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Difference equation satisfied by the Stieltjes transform of a sequence

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1 Introduction

Let $\mathbb{N}_0$ denote the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = 0, 1, 2, \ldots$. We say that the sequence of polynomials $\{q_n\}$ is a *monic basis* of $\mathbb{C}[x]$ if $q_n(x) \in \mathbb{C}[x]$ is a monic polynomial and $\deg(q_n) = n$ for all $n \in \mathbb{N}_0$. If $\{q_n\}$ is a monic basis and $\{\mu_n\}$ is a sequence of complex numbers, then the linear functional $L : \mathbb{C}[x] \to \mathbb{C}$ defined by

$$L[q_n] = \mu_n, \quad n \in \mathbb{N}_0, \tag{1}$$

is called the *moment functional* determined by q_n and μ_n [2]. The number μ_n is called the (generalized) *moment* of order n. In the literature, the chosen standard basis is the monomial basis, $q_n(x) = x^n$. If the sequence of polynomials $\{P_n\}$ satisfies

$$L[P_n P_m] = h_n \delta_{n,m}, \quad n, m \in \mathbb{N}_0, \tag{2}$$

where $h_0 = \mu_0$, $h_n \neq 0$ and $\delta_{n,m}$ is Kronecker's delta, then $\{P_n\}$ is called an *orthogonal polynomial sequence* with respect to L.

If the linear functional L admits an extension to the field of rational functions $\mathbb{C}(x)$, then we can define the *Stieltjes transform* of L by

$$S(t) = L\left[\frac{1}{t-x}\right], \tag{3}$$

where L is always assumed to act on the variable x. Note that, at least as formal power series, we have

$$\frac{1}{t-x} = \frac{1}{t}\sum_{n=0}^{\infty}\left(\frac{x}{t}\right)^n,$$

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and therefore (formally)

$$S(t) = \sum_{n=0}^{\infty} \frac{L[x^n]}{t^{n+1}}. \tag{4}$$

Thus, if we use the monomial basis $q_n(x) = x^n$, we can write

$$S(t) = \sum_{n=0}^{\infty} \frac{L[q_n]}{q_{n+1}(t)} = \sum_{n=0}^{\infty} \frac{\mu_n}{q_{n+1}(t)}, \tag{5}$$

and interpret $S(t)$ as a generating function of the sequence of moments $\{\mu_n\}$.

One could be tempted to define $S(t)$ by (5), but then (3) won't be true in general. However, if the monic basis $\{q_n\}$ satisfies

$$\frac{q_{n+1}(t)}{q_n(t)} - \frac{q_{n+1}(x)}{q_n(x)} = t - x, \tag{6}$$

then we obtain the telescoping identity $\frac{q_n(x)}{q_{n+1}(t)}(t-x) = \frac{q_n(x)}{q_n(t)} - \frac{q_{n+1}(x)}{q_{n+1}(t)}$. Hence (at least formally) we have

$$\frac{1}{t-x} = \sum_{n=0}^{\infty} \frac{q_n(x)}{q_{n+1}(t)},$$

and we see that (3) and (5) define the same function $S(t)$. The general solution of (6) is given by $q_n(x) = \prod_{j=0}^{n-1} \left(x + f(j)\right)$, for some function $f(j)$. In particular, if $f = 0$, we recover the monomial basis. If $f(j) = -j$, we obtain the basis of falling factorials

$$\phi_n(x) = \prod_{j=0}^{n-1} (x - j), \tag{7}$$

that was studied by Bracciali, Pérez and Piñar in [1].

In this paper, we consider the sequence operator $\Psi : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}((t))$, defined by

$$\Psi[c_n] = \sum_{n=0}^{\infty} \frac{c_n}{\phi_{n+1}(t)}. \tag{8}$$

Note that

$$\Psi[c_n] = \sum_{n=-1}^{\infty} d_n t^n, \tag{9}$$

where $d_{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_n$. Moreover, if $\nu_n = L[\phi_n]$, then $\Psi[\nu_n] = S(t)$.

Our objective is to obtain formulas relating (a) shifts in n and multiplication by powers of n in the sequence ν_n, with (b) shifts in the variable t for the function $S(t)$. This way, difference equations satisfied by $S(t)$ can be computed automatically from linear recurrences of the sequence ν_n.

The paper is structured as follows: in Section 2, we introduce all the concepts needed to prove our main result (Theorem 8). In Section 3, we define the class of discrete semiclassical orthogonal polynomials. We consider several examples, and use Theorem 8 to obtain the difference equation satisfied by their Stieltjes transform $S(t)$.

2 Main results

Definition 1 *We define the* falling factorial (or binomial) polynomials *by (7) or by*

$$\phi_n(x) = n! \binom{x}{n}, \tag{10}$$

and the Pochhammer (or rising factorial) polynomials *by [8, 5.2(iii)]*

$$(x)_n = \prod_{j=0}^{n-1} (x+j). \tag{11}$$

Remark 2 *From the definition (7), the following recurrences and relations are immediate [7, 18:5:8]*

$$\phi_{n+1}(x) = (x-n)\,\phi_n(x), \tag{12}$$

$$\phi_n(x) = (-1)^n (-x)_n = (x-n+1)_n, \tag{13}$$

$$(x)_{n+m} = (x)_n (x+n)_m, \tag{14}$$

$$\frac{(x)_n}{(x)_m} = \begin{cases} (x+m)_{n-m}, & n \geq m \\ \dfrac{1}{(x+n)_{m-n}}, & n \leq m \end{cases}, \tag{15}$$

where $n, m \in \mathbb{N}_0$.

Using the previous relations and the definition of the falling factorial polynomials, the following identities can be derived easily.

Lemma 3 *For all $k, n \in \mathbb{N}_0$, we have*

$$\phi_{n+1}(t+k) = (t+1)_k \;\; \phi_{n+1-k}(t), \quad k \leq n+1, \tag{16}$$

$$\phi_{n+1}(t+k) = \frac{(t+1)_k}{(t+1)_{k-1-n}}, \quad k \geq n+1. \tag{17}$$

We now can establish the first relations between transformed sequences and the difference equation of the associated Stieltjes function $S(t)$.

Proposition 4 *For all $k \in \mathbb{N}$, we have*

$$\Psi[\nu_{n+k}] = (t+1)_k \;\; S(t+k) - \sum_{j=0}^{k-1} (t+1)_{k-1-j} \;\; \nu_j. \tag{18}$$

Proof. Using (8), we get

$$\Psi[\nu_{n+k}] = \sum_{n=0}^{\infty} \frac{\nu_{n+k}}{\phi_{n+1}(t)} = \sum_{n=k}^{\infty} \frac{\nu_n}{\phi_{n-k+1}(t)}.$$

Since $n \geq k$, we can use (16) and obtain

$$\begin{aligned}
\sum_{n=k}^{\infty} \frac{\nu_n}{\phi_{n-k+1}(t)} &= (t+1)_k \sum_{n=k}^{\infty} \frac{\nu_n}{\phi_{n+1}(t+k)} \\
&= (t+1)_k \left[\sum_{n=0}^{\infty} \frac{\nu_n}{\phi_{n+1}(t+k)} - \sum_{n=0}^{k-1} \frac{\nu_n}{\phi_{n+1}(t+k)} \right].
\end{aligned}$$

In the second sum we now have $n+1 \leq k$. Hence, we can use (17) and conclude that

$$\sum_{n=0}^{k-1} \frac{(t+1)_k}{\phi_{n+1}(t+k)} \nu_n = \sum_{n=0}^{k-1} (t+1)_{k-1-n} \;\; \nu_n.$$

■

Proposition 5 *We have*

$$\Psi[n\nu_n] = tS(t) - (t+1) S(t+1) = -\Delta_t (tS(t)), \tag{19}$$

where Δ_t denotes the forward difference operator, defined by $\Delta_t f(t) = f(t+1) - f(t)$.

Proof. Using (16) with $k=1$, we get $\phi_{n+1}(t+1)=(t+1)\phi_n(t)$. Thus,

$$-\Delta_t\left[\frac{t}{\phi_{n+1}(t)}\right]=\frac{t}{\phi_{n+1}(t)}-\frac{t+1}{\phi_{n+1}(t+1)}=\frac{t}{\phi_{n+1}(t)}-\frac{1}{\phi_n(t)}.$$

From (12), we obtain

$$\frac{1}{\phi_n(t)}=\frac{t-n}{\phi_{n+1}(t)}.$$

Therefore,

$$-\Delta_t\left[\frac{t}{\phi_{n+1}(t)}\right]=\frac{t}{\phi_{n+1}(t)}-\frac{t-n}{\phi_{n+1}(t)}=\frac{n}{\phi_{n+1}(t)},$$

and we conclude that

$$\begin{aligned}-\Delta_t(tS)&=\sum_{n=0}^{\infty}-\Delta_t\left[\frac{t}{\phi_{n+1}(t)}\right]\nu_n=\sum_{n=0}^{\infty}\frac{n\nu_n}{\phi_{n+1}(t)}\\&=\sum_{n=1}^{\infty}\frac{n\nu_n}{\phi_{n+1}(t)}=\sum_{n=0}^{\infty}\frac{(n+1)\nu_{n+1}}{\phi_{n+2}(t)}=\sum_{n=0}^{\infty}\frac{(n+1)\nu_{n+1}}{\phi_{n+2}(t)}.\end{aligned}$$

■

Corollary 6 *For all $p\in\mathbb{N}_0$, we have*

$$\Psi[n^p\nu_n]=(-\Delta_t t)^p S(t)=\sum_{i=0}^{p}(-1)^i(t+1)_i u_{p,i}(t)S(t+i), \qquad (20)$$

where the polynomials $u_{p,i}(t)\in\mathbb{Z}[t]$, $\deg(u_{p,i})=p-i$, are given by

$$u_{p,i}(t)=\sum_{j=0}^{p-i}\binom{j+i}{i}(-1)^{j+i+p}\left\{\begin{matrix}p+1\\j+i+1\end{matrix}\right\}(t+i+1)_j, \qquad (21)$$

and $\left\{\begin{smallmatrix}n\\k\end{smallmatrix}\right\}$ denote the Stirling numbers of the second kind, defined by [8, 26.8.6]

$$\left\{\begin{matrix}n\\k\end{matrix}\right\}=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

Proof. The result follows from the formula [10, Ch.6, eq.11]

$$(\Delta_t t)^p f(t)=\sum_{j=0}^{p}\left\{\begin{matrix}p+1\\j+1\end{matrix}\right\}\phi_j(t+j)\Delta_t^j f(t).$$

From (13), we have $\phi_j(t+j) = (t+1)_j$, and using the formula for higher order differences [9, 6.1]

$$\Delta_t^j f(t) = \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} f(t+i), \tag{22}$$

we get

$$\begin{aligned}(\Delta_t t)^p f(t) &= \sum_{j=0}^{p} \left\{ {p+1 \atop j+1} \right\} (t+1)_j \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} f(t+i) \\ &= \sum_{i=0}^{p} f(t+i) \sum_{j=i}^{p} \binom{j}{i} (-1)^{j-i} \left\{ {p+1 \atop j+1} \right\} (t+1)_j .\end{aligned}$$

Shifting the summation index in the inner sum and using the factorization $(t+1)_{j+i} = (t+1)_i (t+i+1)_j$, we obtain

$$\begin{aligned}(-\Delta_t t)^p S(t) = \sum_{i=0}^{p} & (-1)^i S(t+i)(t+1)_i \\ & \sum_{j=0}^{p-i} \binom{j+i}{i} (-1)^{j+i+p} \left\{ {p+1 \atop j+i+1} \right\} (t+i+1)_j .\end{aligned}$$

■

Let us recall the well-known property that the forward difference acts on the raising factorials in a similar way as the derivative does on the monomial basis, i.e.,

$$\Delta_x [(x)_n] = n (x+1)_{n-1} . \tag{23}$$

As an extension of this, we obtain the following result.

Corollary 7 *For all $p, m \in \mathbb{N}_0$, we have*

$$(\Delta_t t)^p (t+1)_m = (m+1)^p (t+1)_m . \tag{24}$$

Proof. We use induction on p. The case $p = 0$ is an identity. Assuming the result to be true for $p \geq 0$, we have

$$(\Delta_t t)^{p+1} (t+1)_m = \Delta_t \left[(m+1)^p t (t+1)_m \right] = (m+1)^p \Delta_t \left[(t)_{m+1} \right],$$

where we have used (15) with $n = 1$,

$$(t)_{m+1} = t (t+1)_m .$$

However,

$$\Delta_t\left[\left(t\right)_{m+1}\right] = \left(m+1\right)\left(t+1\right)_m$$

by (23) and therefore

$$\left(\Delta_t t\right)^{p+1}\left(t+1\right)_m = \left(m+1\right)^{p+1}\left(t+1\right)_m,$$

and the result is proved.

■

Next, we put all the pieces together and give a formula to translate recurrence equations with polynomial coefficients for the sequence ν_n, into difference equations satisfied by $S(t)$.

Theorem 8 *For all $p, k \in \mathbb{N}_0$, we have*

$$\begin{aligned}\Psi\left[n^p \nu_{n+k}\right] = & \sum_{j=0}^{p} (-1)^j \, (t+1)_{k+j} \, u_{p,j}(t) \, S(t+j+k) \\ & - \sum_{j=0}^{k-1} (j-k)^p \, (t+1)_{k-1-j} \; \nu_j,\end{aligned}$$

where the polynomials $u_{p,j}(t)$ were defined in (21).

Proof. Using (20) in (18), we get

$$\begin{aligned}\Psi\left[n^p \nu_{n+k}\right] &= (-\Delta_t t)^p \, \Psi\left[\nu_{n+k}\right] \\ &= (-\Delta_t t)^p \left[(t+1)_k \; S(t+k) - \sum_{j=0}^{k-1} (t+1)_{k-1-j} \; \nu_j\right].\end{aligned}$$

From (20), we obtain

$$\begin{aligned}&(-\Delta_t t)^p \left[(t+1)_k \; S(t+k)\right] = \\ &\qquad \sum_{j=0}^{p} (-1)^j \, (t+j+1)_k \; (t+1)_j \, u_{p,j}(t) \, S(t+j+k).\end{aligned}$$

Again we factor using (16), i.e., $(t+j+1)_k \; (t+1)_j = (t+1)_{k+j}$, and therefore

$$(-\Delta_t t)^p \left[(t+1)_k \; S(t+k)\right] = \sum_{j=0}^{p} (-1)^j \, (t+1)_{k+j} \, u_{p,j}(t) \, S(t+j+k).$$

From (24), we get

$$(-\Delta_t t)^p \left[(t+1)_{k-1-j}\right] = (-1)^p (k-j)^p (t+1)_{k-1-j},$$

and we conclude that

$$(-\Delta_t t)^p \sum_{j=0}^{k-1} (t+1)_{k-1-j} \ \nu_j = \sum_{j=0}^{k-1} (j-k)^p (t+1)_{k-1-j} \ \nu_j.$$

■

3 Discrete semiclassical orthogonal polynomials

In this section, we consider functionals of the form

$$L[q] = \sum_{x=0}^{\infty} q(x)\,\rho(x), \quad q \in \mathbb{C}[x], \tag{25}$$

where the weight function $\rho : \mathbb{N}_0 \to \mathbb{C}$ is given by

$$\rho(x) = \frac{(a_1)_x (a_2)_x \cdots (a_p)_x}{(b_1+1)_x \cdots (b_q+1)_x} \frac{z^x}{x!}. \tag{26}$$

Using (14), we see that the weight function $\rho(x)$ satisfies the *Pearson equation*

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\lambda(x)}{\sigma(x+1)}, \tag{27}$$

where the polynomials $\lambda(x), \sigma(x)$ are given by

$$\begin{aligned} \lambda(x) &= z(x+a_1)(x+a_2)\cdots(x+a_p), \\ \sigma(x) &= x(x+b_1-1)\cdots(x+b_q-1). \end{aligned} \tag{28}$$

We call L *discrete semiclassical* and define the class of L to be the number

$$s = \max\{\deg(\lambda)-2,\ \deg(\lambda-\sigma)-1\}. \tag{29}$$

The functionals of class $s=0$ are called *discrete classical [4]*.

Note that the moments $\nu_n(z)$ are given by

$$\nu_n(z) = \sum_{x=0}^{\infty} \phi_n(x)\,\rho(x).$$

In particular, for the first moment $\nu_0(z)$ we get

$$\nu_0(z) = \sum_{x=0}^{\infty} \frac{(a_1)_x (a_2)_x \cdots (a_p)_x}{(b_1)_x \cdots (b_q)_x} \frac{z^x}{x!} = {}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z \right),$$

where ${}_pF_q$ denotes the generalized hypergeometric function [8, 16.2].

Since the falling factorial polynomials $\phi_n(x)$ are eigenvalues of the differential operator $z^n \frac{d^n}{dz^n}$,

$$z^n \frac{d^n}{dz^n} z^x = \phi_n(x) z^x,$$

we have

$$\nu_n(z) = \sum_{x=0}^{\infty} \frac{(a_1)_x (a_2)_x \cdots (a_p)_x}{x! (b_1)_x \cdots (b_q)_x} \phi_n(x) z^x = z^n \frac{d^n}{dz^n} \nu_0(z). \tag{30}$$

Using the identity [8, 16.3.1]

$$\frac{d^n}{dz^n} {}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z \right) = \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} {}_pF_q\left(\begin{array}{c} a_1 + n, \ldots, a_p + n \\ b_1 + n, \ldots, b_q + n \end{array} ; z \right),$$

we obtain

$$\nu_n(z) = z^n \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} {}_pF_q\left(\begin{array}{c} a_1 + n, \ldots, a_p + n \\ b_1 + n, \ldots, b_q + n \end{array} ; z \right). \tag{31}$$

From (30), we see that the exponential generating function of the moments $\nu_n(z)$ is given by

$$G(u; z) = \sum_{n=0}^{\infty} \nu_n(z) \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{(uz)^n}{n!} \frac{d^n}{dz^n} \nu_0(z) = \nu_0[(u+1) z]. \tag{32}$$

Since $\nu_0(z)$ satisfies a differential equation of order $\max\{p, q+1\}$ with polynomial coefficients (see [8, 16.8(ii)]), it follows that the sequence $\{\nu_n\}$ is holonomic (or P-recursive) [5]. Note that from (32) we see that

$$\sum_{n=0}^{\infty} \nu_n(z) \frac{(-1)^n}{n!} = \nu_0(0) = 1,$$

and therefore the coefficient d_{-1} in (9) is always equal to 1 for these moment sequences.

The classical discrete orthogonal families are the Charlier, Meixner, Kravchuk, and Hahn polynomials. Their moments satisfy the recurrences

$$\begin{aligned}
\nu_{n+1} - z\nu_n &= 0, \quad \text{(Charlier polynomials)} \\
(1-z)\,\nu_{n+1} - z\,(n+a)\,\nu_n &= 0, \quad \text{(Meixner polynomials)} \\
(1-z)\,\nu_{n+1} - z\,(n-N)\,\nu_n &= 0, \quad \text{(Kravchuk polynomials)} \\
(a-b-N+1+n)\,\nu_{n+1} + (n-N)\,(n+a)\,\nu_n &= 0, \quad \text{(Hahn polynomials)}
\end{aligned}$$

where $N \in \mathbb{N}_0$. Using Theorem 8, we derive the following difference equations for their Stieltjes transforms

$$\begin{aligned}
(t+1)\,S\,(t+1) - zS\,(t) &= \nu_0, \quad \text{(Charlier)} \\
(t+1)\,S\,(t+1) - z\,(t+a)\,S(t) &= (1-z)\,\nu_0, \quad \text{(Meixner)} \\
(t+1)\,S\,(t+1) - z\,(t-N)\,S(t) &= (1-z)\,\nu_0, \quad \text{(Kravchuk)} \\
(t+1)\,(t+b)\,S\,(t+1) - (t-N)\,(t+a)\,S\,(t) &= (b-a+N)\,\nu_0. \quad \text{(Hahn)}
\end{aligned}$$

Equivalent difference equations were obtained in [1, 3.1] using a different technique.

In [3], we classified the weight functions satisfying (27), with $\deg(\lambda - \sigma) = 2$ and $1 \leq \deg(\sigma) \leq 3$. Below, we use the main cases as examples.

Example 9 *Let's consider the families of polynomials orthogonal with respect to the weight functions*

1. *Generalized Charlier polynomials*

$$\rho\,(x) = \frac{1}{(b)_n}\,\frac{z^x}{x!},$$

2. *Generalized Meixner polynomials*

$$\rho\,(x) = \frac{(a)_x}{(b)_x}\,\frac{z^x}{x!},$$

3. *Generalized Kravchuk polynomials*

$$\rho\,(x) = (-N)_x\,(a)_x\,\frac{z^x}{x!}, \quad N \in \mathbb{N}_0,$$

4. *Generalized Hahn polynomials of type I*

$$\rho\,(x) = \frac{(a_1)_x\,(a_2)_x}{(b)_x}\,\frac{z^x}{x!},$$

5. *Generalized Hahn polynomials of type II*

$$\rho(x) = \frac{(-N)_x (a_1)_x (a_2)_n}{(b_1)_x (b_2)_x} \frac{1}{x!}, \quad N \in \mathbb{N}_0.$$

We see from (31) that the moments are given by

1. *Generalized Charlier polynomials*

$$\nu_n(z) = \tfrac{z^n}{(b)_x} \, {}_0F_1 \left[\begin{array}{c} - \\ b+n \end{array} ; z \right],$$

2. *Generalized Meixner polynomials*

$$\nu_n(z) = z^n \frac{(a)_n}{(b)_n} \, {}_1F_1 \left[\begin{array}{c} a+n \\ b+n \end{array} ; z \right],$$

3. *Generalized Kravchuk polynomials*

$$\nu_n(z) = (-N)_n (a)_n \, {}_2F_0 \left[\begin{array}{c} -N+n, a+n \\ - \end{array} ; z \right],$$

4. *Generalized Hahn polynomials of type I*

$$\nu_n(z) = z^n \frac{(a_1)_n (a_2)_n}{(b)_n} \, {}_2F_1 \left[\begin{array}{c} a_1+n, a_2+n \\ b+n \end{array} ; z \right],$$

5. *Generalized Hahn polynomials of type II*

$$\nu_n = \frac{(-N)_n (a_1)_n (a_2)_n}{(b_1)_n (b_2)_n} \, {}_3F_2 [\begin{array}{c} -N+n, a_1+n, a_2+n \\ b_1+n, b_2+n \end{array} ; 1].$$

Using the Mathematica package HolonomicFunctions [6], we get the recurrence relations

1. *Generalized Charlier polynomials*

$$\nu_{n+2} + (n+b)\, \nu_{n+1} - z\nu_n = 0,$$

2. *Generalized Meixner polynomials*

$$\nu_{n+2} + (n+b-z)\, \nu_{n+1} - z(n+a)\, \nu_n = 0,$$

3. Generalized Kravchuk polynomials

$$z\nu_{n+2} + [(2n - N + a + 1)\, z - 1]\, \nu_{n+1} + z\,(n + a)\,(n - N)\,\nu_n = 0,$$

4. Generalized Hahn polynomials of type I

$$(1 - z)\,\nu_{n+2} + [n + b - (2n + a_1 + a_2 + 1)\, z]\,\nu_{n+1} - z\,(n + a_1)\,(n + a_2)\,\nu_n = 0,$$

5. Generalized Hahn polynomials of type II

$$(N - n + b_1 + b_2 - a_1 - a_2 - 2)\,\nu_{n+2} + [(n + b_1)(n + b_2) + N\,(2n + a_1 + a_2 + 1)]\,\nu_{n+1} \\ - \left(3n^2 + 3n + 1 + a_1 + a_2 + 2na_1 + 2na_2 + a_1a_2\right)\nu_{n+1} + (N - n)\,(a_1 + n)\,(a_2 + n)\,\nu_n = 0.$$

Using Theorem 8, we conclude that their Stieltjes transforms satisfies the difference equations

1. Generalized Charlier polynomials

$$(t + 1)\,(t + b)\,S\,(t + 1) - zS\,(t) = (t + b)\,\nu_0 + \nu_1,$$

2. Generalized Meixner polynomials

$$(t + 1)\,(t + b)\,S\,(t + 1) - z\,(t + a)\,S\,(t) = (t + b - z)\,\nu_0 + \nu_1,$$

3. Generalized Kravchuk polynomials

$$(t + 1)\,S\,(t + 1) - z\,(t + a)\,(t - N)\,S\,(t) = [1 - (t + a - N)\, z]\,\nu_0 - z\nu_1,$$

4. Generalized Hahn polynomials of type I

$$(t + 1)\,(t + b + 1)\,S\,(t + 1) - z\,(t + a_1)\,(t + a_2)\,S\,(t) \\ = [(1 - z)\,t + b + 1 - z\,(a_1 + a_2)]\,\nu_0 + (1 - z)\,\nu_1,$$

5. Generalized Hahn polynomials of type II

$$(t + 1)\,(t + b_1 + 1)\,(t + b_2 + 1)\,S\,(t + 1) - (t + a_1)\,(t + a_2)\,(t - N)\,S\,(t) \\ = [(2 + b_1 + b_2 + N - a_1 - a_2)t + N\,(a_1 + a_2) - a_1a_2 + (b_1 + 1)\,(b_2 + 1)]\,\nu_0 \\ + (1 + b_1 + b_2 + N - a_1 - a_2)\,\nu_1.$$

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