birational (see subsection on systems in one indeterminate)

$$\dim(\mathbb{V}_{\mathbb{K}}(\mathcal{S}_i^*)) = \dim(\mathbb{V}_{\mathbb{K}}(\mathcal{S}_i))$$

follows for every $1 \leq i \leq p$.

Theorem 4.2.11. *Let $\mathbb{K} = \mathbb{C}$. Then all components of formal Puiseux series solution vectors of system (4.22), expanded around a finite point or at infinity, are convergent.*

Proof. First we consider in (4.22) the subsystem

$$\mathcal{S}_1 = \{G_1(y_1, y_1') = \cdots = G_{m_1-1}(y_1, \ldots, y_1^{(m_1-1)}) = 0\}$$

involving only the differential indeterminate y_1. Since $\mathcal{S}_1$ is the first part of a regular chain and has dimension one, we can use Theorem 4.2.4 for $\mathcal{S}_1$ and hence, the first component $y_1(x)$ is convergent.

Let us now evaluate the equations in (4.22) at $y_1(x)$ and its derivatives and consider equation

$$P_2(x, y_2) = G_{m_1}(y_1(x), \ldots, y_1^{(m_1-1)}(x), y_2) = 0$$

with $P_2 \in \mathbb{K}\langle\langle x\rangle\rangle[y_2]$ and the coefficients of P_2 are convergent Puiseux series in x. Since (4.22) is a regular chain and non-constant solutions define a regular zero, by Theorem C.3.2,

$$\text{init}(G_{m_1})(y_1(x), \ldots, y_1^{(m_1-1)}(x)) = G_{m_1,r_{m_1}}(y_1(x), \ldots, y_1^{(m_1-1)}(x) \neq 0$$

and therefore, P_2 is non-trivial in y_2. Then, by Puiseux's Theorem, all solutions $y_2(x) \in \mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(P_2)$ are convergent.

Then we can evaluate (4.22) also at $y_2(x)$ and its derivatives to obtain an equation

$$P_3(x, y_3) = G_{m_1+m_2}(y_1(x), \ldots, y_1^{(m_1-1)}(x), y_2(x), \ldots, y_2^{(m_2-1)}(x), y_3) = 0$$

with $P_3 \in \mathbb{K}\langle\langle x\rangle\rangle[y_3] \setminus \mathbb{K}\langle\langle x\rangle\rangle$ and the coefficients of P_3 are convergent Puiseux series in x.

We can continue this process up to $G_{m-m_p} = 0$ and its Puiseux expansion for $y_p(x)$ and obtain that all components of the solution vector $(y_1(x), \ldots, y_p(x))$ are convergent Puiseux series. □

In the proof of Theorem 4.2.11 we have constructed polynomials $P_2(x, y_2), \ldots, P_p(x, y_p)$ with formal Puiseux series coefficients. In general these polynomials cannot be expressed in a closed form and therefore the roots can not be computed algorithmically. If $y_1(x)$ is algebraic over $\mathbb{K}(x)$, however, this is possible as we show in the following.

Let us set in the following $K_0 = \mathbb{K}(x)$ and let $y_1(x), y_2(x), \ldots \in \mathbb{K}\langle\langle x\rangle\rangle$ be algebraic over $\mathbb{K}(x)$. For $j \in \mathbb{N}^*$ we iteratively define K_j as the field obtained by adjoining to K_{j-1} the element $y_j(x)$, i.e.

$$K_j = K_{j-1}(y_j(x)).$$

Obviously the relation

$$\mathbb{K}(x) = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_j \subseteq \cdots \subseteq \mathbb{K}\langle\langle x\rangle\rangle$$

holds.

Theorem 4.2.12. *Let $(y_1(x), \ldots, y_p(x)) \in \mathbb{K}\langle\langle x\rangle\rangle^p$ be a solution vector of system (4.22) such that $y_1(x)$ is algebraic over $\mathbb{K}(x)$. Then $y_2(x), \ldots, y_p(x)$ are algebraic over $\mathbb{K}(x)$.*

Proof. Let us use the notation introduced in the proof of Theorem 4.2.11. Let $Q_1(x, y_1) \in \mathbb{K}[x, y_1]$ be the minimal polynomial of $y_1(x) \in \mathbb{K}\langle\langle x\rangle\rangle$. Now $y_2(x)$ is a root of the non-zero polynomial

$$P_2(x, y_2) = G_{m_1}(y_1(x), \dots, y_1^{(m_1-1)}(x), y_2) \in K_1[y_2].$$

By computing the derivative of $Q_1(x, y_1(x)) = 0$ with respect to x, we obtain

$$\frac{\partial Q_1}{\partial y_1}(x, y_1(x)) \cdot y_1'(x) + \frac{\partial Q_1}{\partial x}(x, y_1(x)) = 0. \tag{4.23}$$

Since $\frac{\partial Q_1}{\partial y_1} \in \mathbb{K}[x, y_1]$ is a non-zero polynomial of less degree than the minimal polynomial of $y_1(x)$, it follows that

$$\frac{\partial Q_1}{\partial y_1}(x, y_1(x)) \neq 0.$$

Hence, $y_1'(x)$ is a solution of (4.23) and $y_1'(x) \in K_1$. Iteratively this can be continued in order to see that $y_1^{(2)}(x), \dots, y_1^{(m_1-1)}(x) \in K_1$. Therefore,

$$P_2(x, y_2) = G_{m_1}(y_1(x), \dots, y_1^{(m_1-1)}(x), y_2) \in K_1[y_2]$$

with $P_2(x, y_2(x)) = 0$. As we have seen in the proof of Theorem 4.2.11, P_2 effectively depends on y_2 and therefore, $y_2(x)$ is algebraic over $\mathbb{K}(x)$. By continuing this process iteratively, also the next solution components are algebraic over $\mathbb{K}(x)$ and the statement follows. □

Proposition 4.2.13. Let $(y_1(x), \dots, y_p(x)) \in \mathbb{K}\langle\langle x\rangle\rangle^p$ be a solution vector of system (4.22) such that for every $1 \leq j \leq p$ the component $y_j(x)$ is algebraic over $\mathbb{K}(x)$ with minimal polynomial $Q_j \in \mathbb{K}[x, y_j]$. Then

$$\deg_{y_j}(Q_j) \leq (\deg_{y_1}(G_1) + r_1)\, r_{m_1} \cdots r_{m_1+\dots+m_{j-1}}.$$

Proof. Using Theorem 4.2.6 and the fact that the reduced differential equation of $\mathcal{S}_1$ is a factor of G_1, we have for $Q_1(x, y_1)$ the degree bound

$$\deg_{y_1}(Q_1) \leq \deg_{y_1}(G_1) + r_1.$$

For $1 < j \leq p$ and $K_j = K_{j-1}(y_j(x))$ defined as above, the degree of the field extension fulfills

$$[K_j : K_{j-1}] \leq r_{m_1+\dots+m_{j-1}}.$$

Applying the multiplicative formula for the degree to the tower of field extensions we obtain

$$[K_j : \mathbb{K}(x)] = \deg_{y_j}(Q_j) \leq (\deg_{y_1}(G_1) + r_1)\, r_{m_1} \cdots r_{m_1+\dots+m_{j-1}}.$$

□

In order to construct the algebraic solutions of system (4.21) by an ansatz of unknown coefficients, we would also need a bound of the minimal polynomials with respect to x. Unfortunately we were not able construct such a bound in terms of the data directly obtained from the system (4.22).
Additionally we have the following problem from an algorithmic point of view. We have assumed that $(y_1(x), \dots, y_p(x))$ is indeed a solution vector of the original system $\mathcal{S}$, but this is unknown in advance. In the following we present a solution of this problem which leads to Algorithm 8.

Based on Theorem 4.2.12, let us assume that $y_1(x), \dots, y_{j-1}(x) \in \mathbb{K}\langle\langle x \rangle\rangle$ are algebraic with minimal polynomials

$$Q_1(x, y_1) \in \mathbb{K}[x, y_1], \dots, Q_{j-1}(x, y_{j-1}) \in \mathbb{K}[x, y_{j-1}]$$

and let us define

$$P_j(x, y_j) = \operatorname{prem}(G_{m_1+\dots+m_{j-1}}, \{Q_1, \dots, Q_{j-1}\})(y_1(x), \dots, y_{j-1}(x), y_j) \tag{4.24}$$

and $\mathscr{Q}_j$ as the set of irreducible factors of P_j in $K_{j-1}[y_j]$. We call the elements in $\mathscr{Q}_j$ the *solution candidates* of $\mathcal{S}_j$.

Theorem 4.2.14. *Let $(y_1(x), \dots, y_{j-1}(x)) \in \mathbb{K}\langle\langle x \rangle\rangle^{j-1}$ be a solution vector of the subsystem $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{j-1}$ of (4.21) such that all components are algebraic over $\mathbb{K}(x)$ with minimal polynomials $Q_1 \in \mathbb{K}[x, y_1], \dots, Q_{j-1} \in \mathbb{K}[x, y_{j-1}]$ and let $\widehat{Q}_j \in \mathscr{Q}_j$ be a solution candidate of $\mathcal{S}_j$ with $y_j(x)$ as a root. Let us define for $0 \leq i < m_j$*

$$T_{j,i} = \operatorname{prem}(G_{m_1+\dots+m_{j-1}+i}, \{Q_1, \dots, Q_{j-1}\}) \in \mathbb{K}[x, y_1, \dots, y_{j-1}, y_j, \dots, y_j^{(i)}]$$

and

$$H_{j,i} = \operatorname{prem}(T_{j,i}(x, y_1(x), \dots, y_{j-1}(x), y_j, \dots, y_j^{(i)}), \{\widehat{Q}_j\}) \in K_{j-1}[y_j].$$

Then $(y_1(x), \dots, y_j(x))$ is a solution vector of $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_j$ if and only if for all $0 \leq i < m_j$ it holds that

$$H_{j,i}(x, y_j) = 0.$$

Proof. Let us recall that

$$T_{j,i} = \operatorname{init}(Q_1)^{d_1} \cdots \operatorname{init}(Q_{j-1})^{d_{j-1}}\, S_{Q_1}^{e_1} \cdots S_{Q_{j-1}}^{e_{j-1}}\, G_{m_1+\dots+m_{j-1}+i} - \sum_{\substack{k \geq 0 \\ 0 \leq \ell < j}} R_{\ell,k}\, Q_\ell^{(k)}$$

for some $d_1, \dots, d_{j-1}, e_1, \dots, e_{j-1} \in \mathbb{N}$ and $R_{\ell,k} \in \mathbb{K}[x]\{y_1, \dots, y_{j-1}\}$. Similarly,

$$H_{j,i} = \operatorname{init}(\widehat{Q}_j)^{d_j}\, S_{\widehat{Q}_j}^{e_j}\, T_{j,i}(x, y_1(x), \dots, y_{j-1}(x), y_j, \dots, y_j^{(i)}) - \sum_{k \geq 0} R_{j,k}\, \widehat{Q}_j^{(k)}$$

for some $d_j, e_j \in \mathbb{N}$, $R_{j,k} \in K_{j-1}\{y_j\}$ and

$$\operatorname{ord}_{y_j}(H_{j,i}) < \operatorname{ord}_{y_j}(\widehat{Q}_j).$$

Assume that $(y_1(x), \dots, y_j(x))$ is a solution of $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_j$. Then

$$T_{j,i}(x, y_1(x), \dots, y_j(x), \dots, y_j^{(i)}(x) = 0$$

and $H_{j,i}(x, y_j(x)) = 0$ for every $0 \le i < m_j$. Since $\operatorname{ord}_{y_j}(H_{j,i}) < \operatorname{ord}_{y_j}(\widehat{Q}_j)$ and $\widehat{Q}_j$ is the minimal polynomial of y_j, it follows that $H_{j,i}(x, y_j) = 0$.
For the converse direction, let $H_{j,i}(x, y_j) = 0$ for every $0 \le i < m_j$. Then also $H_{j,i}(x, y_j(x)) = 0$ Taking into account that the initials and separants of the minimal polynomials and the irreducible polynomial $\widehat{Q}_j$, respectively, are non-zero after evaluation, we obtain that

$$G_{m_1+\dots+m_{j-1}+i}(x, y_1(x), \dots, y_j^{(i)}(x)) = 0.$$

Consequently, $(y_1(x), \dots, y_j(x))$ is a solution of $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_j$. □

We will avoid the construction of $P_j(x, y_j), H_{j,i} \in K_{j-1}[y_j]$ by using the theory on primitive elements and the arithmetic performance, see [Win12, YNT89]. By the theory on primitive elements there exists $\gamma_{j-1}(x) \in \mathbb{K}\langle\langle x \rangle\rangle$, algebraic over $\mathbb{K}(x)$, such that

$$K_{j-1} = \mathbb{K}(x)(\gamma_{j-1}(x)).$$

Let $R_{j-1} \in \mathbb{K}[x, y]$ be the minimal polynomial of γ_{j-1}. Then the field K_{j-1} is (differentially) isomorphic to

$$\mathscr{K}_{j-1} = \mathbb{K}(x)[y] \Big/ R_{j-1}.$$

Let $\widetilde{P}_j$ be the corresponding polynomial of P_j in $\mathscr{K}_{j-1}[y_j]$. In the following we perform computations in $\mathscr{K}_{j-1}[y_j]$ and hence, we will work with $\widetilde{P}_j$ instead of P_j. In particular, the set of solution candidates can be determined by factoring $\widetilde{P}_j$, which is described more detailed in [Lan85]. In addition, the pseudo remainders can be computed in $\mathscr{K}_{j-1}[y_j]$ in order to obtain $\widetilde{H}_{j,i}$ corresponding to $H_{j,i}$. Observe that in order to check whether a solution candidate is indeed a solution component, one has to check whether $H_{j,i}(x, y_j) = 0$ (see Theorem 4.2.14). However,

$$H_{j,i}(x, y_j) = 0 \ \text{ if and only if } \ \widetilde{H}_{j,i} = 0$$

and this last zero test is an ideal membership problem.
Let us note that every variable $y_1, \dots, y_j$ occurs in the set $\{Q_1, \dots, Q_{j-1}, \widehat{Q}_j\}$ exactly once. This property allows to combine alternately the computation of the pseudo-remainder with respect to one minimal polynomial and the formal evaluation at the solution component. More precisely, the polynomial $\widetilde{H}_{j,i} \in \mathscr{K}_{j-1}[y_j]$ is equal to

$$\operatorname{prem}(\operatorname{prem}(\cdots \operatorname{prem}(G_{m_1+\dots+m_{j-1}}, \{\widetilde{Q}_1\})(y_1(x)), \dots, \{\widetilde{Q}_{j-1}\})(y_{j-1}(x)), \{\widehat{Q}_j\}),$$

where $\widetilde{Q}_1 \in \mathscr{K}_0[y_1], \dots, \widetilde{Q}_{j-1} \in \mathscr{K}_{j-2}[y_{j-1}]$ are the corresponding minimal polynomials of $y_1(x), \dots, y_{j-1}(x)$. This enables to iteratively compute the solution candidates and check whether they indeed define a solution component.

The next algorithm decides whether a system $\widetilde{\mathcal{S}}$ as in (4.21) has an algebraic solution vector and describes all of them in the affirmative case. As described above, we perform computations in $\mathscr{K}_{j-1}[y_j]$ instead of $K_{j-1}[y_j]$. The correctness of the algorithm is based on Theorem 4.2.12, where it is shown that if the subsystem $\mathcal{S}_1$ has an algebraic solution, then all other solution components are algebraic, and on Theorem 4.2.14, where it is shown how to check whether solution candidates are indeed solutions.

Algorithm 8 AlgSolutionSystemSeveral

Input: A finite system $\widetilde{\mathcal{S}}$ as in (4.21).

Output: A set Σ of algebraic solution vectors of $\widetilde{\mathcal{S}}$ represented by their minimal polynomials or the emptyset such that system $\widetilde{\mathcal{S}}$ has an algebraic solution vector if and only if $\Sigma \neq \emptyset$.

Compute a regular chain decomposition of $\widetilde{\mathcal{S}}$.
for every regular chain $\mathcal{S}$ in the decomposition **do**
  Apply Algorithm AlgSolutionSystem to the subsystem $\mathcal{S}_1$. Let Σ consist of p-tuples where the first components are the minimal polynomials $\widetilde{Q}_1$ corresponding to the non-constant solutions of $\mathcal{S}_1$.
  for every constant solution $y_1(x)$ of $\mathcal{S}_1$ **do**
    Plug $y_1(x)$ into $\mathcal{S}$ and restart the algorithm for the resulting system.
  end for
  for $2 \leq j \leq p$ and a solution $(\widetilde{Q}_1, \ldots, \widetilde{Q}_{j-1})$ in Σ **do**
    Compute the set of solution candidates $\mathscr{Q}_j$.
    for every $\widehat{Q}_j \in \mathscr{Q}_j$ **do**
      Compute $\widetilde{H}_{j,i} \in \mathscr{K}_{j-1}[y_j]$ corresponding to $H_{j,i}$ as defined in Theorem 4.2.14.
      if all $\widetilde{H}_{j,i}$ are zero **then**
        Add $(\widetilde{Q}_1, \ldots, \widetilde{Q}_{j-1}, \widehat{Q}_j)$ to Σ.
      end if
    end for
    if none of the solution candidates provide a solution component **then**
      $(\widetilde{Q}_1, \ldots, \widetilde{Q}_{j-1})$ does not extend to a solution vector of $\mathcal{S}$.
    end if
  end for
end for
for every $\emptyset \neq I \subseteq \{1, \ldots, p\}$ **do**
  Apply for $i \in I$ the change of variables $\tilde{y}_i = 1/y_i$ and let $\hat{\mathcal{S}}$ be the resulting system.
  Perform all the previous steps to $\hat{\mathcal{S}}$; in this way one obtains the outputs corresponding to solutions where y_i is of negative order for every $i \in I$.
end for
return Σ.

Let us illustrate in the following example the previous ideas and results.

Example 4.2.15. Let us consider the system of differential equations given by

$$\widetilde{\mathcal{S}} = \begin{cases} yy'y'' + y'^3 - yy'' - y'^2 = 0 \\ z^3 - 2y'^2 + yy' - 1 = 0 \\ z^3 + yy'' - y'^2 = 0 \\ 3z^2z' - 4y'y'' = 0 \end{cases} \tag{4.25}$$

Similar to Example 4.2.7, the system $\widetilde{\mathcal{S}}$ has a regular chain decomposition

$$\mathcal{S} = \begin{cases} G_1 = yy' - 1 = 0 \\ G_2 = y'^2 + yy'' = 0 \\ G_3 = z^3 - 2y'^2 + yy' - 1 = 0 \\ G_4 = 3z^2z' - 4y'y'' = 0 \end{cases} \quad \text{and} \quad \begin{cases} y' - 1 = 0 \\ 2 - y + yy'' = 0 \\ z^3 + y - 3 = 0 \\ 3z^2z' - 4y'' = 0 \end{cases}$$

and systems where the first equation depends only on y. The system on the right does not have any solution vector. The system $\mathcal{S}$ does not have a solution vector where the first component is constant. From Example 4.2.8 we know that the subsystem

$$\mathcal{S}_1 = \{G_1 = 0, G_2 = 0\}$$

of $\mathcal{S}$ has the algebraic solutions given by the minimal polynomial

$$Q_1(x, y) = y^2 - 2(x + c),$$

where $c \in \mathbb{K}$.
Let us compute $\widetilde{P}_2(x, y_2)$ as in (4.24) by considering

$$\operatorname{prem}(G_3, \{Q_1\}) = y^2\, z^3 - 2$$

in $\left(\mathbb{K}(x)[y] \big/ Q_1\right)[z]$, namely as

$$\widetilde{P}_2 = (x + c + \langle Q_1\rangle)\, z^3 + (-1 + \langle Q_1\rangle).$$

Since $\widetilde{P}_2$ is already irreducible, the set of solution candidates is equal to $\{\widetilde{P}_2\}$. Now we check whether $\widetilde{P}_2$ is indeed the minimal polynomial of a solution component. We obtain

$$\begin{aligned} \widetilde{H}_{2,1} &= \operatorname{prem}(\operatorname{prem}(G_4, \{Q_1\}), \{\widetilde{P}_2\}) \\ &= \operatorname{prem}((12(x + c)^2 + \langle Q_1\rangle)\, z^2\, z' + (4 + \langle Q_1\rangle), \{\widetilde{P}_2\}) \\ &= (-12(x + c)^2 + \langle Q_1\rangle)\, z^3 + (12(x + c) + \langle Q_1\rangle) + \langle \widetilde{P}_2\rangle \\ &= 0 + \langle \widetilde{P}_2\rangle. \end{aligned}$$

By construction, $\widetilde{H}_{2,0} = 0$ holds as well and we know from Theorem 4.2.14 that $(Q_1, \widetilde{P}_2)$ indeed defines a solution vector of $\mathcal{S}$.

Alternatively the computations in the previous example could be approached by resultants as we illustrate in the sequel. The problem with this second approach is that in general we cannot ensure that the involved resultants are non-zero.

Example 4.2.16. Let us consider $\mathcal{S}$ from Example 4.2.15. Then

$$P_2(x,z) = \mathrm{Res}_y(\mathrm{prem}(G_3, \{Q_1\}), Q_1) = (2(x+c)\, z^3 - 2)^2.$$

Hence, a possible algebraic Puiseux series solution $z(x)$ is a root of

$$Q_2(x,z) = (x+c)\, z^3 - 1.$$

In order to check whether Q_2 implicitly defines a solution component, let us compute

$$H_{2,1} = \mathrm{prem}(G_4, \{Q_1, Q_2\}) = -4(x+c)\, z^3 + 4$$

and

$$\mathrm{Res}_y(H_{2,1}, Q_1) = (4(x+c)\, z^3 - 4)^2.$$

Since

$$(4(x+c)\, z^3 - 4)^2 \in \langle Q_2 \rangle,$$

the pair of minimal polynomials (Q_1, Q_2) indeed defines a solution vector of $\mathcal{S}_1$.

4.3 Non-consecutive Derivatives

In this section we consider differential equations of the type

$$F(y, y^{(r)}) = 0, \tag{4.26}$$

where $F \in \mathbb{K}[y, y^{(r)}]$ and $r > 1$. As for the case of $r = 1$, see equation (4.1) in Section 4.1, we may assume that F is square-free and has no factor in $\mathbb{K}[y]$ or $\mathbb{K}[y^{(r)}]$. Associated to F we consider the affine plane curve $\mathcal{C}(F) = \{(a, b) \in \mathbb{K}^2 \mid F(a, b) = 0\}$ and its Zariski closure $\mathscr{C}(F)$ in $\mathbb{K}^2_\infty$.

As described in Section 1.3, since F is autonomous, we can assume without loss of generality that the expansion point is either equal to zero or equal to infinity. Let us fix $\mathbf{p}_0 = (y_0, p_0) \in \mathbb{K}^2_\infty$. Then we denote by $\mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ the set of solutions of (4.26) in $\mathbb{K}\langle\langle x\rangle\rangle$ with $y(0) = y_0, y^{(r)}(0) = p_0$ as initial values and $y^{(r)}(x) \neq 0$. Hence, elements in $\mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ cannot be polynomials of degree strictly less than r.
In contrast to the case where $r = 1$, the assumption $\operatorname{ord}_x(y(x)) \geq 0$ would indeed be a restriction, because if $y(x)$ is a solution with negative order, then $1/y(x)$ is in general not a solution of another AODE of the form (4.26).
Moreover, we denote by $\mathbf{IFP}(\mathbf{p}_0)$ the set of all irreducible formal parametrizations of $\mathscr{C}(F)$ centered at $\mathbf{p}_0$ and by $\mathbf{Places}(\mathbf{p}_0)$ its places. Then, we introduce the maps

$$\begin{aligned} \Delta_r : \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0) &\longrightarrow \mathbf{IFP}(\mathbf{p}_0) \\ y(x) &\mapsto \Delta_r(y(x)) = \left(y(t^m), \frac{d^r\, y}{d\, x^r}(t^m) \right), \\ \delta_r : \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0) &\longrightarrow \mathbf{Places}(\mathbf{p}_0) \\ y(x) &\mapsto [\Delta_r(y(x))], \end{aligned}$$

where m is the ramification index of $y(x)$. As in the case of $r = 1$, the map Δ_r is well defined and we call $\mathcal{P} \in \mathbf{Places}(\mathbf{p}_0)$ a *(Puisuex) solution place* of (4.26) if there exists $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ such that $\delta_r(y(x)) = \mathcal{P}$. Similarly, every element $A(t) \in \mathcal{P}$ in the image of δ_r is called a *solution parametrization* of (4.26).

Throughout the whole section we will use for $a(t) \in \mathbb{K}((t))$ and $j \in \mathbb{N}^*$ the index notation a_{k_j} for the j-th non-zero coefficient of order k_j of $a(t)$ not equal to zero, i.e.

$$a(t) = \sum_{j \geq p} a_j t^j = a_0 + a_{k_1} t^{k_1} + a_{k_2} t^{k_2} + \cdots ,$$

where $a_{k_1}, a_{k_2}, \ldots \neq 0,\ k_1, k_2, \ldots \neq 0$ and $-\infty < k_1 < k_2 < \cdots$. If there are finitely many such k_j and k_i is the biggest, then we set $k_{i+j} = \infty$ and $a_{k_{i+j}} = 0$ for every $j \in \mathbb{N}^*$. The relation of the orders we will later need is

$$\begin{aligned} &\operatorname{ord}(a(t)) = p = \min(0, k_1),\ \ \operatorname{ord}(a(t) - a_0) = \min(p, 1) = k_1, \\ &\operatorname{ord}(a(t) - a_0 - a_{k_1} t^{k_1}) = k_2. \end{aligned}$$

Example 4.3.1. In the notation above, for the formal Laurent series

$$a(t) = 2t^{-2} + 1 + 4t^3$$

we have $p = -2 = k_1, k_2 = 3, k_3 = \infty$, whereas for

$$b(t) = 1 + 2t^2 + 4t^3$$

we obtain $p = 0, k_1 = 2, k_2 = 3, k_3 = \infty$.

Equations involving y, y''

In this part of the section we study equation (4.26) with $r = 2$ and formal Puiseux series solution expanded around zero.

Lemma 4.3.2. *Let $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ be of ramification index m, let $(a(t), b(t)) = \Delta_2(y(x))$ and let $a(t) = a_0 + \sum_{j\geq 1} a_{k_j} t^{k_j}$ with $-\infty < k_1 < k_2 < \cdots$ and $k_j, a_{k_j} \neq 0$. Then*

$$t\, a''(t) = m^2\, t^{2m-1}\, b(t) + (m-1)\, a'(t) \tag{4.27}$$

and one of the statements

1. $2m = k_1 - \operatorname{ord}_t(b(t))$ *and* $k_1 \neq m$*; or*
2. $2m = k_2 - \operatorname{ord}_t(b(t)),\ k_1 = m$ *and* $k_2 < \infty$*;*

hold.

Proof. By the chain rule we have $a'(t) = mt^{m-1}y'(t^m)$ and therefore,

$$\begin{aligned} a''(t) &= m^2t^{2m-2}y''(t^m) + m(m-1)t^{m-2}y'(t^m) \\ &= m^2t^{2m-2}b(t) + (m-1)t^{-1}a'(t). \end{aligned}$$

Equation (4.27) then follows by multiplying both sides by t. By expanding expressions we obtain

$$\begin{aligned} t\, a''(t) &= k_1(k_1-1)a_{k_1}t^{k_1-1} + k_2(k_2-1)a_{k_2}t^{k_2-1} + \cdots \\ (m-1)a'(t) &= k_1(m-1)a_{k_1}t^{k_1-1} + k_2(m-1)a_{k_2}t^{k_2-1} + \cdots . \end{aligned}$$

The term $m^2t^{2m-1}b(t)$ has to be of the same order as

$$t\, a''(t) - (m-1)a'(t) = k_1(k_1-m)a_{k_1}t^{k_1-1} + k_2(k_2-m)a_{k_2}t^{k_2-1} + \cdots .$$

There are several cases now:
If $k_1 \neq m$, then $2m + \operatorname{ord}_t(b(t)) = k_1$.
If $k_1 = m$ and $k_2 < \infty$, then $k_2 \neq m$ and therefore, $2m + \operatorname{ord}_t(b(t)) = k_2$.
If $k_1 = m$ and $k_2 = \infty$, then $ta''(t) - (m-1)a'(t)$ is zero and therefore also $b(t) = 0$. This is only possible if $y^{(2)}(x) = 0$. Then $y(x)$ has to be of the form $y_0 + c\,x$ in contradiction to the assumption that $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$. □

Proposition 4.3.3. Let $(a(t), b(t)) \in \mathbf{IFP}(\mathbf{p}_0)$. Then $(a(t), b(t))$ is a solution parametrization if and only if there exists $m \in \mathbb{N}^*$ such that (4.27) holds. In the affirmative case, m is the ramification index of the generating Puiseux series solution. As a consequence, the map Δ_2 is injective.

Proof. The first implication directly follows from Lemma 4.3.2. For the other direction let us now assume that (4.27) holds for an $m \in \mathbb{N}^*$ and write $a(t) = y_0 + \sum_{j\geq p} a_j\, t^j$ with $a_p \neq 0$, and $b(t) = \sum_{j\geq q} b_j\, t^j$. Let us consider $y(x) = y_0 + \sum_{j\geq p} a_j\, x^{j/m}$. By assumption, $y^{(2)}(x) = b(x^{1/m})$ and

$$F(y(x), y^{(2)}(x)) = F(a(x^{1/m}), b(x^{1/m})) = 0.$$

Thus, $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$ and $(a(t), b(t))$ is a solution parametrization.
It remains to show that m is the ramification index of $y(x)$. Otherwise there exists a natural number $n \geq 2$, such that n divides m and satisfying that if $a_j \neq 0$ then n divides j. If we expand equation (4.27) we obtain

$$\sum_{j\geq p} j(j-m)a_j t^{j-1} = \sum_{j\geq q} m^2 b_j t^{2m-1+j}.$$

Hence, $b_{j-2m} \neq 0$ if and only if $j \neq m$ and $a_j \neq 0$. Therefore, $b_j \neq 0$ implies that n divides $j+2m$ and thus, n divides j in contradiction to the irreducibility of $(a(t), b(t))$. Therefore, m is the ramification index of $y(x)$ and $\Delta_2(y(x)) = (a(t), b(t))$. □

In the case of $r = 1$ we were able to show that in every solution place the ramification indexes of the solutions generating the solution parametrizations are the same. For $r > 1$, however, this is still an open problem which we phrase as a conjecture here.

Conjecture 4.3.4. All Puiseux series solutions in $\mathbf{Sol}^*_{\mathbb{K}\langle\langle x\rangle\rangle}(\mathbf{p}_0)$, generating the same solution place in $\mathbf{Places}(\mathbf{p}_0)$, have the same ramification index.

Proposition 4.3.3 characterizes the solution parametrizations. However, in order to detect whether a place $[(a(t), b(t)]$ contains a solution parametrization, one needs to decide whether there exists a reparametrization $s(t) \in \mathbb{K}[[t]]$ of order one such that $(a(s(t)), b(s(t)))$ satisfies equation (4.27). For this purpose, it is enough to consider parametrizations $(a(t), b(t)) \in \mathbf{IFP}(\mathbf{p}_0)$ such that either item 1 or item 2 in Lemma 4.3.2 is fulfilled. Then for every formal power series solution $s(t) \in \mathbb{K}[[t]]$ with $\operatorname{ord}_t(s(t)) = 1$ satisfying the *associated differential equation*

$$\begin{aligned} &t\, a'(s(t))\, s''(t) + t\, a''(s(t))\, s'(t)^2 - (m-1)\, a'(s(t))\, s'(t) \\ &- m\, t^{2m-1}\, b(s(t)) = 0, \end{aligned} \tag{4.28}$$

the formal parametrization $(a(s(t)), b(s(t)))$ is in the image of Δ_2. With the following lemmas we analyze equation (4.28).

Lemma 4.3.5. *Let $\mathbb{L}$ be a subfield of $\mathbb{K}$. Let $(a(t), b(t)) \in \mathbf{IFP}(\mathbf{p}_0)$ with*

$$a(t) = a_0 + \sum_{j\geq 1} a_{k_j}\, t^{k_j},\ \ b(t) = \sum_{j\geq q} b_j\, t^j \in \mathbb{L}((t)),$$

where $-\infty < k_1 < k_2 < \cdots$ and $k_{j+1}, a_{k_j}, b_q \neq 0$, be such that item 1 or 2 in Lemma 4.3.2 holds for an $m \in \mathbb{N}^*$. Let us denote

$$\nu_1 = -k_1 + 1,\ \ \nu_2 = -k_1 + m + 1.$$

Let $\mathcal{S} \subseteq \mathbf{Sol}_{\mathbb{K}[[t]]}$ be the solution set of (4.28) with elements of the form $s(t) = \sum_{i=1}^{\infty} \sigma_i\, t^i$. Then the cardinality of $\mathcal{S}$ fulfills the following.

1. *Let* $2m = k_1 - q$.

   (a) *If* $\nu_2 \leq 0$, *there are exactly* $2m$ *many elements in* $\mathcal{S}$ *with* $\sigma_1^{2m} \in \mathbb{L}$ *and* $\sigma_i \in \mathbb{L}(\sigma_1)$.

   (b) *If* $\nu_1 \leq 0 < \nu_2$, *there are at most* $2m$ *many one-parameter families of elements in* $\mathcal{S}$ *with* $\sigma_1^{2m} \in \mathbb{L}$, $\sigma_2, \ldots, \sigma_{\nu_2 - 1} \in \mathbb{L}(\sigma_1)$, $\sigma_{\nu_2} \in \mathbb{K}$ *is a free parameter and* $\sigma_{\nu_2+1}, \ldots \in \mathbb{L}(\sigma_1, \sigma_{\nu_2})$.

   (c) *If* $0 < \nu_1 < \nu_2$, *there are up to* $2m$ *many two-parameter families of elements in* $\mathcal{S}$ *with* $\sigma_1^{2m} \in \mathbb{L}$, $\sigma_2, \ldots, \sigma_{\nu_1 - 1} \in \mathbb{L}(\sigma_1)$, $\sigma_{\nu_1} \in \mathbb{K}$ *is a free parameter,* $\sigma_{\nu_1+1}, \ldots, \sigma_{\nu_2-1} \in \mathbb{L}(\sigma_1, \sigma_{\nu_1})$, $\sigma_{\nu_2} \in \mathbb{K}$ *is a free parameter and* $\sigma_{\nu_2+1}, \ldots \in \mathbb{L}(\sigma_1, \sigma_{\nu_2})$.

2. *If* $2m = k_2 - q$, $k_1 = m$, $k_2 < \infty$, *there is at most* 1 *one-parameter family of elements in* $\mathcal{S}$ *with* $\sigma_1 \in \mathbb{K}$ *as a free parameter and* $\sigma_i \in \mathbb{L}(\sigma_1)$.

Moreover, if $\mathbb{K} = \mathbb{C}$ *and* $a(t)$, $b(t)$ *are convergent, then all elements in* $\mathcal{S}$ *are convergent.*

Proof. Let $(a(t), b(t))$ and $m \in \mathbb{N}^*$ be such that item 1 or item 2 in Lemma 4.3.2 is fulfilled. We are looking for a solution $s(t) \in \mathbb{K}[[t]]$ with $\mathrm{ord}(s(t)) = 1$ of (4.28) or equivalently of

$$F(t, s, s', s'') = t\, a'(s)\, s'' + t\, a''(s)\, s'^2 - (m-1) a'(s)\, s' - m\, t^{2m-1}\, b(s) = 0. \quad (4.29)$$

The Newton polygon $\mathcal{N}(F)$ corresponding to equation (4.29) contains the two vertices $P_1 = (-1, k_1)$, corresponding to the first and third term, and $P_2 = (2m-1, q)$, corresponding to the fourth term, and points in the lines above P_1 and P_2. The second term contributes to P_1 if and only if $k_1 \neq 1$. In the case of $k_1 = 1$, the contribution is just given by the points above P_1. Note that by assumption $m > 0$ and therefore P_1 always lies above and left of P_2.

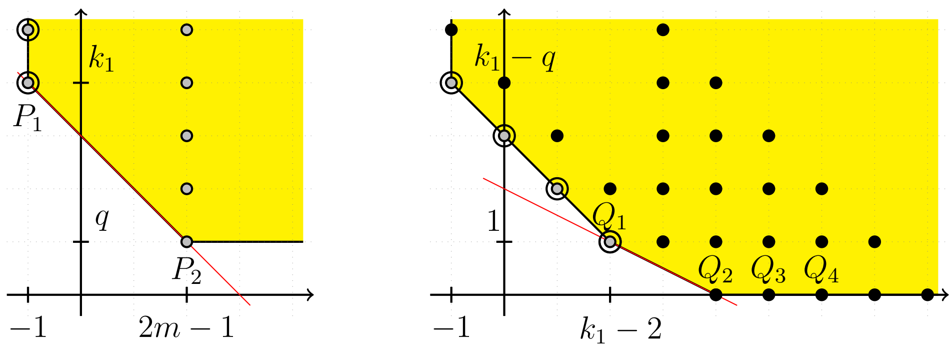

Figure 4.3: The Newton polygon $\mathcal{N}(F)$ (left) and $\mathcal{N}(F_2)$ (right).

Case 1: $\mathbf{2m = k_1 - q}$. The inclination of the side defined by P_1 and P_2 is equal to $\mu = \frac{2m}{k_1 - q} = 1$. The associated characteristic polynomial is equal to

$$\Phi_{(F;1)}(C) = a_{k_1}\, k_1\, (k_1 - m)\, C^{k_1} - m^2\, b_q\, C^q$$

and has the $2m$ distinct roots

$$\sigma_1 = \sqrt[2m]{\frac{m^2\, b_q}{k_1\,(k_1 - m)\, a_{k_1}}}.$$

We perform the change of variable $s(t) = \sigma_1\, t + s_2(t)$ in equation (4.29) to obtain

$$F_2(t, s_2, s_2', s_2'') = F(t, \sigma_1\, t + s_2, \sigma_1 + s_2', s_2'') = 0. \tag{4.30}$$

The Newton polygon $\mathcal{N}(F_2)$ is sketched in the right picture of Figure 4.3. By expanding (4.30), it can be seen that the vertex

$$Q_1 = (k_1 - 2, 1) \in \mathcal{N}(F_2)$$

is guaranteed. The monomials corresponding to Q_1 are $A\, t^{k_1}\, s_2''$, $2A\,(k_1{-}1)\, t^{k_1-1}\, s_2'$, $A\,(k_1-1)(k_1-2)\, t^{k_1-2}\, s_2$, $-A\,(m-1)(k_1-1)\, t^{k_1-1}\, s_2'$ and $-A\,(m-1)\,(k_1-1)\, t^{k_1-2}\, s_2$ with the factor $A = k_1\, a_{k_1}\, \sigma_1^{k_1-1}$. The indicial polynomial corresponding to Q_1 is

$$\Psi_{(F_2;Q_1)}(\mu) = A\,(\mu + k_1 - 1)\,(\mu + k_1 - m - 1)$$

that has the two distinct roots $\nu_1 = -k_1 + 1 < \nu_2 = -k_1 + m + 1$ with $\nu_1, \nu_2 \in \mathbb{Z}$. The possible points of height zero $Q_2, Q_3, \ldots$ in $\mathcal{N}(F_2)$ are of the form

$$Q_j = (k_1 + j - 2, 0).$$

Note that the terms corresponding to a vertex $(k_1 - 1, 0)$ have to cancel out by the choice of σ_1. In a Newton polygon sequence $F_2, F_3, \ldots$ corresponding to a solution $s_2(t)$ the vertices Q_i, with $i \geq j \geq 2$, are also the possible points of height zero in $\mathcal{N}(F_j)$ and the exponents of $s_2(t)$ grow by integers. Hence, $s_2(t)$ is indeed a formal power series and we set $s_2(t) = \sum_{i\geq 2} \sigma_j\, t^j$ and $\mu_j = j$ in the Newton polygon sequence and allow $\sigma_j = 0$. The side $L(F_j; \mu_j = j)$ is the only valid choice and corresponds to the characteristic polynomial

$$\begin{aligned}\Phi_{(F_j;j)}(C) &= \Psi_{(F_j;Q_1)}(j)\, C + \Psi_{(F_j;Q_j)}(j) \\ &= A\,(j + k_1 - 1)\,(j + k_1 - m - 1)\, C + \Psi_{(F_j;Q_j)}(j).\end{aligned} \tag{4.31}$$

Let us note that if Q_j does not exist, we directly obtain for the root σ_j of (4.31) that $\sigma_j = 0$. In the opposite case we have to distinguish among several cases. In the following discussion, when distinguishing cases, we will use the terminology introduced in Chapter 3.

Case 1.1: $\nu_1 < \nu_2 \leq 0$. Since the indicial polynomial corresponding to Q_1 never vanishes, case (Ia) occurs in every step and σ_j are uniquely determined from (4.31) by

$$\sigma_j = \frac{-\Psi_{(F_j;Q_j)}(j)}{A\,(j + k_1 - 1)\,(j + k_1 - m - 1)}. \tag{4.32}$$

Case 1.2: $\nu_1 \leq 0 < \nu_2$. Then the indicial polynomial corresponding to Q_1 has the valid root $\nu_2 = -k_1 + m + 1 \in \mathbb{Q}_{\geq 1}$. For $1 < j < \nu_2$ it holds that $\Psi_{(F_j;Q_1)}(j) \neq 0$

and $\sigma_2, \ldots, \sigma_{\nu_2-1}$ are uniquely determined by (4.32). For the critical value $j = \nu_2$ either the coefficient σ_{ν_2} is a free parameter (case Ib) or no solution exists (case Ic). In the affirmative case, choose $\sigma_{\nu_2} \in \mathbb{K}$ and for $j > \nu_2$ the coefficients σ_j are again uniquely determined by (4.32).

Case 1.3: $\mathbf{0 < \nu_1 < \nu_2}$. Then the indicial polynomial corresponding to Q_1 has the valid roots $\nu_1 = -k_1+1,\ \nu_2 = -k_1+m+1 \in \mathbb{Q}_{\geq 1}$. Similarly as before, for $1 < j < \nu_1$ the coefficients σ_j are uniquely determined by 4.32. For the critical value $j = \nu_1$ either case (Ib) or case (Ic) occurs. In the first case let us choose a $\sigma_{\nu_1} \in \mathbb{K}$. Then for $\nu_1 < j < \nu_2$ the coefficients σ_j are again uniquely determined. For the second critical value $j = \nu_2$ either case (Ib) or case (Ic) occurs and in the first case, after choosing $\sigma_{\nu_2} \in \mathbb{K}$, the following coefficients $\sigma_{\nu_2+1}, \ldots$ are uniquely determined by (4.32).

Case 2: $\mathbf{2m = k_2 - q}$. The inclination of the side defined by $P_1 = (-1, k_1)$ and $P_2 = (2m-1, q)$ is equal to $\mu = \frac{k_2-q}{k_1-q} > 1$. Hence we have to consider only the vertex P_1 and its associated indicial polynomial

$$\Psi_{(F;P_1)}(\mu) = a_{k_1}\, k_1\, mu\, (k_1\, \mu - m),$$

which has $\mu_1 = m/k_1 = 1$ as a root. Thus, σ_1 can be chosen arbitrarily in $\mathbb{K}$.
Let us perform the change of variables $s(t) = \sigma_1\, t + s_2$ in $F = 0$ to obtain the differential equation (4.30) having the same Newton polygon as depicted in Figure 4.3. Using the notation introduced above, the indicial polynomial $\Psi_{(F_2;Q_1)}(\mu)$ with $Q_1 = (k_1-2, 1)$ has the non-valid roots $\nu_1 = -k_1+1 = 1-m < \nu_2 = -k_1+m+1 = 1$. Hence, the characteristic polynomial corresponding to the only valid side containing Q_1 and possibly points of height zero $Q_j = (k_1+j-2, 0)$ is equal to (4.31), which has a unique root σ_j for every $j > 1$.

In all cases we have the following: The indicial polynomial $\Psi_{(F_j;Q_j)}(j)$ is a polynomial expression in the coefficients of $a(t)$ and $b(t)$. Hence, it follows that $\sigma_j \in \mathbb{L}(\sigma_1)$ and in Case 1.2 and $j > \nu_2$ or Case 1.3 and $\nu_1 < j < \nu_2$ that $\sigma_j \in \mathbb{L}(\sigma_1, \sigma_{\nu_2})$ and in Case 1.3 and $j > \nu_2$ that $\sigma_j \in \mathbb{L}(\sigma_1, \sigma_{\nu_1}, \sigma_{\nu_2})$.
Moreover, there is a monomial corresponding to the pivot point Q_1 containing a factor of differential order two, the highest occurring derivative. Hence, in the case of $\mathbb{K} = \mathbb{C}$ and convergent $a(t)$ and $b(t)$, convergence of $s_2(t)$ and therefore also of $s(t)$ follow from Theorem 2 in [Can93b]. □

Theorem 4.3.6. *Let $\mathbb{K} = \mathbb{C}$ and $r = 2$. Then every formal Puiseux series solution of* (4.26)*, expanded around a finite point, is convergent.*

Proof. For linear solutions the statement trivially holds. Let $y(x) \in \mathbf{Sol}^*_{\mathbb{K}\langle\langle x \rangle\rangle}(\mathbf{p}_0)$ have the ramification index $m \in \mathbb{N}^*$ and let $\Delta(y(x)) = (a(t), b(t))$. By Lemma 4.3.2, equations (4.3) and (4.4) hold. Let $p = \text{ord}(a(t) - y_0) \neq 0$. There exists a reparametrization $s(t) \in \mathbb{C}[[t]]$, with $\text{ord}(s(t)) = 1$, to bring $a(t)$ into the form

$$a(s(t)) - y_0 = t^p.$$

Note that in general $a(s(t))$ is not a classical Puisuex parametrization, since p can be negative. Let $\bar{a}(t) = a(s(t))$ and $\bar{b}(t) = b(s(t))$. Then $(\bar{a}(t) - y_0, \bar{b}(t)) = (t^p, \bar{b}(t))$ is a local parametrization of the algebraic curve defined by $F(y - y_0, p)$. If $p > 0$, by Puiseux's theorem, $\bar{b}(t)$ is convergent. If $p < 0$, we can substitute t by t^{-1} and obtain that $(t^{|p|}, \bar{b}(t^{-1}))$ is a classical Puiseux parametrization and again by Puiseux's theorem, $\bar{b}(t^{-1})$ is convergent. Then $\bar{b}(t)$ is convergent as well.
Let $r(t)$ be the compositional inverse of $s(t)$, i.e. $r(s(t)) = t = s(r(t))$. Then $r(t)$ is a formal power series of order one and $a(t) = \bar{a}(r(t)), b(t) = \bar{b}(r(t))$. Since equation (4.28) holds for $(\bar{a}(t), \bar{b}(t))$ and $r(t)$, by Lemma 4.3.5, $r(t)$ is convergent. This implies that $a(t)$ is convergent and therefore, $y(x) = a(x^{1/m})$ is convergent as a Puiseux series. □

Let us illustrate the observations from above in the following example.

Example 4.3.7. Let us consider

$$F(y, y'') = y''^3 - y = 0.$$

For the initial tuple $\mathbf{p}_0 = (0, 0) \in \mathscr{C}(F)$ we obtain the local parametrization $(a(t), b(t)) = (t^3, t)$, where item 1 in Lemma 4.3.2 is fulfilled with $m = 1$. Then the associated differential equation is

$$3s(t)\, s''(t) + 6s'(t)^2 - 1 = 0$$

having the solutions $s_1(t) = t/\sqrt{6}$ and $s_2(t) = -t/\sqrt{6}$. The uniqueness of this solution follows from Lemma 4.3.5 and the fact that $\nu_1 = -2,\ \nu_2 = -1 < 1$. Then we obtain

$$a(s_1(x)) = \frac{x^3}{6\sqrt{6}},\ a(s_2(x)) = \frac{-x^3}{6\sqrt{6}}$$

as the only non-linear solution with $\mathbf{p}_0$ as initial tuple. Hence,

$$\mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F; \mathbf{p}_0) = \{0, \pm x^3/(6\sqrt{6})\}.$$

For the initial tuple $(1, 1) \in \mathscr{C}(F)$ we obtain the local parametrization $((t+1)^3, t+1)$. Here item 2 in Lemma 4.3.2 is fulfilled with $m = 1$. The associated differential equation is

$$3(s(t) + 1)^2\, s''(t) + 6(s(t) + 1)\, s'(t)^2 - s(t) = 0.$$

From the Newton polygon method for differential equations or alternatively by the direct approach we obtain the solution family

$$s(t) = c\,t - c\,t^2 + \frac{c\,(13 + 6c)\,t^3}{18} + \frac{c\,(-7 - 13c - 3c^2)\,t^4}{18} + \mathcal{O}(t^5)$$

and therefore,

$$a(s(t)) = 1 + 3c\,x + 3c\,(c-1)\,x^2 + c\,(c^2 - 5c + 13/6)\,x^3 - c\,(9c^2 - 31c + 7)\,x^4/6 + \mathcal{O}(x^5).$$

Then $\mathbf{Sol}_{\mathbb{K}\langle\langle x\rangle\rangle}(F; (1, 1)) = \{a(s(t)) \mid c \in \mathbb{C}\}$.

Note that the specific form of equation (4.26) involving y and only one derivative, namely $y^{(r)}$, is crucial for showing for instance convergence of the solutions. In Example 4.2.9 we have already seen that equations involving intermediate derivatives in general have non-convergent solutions as well.

For the case of $F \in \mathbb{K}[y, y^{(3)}]$ it is possible to show a similar result as that in Lemma 4.3.2. Moreover, for $\mathbb{K} = \mathbb{C}$, all formal Puiseux series solutions expanded around a finite point are again convergent. Since the proof is very long and tedious, we do not include it in here. The statement for $r > 3$ remains as an open problem and is stated as a conjecture here.

Conjecture 4.3.8. Let $\mathbb{K} = \mathbb{C}$ and $r \geq 4$. Then every formal Puiseux series solution of (4.26), expanded around a finite point, is convergent.

Chapter 5

Conclusion

In this thesis several procedures to compute all formal power series, formal Puiseux series or algebraic solutions of different kind of (systems of) differential equations are presented. The underlying theory, in particular on existence, uniqueness, convergence and computation of the series solutions, is of the essence of this work. For this purpose, we study three different approaches: the direct approach by comparison of coefficients, the Newton polygon method for differential equations and the algebro-geometric approach.

For the first two approaches we were able to prove existence and uniqueness for certain families of differential equations. In the algebro-geometric approach we relate, by using tools from algebraic geometry, the original differential problem to another one where the direct approach and the Newton polygon method for differential equations can be applied. In this way we were able to show existence and uniqueness of solutions of autonomous first-order AODEs (Theorem 4.1.8) and their convergence (Theorem 4.1.9). The unique description of the set of all formal Puiseux series solutions expanded around a finite point by the set of their truncations is proven in Theorem 4.1.14 and presented in Algorithm 3. In the case of solutions expanded around infinity (Algorithm 4) we are able to describe all solutions, but the correspondence to their truncations is not necessarily unique.
The result on convergence gets generalized in Theorem 4.2.11 to systems of dimension one. In the case of existence and uniqueness we are limited to one differential indeterminate (Theorem 4.2.2) or algebraic solutions (Proposition 4.2.14). The computations of the solutions are illustrated in Algorithm 6 and Algorithm 8, respectively.
The convergence of solutions expanded around a finite point, in the case where the second derivative instead of the first derivative appears, is shown in Theorem 4.3.6.

Throughout the thesis we already present open problems and conjectures. Let us briefly recall them here and give a short description of current investigations and plans for future research.
For the direct approach we are working on the following tasks.

- The jet space is in general not an algebraic set. We still expect that; by considering more equations and projecting, which means reducing the jet space; there can be given an (effective) bound on the number of derivatives of the defining differential polynomial has to be computed until the jet space stabilizes, see Cojecture 2.1.5.

- In the direct approach we classify the initial tuples into sets where existence and uniqueness can be guaranteed after a certain number of computational steps. A different classification would be into sets where the corresponding solutions are convergent in a certain neighborhood or nowhere convergent.
- We expect that a similar classification on initial tuples regarding existence and uniqueness of solutions might be possible for algebraic partial differential equations and systems of AODEs.

Although the Newton polygon method for differential equations has been investigated extensively, there is still a big variety of open questions and generalizations to reach.

- By using the computational bounds in the algebro-geometric approach, we expect to find similar computational bounds for directly applying the Newton polygon method for differential equations to the same kind of equations. Since there is in principle not a big difference between using this method for autonomous and non-autonomous equations, these bounds may hold in the non-autonomous case as well.
- For showing convergence of solutions of differential equations with non-consecutive derivatives (4.26), see Conjecture 4.3.8, the main problem lies in constructing the associated differential equation and solving it symbolically. The natural choice for deriving the solutions is the Newton polygon method for differential equations. Since the description of the method itself is getting more and more complicated for bigger derivatives, we aim for a version of the method were symbolical treatment of these equations get simplified.
- In [ACJ03, AILdM10] the Newton polygon method gets extended to the case of partial differential equations and systems of algebraic equations, respectively. We are currently trying to combine the techniques used in there to develop a Newton polygon method for systems of AODEs and systems of partial differential equations.

By using the local version of the algebro-geometric approach introduced in this thesis and the underlying papers, we were able to handle differential problems which algebraic counterpart defines an algebraic curve. But in the global version there are treated more general cases.

- We want to develop the ideas introduced in [GLSW16, VGW18] for our local version of the algebro-geometric approach.
- Algorithm 7 and 8 are using complete polynomial factorization. We expect that this can be avoided and only assumptions as in Section 4.1 are necessary.

Beside the points already mentioned, we think that the relation between the methods used in this thesis are by far not well understood. In particular, we expect that computational bounds from one method can be related to the computational bounds of the other methods.

Appendix A

Algebraic Structures

In this chapter let us introduce some algebraic structures which are commonly used in this thesis such as the ring of formal power series and the field of Puiseux series. These structures are well known and presented for example in [Wal50]. Throughout the chapter let $\mathbb{K}$ be an algebraically closed field of characteristic zero.

Formal Power Series

Let $x_0 \in \mathbb{K}$. Then we use the notation $\mathbb{K}[[x - x_0]]$ for the ring of *formal power series expanded around* x_0, i.e. $\mathbb{K}[[x - x_0]]$ consists of elements of the form

$$\varphi(x) = \sum_{i \geq 0} c_i \, (x - x_0)^i$$

with $c_i \in \mathbb{K}$. Formal power series can be seen as a direct generalization of polynomials by allowing infinitely many terms.
The *order* of $\varphi \in \mathbb{K}[[x - x_0]]$ is defined as the minimal $i \in \mathbb{N}$ such that $c_i \neq 0$ and equal to ∞ for $\varphi = 0$. Additionally to defining formal power series expanded around finite points $x_0 \in \mathbb{K}$, we also define the ring of *formal power series expanded around infinity* consisting of elements of the form

$$\varphi(x) = \sum_{i \leq 0} c_i \, x^i$$

and denoted by $\mathbb{K}[[x^{-1}]]$. For $\varphi \in \mathbb{K}[[x^{-1}]]$ we define the *order* as the maximal $i \in \mathbb{N}$ with $c_i \neq 0$ and the *leading coefficient* as c_i. We will use the notation $\mathrm{ord}_{x-x_0}(\varphi(x))$ and $\mathrm{ord}_{x^{-1}}(\varphi(x))$, respectively, and omit the index if it is clear from the context. The leading coefficient of $\varphi(x)$ is denoted by $\mathrm{lc}(\varphi(x))$.

A formal power series $\varphi(x) \in \mathbb{K}[[x - x_0]]$ expanded around $x_0 \in \mathbb{K}$ can be transformed by the following substitution into a formal power series expanded around zero

$$\tilde{\varphi}(x) = \varphi(x + x_0) \in \mathbb{K}[[x]].$$

Similarly, for $\varphi(x) \in \mathbb{K}[[x^{-1}]]$ the substitution $\tilde{\varphi}(x) = \varphi(1/x)$ can be done to obtain a formal power series expanded around zero. We emphasize that this does not mean that $\varphi(x)$ can be expanded as formal power series around any $x_0 \in \mathbb{K}$. In this thesis we mainly focus on formal power series expanded around $x_0 = 0$ or around infinity.

Example A.0.1. The square root $\sqrt{x}$ can be expanded as formal power series around $x_0 = 1$ by

$$\sqrt{x} = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \cdots,$$

but there is no formal power series expansion of $\sqrt{x}$ around $x_0 = 0$. However, by the substitution described above,

$$\sqrt{x+1} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \in \mathbb{K}[[x]].$$

The elements in $\mathbb{K}[[x]]$ and $\mathbb{K}[[x^{-1}]]$ of order zero are invertible with respect to the multiplication and there are no zero divisors. Let $f, g \in \mathbb{K}[[x]]$ and $\mathrm{ord}(g) > 0$. Then the composition $f(g) \in \mathbb{K}[[x]]$ is meaningful.[1] Moreover, there exists a composition inverse to f if and only if the order of f is equal to one. Here the identity is the formal power series $x \in \mathbb{K}[[x]]$.

Formal Puiseux Series

Let $x_0 \in \mathbb{K}$. Since the ring of formal power series $\mathbb{K}[[x - x_0]]$ is an integral domain, it is possible to consider its fraction field. We call it the field of *formal Laurent series expanded around* x_0 and denote it by $\mathbb{K}((x - x_0))$. Similarly, $\mathbb{K}((x^{-1}))$ denotes the fraction field of $\mathbb{K}[[x^{-1}]]$ and is called the field of *formal Laurent series expanded around infinity*. It turns out that every non-zero formal Laurent series $\varphi(x)$ can be written as a formal power series plus finitely many terms with negative integer exponents or finitely many terms with positive integer exponents in the case of series expanded around infinity. To be precise, we can write

$$\varphi(x) = \sum_{i \geq i_0} c_i \, (x - x_0)^i \quad \text{or} \quad \varphi(x) = \sum_{i \leq i_0} c_i \, x^i,$$

respectively, with $c_i \in \mathbb{K}$ and $i_0 \in \mathbb{Z}$ such that $c_{i_0} \neq 0$. The *order* of φ is defined as $\mathrm{ord}(\varphi(x)) = i_0$ and the *leading coefficient* as c_{i_0}.

From the field of formal Laurent series it is possible to construct an algebraically closed field as we will show in Section A.1.[2] Let us define for $x_0 \in \mathbb{K}$ the field of *formal Puiseux series* expanded around x_0 or infinity as the set of consisting formal Laurent series with fractional exponents with bounded denominator and denote it by $\mathbb{K}\langle\langle x - x_0 \rangle\rangle$ or $\mathbb{K}\langle\langle x^{-1} \rangle\rangle$, respectively. In other words,

$$\mathbb{K}\langle\langle x - x_0 \rangle\rangle = \bigcup_{n \geq 1} \mathbb{K}(((x - x_0)^{1/n})) \quad \text{and} \quad \mathbb{K}\langle\langle x^{-1} \rangle\rangle = \bigcup_{n \geq 1} \mathbb{K}(((x^{-1})^{1/n})).$$

Every non-zero $\varphi(x) \in \mathbb{K}\langle\langle x - x_0 \rangle\rangle$ can be written as

$$\varphi(x) = \sum_{i \geq i_0} c_i \, (x - x_0)^{i/n} \quad \text{or} \quad \varphi(x) = \sum_{i \leq i_0} c_i \, x^{i/n},$$

[1] For example the substitution $g = x - 1$ into $f = \sqrt{x+1} \in \mathbb{K}[[x]]$ from Example A.0.1 would not result in a formal power series expanded around zero.

[2] $\mathbb{K}((x - x_0))$ is not algebraically closed since for example $\sqrt{x} \notin \mathbb{K}((x))$ which is a root of $y^2 - x \in \mathbb{K}((x))[y]$.

respectively, with $c_i \in \mathbb{K}, i_0 \in \mathbb{Z}$ and $n \in \mathbb{N}^*$ such that $c_{i_0} \neq 0$. The *order* and the *leading coefficient* of φ are defined as $\operatorname{ord}(\varphi(x)) = i_0/n$ and $\operatorname{lc}(\varphi(x)) = c_{i_0}$. The minimal natural number n such that $\varphi(x)$ belongs to $\mathbb{K}((x-x_0)^{1/n}))$ or $\mathbb{K}(((x^{-1})^{1/n}))$, respectively, is called the *ramification index* of $\varphi(x)$.

Example A.0.2. The square root $\sqrt{x} = x^{1/2}$ is an element of $\mathbb{C}\langle\langle x \rangle\rangle$ with $\operatorname{ord}(\sqrt{x}) = 1/2$ and ramification index equal to 2.

For $x_0 \in \mathbb{K}$, similar to what we have seen for formal power series, we can perform the change of variables $\tilde{x} = x - x_0$ or $\tilde{x} = 1/x$, respectively, in order to obtain formal Puiseux series expanded around zero. The composition of formal Puiseux series f and g is well defined and again a formal Puiseux series as long as $\operatorname{ord}(g) > 0$.

A.1 Newton Polygon Method for Algebraic Equations

In this section we describe the Newton polygon method for algebraic equations. It is the main tool for proving the algebraic closure of $\mathbb{K}\langle\langle x\rangle\rangle$ and computing formal parametrizations of algebraic curves. The method constructs solutions of algebraic equations term by term and consists of two main steps: finding possible coefficients and exponents, and then substituting them into the previous equation. The differential counterpart, described in Chapter 3, has in general some limitations such as the non-existence of solutions and free parameters. These problems do not occur in the algebraic version.

For a given polynomial $F \in \mathbb{K}\langle\langle x\rangle\rangle[y]$ of the form

$$F(x,y) = \sum a_{\alpha,\rho}\, x^{\alpha}\, y^{\rho} = 0 \quad \text{with } a_{\alpha,\rho} \in \mathbb{K}^*$$

let us make the ansatz $y(x) = c\,x^{\mu} + y_2(x)$ with $\mathrm{ord}_x(y_2(x)) > \mu$ and $\mu \in \mathbb{Q}$. Then

$$\begin{aligned} F(c\,x^{\mu} + y_2(x)) &= \sum a_{\alpha,\rho}\, x^{\alpha}\, (c\,x^{\mu} + y_2(x))^{\rho} \\ &= \sum a_{\alpha,\rho}\, c^{\rho}\, x^{\alpha+\mu\,\rho} + \quad \text{terms in } y_2(x) = 0. \end{aligned}$$

The terms in $y_2(x)$ are of higher order and thus, the coefficients of terms where $\alpha + \mu\,\rho$ is minimal have to cancel out. Based on this observation we define associated to F the point set

$$\mathcal{P}(F) = \{(\alpha,\rho) \mid a_{\alpha,\rho} \neq 0\} \subseteq \mathbb{Q} \times \mathbb{N}. \tag{A.1}$$

The Newton polygon $\mathcal{N}(F)$ of F is defined as the convex hull of the set

$$\bigcup_{P \in \mathcal{P}(F)} (P + \{(a,0) \mid a \geq 0\}),$$

where "$+$" denotes the Minkowski sum[3].

Let us denote by $L(F;\mu) \subset \mathbb{R}^2$ a line with slope $-1/\mu$ (with respect to the x-axis) such that the Newton polygon $\mathcal{N}(F)$ is contained in the right closed half plane defined by $L(F;\mu) \subset \mathbb{R}^2$ and such that

$$L(F;\mu) \cap \mathcal{N}(F) \neq \emptyset.$$

We say that $L(F;\mu)$ has *inclination* μ and corresponding to every $L(F,\mu)$ we define the characteristic polynomial (of $L(F;\mu)$)

$$\Phi_{(F;\mu)}(C) = \sum_{P_{\alpha,\rho} \in L(F;\mu) \cap \mathcal{P}(F)} a_{\alpha,\rho}\, C^{\rho}, \tag{A.2}$$

where $\mathcal{P}(F)$ is defined as in (A.1). In contrast to the differential case, $\mathcal{P}(F)$ consists only of vertices with multiplicity one. Hence, if $L(F;\mu)$ intersects $\mathcal{N}(F)$ only in one

[3] The Minkowski sum of two sets A and B is defined as $A + B = \{a + b \mid a \in A, b \in B\}$

vertex, there cannot be a solution and we only need to consider sides in the Newton polygon. Since every side involves at least two vertices, the characteristic polynomial $(A.2)$ is non-constant and has roots $c_1 \in \mathbb{K}^*$.
After $\mu_1 \in \mathbb{Q}$ and $c_1 \in \mathbb{K}^*$ are determined, the same process is performed on y_2 in the equation

$$F_2(x, y_2) = F(x, c_1\, x^{\mu_1} + y_2) = 0.$$

Now, in $\mathcal{N}(F_2)$, we only consider sides with inclination bigger than μ and iteratively continuing this process, we obtain a sequence of polynomials $F = F_1, F_2, \ldots$ and corresponding roots $(c_i, \mu_i) \in \mathbb{K}^* \times \mathbb{Q}$.

In the notation above, for showing the algebraic closure of $\mathbb{K}\langle\langle x \rangle\rangle$, it still has to be shown that

1. There always exists a side $L(F_i; \mu_i)$ in $\mathcal{N}(F_i)$ with inclination μ_i bigger than the previous ones. In other words, the process can be continued.

2. There exists a common denominator $n \in \mathbb{N}^*$ of all $\mu_i \in \mathbb{Q}$, namely the ramification index of $y(x)$. This means that $y(x)$ is indeed a formal Puiseux series.

For a more detailed description of the Newton polygon method for algebraic equations and proofs of the above facts we refer to [Wal50][Chapter IV, §3].

Appendix B

More Differential Algebra

In this chapter we recall some definitions of differential algebra which are commonly used and useful for a better understanding of the basics. These notions can be found in most standard textbooks on differential algebra such as [Rit50, Kol73].
Throughout the chapter let $\mathcal{R}$ be a commutative field and $\mathbb{K}$ be an algebraically closed field of characteristic zero.

Definition B.0.1. Let $\delta : \mathcal{R} \to \mathcal{R}$ be a mapping fulfilling for all $a, b \in \mathcal{R}$

$$\delta(a + b) = \delta(a) + \delta(b) \quad \text{and} \quad \delta(a\, b) = \delta(a)\, b + a\, \delta(b). \tag{B.1}$$

Then $(\mathcal{R}; \delta)$ is called an (ordinary) *differential ring* with the derivation δ. Moreover, if R is a field, $(R; \delta)$ is called a *differential field*.
If there are several mappings $\delta_1, \ldots, \delta_n$ fulfilling (B.1) whose compositions are commutative, i.e. for all $a \in \mathcal{R}$ and $i, j \in \{1, \ldots, n\}$ it holds that

$$\delta_i(\delta_j(a)) = \delta_j(\delta_i(a)),$$

then $(R; \delta_1, \ldots, \delta_n)$ is called a *ring! partial differential ring*.

In the above definition the second property in (B.1) is often called the *Leibniz product rule*.

Definition B.0.2. Let $(\mathcal{R}; \delta)$ be a differential ring. Then the set

$$C = \{c \in \mathcal{R} \mid \delta(c) = 0\}$$

is called the *set of constants* (of $(\mathcal{R}; \delta)$). If $\mathcal{R}$ is a field, then C is a subfield of $\mathcal{R}$ and called the *field of constants*.

Example B.0.3. The ring of formal power series $\mathbb{K}[[x]]$ equipped with the usual derivative

$$\frac{d}{dx}\left(\sum_{i \geq 0} c_i\, x^i\right) = \sum_{i \geq 0} (i + 1) c_{i+1}\, x^i$$

defines a differential ring with the set of constants $\mathbb{K}$.

For constructing the ring of differential polynomials let us say that the symbols $x_1, \ldots, x_n$ are related to the derivation operators by $\delta_i(x_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta symbol, i.e. $\delta_{ij} = 1$ if and only if $i = j$ and zero otherwise. Thus, δ_i can be seen as the partial derivative with respect to x_i. The symbols $x_1, \ldots, x_n$ will be called *independent variables*.
Differential indeterminates are symbols y viewed as unknown functions in the independent variables. Applying the derivations to them, we obtain the infinite set

$$\{y, \delta_1(y), \delta_2(y), \ldots, \delta_1^2(y), \delta_1\delta_2(y), \delta_2^2(y), \ldots, \delta_1^{k_1} \cdots \delta_n^{k_n}(y), \ldots\}.$$

For $\Delta(y) = \delta_1^{k_1} \cdots \delta_n^{k_n}(y)$ the sum $k = k_1 + \cdots + k_n$ is called the *order* of the derivative operator Δ.
For a given ring $\mathcal{R}$ and differential indeterminates $y_1, \ldots, y_m$, the polynomials over the elements $\Delta(y_i)$ with $\Delta = \delta_1^{k_1} \cdots \delta_n^{k_n}$ for some $k_1, \ldots, k_n \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$ are called *differential polynomials*. The set of differential polynomials form a ring, the *differential polynomial ring*, denoted by $\mathcal{R}\{y_1, \ldots, y_m\}$.

Example B.0.4. Let us continue Example B.0.3. In this setting we will also use the notation $\delta^{(k)}(y) = y^{(k)}$. Some examples of differential polynomials in $\mathbb{K}[[x]]\{y\}$ are

$$F_1 = yy''^2 + xy'', \; F_2 = 2x^2y''^3 - y^2.$$

The order of F_1 and F_2 is equal to 2.

Definition B.0.5. Let $(\mathcal{R}; \delta_1, \ldots, \delta_n)$ be an ordinary or partial differential ring. An ideal $I \subseteq \mathcal{R}\{y_1, \ldots, y_m\}$ is called a *differential ideal* if I is closed under taking derivations, i.e. for all $F \in I$ and $\Delta = \delta_1^{k_1} \cdots \delta_n^{k_n}$ with $k_1, \ldots, k_n \in \mathbb{N}$ we have that $\Delta(F) \in I$.
Furthermore, we call a differential ideal I of $\mathcal{R}\{y_1, \ldots, y_m\}$ *perfect*, if I is equal to its radical $\sqrt{I}$.

For a set $\mathcal{S} \subset \mathcal{R}$ we denote by $[\mathcal{S}]$ the smallest differential ideal containing $\mathcal{S}$ and by $\{\mathcal{S}\}$ the smallest perfect differential ideal containing $\mathcal{S}$.
A differential ideal contains an infinite number of differential polynomials unless it is equal to the trivial differential ideal $\{0\}$ or $\mathcal{R}$. Consequently, it is not possible to represent any non-trivial differential ideal of $\mathcal{R}\{y_1, \ldots, y_m\}$ as an ideal in $\mathcal{R}[y_1, \ldots, y_m, \delta_1(y_1), \ldots, \delta_m(y_1), \ldots]$ by a finite basis.

In this thesis we work mainly with one independent variable x and one derivation δ. The following definition is given for this case but can be generalized to partial differential rings, see for example Chapter 1 in [Kol73].

Definition B.0.6. Let $(\mathcal{R}; \delta)$ be a differential ring and $y_1, \ldots, y_m$ some differential indeterminates. Then a *ranking* (of $y_1, \ldots, y_m$) is a total ordering on the set of all derivatives $\{\delta^k(y_i) \mid k \in \mathbb{N}, i \in \{1, \ldots, m\}\}$ fulfilling for all $\delta^k(y_i) \leq \delta^\ell(y_j)$ and $q \in \mathbb{N}$ that

$$\delta^k(y_i) \leq \delta^{k+1}(y_i) \;\; \text{and} \;\; \delta^{k+q}(y_i) \leq \delta^{\ell+q}(y_j).$$

A ranking is said to be *orderly* if for all $i, j \in \{1, \ldots, m\}$ and $k, \ell \in \mathbb{N}$ with $k \leq \ell$ it holds that

$$\delta^k y_i \leq \delta^\ell y_j.$$

For orderly rankings and for a given $\delta^\ell(y_j)$ it turns out that there are only finitely many smaller elements in $\{\delta^k(y_i) \mid k \in \mathbb{N}, i \in \{1, \ldots, m\}\}$. Consequently, orderly rankings are well-orderings.

For a differential polynomial $F \in \mathcal{R}\{y_1, \ldots, y_m\} \setminus \mathcal{R}$ equipped with an orderly ranking there exists a unique term which has a maximal derivative $u_F \in \{\delta^k(y_i) \mid k \in \mathbb{N}, i \in \{1, \ldots, m\}\}$ with positive exponent. This term is called the *leader* of F and its coefficient is called the *initial* of F. To be more specific, there is a $d \in \mathbb{N}^*$ such that

$$F = \operatorname{init}(F)\, u_F^d + \sum_{j=1}^{d-1} F_j\, u_F^j + \quad \text{terms with smaller derivatives,}$$

where the initial of F, denoted by $\operatorname{init}(F)$, and $F_1, \ldots, F_{d-1}$ only depend on smaller derivatives than the leader u_F of F. Let us take the derivative of the above expression to obtain

$$\delta(F) = \left(d \operatorname{init}(F)\, u_F^{d-1} + \sum_{j=1}^{d-1} j F_j\, u_F^{j-1} \right) \delta(u_F) + \quad \text{terms with smaller derivatives.}$$

The expression $\delta(F)$ has the leader $\delta(u_F)$ and the initial

$$S_F = d \operatorname{init}(F)\, u_F^{d-1} + \sum_{j=1}^{d-1} j F_j\, u_F^{j-1}.$$

The differential polynomial S_F is called the *separant* of F and occurs in every higher derivative of F as initial (see also Lemma 2.1.6). Note that we could also take the formal derivative of F with respect to u_F to obtain the separant.

Example B.0.7. The leader of F_1 and F_2 from above is equal to y''. The initial are $\operatorname{init}(F_1) = y,\ \operatorname{init}(F_2) = 2x^2$ and the separants $S_{F_1} = 2yy'' + x,\ S_{F_2} = 6x^2 y''^2$. Here we see also the linear occurrence of S_{F_2} in the derivative of F_2:

$$\frac{d\,F_2}{dx} = 6x^2 y''^2 y^{(3)} + 4xy''^3 - 2yy' = S_{F_2}\, y^{(3)} + \quad \text{terms in } \{x, y, y', y''\}.$$

Rankings can naturally be generalized from $\{\delta^k(y_i) \mid k \in \mathbb{N}, i \in \{1, \ldots, m\}\}$ to $\mathcal{R}\{y_1, \ldots, y_m\}$. For every $F, G \in \mathcal{R}\{y_1, \ldots, y_m\}$ we set $F < G$ if and only if for the corresponding leaders it holds that $u_F < u_G$ or $u_F = u_G$ and $\deg_{u_F}(F) < \deg_{u_G}(G)$. Using this ranking, we observe that $\operatorname{init}(F), S_F < F$. Moreover, it is possible to compute pseudo remainders in $\mathcal{R}\{y_1, \ldots, y_m\}$ now.

For given $F \in \mathcal{R}\{y_1, \ldots, y_m\}$ and $\mathcal{S} \subseteq \mathcal{R}\{y_1, \ldots, y_m\}$, the pseudo remainders of F and elements in the differential ideal $[\mathcal{S}]$ is called the *differential pseudo remainder* of F with respect to $\mathcal{S}$. The differential pseudo remainder can be uniquely written as

$$\operatorname{prem}(F, \mathcal{S}) = \prod_{G \in [\mathcal{S}]} \operatorname{init}(G)^{d_G}\, S_G^{e_G}\, F - \sum_{G \in [\mathcal{S}]} F_G\, G$$

with $F_G \in \mathcal{R}\{y_1, \ldots, y_m\}$ and $d_G, e_G \in \mathbb{N}$ such that $\operatorname{prem}(F, \mathcal{S}) < G$ for every $G \in [\mathcal{S}]$. For more details on the differential pseudo remainder and the reduction process we refer to [Rit50, Kol73].

Example B.0.8. In the example above, $F_1 = yy''^2 + xy'' < F_2 = 2x^2y''^3 - y^2$ and

$$\operatorname{prem}(F_2, F_1) = \operatorname{init}(F_1)^2\, F_2 - 2(x^2y - x^3)\, F_1 = 2x^4y'' - y^4.$$

Except for Section 4.2 we work in the present thesis with only one differential indeterminate y. Then all rankings coincide, are given by

$$y < \delta(y) < \delta^2(y) < \cdots$$

and are orderly. As differential rings we mostly use $\mathcal{R} = \mathbb{K}[[x]]$ and $\mathcal{R} = \mathbb{K}\langle\langle x \rangle\rangle$ equipped with the usual derivation $\delta = \frac{d}{dx}$. Note that from the appendix A it follows that $\left(\mathbb{K}\langle\langle x \rangle\rangle, \frac{d}{dx}\right)$ is in fact a differential field containing $\mathbb{K}[[x]]$.

Appendix C

More Algebraic Geometry

This appendix consists of some basic notions on algebraic curves, local parametrizations and triangular sets. These notions can be found in some standard textbooks on the corresponding topic. For algebraic curves and formal parametrizations we refer to [Wal50], for the generalizations to space curves to [AMNR92] and for triangular sets and in particular regular chains to [Wan12, Kal93, ALM99].
Throughout the chapter let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $\mathbb{K}_\infty = \mathbb{K} \cup \{\infty\}$ be the projective compactification of $\mathbb{K}$. Moreover, let $\mathscr{K} \supseteq \mathbb{K}$ be a field and let $\mathcal{S} \subset \mathbb{K}[y_0, \ldots, y_m]$. Then the zero set implicitly defined by $\mathcal{S}$ defines the algebraic set

$$\mathbb{V}_{\mathscr{K}}(\mathcal{S}) = \{a \in \mathscr{K}^{m+1} \mid F(a) = 0 \text{ for all } F \in \mathcal{S}\}.$$

C.1 Plane Curves and Formal Parametrizations

Associated to a bivariate polynomial $F \in \mathbb{K}[x, y]$ the algebraic set $\mathbb{V}_\mathbb{K}(F) \subset \mathbb{K}^2$ defines an affine algebraic plane curve denoted by $\mathcal{C}(F)$. The Zariski closure of $\mathcal{C}(F)$ in $\mathbb{K}_\infty^2$ will be denoted by $\mathscr{C}(F)$. It would be possible to work in projective space and with projective plane curves instead, where frankly speaking only one infinity exists, but for our purposes it turns out that the definition of $\mathscr{C}(F)$ presented here is preferable.

A *formal parametrization* of $\mathscr{C}(F)$ centered at $\mathbf{p}_0 \in \mathscr{C}(F)$ is a pair of formal Laurent series $A(t) \in \mathbb{K}((t))^2 \setminus \mathbb{K}^2$ such that $A(0) = \mathbf{p}_0$ and $F(A(t)) = 0$. Note that formal parametrizations can be seen as elements in $\mathbb{V}_{\mathbb{K}\langle\langle x\rangle\rangle}(F)$. From Appendix A.1 we know how to compute formal parametrizations: Let us consider a curve point $\mathbf{p}_0 = (c_1, c_2)$ with $c_1 \in \mathbb{K}$. Then, by the Newton polygon method for algebraic equations, there exists a solution $y(x) \in \mathbb{K}\langle\langle x - c_1\rangle\rangle$ of $F = 0$. By the change of variables $\tilde{y}(x) = y(x - c_1)$, we obtain that $(c_1 + x, \tilde{y}(x))$ parametrizes $\mathscr{C}(F)$. If we set $t = x^n$, where n is the ramification index of y (and of $\tilde{y}$), the pair

$$(c_1 + t^n, \tilde{y}(t^n)) \in \mathbb{K}((t))^2 \tag{C.1}$$

is a formal parametrization of $\mathscr{C}(F)$ with center at $(c_1, y(c_1)) = \mathbf{p}_0$.
For curve points of the form $\mathbf{p}_0 = (\infty, c_2)$, consider $G(x, y) = \mathrm{num}(F(1/x, y))$. Then we can proceed as above to obtain a formal parametrization $(t^n, \tilde{y}(t^n)) \in \mathbb{K}((t))$ of

$\mathscr{C}(G)$ centered at $(0, c_2)$. Changing back variables, we obtain the formal parametrization

$$(t^{-n}, \tilde{y}(t^n)) \in \mathbb{K}((t))^2 \tag{C.2}$$

of $\mathscr{C}(F)$ centered at $\mathbf{p}_0$.
We will refer to formal parametrizations of the form (C.1) or (C.2), respectively, by *classical Puiseux parametrizations*.

In the set of all formal parametrizations of $\mathscr{C}(F)$ the relation $\sim$ defined as $A(t) \sim B(t)$ if and only if there exists a formal power series $s(t) \in \mathbb{K}[[t]]$ of order one such that

$$A(s(t)) = B(t)$$

is an equivalence relation. Such an $s(t) \in \mathbb{K}[[t]]$ is also called a *reparametrization*.
A formal parametrization is said to be *reducible* if it is equivalent to another one in $\mathbb{K}((t^m))^2$ for some $m > 1$. Otherwise, it is called *irreducible*. An equivalence class of an irreducible formal parametrization $(a(t), b(t))$ is called a *place* of $\mathscr{C}(F)$ centered at the common center point $\mathbf{p}_0$ and is denoted by $[(a(t), b(t))]$. The order is an invariant of a place, i.e. every formal parametrization $(b_1(t), b_2(t))$ in a place $[(a_1(t), a_2(t))]$ of $\mathscr{C}(F)$ centered at $\mathbf{p}_0 = (c_1, c_2)$ fulfills

$$\operatorname{ord}(a_i) = \operatorname{ord}(b_i) \quad \text{and} \quad \operatorname{ord}(a_i(t) - c_i) = \operatorname{ord}(b_i(t) - c_i)$$

for $i \in \{1, 2\}$. Moreover, in every place there is, up to the reparametrization with $s(t) = \lambda t$ with $\lambda^{|n|} = 1$ and $n = \operatorname{ord}(a_1(t) - c_1)$, exactly one classical Puiseux parametrization. Therefore, we may also speak about the *order* of $[(a_1(t), a_2(t))]$ and set it to $|n| \in \mathbb{N}^*$.
We define $\mathbf{IFP}(\mathbf{p}_0)$ as the set containing all irreducible formal parametrizations of $\mathscr{C}(F)$ centered at $\mathbf{p}_0$ and $\mathbf{Places}(\mathbf{p}_0)$ as the set containing the places of $\mathscr{C}(F)$ centered at $\mathbf{p}_0$. A representative of a place is for example a classical Puiseux parametrization. The set $\mathbf{CPP}(\mathbf{p}_0) \subset \mathbf{IFP}(\mathbf{p}_0)$ consists of one classical Puiseux parametrization for every place centered at $\mathbf{p}_0$. For a given $\mathbf{p}_0 \in \mathscr{C}(F)$, the sets $\mathbf{CPP}(\mathbf{p}_0), \mathbf{Places}(\mathbf{p}_0)$ are finite, whereas $\mathbf{IFP}(\mathbf{p}_0)$ is infinite.

From Theorem 5.8(i),(ii) in [Wal50] there directly follows an interesting relation between the multiplicity of curve points and the orders of places.

Theorem C.1.1. *Let $\mathbb{K}$ be an algebraically closed field, let $F \in \mathbb{K}[x, y]$. Then the multiplicity of $\mathscr{C}(F)$ at $\mathbf{p}_0$ and the sum of the orders of $\mathbf{Places}(\mathbf{p}_0)$ coincide.*

In the generic case a point $\mathbf{p}_0 \in \mathscr{C}(F)$ is simple, i.e. it has multiplicity one. Then, by Theorem C.1.1, there is exactly one place of $\mathscr{C}(F)$ centered at $\mathbf{p}_0$ and its order is equal to 1. For us are the non-generic points of particular interest.

Let $[(a(t), b(t))] \in \mathbf{Places}(\mathbf{p}_0)$. Then, the tangent vector $\bar{v}$ of $\mathscr{C}(F)$ through the place $[(a(t), b(t))]$ is (see Section 5.3. in [Wal50])

$$\bar{v} = \begin{cases} (\operatorname{lc}(a), \operatorname{lc}(b)), & \text{if } \operatorname{ord}(a - a_0) = \operatorname{ord}(b - b_0), \\ (\operatorname{lc}(a), 0), & \text{if } \operatorname{ord}(a - a_0) < \operatorname{ord}(b - b_0), \\ (0, \operatorname{lc}(b)), & \text{if } \operatorname{ord}(a - a_0) > \operatorname{ord}(b - b_0). \end{cases} \tag{C.3}$$

Note that the tangent vector is uniquely defined up to the multiplication with a constant.
A curve point $\mathbf{p}_0 = (a_0, b_0)$ of $\mathscr{C}(F)$ with a tangent parallel to one of the axes is called a *ramification point*. We distinguish between ramification points with respect to x, where the tangent is parallel to the x-axis, and ramification points w.r.t. y, where the tangent is parallel to the y-axis. Note that our notion of ramification point is compatible with [Sha94][page 144] by considering projections on the x and y-coordinate.
From the characterization above, it directly follows that $\mathbf{p}_0$ is a y-ramification point if and only if $\mathrm{ord}(a(t) - a_0) > \mathrm{ord}(b(t) - b_0)$, and $\mathbf{p}_0$ is a x-ramification point if and only if $\mathrm{ord}(b(t) - b_0) > \mathrm{ord}(a(t) - a_0)$.

C.2 Space Curves

Algebraic sets $\mathbb{V}_{\mathbb{K}}(\mathcal{S})$ corresponding to systems of polynomials $\mathcal{S} \in \mathbb{K}[y_0, \ldots, y_m]$ in $(m+1)$-dimensional space are said to be of dimension one if $\mathbb{V}(\mathcal{S})$ consists more than a finite number of points and intersects a generic hyperplane finitely many times. Algorithmically the dimension can for example be computed by using Groebner bases. Let $\mathbb{V}_{\mathbb{K}}(\mathcal{S})$ be of dimension one now. Then it can be decomposed into a finite union of curves and points. The points are not really of interest for us. Every curve can be locally parametrized by projecting it to $\mathbb{K}^2$ and compute classical Puiseux parametrizations there as it is described in Appendix A.1. Not every projection is working, because it is possible to project the whole curve to a point. Non injective projections, which are the non generic cases, can be avoided by suitable coordinate transformations. After computing all the classical Puiseux parametrizations of projections onto y_0 and every other coordinate y_i with $1 \leq i \leq m$, a formal parametrization of the space curve can be read off by a lifting process as we illustrate in the following example.

Example C.2.1. Let us consider the system of polynomials

$$\mathcal{S} = \{y_0\, y_1\, y_2 + y_1^3 - y_0\, y_2 - y_1^2,\ y_0\, y_1 - y_0\, y_2 - y_1^2 - 1\}.$$

The corresponding algebraic set is of dimension one and can be decomposed into

$$\mathbb{V}_{\mathbb{C}}(\mathcal{S}) = \mathbb{V}_{\mathbb{C}}(y_1 - 1,\ y_0\, y_2 - y_0 + 2) \cup \mathbb{V}_{\mathbb{C}}(y_0\, y_1 - 1,\ y_1^3 + y_2,\ y_0\, y_2 + y_1^2).$$

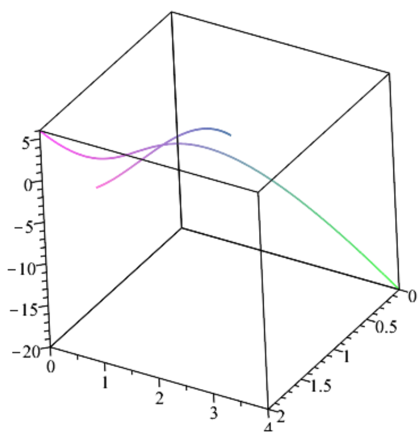

Figure C.1: Plot of $\mathbb{V}_{\mathbb{C}}(\mathcal{S})$.

In the first component the value $y_1 = 1$ is fixed. Projecting to $\mathbb{C}[y_0, y_2]$, we obtain the plane curve $\mathbb{V}_{\mathbb{C}}(y_0\, y_2 - y_0 + 2)$ with the classical Puiseux parametrization

$$(a_0(t), a_2(t)) = (1 + t, -1 + 2t - 2t^2 + 2t^3 + \mathcal{O}(t^4))$$

at $(1, -1)$. Hence,

$$(a_0(t), 1, a_2(t))$$

is a formal parametrization of $\mathbb{V}_{\mathbb{C}}(y_1 - 1,\ y_0 y_2 - y_0 + 2)$ with center at $(1, 1, -1)$.

For the second component we first project to $\mathbb{C}[y_0, y_1]$ to obtain the formal parametrization

$$(b_0(t), b_1(t)) = (1 + t, 1 - t + t^2 - t^3 + \mathcal{O}(t^4))$$

of $\mathbb{V}_{\mathbb{C}}(y_0\, y_1 - 1)$ at $(1, 1)$. The projection onto $\mathbb{C}[y_0, y_2]$ is given by $\mathbb{V}_{\mathbb{C}}(y_0^3\, y_2 + 1)$ having the formal parametrization

$$(b_0(t), b_2(t)) = (1 + t, -1 + 3t - 6t^2 + 10t^3 + \mathcal{O}(t^4))$$

at $(1, -1)$. Then

$$(b_0(t), b_1(t), b_2(t))$$

locally parametrizes the second component $\mathbb{V}_{\mathbb{C}}(y_0\, y_1 - 1,\ y_1^3 + y_2,\ y_0\, y_2 + y_1^2)$ at $(1, 1, -1)$.

We refer to [AMNR92] and references therein for more details on the computation of the dimension of algebraic sets and formal parametrizations of space curves.

C.3 Regular Chains

In this section we recall the notion of regular chains and regular zeros. For further details we refer to [Kal93, Wan12] and [ALM99].
Let us denote for $F \in \mathbb{K}[y_0, \ldots, y_m]$ by $\mathrm{lv}(F)$ the leading variable, by $\mathrm{lc}(F)$ the leading coefficient and by $\mathrm{init}(F)$ the initial of F with respect to the ordering $y_0 < \cdots < y_m$. In addition, we denote for $G \in \mathbb{K}[y_0, \ldots, y_m]$ by $\mathrm{Res}_{y_i}(F, G)$ the resultant of F and G with respect to y_i.
Let $\mathcal{S} = \{F_1, \ldots, F_M\} \subset \mathbb{C}[y_0, \ldots, y_m]$ be a finite system of polynomials in *triangular form*, i.e. $\mathrm{lv}(F_i) < \mathrm{lv}(F_j)$ for any $1 \leq i < j \leq M$. Observe that $M \leq m$ and any leading variable can occur at most one time. Then we define $\mathrm{Res}(F, \mathcal{S})$ as the resultant of F and consecutively $F_M, \ldots, F_1$ with respect to their leading variables, i.e.

$$\mathrm{Res}(F, \mathcal{S}) = \mathrm{Res}_{\mathrm{lv}(F_1)}(\cdots \mathrm{Res}_{\mathrm{lv}(F_M)}(F, F_M), \cdots, F_1).$$

Moreover, we define the initial of a system as

$$\mathrm{init}(\mathcal{S}) = \{\mathrm{init}(F_j) \mid 1 \leq j \leq M\}$$

and their product as

$$\mathrm{pinit}(\mathcal{S}) = \prod_{j=1}^{M} \mathrm{init}(F_j).$$

A *regular chain* is a system of algebraic equations $\mathcal{S}$ in triangular form with the additional property that

$$\mathrm{Res}(F, \mathcal{S}) \neq 0 \quad \text{for every} \quad F \in \mathrm{init}(\mathcal{S}).$$

Let $\mathscr{K} \supseteq \mathbb{K}$ be a field. Then for a regular chain $\mathcal{S} \subset \mathbb{K}[y_0, \ldots, y_m]$ we define a *regular zero* of $\mathcal{S}$ as an element $a = (a_0, \ldots, a_m) \in \mathbb{V}_{\mathscr{K}}(\mathcal{S})$ such that

- if $(\mathcal{S} \cap \mathbb{K}[y_0]) \setminus \mathbb{K} = \emptyset$, the component a_0 is transcendental over $\mathbb{K}$; and
- for every $1 \leq k \leq m$ with $(\mathcal{S} \cap \mathbb{K}[y_0, \ldots, y_k]) \setminus \mathbb{K}[y_0, \ldots, y_{k-1}] = \emptyset$ the component a_k is transcendental over $\mathbb{K}(a_0, \ldots, a_{k-1})$.

In other words, if a variable does not occur as a leading variable in the system, then the regular zero contains a transcendental element in the corresponding component. Let us illustrate the above definitions by an example.

Example C.3.1. Let us consider

$$\mathcal{S} = \{y_1^2 - y_0^2,\ y_1\,(y_2 - y_0)\}.$$

The system $\mathcal{S}$ is in triangular form and $\mathrm{init}(\mathcal{S}) = \{y_1^2,\ y_2\,y_1\}$. Then

$$\begin{aligned}
&\mathrm{Res}(y_1^2, \mathcal{S}) = \mathrm{Res}_{y_1}(\mathrm{Res}_{y_2}(y_1^2, y_1\,(y_2 - y_0)), y_1^2 - y_0^2) = x^4,\\
&\mathrm{Res}(y_2\,y_1, \mathcal{S}) = x^6,
\end{aligned}$$

are both non-zero and $\mathcal{S}$ defines a regular chain. The regular zeros are $(c, c, c),\ (c, -c, c)$, where c is a transcendental element.

There are several equivalent definitions of regular chains and regular zeros, see for example [Wan12][Proposition 5.1.5, Corollary 5.1.6]. In [YZ90] regular chains are called "proper ascending chains", but are defined exactly as here. Another equivalent definition can be found in [Kal93], which turns out to be useful in our reasonings in Section 4.2. Let us present it in the following theorem.

Theorem C.3.2. *Let $\mathcal{S} = \{F_1, \dots, F_M\} \subset \mathbb{K}[y_0, \dots, y_m]$ be a finite system of polynomials in triangular form and denote by $\mathcal{S}_k$ the first k polynomials of $\mathcal{S}$. Then the following are equivalent:*

1. *$\mathcal{S}$ is a regular chain.*
2. *$|\mathcal{S}| = 1$ or for any $k = 2, \dots, M$ the subsystem $\mathcal{S}_{k-1}$ is a regular chain and for any regular zero a of $\mathcal{S}_{k-1}$ and $F \in \mathcal{S}_k$ it holds that $\operatorname{init}(F)(a) \neq 0$.*

Regular chains can be helpful in order to represent algebraic sets as Theorem 5.2.2 in [Wan12] shows:

Theorem C.3.3. *Let $\mathcal{S} \subseteq \mathbb{K}[y_0, \dots, y_m]$ be a polynomial system. Then there exists a finite set of regular chains $\mathcal{S}_1, \dots, \mathcal{S}_N \subseteq \mathbb{K}[y_0, \dots, y_m]$ such that*

$$\mathbb{V}_{\mathscr{K}}(\mathcal{S}) = \bigcup_{j=1}^{N} \mathbb{V}_{\mathscr{K}}(\mathcal{S}_j) \setminus \mathbb{V}_{\mathscr{K}}(\operatorname{pinit}(\mathcal{S}_j)), \tag{C.4}$$

We note that, with the notation of [Wan12],

$$\mathbb{V}_{\mathscr{K}}(\mathcal{S}_j / \operatorname{init}(\mathcal{S}_j)) = \mathbb{V}_{\mathscr{K}}(\mathcal{S}_j) \setminus \mathbb{V}_{\mathscr{K}}(\operatorname{pinit}(\mathcal{S}_j)),$$

as it is also mentioned in Chapter 1.5 therein. Let us recall that $\mathbb{V}_{\mathbb{K}}(\mathcal{S}_j)$ is an algebraic set of dimension $m - |\mathcal{S}_j|$.

There are several implementations for performing computations with regular chains and in particular computing regular chain decomposition as in (C.4) such as in the `Maple`-package `RegularChains`.

Appendix D

Alternative Proof of Key Lemma

In Section 4.1 we highly used the results from Lemma 3.3.1 applied to the associated differential equation 4.7. In fact, we concentrate on the results on reparametrizations, i.e. formal power series of order one. For them we are able to give an alternative proof without using the Newton polygon method for differential equations as we show here. More precisely, we relate the associated differential equation to Briot-Bouquet equations considered in [BB56].

Lemma D.0.1 (Briot-Bouquet). *Let $g(t,z), f(t,z) \in \mathbb{L}[[t,z]]$, where $\mathbb{L}$ is a subfield of the algebraically closed field $\mathbb{K}$. Let us consider the differential equation*

$$F(t,z,z') = f(t,z)\,t\,z' - g(t,z) = 0 \tag{D.1}$$

with $f(0,0) \neq 0$ and $g(0,0) = 0$. Let us denote

$$\lambda = \frac{1}{f(0,0)} \frac{\partial g}{\partial z}(0,0).$$

Then it holds that

1. *If $\lambda \notin \mathbb{N}^*$, then $\mathbf{Sol}_{\mathbb{K}[[t]]}(F)$ has exactly one element of positive order $z(t) = \sum_{i=1}^{\infty} \zeta_i\, t^i$ and all the coefficients $\zeta_i \in \mathbb{L}$.*

2. *If $\lambda \in \mathbb{N}^*$, then there is either no solution or a one-parameter family of solutions the form $z(t) = \sum_{i=1}^{\infty} \zeta_i\, t^i \in \mathbf{Sol}_{\mathbb{K}[[t]]}(F)$, where $\zeta_1, \ldots, \zeta_{\lambda-1}$ are uniquely determined elements in $\mathbb{L}$, $\zeta_\lambda \in \mathbb{K}$ is a free parameter and for $i > \lambda$ the coefficients $\zeta_i \in \mathbb{L}(\zeta_\lambda)$ are determined by ζ_λ. Moreover, the existence of such a solution can be decided algorithmically and two solutions are equal if they coincide up to order λ.*

Furthermore, the following statements hold in both cases:

(a) *Let $\mathbb{K} = \mathbb{C}$. If f and g are convergent power series, then any element of $\mathbf{Sol}_{\mathbb{K}[[t]]}(F)$ of order greater or equal to one is convergent.*

(b) *Let us write* $f = \sum_{i+j\geq 1} f_{i,j}\, t^i\, z^j$, $g = \sum_{i+j\geq 0} g_{i,j}\, t^i\, z^j$, $f_{i,j}, g_{i,j} \in \mathbb{L}$ *and let* $z(t) = \sum_{i=1}^{\infty} \zeta_i\, t^i \in \mathbf{Sol}_{\mathbb{K}[[t]]}(F)$. *For any* $k \in \mathbb{N}^*$, *the coefficient* ζ_k *is completely determined by the coefficients of* $f_{i,j}$ *and* $g_{i,j}$ *such that* $i+j \leq k$ *and, in case* $\lambda \in \mathbb{N}^*$ *and* $k > \lambda$, ζ_k *also depends on the coefficient* ζ_λ.

Proof. These results are partly consequences of the Theorem XXVIII in Section 80 and Section 86 in [BB56]. Nevertheless we will prove existence and uniqueness of the solutions independently in order to show precisely the dependency of the coefficients of the solutions with respect to the coefficients of $f(t,z)$ and $g(t,z)$.
We may assume that $f(0,0) = 1$ and $\frac{\partial g}{\partial z}(0,0) = \lambda$ by multiplying both sides of equation (D.1) with $1/f(0,0)$. Let us write $f = 1 - \tilde{f}$ and $g = \lambda\, z + g_{1,0}\, t + \tilde{g}$, where $\tilde{f} = \sum_{i+j\geq 1} f_{i,j}\, t^i\, z^j$ and $\tilde{g} = \sum_{i+j\geq 2} g_{i,j}\, t^i\, z^j$. Then, the differential equation (D.1) is equivalent to the following one

$$t\, z' - \lambda\, z = g_{1,0}\, t + \tilde{g}(t,z) + t\, z'\, \tilde{f}(t,z). \tag{D.2}$$

Let us substitute $z(t) = \sum_{i=1}^{\infty} \zeta_i\, t^i$ into equation (D.2) and let us compute the coefficient of t^k for $k \in \mathbb{N}^*$ in both sides. On the left hand side we obtain $(k-\lambda)\,\zeta_k$. By expanding the right hand side of (D.2), we obtain a polynomial expression in $t, \zeta_i, f_{i,j}$ and $g_{i,j}$ with non-negative integer coefficients. More precisely, since $\mathrm{ord}(z(t)) \geq 1$, the coefficient

$$[t^k](t\, z'(t)\, \tilde{f}(t, z(t)))$$

is a polynomial expression in ζ_i, $1 \leq i \leq k-1$, and $f_{i,j}$ with $1 \leq i+j \leq k-1$. The coefficient of $[t^k](\tilde{g}(t,z(t)))$ is a polynomial expression in ζ_i, $1 \leq i \leq k-1$ and $g_{i,j}$ for $2 \leq i+j \leq k-1$. Let us define P_k as the coefficient of t^k of the right hand side of (D.2). Then $z(t)$ is a solution of equation (D.2) if and only if for every $k \in \mathbb{N}^*$ the following relations hold

$$(k-\lambda)\,\zeta_k = P_k(\zeta_1, \ldots, \zeta_{k-1}, g_{1,0}, f_{1,0}, f_{0,1}, g_{i,j}, f_{i,j};\ 2 \leq i+j \leq k-1). \tag{D.3}$$

If $\lambda \notin \mathbb{N}^*$, equation (D.3) can be solved uniquely for every ζ_k, $k \geq 1$. If $\lambda \in \mathbb{N}^*$, then $\zeta_1, \ldots, \zeta_{\lambda-1}$ satisfying equations (D.3) are uniquely determined as well. If equation (D.2) has at least one formal power series solution of order greater than or equal to one, then necessarily $P_\lambda(\zeta_1, \ldots, \zeta_{\lambda-1}, f_{i,j}, g_{i,j}) = 0$ and arbitrary ζ_λ satisfies equation (D.3). Once that ζ_λ has been chosen, there exists a unique sequence of ζ_i, $i > \lambda$, solving the equation (D.3) for $k > \lambda$. Since equation (D.3) is linear in ζ_k and P_k is a polynomial expression, no field extensions are necessary.
In order to show item (a), let f and g be convergent power series in a neighborhood of the origin. In the case $\lambda \notin \mathbb{N}^*$ the convergence of the solution follows by the majorant series method using the fact that the coefficients of P_k are non negative (see for instance Section 12.6 in [Inc26]). In the case $\lambda \in \mathbb{N}^*$ we perform for a solution $z(t) = \sum \zeta_i\, t^i$ the change of variables $z(t) = \zeta_1 t + \cdots + \zeta_\lambda t^\lambda + t^\lambda\, w(t)$ and reduce it to the previous case, see for instance Section 86 of [BB56]. □

Lemma D.0.2. *Let* $\mathbf{p}_0 = (y_0, p_0) \in \mathbb{K} \times \mathbb{K}_\infty$ *and* $\mathbb{L}$ *be a subfield of* $\mathbb{K}$. *Let* $\mathcal{P} = [(a(t), b(t))] \in \mathbf{Places}(\mathbf{p}_0)$ *with* $a(t), b(t) \in \mathbb{L}((t))$ *be such that equation (4.4) holds for an* $m \in N^*$, *i.e.* $m = \frac{p+1-q}{1-h} \geq 1$ *with* $p = \operatorname{ord}(a'(t)) \in \mathbb{N}$, $q = \operatorname{ord}(b(t)) \in \mathbb{Z}$. *Then, it holds that*

1. *If* $h \leq 0$, *then* $\mathbf{Sol}_{\mathbb{K}[[t]]}(F)$ *consists of exactly* $m\,(1-h)$ *elements of the form* $s(t) = \sum_{i=1}^{\infty} \sigma_i\, t^i$, *where* $\sigma_1^{m(1-h)} \in \mathbb{L}$ *and all the other coefficients* $\sigma_i \in \mathbb{L}(\sigma_1)$.
2. *If* $h \geq 2$, *then* $\mathbf{Sol}_{\mathbb{K}[[t]]}(F)$ *consists of up to* $m\,(h-1)$ *one-parameter families of the form* $s(t) = \sum_{i=1}^{\infty} \sigma_i\, t^i$, *where* $\sigma_1^{m(h-1)} \in \mathbb{L}$, $\sigma_2, \ldots, \sigma_{q-p-2}$ *are uniquely determined elements in* $\mathbb{L}(\sigma_1)$, $\sigma_{q-p-1} \in \mathbb{K}$ *is a free parameter and for* $i \geq q-p$ *the coefficients* $\sigma_i \in \mathbb{L}(\sigma_1, \sigma_{q-p-1})$ *are determined by* σ_1 *and* σ_{q-p-1}.

Moreover, the following statements hold in both cases

(a) *Let* $\mathbb{K} = \mathbb{C}$. *If* $a(t)$ *and* $b(t)$ *are convergent as Puiseux series, then any element in* $\mathbf{Sol}_{\mathbb{K}[[t]]}(F)$ *of positive is convergent.*

(b) *Let us write* $a(t) = y_0 + \sum_{i\geq 0} a_i\, t^{p+1+i}$, $b(t) = \sum_{i \geq 0} b_i\, t^{q+i}$ *with* $a_i, b_i \in \mathbb{L}$. *Then for any* $k \in \mathbb{N}^*$, *the coefficient* σ_k *is completely determined by the coefficients* σ_1, a_i *and* b_i *such that* $0 \leq i \leq k-1$ *and, in case* $q-p-1 \in \mathbb{N}^*$ *and* $k \geq q-p$, σ_k *also depends on the coefficient* σ_{p-q-1}.

Proof. Let us define $\nu = |m\,(1-h)| = |q-p-1| \geq 1$.
First we prove the case $h \leq 0$. Multiplying both sides of the associated differential equation (4.7) by $s(t)^{-q}\, t^{-\nu+1}$, we obtain the equivalent differential equation

$$t^{-\nu+1}\, \tilde{a}(s(t)) \cdot s'(t) = m\, \tilde{b}(s(t)), \tag{D.4}$$

where $\tilde{a}(s) = s^{-q}\, a'(s) = \sum_{i\geq 0} \tilde{a}_i\, s^{i+\nu-1}$ with $\tilde{a}_i = (p+1+i)\, a_i$, $\tilde{a}_0 = (p+1)\, a_{p+1} \neq 0$ and $\tilde{b}(s) = s^{-q}\, b(s) = \sum_{i\geq 0} b_i\, s^i$. Let us fix a non-zero $\sigma \in \mathbb{K}$ and perform the change of variables $s(t) = t\,(\sigma + z(t))$ in the differential equation (D.4) to obtain

$$\sum_{i\geq 0} \tilde{a}_i\, t^i\, (\sigma + z(t))^{i+\nu-1}\, (\sigma + z(t) + t\, z'(t)) = m \sum_{i \geq 0} b_i\, t^i\, (\sigma + z(t))^i.$$

Let us move the terms of the left hand side not involving z' to the right hand side to obtain

$$\left(\sum_{i\geq 0} \tilde{a}_i\, t^i\, (\sigma + z(t))^{i+\nu-1} \right) t\, z'(t) = \sum_{i\geq 0} t^i \left(m\, b_i\, (\sigma + z(t))^i - \tilde{a}_i\, (\sigma + z(t))^{i+\nu} \right). \tag{D.5}$$

Hence, a formal (respectively convergent) power series $s(t) = \sum_{i=1}^{\infty} \sigma_i\, t^i$ is a solution of (4.7) if and only if $z(t) = \sum_{i=2}^{\infty} \sigma_i t^{i-1}$ is a formal (respectively convergent) power series solution of (D.5) for $\sigma = \sigma_1$.
By considering in equation (D.5) the terms independent of t, we obtain $0 = m\, b_0 - \tilde{a}_0\, \sigma^\nu$. As a consequence, if $s(t) = \sum_{i=1}^{\infty} \sigma_i\, t^i$ is a formal power solution of equation

(4.7), then $\sigma_1^{\nu} = \frac{m\,b_0}{\tilde{a}_0}$. Since $\tilde{a}_0,\ b_0 \neq 0$, there are exactly ν possibilities for σ_1. It remains to prove that for any such $\sigma_1 = \sigma$, there exists a unique formal power series solution $z(t)$ of (D.5) with $\mathrm{ord}(z(t)) \geq 1$ satisfying the properties specified in the statement of the lemma. As described below, this is a direct consequence of Lemma D.0.1 applied to the differential equation (D.5).

First, let us show that (D.5) satisfies the hypothesis of the first case of Lemma D.0.1. Let us denote by $g(t, z(t))$ the right hand side of equation (D.5) and by $f(t, z(t))\,t\,z'(t)$ the left hand side. We have that $f(0,0) = \tilde{a}_0\,\sigma_1^{\nu-1} \neq 0,\ g(0,0) = m\,b_0 - \tilde{a}_0\sigma_1^{\nu} = 0$ and $\frac{\partial g}{\partial z}(0,0) = -\nu\,\tilde{a}_0\sigma_1^{\nu-1}$. Hence $\frac{1}{f(0,0)}\frac{\partial g}{\partial z}(0,0) = -\nu = -m\,(1-h) < 0$. Then, by Lemma D.0.1, the following statements hold. There exists a unique formal power series solution $z(t) = \sum_{i=2}\sigma_i t^{i-1}$ of (D.5) and consequently, $s(t) = \sum_{i=1}\sigma_i\,t^i$ is a solution of (4.7). If $a(s)$ and $b(s)$ are convergent, the series $f(t,z)$ and $g(t,z)$ are convergent and then $z(t)$ and $s(t)$ are convergent. Moreover, the coefficients of $g(t,z), f(t,z)$ and therefore of $z(t)$ and $s(t)$ belong to the field $\mathbb{L}(\sigma_1)$.

It remains to prove item (b) for the case of $h \leq 0$. Since $\nu - 1 \geq 0$, the coefficient $f_{i,j}$ in $f(t,z) = \sum_{i+j\geq 0} f_{i,j}\,t^i\,z^j$ depends only on σ_1 and $\tilde{a}_i$. Similarly, the coefficient $g_{i,j}$ in $g(t,z) = \sum_{i+j\geq 1} g_{i,j}\,t^i\,z^j$ depends only on $\sigma_1,\ \tilde{a}_i$ and b_i. For $k \in \mathbb{N}^*$, by Lemma D.0.1, σ_k depends only on $g_{i,j}$ and $f_{i,j}$ with $i+j \leq k-1$, which in their turn depend on σ_1 and $\tilde{a}_i$ and b_i for $0 \leq i \leq k-1$. Since $\tilde{a}_i = (p+1+i)\,a_i$, item (b) is proven.

Let us now consider the case $h \geq 2$. Multiplying both sides of (4.7) by $s(t)^{-q}$, we obtain the equivalent differential equation

$$\tilde{a}(s(t))\,s'(t) = m\,t^{\nu+1}\,\tilde{b}(s(t)), \tag{D.6}$$

where $\tilde{a}(s) = s^{-p}\,a'(s) = \sum_{i\geq 0}\tilde{a}_i\,s^i$ with $\tilde{a}_i = (p+1+i)\,a_i,\ \tilde{a}_0 = k\,a_0 \neq 0$ and $\tilde{b}(s) = s^{-p}\,b(s) = \sum_{i\geq 0} b_i\,s^{i+\nu+1}$. After performing for $\sigma \in \mathbb{K}$ the change of variable $s(t) = t\,(\sigma + z(t))$ in (D.6), we obtain the equivalent differential equation

$$\left(\sum_{i\geq 0}\tilde{a}_i\,t^i\,(\sigma + z(t))^i\right)t\,z'(t) = \sum_{i\geq 0} t^i\left(m\,b_i\,(\sigma + z(t))^{i+\nu+1} - \tilde{a}_i\,(\sigma + z(t))^{i+1}\right). \tag{D.7}$$

By considering in equation (D.7) the terms independent of t, we again obtain $\sigma_1^{\nu} = \frac{\tilde{a}_0}{m\,b_0}$ as necessary condition for a solution $s(t) = \sum_{i=1}^{\infty}\sigma_i\,t^i$ of (4.7). For $\sigma = \sigma_1$, by setting $g(t,z)$ equal to the right hand side and $f(t,z)\,t\,z'$ equal to the left hand side in equation (D.7), the hypothesis of the second case of Lemma D.0.1 are fulfilled: $f(0,0) = \tilde{a}_0 \neq 0,\ g(0,0) = 0,\ \frac{\partial g}{\partial z}(0,0) = m\,(\nu+1)\,b_0\,\sigma_1^{\nu} - \tilde{a}_0$ and $\lambda = \frac{1}{f(0,0)}\frac{\partial g}{\partial z}(0,0) = \nu \in \mathbb{N}^*$. As a consequence of Lemma D.0.1 and the equivalence between equations (4.7) and (D.7), we obtain that for every σ_1 with $\sigma_1^{\nu} = \frac{\tilde{a}_0}{m\,b_0}$ equation (4.7) either has no formal power series solutions of order one or it has a one-parametric family. In the affirmative case, two of the solutions are equal if they coincide up to order $\nu+1$. The others properties of these solutions are proven analogously as in the preceding case. □

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