ÖAW
Austrian Academy of Sciences

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

JOHANNES KEPLER UNIVERSITY LINZ

A sequence of polynomials generated by a Kapteyn series of the second kind

Diego Dominici   Veronika Pillwein

DK-Report No. 2019-05   05 2019

A–4040 LINZ, ALTENBERGERSTRASSE 69, AUSTRIA

Supported by

Austrian Science Fund (FWF)   Upper Austria

FWF Der Wissenschaftsfonds.   LAND OBERÖSTERREICH

DK sponsors:

- **Johannes Kepler University Linz** (JKU)
- **Austrian Science Fund** (FWF)
- **Upper Austria**

A sequence of polynomials generated by a Kapteyn series of the second kind

Diego Dominici* Veronika Pillwein†

Abstract

In this paper, we find an explicit representation for a Kapteyn series of the second kind in terms of a family of polynomials $P_n(x)$. We also use symbolic computation methods to find a recurrence relation that allows fast calculation of the coefficients of $P_n(x)$.

1 Introduction

Series of the form

$$\sum_{n=0}^{\infty} \alpha_n^{\nu} \mathrm{J}_{\nu+n} \left[(\nu+n) z \right], \tag{1}$$

and

$$\sum_{n=0}^{\infty} \alpha_n^{\mu,\nu} \mathrm{J}_{\mu+n} \left[(\mu+\nu+2n) z \right] \mathrm{J}_{\nu+n} \left[(\mu+\nu+2n) z \right], \tag{2}$$

where $\mu, \nu \in \mathbb{C}$ and $J_n(z)$ is the Bessel function of the first kind [18, 10.2.2]

$$\mathrm{J}_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu+n+1)\, n!} \left(\frac{z}{2} \right)^{\nu+2n},$$

are called *Kapteyn series of the first kind* and *Kapteyn series of the second kind* respectively.

The first researcher to investigate such series in a systematic way was Willem Kapteyn (not to be confused with his brother Jacobus Cornelius Kapteyn) in the articles [8] and [9]. Most of the early work on Kapteyn series can be found in the books by Niels Nielsen [16, Chapter XXII] and George Neville Watson [21, Chapter 17]. For additional properties, see [3], [4], [19], [20], and especially the very recent book [1].

Applications of Kapteyn series to problems in physics can be found in [6], [13], [14], and [15].

In [5], we considered the functions $g_n(z)$ defined by

$$\sum_{k=0}^{\infty} k^{2n}\, \mathrm{J}_k^2(2kz) = g_n(z) = \sum_{k=0}^{\infty} b_{n,k}\, z^{2k}, \quad n \in \mathbb{N}_0, \tag{3}$$

*e-mail: diego.dominici@dk-compmath.jku.at, Johannes Kepler University Linz, Doktoratskolleg "Computational Mathematics", Altenberger Straße 69, 4040 Linz, Austria; Permanent address: Department of Mathematics, State University of New York at New Paltz, 1 Hawk Dr., New Paltz, NY 12561-2443, USA

†e-mail: veronika.pillwein@risc.jku.at, RISC, JKU Linz, Altenberger Straße 69, 4040 Linz, Austria

where $\mathbb{N}$ denotes the set of natural numbers and

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

We computed the first few $g_n(z)$ and obtained

$$\begin{aligned}
g_0(z) &= \frac{1}{2} + \frac{1}{2\sqrt{1-4z^2}}, \quad g_1(z) = \frac{z^2\left(1+z^2\right)}{\left(1-4z^2\right)^{\frac{7}{2}}}, \\
g_2(z) &= \frac{z^2\left(1+37z^2+118z^4+27z^6\right)}{\left(1-4z^2\right)^{\frac{13}{2}}}, \\
g_3(z) &= \frac{z^2\left(1+217z^2+5036z^4+23630z^6+22910z^8+2250z^{10}\right)}{\left(1-4z^2\right)^{\frac{19}{2}}},
\end{aligned}$$

which seemed to suggest that $g_n(z)$ should be of the form

$$g_n(z) = \frac{P_n\left(z^2\right)}{\left(1-4z^2\right)^{3n+\frac{1}{2}}} + \frac{1}{2}\delta_{n,0}, \quad n \in \mathbb{N}_0, \tag{4}$$

where $P_n(x) \in \mathbb{R}[x]$, $\deg(P_n) = 2n$, and $\delta_{n,k}$ is Kronecker's delta, defined by

$$\delta_{n,k} = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}.$$

The purpose of this paper is to show that this conjecture is true.

2 The coefficients $b_{n,k}$

The Bessel functions of the first kind have the hypergeometric representation [18, 10.16.9]

$$\mathrm{J}_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(\begin{array}{c} - \\ \nu+1 \end{array}; -\frac{z^2}{4}\right),$$

where ${}_pF_q$ denotes the generalized hypergeometric function defined by [18, 16.2.1]

$${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!},$$

and $(x)_n$ is the Pochhammer symbol (also called shifted or rising factorial) defined by [18, 5.2(iii)] $(x)_0 = 1$ and

$$(x)_n = x(x+1)\cdots(x+n-1), \quad n \in \mathbb{N},$$

or as the ratio of two Gamma functions

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad -(x+n) \notin \mathbb{N}_0.$$

To begin, we find some representations of the coefficients $b_{n,k}$ appearing in the Taylor series (3).

Proposition 1 *Let $b_{n,k}$ be defined by*

$$\sum_{k=0}^{\infty} k^{2n} \mathrm{J}_k^2\left(2kz\right) = \sum_{k=0}^{\infty} b_{n,k}\ z^{2k}, \quad |z| < \frac{1}{2}.$$

Then,

$$b_{n,k} = \binom{2k}{k} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k+j)!\,(k-j)!} j^{2k+2n}, \quad n,k \in \mathbb{N}_0. \tag{5}$$

Proof. If we use the identity [18, 16.12.1]

$${}_0F_1\left(\begin{matrix} - \\ a \end{matrix};z\right)\ {}_0F_1\left(\begin{matrix} - \\ b \end{matrix};z\right) = {}_2F_3\left(\begin{matrix} \frac{a+b}{2}, \frac{a+b-1}{2} \\ a, b, a+b-1 \end{matrix};4z\right),$$

we see that

$$\begin{aligned}
\mathrm{J}_\nu^2\left(z\right) &= \frac{1}{\Gamma^2\left(\nu+1\right)} \left(\frac{z}{2}\right)^{2\nu}\ {}_2F_3\left(\begin{matrix} \nu+1, \nu+\frac{1}{2} \\ \nu+1, \nu+1, 2\nu+1 \end{matrix};-z^2\right) \\
&= \frac{1}{\Gamma^2\left(\nu+1\right)} \left(\frac{z}{2}\right)^{2\nu}\ {}_1F_2\left(\begin{matrix} \nu+\frac{1}{2} \\ \nu+1, 2\nu+1 \end{matrix};-z^2\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
g_n\left(z\right) &= \sum_{k=0}^{\infty} k^{2k+2n} \frac{z^{2k}}{(k!)^2}\ {}_1F_2\left(\begin{matrix} k+\frac{1}{2} \\ k+1, 2k+1 \end{matrix};-4k^2z^2\right) \\
&= \sum_{k=0}^{\infty} k^{2k+2n} \frac{z^{2k}}{(k!)^2} \sum_{l=0}^{\infty} \frac{\left(k+\frac{1}{2}\right)_l}{(k+1)_l\,(2k+1)_l} \frac{\left(-4k^2z^2\right)^l}{l!} \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} z^{2(k+l)}\ \frac{\left(k+\frac{1}{2}\right)_l}{(k!)^2\,(k+1)_l\,(2k+1)_l} \frac{(-4)^l}{l!} k^{2(k+l+n)}.
\end{aligned}$$

Setting $k+l=j$, we get

$$g_n\left(z\right) = \sum_{k=0}^{\infty} z^{2k} \sum_{j=0}^{k} \frac{1}{(j!)^2} \frac{\left(j+\frac{1}{2}\right)_{k-j}}{(j+1)_{k-j}\,(2j+1)_{k-j}} \frac{(-4)^{k-j}}{(k-j)!} j^{2k+2n}$$

Thus,

$$b_{n,k} = \sum_{j=0}^{k} \frac{1}{(j!)^2} \frac{\left(j+\frac{1}{2}\right)_{k-j}}{(j+1)_{k-j}\,(2j+1)_{k-j}} \frac{(-4)^{k-j}}{(k-j)!} j^{2k+2n}.$$

To simplify the expression, let's write

$$w\left(j\right) = \frac{1}{(j!)^2} \frac{\left(j+\frac{1}{2}\right)_{k-j}}{(j+1)_{k-j}\,(2j+1)_{k-j}} \frac{(-4)^{k-j}}{(k-j)!}.$$

Then,

$$w(0) = \frac{\left(\frac{1}{2}\right)_k (4)^k}{(k!)^2},$$

and

$$\frac{w(j+1)}{w(j)} = \frac{(j+1)^2}{k+j+1}.$$

Using the identity [18, 5.2.8]

$$(x)_{2n} = 4^n \left(\frac{x}{2}\right)_n \left(\frac{x+1}{2}\right)_n,$$

with $x=1$, we obtain

$$\frac{\left(\frac{1}{2}\right)_k (4)^k}{(k!)^2} = \frac{1}{(k!)^2}\frac{(1)_{2k}}{(1)_k} = \frac{(2k)!}{(k!)^3} = \binom{2k}{k}\frac{1}{k!}.$$

We conclude that

$$w(j) = \binom{2k}{k}\frac{1}{k!}\prod_{i=0}^{j-1}\frac{(i+1)^2}{k+i+1} = \binom{2k}{k}\frac{(j!)^2}{(k+j)!},$$

and the result follows. ■

Remark 2 *Note that*

$$b_{n,0} = 0^{2n} = \delta_{n,0}.$$

Proposition 3 *Let $b_{n,k}$ be defined by (5). Then,*

$$b_{n,k} = \frac{1}{2}\binom{2k}{k}\sum_{j=0}^{2k}\frac{(-1)^{2k-j}}{j!\,(2k-j)!}(k-j)^{2k+2n} + \frac{1}{2}\delta_{n+k,0}. \tag{6}$$

Proof. We have

$$\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k+j)!\,(k-j)!}j^{2k+2n} = \sum_{j=0}^{k}\frac{(-1)^{j}}{(2k-j)!j!}(k-j)^{2k+2n}, \quad j\to k-j.$$

Also,

$$\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k+j)!\,(k-j)!}j^{2k+2n} = \sum_{j=k}^{2k}\frac{(-1)^{2k-j}}{j!\,(2k-j)!}(j-k)^{2k+2n}.$$

Thus,

$$2\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k+j)!\,(k-j)!}j^{2k+2n} = \sum_{j=0}^{2k}\frac{(-1)^{j}}{(2k-j)!j!}(k-j)^{2k+2n} + \frac{(-1)^k}{(k!)^2}0^{2(k+n)},$$

and we obtain

$$b_{n,k} = \frac{1}{2}\binom{2k}{k}\sum_{j=0}^{2k}\frac{(-1)^{j}}{(2k-j)!j!}(k-j)^{2k+2n} + \frac{1}{2}\delta_{n+k,0}.$$

■

Next, we analyze the summand in the representation (6).

Lemma 4 *Let the functions* $q_n(k)$ *be defined by*

$$q_n(k) = \sum_{j=0}^{2k} \frac{(-1)^{2k-j}}{j!\,(2k-j)!} (k-j)^{2k+2n}, \quad n,k \in \mathbb{N}_0. \tag{7}$$

Then, we can write $q_n(k)$ *as the forward difference of a polynomial*

$$q_n(k) = \frac{1}{(2k)!} \Delta_x^{2k} \left[(x-k)^{2k+2n} \right]_{x=0}. \tag{8}$$

Proof. The forward difference operator (with respect to x) Δ_x is defined by

$$\Delta_x f(x) = f(x+1) - f(x). \tag{9}$$

Iterating (9), one obtains an expression for the m-th order forward difference of a function

$$\Delta_x^m f(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(x+j). \tag{10}$$

Comparing (7) with (10), the result follows. ■

The Stirling numbers of the second kind are defined by [18, 26.8.6]

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n = \frac{1}{k!} \left[\Delta_x^k x^n \right]_{x=0}.$$

They have many amazing properties, including:

1) The exponential generating function [18, 26.8.12]

$$\sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} \frac{t^n}{n!} = \frac{\left(e^t - 1\right)^k}{k!}.$$

Since $\left\{ {n \atop k} \right\} = 0$, for $k > n$, we can write

$$\sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} \frac{t^n}{n!} = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left\{ {n+k \atop k} \right\} \frac{t^{n+k}}{(n+k)!},$$

and therefore

$$\sum_{n=0}^{\infty} \left\{ {n+k \atop k} \right\} \frac{t^n}{(n+k)!} = \frac{1}{k!} \left(\frac{e^t - 1}{t} \right)^k. \tag{11}$$

2) The difference-differential transformation [18, 26.8.37]

$$\frac{1}{k!} \Delta_x^k = \sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} \frac{1}{n!} \frac{d^n}{dx^n}. \tag{12}$$

Remark 5 *In the next results, we will need some material from the theory of generating functions (see [22] for additional information).*

1) Given a generating function

$$F(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{13}$$

we define $[z^n] F(z)$ *to be the coefficient of* z^n *in the Maclaurin series of* $F(z)$, *i.e.,*

$$[z^n] F(z) = a_n. \tag{14}$$

2) The even part of the generating function (13) is given by

$$\frac{F(z) + F(-z)}{2} = \sum_{n=0}^{\infty} a_{2n} z^{2n}. \tag{15}$$

3) Given two sequences defined by their generating functions

$$F(z) = \sum_{n=0}^{\infty} a_n z^n, \quad G(z) = \sum_{n=0}^{\infty} b_n z^n,$$

the Cauchy product of the sequences is defined by $(a_j * b_j)_n = \sum_{j=0}^{n} a_j b_{n-j}$. *The generating function of the Cauchy product of two sequences is the product of their generating functions,*

$$\sum_{n=0}^{\infty} (a_j * b_j)_n \; z^n = F(z) G(z). \tag{16}$$

We have now all the elements to get new representations of the functions $q_n(k)$.

Proposition 6 *Let* $q_n(k)$ *be defined by (7). Then,*

$$q_n(k) = \sum_{j=0}^{2n} \left\{ {j+2k \atop 2k} \right\} \binom{2n+2k}{2n-j} (-k)^{2n-j}. \tag{17}$$

Proof. Using (12) in (8), we have

$$q_n(k) = \sum_{j=0}^{\infty} \left\{ {j \atop 2k} \right\} \frac{1}{j!} \left[\frac{d^j}{dx^j} (x-k)^{2k+2n} \right]_{x=0}.$$

But

$$\frac{1}{j!} \left[\frac{d^j}{dx^j} (x-k)^{2k+2n} \right]_{x=0} = [x^j] (x-k)^{2k+2n}$$
$$= [x^j] \sum_{j=0}^{2k+2n} \binom{2k+2n}{j} x^j (-k)^{2k+2n-j} = \binom{2k+2n}{j} (-k)^{2k+2n-j},$$

where $\left[x^{j}\right]$ was defined in (14).

Therefore,

$$q_{n}\left(k\right)=\sum_{j=0}^{\infty}\left\{ \begin{matrix} j \\ 2k \end{matrix} \right\} \frac{1}{j!}\binom{2k+2n}{j}\left(-k\right)^{2k+2n-j}.$$

However, since

$$\left\{ \begin{matrix} j \\ 2k \end{matrix} \right\}\binom{2k+2n}{j}=0,\quad j>2k+2n,$$

we have

$$q_{n}\left(k\right)=\sum_{j=2k}^{2k+2n}\left\{ \begin{matrix} j \\ 2k \end{matrix} \right\} \frac{1}{j!}\binom{2k+2n}{j}\left(-k\right)^{2k+2n-j}=\sum_{j=0}^{2n}\left\{ \begin{matrix} j+2k \\ 2k \end{matrix} \right\}\binom{2n+2k}{j+2k}\left(-k\right)^{2n-j},$$

and the result follows from the identity [18, 26.3.1]

$$\binom{n}{k}=\binom{n}{n-k}.$$

■

Corollary 7 *Let $q_{n}\left(k\right)$ be defined by (7). Then, $q_{n}\left(k\right)=\left(2k+1\right)_{2n}\ r_{n}\left(k\right),$ where $r_{n}\left(k\right)$ is defined by*

$$r_{n}\left(k\right)=\sum_{j=0}^{2n}\left\{ \begin{matrix} 2k+j \\ 2k \end{matrix} \right\} \frac{\left(2k\right)!}{\left(2k+j\right)!}\frac{\left(-k\right)^{2n-j}}{\left(2n-j\right)!}. \tag{18}$$

In particular, the first few $r_{n}\left(k\right)$ are given by,

$$r_{0}\left(k\right)=1,\quad r_{1}\left(k\right)=\frac{k}{12},\quad r_{2}\left(k\right)=\frac{k\left(5k-1\right)}{360}. \tag{19}$$

Proof. Since

$$\binom{2n+2k}{2n-j}=\frac{\Gamma\left(2n+2k+1\right)}{\left(2k+j\right)!\left(2n-j\right)!}=\left(2k+1\right)_{2n}\frac{\Gamma\left(2k+1\right)}{\left(2k+j\right)!\left(2n-j\right)!},$$

the result follows from (17). ■

Next, we find a generating function for the sequence $r_{n}\left(k\right)$.

Proposition 8 *Let $r_{n}\left(k\right)$ be defined by (18). Then, $r_{n}\left(k\right)$ has the ordinary generating function*

$$\sum_{n=0}^{\infty}r_{n}\left(k\right)t^{2n}=\left[\frac{2}{t}\sinh\left(\frac{t}{2}\right)\right]^{2k}. \tag{20}$$

Proof. Let

$$R_{k}\left(t\right)=\sum_{n=0}^{\infty}r_{n}\left(k\right)t^{2n}.$$

From (18), we see that we can write $r_n(k)$ as a Cauchy product

$$r_n(k) = (2k)! \left(\left\{ {2k+j \atop 2k} \right\} \frac{1}{(2k+j)!} * \frac{(-k)^j}{j!} \right)_{2n}.$$

Using (15), we get

$$\frac{1}{(2k)!} R_k(t) = \sum_{n=0}^{\infty} t^{2n} \left(\left\{ {2k+j \atop 2k} \right\} \frac{1}{(2k+j)!} * \frac{(-k)^j}{j!} \right)_{2n} = \frac{G_k(t) + G_k(-t)}{2},$$

where

$$G_k(t) = \sum_{n=0}^{\infty} t^n \left(\left\{ {2k+j \atop 2k} \right\} \frac{1}{(2k+j)!} * \frac{(-k)^j}{j!} \right)_n = \left[\sum_{j=0}^{\infty} \left\{ {2k+j \atop 2k} \right\} \frac{t^j}{(2k+j)!} \right] \left[\sum_{j=0}^{\infty} \frac{(-k)^j}{j!} t^j \right],$$

after using (16).

From (11), we have

$$\sum_{n=0}^{\infty} \left\{ {n+2k \atop 2k} \right\} \frac{t^n}{(n+2k)!} = \frac{1}{(2k)!} \left(\frac{e^t - 1}{t} \right)^{2k},$$

and clearly

$$\sum_{j=0}^{\infty} \frac{(-k)^j}{j!} t^j = e^{-kt}.$$

Thus,

$$(2k)! G_k(t) = \left(\frac{e^t - 1}{t} \right)^{2k} e^{-kt} = \left(\frac{e^t - 1}{t} \right)^{2k} e^{-2k\frac{t}{2}} = \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{t} \right)^{2k},$$

and we conclude that

$$G_k(t) = \frac{1}{(2k)!} \left[\frac{2}{t} \sinh\left(\frac{t}{2} \right) \right]^{2k}.$$

Since $\frac{2}{t} \sinh\left(\frac{t}{2}\right)$ is an even function, we get

$$R_k(t) = (2k)! \frac{G_k(t) + G_k(-t)}{2} = \left[\frac{2}{t} \sinh\left(\frac{t}{2} \right) \right]^{2k}.$$

■

Corollary 9 *Let $r_n(k)$ be defined by (18). Then, $r_n \in \mathbb{Q}[k]$ and* $\deg(r_n) = n$.

Proof. From (20), we have

$$\sum_{n=0}^{\infty} r_n(x+y) t^{2n} = \left[\frac{2}{t} \sinh\left(\frac{t}{2} \right) \right]^{2(x+y)} = \left[\frac{2}{t} \sinh\left(\frac{t}{2} \right) \right]^{2x} \left[\frac{2}{t} \sinh\left(\frac{t}{2} \right) \right]^{2y},$$

and using (16) we get

$$r_n(x+y) = \sum_{j=0}^{n} r_j(x)\, r_{n-j}(y). \tag{21}$$

In particular, setting $y = 1$

$$\Delta_x r_n(x) = r_n(x+1) - r_n(x) = \sum_{j=0}^{n-1} r_{n-j}(1)\, r_j(x),$$

where we have used (19). Using induction, the result follows. ■

To summarize, in this section, we have shown that

$$b_{n,k} = \frac{1}{2}\binom{2k}{k} q_n(k) + \frac{1}{2}\delta_{n+k,0}$$

and

$$q_n(k) = (2k+1)_{2n}\; r_n(k),$$

where $r_n \in \mathbb{Q}[k]$ and $\deg(r_n) = n$.

3 Main result

In this section, we use our previous results to prove (4). We start with a few formulas that we will need in the sequel.

Lemma 10 *For all $j, k, n \in \mathbb{N}_0$, we have*

$$\begin{aligned} &\binom{3n+\frac{1}{2}}{k-j} (-4)^{k-j} \binom{2j}{j} (2j+1)_{2n} \\ &= \frac{4^{n+k}}{k!} \left(\frac{1}{2}\right)_{3n+1} \binom{k}{j} (-1)^{k-j} \left(j-k+3n+\frac{3}{2}\right)_{k-2n-1} (j+1)_n. \end{aligned} \tag{22}$$

Proof. We have

$$\binom{3n+\frac{1}{2}}{k-j} (-4)^{k-j} \binom{2j}{j} (2j+1)_{2n} = (-4)^{k-j} \frac{\Gamma\left(3n+\frac{3}{2}\right)\Gamma(2j+2n+1)}{j!\,(k-j)!\,\Gamma\left(3n+\frac{3}{2}-k+j\right)\Gamma(j+1)}$$

and

$$\begin{aligned} &\frac{4^{n+k}}{k!} \left(\frac{1}{2}\right)_{3n+1} \binom{k}{j} (-1)^{k-j} \left(j-k+3n+\frac{3}{2}\right)_{k-2n-1} (j+1)_n \\ &= (-1)^{k-j}\, 4^{n+k} \frac{\Gamma\left(3n+\frac{3}{2}\right)\Gamma\left(j+n+\frac{1}{2}\right)\Gamma(j+n+1)}{j!\,(k-j)!\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(j-k+3n+\frac{3}{2}\right)\Gamma(j+1)}. \end{aligned}$$

Hence, we need to show that

$$4^{-j}\Gamma(2j+2n+1) = 4^n \frac{\Gamma\left(j+n+\frac{1}{2}\right)\Gamma(j+n+1)}{\Gamma\left(\frac{1}{2}\right)}.$$

But this is a direct consequence of the duplication formula for the Gamma function [18, 5.5.5]

$$\Gamma\left(2z\right)=\frac{2^{2z-1}}{\Gamma\left(\frac{1}{2}\right)}\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right). \tag{23}$$

∎

Proposition 11 *For all $j\in\mathbb{N}_0$ we have*

$$\sum_{k=0}^{\infty}\binom{2k}{k}\binom{k}{j}\ z^{k}=\binom{2j}{j}\frac{z^{j}}{\left(1-4z\right)^{j+\frac{1}{2}}},\qquad\left|4z\right|<1. \tag{24}$$

Proof. Since $\binom{k}{j}=0$ for $k<j$, we can write

$$\sum_{k=0}^{\infty}\binom{2k}{k}\binom{k}{j}\ z^{k}=\sum_{k=j}^{\infty}\binom{2k}{k}\binom{k}{j}\ z^{k}$$

$$=\sum_{k=0}^{\infty}\binom{2k+2j}{k+j}\binom{k+j}{j}\ z^{k+j}=\frac{z^{j}}{j!}\sum_{k=0}^{\infty}\frac{\Gamma\left(2k+2j+1\right)}{\Gamma\left(k+j+1\right)}\ \frac{z^{k}}{k!}.$$

From (23), we get

$$\frac{\Gamma\left(2k+2j+1\right)}{\Gamma\left(k+j+1\right)}=4^{k+j}\frac{\Gamma\left(k+j+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}=4^{k+j}\left(\frac{1}{2}\right)_{k+j}. \tag{25}$$

Using the identity [17, 18:5:12]

$$\left(x\right)_{m+n}=\left(x\right)_{m}\left(x+m\right)_{n}$$

we have

$$\left(\frac{1}{2}\right)_{k+j}=\left(j+\frac{1}{2}\right)_{k}\left(\frac{1}{2}\right)_{j}$$

and therefore [17, 18:3:4]

$$\sum_{k=0}^{\infty}\binom{2k}{k}\binom{k}{j}\ z^{k}=z^{j}\frac{\left(4\right)^{j}}{j!}\left(\frac{1}{2}\right)_{j}\ \sum_{k=0}^{\infty}\left(j+\frac{1}{2}\right)_{k}\ \frac{\left(4z\right)^{k}}{k!}=z^{j}\frac{\left(4\right)^{j}}{j!}\left(\frac{1}{2}\right)_{j}\ \left(1-4z\right)^{j+\frac{1}{2}}.$$

But if we read (25) as

$$\left(k+j+1\right)_{k+j}=4^{k+j}\left(\frac{1}{2}\right)_{k+j},$$

we get

$$\frac{\left(4\right)^{j}}{j!}\left(\frac{1}{2}\right)_{j}=\frac{\left(j+1\right)_{j}}{j!}=\binom{2j}{j}$$

and the result follows. ∎

Corollary 12 *Let $u_m(k)$ be a polynomial in k of degree m. Then,*

$$\sum_{k=0}^{\infty}\binom{2k}{k}u_m(k)\ z^k=\frac{U_m(z)}{(1-4z)^{m+\frac{1}{2}}},$$

where $U_m(z)$ is a polynomial in z with $\deg(U_m)\leq m$.

Proof. Let's write $u_m(k)$ in the basis of binomial polynomials.

$$u_m(k)=\sum_{j=0}^{m}a_{m,j}\binom{k}{j}.$$

Using (24), we get

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{2k}{k}u_m(k)\ z^k&=\sum_{j=0}^{m}a_{m,j}\binom{2j}{j}\frac{z^j}{(1-4z)^{j+\frac{1}{2}}}\\&=\frac{1}{(1-4z)^{m+\frac{1}{2}}}\sum_{j=0}^{m}a_{m,j}\binom{2j}{j}z^j(1-4z)^{m-j},\end{aligned}$$

and we conclude that

$$U_m(z)=\sum_{j=0}^{m}a_{m,j}\binom{2j}{j}z^j(1-4z)^{m-j}.$$

■

We can now prove our main result.

Theorem 13 *Let $r_n(k)$ be a polynomial in k of degree n and $P_n(z)$ be defined by*

$$P_n(z)=(1-4z)^{3n+\frac{1}{2}}\sum_{k=0}^{\infty}\binom{2k}{k}(2k+1)_{2n}\ r_n(k)\ z^k.$$

Then, $P_n(z)$ is a polynomial in z of degree $2n$.

Proof. We know from Corollary 12 that $P_n(z)$ is a polynomial with $\deg(P_n)\leq 3n$. Thus, we write

$$P_n(z)=\sum_{j=0}^{3n}c_{n,j}\ z^j.$$

Using the Cauchy product between power series, we have

$$c_{n,k}=\sum_{j=0}^{k}\binom{3n+\frac{1}{2}}{k-j}(-4)^{k-j}\binom{2j}{j}(2j+1)_{2n}\ r_n(j).$$

From (22), we get

$$c_{n,k}=\frac{4^{n+k}}{k!}\left(\frac{1}{2}\right)_{3n+1}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(j-k+3n+\frac{3}{2}\right)_{k-2n-1}(j+1)_n\ r_n(j),$$

which we can write as the finite difference

$$c_{n,k} = \frac{4^{n+k}}{k!} \left(\frac{1}{2}\right)_{3n+1} \left[\Delta_x^k C_{n,k}(x)\right]_{x=0},$$

where

$$C_{n,k}(x) = \left(x - k + 3n + \frac{3}{2}\right)_{k-2n-1} (x+1)_n \; r_n(x).$$

$C_{n,k}(x)$ is a polynomial of degree $k-1$ for $k \geq 2n+1$ and therefore

$$\Delta_x^k C_{n,k}(x) = 0, \quad k \geq 2n+1.$$

We conclude that $c_{n,k} = 0$ for $k > 2n$, and the result is proved. ■

4 Symbolic Computation

In this section we apply computer algebra methods to derive furter results about the coefficient sequence $c_{n,k}$. Using algorithms for symbolic summation, it is possible to discover and prove a recurrence relation for fast computation of these coefficients. As a side result, we obtain a simple closed form for the leading coefficients that would otherwise not be easily discoverd.

Holonomic functions form a class of functions for which a wide variety of algorithms is available to discover and prove non-trivial identities. In one variable, they are functions satisfying a linear difference or differential equation with polynomial coefficients. The classical (continuous) orthogonal polynomials are honolomic both in the degree n (satisfying a three term recurrence) and in the variable x (satisfying a second ordinary differential equation) and also holonomic as multivariate functions in n and x. For a non-expert introduction to holonomic functions in one and several variables as well as some algorithms for them, see [11].

Stirling numbers are an example for a sequence that is just outside the class of holonomic functions. They also satisfy recurrence relations, but of a different type. Methods like automated guessing of recurrence based on given data can certainly also be applied to Stirling-type sequences, however tools for symbolic summation will not work the same way. There has been work on extending these algorithms [2] and these methods are also implemented in the Mathematica package HolonomicFunctions [12] by Christoph Koutschan. The sequence $b_{n,k}$ defined in (3) is of this Stirling-type and below we use automated guessing and a variation of Zeilberger's algorithm [23] to derive recurrence relations for it. The Mathematica notebook containing all computations carried out in this notebook can be found at https://www3.risc.jku.at/people/vpillwei/kapteyn/.

As a first step, we compute a recurrence relation for $b_{n,k}$ using HolonomicFunctions. There are different ways to write the sequence and it does make a difference for the algorithm. We use the definition (5),

$$b_{n,k} = \binom{2k}{k} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k+j)!\,(k-j)!} j^{2k+2n}, \quad n, k \in \mathbb{N}_0,$$

instead of one involving Stirling numbers. Using the command

$$\text{Annihilator}[b[n,k], \{S[n], S[k]\}]$$

in HolonomicFunctions gives the recurrence,

$$(-k-1)S_nS_k + 2(2k+1)S_n + (k+1)^3S_k = 0.$$

The output is in operator form, where S_m denotes the forward shift in the variable m. The recurrence then reads as stated in the following lemma. To avoid the case distinction with the Kronecker delta for the case of both n and k being zero, in the following we always assume that $k \geq 1$. Note that $b_{n,0} = 0$ for $n \geq 1$. Hence, we may even consider $n, k \geq 1$.

Lemma 14 *Let the sequence $b_{n,k}$ be defined by (5). Then,*

$$(k+1)b_{n+1,k+1} = 2(2k+1)b_{n+1,k} + (k+1)^3 b_{n,k+1}, \quad n \geq 0, \quad k \geq 1,$$

with initial values

$$b_{n,1} = 1, \quad b_{0,k} = \frac{1}{2}\binom{2k}{k}.$$

Proof. The recurrence can be derived as shown above and the initial values $b_{n,1}$ are trivially verified for $k \geq 1$. It remains to show that $b_{0,k} = \frac{1}{2}\binom{2k}{k}$. For this first observe that for $n = 0$ and $k \geq 1$ we can rewrite

$$\sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k+j)!\,(k-j)!} j^{2k+2n} = \frac{1}{2}\frac{1}{(2k)!}\sum_{j=0}^{2k}\binom{2k}{k}(-1)^j(k-j)^{2k}.$$

Here we first reverse the order of summation and then using the fact that $k \geq 1$ extend the summation symmetrically to go up to $j = 2k$. Note that for $j = k$ the summand vanishes if $k \geq 1$. Using [7, (5.42)]

$$\sum_{j}^{m}\binom{m}{j}(-1)^j(a_0 + a_1 j + \cdots + a_m j^m) = (-1)^m m! a_m,$$

the result follows with $m = 2k$ and $a_{2k} = 1$. ■

The objects we are actually interested in are the polynomials $P_n(z)$ in the numerator of $g_n(z)$. Recall that they were defined as

$$P_n(z) = (1-4z)^{3n+1/2}\sum_{k\geq 1} b_{n,k}z^k = \sum_{k\geq 1} c_{n,k}z^k,$$

with

$$c_{n,k} = \sum_{j=1}^{k} \frac{(-3n-\frac{1}{2})_{k-j}}{(k-j)!} 4^{k-j} b_{n,j} = \sum_{j=1}^{k} a_{n,k-j}b_{n,j}. \tag{26}$$

In order to derive a recurrence relation for the coefficient sequence $c_{n,k}$ we employ *creative telescoping* [24]. The basic principle is as follows: given the summand

$$f(n,k,j) = a_{n,k-j}b_{n,j},$$

an operator of the form

$$\mathcal{A} + (S_j - 1)\mathcal{D}$$

is determined that annihilates the input, i.e., when applied to the summand $f(n,k,j)$ gives zero. Moreover, $\mathcal{A}$ has coefficients depending only on n and k and not on the summation variable j and uses only shifts of f in n and k, i.e.,

$$\mathcal{A} = \sum_{a,b} \gamma_{a,b}(n,k) S_n^a S_k^b.$$

Because of the nature of this operator and the factor $\Delta_j = S_j - 1$ in front of the second operator $\mathcal{D}$, one can sum over the equation and the delta-part can be evaluated using telescoping. In the ideal case, the summand has natural boundaries and the delta-part telescopes to zero. In this case the final recurrence for the sum is just $\mathcal{A} \cdot c_{n,k} = 0$. But even in a less lucky case, at least an inhomogeneous recurrence can be determined that possibly can be simplified further. Indeed, this is the case in our application.

The method of creative telescoping is implemented in the package HolonomicFunctions, even for the non-holonomic case. However, the size of the input for $c_{n,k}$ is too large and the computations are very expensive. However, it is possible to guess a recurrence for $c_{n,k}$ first and use the support of the guessed recurrence as an input for creative telescoping. This speeds up the process considerably as the ansatz becomes much smaller. Of course the procedure is still rigorous - if there would not be an operator of this form, HolonomicFunctions will return the empty set.

For guessing we use the Mathematica implementation of Manuel Kauers [10] and find that

$$\begin{aligned}
&(k+3)c_{n+1,k+3} - (k+3)^3 c_{n,k+3} - 4(k-3n-1)c_{n+1,k+2} \\
&+ 2\left(6k^3 - 18k^2 n + 33k^2 - 90kn + 57k - 114n + 29\right) c_{n,k+2} \\
&- 4\left(12k^3 - 72k^2 n + 24k^2 + 108kn^2 - 144kn + 9k + 216n^2 - 24n + 2\right) c_{n,k+1} \\
&+ 8(2k-6n-1)^3 c_{n,k} = 0, \qquad n,k \geq 1.
\end{aligned} \tag{27}$$

From this we obtain an input for the support of the shifts in n and k in the method CreativeTelescoping of HolonomicFunctions. Once more note that a notebook with all these calculations can be downloaded and checked.

Given the summand as $a_{n,k-j} b_{n,k}$ in terms of their defining annihilators and the support

$$\{1, S(k), S(k)^2, S(k)^3, S(k)^2 S(n), S(k)^3 S(n)\}$$

as an input, CreativeTelescoping returns the two following operators

$$\begin{aligned}
\mathcal{A} = {} & (k+3)S_k^3 S_n - (k+3)^3 S_k^3 - 4(k-3n-1)S_k^2 S_n \\
&+ 2\left(6k^3 - 18k^2 n + 33k^2 - 90kn + 57k - 114n + 29\right) S_k^2 \\
&- 4\left(12k^3 - 72k^2 n + 24k^2 + 108kn^2 - 144kn + 9k + 216n^2 - 24n + 2\right) S_k \\
&+ 8(2k-6n-1)^3,
\end{aligned}$$

which is the operator form of recurrence (26) above, and

$$\begin{aligned}
\mathcal{D} = {} & \frac{8j(2j-2k+6n+3)(2j-2k+6n+5)(2j-2k+6n+7)}{(j-k-3)(j-k-2)(j-k-1)} S_n \\
& - \frac{24j^3(2n+1)(6n+5)(6n+7)}{(j-k-3)(j-k-2)(j-k-1)}
\end{aligned}$$

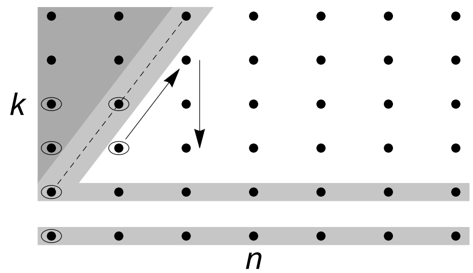

Figure 1: Recurrence for $c_{n,k}$.

for the delta part. In this case, we run into two difficulties. First, the summand $c_{n,k}$ does not have natural bounds for summation, i.e., it does not vanish outside the range of summation. On the other hand, we cannot sum j up to $k+3$ as we run into poles. Hence, we proceed by summing from $j=1$ to $k-1$ over the equation

$$\mathcal{A}\cdot(a_{n,k-j}b_{n,j})+(S_j-1)\mathcal{D}\cdot(a_{n,k-j}b_{n,j})=0.$$

In order to obtain a recurrence for $c_{n,k}$ we have to add and subtract the missing summands in the first part $\sum_{j=1}^{k-1}\mathcal{A}\cdot(a_{n,k-j}b_{n,j})$. As $b_{n,k}$ is given as a sum itself, it is easier to plug in only $a_{n,k-j}$ explicitely and use the recurrence satisfied by $b_{n,k}$ to simplify the equations. All this can be executed automatically in HolonomicFunctions. However all steps can also be easily veryfied using paper and pencil.

Theorem 15 *Let $c_{n,k}$ be defined by* (26)*, then for $n,k\geq 1$, the sequence satisfies the recurrence (26) with initial values*

$$c_{n,1}=1,\quad c_{n,2}=2^{2n+2}-3(4n+1),\quad c_{n,2n}=2^{4n-1}\frac{(1/2)_n^3}{n!},\quad c_{n,k}=0,\quad k\geq 1.$$

Proof. The recurrence can be computed as described above with computational details in the accompanying Mathematica notebook available at https://www3.risc.jku.at/people/vpillwei/kapteyn/. The initial values $c_{n,1}$ and $c_{n,2}$ follow easily by plugging in the formula (26). In order to compute $c_{n,2n}$, first we plug in $k=2n$ in the recurrence relation (26) and obtain

$$\begin{aligned}0=&-8\left(12n^3+12n^2-3n+1\right)c_{n,2n+1}-2\left(24n^3+48n^2-29\right)c_{n,2n+2}\\&-8(2n+1)^3c_{n,2n}-(2n+3)^3c_{n,2n+3}+4(n+1)c_{n+1,2n+2}+(2n+3)c_{n+1,2n+3}.\end{aligned}$$

Next, observe that $c_{m,k}=0$ for $k\geq 2m+1$ by Theorem 13. Hence, $c_{n,2n+1}$, $c_{n,2n+2}$, $c_{n,2n+3}$, and $c_{n+1,2n+3}$ are all zero and the relation above simplifies to

$$4(n+1)c_{n+1,2n+2}-8(2n+1)^3c_{n,2n}=0.$$

This recurrence can easily be solved and with $c_{1,2}=1$ we obtain the result above. ∎

Note that this recurrence with the given initial values can actually be used to compute the sequence $c_{n,k}$. In Fig. 1 the support of the recurrence is indicated by circles around the dots in the lattice, the dark gray area are the indices for which $c_{n,k} = 0$ and the light gray area depicts the non-zero initial values. The lattice is centered at $(1, 1)$. The first value to compute is $c_{2,3}$ and from there one always continues first along the $(n, 2n - 1)$-line and then downwards (n, i) for $2n - 2 \geq i \geq 3$. This way all values of the sequences can be computed recursively.

Remark 16 *It is worth remarking that the closed form $c_{n,2n}$ is not easily proven without the recurrence relation and really gives the double sum evaluation*

$$\sum_{j=1}^{2n} \frac{(-3n - \frac{1}{2})_{2n-j}}{(2n - j)!} 4^{2n-j} \binom{2j}{j} \sum_{i=0}^{j} \frac{(-1)^{j-i}}{(j + i)! \, (j - i)!} i^{2j+2n} = 2^{4n-1} \frac{(\frac{1}{2})_n^3}{n!}.$$

5 Conclusions

We have proved that the Kapteyn series of the second kind

$$g_n(z) = \sum_{k=0}^{\infty} k^{2n} \ \mathrm{J}_k^2(2kz)$$

can be represented as

$$g_n(z) = \frac{P_n(z^2)}{(1 - 4z^2)^{3n+\frac{1}{2}}} + \frac{1}{2}\delta_{n,0}, \quad n \in \mathbb{N}_0,$$

where $P_n(x)$ is a polynomial of degree $2n$.

Writing

$$P_n(z) = (1 - 4z)^{3n+\frac{1}{2}} \sum_{k=1}^{\infty} b_{n,k} z^k = \sum_{k=1}^{\infty} c_{n,k} z^k,$$

we have obtained several properties of the coefficients $b_{n,k}$, and a recurrence for the coefficients $c_{n,k}$.

Numerical evidence suggests that all coefficients $c_{n,k}$ should be nonnegative integers, but so far we haven't been able to prove this. Note that even for the closed form of the leading coefficients $c_{n,2n}$ positivity is obvious, but not that they are integers. Thus, we propose the following conjecture.

Conjecture 17 *Let the polynomials $P_n(z)$ be defined by*

$$\sum_{k=0}^{\infty} k^{2n} \ \mathrm{J}_k^2(2kz) = \frac{P_n(z^2)}{(1 - 4z^2)^{3n+\frac{1}{2}}}, \quad n \in \mathbb{N}.$$

Then, $P_n(x) \in \mathbb{N}_0[x]$.

Acknowledgement 18 *The first author was supported by the strategic program "Innovatives OÖ– 2010 plus" from the Upper Austrian Government. The second author was partly funded by the Austrian Science Fund (FWF): W1214-N15.*

References

[1] A. Baricz, D. Jankov Maširević, and T. K. Pogány. *Series of Bessel and Kummer-type functions*, volume 2207 of *Lecture Notes in Mathematics*. Springer, Cham, 2017.

[2] F. Chyzak, M. Kauers, and B. Salvy. A Non-Holonomic Systems Approach to Special Function Identities. In J. May, editor, *Proceedings of ISSAC'09*, pages 111–118, 2009.

[3] D. Dominici. A new Kapteyn series. *Integral Transforms Spec. Funct.*, 18(5-6):409–418, 2007.

[4] D. Dominici. An application of Kapteyn series to a problem from queueing theory. *PAMM*, 7(1):2050005–2050006, 2008.

[5] D. E. Dominici. On Taylor series and Kapteyn series of the first and second type. *J. Comput. Appl. Math.*, 236(1):39–48, 2011.

[6] A. Eisinberg, G. Fedele, A. Ferrise, and D. Frascino. On an integral representation of a class of Kapteyn (Fourier-Bessel) series: Kepler's equation, radiation problems and Meissel's expansion. *Appl. Math. Lett.*, 23(11):1331–1335, 2010.

[7] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company, Reading, MA, second edition, 1994.

[8] W. Kapteyn. Recherches sur les fonctions de Fourier-Bessel. *Ann. Sci. École Norm. Sup. (3)*, 10:91–122, 1893.

[9] W. Kapteyn. On an expansion of an arbitrary function in a series of Bessel functions. *Messenger of Math.*, 35:122–125, 1906.

[10] M. Kauers. Guessing Handbook. RISC Report Series 09-07, Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, Schloss Hagenberg, 4232 Hagenberg, Austria, 2009.

[11] M. Kauers. The Holonomic Toolkit. In J. Blümlein and C. Schneider, editors, *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions*, pages 119–144. Springer, 2013.

[12] C. Koutschan. *Advanced Applications of the Holonomic Systems Approach*. PhD thesis, RISC-Linz, Johannes Kepler University, September 2009.

[13] I. Lerche and R. C. Tautz. Kapteyn series arising in radiation problems. *J. Phys. A*, 41(3):035202, 10, 2008.

[14] I. Lerche and R. C. Tautz. Kapteyn series in high intensity Compton scattering. *J. Phys. A*, 43(11):115207, 9, 2010.

[15] I. Lerche, R. C. Tautz, and D. S. Citrin. Terahertz-sideband spectra involving Kapteyn series. *J. Phys. A*, 42(36):365206, 9, 2009.

[16] N. Nielsen. *Handbuch der Theorie der Zylinderfunktionen*. Druck und Verlag von B. G. Teubner, Leipzig, 1904.

[17] K. Oldham, J. Myland, and J. Spanier. *An atlas of functions*. Springer, New York, second edition, 2009.

[18] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

[19] R. C. Tautz and D. Dominici. The analytical summation of a new class of Kapteyn series. *Phys. Lett. A*, 374(13-14):1414–1419, 2010.

[20] R. C. Tautz, I. Lerche, and D. Dominici. Methods for summing general Kapteyn series. *J. Phys. A*, 44(38):385202, 14, 2011.

[21] G. N. Watson. *A treatise on the theory of Bessel functions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.

[22] H. S. Wilf. *generatingfunctionology*. A K Peters, Ltd., Wellesley, MA, third edition, 2006.

[23] D. Zeilberger. A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.*, 32(3):321–368, 1990.

[24] D. Zeilberger. The method of creative telescoping. *J. Symbolic Computation*, 11:195–204, 1991.

Technical Reports of the Doctoral Program

"Computational Mathematics"

2019

2019-01 A. Seiler, B. Jüttler: *Approximately $\mathcal{C}^1$-smooth Isogeometric Functions on Two-Patch Domains* Jan 2019. Eds.: J. Schicho, U. Langer

2019-02 A. Jiménez-Pastor, V. Pillwein, M.F. Singer: *Some structural results on D^n-finite functions* Feb 2019. Eds.: M. Kauers, P. Paule

2019-03 U. Langer, A. Schafelner: *Space-Time Finite Element Methods for Parabolic Evolution Problems with Non-smooth Solutions* March 2019. Eds.: B. Jüttler, V. Pillwein

2019-04 D. Dominici, F. Marcellán: *Discrete semiclassical orthogonal polynomials of class 2* April 2019. Eds.: P. Paule, V. Pillwein

2019-05 D. Dominici, V. Pillwein: *A sequence of polynomials generated by a Kapteyn series of the second kind* May 2019. Eds.: P. Paule, J. Schicho

2018

2018-01 D. Dominici: *Laguerre-Freud equations for Generalized Hahn polynomials of type I* Jan 2018. Eds.: P. Paule, M. Kauers

2018-02 C. Hofer, U. Langer, M. Neumüller: *Robust Preconditioning for Space-Time Isogeometric Analysis of Parabolic Evolution Problems* Feb 2018. Eds.: U. Langer, B. Jüttler

2018-03 A. Jiménez-Pastor, V. Pillwein: *Algorithmic Arithmetics with DD-Finite Functions* Feb 2018. Eds.: P. Paule, M. Kauers

2018-04 S. Hubmer, R. Ramlau: *Nesterov's Accelerated Gradient Method for Nonlinear Ill-Posed Problems with a Locally Convex Residual Functional* March 2018. Eds.: U. Langer, R. Ramlau

2018-05 S. Hubmer, E. Sherina, A. Neubauer, O. Scherzer: *Lamé Parameter Estimation from Static Displacement Field Measurements in the Framework of Nonlinear Inverse Problems* March 2018. Eds.: U. Langer, R. Ramlau

2018-06 D. Dominici: *A note on a formula of Krattenthaler* March 2018. Eds.: P. Paule, V. Pillwein

2018-07 C. Hofer, S. Takacs: *A parallel multigrid solver for multi-patch Isogeometric Analysis* April 2018. Eds.: U. Langer, B. Jüttler

2018-08 D. Dominci: *Power series expansion of a Hankel determinant* June 2018. Eds.: P. Paule, M. Kauers

2018-09 P. Paule, S. Radu: *A Unified Algorithmic Framework for Ramanujan's Congruences Modulo Powers of 5, 7, and 11* Oct 2018. Eds.: M. Kauers, V. Pillwein

2018-10 D. Dominici: *Matrix factorizations and orthogonal polynomials* Nov 2018. Eds.: P. Paule, M. Kauers

2018-11 D. Dominici, V. Pillwein: *Difference equation satisfied by the Stieltjes transform of a sequence* Dec 2018. Eds.: P. Paule, M. Kauers

2018-12 N.A. Smoot: *A Family of Congruences for Rogers–Ramanujan Subpartitions* Dec 2019. Eds.: P. Paule, V. Pillwein

The complete list since 2009 can be found at
`https://www.dk-compmath.jku.at/publications/`

Technical Reports of the Doctoral Program
"Computational Mathematics"

2019

2019-01 A. Seiler, B. Jüttler: *Approximately $\mathcal{C}^1$-smooth Isogeometric Functions on Two-Patch Domains* Jan 2019. Eds.: J. Schicho, U. Langer

2019-02 A. Jiménez-Pastor, V. Pillwein, M.F. Singer: *Some structural results on D^n-finite functions* Feb 2019. Eds.: M. Kauers, P. Paule

2019-03 U. Langer, A. Schafelner: *Space-Time Finite Element Methods for Parabolic Evolution Problems with Non-smooth Solutions* March 2019. Eds.: B. Jüttler, V. Pillwein

2019-04 D. Dominici, F. Marcellán: *Discrete semiclassical orthogonal polynomials of class 2* April 2019. Eds.: P. Paule, V. Pillwein

2019-05 D. Dominici, V. Pillwein: *A sequence of polynomials generated by a Kapteyn series of the second kind* May 2019. Eds.: P. Paule, J. Schicho

2018

2018-01 D. Dominici: *Laguerre-Freud equations for Generalized Hahn polynomials of type I* Jan 2018. Eds.: P. Paule, M. Kauers

2018-02 C. Hofer, U. Langer, M. Neumüller: *Robust Preconditioning for Space-Time Isogeometric Analysis of Parabolic Evolution Problems* Feb 2018. Eds.: U. Langer, B. Jüttler

2018-03 A. Jiménez-Pastor, V. Pillwein: *Algorithmic Arithmetics with DD-Finite Functions* Feb 2018. Eds.: P. Paule, M. Kauers

2018-04 S. Hubmer, R. Ramlau: *Nesterov's Accelerated Gradient Method for Nonlinear Ill-Posed Problems with a Locally Convex Residual Functional* March 2018. Eds.: U. Langer, R. Ramlau

2018-05 S. Hubmer, E. Sherina, A. Neubauer, O. Scherzer: *Lamé Parameter Estimation from Static Displacement Field Measurements in the Framework of Nonlinear Inverse Problems* March 2018. Eds.: U. Langer, R. Ramlau

2018-06 D. Dominici: *A note on a formula of Krattenthaler* March 2018. Eds.: P. Paule, V. Pillwein

2018-07 C. Hofer, S. Takacs: *A parallel multigrid solver for multi-patch Isogeometric Analysis* April 2018. Eds.: U. Langer, B. Jüttler

2018-08 D. Dominci: *Power series expansion of a Hankel determinant* June 2018. Eds.: P. Paule, M. Kauers

2018-09 P. Paule, S. Radu: *A Unified Algorithmic Framework for Ramanujan's Congruences Modulo Powers of 5, 7, and 11* Oct 2018. Eds.: M. Kauers, V. Pillwein

2018-10 D. Dominici: *Matrix factorizations and orthogonal polynomials* Nov 2018. Eds.: P. Paule, M. Kauers

2018-11 D. Dominici, V. Pillwein: *Difference equation satisfied by the Stieltjes transform of a sequence* Dec 2018. Eds.: P. Paule, M. Kauers

The complete list since 2009 can be found at
`https://www.dk-compmath.jku.at/publications/`