AN ALGORITHM TO PROVE HOLONOMIC DIFFERENTIAL EQUATIONS FOR MODULAR FORMS

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Abstract. Express a modular form g of positive weight locally in terms of a modular function h as $y(h)$, say. Then $y(h)$ as a function in h satisfies a holonomic differential equation; i.e., one which is linear with coefficients being polynomials in h. This fact traces back to Gauß and has been popularized prominently by Zagier. Using holonomic procedures, computationally it is often straightforward to derive such differential equations as conjectures. In the spirit of the "first guess, then prove" paradigm, we present a new algorithm to prove such conjectures.

1. Description of Contents

The study of holonomic functions and sequences satisfying linear differential and difference equations, respectively, with polynomial coefficients has roots tracing back (at least) to the time of Gauß.

Besides holonomic functions and sequences, the second major class of objects considered in this article are modular forms and functions which are non-holonomic: any modular form satisfies a *non-linear* third order differential equation with constant coefficients; see, for instance, [14, Prop. 16].

Neverthess, there is a connection between holonomic functions and modular forms which also traces back to Gauß. Namely, express a modular form g of positive weight locally in terms of a modular function h as $y(h)$, say; then $y(h)$ as a function in h satisfies a holonomic differential equation.

Zagier in his classical exposition [14, Prop. 21] introduces to this fact as follows: "... it is at the heart of the original discovery of modular forms by Gauss and

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of the later work of Fricke and Klein and others, and appears in modern literature as the theory of Picard-Fuchs differential equations or of the Gauss-Manin connection but it is not nearly as well known as it ought to be."

In [10] we began an algorithmic study of this connection which, on the holonomic side, utilizes aspects of Zeilberger's "holonomic systems approach" to special functions identities [15]. Ibid., on the modular functions and forms side, we sketched a contribution to theme of differential equations and modular forms, which follows the "first guess, then prove" paradigm. In this article we present the full mathematical details and derivations leading up to this new algorithmic tool. The algorithm, ModFormDE, which provides non-trivial computer support to prove claims of the following kind: given a modular function h and a modular form g of positive weight, both for a fixed congruence subgroup, prove with regard to the local expansion $g = y(h)$ that the function $y(h)$ in h satisfies a differential equation which is linear and with coefficients being polynomials in h. This article can be read completely independently from [10]; there are some natural overlaps, but those are kept to a minimum.

Our article is structured as follows. In Section 2 we present two examples to introduce in a concrete fashion to the holonomic paradigm in connection with modular forms and functions. Section 3 contains basic notions and facts about modular forms and functions needed; readers familiar with these notions will skip this section. In Section 4 we describe our algorithm ModFormDE which is based on work of Yifan Yang [13]; an illustrating example traces through its steps. In Section 5 we present the two main theorems of the paper, Thm. 5.2 and Thm. 5.3; they specify bounds for the total number of poles of modular functions which are essential for the ModFormDE algorithm. In Section 6 we introduce local expansions; they give rise to a notion of orders of modular forms of even weight, which will be used in a crucial way. The Sections 7, 8, and 9 derive and prove the bounds given in the main theorems; Section 10 gives a summary of how these things are related. The Appendix Section 11 contains proofs, computational aspects, and basic facts of meromorphic functions on Riemann surfaces. All this material is of relevance, but if presented within the main text, would disturb the flow of the presentation.

Conventions used throughout this paper: N denotes a positive integer, k is a fixed non-negative integer (the weight of a modular form),

$$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\},\ \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\},\ \text{and}\ \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}.$$

The ring of univariate polynomials with complex coefficients is denoted by $\mathbb{C}[X]$, its quotient field, the field of rational functions, is $\mathbb{C}(X)$.

Throughout, Γ stands for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$; see Section 3.

2. Introductory Examples

Holonomic differential equations are linear with polynomial coefficients. To illustrate their fundamental role for this paper, we consider concrete examples.

Example 2.1. Given

$$(1)\quad G(t) := {}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix};t\right) = \sum_{n=0}^{\infty}\frac{(1/2)_n(1/2)_n}{(1)_n}\frac{t^n}{n!} = 1+\frac{t}{4}+\frac{9t^2}{64}+\frac{25t^3}{256}+\frac{1225t^4}{16384}+\dots$$

where $(a)_n := a(a+1)\dots(a+n-1)$, $n \geq 1$, and $(a)_0 = 1$.

Problem. Determine coefficients $c(n)$ such that

$$(2)\qquad G(t) = \sum_{n=0}^{\infty} c(n)H(t)^n \text{ where } H(t) := 4t(1-t).$$

Using the holonomic tool-box, e.g., the RISC package `GeneratingFunctions` as described in more detail in [10], one can solve this problem as follows:[1]

Step 1: Take as input sufficiently many coefficients in the expansion (1) of $G(t)$.[2]

Step 2: With this input, compute sufficiently many values of the $c(n)$:

$$c(0) = 1, c(1) = \frac{1}{16}, c(2) = \frac{25}{1024}, \dots, c(11) = \frac{2363152308430225}{1152921504606846976}.$$

"Sufficiently many" is meant with regard to the next step.

Step 3: Using a package like `GeneratingFunctions`, guess a recurrence for the sequence $(c(n))_{n\geq 0}$:

In[1]:= **<< RISC'GeneratingFunctions'**

Package GeneratingFunctions version 0.8 written by Christian Mallinger © RISC-JKU

In[2]:= $\mathbf{cRec = GuessRE[\{1, \frac{1}{16}, \frac{25}{1024}, \dots, \frac{2363152308430225}{1152921504606846976}\}, c[n]][[1]]}$

Out[2]= $\{16(n+1)^2 c(n+1) - (4n+1)^2 c(n) = 0, c(0) = 1\}$

In other words, we have guessed that

$$(3)\qquad G(t) = Y(H(t)) \text{ where } Y(t) := \sum_{n=0}^{\infty} c(n)t^n \text{ with } c(n) = \frac{(1/4)_n(1/4)_n}{(1)_n n!}.$$

Step 4: To prove (3), we derive holonomic differential equations satisfied by $G(t)$, respectively by $Y(H(t))$. To derive the differential equation for $G(t)$, we input the first 12 coefficients of the power series expansion (1):

[1] The package, written in the Mathematica system, is available upon password request to the first-named author.

[2] It turns out that taking the first 12 coefficients is sufficient.

In[3]:= $\mathbf{GDE = GuessDE[\{1, \frac{1}{4}, \frac{9}{64}, \frac{25}{256}, \frac{1225}{16384}, \ldots, \frac{7775536041}{274877906944}\}, G[t]]}$

Out[3]= $\{4\left(t^2-t\right)G''(t)+4(2t-1)G'(t)+g(t)=0, G(0)=1, G'(0)=\frac{1}{4}\}$

This was derived as a guess. But using the power series expansion (1), one can easily verify that this equation is indeed satisfied by $G(t)$.

To derive the differential equation for $Y(H(t))$, we first derive a differential equation for $Y(t)$ by converting the recurrence for the $c(n)$ into a differential equation for their generating function $Y(t)=\sum_{n\geq 0} c(n)t^n$:

In[4]:= $\mathbf{YDE = RE2DE[cRec, Y[t]]}$

Out[4]= $\{-16\left(t^2-t\right)Y''(t)-8(3t-2)Y'(t)-Y(t)=0, Y(0)=1, Y'(0)=\frac{1}{16}\}$

Finally the differential equation for $Y(H(t))$ can be computed by exploiting the holonomic closure property of algebraic composition; see [7, Thm. 7.2.5]:

In[5]:= $\mathbf{ACompose[yDE, y[t] == 4t(1-t), y[t]]}$

Out[5]= $\{4\left(t^2-t\right)y''(t)+4(2t-1)y'(t)+y(t)=0, y(0)=1, y'(0)=\frac{1}{4}\}$

This differential equation for $Y(H(t))$ is the same as GDE in `Out[3]` for $G(t)$; also the initial values coincide, which proves (3).

Remark 2.2. The identity in (3) is the special case $a=b=1/4$ of

$$ {}_2F_1\left(\begin{matrix} 2a\ 2b \\ a+b+\frac{1}{2} \end{matrix}; t\right) = {}_2F_1\left(\begin{matrix} a\ b \\ a+b+\frac{1}{2} \end{matrix}; 4t(1-t)\right), \tag{4} $$

a classical identity in the theory of hypergeometric series; e.g., [1, (3.1.3)].

Example 2.3. Given

$$ g(\tau) := \theta_3(\tau)^2 = 1+4x+4x^2+4x^4+8x^5+4x^8+4x^9+\ldots \text{ with } x=e^{\pi i\tau}, \tag{5} $$

where $\tau \in \mathbb{H}$. In addition to $\theta_3(\tau)$, we also need another Jacobi function $\theta_2(\tau)$:

$$ \theta_2(\tau) := \sum_{n\in\mathbb{Z}+1/2} e^{\pi i n^2\tau} \text{ and } \theta_3(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} = 1+2\sum_{n=1}^{\infty} e^{\pi i n^2\tau}, \ \tau\in\mathbb{H}. \tag{6} $$

Problem. Determine coefficients $c(n)$ such that for all $\tau\in\mathbb{H}$ with $\mathrm{Im}(\tau)$ sufficiently big:

$$ g(\tau) = \sum_{n=0}^{\infty} c(n)h(\tau)^n \text{ where } h(\tau) := \frac{1}{16}\lambda(\tau)(1-\lambda(\tau)) = x-24x^2+\ldots \tag{7} $$

where

$$ \lambda(\tau) := \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4} = 16x(1-8x+44x^2+\ldots) \text{ with } x=e^{\pi i\tau} \text{ and } \tau\in\mathbb{H}. $$

As explained in [10, Ex. 2.5], one can verify that for all $\gamma \in \Gamma(2,4,4)$, a congruence subgroup defined in (19),

$$\theta_2(\gamma\tau)^2 = (c\tau + d)\,\theta_2(\tau)^2,$$

and for all $\gamma \in \Gamma(2,4,2)$, another congruence subgroup defined in (19),

$$\theta_3(\gamma\tau)^2 = (c\tau + d)\,\theta_3(\tau)^2.$$

In other words, θ_2^2 and θ_3^2 are modular forms of weight 1 for $\Gamma(2,4,4) \subseteq \Gamma(2,4,2)$;[3] consequently, λ is a modular function for $\Gamma(2,4,4)$.[4]

The problem to find an expansion as in (7) is similar to the expansion problem (3). Differences are: in (3), $G(t)$ is a hypergeometric function, in contrast to a modular form $g(\tau)$ in (7); in addition, in (3), $H(t)$ is a rational function (actually, a polynomial) in contrast to a modular function $h(\tau)$ in (7).

Power series expansion (and, more generally, Puiseux series expansion) of modular forms in terms of modular functions is the central theme in [10]. Namely, one has the crucial fact, Prop. 4.3 below, that in expansions like (7) the coefficients $c(n)$ constitute a holonomic sequence. As a consequence, holonomic tools as in the situation of (3) can be applied. More concretely, the holonomic approach to solve problem (7), can be summarized as follows:

• By Prop. 4.3 we know that there exists a power series,

$$y(z) := \sum_{n=0}^{\infty} c(n) z^n, \tag{8}$$

with a *holonomic* coefficient sequence $(c(n))_{n\geq 0}$, such that locally

$$g(\tau) = y(h(\tau)). \tag{9}$$

• Also by Prop. 4.3, $y(h)$ must satisfy a *holonomic* differential equation of the form

$$P_m(h)y^{(m)}(h) + P_{m-1}(h)y^{(m-1)}(h) + \cdots + P_0(h)y(h) = 0, \tag{10}$$

with polynomials $P_j(X) \in \mathbb{C}[X]$ with $P_m(X) \neq 0$.

• A fundamental holonomic fact says:[5] the differential equation (10) can be converted into a recurrence for $(c(n))_{n\geq 0}$, and vice versa.

• Our algorithm ModFormDE, described in Section 4, can be used to prove conjectured differential equations of the form (10).

Consequently, to solve problem (7), one can proceed as follows:

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[3]For the definition of modular form see Section 3.1.

[4]For the definition of modular function see Section 3.2.

[5]E.g., in the given context, used systematically in [10].

- First use holonomic tools to *guess* a differential equation of the form (10), then *prove* the differential equation using the algorithm ModFormDE. Finally, to determine the desired coefficient sequence $c(n)$, convert the proven differential equation into a recurrence for the $c(n)$.

To put this strategy into action, we begin by holonomic *guessing* of a recurrence for the $(c(n))_{n\geq 0}$. This way we already can see what is expected as an answer to problem (7).

In the next step, we convert this recurrence into a differential equation for y, which we then prove using the algorithm ModFormDE. Finally, the proven holonomic differential equation is converted back into the recurrence for the $c(n)$, which then gives a valid and proven answer to problem (7).

The computational steps are as follows:

Step 1: Take as input sufficiently many coefficients in the expansion (5) of $g(\tau)$.[6]

Step 2: With this input, as explained in more detail in [10, Ex. 4.2], compute sufficiently many values of the $c(n)$:

$$c(0) = 1, c(1) = 4, c(2) = 100, c(3) = 3600, \ldots, c(8) = 924193822500. \tag{11}$$

"Sufficiently many" is meant with regard to the next step.

Step 3: Using the `GeneratingFunctions` package, guess a recurrence for the sequence $(c(n))_{n\geq 0}$:

In[6]:= $\mathbf{cList = \{1, 4, 100, 3600, 152100, 7033104, 344622096, 17582760000, 924193822500\};}$

In[7]:= $\mathbf{yRec = GuessRE[cList, c[n]][[1]]}$

Out[7]= $\{\{-4(1+4n)^2 c[n] + (1+n)^2 c[1+n] == 0, c[0] == 1\}$

In other words, expressing the solution to this recurrence (of order 1) in terms of rising factorials, we algorithmically derived the following conjecture for $c(n)$ such that (7):

$$c(n) = \frac{(1/4)_n (1/4)_n}{(1)_n} \frac{4^{3n}}{n!}. \tag{12}$$

Step 4: To prove (12), the first step is to transform the recurrence `yRec` from `Out[7]` into a holonomic differential equation satisfied by $y(z) = \sum_{n=0}^{\infty} c(n) z^n$. This is done by using the procedure call `RE2DE` as above:

In[8]:= $\mathbf{yDE = RE2DE[\{yRec, c[n], y[n]]}$

[6] It turns out that taking the first 10 coefficients as given in (5) is sufficient.

Out[8]= $\{-4y[z]-(-1+96z)y'[z]-(-z+64z^2)y''[z]=0, y[0]=1, y'[0]=4\}$

In view of (9), this differential equation rewrites into

$$(64h^2-h)\frac{d^2y}{dz^2}(h)+(96h-1)\frac{dy}{dz}(h)+4y(h)=0 \tag{13}$$

with

$$y(0)=c_0=1 \text{ and } \frac{dy}{dz}(0)=c_1=4. \tag{14}$$

The verification of (14) is straightforward from the x-expansions (5) and (7) of g and h.

Using these x-expansions, also (13) can be verified up to a desired precision; i.e., by checking that the coefficients of x^n in the x-expansion of the left side are zero up to a certain power. But, needless to say, this gives no proof!

Step 5. To prove the correctness of (13), which is the conjectured differential equation of the form (10), we use the algorithm ModFormDE as detailed out in Section 4.4

Step 6. After having proved that (13), resp. `yDE` in `Out[8]`, is correct, in view of (8) we translate it back to a recurrence for the $c(n)$. Using the holonomic tool-box [10, Prop. 3.1] this can be done as follows:

In[9]:= **DE2RE[yDE, y[z], c[n]]**

Out[9]= $-4(1+4n)^2c[n]+(1+n)^2c[1+n]=0, c[0]=1, c[1]=4$

As expected, this recurrence is nothing but recurrence `yRec` from `Out[7]` which we had guessed. But now it comes as a consequence of a *proven* differential equation, consequently we *proved* that (12) is indeed the answer to problem (12).

Remark 2.4 (Existence of (13)). As explained in Section 4, Prop. 4.1 and Ex. 4.2, a holonomic differential equation of order 2,

$$p_2(h)\frac{d^2y}{dz^2}(h)+p_1(h)\frac{dy}{dz}(h)+p_0(h)y(h)=0, \tag{15}$$

with $p_j(X)\in\mathbb{C}[X]$ is guaranteed to exist.

Remark 2.5. Using holonomic tools has led us to guess that

$$\theta_3(\tau)^2={}_2F_1\left(\begin{matrix}\frac14 & \frac14\\ & 1\end{matrix}; 4\lambda(\tau)(1-\lambda(\tau))\right); \tag{16}$$

using our algorithm ModFormDE, this relation is proved algorithmically. In exactly the same manner one can algorithmically derive and prove that

$$\theta_3(\tau)^2={}_2F_1\left(\begin{matrix}\frac12 & \frac12\\ & 1\end{matrix}; \lambda(\tau)\right). \tag{17}$$

Combining these two facts gives an alternative proof of the special instance $a = b = 1/4$ of (4). This alternative proof avoids holonomic algebraic composition, it only uses the differential equation (13) which relates $g(\tau)$ and $h(\tau)$.[7]

Remark 2.6. We also note the following connection to the complete elliptic integral $K = K(\tau)$ of the first kind with modulus $k = k(\tau)$,

$$K(k(\tau)) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k(\tau)^2 \sin(\varphi)^2}} = \frac{\pi}{2}\theta_3(\tau)^2 \text{ where } k(\tau)^2 = \lambda(\tau). \tag{18}$$

This equality involving the theta function is immediate from (17) by series expansion of the integrand in terms of powers of $\lambda(\tau) = k(\tau)^2$. Besides other applications, the Borweins in their famous monograph [2] used this identity together with 11 similar ones [2, Thm. 5.6 and Thm. 5.7] to derive and explain identities which Ramanujan [12] gave (without too many details) to establish formulas to approximate π, respectively $1/\pi$.

3. Modular Functions and Forms: Basic Facts and Notions

This section contains basic notions and facts about modular forms and functions needed. Readers familiar with these notions will skip this section, and use it only as a dictionary concerning the notation used.

The group $\mathrm{SL}_2(\mathbb{Z}) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathbb{Z}^{2\times 2} : ad - bc = 1\}$ acts on elements $\tau \in \mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}.$$

As usual, the normal subgroup $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$, called principal congruence subgroup, is defined as

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

In this article, a congruence subgroup Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma(N) \subseteq \Gamma$ for some N.[8] Besides $\Gamma(N)$, important congruence subgroups are

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \pmod N \right\}$$

[7] Via the chain rule, $\frac{dg(\tau)}{d\tau} = \frac{dy(h(\tau))}{d\tau} = \frac{dy}{dz}(h) \cdot \frac{dh}{d\tau}(\tau)$.

[8] More generally, a congruence subgroup is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ such that $\Gamma \supseteq \Gamma(N)$ for some N and, in addition, Γ is commensurable with $\mathrm{SL}_2(\mathbb{Z})$. I.e., $\Gamma \cap \mathrm{SL}_2(\mathbb{Z})$ has finite index in Γ and $\mathrm{SL}_2(\mathbb{Z})$. Yang's setting [13] allows such Γ; hence our algorithm ModFormsDE also extends to this case.

and

$$\Gamma_1(N) := \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \pmod N \Big\}.$$

There are many refinements, for instance, for positive integers N, M, P such that $M \mid NP$:

$$\Gamma(N, M, P) := \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \\ a \equiv d \equiv 1 \,(\mathrm{mod}\, M), b \equiv 0 \,(\mathrm{mod}\, P), c \equiv 0 \,(\mathrm{mod}\, N) \Big\}. \tag{19}$$

This congruence subgroup already appeared in Example 2.3.

Throughout this paper, Γ stands for a congruence subgroup.

3.1. **Modular Forms.** For a meromorphic function, $f : \mathbb{H} \to \hat{\mathbb{C}}$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the weight-k operator is defined as usual by

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau), \;\; \tau \in \mathbb{H}.$$

Also as usual, we define (e.g., [9, Def. 1.8]): Let $f : \mathbb{H} \to \hat{\mathbb{C}}$ be meromorphic, and Γ a congruence subgroup. Then f is called a (meromorphic) *modular form* of weight k for Γ, if for all $\gamma \in \Gamma$,

$$(f|_k\gamma)(\tau) = f(\tau), \;\; \tau \in \mathbb{H}, \tag{20}$$

and if for each $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ there exist $w_0 = w_0(\gamma_0) \in \mathbb{Z}_{>0}$ and $n_0 = n_0(\gamma_0) \in \mathbb{Z}$ such that $f|_k\gamma_0$ admits a Fourier expansion (with coefficients in $\mathbb{C}$) of the form,

$$(f|_k\gamma_0)(\tau) = \sum_{n \geq n_0} a_{\gamma_0}(n) q_{w_0}^n, \;\; \tau \in \mathbb{H} \text{ such that } \mathrm{Im}(\tau) \text{ sufficiently big}, \tag{21}$$

where $q_{w_0} := e^{2\pi i \tau / w_0}$ and $a_{\gamma_0}(n_0) \neq 0$. If these conditions hold, one can show that $w_0(\gamma_0) = w_{\gamma_0}(\Gamma)$, where[9]

$$w_{\gamma_0}(\Gamma) := \min_{m \in \mathbb{Z}_{>0}} \Big\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \gamma_0^{-1}\Gamma\gamma_0 \;\text{ or }\; \begin{pmatrix} -1 & m \\ 0 & -1 \end{pmatrix} \in \gamma_0^{-1}\Gamma\gamma_0 \Big\}. \tag{22}$$

If $\gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $\gamma_0\infty = a/c$.[10] Considering an expansion as (21) for any $\gamma_1 \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_1\infty = a/c$, the values w_0 and n_0 do not change; i.e.,

$$(f|_k\gamma_1)(\tau) = \sum_{n \geq n_0} b_{\gamma_0}(n) q_{w_0}^n \;\text{ if }\; (f|_k\gamma_0)(\tau) = \sum_{n \geq n_0} a_{\gamma_0}(n) q_{w_0}^n. \tag{23}$$

As consequence, we can define the order ("of vanishing") of a modular form of weight k at $a/c \in \hat{\mathbb{Q}}$ as follows:

$$\mathrm{ord}_{a/c} f := n_0, \;\text{ where } n_0 \text{ is taken as in the expansion (21).} \tag{24}$$

[9] See, e.g., [4, Sect. 3.2].

[10] $a/0 := \infty$.

Another invariance of w_0 and n_0 occurs when γ_0 is replaced by $\gamma\gamma_0$ where $\gamma \in \Gamma$; i.e.,

$$(f|_k(\gamma\gamma_0))(\tau) = (f|_k\gamma_0)(\tau) = \sum_{n\geq n_0} a_{\gamma_0}(n) q_{w_0}^n \text{ if } (f|_k\gamma_0)(\tau). \tag{25}$$

Given a congruence subgroup Γ, we define

$$M_k(\Gamma) := \{f : \mathbb{H} \to \hat{\mathbb{C}} : f \text{ a modular form of weight } k \text{ for } \Gamma\}.$$

3.2. **Modular Functions.** If the modular form f has weight $k = 0$, it is called a *modular function*; we write $f|\gamma_0$ instead of $f|_0\gamma_0$, and define,

$$M(\Gamma) := \{f : \mathbb{H} \to \hat{\mathbb{C}} : f \text{ a modular function for } \Gamma\}.$$

For algorithmic zero recognition, modular functions $f \in M(\Gamma)$ are fundamental objects behaving in this regard like polynomials. Namely, owing to $k = 0$, one has the invariance $f(\gamma\tau) = f(\tau)$ for all $\gamma \in \Gamma$, and the expansions (21) then allow to extend f meromorphically to all the points $a/c \in \hat{\mathbb{Q}}$.[11] To this end, the first step is to extend the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ to an action on $\hat{\mathbb{H}} = \mathbb{H} \cup \hat{\mathbb{Q}}$. Notation: for any congruence subgroup Γ the orbit of $\tau \in \hat{\mathbb{H}}$ with respect to this action is written as $[\tau]_\Gamma := \{\gamma\tau : \gamma \in \Gamma\}$. We will write $[\tau]$ instead of $[\tau]_\Gamma$, if the subgroup Γ is clear from the context.

After extending the action, the meromorphic extension of a modular function f from $\mathbb{H}$ to a function on $\hat{\mathbb{H}}$ is done by choosing $\gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_0\infty = a/c$, and one defines

$$f(a/c) := (f|\gamma_0)(\infty) := \begin{cases} \infty & \text{if } n_0 < 0, \\ a_{\gamma_0}(0) & \text{if } n_0 = 0 \\ 0 & \text{if } n_0 > 0 \end{cases}.$$

Owing to the invariance (23), this definition is independent from the choice of γ_0. The invariance (25) implies $f(a/c) = f(\gamma\frac{a}{c})$ for any $\gamma \in \Gamma$. Together with the invariance (20), this means, a modular function is constant on all the Γ-orbits $[\tau]_\Gamma$. The set of all such orbits, denoted by $X(\Gamma)$, can be equipped with the structure of a compact Riemann surface.[12] Hence a modular function f with respect to Γ can be interpreted as a function $\hat{f} : X(\Gamma) \to \hat{\mathbb{C}}$; in fact, such $\hat{f}$ are meromorphic functions on $X(\Gamma)$.

[11] Recall $a/0 := \infty$.

[12] Charts are given explicitly in Section 6.1.

3.3. **Cusps.** If $\tau = a/c \in \hat{\mathbb{Q}}$ the orbits $[\tau]_\Gamma \in X(\Gamma)$ are called cusps. Congruence subgroups have only a finite index in $\mathrm{SL}_2(\mathbb{Z})$, hence $\hat{\mathbb{Q}} = \{\gamma\infty : \gamma \in \mathrm{SL}_2(\mathbb{Z})\} = \mathrm{SL}_2(\mathbb{Z})\infty = \cup_j \Gamma\gamma_j\infty = \cup_j [\gamma_j\infty]_\Gamma$ is a disjoint union of *finitely many* cusps. In view of the invariance (25), a Fourier expansion for $f \in M(\Gamma)$ as in (21) is called an expansion of f at the cusp $[a/c]_\Gamma$, or simply at a/c.

In various contexts, special attention is given to the case $a/c = \infty$. Then one can exploit the fact that each congruence subgroup Γ contains a translation matrix $\left(\begin{smallmatrix} 1 & w_0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & w_0 \\ 0 & -1 \end{smallmatrix}\right)$ with $w \in \mathbb{Z}_{>0}$ minimal; notice that $w_0 = w_I(\Gamma)$ with $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in (22). As a consequence of (20), any modular function $f \in M(\Gamma)$ has minimal period $w_0 \geq 1$. As a consequence, for $f \in M(\Gamma)$ we can uniquely define an expansion at infinity[13] by singling out the Fourier expansion (21) with the choice $\gamma_0 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = I$: in other words, for all $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau)$ sufficiently large,

$$f(\tau) = \sum_{n \geq n_0} a_I(n) q^{n/w_0}. \tag{26}$$

If $\gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then $\gamma_0\infty = a/c$, and an expansion as in (21) is called expansion of f a the cusp $[a/c]$ or, in short, at a/c.

Owing to (23) and (25), the minimal period $w_0 = w_{\gamma_0}(\Gamma)$ is independent from choice of the representative $\gamma_0\infty$ of the cusp $[a/c]_\Gamma = [\gamma_0\infty]_\Gamma$; it called the width of the cusp $[a/c]_\Gamma$.

The order $\mathrm{ord}_{a/c} f$, as defined in (24), is also called the order of f at the cusp $[a/c]$. For the order of f at the cusp $[\infty]$ (in short, at infinity) one often uses the short hand notation,

$$\mathrm{ord}\, f := \mathrm{ord}_\infty f.$$

3.4. **Zero recognition of modular functions.** For zero recognition of a modular function f, one exploits its extension to a meromorphic function $\hat{f} : X(\Gamma) \to \hat{\mathbb{C}}$ on the compact Riemann surface $X(\Gamma)$. Namely, if such functions are non-constant they have the property that

$$\text{number of poles of } \hat{f} = \text{number of zeros of } \hat{f}, \tag{27}$$

counting multiplicities; for further details see Lemma 11.3 in the Appendix Section 11.3.

In practice, there are various ways to bring the fact (27) into action. With regard to our algorithm ModFormDE, the strategy for zero recognition will be this.

Given $\alpha \in M(\Gamma)$, decide whether $\alpha = 0$. It is important what "given" for a modular function $\alpha \in M(\Gamma)$ means in our context; namely,

[13] I.e., at the point $\infty \in \hat{\mathbb{Q}}$

($\star$) it is possible to compute as many coefficients $a(n)$ in its expansion at infinity,

$$\alpha(\tau) = \sum_{n \geq n_0} a(n) q_{w_0}^n \text{ with } q_{w_0} := e^{2\pi i\tau/w_0}.$$

Here $n_0 = \operatorname{ord}\alpha = \operatorname{ord}_\infty \alpha$, and w_0 is the width of the cusp $[\infty]$; in terms of charts (61): $q_{w_0} = z_\infty(\tau)$ when $\gamma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Now suppose we have a bound on the number of poles of α,

$$\mathrm{NofPoles}(\alpha) \leq N.$$

Then, as a consequence of (27), α is proven to be 0, if all the coefficients $a(n_0), \ldots, a(N)$ are 0.

It is important to notice, that "number of poles/zeros" has to be taken in the interpretation of α as the induced meromorphic function $\hat{\alpha} : X(\Gamma) \to \mathbb{C}$; in other words, $\mathrm{NofPoles}(\alpha)$ is the number of poles of $\hat{\alpha}$, multiplicities counted.

Remark 3.1. Notions like NofPoles will be used heavily when describing the mathematical fundament of the algorithm ModFormDE. They are defined explicitly in (50) for modular functions, and in (114) for modular forms of even weight.

4. The Algorithm ModFormDE

4.1. **Existence of Holonomic Differential Equations for Modular Forms.** In Section 2 we showed how holonomic differential equations like (13) can be derived, as a *guess*, in computer-supported fashion. Modular form theory guarantees the existence of such differential equations. Zagier [14, Prop. 21] introduces to this fact as follows: "... it is at the heart of the original discovery of modular forms by Gauss and of the later work of Fricke and Klein and others, and appears in modern literature as the theory of Picard-Fuchs differential equations or of the Gauss-Manin connection — but it is not nearly as well known as it ought to be. Here is a precise statement:"

Proposition 4.1. *Let $g(\tau)$ be a modular form of weight $k > 0$ and $h(\tau)$ a modular function, both with respect to the congruence subgroup Γ. Express $g(\tau)$ locally as $y(h(\tau))$. Then the function $y(h)$ satisfies a linear differential equation of order $k+1$ with algebraic coefficients, or with polynomial coefficients if the compact Riemann surface $X(\Gamma)$ has genus 0 and $\mathrm{ord}(h(\tau)) = 1$.*

Example 4.2. The *existence* of the differential equation (13) is owing to the following facts: g is a modular form of weight 1 for $\Gamma(2,4,2)$, h is a modular function for $\Gamma(2,4,2)$ with $\mathrm{ord}(h) = 1$[14], and $X(\Gamma(2,4,2))$ has genus 0.

[14] Such a modular function is called a Hauptmodul.

An important fact in the light of the holonomic paradigm: if one drops to require minimality of the order of the differential equation, $y(h)$ always satisfies a *holonomic* differential equation:

Proposition 4.3 ([10], Prop. 6.2)**.** *In the setting of Prop. 4.1, the function $y(h)$ satisfies a linear differential equation with rational coefficients also when the genus of $X(\Gamma)$ is non-zero or when* $\operatorname{ord}(h(\tau)) > 1$*. — In these cases, the order of the differential equation in general will be larger than $k+1$.*

After stating Prop. 4.1, Zagier [14, p. 21] continues: "This proposition is perhaps the single most important source of applications of modular forms in other branches of mathematics, so with no apology we sketch three different proofs, ..."

Zagier's third proof is constructive; i.e., given g and h, it constructs the corresponding differential equation. Following the holonomic paradigm, we take a different approach: we first *guess* the corresponding holonomic differential equation algorithmially, and then prove it using our algorithm ModFormDE.

4.2. **The Mathematical Fundament of Algorithm ModFormDE.** Our algorithm ModFormDE solves the following problem:

GIVEN a modular form $g \in M_k(\Gamma)$ with weight $k \in \mathbb{Z}_{\geq 1}$ for the congruence subgroup Γ, and a modular function $h \in M(\Gamma)$ such that $\operatorname{ord} h = 1$ where $q_{w_0} = q^{1/w_0}$ with $q = e^{2\pi i\tau}$ is the local expansion variable at infinity. Moreover, suppose g has a local expansion of the form

$$g(\tau) = y(h(\tau)) \text{ where } y(z) := \sum_{n=\operatorname{ord} g}^{\infty} c(n)z^n;$$

$y(z)$ exists with uniquely determined holonomic coefficients according to Prop. 4.1 and Prop. 4.3.

PROVE that $y(h)$ satisfies a holonomic differential equation of the form

$$P_m(h)y^{(m)}(h) + P_{m-1}(h)y^{(m-1)}(h) + \cdots + P_0(h)y(h) = 0, \tag{28}$$

where the $P_j(X)$ are given polynomials in $\mathbb{C}[X]$ with $P_m(X) \neq 0$.

Note. Notice that $y^{(n)}(h) := \frac{d^n y}{dz^n}(z)|_{z=h}$.

Our algorithm ModFormDE is based on work of Yifan Yang [13]. So, before describing the steps of ModFormDE, we recall notation and notions used there.

• Differential operators [13, p. 4]: Let $\varphi(\tau)$ be a function defined on $\mathbb{H}$ having an x-expansion[15] $\tilde{\varphi}(x) = \sum_{n \geq n_0} a(n)x^n$ with $x = q^{1/w_0}$ where $q = e^{2\pi i\tau}$:

$$D_x\varphi = D_x\varphi(\tau) := \frac{w_0}{2\pi i} \cdot \frac{d\varphi}{d\tau}(\tau) = \frac{w_0}{2\pi i} \cdot \varphi'(\tau) = x\, \tilde{\varphi}'(x). \tag{29}$$

[15] Here we extend the setting in [13] from $q = e^{2\pi i\tau}$ to $x = q^{1/N_0}$ to adapt to applications.

Let $\psi = \psi(z)$ be a function analytic in a neighborhood of 0, let $h = h(\tau) \in M(\Gamma)$:

$$D_h\psi = D_h\psi(h) := h\frac{d\psi}{dz}(h) = h\psi'(h).$$

Fundamental functions on $\mathbb{H}$ [13, Thm. 1]:

Let Γ be a congruence subgroup, and let $g \in M_k(\Gamma)$, $k \geq 1$, and $h \in M(\Gamma)$ be fixed:

$$G_1 := \frac{D_x h}{h} = \frac{N_0}{2\pi i} \cdot \frac{h'}{h} \text{ and } G_2 := \frac{D_x g}{g} = \frac{N_0}{2\pi i} \cdot \frac{g'}{g}; \tag{30}$$

notice that $h' = h'(\tau) = \frac{dh(\tau)}{d\tau}$ and $g' = g'(\tau) = \frac{dg(\tau)}{d\tau}$.

• Fundamental modular functions [13, Thm. 1]:

$$p_1 := \frac{D_x G_1 - 2G_1G_2/k}{G_1^2} \text{ and } p_2 := -\frac{D_x G_2 - G_2^2/k}{G_1^2}. \tag{31}$$

As proved in [13, Lemma 1], the p_j are modular functions in $M(\Gamma)$. Moreover, they are also algebraic functions in $h \in M(\Gamma)$. This means, for fixed $j \in \{1, 2\}$, p_j and h satisfy an algebraic relation; see the remark after Thm. 8.1 in [10].

Because of the chain rule we have,

$$D_h y = D_h y(h) = hy'(h) = h\frac{g'}{h'} = g\frac{G_2}{G_1}. \tag{32}$$

Yang [13, p. 9] also computed that

$$D_h^2 y = \Big(1 - \frac{1}{k}\Big) \cdot g\frac{G_2^2}{G_1^2} + (-p_1) \cdot g\frac{G_2}{G_1} + (-p_2) \cdot g, \tag{33}$$

and

$$\begin{aligned} D_h^3 y = {} & \Big(1 - \frac{1}{k}\Big)\Big(1 - \frac{2}{k}\Big) \cdot g\frac{G_2^3}{G_1^3} - 3\Big(1 - \frac{1}{k}\Big)p_1 \cdot g\frac{G_2^2}{G_1^2} \\ & + \Big(p_1^2 - \Big(3 - \frac{2}{k}\Big)p_2 - D_h p_1\Big) \cdot g\frac{G_2}{G_1} + (p_1 p_2 - D_h p_2) \cdot g, \end{aligned} \tag{34}$$

where the $p_j \in M(\Gamma)$ are the fundamental modular functions defined in (31) and $D_h p_j = h\frac{dp_j}{dh}$. Mathematical induction [13, p. 10] on $m \geq 0$ leads to[16],

$$D_h^m y = \prod_{j=0}^{m-1}(1 - \frac{j}{k}) \cdot g\frac{G_2^m}{G_1^m} + a_{m,m-1} \cdot g\frac{G_2^{m-1}}{G_1^{m-1}} + \cdots + a_{m,1} \cdot g\frac{G_2}{G_1} + a_{m,0} \cdot g, \tag{35}$$

with the $a_{m,j}$ being multivariate polynomials from the polynomial ring

$$R := \mathbb{C}\Big[h, p_1, p_2, \frac{dp_1}{dh}, \frac{dp_2}{dh}, \ldots, \frac{d^m p_1}{dh^m}, \frac{d^m p_2}{dh^m}, \ldots\Big]. \tag{36}$$

[16] Notice that here we assume $D_h^0 y = y(h) = g$.

Lemma 4.4. *Let Γ be a congruence subgroup. Let $g \in M_k(\Gamma)$ with $k \geq 1$, and $h \in M(\Gamma)$. Then the elements of R are modular functions in $M(\Gamma)$.*

Proof. Let $p \in \{p_1, p_2\}$. By [13, Lemma 1], $p \in M(\Gamma)$. In addition, p is an algebraic function in h; see, e.g., the remark after Thm. 8.1 in [10]. This means, there exists a polynomial

$$R(X,Y) := Y^n + c_1(X)Y^{n-1} + \cdots + c_n(X)$$

with rational function coefficients $c_j(X) \in \mathbb{C}(X)$ such that $R(h,p) = 0$. By the implicit function theorem one has that locally there exists an meromorphic function $r(z)$ such that $R(h,p) = 0$ iff $p = r(h)$. Moreover,

$$\frac{dp}{dh} = r'(h) = -\frac{\partial R}{\partial X}(h,p) / \frac{\partial R}{\partial Y}(h,p),$$

which, as a rational function in h and p, is in $M(\Gamma)$. Applying the same argument to p' and h, etc., completes the proof also for the higher derivatives of p. □

Remark 4.5. In the proof we introduced a new function symbol r when writing p as a function in h; i.e., $p = r(h)$. However, in order to keep notation as lean as possible, whenever things are clear from the context we will follow Yang, and write $p = p(h)$ instead of $p = r(h)$ when referring to p as a function in h.

By relation (35) we are led to the following fact.

Lemma 4.6. *Let Γ be a congruence subgroup. Let $g \in M_k(\Gamma)$ with $k \geq 1$, and $h \in M(\Gamma)$. Then: (1) any expression of the form,*

$$Y := Q_m(h)D_h^m y + Q_{m-1}(h)D_h^{m-1} y + \cdots + Q_0(h)y, \tag{37}$$

with polynomials $Q_j(X) \in \mathbb{C}[X]$ can be written into "Yang form" as

$$Y = \alpha_m \cdot g\frac{G_2^m}{G_1^m} + \alpha_{m-1} \cdot g\frac{G_2^{m-1}}{G_1^{m-1}} + \cdots + \alpha_0 \cdot g \;\; \text{with} \;\; \alpha_j \in R; \tag{38}$$

(2) these coefficients α_j are uniquely determined.

Proof. Part (1) is immediate from (35). Part (2) is a consequence of the fact that the $\frac{G_2^m}{G_1^m}$ are linearly independent over $M(\Gamma)$; this is proved in Prop. 11.1 in the Appendix Section 11. □

4.3. **Input, Output, and Steps of the Algorithm ModFormDE.**

INPUT. (1) $g \in M_k(\Gamma)$ and $h \in M(\Gamma)$ such that $\operatorname{ord} h = 1$; both functions are given in the form of their x-expansions, where $x = q^{1/w_0}$ with $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, is the local expansion variable at infinity. More precisely, we assume that sufficiently many coefficients of

$$g(\tau) = \sum_{n=M}^{\infty} g(n)x^n, \ M \in \mathbb{Z} \text{ fixed},$$

and

$$h(\tau) = x(1 + h_1 x + h_2 x^2 + \dots)$$

can be computed.

(2) Polynomials $P_0(X), \dots, P_m(X)$ in $\mathbb{C}[X]$ with $P_m(X) \neq 0$.

(3) NofPoles(h): the number of poles of $\hat{h}$, defined in (50).

(4) If k is even, NofPoles(g): a pole number defined in (114); if k is odd, NofPoles(g^2).

(5) If k is odd, NofCusps(Γ) and NofElliptic(Γ): the number of cusps and of elliptic points, defined in (48) and (49).

OUTPUT. Bounds for

$$\text{NofPoles}(p_1), \text{NofPoles}(p_2), \text{ and NofPoles}\left(\frac{d^j p_i}{dh^j}\right), \ i = 1, 2, j \geq 1. \tag{39}$$

As a consequence of the steps of the algorithm, these bounds as part of the strategy described in Section 3.4, enable a proof of the correctness of the differential relation,

$$P_m(h)y^{(m)}(h) + P_{m-1}(h)y^{(m-1)}(h) + \dots + P_0(h)y(h) = 0. \tag{40}$$

In case (40) is not valid, the algorithm detects this. The output bounds for (39) are specified in the Theorems 5.2 and 5.3 in Section 5.

THE STEPS OF THE ALGORITHM “ModFormDE”:

Step 0: Rewrite the left side of (40) into the form (37). — This is done by using the relations $hy'(h) = D_h y$,

$$h^2 y''(h) = D_h^2 y - D_h y, \ h^3 y^{(3)}(h) = D_h^3 y - 3hD_h^2 y + (3h - 1)D_h y, \ \text{a.s.o.},$$

which, for example, can be precomputed.

Step 1: Transform the expression (37) into Yang form (38). — This is done by using the relations (32), (33), (34), and (35) for $m \geq 4$, which, for example, can be precomputed.

Step 2: Owing to the uniqueness of the coefficients α_j in (38), the proof of (28) finally is reduced to prove that

$$\alpha_m = 0, \alpha_{m-1} = 0, \ldots, \alpha_0 = 0. \tag{41}$$

Since the α_j are modular functions in $M(\Gamma)$, this task, owing to (27), reduces to determine upper bounds for the number of poles of each α_j. Because of $\alpha_j \in R$, by definition (36) it is sufficient to provide such bounds for h and for the $\frac{d^j p_j}{dh^j}$, $j \geq 0$, which is done in the Sections 7, 8, and 9, together with the summary given in Section 10.

Finally, each zero test, $\alpha_j = 0$, is completed by computing sufficiently many coefficients in the x-expansion of α_j, which is derived from those of g and h.[17] In Section 4.4 we exemplify the steps of the algorithm by proving (13).

4.4. **Proving** (13) **with the ModFormDE Algorithm.** To illustrate the ModFormDE algorithm, we prove the validity of (13).

As in (5) and (7), we are given[18] $g \in M_1(\Gamma)$ and $h \in M(\Gamma)$ with $\operatorname{ord} h = 1$; here $\Gamma = \Gamma(2,4,2)$, and we note that $X(\Gamma(2,4,2))$ has genus $g_\Gamma = 0$. Noticing that $\operatorname{ord} g = 1$, by Prop. 4.1 and Prop. 4.3 we know that g has a local expansion of the form

$$g(\tau) = y(h(\tau)) \text{ where } y(z) := \sum_{n=0}^{\infty} c(n) z^n,$$

where $y(z)$ has uniquely determined *holonomic* coefficients. In Ex. 2.3(11) we computed several of these coefficients; in addition, using software we conjectured the holonomic differential equation (13)

$$(64h^2 - h)\frac{d^2y}{dz^2}(h) + (96h - 1)\frac{dy}{dz}(h) + 4y(h) = 0$$

Using ModFormDE, its validity is proved as follows.

Concerning the input data (1), the x-expansions, $x = q^{\pi i \tau}$, are immediate from (6).

The polynomials (2) are read off from the differential equation: $P_0(X) = 4, P_1(X) = 96X - 1$, and $P_2(X) = X(64X - 1)$.

Concerning (3): NofPoles$(h) = 1$, since h is a Hauptmodul.

Concerning (4): Since $k = 1$, we have to determine NofPoles(g^2). The product expansion of g is classical; it gives,

$$g(\tau)^2 = \frac{\eta(\tau)^{20}}{\eta(\tau/2)^8 \eta(2\tau)^8}, \tag{42}$$

[17] Recall $x = q^{1/w_0}$ with $q = e^{2\pi i \tau}$.

[18] Given, in the sense of $(\star)$ in Section 3.4

using the Dedekind eta function $\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{j=1}^{\infty}(1-q^j)$, $q = e^{2\pi i \tau}$. This representation tells that g^2 has no pole in $\mathbb{H}$. So we need to inspect the cusps of $X(\Gamma)$, which are 3 in total; namely $[0]$, $[1]$ and $[\infty]$. To see this, and to find the widths of each of these cusps, which is 2 in each instance, one can, e.g., run Magma as described in Section 11.2.1.

Since $g^2 \in M_2(\Gamma)$, according to Definition 6.12,

$$\begin{aligned} \mathrm{NofPoles}(g^2) &= - \sum_{\substack{\text{cusps } [a/c] \in X(\Gamma) \\ \mathrm{ord}\, \tilde{F}_{a/c}(z) < 0}} \mathrm{ord}\, \tilde{F}_{a/c}(z) \\ &= - \sum_{\substack{\text{cusps } [a/c] \in X(\Gamma) \\ \mathrm{ord}_{a/c}(g^2) \leq 0}} \Big(\mathrm{ord}_{a/c}(g^2) - 1 \Big), \end{aligned} \tag{43}$$

where $\tilde{F}_p(z)$ is the Laurent series for $F := g^2$ defined in Lemma 6.6; the last equality is by (77). Hence, in view of the three cusps $[\infty]$, $[0]$, and $[1]$, we have to determine the orders of the x-expansions, $x = e^{\pi i \tau}$, of

$$(g^2)(\tau), (g^2|_2\gamma_0)(\tau), \text{ and } (g^2|_2\gamma_1)(\tau),$$

with $\gamma_0 := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\gamma_1 := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, such that $\gamma_0 \infty = 0$ and $\gamma_1 \infty = 1$.

At the cusp $[\infty]$:

$$g(\tau)^2 = 1 + 8x + 24x^2 + O(x^3), \quad \text{hence } \mathrm{ord}_\infty(g^2) = 0.$$

At the cusp $[0]$:

$$(g^2|_2\gamma_0)(\tau) = (1 \cdot \tau + 0)^{-2} g(-1/\tau)^2 = g(\tau)^2, \quad \text{hence } \mathrm{ord}_0(g^2) = 0;$$

here one applies the classic transformation formula $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$.

At the cusp $[1]$: as explained in Section 11.2.2, one finds that

$$(g^2|_2\gamma_1)(\tau) = u \cdot x + O(x^2), \quad \text{hence } \mathrm{ord}_1(g^2) = 1.$$

Summarizing, using this information in (114), gives,

$$\mathrm{NofPoles}(g^2) = -\Big(\mathrm{ord}_\infty(g^2) - 1\Big) - \Big(\mathrm{ord}_0(g^2) - 1\Big) = 1 + 1 = 2.$$

Concerning input data (5): Since k is odd, we have to consider this case, and note that

$$\mathrm{NofCusps}(\Gamma) = 3 \text{ and } \mathrm{NofElliptic}(\Gamma) = 0;$$

to obtain this information is routine and can be left to software.

Remark 4.7. We have seen that the non-routine part in providing the required input data (1) to (5) to algorithm ModFormDE, is to determine NofPoles(g^2). Nevertheless, for certain classes of modular functions and modular forms (e.g., those representable by eta quotients), also this step can be turned into algorithmics.

After providing all the required input data (1) to (5), we turn to the steps of algorithm ModFormDE:

Step 0: The conjectured differential equation rewrites into,

$$Y := (64h-1)D_h^2 y + 32hD_h y + 4h\, g = 0. \tag{44}$$

Step 1: Owing to $g \in M_1(\Gamma)$ we have $k = 1$, and the Yang form of Y becomes

$$Y = \alpha_1 \cdot g\frac{G_2}{G_1} + \alpha_0 \cdot g, \tag{45}$$

with

$$\alpha_1 = 32h - (64h-1)p_1 \in M(\Gamma) \;\text{ and }\; \alpha_0 = 4h - (64h-1)p_2 \in M(\Gamma). \tag{46}$$

Step 2: For this step, since $k = 1$, we need to use Theorem 5.3. This theorem requires the notion NofPoles(f) for $f \in M(\Gamma)$, defined in (50), and the extended notion (114) for modular forms $f \in M_{2k}(\Gamma)$ defined in Section 6.

Step 2a. We first prove $\alpha_1 = 0$. By (55) of Theorem 5.3,

$$\begin{aligned}
\mathrm{NofPoles}(\alpha_1) \leq& \mathrm{NofPoles}(h) + \mathrm{NofPoles}(h) + \mathrm{NofPoles}(p_1)\\
=&2\,\mathrm{NofPoles}(h) + \mathrm{NofPoles}(p_1)\\
\leq&2\,\mathrm{NofPoles}(h) + (2k+4)(g-1) + 8\,\mathrm{NofPoles}(h) + 3\,\mathrm{NofPoles}(g^2)\\
=&10\,\mathrm{NofPoles}(h) + 3\,\mathrm{NofPoles}(g^2) = 10\cdot 1 + 3\cdot 2 = 16.
\end{aligned}$$

Computing the x-expansion up to the power of x^{16} shows that $\alpha_1(\tau) = 0 + 0x + 0x^2 + \cdots + 0x^{16} + \ldots$ This implies that α_1 has at least 17 zeros and 16 poles or less; so α_1 has to be 0.

Step 2b. Second, we prove $\alpha_2 = 0$. By (56) of Theorem 5.3,

$$\begin{aligned}
\mathrm{NofPoles}(\alpha_2) \leq&2\,\mathrm{NofPoles}(h) + \mathrm{NofPoles}(p_2)\\
\leq&2\,\mathrm{NofPoles}(h) + (6k+4)(g-1) + 6\,\mathrm{NofPoles}(h) + 10\,\mathrm{NofPoles}(g^2)\\
&+ 2\,\mathrm{NofCusps}(\Gamma) + 2\,\mathrm{NofElliptic}(\Gamma)\\
=&8\,\mathrm{NofPoles}(h) + 10\,\mathrm{NofPoles}(g^2) + 2\cdot 3 + 2\cdot 0 = 8 + 20 + 6 = 34.
\end{aligned}$$

Computing the x-expansion up to the power of x^{34} shows that $\alpha_2(\tau) = 0 + 0x + 0x^2 + \cdots + 0x^{34} + \ldots$ This implies that α_2 has at least 35 zeros and 34 poles or less; so α_2 has to be 0.

5. Main theorems of even and odd weight

Our algorithm ModFormDE consists of two steps: after Step 1, which transforms the conjectured differential equation into Yang form (38), in Step 2 one has to prove (41) for the coefficients $\alpha_j \in R \subseteq M(\Gamma)$. To this end, one invokes bounds for the number of poles of the elements in R. In this section we state the two main theorems, one for k even and one for k odd, which provide such bounds by determining bounds for the number of poles of the generators of R,

$$h, p_1, p_2, \frac{dp_1}{dh}, \frac{dp_2}{dh}, \dots, \frac{d^m p_1}{dh^m}, \frac{d^m p_2}{dh^m}, \quad \text{a.s.o.}$$

To state the main results of this section, Thm. 5.2 and Thm. 5.3, we need to make some preparations.

In Section 3.3 we defined cusps. We need to recall another standard notion from modular group actions.

Definition 5.1. *Let $P = [p] \in X(\Gamma)$ with $p \in \mathbb{H}$. Then P is an elliptic point of $X(\Gamma)$, if*

$$\{\gamma \in \Gamma : \gamma p = p\} \not\subseteq \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}. \tag{47}$$

One also says that p is an elliptic point for Γ.

Define

$$\text{NofCusps}(\Gamma) := \text{ no. of cusps of } X(\Gamma), \tag{48}$$

$$\text{NofElliptic}(\Gamma) := \text{ no. of elliptic points of } X(\Gamma). \tag{49}$$

In addition, for a modular function $f \in M(\Gamma)$ define,

$$\text{NofPoles}(f) := \text{ number of poles } P \in X(\Gamma) \text{ of } \hat{f}. \tag{50}$$

multiplicities of poles are counted.

To state the following theorem, we need to extend definition (50) to modular forms with even weight; this is done by the relation (114) in Definition 6.12. Using these definitions, (50) and (114), the bounds we obtained for even weight k are as follows.

Theorem 5.2. *For a congruence subgroup Γ let $t \in M(\Gamma)$ and $F \in M_k(\Gamma)$. Let g_Γ be the genus of $X(\Gamma)$. Then, if k is even, one has,*

$$\text{NofPoles}(p_1) \leq 8\,\text{NofPoles}(t) + 3\,\text{NofPoles}(F) + (k+4)(g_\Gamma - 1), \tag{51}$$

$$\begin{aligned} \text{NofPoles}(p_2) \leq{}& 2\,\text{NofElliptic}(\Gamma) + 2\,\text{NofCusps}(\Gamma) \\ &+ 6\,\text{NofPoles}(t) + 10\,\text{NofPoles}(F) + (3k+4)(g_\Gamma - 1), \end{aligned} \tag{52}$$

and for the derivatives where $j \geq 1$,

$$\operatorname{NofPoles}\left(\frac{d^j p_1}{dt^j}\right) \leq (12j+6)\operatorname{NofPoles}(t) + 3(j+1)\operatorname{NofPoles}(F) + (jk+8j+k+2)(g_\Gamma - 1), \tag{53}$$

$$\operatorname{NofPoles}\left(\frac{d^j p_2}{dt^j}\right) \leq 2(j+1)\operatorname{NofElliptic}(\Gamma) + 2(j+1)\operatorname{NofCusps}(\Gamma) + 2(5j+2)\operatorname{NofPoles}(t) + 10(j+1)\operatorname{NofPoles}(F) + (3jk+8j+3k+2)(g_\Gamma - 1). \tag{54}$$

The bounds we obtained for odd weight k are as follows.

Theorem 5.3. *For a congruence subgroup Γ let $t \in M(\Gamma)$ and $F \in M_k(\Gamma)$. Let g_Γ be the genus of $X(\Gamma)$. Then, if k is odd, one has,*

$$\operatorname{NofPoles}(p_1) \leq 8\operatorname{NofPoles}(t) + 3\operatorname{NofPoles}(F^2) + (2k+4)(g_\Gamma - 1), \tag{55}$$

$$\operatorname{NofPoles}(p_2) \leq 2\operatorname{NofElliptic}(\Gamma) + 2\operatorname{NofCusps}(\Gamma) + 6\operatorname{NofPoles}(t) + 10\operatorname{NofPoles}(F^2) + (6k+4)(g_\Gamma - 1), \tag{56}$$

and for the derivatives where $j \geq 1$,

$$\operatorname{NofPoles}\left(\frac{d^j p_1}{dt^j}\right) \leq (12j+6)\operatorname{NofPoles}(t) + 3(j+1)\operatorname{NofPoles}(F^2) + (2jk+8j+2k+2)(g_\Gamma - 1), \tag{57}$$

$$\operatorname{NofPoles}\left(\frac{d^j p_2}{dt^j}\right) \leq 2(j+1)\operatorname{NofElliptic}(\Gamma) + 2(j+1)\operatorname{NofCusps}(\Gamma) + 2(5j+2)\operatorname{NofPoles}(t) + 10(j+1)\operatorname{NofPoles}(F^2) + (6jk+8j+6k+2)(g_\Gamma - 1). \tag{58}$$

The rest of our paper is devoted to the proofs of these statements. Section 6 introduces the mathematical requirements needed. Section 7 proves the bound for p_1, Section 8 for p_2. The bounds for the derivatives of p_1 and p_2 are given in Section 9. These considerations are completed by a proof summary in Section 10.

6. Locals expansions and orders

The rest of this section is devoted to proving these bounds. To this end, we first consider local expansions which are used in a crucial way, also for defining $\operatorname{NofPoles}(F)$ for modular forms F with even weight; see Def. 6.12.

6.1. **Local expansions and orders.** To bound the possible number of poles of modular functions we use local series expansions in terms of charts. These charts z_P, as defined below, are homeomorphisms between open subsets of $X(\Gamma)$ and of $\mathbb{C}$. More details on the topology used, in particular, why these charts make $X(\Gamma)$ into a Riemann surface, can be found, for instance, in [4, Sect. 2.2, 2.3 and 2.4].

Given $P = [p] \in X(\Gamma)$ for some $p \in \hat{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, we consider charts $z_p : U_P \to \mathbb{C}$ with $z_p([\tau]) := z_p(\tau)$ defined as usual by

$$z_p(\tau) := \tau - p, \text{ if } p \in \mathbb{H} \text{ is no elliptic point,} \tag{59}$$

or by

$$z_p(\tau) := \left(\frac{\tau - p}{\tau - \overline{p}}\right)^{h(p)}, \text{ if } p \in \mathbb{H} \text{ is an elliptic point (cf. Def. 5.1),} \tag{60}$$

or, by

$$z_p(\tau) := e^{2\pi i \gamma_0^{-1}\tau/w}, \text{ if } p = \frac{a_0}{c_0} = \gamma_0\infty \in \mathbb{Q} \cup \{\infty\}, \tag{61}$$

where $\gamma_0 = \left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $w = w_{\gamma_0}(\Gamma)$; see (22). Here $U_P \subseteq X(N)$ is a neighborhood of $P = [p]$ such that $U_P = \{[\tau] : \tau \in V_0\}$ where $V_0 \subseteq \hat{\mathbb{H}}$ is suitable open neighborhood of p in the given topology of $\hat{\mathbb{H}}$. Notice that defining $z_p([\tau]) := z_p(\tau)$, we overloaded meaning: besides being a map on open subsets of $X(\Gamma)$, z_p is also an analytic function on open subsets of $\hat{\mathbb{H}}$.

Furthermore, the periods $h(p)$ equal either 2 or 3; we also note explicitly that all these charts are centered at 0; i.e.,

$$z_p(P) = z_p(p) = 0. \tag{62}$$

We need to describe the behavior of charts under the change of orbit representatives.

Lemma 6.1. *Given $p \in \hat{\mathbb{H}}$, let $r := \gamma p$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then the charts $z_p : V_0 \to \mathbb{C}$ relate to the charts $z_r : \gamma(V_0) \to \mathbb{C}$ as follows:*

(i) if $p \in \mathbb{H}$ is no elliptic point, then r is no elliptic point, and for $\tau \in V_0$,

$$z_r(\gamma\tau) = \gamma\tau - r = \frac{1}{cp + d} \cdot \frac{z_p(\tau)}{c\tau + d}; \tag{63}$$

(ii) if $p \in \mathbb{H}$ is an elliptic point, then r is an elliptic point,[19] *and for $\tau \in V_0$,*

$$z_r(\gamma\tau) = \left(\frac{\gamma\tau - r}{\gamma\tau - \overline{r}}\right)^{h(r)} = \left(\frac{c\overline{p} + d}{cp + d}\right)^{h(p)} \cdot z_p(\tau); \tag{64}$$

(iii) if $p = \gamma_0\infty \in \hat{\mathbb{Q}}$, $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$, is a cusp, then r is a cusp, and for $\tau \in V_0$,

$$z_r(\gamma\tau) = e^{2\pi i (\gamma\gamma_0)^{-1}(\gamma\tau)/w} = z_p(\tau). \tag{65}$$

[19] Notice that $h(p) = h(r)$.

Proof. By straight-forward verifications. □

Before turning to local series representations, we recall the notion of order ("of vanishing") of Laurent series.

Definition 6.2 (order of a Laurent series)**.** *Let $\varphi(z) = \sum_{n=M}^{\infty} c(n)z^n$ be a Laurent series with $M \in \mathbb{Z}$ and $c(M) \neq 0$:*

$$\operatorname{ord} \varphi(z) := M.$$

Remark 6.3. In this paper we use several notions of order. When referring to the order of a Laurent series in powers of z, we always include the argument z explicitly; i.e., we write $\operatorname{ord} \varphi(z)$ instead of $\operatorname{ord} \varphi$.

The first part of the following lemma is implied by the fact that the z_p are local charts; the second part, the order invariance, from Riemann surface point of view is by connecting $\operatorname{Ord}_x f$ with the notion of multiplicity $\operatorname{mult}_x f$.[20] For an alternative proof of this invariance one can use the argument from the proof of Lemma 6.6 when $k = 0$.

Lemma 6.4. *Given a non-zero $t \in M(\Gamma)$, and $P = [p] \in X(\Gamma)$, $p \in \hat{\mathbb{H}}$. Then there exists a Laurent series*

$$\tilde{t}_p(z) := \sum_{n=M_p}^{\infty} a_p(n) z^n \tag{66}$$

such that

$$t(\tau) = \tilde{t}_p(z_p(\tau)), \quad \tau \in V_0,$$

where $V_0 \subseteq \hat{\mathbb{H}}$ is a suitable neighborhood of p. Moreover, for any $\gamma \in \Gamma$,

$$M_p = \operatorname{ord} \tilde{t}_p(z) = \operatorname{ord} \tilde{t}_{\gamma p}(z) = M_{\gamma p};$$

i.e., the order, $\operatorname{ord} \tilde{t}_p(z)$, is independent from the choice of p as a representative of $P = [p]$.

The following definition generalizes (24) from cusps $P = [a/c]$, $a/c \in \hat{\mathbb{Q}}$, to general points (Γ-orbits) $P = [p] \in X(\Gamma)$:

Definition 6.5 (order of a modular function at at point)**.** *For non-zero $t \in M(\Gamma)$ and $P = [p] \in X(\Gamma)$, $p \in \hat{\mathbb{H}}$:*

$$\operatorname{Ord}_P t := \operatorname{ord} \tilde{t}_p(z), \tag{67}$$

where $\tilde{t}_p(z)$ is as in Lemma 6.4.

The next two lemmas generalize Lemma 6.4 to modular forms.

[20] See the Section 11.3 for further details.

Lemma 6.6. *Given a non-zero $F \in M_k(\Gamma)$ with k even, and $P = [p] \in X(\Gamma)$, $p \in \hat{\mathbb{H}}$. Then there exists a Laurent series*

$$\tilde{F}_p(z) := \sum_{n=N_p}^{\infty} b_p(n) z^n \tag{68}$$

such that

$$F(\tau) = z_p'(\tau)^{k/2}\, \tilde{F}_p(z_p(\tau)), \quad \tau \in V_0, \tag{69}$$

where $V_0 \subseteq \hat{\mathbb{H}}$ is a suitable neighborhood of p. Moreover, for any $\gamma \in \Gamma$,

$$N_p = \operatorname{ord} \tilde{F}_p(z) = \operatorname{ord} \tilde{F}_{\gamma p}(z) = N_{\gamma p};$$

i.e., the order, $\operatorname{ord} \tilde{F}_p(z)$, is independent from the choice of p as a representative of $P = [p]$.

Proof. Take $t \in M(\Gamma)$ with $\tilde{t}_p(z)$ as in (66). Consequently,

$$t'(\tau) = z_p'(\tau) \sum_{n=M_p}^{\infty} n\, a_p(n) z_p(\tau)^{n-1} \in M_2(\Gamma), \tag{70}$$

and applying Lemma 6.4 to the modular function $F/(t')^{k/2} \in M(\Gamma)$ proves the first part of the statement. To prove the invariance of the order when choosing different orbit representatives, assume that $r = \gamma p$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. If $p \in \mathbb{H}$ is not an elliptic point, for the expansion with respect to r we have by (63),

$$F(\gamma\tau) = z_r'(\gamma\tau)^{k/2}\, \tilde{F}_r(z_r(\gamma\tau)) = \tilde{F}_r\Big(\frac{1}{cp+d} \cdot \frac{z_p(\tau)}{c\tau+d}\Big).$$

This, together with the modular transformation property, implies

$$\begin{aligned} F(\tau) &= \frac{1}{(c\tau+d)^k} F(\gamma\tau) = \frac{1}{(c\tau+d)^k} \tilde{F}_r\Big(\frac{1}{cp+d} \cdot \frac{z_p(\tau)}{c\tau+d}\Big) \\ &= \tilde{F}_p(z_p(\tau)), \end{aligned}$$

where the last line is by (69) with $z_p'(\tau) = 1$. Hence by $-d/c \notin \mathbb{H}$, and observing that

$$\frac{1}{c\tau+d} = \frac{1}{cp+d} - \frac{c(\tau-p)}{(cp+d)^2} + O\left((\tau-p)^2\right) = \frac{1}{cp+d} + O(z_p(\tau)),$$

we have $\operatorname{ord} \tilde{F}_r = \operatorname{ord} \tilde{F}_p$. Second, suppose $p \in \mathbb{H}$ is an elliptic point. Then by (64),

$$z_r(\gamma\tau) = \xi(p) z_p(\tau) \text{ for } \xi(p) := \Big(\frac{c\overline{p}+d}{cp+d}\Big)^{h(p)}.$$

Hence,

$$F(\gamma\tau) = z_r'(\gamma\tau)^{k/2}\,\tilde{F}_r(z_r(\gamma\tau)) = \Big(\frac{\xi(p)}{(\gamma\tau)'}\Big)^{k/2} z_p'(\tau)^{k/2}\tilde{F}_r(\xi(p)z_p(\tau))$$
$$= \xi(p)^{k/2}(c\tau+d)^k z_p'(\tau)^{k/2}\tilde{F}_r(\xi(p)z_p(\tau)).$$

This implies, similarly to above,

$$F(\tau) = \frac{1}{(c\tau+d)^k}F(\gamma\tau) = \xi(p)^{k/2}z_p'(\tau)^{k/2}\tilde{F}_r(\xi(p)z_p(\tau))$$
$$= z_p'(\tau)^{k/2}\tilde{F}_p(z_p(\tau));$$

which gives $\operatorname{ord}\tilde{F}_r = \operatorname{ord}\tilde{F}_p$. Finally, suppose $p = \gamma_0\infty \in \hat{\mathbb{Q}}$, $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$. Then applying (65) gives,

$$F(\gamma\tau) = z_r'(\gamma\tau)^{k/2}\,\tilde{F}_r(z_r(\gamma\tau)) = \Big(\frac{z_p'(\tau)}{(\gamma\tau)'}\Big)^{k/2}\tilde{F}_r(z_p(\tau))$$
$$= (c\tau+d)^k\, z_p'(\tau)^{k/2}\tilde{F}_r(z_p(\tau)).$$

Invoking the modular transformation property as above, we obtain

$$F(\tau) = \frac{1}{(c\tau+d)^k}F(\gamma\tau) = z_p'(\tau)^{k/2}\tilde{F}_r(z_p(\tau))$$
$$= z_p'(\tau)^{k/2}\tilde{F}_p(z_p(\tau));$$

which implies $\operatorname{ord}\tilde{F}_r = \operatorname{ord}\tilde{F}_p$, and which completes the proof of the lemma. □

To state the analogue of Lemma 6.6 for odd k, we need the square root of a Laurent series, which is a Puiseux series defined as follows.

Definition 6.7. *Let $G(z) = \sum_{n=N}^{\infty} c(n)z^n$ with $\operatorname{ord} G(z) = N$. Then*

$$G(z)^{1/2} := \sqrt{c(N)}\, z^{N/2}(1+\psi(z))^{1/2} = \sqrt{c(N)}\, z^{N/2}\sum_{\ell=0}^{\infty}\binom{1/2}{\ell}\psi(z)^{\ell},$$

where $\psi(z) = \frac{1}{c(N)}\sum_{n=1}^{\infty} c(N+n)z^n$, and where for $\sqrt{c(N)}$ we choose the principal branch.

Lemma 6.8. *Given a non-zero $F \in M_k(\Gamma)$ with k odd, and $P = [p] \in X(\Gamma)$, $p \in \hat{\mathbb{H}}$. Then*

$$F(\tau) = z_p'(\tau)^{k/2}\,\tilde{F}_p(z_p(\tau)), \qquad \tau \in V_0, \tag{71}$$

with

$$\tilde{F}_p(z) := \tilde{G}_p(z)^{1/2};$$

where $\tilde{G}_p(z)$ is the Laurent series such that, according to Lemma 6.6,

$$G(\tau) := F(\tau)^2 = z_p'(\tau)^k\,\tilde{G}_p(z_p(\tau)) \in M_{2k}(\Gamma), \qquad \tau \in V_0, \tag{72}$$

where $V_0 \subseteq \hat{\mathbb{H}}$ is a suitable neighborhood of p, and where the root $z_p'(z)^{1/2}$ is chosen as the appropriate branch.

Proof. The existence of the Laurent series $\tilde{G}_(z)$ such that (72) is owing to Lemma 6.6 since $G(\tau) := F(\tau)^2 \in M_{2k}(\Gamma)$. The rest follows from Def. 6.7. □

For our analysis of poles it will be convenient to extend Def. 6.5 to modular forms.

Definition 6.9 (order of a modular form at a point). *For non-zero $F \in M_k(\Gamma)$ with k even, and $P = [p] \in X(\Gamma)$, $p \in \hat{\mathbb{H}}$:*

$$\operatorname{Ord}_P F := \operatorname{ord} \tilde{F}_p(z), \tag{73}$$

where $\tilde{F}_p$ is as in Lemma 6.6.

Remark 6.10. This definition can be extended to k odd, but we do not need it here.

Remark 6.11. A more important remark concerns the fact that if $F \in M(\Gamma)$ (i.e., if $k = 0$), one has,

$$\operatorname{Ord}_P F = \operatorname{Ord}_P \hat{F}, \tag{74}$$

where $P = [p] \in X(\Gamma)$. Here the order on the right side is the order at P of the induced meromorphic function $\hat{F} : X(\Gamma) \to \hat{\mathbb{C}}$ defined on the Riemann surface $X = X(\Gamma)$; see Section 11.3. In this case, one has, according to Lemma 11.3,

$$\sum_{P \in X(\Gamma)} \operatorname{Ord}_P F = 0. \tag{75}$$

We will need the generalization (80) for modular forms $F \in M_k(\Gamma)$ with $\operatorname{ord}_p F$; this is the main motivation to introduce the order as in Def. 6.9

Before stating and proving (80), we conclude this section by extending the notion $\operatorname{NofPoles}(F)$ from modular functions to modular forms.

Definition 6.12. *Let $F \in M_k(\Gamma)$ with k even.*

$$\operatorname{NofPoles}(F) := - \sum_{\substack{[p] \in X(\Gamma) \\ \operatorname{ord} \tilde{F}_p(z) < 0}} \operatorname{Ord}_{[p]} F, \tag{76}$$

where $\tilde{F}_p(z)$ is the Laurent series defined in Lemma 6.6.

Remark 6.13. When $k = 0$; i.e., if F is a modular function in $M(\Gamma)$, then this coincides with the definition (50). This means, then $\operatorname{NofPoles}(F)$ is nothing but the number of poles of the induced function $\hat{F}$ on $X(\Gamma)$.

We conclude this section by a fact which is useful in applying the algorithm ModFormDE; see, for example, Section 4.4. Given a modular forms of even weight, it relates our two different notions of orders taken at cusps.

Lemma 6.14. *Let $F \in M_{2k}(\Gamma)$ and $P = [a/c] \in X(\Gamma)$, $p = a/c \in \hat{\mathbb{Q}}$. Then,*

$$\operatorname{ord}_{a/c} F = \operatorname{Ord}_{[a/c]} F + k. \tag{77}$$

Proof. By Lemma 6.6,

$$F(\tau) = z'_{a/c}(\tau)^k \sum_{n=N_{a/c}}^{\infty} b(n) z_{a/c}(\tau)^n, \tag{78}$$

where $N_{a/c} = \operatorname{ord}_{[a/c]} F \neq 0$. If $\gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_0 \infty = a/c$, then

$$z_p(\tau) := e^{2\pi i \gamma_0^{-1}\tau/w_0} \text{ where } w_0 = w_{\gamma_0}(\Gamma).$$

Define,

$$q_{w_0} := z_{a/c}(\gamma_0 \tau) = e^{2\pi i \tau/w_0}, \text{ and thus, } (c_0\tau + d_0)^{-2} z'_{a/c}(\gamma_0\tau) = \frac{2\pi i}{w_0} q_{w_0}. \tag{79}$$

The weight-$2k$ action applied to F, and using (78) gives,

$$\begin{aligned} (F|_{2k}\gamma_0)(\tau) &= (c_0\tau + d_0)^{-2k} z'_{a/c}(\gamma_0\tau)^k \sum_{n=N_{a/c}}^{\infty} b(n) q_{w_0}^n \\ &= \left(\frac{2\pi i}{w_0}\right)^k q_{w_0}^k \sum_{n=N_{a/c}}^{\infty} b(n) q_{w_0}^n, \end{aligned}$$

where the last equality is by (79). Comparing this to the definition (24), proves the statement. ☐

Notice that for $k = 0$, Lemma 6.14 turns into (74) if P is a cusp.

6.2. **A bound from the Riemann-Hurwitz formula.** A last major ingredient for our analysis of poles is the following proposition which is an application of the Riemann-Hurwitz formula.

Proposition 6.15. *Let $F \in M_k(\Gamma)$ be non-zero with k even. Then*

$$\sum_{P \in X(\Gamma)} \operatorname{Ord}_P F = k(g_\Gamma - 1), \tag{80}$$

where g_Γ is the genus of the compact Riemann surface $X(\Gamma)$.

Remark 6.16. Notice that the sum on the left side is well-defined. Namely, suppose that $\mathrm{ord}_P F \neq 0$ for infinitely finitely many $P \in X(\Gamma)$. Choosing a bounded fundamental domain $\mathcal{F} \subset \hat{\mathbb{H}}$ for $X(\Gamma)$, this would imply that F would have infinitely many poles or zeros in $\mathcal{F}$. Let us assume the latter.[21] Then this set has a limit point in $\hat{\mathbb{H}}$. Since F is non-zero, this limit point must be in $\hat{\mathbb{Q}}$. But property (21) in the definition of modular forms implies that the poles of F cannot cluster at any $a/c \in \hat{\mathbb{Q}}$.[22]

We will apply Prop. 6.15 in the following form.

Corollary 6.17. *Let $F \in M_k(\Gamma)$, k even, such that for $P = [p] \in X(\Gamma)$,*

$$F(\tau) = z_p'(\tau)^{k/2} \tilde{F}_p(z_p(\tau)),$$

where $\tilde{F}_p(z)$ is a Laurent series as in Lemma 6.6. Then

$$\sum_{[p]\in X(\Gamma)} \mathrm{ord}\, \tilde{F}_p(z) = k(g_\Gamma - 1), \tag{81}$$

where g_Γ is the genus of the compact Riemann surface $X(\Gamma)$.

Convention. When here and in the following the domain of a sum is specified as "$[p] \in X(\Gamma)$", then this is understood as follows: For each $P = [p] \in X(\Gamma)$ take exactly one representative $p \in \hat{\mathbb{H}}$; the sum then runs over all such p.

Before we prove Prop. 6.15, we prepare with a few facts.

Lemma 6.18. *Let F and G be non-zero modular forms in $M_k(\Gamma)$ with k even. Then for $P \in X(\Gamma)$,*

$$\mathrm{Ord}_P F - \mathrm{Ord}_P G = \mathrm{Ord}_P(F/G). \tag{82}$$

Proof. Apply Lemma 6.6 together with Lemma 6.4. □

Lemma 6.19. *Let F and G be non-zero modular forms in $M_k(\Gamma)$ with k even. Then*

$$\sum_{P\in X(\Gamma)} \mathrm{Ord}_P F = \sum_{P\in X(\Gamma)} \mathrm{Ord}_P G. \tag{83}$$

Proof. In view of $F/G \in M(\Gamma)$, apply Lemma 11.3 together with Lemma 6.18 □

[21]Otherwise, consider $1/F$.
[22]See, for instance, the remark in [4, Def. 3.2.1].

As already mentioned, for the proof of Prop. 6.15 we utilize the Riemann-Hurwitz formula for a non-constant meromorphic function $\varphi : X(\Gamma) \to \hat{\mathbb{C}}$ which has exactly n poles in $X(\Gamma)$, counting multiplicities; see, e.g., [8, Thm. 4.16]:[23]

$$\sum_{P \in X(\Gamma)} (\operatorname{mult}_P \varphi - 1) = 2\, g_\Gamma + 2\, n - 2. \tag{84}$$

Here g_Γ denotes the genus of $X(\Gamma)$, and $\operatorname{mult}_P \varphi$ is the usual multiplicity of φ at P; we recall its exact definition in Section 11.3.

Now we are ready to prove the sum estimate (80).

Proof of Prop. 6.15. For $k = 0$ this is (123). To prove the statement for weight $k = 2$, take some $t \in M(\Gamma)$ and suppose $\hat{t}$ has exactly n poles in $X(\Gamma)$. For $P \in X(\Gamma)$, notice that

$$\operatorname{mult}_P \hat{t} = \begin{cases} -\operatorname{Ord}_P t, & \text{if } P \text{ is a pole of } \hat{t} \\ \operatorname{Ord}_P t' + 1, & \text{otherwise} \end{cases}. \tag{85}$$

The second equality is a consequence of using Def. 6.9 on $t' \in M_2(\Gamma)$ together with (70). In addition, we need that at a pole $P \in X(\Gamma)$ of $\hat{t}$,

$$\operatorname{Ord}_P t' - \operatorname{Ord}_P t = -1.$$

Using these properties and defining,

$$\operatorname{Poles}(t) := \{P \in X(\Gamma) : P \text{ is a pole of } \hat{t}\}, \tag{86}$$

we obtain by (85),

$$\begin{aligned}\sum_{P \in \operatorname{Poles}(t)} (\operatorname{mult}_P t - 1) &= \sum_{P \in \operatorname{Poles}(t)} (-\operatorname{Ord}_P t - 1) \\ &= -2 \sum_{P \in \operatorname{Poles}(t)} \operatorname{Ord}_P t + \sum_{P \in \operatorname{Poles}(t)} \operatorname{Ord}_P t' = 2n + \sum_{P \in \operatorname{Poles}(t)} \operatorname{Ord}_P t',\end{aligned}$$

and,

$$\sum_{P \in X(\Gamma) \setminus \operatorname{Poles}(t)} (\operatorname{mult}_P t - 1) = \sum_{P \in X(\Gamma) \setminus \operatorname{Poles}(t)} \operatorname{ord}_P t'.$$

[23] Actually the special case we need, $Y = \hat{\mathbb{C}}$, was given by Riemann; e.g., [3].

Combining these two sums we obtain, as a consequence of Lemma 11.3 and Lemma 6.18,

$$\begin{aligned}0 &= \sum_{P\in X(\Gamma)} \mathrm{Ord}_P(F/t') = \sum_{P\in X(\Gamma)} \mathrm{Ord}_P F - \sum_{P\in X(\Gamma)} \mathrm{Ord}_P t' \\ &= \sum_{P\in X(\Gamma)} \mathrm{Ord}_P F + 2n - \sum_{P\in X(\Gamma)} (\mathrm{mult}_P t - 1) \\ &= \sum_{P\in X(\Gamma)} \mathrm{Ord}_P F - 2g_\Gamma + 2.\end{aligned}$$

This proves (6.15) for $F \in M_2(\Gamma)$; for the last equality we invoked the Riemann-Hurwitz formula (84).

To prove (6.15) for $F \in M_k(\Gamma)$ where $k \geq 2$ is even, take some $g \in M_2(\Gamma)$. Then $f := F/g^{k/2} \in M(\Gamma)$, and one has,

$$\begin{aligned}0 &\overset{(123)}{=} \sum_{P\in X(\Gamma)} \mathrm{Ord}_P f \overset{(82)}{=} \sum_{P\in X(\Gamma)} \mathrm{Ord}_P F - \frac{k}{2} \sum_{P\in X(\Gamma)} \mathrm{Ord}_P g \\ &= \sum_{P\in X(\Gamma)} \mathrm{Ord}_P F - \frac{k}{2}(2g_\Gamma - 2)),\end{aligned}$$

where the last line is by the $k = 2$ part already proven. This completes the proof of Prop. 6.15. □

7. Bounds for p_1

7.1. **Rewriting p_1.** We need to rewrite $p_1 \in M(\Gamma)$ in a form which is convenient for the analysis of poles.

Let $P = [p] \in X(\Gamma)$ be fixed. For $t \in M(\Gamma)$ let

$$t(\tau) = T_p(z_p(\tau)), \quad \text{where } T_p(z) := \tilde{t}_p(z) \tag{87}$$

is the Laurent series as defined in (66).

Derivatives of a Laurent series $T(z) = \sum_{n=M}^{\infty} a(n)z^n$ are defined as usual by

$$T^{(j)}(z) := \sum_{n=M}^{\infty} (n)_j\, a(n) z^{n-j},\ j = 0, 1, \ldots$$

In particular, $T(z) = T^{(0)}(z)$, $T'(z) = T^{(1)}(z)$, and $T''(z) = T^{(2)}(z)$. Hence, relation (70) turns into,

$$D_q t(\tau) = \frac{1}{2\pi i} t'(\tau) = \frac{1}{2\pi i} z_p'(\tau) T_p^{(1)}(z_p(\tau)). \tag{88}$$

For $D_q^2 t$ one has,

$$D_q^2 t = \frac{1}{(2\pi i)^2}\Big(z_p'' T_p^{(1)}(z_p) + (z_p')^2 T_p^{(2)}(z_p)\Big), \tag{89}$$

where we omitted the argument τ.

Given the same fixed $p \in \hat{\mathbb{H}}$ as above for $t \in M(\Gamma)$:

Case A: For $F \in M_k(\Gamma)$, k even, let

$$F(\tau) = z_p'(\tau)^{k/2} f_{0,p}(z_p(\tau)), \quad \text{where } \ f_{0,p}(z) := \tilde{F}_p(z) \tag{90}$$

is the Laurent series defined as in (68).

Case B: For $F \in M_k(\Gamma)$, k odd, let

$$F(\tau) = z_p'(\tau)^{k/2} f_{1,p}(z_p(\tau)), \quad \text{where } \ f_{1,p}(z) := \tilde{G}_p(z)^{1/2} \tag{91}$$

is the square root of the Laurent series $\tilde{G}_p(z)$ defined as in (72) such that

$$G(\tau) := F(\tau)^2 = z_p'(\tau)^k\, \tilde{G}_p(z_p(\tau)) \in M_{2k}(\Gamma). \tag{92}$$

Combining both Cases A and B, one has[24]

$$\frac{D_q F}{F} = \frac{1}{2\pi i}\Big(\frac{z_p'\, f_{\delta,p}^{(1)}(z_p)}{f_{\delta,p}(z_p)} + \frac{k}{2}\cdot\frac{z_p''}{z_p'}\Big), \tag{93}$$

where $\delta = 0$ if k is even, and $\delta = 1$ if k is odd.

Lemma 7.1. *Given $t \in M(\Gamma)$, $F \in M_k(\Gamma)$, and $P = [p] \in X(\Gamma)$, $p \in \hat{\mathbb{H}}$. Then there exists a Laurent series,*

$$P_{1,p}(z) := -1 + \frac{T_p(z)}{T_p^{(1)}(z)}\Big(\frac{T_p^{(2)}(z)}{T_p^{(1)}(z)} - \frac{2}{k}\frac{f_{\delta,p}^{(1)}(z)}{f_{\delta,p}(z)}\Big), \tag{94}$$

where $\delta = 0$ if k is even, and $\delta = 1$ if k is odd, such that

$$p_1(\tau) = P_{1,p}(z_p(\tau)), \quad \tau \in V_0, \tag{95}$$

V_0 being a suitable open neighborhood of p.

Proof. By (31)[25] and using,

$$D_q G_1 = \frac{D_q^2 t}{t} - \frac{(D_q t)^2}{t^2},$$

[24] We again omit the argument τ.

[25] With t instead of h, and with F instead of g.

one has, applying (88), (88), and (102),

$$
\begin{aligned}
p_1 &= \Big(\frac{t}{D_q t}\Big)^2\Big(\frac{D_q^2 t}{t} - \frac{(D_q t)^2}{t^2}\Big) - \frac{2}{k}\frac{t}{D_q t}\frac{D_q F}{F} \\
&= -1 + \frac{t}{D_q t}\Big(\frac{D_q^2 t}{D_q t} - \frac{2}{k}\frac{D_q F}{F}\Big) = -1 + (2\pi i)\frac{T_p(z_p)}{z_p' T_p^{(1)}(z_p)} \\
&\quad \times \left(\frac{z_p'' T_p^{(1)}(z_p) + (z_p')^2 T_p^{(2)}(z_p)}{(2\pi i) z_p' T_p^{(1)}(z_p)} - \frac{2}{k}\frac{1}{2\pi i}\Big(\frac{z_p'\, f_{\delta,p}^{(1)}(z_p)}{f_{\delta,p}(z_p)} + \frac{k}{2}\cdot\frac{z_p''}{z_p'}\Big)\right) \\
&= -1 + (2\pi i)\frac{T_p(z_p)}{z_p' T_p^{(1)}(z_p)}\Big(\frac{z_p' T_p^{(2)}(z_p)}{(2\pi i) T_p^{(1)}(z_p)} - \frac{2}{k}\frac{z_p'\, f_{\delta,p}^{(1)}(z_p)}{(2\pi i) f_{\delta,p}(z_p)}\Big).
\end{aligned}
$$

□

7.2. **Bounds for the poles of p_1.** To estimate the number of poles of $p_1 \in M(\Gamma)$, Lemma 7.1 implies,

$$
\begin{aligned}
\mathrm{NofPoles}(p_1) &= -\sum_{P\in \mathrm{Poles}(p_1)} \mathrm{Ord}_P\, p_1 = -\sum_{[p]\in \mathrm{Poles}(p_1)} \mathrm{ord}\, P_{1,p}(z) \\
&\leq -\sum_{[p]\in \mathrm{Poles}(p_1)} \mathrm{ord}\,\frac{T_p(z)}{T_p^{(1)}(z)} - \sum_{[p]\in \mathrm{Poles}(p_1)} \mathrm{ord}\,\frac{T_p^{(2)}(z)}{T_p^{(1)}(z)} \\
&\quad - \sum_{[p]\in \mathrm{Poles}(p_1)} \mathrm{ord}\,\frac{f_{\delta,p}^{(1)}(z)}{f_{\delta,p}(z)},
\end{aligned} \tag{96}
$$

with $\delta = 0$ if k is even, and $\delta = 1$ if k is odd.

To proceed with our pole estimation, we treat each of the sums in (96) separately. To this end, it will be convenient to define

$$
\pi(x) := \begin{cases} x, & \text{if } x < 0 \\ 0, & \text{if } x \geq 0 \end{cases}, \quad \text{and } \zeta(x) := \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}.
$$

In addition, the following two facts will be useful.

Lemma 7.2.

$$
\sum_{[p]\in X(\Gamma)} \mathrm{ord}\, T_p^{(1)}(z) = \sum_{[p]\in X(\Gamma)} \pi(\mathrm{ord}\, T_p^{(1)}(z)) + \sum_{[p]\in X(\Gamma)} \zeta(\mathrm{ord}\, T_p^{(1)}(z)) = 2(g_\Gamma - 1).
$$

Proof. In view of (88), the statement is proved by applying Corollary 6.17 to $t' \in M_2(\Gamma)$. □

The second lemma we want to list explicitly is trivial, but useful.

Lemma 7.3. *Let $f(z) = \sum_{n=M}^{\infty} c(n)z^n$ be a Laurent series, then:*

$$\operatorname{ord} f(z) < 0 \Rightarrow \operatorname{ord} f^{(j)}(z) = \operatorname{ord} f(z) - j \quad \textit{for } j \in \mathbb{Z}_{\geq 0}.$$

We begin with bounds for the first two sums on the right side of (96).

Lemma 7.4. *For $j \in \mathbb{Z}_{\geq 0}$, $j \neq 1$, $i \in \{1,2\}$,*

$$-\sum_{[p]\in \mathrm{Poles}(p_i)} \operatorname{ord} \frac{T_p^{(j)}(z)}{T_p^{(1)}(z)} \leq 2g_\Gamma - 2 - 2\sum_{[p]\in X(\Gamma)} \pi(\operatorname{ord} T_p(z)) + (j+1) \sum_{\substack{[p]\in X(\Gamma)\\ \operatorname{ord} T_p(z)<0}} 1.$$

Proof.

$$\begin{aligned}
-\sum_{[p]\in \mathrm{Poles}(p_i)} \operatorname{ord} \frac{T_p^{(j)}(z)}{T_p^{(1)}(z)} &\leq -\sum_{\substack{[p]\in X(\Gamma)\\ \operatorname{ord} T_p^{(j)}(z)/T_p^{(1)}(z)<0}} \operatorname{ord} \frac{T_p^{(j)}(z)}{T_p^{(1)}(z)} \\
&= -\sum_{[p]\in X(\Gamma)} \pi\Big(\operatorname{ord} \frac{T_p^{(j)}(z)}{T_p^{(1)}(z)}\Big) = -\sum_{[p]\in X(\Gamma)} \pi(\operatorname{ord} T_p^{(j)}(z)) + \sum_{[p]\in X(\Gamma)} \zeta(\operatorname{ord} T_p^{(1)}(z)) \\
&= -\sum_{[p]\in X(\Gamma)} \pi(\operatorname{ord} T_p^{(j)}(z)) + 2g_\Gamma - 2 - \sum_{[p]\in X(\Gamma)} \pi(\operatorname{ord} T_p^{(1)}(z)),
\end{aligned}$$

where the last equality is by Lemma 7.2. Now the statement follows by Lemma 7.3 in the version,

$$-\sum_{[p]\in X(\Gamma)} \pi(\operatorname{ord} T_p^{(j)}(z)) = -\sum_{[p]\in X(\Gamma)} \pi(\operatorname{ord} T_p(z)) + j \sum_{\substack{[p]\in X(\Gamma)\\ \operatorname{ord} T_p(z)<0}} 1. \tag{97}$$

∎

The following simplification of the upper bound is straightforward.

Corollary 7.5. *For $j \in \mathbb{Z}_{\geq 0}$, $j \neq 1$, $i \in \{1,2\}$,*

$$-\sum_{[p]\in \mathrm{Poles}(p_i)} \operatorname{ord} \frac{T_p^{(j)}(z)}{T_p^{(1)}(z)} \leq 2g_\Gamma - 2 + (j+3)\ \mathrm{NofPoles}(t).$$

Proof. Obviously,

$$\sum_{\substack{[p]\in X(\Gamma)\\ \operatorname{ord} T_p(z)<0}} 1 \leq -\sum_{[p]\in X(\Gamma)} \pi(\operatorname{ord} T_p(z))$$

and, by definition (87),

$$-\sum_{[p]\in X(\Gamma)} \pi(\operatorname{ord} T_p(z)) = \mathrm{NofPoles}(t).$$

This implies the corollary. ∎

Next we treat the even case of the third sum in (96).

Lemma 7.6. *For $j \in \mathbb{Z}_{\geq 1}$, $i \in \{1, 2\}$,*

$$-\sum_{[p]\in \mathrm{Poles}(p_i)} \mathrm{ord} \frac{f_{0,p}^{(j)}(z)}{f_{0,p}(z)} \leq k(g_\Gamma - 1) - 2 \sum_{[p]\in X(\Gamma)} \pi(\mathrm{ord}\, f_{0,p}(z)) + j \sum_{\substack{[p]\in X(\Gamma) \\ \mathrm{ord}\, f_{0,p}(z)<0}} 1.$$

Proof. The proof works analogously to that of Lemma 7.4; the only difference is that one uses Lemma 7.2 for general even k in the version,

$$\sum_{[p]\in X(\Gamma)} \mathrm{ord}\, f_{0,p}(z) = \sum_{[p]\in X(\Gamma)} \pi(\mathrm{ord}\, f_{0,p}(z)) + \sum_{[p]\in X(\Gamma)} \zeta(\mathrm{ord}\, f_{0,p}(z)) = k(g_\Gamma - 1).$$

□

Corollary 7.7. *For $j \in \mathbb{Z}_{\geq 1}$, $i \in \{1, 2\}$,*

$$-\sum_{[p]\in \mathrm{Poles}(p_i)} \mathrm{ord} \frac{f_{0,p}^{(j)}(z)}{f_{0,p}(z)} \leq k(g_\Gamma - 1) + (j + 2)\, \mathrm{NofPoles}(F).$$

Proof. Obviously,

$$\sum_{\substack{[p]\in X(\Gamma) \\ \mathrm{ord}\, f_{0,p}(z)<0}} 1 \leq -\sum_{[p]\in X(\Gamma)} \pi(\mathrm{ord}\, f_{0,p}(z)).$$

By Def. (6.12),

$$-\sum_{[p]\in X(\Gamma)} \pi(\mathrm{ord}\, f_{0,p}(z)) = \mathrm{NofPoles}(F).$$

This implies the corollary. □

Case (1a), k even (i.e., $\delta = 0$): Applying Corollary 7.5 and 7.7 to (96) gives,

$$\begin{aligned} \mathrm{NofPoles}(p_1) &\leq 2g_\Gamma - 2 + 3\,\mathrm{NofPoles}(t) + 2g_\Gamma - 2 + 5\,\mathrm{NofPoles}(t) \\ &\quad + k(g_\Gamma - 1) + 3\,\mathrm{NofPoles}(F) \\ &= (k + 4)(g_\Gamma - 1) + 8\,\mathrm{NofPoles}(t) + 3\,\mathrm{NofPoles}(F). \end{aligned} \tag{98}$$

For Case (1b) we need one more observation. Recalling the definition (91), one has,

$$\frac{f_{1,p}^{(1)}(z)}{f_{1,p}(z)} = \frac{1}{2} \frac{\tilde{G}_p^{(1)}(z)}{\tilde{G}_p(z)}, \tag{99}$$

with $f_{1,p}^{(1)}(z) = \tilde{G}_p(z)^{1/2}$, where $\tilde{G}_p(z)$ is the Laurent series such that for $G := F^2 \in M_{2k}(\Gamma)$:

$$G(\tau) = z_p'(\tau)^k\, \tilde{G}_p(z_p(\tau)).$$

Consequently, Cor. 7.7 with $j = 1$ carries over to this situation with $2k$ instead of k and $G = F^2$ instead of F:

Corollary 7.8. *For $j \in \mathbb{Z}_{\geq 1}$, $i \in \{1, 2\}$,*

$$-\sum_{[p]\in \mathrm{Poles}(p_i)} \operatorname{ord} \frac{f_{1,p}^{(1)}(z)}{f_{1,p}(z)} \leq 2k(g_\Gamma - 1) + 3\,\mathrm{NofPoles}(F^2).$$

Now we establish the bound estimate for odd k.

Case (1b), k odd (i.e., $\delta = 1$): Applying Corollary 7.5 and 7.8 to (96) gives,

$$\begin{aligned}\mathrm{NofPoles}(p_1) &\leq 2g_\Gamma - 2 + 3\,\mathrm{NofPoles}(t) + 2g_\Gamma - 2 + 5\,\mathrm{NofPoles}(t)\\ &\quad + 2k(g_\Gamma - 1) + 3\,\mathrm{NofPoles}(F^2)\\ &= (2k + 4)(g_\Gamma - 1) + 8\,\mathrm{NofPoles}(t) + 3\,\mathrm{NofPoles}(F^2). \end{aligned} \tag{100}$$

8. Bounds for p_2

8.1. **Rewriting p_2.** As with p_1, we first need to rewrite $p_2 \in M(\Gamma)$ in a form which is convenient for the analysis of poles.

Let $P = [p] \in X(\Gamma)$ be fixed. We assume the same setting as in Section 7: i.e., for $t \in M(\Gamma)$ the Laurent series $T_p(z)$ is defined as in (87) such that

$$t(\tau) = T_p(z_p(\tau)), \quad \text{where } T_p(z) := \tilde{t}_p(z); \tag{101}$$

for $F \in M_k(\Gamma)$, the Laurent series $f_{\delta,p}(z_p(\tau))$, $\delta \in \{0, 1\}$,

$$F(\tau) = z_p'(\tau)^{k/2} f_{\delta,p}(z_p(\tau)), \quad \text{where } f_{\delta,p}(z) := \tilde{F}_p(z)$$

are defined as in (90) if k is even, and as in (91) if k is odd. In the even case, we take $\delta = 0$, in the odd case $\delta = 1$.

$$\frac{D_q F}{F} = \frac{1}{2\pi i}\Big(\frac{z_p' f_{\delta,p}^{(1)}(z_p)}{f_{\delta,p}(z_p)} + \frac{k}{2} \cdot \frac{z_p''}{z_p'}\Big), \tag{102}$$

For p_1 we used the formula (102) for D_qF/F. For p_2 we need,

$$\frac{D_q^2 F}{F} = \frac{1}{(2\pi i)^2}\Big(\frac{(z_p')^2 f_{\delta,p}^{(2)}(z_p)}{f_{\delta,p}(z_p)} + (k+1)\cdot\frac{z_p'' f_{\delta,p}^{(1)}(z_p)}{f_{\delta,p}(z_p)} + \frac{k}{2}\cdot\frac{z_p^{(3)}}{z_p'} + \frac{k(k-2)}{4}\cdot\frac{(z_p'')^2}{(z_p')^2}\Big),$$

which, as (102), can be derived by straightforward computation.

This relation together with (102) gives,

$$
\begin{aligned}
p_2 &= \frac{1}{G_1^2}\Big(D_qG_2-\frac{1}{k}G_2^2\Big)=\Big(\frac{t}{D_qt}\Big)^2\Big(D_q\frac{D_qF}{F}-\frac{1}{k}\Big(\frac{D_qF}{F}\Big)^2\Big)\\
&= \Big(\frac{t}{D_qt}\Big)^2\Big(\frac{D_q^2F}{F}-(1+\frac{1}{k})\Big(\frac{D_qF}{F}\Big)^2\Big)=\frac{T_p(z_p)^2}{T_p^{(1)}(z_p)^2}\\
&\times\Big(\frac{f_{\delta,p}^{(2)}(z_p)}{f_{\delta,p}(z_p)}-(1+\frac{1}{k})\frac{f_{\delta,p}^{(1)}(z_p)^2}{f_{\delta,p}(z_p)^2}+\frac{kz_p^{(3)}}{2(z_p')^3}-\frac{3k(z_p'')^2}{4(z_p')^4}\Big)
\end{aligned}
\tag{103}
$$

We need to consider in more detail the expression,

$$\ell(z_p^{(1)}(\tau),z_P^{(2)}(\tau),z_p^{(3)}(\tau)):=\frac{z_p^{(3)}(\tau)}{z_p'(\tau)^3}-\frac{3kz_p''(\tau)^2}{2z_p'(\tau)^4}.$$

If $[p]\in X(\Gamma)$ is an ordinary point, $z_p(\tau)=\tau-p$ and

$$\ell(z_p^{(1)}(\tau),z_P^{(2)}(\tau),z_p^{(3)})=0 \ \text{ for all } \ \tau\in V_0.$$

If $[p]\in X(\Gamma)$ is an elliptic point of order 2, $z_p(\tau)=(\frac{\tau-p}{\tau-\overline{p}})^2$, and one has,

$$\ell(z_p^{(1)}(\tau),z_P^{(2)}(\tau),z_p^{(3)})=-\frac{3}{8}\frac{1}{z_p(\tau)^2}.$$

If $[p]\in X(\Gamma)$ is an elliptic point of order 3, $z_p(\tau)=(\frac{\tau-p}{\tau-\overline{p}})^3$, and one has,

$$\ell(z_p^{(1)}(\tau),z_P^{(2)}(\tau),z_p^{(3)})=-\frac{4}{9}\frac{1}{z_p(\tau)^2}.$$

If $[p]\in X(\Gamma)$ is a cusp; then $p=a/c=\gamma\infty$, $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, and $z_p(\tau):=e^{2\pi i\gamma^{-1}\tau/w}$ with $w=w_\gamma(\Gamma)$ as in (22). In this case,

$$\ell(z_p^{(1)}(\tau),z_P^{(2)}(\tau),z_p^{(3)})=-\frac{1}{2}\frac{1}{z_p(\tau)^2}.$$

This leads us to summarize in a definition.

Definition 8.1. *Define $c:X(\Gamma)\to\mathbb{C}$ for $P=[p]\in X(\Gamma)$ as follows:*

$$c(P):=c(p):=\begin{cases}0, & \text{if } P \text{ is an ordinary point}\\ -3/8, & \text{if } P \text{ is elliptic of order } 2\\ -4/9, & \text{if } P \text{ is elliptic of order } 3\\ -1/2, & \text{if } P \text{ is a cusp, } P=[a/c].\end{cases}.$$

Summarizing, we obtained a Laurent series representation of p_2.

Lemma 8.2. *Given $t \in M(\Gamma)$, $F \in M_k(\Gamma)$, and $P = [p] \in X(\Gamma)$, $p \in \hat{\mathbb{H}}$. Then there exists a Laurent series,*

$$P_{2,p}(z) := \frac{T_p(z)^2}{T_p^{(1)}(z)^2}\Big(\frac{f_{\delta,p}^{(2)}(z)}{f_{\delta,p}(z)} - (1+\frac{1}{k}) \cdot \frac{f_{\delta,p}^{(1)}(z)^2}{f_{\delta,p}(z)^2} + \frac{k}{2} \cdot \frac{c(p)}{z^2}\Big) \tag{104}$$

where c is as in Def. 8.1, and where $\delta = 0$ if k is even, and $\delta = 1$ if k is odd, such that

$$p_2(\tau) = P_{2,p}(z_p(\tau)), \quad \tau \in V_0, \tag{105}$$

V_0 being a suitable open neighborhood of p.

8.2. **Bounds for the poles of p_2.** To estimate the number of poles of $p_2 \in M(\Gamma)$, Lemma 8.2 implies,

$$\begin{aligned} \mathrm{NofPoles}(p_2) &= -\sum_{P \in \mathrm{Poles}(p_2)} \mathrm{Ord}_P\, p_2 = -\sum_{[p] \in \mathrm{Poles}(p_2)} \mathrm{ord}\, P_{2,p}(z) \\ &\le -\sum_{[p] \in \mathrm{Poles}(p_2)} \mathrm{ord}\, \frac{T_p(z)^2}{T_p^{(1)}(z)^2} - \sum_{[p] \in \mathrm{Poles}(p_2)} \mathrm{ord}\, \frac{f_{\delta,p}^{(2)}(z)}{f_{\delta,p}(z)} \\ &\quad - \sum_{[p] \in \mathrm{Poles}(p_2)} \mathrm{ord}\, \frac{f_{\delta,p}^{(1)}(z)^2}{f_{\delta,p}(z)^2} + \sum_{\substack{[p] \in \mathrm{Poles}(p_2) \\ c(P) \neq 0}} 2, \end{aligned} \tag{106}$$

with $\delta = 0$ if k is even, and $\delta = 1$ if k is odd.

As a consequence, similarly to the Cases (1a) and (1b), we obtain bounds for p_2:

Case (2a), k even (i.e., $\delta = 0$): Applying Corollary 7.5 and 7.7 to (106) gives,

$$\begin{aligned} \mathrm{NofPoles}(p_2) &\le 4g_\Gamma - 4 + 6\,\mathrm{NofPoles}(t) + k(g_\Gamma - 1) + 4\,\mathrm{NofPoles}(F) \\ &\quad + 2k(g_\Gamma - 1) + 6\,\mathrm{NofPoles}(F) \\ &\quad + 2\ \mathrm{NofCusps}(\Gamma) + 2\ \mathrm{NofElliptic}(\Gamma). \\ &= (3k+4)(g_\Gamma - 1) + 6\,\mathrm{NofPoles}(t) + 10\,\mathrm{NofPoles}(F) \\ &\quad + 2\ \mathrm{NofCusps}(\Gamma) + 2\ \mathrm{NofElliptic}(\Gamma). \end{aligned} \tag{107}$$

For Case (2b) we need one more observation. As with (99), we recall the definition (91) and obtain,

$$\frac{f_{1,p}^{(2)}(z)}{f_{1,p}(z)} = -\frac{1}{4}\frac{\tilde{G}_p^{(1)}(z)^2}{\tilde{G}_p(z)^2} + \frac{1}{2}\frac{\tilde{G}_p^{(2)}(z)}{\tilde{G}_p(z)}, \tag{108}$$

with $f_{1,p}^{(1)}(z) = \tilde{G}_p(z)^{1/2}$, where $\tilde{G}_p(z)$ is the Laurent series such that for $G := F^2 \in M_{2k}(\Gamma)$:

$$G(\tau) = z_p'(\tau)^k\, \tilde{G}_p(z_p(\tau)).$$

Owing to Cor. 7.7, relation (108) implies,

$$\begin{aligned} -\sum_{[p]\in\mathrm{Poles}(p_2)} \mathrm{ord}\, \frac{f_{1,p}^{(2)}(z)}{f_{1,p}(z)} &\leq 4k(g-1) + 2(1+2)\,\mathrm{NofPoles}(F^2) \\ &\quad + 2k(g-1) + (2+2)\,\mathrm{NofPoles}(F^2) \\ &= 6k(g-1) + 10\,\mathrm{NofPoles}(F^2). \end{aligned} \tag{109}$$

Now we are ready to establish the bound estimate for p_2 for odd k.

Case (2b), k odd (i.e., $\delta = 1$): Applying (109), Corollary 7.5 and 7.8 to (106) gives,

$$\begin{aligned} \mathrm{NofPoles}(p_2) &\leq 4g_\Gamma - 4 + 6\,\mathrm{NofPoles}(t) + 6k(g_\Gamma - 1) + 10\,\mathrm{NofPoles}(F^2) \\ &\quad + 4k(g_\Gamma - 1) + 6\,\mathrm{NofPoles}(F^2). \\ &\quad + 2\ \mathrm{NofCusps}(\Gamma) + 2\ \mathrm{NofElliptic}(\Gamma). \\ &= (10k+4)(g_\Gamma - 1) + 6\,\mathrm{NofPoles}(t) + 16\,\mathrm{NofPoles}(F^2) \\ &\quad + 2\ \mathrm{NofCusps}(\Gamma) + 2\ \mathrm{NofElliptic}(\Gamma). \end{aligned} \tag{110}$$

We want to point out that, when setting up such bound estimates, a proper organization of the terms involved can be important. For example, we can improve upon (110) as follows.

Instead of treating $\phi_p(z) := \frac{f_{1,p}^{(2)}(z)}{f_{1,p}(z)}$ and $\psi_p(z) := \frac{f_{1,p}^{(1)}(z)^2}{f_{1,p}(z)^2}$ separately to obtain,

$$-\sum_{[p]\in\mathrm{Poles}(p_2)} \mathrm{ord}\,\phi_p(z) - \sum_{[p]\in\mathrm{Poles}(p_2)} \mathrm{ord}\,\psi_p(z) \leq 10k(g_\Gamma - 1) + 32\,\mathrm{NofPoles}(F),$$

one can combine them,

$$\phi_p(z) - \Big(1 + \frac{1}{k}\Big)\psi_p(z) \overset{(99),(108)}{=} -\frac{2k+1}{4k}\frac{\tilde{G}_p^{(1)}(z)^2}{\tilde{G}_p(z)^2} + \frac{1}{2}\frac{\tilde{G}_p^{(2)}(z)}{\tilde{G}_p(z)},$$

to obtain, using Cor. 7.7 in its version for F^2 instead of F,

$$\begin{aligned} -\sum_{[p]\in\mathrm{Poles}(p_2)} \mathrm{ord}\,\Big(\phi_p(z) - (1+\frac{1}{k})\psi_p(z)\Big) &\leq 4k(g_\Gamma - 1) + 2(1+2)\,\mathrm{NofPoles}(F^2) \\ &\quad + 2k(g_\Gamma - 1) + (2+2)\,\mathrm{NofPoles}(F^2) \\ &= 6k(g_\Gamma - 1) + 10\,\mathrm{NofPoles}(F^2). \end{aligned}$$

This simple reorganization led us to an improvement of the bound estimate (110).

Case (2b), k odd, improved version:

$$\begin{aligned} \mathrm{NofPoles}(p_2) &\leq (6k+4)(g_\Gamma - 1) + 6\,\mathrm{NofPoles}(t) + 10\,\mathrm{NofPoles}(F^2) \\ &\quad + 2\ \mathrm{NofCusps}(\Gamma) + 2\ \mathrm{NofElliptic}(\Gamma). \end{aligned} \tag{111}$$

9. Bounds for the derivatives of p_1 and p_2

To complete the proof of Theorem 5.2 we need to bound the number of poles of the derivatives $\frac{d^j p_i}{dt^j} \in M(\Gamma)$, $i \in \{1,2\}$ and $j \in \mathbb{Z}_{\geq 1}$. To this end, given H and t in $M(\Gamma)$ where $H = r(t)$ is a function in t, we will show that such bounds for $\frac{d^j H}{dt^j}$ can be expressed in terms of pole bounds for H and t.

We begin with the case $j = 1$, noting that

$$\frac{dH}{dt} = r'(t) = \frac{H'}{t'} \in M(\Gamma) \quad \text{owing to} \quad H'(\tau) = r'(t(\tau))t'(\tau). \tag{112}$$

To state our first lemma, we need define a number of "different" poles of a modular form F with even weight. Via the relation (69), this number is related to the number of pairwise different poles of F contained in a fundamental domain.

Definition 9.1. *Let $F \in M_k(\Gamma)$ with k even. Then*

$$\mathrm{NofDiffPoles}(F) := \sum_{\substack{P \in X(\Gamma) \\ \mathrm{Ord}_P F < 0}} 1 = \sum_{\substack{[p] \in X(\Gamma) \\ \mathrm{ord}\, \tilde{F}_p(z) < 0}} 1, \tag{113}$$

where $\tilde{F}_p(z)$ is the Laurent series defined in Lemma 6.6.

Lemma 9.2.

$$\begin{aligned}\mathrm{NofPoles}\left(\frac{dH}{dt}\right) \leq 2g_\Gamma - 2 &+ \mathrm{NofPoles}(H) + \mathrm{NofPoles}(t) \\ &+ \mathrm{NofDiffPoles}(H) + \mathrm{NofDiffPoles}(t).\end{aligned}$$

Proof.

$$\begin{aligned}
-\sum_{P \in \mathrm{Poles}(\frac{dH}{dt})} \mathrm{Ord}_P\left(\frac{dH}{dt}\right) &\leq -\sum_{P \in X(\Gamma)} \pi(\mathrm{Ord}_P\left(\frac{dH}{dt}\right)) \\
&\overset{(112)}{=} -\sum_{P \in X(\Gamma)} \pi(\mathrm{Ord}_P\, H') + \sum_{P \in X(\Gamma)} \zeta(\mathrm{Ord}_P\, t') \\
&\overset{(80)}{=} -\sum_{P \in X(\Gamma)} \pi(\mathrm{Ord}_P\, H') + 2(g_\Gamma - 1) - \sum_{P \in X(\Gamma)} \pi(\mathrm{Ord}_P\, t').
\end{aligned}$$

Observing that, by Lemma 7.3,

$$-\sum_{P \in X(\Gamma)} \pi(\mathrm{Ord}_P\, H') = -\sum_{P \in X(\Gamma)} \pi(\mathrm{Ord}_P\, H) + \sum_{\substack{P \in X(\Gamma) \\ \mathrm{Ord}_P H < 0}} 1,$$

together with the analogous relation for t instead of H, completes the proof of the lemma. $\square$

Lemma 9.3. *For $j \in \mathbb{Z}_{\geq 1}$,*

$$\begin{aligned}\mathrm{NofPoles}\left(\frac{d^j H}{dt^j}\right) \leq 2j(g_\Gamma - 1) &+ \mathrm{NofPoles}(H) + j\ \mathrm{NofPoles}(t)\\ &+ \sum_{m=0}^{j-1} \mathrm{NofDiffPoles}\left(\frac{d^m H}{dt^m}\right) + j\ \mathrm{NofDiffPoles}(t).\end{aligned}$$

Proof. The case $j=1$ is Lemma 9.2. Assuming the statement holds for $j-1$, we show it holds for j. By Lemma 9.2,

$$\begin{aligned}\mathrm{NofPoles}\left(\frac{d^j H}{dt^j}\right) &\leq 2g_\Gamma - 2 + \mathrm{NofPoles}\left(\frac{d^{j-1} H}{dt^{j-1}}\right) + \mathrm{NofPoles}(t)\\ &+ \mathrm{NofDiffPoles}\left(\frac{d^{j-1} H}{dt^{j-1}}\right) + \mathrm{NofDiffPoles}(t)\\ &\leq 2g_\Gamma - 2 + \Big(2(j-1)(g_\Gamma - 1) + \mathrm{NofPoles}(H)\\ &+ (j-1)\,\mathrm{NofPoles}(t) + \sum_{m=0}^{j-2} \mathrm{NofDiffPoles}\left(\frac{d^m H}{dt^m}\right)\\ &+ (j-1)\,\mathrm{NofDiffPoles}(t)\Big) + \mathrm{NofPoles}(t)\\ &+ \mathrm{NofDiffPoles}\left(\frac{d^{j-1} H}{dt^{j-1}}\right) + \mathrm{NofDiffPoles}(t);\end{aligned}$$

for the equality we applied the induction hypothesis. □

For the next lemma we need the counterpart to Def. (9.1).

Definition 9.4. *Let $F \in M_k(\Gamma)$ with k even. Then*

$$\mathrm{NofDiffZeros}(F) := \sum_{\substack{P \in X(\Gamma)\\ \mathrm{Ord}_P F > 0}} 1 = \sum_{\substack{[p] \in X(\Gamma)\\ \mathrm{ord}_p \tilde{F}_p(z) > 0}} 1, \tag{114}$$

where $\tilde{F}_p(z)$ is the Laurent series defined in Lemma 6.6.

Lemma 9.5. *For $j \in \mathbb{Z}_{\geq 1}$,*

$$\mathrm{NofDiffPoles}\left(\frac{d^j H}{dt^j}\right) \leq \mathrm{NofDiffPoles}(H) + \mathrm{NofDiffZeros}(t'). \tag{115}$$

Proof. Given $P = [p] \in X(\Gamma)$, let $H(\tau) = \tilde{H}_p(z_p(\tau))$ and $t(\tau) = \tilde{t}_p(z_p(\tau))$ with Laurent series on the right hand sides as defined in Lemma 6.4. Then $H'(\tau) =$

$z_p'(\tau)(\tilde{H}_p)'(z_p(\tau))$, and owing to $\frac{dH}{dt} = H'/t'$,

$$\mathrm{NofDiffPoles}\Big(\frac{dH}{dt}\Big) = \sum_{\substack{P\in X(\Gamma)\\ \mathrm{Ord}_P\, H'/t'<0}} 1 \le \sum_{\substack{[p]\in X(\Gamma)\\ \mathrm{ord}(\tilde{H}_p)'(z)<0}} 1 + \sum_{\substack{[p]\in X(\Gamma)\\ \mathrm{ord}(\tilde{t}_p)'(z)>0}} 1$$

$$= \sum_{\substack{[p]\in X(\Gamma)\\ \mathrm{ord}(\tilde{H}_p)(z)<0}} 1 + \mathrm{NofDiffZeros}(t') = \mathrm{NofDiffPoles}(H) + \mathrm{NofDiffZeros}(t').$$

The general case $j \ge 1$ follows by mathematical induction. We omit its (technical) details; instead we sketch the essential structure underlying the induction step. Because of $\frac{d^jH}{dt^j} \in M(\Gamma)$, for each fixed $j \in \mathbb{Z}_{\ge 1}$ and $[p] \in X(\Gamma)$ there is a representation $\frac{d^jH}{dt^j}(t(\tau)) = L_p(z_p(\tau))$, where $L_p(z)$ is a Laurent series. To obtain further insight into this representation, we use Faá Di Bruno's Formula,

$$H^{(j)}(\tau) = \frac{d^j}{d\tau^j}H(t(\tau)) = \sum_{i=1}^{j} \frac{d^iH(t(\tau))}{dt^i} \cdot B_{i,k}(t'(\tau), \dots, t^{(i-k+1)}(\tau)),$$

with $B_{i,k}(x_1, \dots, x_{i-k+1})$ being the (partial) Bell polynomials. With this formula one finds that

$$L_p(z) = \sum_{i=1}^{j} c_{i,p}(z) \cdot (\tilde{H}_p)^{(i)}(z) \ \text{ with } \ c_{i,p}(z) = \frac{C_{i,p}}{(\tilde{t})_p'(z)^{2j-1}},$$

where the $C_{i,p}$ are polynomials in $(\tilde{t}_p)'(z), (\tilde{t}_p)''(z), \dots$, such that for each monomial, *constant* $\cdot (\tilde{t}_p)'(z)^{\alpha_1}(\tilde{t}_p)''(z)^{\alpha_2} \dots$, occurring as a summand in $C_{i,p}$, one has $1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \cdots \le 2j - 2$. This property guarantees that no further poles are introduced. We give the L_p for $j = 1, 2, 3$ explicitly:

$$L_p(z) = \frac{1}{(\tilde{t}_p)'(z)} \cdot (\tilde{H}_p)'(z), \quad \text{if } j = 1;$$

$$L_p(z) = -\frac{(\tilde{t}_p)''(z)}{(\tilde{t}_p)'(z)^3} \cdot (\tilde{H}_p)'(z) + \frac{(\tilde{t}_p)'(z)}{(\tilde{t}_p)'(z)^3} \cdot (\tilde{H}_p)''(z), \quad \text{if } j = 2;$$

$$\begin{aligned} L_p(z) = {} & \frac{3(\tilde{t}_p)''(z)^2 + (\tilde{t}_p)'(z)(\tilde{t}_p)^{(3)}(z)}{(\tilde{t}_p)'(z)^5} \cdot (\tilde{H}_p)'(z) \\ & - \frac{3(\tilde{t}_p)'(z)(\tilde{t}_p)''(z)}{(\tilde{t}_p)'(z)^5} \cdot (\tilde{H}_p)''(z) + \frac{(\tilde{t}_p)'(z)^2}{(\tilde{t}_p)'(z)^5} \cdot (\tilde{H}_p)^{(3)}(z), \quad \text{if } j = 3. \end{aligned}$$

Further details of the induction proof are left to the reader. □

Lemma 9.5 implies for $j \in \mathbb{Z}_{\ge 1}$,

$$\sum_{m=1}^{j-1} \mathrm{NofDiffPoles}\Big(\frac{d^mH}{dt^m}\Big) \overset{(115)}{\le} (j-1)(\mathrm{NofDiffPoles}(H) + \mathrm{NofDiffZeros}(t')).$$

We use this to simplify the right side of Lemma 9.3,

$$\begin{aligned}\operatorname{NofPoles}\left(\frac{d^jH}{dt^j}\right) &\le 2j(g_\Gamma-1)+\operatorname{NofPoles}(H)+j\ \operatorname{NofPoles}(t)\\ &\quad + j\ \operatorname{NofDiffPoles}(H)+(j-1)\operatorname{NofDiffZeros}(t')\\ &\quad + j\operatorname{NofDiffPoles}(t).\end{aligned}\tag{116}$$

For further simplification, observe that

$$\begin{aligned}\operatorname{NofDiffZeros}(t') &\le \sum_{P\in X(\Gamma)}\zeta(\operatorname{Ord}_P t') \overset{(80)}{=} 2(g_\Gamma-1)-\sum_{P\in X(\Gamma)}\pi(\operatorname{Ord}_P t')\\ &= 2(g_\Gamma-1)+\operatorname{NofPoles}(t') = 2(g_\Gamma-1)+\operatorname{NofDiffPoles}(t)+\operatorname{NofPoles}(t)\end{aligned}$$

where the last equality is by Lemma 7.3. Using this on (116) one obtains,

$$\begin{aligned}\operatorname{NofPoles}\left(\frac{d^jH}{dt^j}\right) &\le 2(2j-1)(g_\Gamma-1)+\operatorname{NofPoles}(H)+(2j-1)\operatorname{NofPoles}(t)\\ &\quad + j\ \operatorname{NofDiffPoles}(H)+(2j-1)\operatorname{NofDiffPoles}(t).\end{aligned}\tag{117}$$

Finally, as another step of simplification, we apply

$$\operatorname{NofDiffPoles}(H)\le\operatorname{NofPoles}(H)\ \text{ and }\ \operatorname{NofDiffPoles}(t)\le\operatorname{NofPoles}(t),$$

which reduces (117) to

Lemma 9.6. *For $H, t \in M(\Gamma)$ and $j \in \mathbb{Z}_{\ge 1}$,*

$$\begin{aligned}\operatorname{NofPoles}\left(\frac{d^jH}{dt^j}\right) &\le (2j-1)(2g_\Gamma-2)\\ &\quad + (j+1)\operatorname{NofPoles}(H)+(4j-2)\operatorname{NofPoles}(t).\end{aligned}\tag{118}$$

10. The proof of Theorem 5.2 and Theorem 5.3 summarized

First we collect all the ingredients to prove Theorem 5.2 whose statements are for the case of even weight k.

The derivations resulting in (98), resp. (107), prove the bounds (51) for p_1, resp. (52) for p_2, of Theorem 5.2. Applying Lemma 9.6 for even k to $H := p_1$ gives,

$$\begin{aligned}\operatorname{NofPoles}\left(\frac{d^jp_1}{dt^j}\right) &\le (2j-1)(2g_\Gamma-2)\\ &\quad + (j+1)\operatorname{NofPoles}(p_1)+(4j-2)\operatorname{NofPoles}(t)\\ &\overset{(98)}{\le} (2j-1)(2g_\Gamma-2)+(4j-2)\operatorname{NofPoles}(t)\\ &\quad + (j+1)\Big((k+4)(g_\Gamma-1)+8\operatorname{NofPoles}(t)+3\operatorname{NofPoles}(F)\Big)\\ &= (jk+8j+k+2)(g_\Gamma-1)\\ &\quad + 6(2j+1)\operatorname{NofPoles}(t)+3(j+1)\operatorname{NofPoles}(F);\end{aligned}$$

which is (53) of Theorem 5.2 For even k, applying Lemma 9.6 to $H := p_2$, using (107), gives (54) of Theorem 5.2.

Finally, we collect all the ingredients to prove Theorem 5.3 whose statements are for the case of odd weight k.

The derivations resulting in (100), resp. (111), prove the bounds (55) for p_1, resp. (56) for p_2, of Theorem 5.3. Applying Lemma 9.6 for odd k to $H := p_1$ gives,

$$
\begin{aligned}
\operatorname{NofPoles}\left(\frac{d^j p_1}{dt^j}\right) &\leq (2j-1)(2g_\Gamma - 2) \\
&\quad + (j+1)\operatorname{NofPoles}(p_1) + (4j-2)\operatorname{NofPoles}(t) \\
&\overset{(100)}{\leq} (2j-1)(2g_\Gamma - 2) + (4j-2)\operatorname{NofPoles}(t) \\
&\quad + (j+1)\Big(8\operatorname{NofPoles}(t) + 3\operatorname{NofPoles}(F^2) + (2k+4)(g_\Gamma - 1)\Big) \\
&= 2(1+4j+k+jk)(g_\Gamma - 1) \\
&\quad + 6(2j+1)\operatorname{NofPoles}(t) + 3(j+1)\operatorname{NofPoles}(F^2);
\end{aligned}
$$

which is (57) of Theorem 5.2 For odd k, applying Lemma 9.6 to $H := p_2$, using (111), gives (58) of Theorem 5.3.

This completes the proofs of Theorem 5.2 and Theorem 5.3.

11. Appendix

11.1. **Linear independence of Yang functions.**

Proposition 11.1. *Let Γ be a congruence subgroup. Let $g \in M_k(\Gamma)$ with $k \geq 1$, and $h \in M(\Gamma)$. Then the Yang functions,*

$$\Big\{1, \frac{G_2}{G_1}, \frac{G_2^2}{G_1^2}, \dots, \frac{G_2^m}{G_1^m}\Big\},\ m \geq 0,$$

as functions on $\mathbb{H}$ are linearly independent over the field $M(\Gamma)$ of modular functions for Γ.

Before proving Prop. 11.1 we prove a lemma.

Lemma 11.2. *Let $a_0(\tau), \dots, a_m(\tau)$ be functions on $\mathbb{H}$ with period r; i.e.,*

$$a_j(\tau + r) = a(\tau),\ \tau \in \mathbb{H},\ j = 0, \dots, m.$$

If for fixed integers $c, d \in \mathbb{Z}$, $c \neq 0$,

$$a_0(\tau) + \frac{a_1(\tau)}{c\tau + d} + \frac{a_2(\tau)}{(c\tau+d)^2} + \cdots + \frac{a_1(\tau)}{(c\tau+d)^m} = 0,\ \tau \in \mathbb{H}, \tag{119}$$

then

$$a_0(\tau) = a_1(\tau) = a_2(\tau) = \cdots = a_m(\tau),\ \tau \in \mathbb{H}.$$

Proof. Consider the Vandermonde matrix

$$A = \begin{pmatrix} 1 & \frac{1}{c\tau+d} & \frac{1}{(c\tau+d)^2} & \cdots & \frac{1}{(c\tau+d)^m} \\ 1 & \frac{1}{c(\tau+r)+d} & \frac{1}{(c(\tau+r)+d)^2} & \cdots & \frac{1}{(c(\tau+r)+d)^m} \\ 1 & \frac{1}{c(\tau+2r)+d} & \frac{1}{(c(\tau+2r)+d)^2} & \cdots & \frac{1}{(c(\tau+2r)+d)^m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{c(\tau+mr)+d} & \frac{1}{(c(\tau+mr)+d)^2} & \cdots & \frac{1}{(c(\tau+mr)+d)^m} \end{pmatrix},$$

which is invertible since it has a non-zero determinant:

$$\det(A) = \prod_{1\leq i<j\leq m+1} \Big(\frac{1}{c(\tau+jr)+d} - \frac{1}{c(\tau+ir)+d}\Big) \neq 0.$$

Hence a relation like (119) with $a_j(\tau)$ not all zero cannot exist. □

Proof of Prop. 11.1. Let $a_0(\tau), \ldots, a_m(\tau)$ be modular functions in $M(\Gamma)$. By induction on $m \geq 0$, we will prove the statement in the following form: Suppose that

$$a_0(\tau) + a_1(\tau)\frac{G_2(\tau)}{G_1(\tau)} + \cdots + a_m(\tau)\frac{G_2(\tau)^m}{G_1(\tau)^m} = 0, \;\tau \in \mathbb{H}, \tag{120}$$

then $a_j = 0$ for $0 \leq j \leq m$.

The statement is true for $m = 0$. Assuming its truth up to $m = N - 1$, we prove it is true also for $m = N$.

In our argument we use as a crucial fact that there exists an common period $r \in \mathbb{Z}_{\geq 1}$ such that $a_j(\tau + r) = a_j(\tau)$ and $G_j(\tau + r) = G_j(\tau)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$\frac{G_2(\gamma\tau)}{G_1(\gamma\tau)} = \frac{G_2(\tau)}{G_1(\tau)} + \frac{ck(c\tau+d)^{-1}}{G_1(\tau)}$$

with $c' := c/(2\pi i)$. Now suppose a relation of type (120) holds for $m = N$; i.e.,

$$a_0(\tau) + a_1(\tau)\frac{G_2(\tau)}{G_1(\tau)} + \cdots + a_N(\tau)\frac{G_2(\tau)^N}{G_1(\tau)^N} = 0, \;\tau \in \mathbb{H}. \tag{121}$$

Applying $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ to this equation yields,

$$a_0(\tau)+a_1(\tau)\Big(\frac{G_2(\tau)}{G_1(\tau)}+\frac{c'k(c\tau+d)^{-1}}{G_1(\tau)}\Big)+\cdots+a_N(\tau)\Big(\frac{G_2(\tau)}{G_1(\tau)}+\frac{c'k(c\tau+d)^{-1}}{G_1(\tau)}\Big)^N = 0.$$

This can be rewritten into the form,

$$b_0(\tau) + \frac{b_1(\tau)}{c\tau+d} + \cdots + \frac{b_{N-1}(\tau)}{(c\tau+d)^{N-1}} + \frac{(c'k)^N a_N(\tau)}{G_1(\tau)(c\tau+d)^N} = 0, \;\tau \in \mathbb{H}.$$

Here we use the common periodicity r, $a_j(\tau + r) = a_j(\tau)$ and $G_j(\tau + r) = G_j(\tau)$, which produces periodic coefficients; i.e., $b_j(\tau + r) = b_j(\tau)$. Hence, by

Lemma 11.2, the $b_0, \ldots, b_{N-1}$ are all 0, which implies that also $a_N = 0$. This reduces (121) to

$$a_0(\tau) + a_1(\tau)\frac{G_2(\tau)}{G_1(\tau)} + \cdots + a_{N-1}(\tau)\frac{G_2(\tau)^{N-1}}{G_1(\tau)^{N-1}} = 0, \ \tau \in \mathbb{H},$$

which by the induction hypothesis implies $a_0 = \cdots = a_{N-1} = 0$. This completes the proof of Prop. 11.1. $\square$

11.2. **Computational details for the ModFormDE example in Sect. 4.4.** This section presents computational parts of our exemplification of algorithm ModFormDE in Section 4.4.

11.2.1. *Cusps of the congruence group* $\Gamma(2,4,2)$*.* With the Magma system, the cusps $[0]$, $[1]$, and $[\infty]$ of $\Gamma = \Gamma(2,4,2)$, together with each width, can be computed as follows:

```
> G:=CongruenceSubgroup([2,4,2]);
> Cusps(G);
[
    oo,
    0,
    1
]
> Widths(G);
[ 2, 2, 2 ]
```

11.2.2. *Expansion of* g^2 *at the cusp* $[1]$ *of* $X(\Gamma(2,4,2)$*.* By Lemma 1.13 in [11] and because of $(c\tau+d)^{-1/2}\eta\Big(\frac{a\tau+b}{c\tau+d}\Big) = \epsilon(a,b,c,d)\eta(\tau)$, where $\epsilon(a,b,c,d)$ is a 24-th root of unity, we have for all $A, B, C, D \in \mathbb{Z}$ such that $AD - BC \neq 0$:

$$\begin{aligned}&\Big(\frac{\gcd(A,C)}{AD-BC}(C\tau+D)\Big)^{-1/2}\\ =&\epsilon(A/\gcd(A,C), -y, C/\gcd(A,C), x)\eta\Big(\frac{\gcd(A,C)\tau + Bx + Dy}{(AD-BC)\gcd(A,C)^{-1}}\Big).\end{aligned}$$

Here x, y are any integers such that $Ax + Cy = \gcd(A,C)$. This formula together with $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ implies that

$$(C\tau+D)^{-1/2}\eta\Big(\frac{A\tau+B}{C\tau+D}\Big) = q^{\frac{\gcd(A,C)^2}{24(AD-BC)}}(u + O(q^{\frac{\gcd(A,C)^2}{AD-BC}})). \tag{122}$$

Now since

$$\tau^{-2}g\Big(\frac{\tau-1}{\tau}\Big)^2 = 2^4\frac{\Big(\tau^{-1/2}\eta\Big(\frac{\tau-1}{\tau}\Big)\Big)^{20}}{\Big((2\tau)^{-1/2}\eta\Big(\frac{\tau-1}{2\tau}\Big)\Big)^8\Big(\tau^{-1/2}\eta\Big(\frac{2\tau-2}{\tau}\Big)\Big)^8}$$

we have by (122):

$$\begin{aligned}\tau^{-2}g\Big(\frac{\tau-1}{\tau}\Big)^2 &= \frac{(q^{1/24}(u_1+O(q)))^{20}}{(q^{1/(24\cdot 2)}(u_2+O(q^{1/2})))^8(q^{1/(24\cdot 2)}(u_3+O(q^{1/2})))^8}\\ &= q^{20/24-8/48-8/48}(u+O(q^{1/2})) = q^{1/2}(u+O(q^{1/2})),\end{aligned}$$

where u and the u_i are non-zero complex numbers.

11.3. **Meromorphic Functions on Riemann Surfaces - Basic Notions.**

To make this article as much self-contained as possible, in this second appendix section we recall most of the facts we need about meromorphic functions on Riemann surfaces. For the terminology we basically follow [6]; other classic texts are [5] and [8].

Lemma 11.3 states a fundamental fact why implies as an immediate but important corollary that any analytic function on a compact Riemann surface is constant. For its proof see, for instance, [8, Prop. 4.12]:

Lemma 11.3. *Let f be a non-constant meromorphic function on a compact Riemann surface X. Then*

$$\sum_{x\in X}\operatorname{Ord}_x f = 0. \tag{123}$$

.

Here the order of f at $x_0 \in X$, $\operatorname{Ord}_{x_0} f$, is defined as follows.

Definition 11.4. *Suppose*

$$f(x) = \sum_{n=m} c_n(\phi(x)-\phi(x_0))^n,\ c_m \neq 0,$$

is the local Laurent expansion of f at $x_0 \in X$ using the local coordinate chart $\phi : U_0 \to \mathbb{C}$ which homeomorphically maps a neighborhood U_0 of x_0 to an open set $V_0 \subseteq \mathbb{C}$. Then,

$$\operatorname{Ord}_{x_0} f := m.$$

In our context, $X = X(\Gamma)$ and $f = \hat{t} : X(\Gamma) \to \hat{\mathbb{C}}$ where $\hat{t}$ is induced by the modular function $t \in M(\Gamma)$; moreover, $\phi = z_p$ as described in Section 6.1 serve as the local charts at $x_0 = P = [p] \in X(\Gamma)$.

Let $\mathcal{M}(X)$ denote the field of meromorphic functions $f : X \to \hat{\mathbb{C}}$ on a Riemann surface X.[26] Let $f \in \mathcal{M}(X)$ be non-constant and $x \in X$: then for every neighborhood U of x there exist neighborhoods $U_x \subseteq U$ of x and V of $f(x)$ such that the set $f^{-1}(v) \cap U_x$ contains exactly k elements for every $v \in V \setminus \{f(x)\}$. This number k is called the multiplicity of f at x; notation: $k = \operatorname{mult}_x f$.[27] If X is compact, $f \in \mathcal{M}(X)$ is surjective and each $v \in \hat{\mathbb{C}}$ has the same number of preimages, say n, counting multiplicities; i.e., $n = \sum_{x \in f^{-1}(v)} \operatorname{mult}_x f$; see, e.g., [6, Thm. 4.24]. This number n is called the degree of f; notation: $n = \operatorname{Deg} f$. One of the consequences is that non-constant functions on compact Riemann surfaces have as many (finitely many) zeros as poles counting multiplicities; this is Lemma 11.3.

12. Conclusion

In this paper we focused on the mathematics underlying our algorithm ModFormDE. With regard to possible applications, we feel there is quite some potential waiting for further exploration. There will be also the need to supplement such investigations by algorithmic developments, in particular, by supporting software. For instance, in our illustrating example in Section 4.4 we have seen that we still need software to determine $\operatorname{NofPoles}(g^2)$ automatically.

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References

[1] George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*. Cambridge University Press, 1999.
[2] Jonathan M. Borwein and Peter B. Borwein. *Pi and the AGM*. John Wiley & Sons, 1987.
[3] Richard Dedekind and Heinrich Weber. *Theory of Algebraic Functions in One Variable. Translated and introduced by John Stillwell*. History of Mathematics, Vol. 39. American Mathematical Society, 2012.
[4] Fred Diamond and Jerry Shurman. *A First Course in Modular Forms*. Springer, 2005.
[5] Hershel M. Farkas and Irwin Kra. *Riemann Surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer, 1980.
[6] Otto Forster. *Lectures on Riemann Surfaces*. Springer, 1981.
[7] Manuel Kauers and Peter Paule. *The Concrete Tetrahedron*. Springer, 2011.
[8] Rick Miranda. *Algebraic Curves and Riemann Surfaces*, volume 5 of *Grad. Stud. Math.* AMS, 1995.
[9] Ken Ono. *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*. CBMS Regional Conference Series in Mathematics, Number 102, 2004.

[26] In this context $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is understood to be a compact Riemann surface isomorphic to the Riemann sphere.

[27] If x is a pole of f: $\operatorname{mult}_x f = -\operatorname{ord}_x f$; otherwise, $\operatorname{mult}_x f = \operatorname{ord}_x(f - f(x))$.

[10] Peter Paule and Cristian-Silviu Radu. Holonomic Relations for Modular Functions and Forms: First Guess, then Prove. 2020. In preparation.
[11] Cristian-Silviu Radu. An algorithmic approach to Ramanujan-Kolberg identities. *Journal of Symbolic Computation*, 68(1):225–253, 2015.
[12] Srinivasa Ramanujan. Modular equations and approximations to π. *Quart. J. Math.*, 45:350–372, 1914.
[13] Yifan Yang. On differential equations satisfied by modular forms. *Math. Z*, 246:1–19, 2004.
[14] Don Zagier. Elliptic Modular Forms and Their Applications. In Ranestad K. (eds), editor, *The 1-2-3 of Modular Forms*, pages 1–103. Universitext Springer, 2008.
[15] Doron Zeilberger. A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.*, 32:321–368, 1990.

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