ON THE PARITY OF SOME PARTITION FUNCTIONS

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Abstract. Recently, Andrews carried out a thorough investigation of integer partitions in which all parts of a given parity are smaller than those of the opposite parity. Further, considering a subset of this set of partitions, he obtains several interesting arithmetic and combinatorial properties and its connections to the third order mock theta function $\nu(q)$. In fact, he shows the existence of a Dyson-type crank that explains a mod 5 congruence in this subset. At the end of his paper, one of the problems he poses is to undertake a more extensive investigation on the properties of the subset of partitions. Since then there have been several investigations in various ways, including works of Jennings-Shaffer and Bringmann (Ann. Comb. 2019), Barman and Ray (2019), and Uncu (2019). In this paper, we study certain congruences satisfied by the above set of partitions (and the subset above) along with a certain subset of partitions (of Andrews' partitions above) studied by Uncu and also establish a connection between one of Andrews' partition function above with $p(n)$, the number of unrestricted partitions of n. Besides, we provide a combinatorial description of Uncu's partition function.

1. Introduction

In a recent work, Andrews [1] investigated integer partitions in which each even part is less than each odd part. For $n \geq 1$, he denotes by $\mathcal{EO}(n)$ the number of such partitions of n. Among several arithmetic results, he shows that the generating function for $\mathcal{EO}(n)$ is:

$$\sum_{n\geq 0} \mathcal{EO}(n)q^n = \frac{1}{(1-q)(q^2;q^2)_\infty}. \tag{1.1}$$

Andrews then delves into a subset of partitions which are enumerated by $\mathcal{EO}(n)$. For $n \geq 1$, he denotes by $\overline{\mathcal{EO}}(n)$ the number of partitions counted by $\mathcal{EO}(n)$ in which <u>only</u> the largest part appears an odd number of times. It turns out that $\overline{\mathcal{EO}}(n)$ satisfies several arithmetic and combinatorial properties and has connections to the third order mock theta function, $\nu(q)$ which is defined by:

$$\nu(q) = \sum_{n\geq 0} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}, \tag{1.2}$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{1.3}$$

In fact, Andrews shows that the generating function for $\overline{\mathcal{EO}}(n)$ is:

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Theorem 1.1. *We have*

$$\sum_{n\geq 0}\overline{\mathcal{EO}}(n)q^n=\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}=\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2}. \tag{1.4}$$

Besides, he proves the following congruence for $\overline{\mathcal{EO}}(n)$, similar in vein to Ramanujan's partition congruence mod 5:

Theorem 1.2.

$$\overline{\mathcal{EO}}(10n+8)\equiv 0 \pmod 5.$$

More so, he proves Theorem 1.2 combinatorially by providing a Dyson-type crank for partitions enumerated by $\mathcal{EO}(n)$. At the end of his paper, Andrews proposes to undertake a more extensive investigation of $\overline{\mathcal{EO}}(n)$.

Actually, the partition class enumerated by $\mathcal{EO}(n)$ above is one of eight different classes of partitions which Andrews considered in his recent paper [2]. Recently, Jennings-Shaffer and Bringmann [5] obtained new identities for three of these partition classes. By treating a different subset of $\mathcal{EO}(n)$, Uncu [11] has studied several interesting properties of 4-decorated diagrams. If we denote by $\mathcal{EO}_u(n)$ the partition function defined by Uncu, then the generating function is given by

$$\sum_{n\geq 0}\mathcal{EO}_u(n)q^n=\frac{1}{(q^2;q^4)_\infty^2}=\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2}. \tag{1.5}$$

Recently, among other things, Barman and Ray [4] studied various congruences satisfied by $\overline{\mathcal{EO}}(n)$. By using the theory of Hecke operators, they proved an infinite family of congruences satisfied by $\overline{\mathcal{EO}}(n)$. More precisely, they proved the following:

Theorem 1.3. *Let k,n be non-negative integers. For each i with $1\leq i\leq k+1$, if $p_i\geq 5$ is prime such that $p_i\equiv 2 \pmod 3$, then for any integer $j\equiv 0 \pmod{p_{k+1}}$*

$$\overline{\mathcal{EO}}\left(p_1^2\cdots p_{k+1}^2 n+\frac{p_1^2\cdots p_k^2 p_{k+1}(3j+p_{k+1})-1}{3}\right)\equiv 0 \pmod 2.$$

For a prime $p\geq 5$ with $p\equiv 2 \pmod 3$, by specializing $p_1=p_2=\cdots=p_{k+1}=p$, we see from Theorem 1.3 that

$$\overline{\mathcal{EO}}\left(p^{2(k+1)}n+\frac{p^{2k+1}(3j+p)-1}{3}\right)\equiv 0 \pmod 2. \tag{1.6}$$

In particular, by choosing $p=5$ and $k=0$ in (1.6) we obtain

$$\overline{\mathcal{EO}}(25n+5j+8)\equiv 0 \pmod 2. \tag{1.7}$$

They further prove that Andrews' congruence in Theorem 1.2 is also true modulo 4, except when $n\equiv 0 \pmod 5$.

Theorem 1.4. *Let $t\in\{1,2,3,4\}$. Then for all $n\geq 0$ we have*

$$\overline{\mathcal{EO}}(10(5n+t)+8)\equiv 0 \pmod 4.$$

In this paper, we prove several congruences modulo 2 for each of $\mathcal{EO}(n),\overline{\mathcal{EO}}(n)$ and $\mathcal{EO}_u(n)$. We prove these congruences using tools from modular forms and a 2-dissection formula from Ramanujan's notebook.

We now give a brief outline of the paper. In Sect. 3 we recall the definition and some basic results from modular forms and Hecke theory. In Sect. 4 we state a 2-dissection formula from Ramanujan's notebooks [3, Entry 25, p. 40] and also state a few classical results of Euler and Jacobi. In Sect. 5 we discuss some new results for Andrews' $\mathcal{EO}(n)$. In Sect. 6 we discuss some new results related to the parity of $\overline{\mathcal{EO}}(n)$ and also prove infinitely many congruences satisfied by $\overline{\mathcal{EO}}(n)$. In Sect. 7 we discuss some new results related to the parity of $\mathcal{EO}_u(n)$ and prove two results that yield infinitely many congruences satisfied by $\mathcal{EO}_u(n)$. Finally, in Sect. 8 we discuss a few conjectural congruences that we observed via computation and the limitations of our method.

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3. Notations and preliminaries

In this section, we recall some basic facts from modular forms and Hecke theory (see, for example [7, 9] for more details). The full modular group $SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

For a positive integer N, we denote the congruence subgroup $\Gamma_0(N)$ of level N of $SL_2(\mathbb{Z})$ as follows.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

The group $GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a.b.c, d \in \mathbb{R},\ ad - bc > 0 \right\}$ acts on $\mathcal{H}$ via fractional linear transformations, i.e., for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and $\tau \in \mathcal{H}$

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

Let $k \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For a complex-valued function f define the slash operator as follows;

$$f \mid_k \gamma(\tau) := (c\tau + d)^{-k} f(\gamma \cdot \tau).$$

Definition 3.1 (Modular Forms)**.** Let k be an integer and χ a Dirichlet character modulo N. A holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight k, level N and character χ if

(1) $f \mid_k \gamma(\tau) \ = \ \chi(d) f(\tau)\ \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, i.e,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau), \quad \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

(2) f is holomorphic at all the cusps of $\Gamma_0(N)$.

Further, we say that f is a cusp form if f vanishes at all the cusps of $\Gamma_0(N)$.

We denote the space of modular forms and the subspace of cusp forms of weight k and character χ for $\Gamma_0(N)$, by $M_k(N,\chi)$ and $S_k(N,\chi)$ respectively. If χ is the trivial character, then we write the spaces as $M_k(N)$ and $S_k(N)$.

A modular form f has Fourier series expansion as follows;

$$f(\tau) = \sum_{n\geq 0} a(n)q^n, \ q := e^{2\pi i\tau}.$$

We now define certain linear operators called *Hecke operators* on the space of modular forms.

Definition 3.2. Let m be a positive integer and $f(\tau) = \sum_{n\geq 0} a(n)q^n \in M_k(N,\chi)$. Then the mth Hecke operator is defined by

$$f(\tau)|T_m := \sum_{n\geq 0}\left(\sum_{d|(n,m)} \chi(d)d^{k-1}a\left(\frac{mn}{d^2}\right)\right)q^n. \tag{3.1}$$

Definition 3.3. A modular form $f(\tau) = \sum_{n\geq 0} a(n)q^n \in M_k(N,\chi)$ is called a Hecke eigenform if it is an eigenfunction for all the Hecke operators $T_m,\ m\geq 2$. i.e., for every $m\geq 2$ there exists $\lambda(m)\in\mathbb{C}$ such that $f(\tau)|T_m = \lambda(m)f(\tau)$.

Let $a,q\in\mathbb{C}$ and $n\in\mathbb{N}$. Throughout we use the following notation defined below:

$$(a;q)_\infty \quad := \quad \prod_{n=1}^{\infty}(1-aq^{n-1}).$$

We now define the Dedekind η-function which is a modular form of weight $\frac{1}{2}$ over $SL_2(\mathbb{Z})$ with a certain character of order 24.

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = q^{1/24}(q;q)_\infty. \tag{3.2}$$

4. Known results

For a positive integer k, we denote by T_k the kth triangular number defined by

$$T_k := \frac{k(k+1)}{2}, \tag{4.1}$$

and by ω_k the kth pentagonal number defined by

$$\omega_k := \frac{3k^2+k}{2}, \tag{4.2}$$

so that replacing k by $-k$ we get the pentagonal number ω_{-k} defined by:

$$\omega_{-k} := \frac{3k^2-k}{2}. \tag{4.3}$$

We next state the following classical results of Euler and Jacobi.

Theorem 4.1 (Euler). *We have*

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n,$$

where $p(n)$ is the number of unrestricted partitions of n.

Theorem 4.2 (Euler Pentagonal Number theorem). *We have*

$$(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\omega_k}.$$

Theorem 4.3 (Jacobi). *We have*

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{T_n}.$$

Let us define the following theta series:

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \ \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{4.4}$$

Then by simple application of Jacobi triple product identity (see [3, Entry 25, p.40]), one obtains:

Lemma 4.4. *We have*

$$\phi(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \qquad \psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty},$$

$$\phi(q) - \phi(q^4) = 2q\psi(q^8).$$

Lemma 4.4 yields the following 2-dissection formula, which would be very crucial in our proofs later.

Lemma 4.5. *We have*

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5 (q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2 (q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5 (q^8;q^8)_\infty}. \tag{4.5}$$

5. Some results for $\mathcal{EO}(n)$

In this section, we state and prove some results for $\mathcal{EO}(n)$. The following theorem describes the connection of $\mathcal{EO}(n)$ with $p(n)$.

Theorem 5.1. *For all $n \geq 0$ we have*

$$p(n) = \mathcal{EO}(2n) - \mathcal{EO}(2n-1),$$

where $p(n)$ is the number of unrestricted partitions of n.

The next theorem asserts that the odd and the even parts of the generating function for $\mathcal{EO}(n)$ are exactly same. More precisely

Theorem 5.2. *We have*

$$\sum_{n\geq 0} \mathcal{EO}(2n)q^n = \sum_{n\geq 0} \mathcal{EO}(2n+1)q^n = \frac{1}{(1-q)(q;q)_\infty}.$$

Proof of Theorem 5.1. In (1.1), we multiply both sides by $(1-q)$ and rearrange the sum on the left-hand side to obtain:

$$1+\sum_{n\geq 1}(\mathcal{EO}(n)-\mathcal{EO}(n-1))q^n = \frac{1}{(q^2;q^2)_\infty}. \tag{5.1}$$

Since the right-hand side of (5.1) clearly has all even powers of q when expanded as a power series, by replacing $q \mapsto \sqrt{q}$ we immediately obtain:

$$1+\sum_{n\geq 1}(\mathcal{EO}(2n)-\mathcal{EO}(2n-1))q^n = \frac{1}{(q;q)_\infty}. \tag{5.2}$$

The result now follows from Theorem 4.1. □

Proof of Theorem 5.2. We again start with (1.1). Multiply and divide the numerator and denominator of the right-hand side of (1.1) by $(1+q)$ to get

$$\begin{aligned}\sum_{n\geq 0}\mathcal{EO}(n)q^n &= \frac{1+q}{(1-q^2)(q^2;q^2)_\infty},\\ &= \frac{1}{(1-q^2)(q^2;q^2)_\infty}+q\frac{1}{(1-q^2)(q^2;q^2)_\infty}. \end{aligned}\tag{5.3}$$

Note that the first sum in the right-hand side of (5.3) has all the even exponents of q and the second sum has all the odd exponents of q. Next we split the sum in the left-hand side of (5.3) into two sums, one involving all the even powers of q and the other involving all the odd powers of q. Finally, comparing both sides, the result follows with $q \mapsto \sqrt{q}$. □

As corollaries of Theorems 5.1 and 5.2 we obtain:

Corollary 5.2.1. *For all $n \geq 0$ we have*

$$\mathcal{EO}(2n)=\mathcal{EO}(2n+1).$$

Corollary 5.2.2. *For all $n \geq 1$ we have*

$$p(n)=\mathcal{EO}(2n+1)-\mathcal{EO}(2n-1)=\mathcal{EO}(2n)-\mathcal{EO}(2n-2).$$

Proofs of Corollaries 5.2.1 *and* 5.2.2. Corollary 5.2.1 follows from Theorem 5.2 and Corollary 5.2.2 follows from Theorem 5.1 and Corollary 5.2.1. □

We end this section with the following congruences satisfied by $\mathcal{EO}(n)$.

Theorem 5.3. *For all $n \geq 1$ we have*

$$\begin{aligned}\mathcal{EO}(10n+7)\equiv\mathcal{EO}(10n+8)\equiv\mathcal{EO}(10n+9) \pmod 5,\\ \mathcal{EO}(14n+9)\equiv\mathcal{EO}(14n+10)\equiv\mathcal{EO}(14n+11) \pmod 7,\\ \mathcal{EO}(22n+11)\equiv\mathcal{EO}(22n+12)\equiv\mathcal{EO}(22n+13) \pmod{11}.\end{aligned}$$

Proof of Theorem 5.3. We recall Ramanujan's partition congruences modulo $5, 7$ and 11.

$$\begin{aligned}p(5n+4) &\equiv 0 \pmod 5,\\ p(7n+5) &\equiv 0 \pmod 7,\\ p(11n+6) &\equiv 0 \pmod{11}.\end{aligned}$$

Next invoking Theorem 5.1 and Corollary 5.2.2, the result follows. □

6. Some results for $\overline{\mathcal{EO}}(n)$

Here we discus some congruences for $\overline{\mathcal{EO}}(n)$. To do that we first let $d_{\ell,6}(n)$ to denote the number of positive divisors, d of n such that $d \equiv \ell \pmod 6$. Also let $r_{\Delta,\Pi}(n)$ be the number of representations of n as a sum of a triangular and a pentagonal number. Then Hirschhorn [6] proves:

Theorem 6.1. *We have*

$$r_{\Delta,\Pi}(n) = d_{1,6}(n) - d_{5,6}(n).$$

We now state our first result below which asserts the even-odd parity of $\overline{\mathcal{EO}}(2n)$.

Theorem 6.2. *For every $n \geq 1$ we have*

$$\overline{\mathcal{EO}}(2n) \equiv \begin{cases} 1 \pmod 2 & \text{if } d_{1,6}(n) \not\equiv d_{5,6}(n) \pmod 2, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

Next, we obtain the following generating functions over certain arithmetic progressions for $\overline{\mathcal{EO}}(n)$.

Theorem 6.3. *We have*

$$\begin{aligned} \sum_{n\geq 0} \overline{\mathcal{EO}}(8n+2)q^n &= 2\frac{(q^4;q^4)_\infty^7}{(q;q)_\infty^3(q^2;q^2)_\infty(q^8;q^8)_\infty^2}, \\ \sum_{n\geq 0} \overline{\mathcal{EO}}(8n+4)q^n &= 2\frac{(q^2;q^2)_\infty^7(q^8;q^8)_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^3}, \\ \sum_{n\geq 0} \overline{\mathcal{EO}}(8n+6)q^n &= 4\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^3}. \end{aligned}$$

Proof of Theorem 6.2. Recall from Theorem 1.1 that the product on the right-hand side of the identity involves only the even powers of q, so that upon replacing $q \mapsto \sqrt{q}$, we obtain:

$$\sum_{n\geq 0} \overline{\mathcal{EO}}(2n)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}. \tag{6.1}$$

Modulo 2, (6.1) becomes

$$\sum_{n\geq 0} \overline{\mathcal{EO}}(2n)q^n \equiv (q;q)_\infty^4 \pmod 2. \tag{6.2}$$

Write $(q;q)_\infty^4 = (q;q)_\infty^3 \cdot (q;q)_\infty$ and invoke Theorems 4.2 and 4.3 to obtain:

$$\begin{aligned} (q;q)_\infty^4 &= \left(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{T_m}\right)\left(\sum_{k=-\infty}^{\infty}(-1)^k q^{\omega_k}\right), \\ &= \sum_{m=0}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^{m+k}(2m+1)q^{T_m+\omega_k}. \end{aligned}$$

Now, the coefficient of q^n in the right-hand side of (6.3) is essentially the sum of numbers of the form $(-1)^{m+k}(2m+1)$ (which is odd) such that such $n = T_m + \omega_k$ for some $m \in \mathbb{N}$ and $k \in \mathbb{Z}$. Speaking differently, the coefficient of q^n is the sum of odd numbers of the above form added $r_{\Delta,\Pi}(n)$ times. Hence the result follows from this and (6.2). $\square$

Proof of Theorem 6.3. Recall from Theorem 1.1 that the product on the right-hand side of the identity involves only the even powers of q, so that we have

$$\sum_{n\geq 0}\overline{\mathcal{EO}}(2n)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}. \tag{6.3}$$

Next, we apply Lemma 4.5 in (6.3) to obtain

$$\begin{aligned}\sum_{n\geq 0}\overline{\mathcal{EO}}(2n)q^n &= (q^2;q^2)_\infty^3\left\{\frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2}+2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty}\right\},\\ &= \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^2(q^{16};q^{16})_\infty^2}+2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty}. \qquad (6.4)\end{aligned}$$

As before, we split the left-hand side of (6.4) into odd and even parts to obtain

$$\sum_{n\geq 0}\overline{\mathcal{EO}}(4n)q^n = \frac{(q^4;q^4)_\infty^5}{(q;q)_\infty^2(q^8;q^8)_\infty^2}, \tag{6.5}$$

$$\sum_{n\geq 0}\overline{\mathcal{EO}}(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^2(q^4;q^4)_\infty}. \tag{6.6}$$

Let us pause for a while and take a closer look at the identities in (6.5) and (6.6). We see that the right-hand sides of each of these identities have a factor $1/(q;q)_\infty^2$ multiplied and this allows us again to apply the 2-dissection formula of Lemma 4.5 in each of them. This yields

$$\sum_{n\geq 0}\overline{\mathcal{EO}}(4n)q^n = \frac{(q^4;q^4)_\infty^5(q^8;q^8)_\infty^3}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2}+2q\frac{(q^4;q^4)_\infty^7(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty^3},$$

$$\sum_{n\geq 0}\overline{\mathcal{EO}}(4n+2)q^n = 2\frac{(q^8;q^8)_\infty^7}{(q^2;q^2)_\infty^3(q^4;q^4)_\infty(q^{16};q^{16})_\infty^2}+4q\frac{(q^4;q^4)_\infty(q^8;q^8)_\infty(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^3}.$$

The result now follows by comparing even and odd parts of the sums on both sides of the above identities and $q\mapsto\sqrt{q}$. □

Corollary 6.3.1. *For all $n\geq 0$ we have*

$$\begin{aligned}\overline{\mathcal{EO}}(4n+2)&\equiv 0 \pmod 2,\\ \overline{\mathcal{EO}}(8n+4)&\equiv 0 \pmod 2,\\ \overline{\mathcal{EO}}(8n+6)&\equiv 0 \pmod 4.\end{aligned}$$

Proof of Corollary 6.3.1. The result is immediate from Theorem 6.3 by comparing coefficients on both sides. □

The next theorem gives us infinitely many congruences for $\overline{\mathcal{EO}}(n)$. It can be thought of as the odd-prime-power counterpart of Theorem 1.3 in the case $k=0$ where instead of a p^2n we have a pn on one side and a p^3n on the other side of the congruence.

Theorem 6.4. *Let $p\geq 5$ be a prime with $p\equiv 2 \pmod 3$. Let $r\in\mathbb{Z}$ be such that $3r+2\equiv 0 \pmod p$. Then*

$$\overline{\mathcal{EO}}\left(p^3n+pr+\frac{2p-1}{3}\right)\equiv\overline{\mathcal{EO}}\left(pn+\frac{3r+2-p}{3p}\right) \pmod 2.$$

For $(p, r) = (5, 11)$ and $(p, r) = (5, 21)$ we obtain, for example, the following congruences from Theorem 6.4 after replacing $n \mapsto 2n$:

$$\overline{\mathcal{EO}}\,(250n + 58) \equiv \overline{\mathcal{EO}}\,(10n + 2) \pmod 2, \tag{6.7}$$

$$\overline{\mathcal{EO}}\,(250n + 108) \equiv \overline{\mathcal{EO}}\,(10n + 4) \pmod 2. \tag{6.8}$$

Proof of Theorem 6.4. We know that

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^n = \frac{(q^4; q^4)_\infty^3}{(q^2; q^2)_\infty^2} \equiv (q; q)_\infty^8 \pmod 2, \tag{6.9}$$

which yields upon substituting $q \mapsto q^3$

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^{3n+1} \equiv \eta^8(3\tau) \pmod 2. \tag{6.10}$$

Now let $\eta^8(3\tau) = \sum_{n=0}^{\infty} a(n) q^n$. Then it is clear that $a(n) = 0$ unless $n \equiv 1 \pmod 3$. Thus

$$\overline{\mathcal{EO}}(n) \equiv a(3n + 1) \pmod 2. \tag{6.11}$$

By [8, Theorems 1.64 and 1.65, p. 18] we see that $\eta^8(3\tau) \in \mathcal{S}_4\,(9)$. From [10, Table I, p. 4852], we know that $\eta^8(3\tau)$ is a Hecke eigenform. Therefore, for a prime p we have

$$\eta^8(3\tau) | T_p = \sum_{n=0}^{\infty} \left(a(pn) + p^3 a\left(\frac{n}{p}\right) \right) q^n = \lambda(p) \sum_{n=0}^{\infty} a(n) q^n. \tag{6.12}$$

By comparing coefficents of q^n on both sides of (6.12) we see that

$$a(pn) + p^3 a\left(\frac{n}{p}\right) = \lambda(p) a(n). \tag{6.13}$$

Now choose $n = 1$ in (6.13) and note that $a(1) = 1$. This gives

$$\lambda(p) = a(p).$$

Let $5 \leq p \equiv 2 \pmod 3$. Thus $a(p) = \lambda(p) = 0$ and from (6.13) we get

$$a(pn) + p^3 a\left(\frac{n}{p}\right) = 0. \tag{6.14}$$

Modulo 2, (6.14) becomes

$$a(pn) \equiv a\left(\frac{n}{p}\right) \pmod 2. \tag{6.15}$$

In (6.15), we set $n \mapsto (3n + 2)$ so that we have

$$a(3pn + 2p) \equiv a\left(\frac{3n + 2}{p}\right) \pmod 2. \tag{6.16}$$

We further set $n \mapsto p^2 n + r$ such that $p|(3r + 2)$ in (6.16). This yields

$$a(3p^3 n + 3pr + 2p) \equiv a\left(3pn + \frac{3r + 2}{p}\right) \pmod 2. \tag{6.17}$$

The result now follows from (6.11) and (6.17). We note here that since $5 \leq p \equiv 2 \pmod 3$ and $p|(3r + 2)$, both $(2p - 1)/3$ and $(3r + 2 - p)/3p$ are integers. $\square$

7. Some results for $\mathcal{EO}_u(n)$

To begin with, we obtain a different interpretation of the function $\mathcal{EO}_u(n)$, and for that we start with the following identity:

Theorem 7.1 (Cauchy). *For $a \in \mathbb{C}$, $|q| < 1, |t| < 1$ we have*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} t^n = \frac{(at;q)_\infty}{(t;q)_\infty}.$$

We will now state our first result related to $\mathcal{EO}_u(n)$ below.

Theorem 7.2. *We have*

$$\sum_{n\geq 0} \mathcal{EO}_u(n) q^n = \sum_{n\geq 0} \frac{q^{2n}}{[(1-q^{2\cdot 2})(1-q^{2\cdot 4})\cdots(1-q^{2\cdot 2n})]^2(1-q^{2(2n+1)})(1-q^{2(2n+2)})\cdots}$$

The combinatorial interpretation of Theorem 7.2 is a follows.

Theorem 7.3. *For $n \geq 1$, $\mathcal{EO}_u(n)$ represents the number of vector partitions (π_1, π_2, π_3) of n with the following properties:*

1. *Every part in π_1 is even and repeats an even number of times, except the largest part which repeats an odd number of times.*
2. *Every part in π_2 is same as in π_1 and they repeat an even number of times.*
3. *Every part in π_3 is bigger than every part of π_1 (and hence π_2) and they repeat an even number of times.*

Remark 7.4. We remark here that Theorem 7.3 can be combinatorially described in terms of two vector partitions (π_1, π_2) where π_1 is a certain weighted partition and π_2 is same as π_3 in Theorem 7.3.

Proof of Theorem 7.2. We first replace $q \mapsto q^4$ in Theorem 7.1 and then put $a = t = q^2$ to obtain:

$$\sum_{n\geq 0} \frac{(q^2;q^4)_n}{(q^4;q^4)_n} q^{2n} = \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty},$$

which yields upon multiplying both sides by $(q^2;q^2)_\infty^{-2}$ the following:

$$\frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} \frac{(q^2;q^4)_n}{(q^4;q^4)_n} q^{2n} = \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2} = \sum_{n\geq 0} \mathcal{EO}_u(n) q^n. \tag{7.1}$$

Finally we show that the left-hand side in (7.1) is what we desire in the theorem. For that we multiply the denominator and numerator of each summand in the left-hand side of (7.1) by $(q^4;q^4)_n$ to get

$$\begin{aligned}\frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} \frac{(q^2;q^4)_n}{(q^4;q^4)_n} q^{2n} &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} \frac{(q^2;q^2)_{2n}}{(q^4;q^4)_n^2} q^{2n} \\ &= \sum_{n\geq 0} \frac{q^{2n}}{(q^4;q^4)_n^2 (q^{2(2n+1)};q^2)_\infty}.\end{aligned}$$

□

Proof of Theorem 7.3. Let π_1 and π_2 denote all partitions generated as follows:

$$\begin{aligned}
\pi_1 : &\qquad \frac{q^{2n}}{(1-q^{2\cdot 2})(1-q^{2\cdot 4})\cdots(1-q^{2\cdot 2n})},\\
\pi_2 : &\qquad \frac{1}{(1-q^{2\cdot 2})(1-q^{2\cdot 4})\cdots(1-q^{2\cdot 2n})},\\
\pi_3 : &\qquad \frac{1}{(1-q^{2(2n+1)})(1-q^{2(2n+2)})\cdots}.
\end{aligned}$$

Then it is clear that π_1 and π_2 satisfies the properties stated in Theorem 7.3. □

Finally we state and prove some congruences for $\mathcal{EO}_u(n)$. We start with the following theorem:

Theorem 7.5.

$$\sum_{n\geq 0}\mathcal{EO}_u(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^3(q^4;q^4)_\infty}.$$

Corollary 7.5.1. *For all $n \geq 1$ we have*

$$\mathcal{EO}_u(4n+2) \equiv 0 \pmod 2.$$

Proof of Theorem 7.5. We start with the generating function in (1.5) and apply the 2-dissection formula with $q \mapsto q^2$ of Lemma 4.5 to the far right-hand side of (1.5) to obtain:

$$\begin{aligned}
\sum_{n\geq 0}\mathcal{EO}_u(n)q^n &= (q^4;q^4)_\infty^2\left\{\frac{(q^{16};q^{16})_\infty^5}{(q^4;q^4)_\infty^5(q^{32};q^{32})_\infty^2}+2q^2\frac{(q^8;q^8)_\infty^2(q^{32};q^{32})_\infty^2}{(q^4;q^4)_\infty^5(q^{16};q^{16})_\infty}\right\},\\
&= \frac{(q^{16};q^{16})_\infty^5}{(q^4;q^4)_\infty^3(q^{32};q^{32})_\infty^2}+2q^2\frac{(q^8;q^8)_\infty^2(q^{32};q^{32})_\infty^2}{(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty}. \qquad (7.2)
\end{aligned}$$

Clearly, all the exponents of q in the right-hand side of (7.2) when expanded as a power series are $\equiv 0,2 \pmod 4$. We now split the sum on the left-hand side of (7.2) into four parts, where in each part all the exponents of q are either $\equiv 0,1,2,3 \pmod 4$. Next, comparing both sides of (7.2) and changing $q \mapsto q^{1/4}$ we get

$$\begin{aligned}
\sum_{n\geq 0}\mathcal{EO}_u(4n)q^n &= \frac{(q^4;q^4)_\infty^5}{(q;q)_\infty^3(q^8;q^8)_\infty^2},\\
\sum_{n\geq 0}\mathcal{EO}_u(4n+2)q^n &= 2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^3(q^4;q^4)_\infty}. \qquad (7.3)
\end{aligned}$$

Hence the result follows. □

Proof of Corollary 7.5.1. This is immediate from Theorem 7.5 by comparing coefficients on both sides of the identity. □

The next theorem describes the odd parity of $\mathcal{EO}_u(4n)$.

Theorem 7.6. *For all $n \geq 1$ we have*

$$\mathcal{EO}_u(4\cdot\omega_n) \equiv \mathcal{EO}_u(4\cdot\omega_{-n}) \equiv 1 \pmod 2.$$

Proof of Theorem 7.6. We start with the first identity in (7.3). Modulo 2, the identity becomes

$$\sum_{n\geq 0} \mathcal{EO}_u(4n)q^n \equiv (q;q)_\infty \pmod 2. \tag{7.4}$$

Using Theorem 4.2 in (7.4) we obtain

$$\sum_{n\geq 0} \mathcal{EO}_u(4n)q^n \equiv 1 + \sum_{n=1}^{\infty}(-1)^n(q^{\omega_n} + q^{\omega_{-n}}) \pmod 2. \tag{7.5}$$

The result now follows from (7.5). □

The following theorem yields infinitely many congruences for $\mathcal{EO}_u$, and is similar in vein to Theorem 1.3 of Barman and Ray [4].

Theorem 7.7. *For non-negative integers k, n, let for each $1 \leq i \leq k+1$, p_i denote a prime such that $5 \leq p_i \equiv 5 \pmod 6$. Then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$ we have*

$$\mathcal{EO}_u\left(p_1^2\cdots p_{k+1}^2 n + \frac{p_1^2\cdots p_k^2 p_{k+1}(6j+p_{k+1})-1}{6}\right) \equiv 0 \pmod 2.$$

For a prime $p \geq 5$ with $p \equiv 5 \pmod 6$, by specializing $p_1 = p_2 = \cdots = p_{k+1} = p$, we see from Theorem 7.7 that

$$\mathcal{EO}_u\left(p^{2(k+1)}n + \frac{p^{2k+1}(6j+p)-1}{6}\right) \equiv 0 \pmod 2. \tag{7.6}$$

In particular, by choosing $p = 5$ and $k = 0$ in (7.6) we obtain

$$\mathcal{EO}_u(25n + 5j + 4) \equiv 0 \pmod 2. \tag{7.7}$$

Proof of Theorem 7.7. We know that

$$\sum_{n=0}^{\infty} \mathcal{EO}_u(n)q^n = \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2} \equiv (q;q)_\infty^4 \pmod 2, \tag{7.8}$$

which yields upon substituting $q \mapsto q^6$

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)q^{6n+1} \equiv \eta^4(6\tau) \pmod 2. \tag{7.9}$$

Now let $\eta^4(6\tau) = \sum\limits_{n=0}^{\infty} b(n)q^n$. Then it is clear that $b(n) = 0$ unless $n \equiv 1 \pmod 6$. Thus

$$\mathcal{EO}_u(n) \equiv b(6n+1) \pmod 2. \tag{7.10}$$

By [8, Theorems 1.64 and 1.65, p. 18] we see that $\eta^4(6\tau) \in \mathcal{S}_2(36)$. From [10, Table I, p. 4852], we know that $\eta^4(6\tau)$ is a Hecke eigenform so that for a prime p if T_p is the Hecke operator, then

$$\eta^4(6\tau)|T_p = \sum_{n=0}^{\infty}\left(b(pn) + pb\left(\frac{n}{p}\right)\right)q^n = \lambda(p)\sum_{n=0}^{\infty} b(n)q^n. \tag{7.11}$$

By comparing coefficients of q^n on both sides of (7.11) we see that

$$b(pn) + pb\left(\frac{n}{p}\right) = \lambda(p)b(n). \tag{7.12}$$

Now choose $n = 1$ in (7.12) and note that $b(1) = 1$. This gives

$$\lambda(p) = b(p).$$

Let $5 \leq p \equiv 5 \pmod 6$. Thus $b(p) = \lambda(p) = 0$ and from (7.12) we get

$$b(pn) + pb\left(\frac{n}{p}\right) = 0. \tag{7.13}$$

We now set $n \mapsto pn + r$ where $(p, r) = 1$. Thus (7.13) yields

$$b(p^2 n + pr) = 0, \tag{7.14}$$

and setting $n \mapsto pn$ yields modulo 2

$$b(p^2 n) \equiv b(n) \pmod 2. \tag{7.15}$$

Again, substituting n by $6n - pr + 1$ in (7.14) and together with (7.10) we get

$$\mathcal{EO}_u\left(p^2 n + \frac{p^2-1}{6} - pr\frac{p^2-1}{6}\right) \equiv 0 \pmod 2, \tag{7.16}$$

and substituting n by $6n + 1$ in (7.15) and using (7.10) we get

$$\mathcal{EO}_u\left(p^2 n + \frac{p^2-1}{6}\right) \equiv \mathcal{EO}_u(n) \pmod 2. \tag{7.17}$$

Since $p \geq 5$, it follows that $6|(1-p^2)$ and $(\frac{1-p^2}{6}, p) = 1$. Hence when r runs over a residue system excluding multiples of p, so does $\frac{1-p^2}{6}r$. Thus (7.16) can be rewritten as

$$\mathcal{EO}_u\left(p^2 n + \frac{p^2-1}{6} + pj\right) \equiv 0 \pmod 2, \tag{7.18}$$

where $(j, p) = 1$. Next, for each $1 \leq i \leq k$ let $p_i \geq 5$ be primes such that $p_i \equiv 5 \pmod 6$. Since

$$p_1^2 \cdots p_k^2 n + \frac{p_1^2 \cdots p_k^2 - 1}{6} = p_1^2\left(p_2^2 \cdots p_k^2 n + \frac{p_2^2 \cdots p_k^2 - 1}{6}\right) + \frac{p_1^2-1}{6},$$

by repeatedly applying (7.17) we obtain

$$\mathcal{EO}_u\left(p_1^2 \cdots p_k^2 n + \frac{p_1^2 \cdots p_k^2 - 1}{6}\right) \equiv \mathcal{EO}_u(n) \pmod 2. \tag{7.19}$$

Finally let $p_{k+1} \geq 5$ be a prime and j be such that $(j, p_{k+1}) = 1$. Then (7.18) and (7.19) yield the theorem. $\square$

Finally, we state and prove the following counterpart to Theorem 7.7, similar in vein to Theorem 6.4.

Theorem 7.8. *Let $p \geq 5$ be any prime such that $p \equiv 5 \pmod 6$. Let $r \in \mathbb{Z}$ be such that $6r + 5 \equiv 0 \pmod p$. Then*

$$\mathcal{EO}_u\left(p^3 n + pr + \frac{5p-1}{6}\right) \equiv \mathcal{EO}_u\left(pn + \frac{6r+5-p}{6p}\right) \pmod 2.$$

For $(p,r)=(5,5)$ and $(p,r)=(5,10)$ we obtain, for example, the following congruences from Theorem 7.8 after replacing $n\mapsto 2n+1$ in the first case and $n\mapsto 2n$ in the second case:

$$\mathcal{EO}_u\left(250n+154\right) \equiv \mathcal{EO}_u\left(10n+6\right) \pmod 2, \tag{7.20}$$

$$\mathcal{EO}_u\left(250n+54\right) \equiv \mathcal{EO}_u\left(10n+2\right) \pmod 2. \tag{7.21}$$

Proof of Theorem 7.8. We know that

$$\sum_{n=0}^{\infty}\mathcal{EO}_u(n)q^n=\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2}\equiv (q;q)^4 \pmod 2, \tag{7.22}$$

which yields upon substituting $q\mapsto q^6$

$$\sum_{n=0}^{\infty}\mathcal{EO}_u(n)q^{6n+1}\equiv \eta^4(6\tau) \pmod 2. \tag{7.23}$$

Now let $\eta^4(6\tau)=\sum\limits_{n=0}^{\infty}b(n)q^n$. Then it is clear that $b(n)=0$ unless $n\equiv 1 \pmod 6$. Thus

$$\mathcal{EO}_u(n)\equiv b(6n+1) \pmod 2. \tag{7.24}$$

By [8, Theorems 1.64 and 1.65, p. 18] we see that $\eta^4(6\tau)\in\mathcal{S}_2\left(36\right)$. From [10, Table I, p. 4852], we know that $\eta^4(6\tau)$ is a Hecke eigenform. Therefore, for a prime p we have

$$\eta^4(6\tau)|T_p=\sum_{n=0}^{\infty}\left(b(pn)+pb\left(\frac{n}{p}\right)\right)q^n=\lambda(p)\sum_{n=0}^{\infty}b(n)q^n. \tag{7.25}$$

By comparing coefficients of q^n on both sides of (7.25) we see that

$$b(pn)+pb\left(\frac{n}{p}\right)=\lambda(p)b(n). \tag{7.26}$$

Now choose $n=1$ in (7.26) and note that $b(1)=1$. This gives

$$\lambda(p)=b(p).$$

Let $5\le p\equiv 5 \pmod 6$. Thus $b(p)=\lambda(p)=0$ and from (7.26) we get

$$b(pn)+pb\left(\frac{n}{p}\right)=0. \tag{7.27}$$

Modulo 2, (7.27) becomes

$$b(pn)\equiv b\left(\frac{n}{p}\right) \pmod 2. \tag{7.28}$$

In (7.28), we set $n\mapsto(6n+5)$ so that we have

$$b(6pn+5p)\equiv b\left(\frac{6n+5}{p}\right) \pmod 2. \tag{7.29}$$

We further set $n\mapsto p^2n+r$ such that $p|(6r+5)$ in (7.29). This yields

$$b(6p^3n+6pr+5p)\equiv b\left(6pn+\frac{6r+5}{p}\right) \pmod 2. \tag{7.30}$$

The result now follows from (7.24) and (7.30). We note here that since $5\le p\equiv 5 \pmod 6$ and $p|(6r+5)$, both $(5p-1)/6$ and $(6r+5-p)/6p$ are integers. $\square$

8. Concluding remarks

Computation suggests the following congruences although we are unable to prove/disprove them at this point:

$$\overline{\mathcal{EO}}(10n+2) \equiv \overline{\mathcal{EO}}(10n+4) \equiv 0 \pmod 2.$$

Similar congruences are suggested for $\mathcal{EO}_u(n)$ as follows:

$$\mathcal{EO}_u(10n+2) \equiv \mathcal{EO}_u(10n+6) \equiv 0 \pmod 2.$$

Our inability to prove the above congruences lies in the very fact that certain type of coefficients in the corresponding powers of η-functions do not necessarily vanish and the above congruences correspond exactly to those coefficients which do not necessarily vanish modulo 2.

Our guess is one might use a higher dissection than the 2-dissection we used here to prove these congruences. In fact, a 5-dissection formula might be needed to prove the above congruences.

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