An Algebraic-Geometric Method for Computing Zolotarev Polynomials — Additional Information

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This report is an appendix to [2] providing two pieces of information in addition to the explicit computations of [2]. On the one hand we treat the problem of explicit construction of proper Zolotarev polynomials of higher degree using explicit expressions for proper Zolotarev polynomials of lower degree. In particular we show how Z_6, as computed in [2], is related to Z_2. Furthermore, we provide some ideas of intervals on the parameter t in which the Zolotarev polynomial of degree 5 is bounded by ± 1 and attains these values at least 5 times.

1 Introduction

Please note, that this report is meant to be an appendix to [2]. This means we use all notation and definitions from there. The current report emphasizes and extends results from [2] and puts them in a wider framework. In this sense it is not self contained. However, we tried to put as much information as needed to recall the most important notions and refer to [2] whenever needed.

In [2] we are dealing with the task of finding rational solutions of the Zolotarev ODE:

$$n^2(x-\beta)^2(1-y^2)-(1-x^2)(x-\gamma)(x-\delta)y'^2=0\,. \tag{1}$$

We call all polynomial solutions of the ODE (1) a Zolotarev polynomial. If additionally the polynomial Z_n fulfills

$$Z_n'(\beta)=0\,, \qquad\qquad Z_n(\gamma)=-Z_n(\delta)=\pm 1\,, \tag{2}$$

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we call it a proper Zolotarev polynomial. As in [2] we concentrate on those.

In [2] we presented an algorithm to compute all rational solutions of the Zolotarev ODE (1), in particular it also finds all polynomial solutions. This however only works subject to the computation of a parametrization of the relation curve.

2 Construction of Zolotarev polynomials using those of lower degree

In [2] we computed the Zolotarev polynomial of degree 6. As one might notice from the algorithm in [2] it is possible to compute Z_2 and Z_6 using the same parametrization for β, γ, δ. In a similar way one can compute Z_{10}, Z_{14} and so on. Furthermore, for Z_3 and Z_9 one can do the same. In this section we show, how this works in general using a known result from Lebedev.

In [3] Lebedev showed the following theorem for a more general setting.

Theorem 2.1. (Lebedev [3])
Let Z_m be a Zolotarev polynomial of degree $m > 1$ and let T_ℓ be the Chebyshev polynomial of degree $\ell > 1$. Then $Z_{\ell m} = T_\ell(Z_m)$ is a Zolotarev polynomial of degree ℓm.

Proof. We give a proof using the Zolotarev ODE. We know that Z_m solved the ODE, hence

$$m^2(x-\beta)^2(1-Z_m^2) - (1-x^2)(x-\gamma)(x-\delta)Z_m'^2 = 0\,.$$

We want to show that

$$\ell^2 m^2(x-\beta)^2(1-T_\ell(Z_m)^2) - (1-x^2)(x-\gamma)(x-\delta)Z_m'^2 T_\ell(Z_m)'^2 = 0\,.$$

This is equivalent to the following equations:

$$\begin{aligned}
\ell^2 m^2(x-\beta)^2(1-T_\ell(Z_m)^2) - (1-x^2)(x-\gamma)(x-\delta)\frac{m^2(x-\beta)^2(1-Z_m^2)}{(1-x^2)(x-\gamma)(x-\delta)}T_\ell(Z_m)'^2 &= 0\,,\\
\ell^2(1-T_\ell(Z_m)^2) - (1-Z_m^2)T_\ell(Z_m)'^2 &= 0\,,\\
\ell^2(1-T_\ell(Z_m)^2) - (1-Z_m^2)\ell^2 U_{\ell-1}(Z_m)'^2 &= 0\,,\\
T_\ell(Z_m)^2 - (Z_m^2-1)U_{\ell-1}(Z_m)^2 - 1 &= 0\,.
\end{aligned}$$

The last equation holds since the Chebyshev polynomials T_ℓ and $U_{\ell-1}$ fulfill the Pell equation which can be easily seen from the trigonometric representation of Chebyshev polynomials (compare [1, §18.5.1–2]). Furthermore, we used that $T_k'(x) = xU_{k-1}$ (compare [1, §18.9.21]), where U_k is the Chebyshev polynomial of second kind of degree k. □

Corollary 2.2.
Let Z_m be a proper Zolotarev polynomial of degree $m > 1$ and let T_ℓ be the Chebyshev polynomial of degree $\ell > 1$. Then $Z_{\ell m} = T_\ell(Z_m)$ is a proper Zolotarev polynomial if and only if $2 \nmid \ell$.

Proof. We have shown in Theorem 2.1 that $Z_{\ell m}$ is a Zolotarev polynomial. We assume that Z_m is a proper one, i. e. $Z'_m(\beta) = 0$ and $Z_m(\gamma) = -Z_m(\delta) = \pm 1$. Then it is easy to see that $Z'_{\ell m}(\beta) = T'_\ell(Z_m(\beta))Z'_m(\beta) = 0$. Furthermore, if $Z_m(\gamma) = 1$, then $Z_{\ell m}(\gamma) = T_\ell(Z_m(\gamma)) = T_\ell(1) = 1$. Similarly, if $Z_m(\gamma) = -1$, then $Z_{\ell m}(\gamma) = T_\ell(Z_m(\gamma)) = T_\ell(-1) = (-1)^\ell$. The same holds for δ. Since for properness we require the evaluations at γ and δ to be different, we know that 2 must not divide ℓ. □

Corollary 2.2 shows that from a proper Zolotarev polynomial of degree m we can construct proper Zolotarev polynomials of degree $(2k + 1)m$.

The following remark shows, that for computing explicit expressions for all proper Zolotarev polynomials, it is enough to know the proper Zolotarev of degree 2^k for all $k \in \mathbb{N}$ and of degree p for all prime numbers p.

Remark 2.3.
Let $n = 2^{\varepsilon_1} \cdot 3^{\varepsilon_2} \cdot p_3^{\varepsilon_3} \cdot \ldots \cdot p_\nu^{\varepsilon_\nu}$ where p_i is the i-th prime and $\nu \in \mathbb{N}$. By Corollary 2.2 we get the following by immediate observations.

- *Assume that $\varepsilon_1 \geq 1$ and $\sum_{i=2}^{\nu} \varepsilon_i \geq 1$:*
  Then $Z_n = T_{p_2^{\varepsilon_2} \ldots p_\nu^{\varepsilon_n u}}(Z_{2^{\varepsilon_1}})$ and it is proper if $Z_{2^{\varepsilon_1}}$ was chosen to be proper.
- *Assume that $\varepsilon_1 = 0$ and let i be the smallest index such that $\varepsilon_{i-1} = 0$ and $\varepsilon_i \neq 0$:*
  Then $Z_n = T_{p_i^{\varepsilon_i - 1} \ldots p_\nu^{\varepsilon_n u}}(Z_{p_i})$ and it is proper if Z_{p_i} was chosen to be proper.

So let us see this on the initially mentioned example of Z_6.

Example 2.4.
In [2] we computed the Zolotarev polynomial of degree 6 to be

$$Z_6 = \frac{1}{(1+t)^3} \sum_{i=0}^{6} a_i x^i \,,$$

with

$$\begin{aligned} &a_0 = 1 - 6t - 3t^2\,, && a_2 = 3(1 + 10t + 5t^2)\,, && a_4 = -12(2t + t^2)\,, \quad a_6 = -4\,, \\ &a_1 = 3(3 + t)(1 - t^2)\,, && a_3 = 4(1 + t)(-5 + 2t + t^2)\,, && a_5 = 12(1 + t)\,. \end{aligned}$$

We show now, that indeed $Z_6 = T_3(Z_2)$ taking into account, that we need to take the same parametrization of the relation curve (compare [2]). Since in [2] we used a different parametrization for Z_2 and Z_6, we recompute Z_2 with the parametrization $\beta = \frac{t+1}{2}$, $\gamma = 2+t$, $\delta = t$. Then we get $Z_2 = \frac{tx - x^2 + x + 1}{t+1}$. Note, that this is just a reparametrized version *of the Z_2 obtained in [2]. The Chebyshev polynomial of degree 3 is $T_3 = -3x + 4x^3$. Hence, we get*

$$T_3(Z_2) = \frac{4\left(tx - x^2 + x + 1\right)^3}{(t+1)^3} - \frac{3\left(tx - x^2 + x + 1\right)}{t+1}.$$

This can be easily checked to be the same as Z_6 as above.

3 Intervals

In this section we investigate the Zolotarev polynomials Z_n of degree $n \in \{2, \dots, 6\}$, respectively. More precisely, we look at intervals for the parameter t such that the polynomial Z_n is bounded for $x \in [-1, 1]$ by ± 1. Furthermore, we would like that the values ± 1 are obtained n times for $x \in [-1, 1]$.

Definition 3.1.
Let $Z_n(x, t)$ be a Zolotarev polynomial and let I be an interval for the parameter t. We say that I fulfills the parameter condition *if the following hold:*

- *For all $x \in [-1, 1]$ we have $Z_n(x) \in [-1, 1]$.*
- *There are $\tau_1, \dots, \tau_n$ with $\tau_i \neq \tau_j$ for $i \neq j$ and for all $i \in \{1, \dots, n\}$ we have $Z_n(\tau_i) = \pm 1$.*

The cases for $n \in \{2, \dots, 4\}$ have already been known for a while. Nevertheless, we briefly show them here to get acquainted with the settings of this paper.

3.1 Degree 2

We consider the parametrization $\beta = \frac{1}{2t}$, $\gamma = \frac{1}{t} + 1$, $\delta = \frac{1}{t} - 1$. Note, that with the description of [2] this is the case which has ID 10. Then the Zolotarev polynomial computed by the algorithm in [2] is given by $Z_2 = t + x - tx^2$. For the interval $t \in [-\frac{1}{2}, \frac{1}{2}]$ the polynomial is bounded by ± 1 as illustrated in gray in Figure 1. Choices for t outside this interval are plotted in red. It is easily computable that these are not bounded by ± 1 within the interval for $x \in [-1, 1]$. The Chebyshev polynomial of degree 1 and 2 are in green and blue respectively.

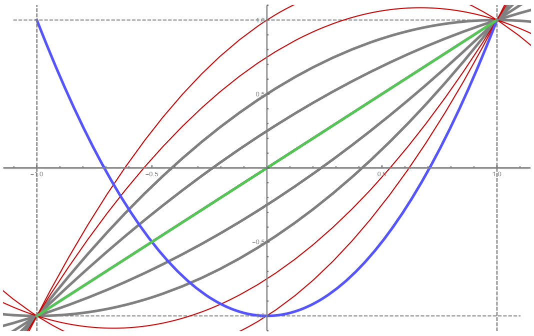

Figure 1: Z_2 for $t \in [-\frac{1}{2}, \frac{1}{2}]$ and Chebyshev polynomials

3.2 Degree 3

We consider a different parametrization $\beta = \frac{1}{t}, \gamma = -\frac{t^2-3}{2t}, \delta = \frac{t^2+9}{6t}$ than in [2]. Note, that with the notation of [2] this is the case which has ID 2. The Zolotarev polynomial in this case is $Z_3 = \frac{-t^4+18t^2(3x^2-2)+108tx(x^2-1)-162x^2+81}{(t^2-9)^2}$. For the interval $t \in [-1, 1]$ the polynomial is bounded by ± 1 as illustrated in gray in Figure 2. Choices for t outside this interval are plotted in red. It is easily computable that these are not bounded by ± 1 within the interval for $x \in [-1, 1]$. The Chebyshev polynomial of degree 2 and 3 are in green and blue respectively.

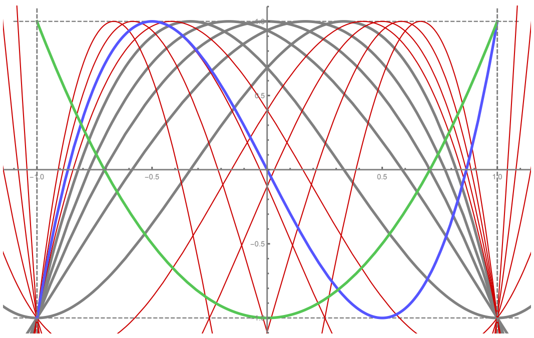

Figure 2: Z_3 for $t \in [-1, 1]$ and Chebyshev polynomials

3.3 Degree 4

We use the same setting as in [2]. For the interval $t \in [-1 + \sqrt{2}, 1 - \sqrt{2}]$ the polynomial is bounded by ± 1 as illustrated in gray in Figure 3. Choices for t outside this interval are plotted in red. It is easily computable that these are not bounded by ± 1 within the interval for $x \in [-1, 1]$. The Chebyshev polynomial of degree 3 and 4 are in green and blue respectively. The Zolotarev polynomial itself can be found for instance in [6] and according to that was already described by Markov. Using a slightly different parametrization Rack [5] investigated Z_4 and the respective interval fulfilling the parameter condition. The reparametrization to our expression can be achieved by $t \longrightarrow \frac{1-t}{1+t}$.

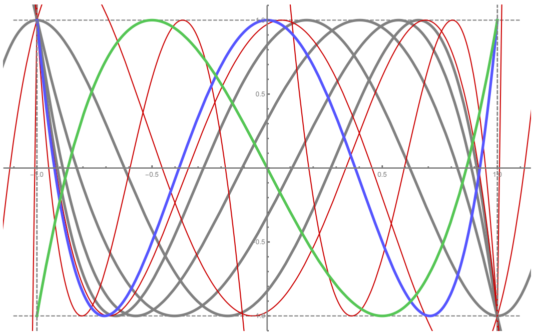

Figure 3: Z_4 for $t \in [-1+\sqrt{2}, 1-\sqrt{2}]$ and Chebyshev polynomials

3.4 Degree 5

We consider the Zolotarev polynomial of degree 5 as computed in [2]. First we investigate the root $\alpha = \sqrt{\frac{2(5t^2-1)}{25t(t+1)^3}}$ which appears in Z_5 (see Figure 4). Note, that α occurs only linearly.

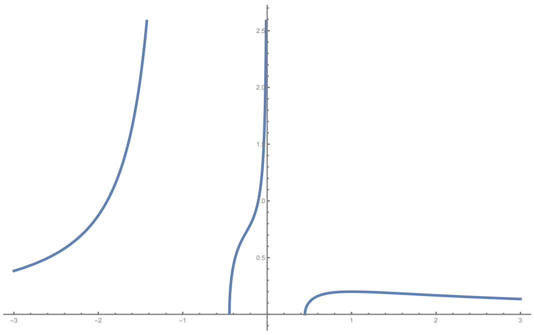

Figure 4: α from Z_5 depending on t

It obtains real values in the intervals $I_1 = (-\infty, -1)$, $I_2 = [-\frac{1}{\sqrt{5}}, 0)$ and $I_3 = [\frac{1}{\sqrt{5}}, \infty)$. In the following subsections we investigate these intervals in detail. Note, that the common denominator of Z_5 has zeros at $t = 1$ and $t = -\frac{1}{3}$. The later one is not in any of the intervals above, whereas $t = 1$ is in I_3 and hence, we need to take care of that.

Before investigating the intervals let us consider $Z_5(0)$ for general t.

$$Z_5(0) = -\frac{c_{10}t^{10} + \ldots + c_0}{(t-1)^6(3t+1)^4},$$

with

$$(c_{10}, \ldots, c_0) = (2581, 8122, 6221, 1128, -966, -500, -174, -56, 17, 10, 1).$$

We first solve $Z_5(0) = 1$ which happens at

$$\tau_{1,1} = -1, \qquad \tau_{1,2} = \frac{1}{11}\left(\frac{2\sqrt[3]{11\sqrt{33}-63}}{3^{2/3}} - \frac{4}{\sqrt[3]{33\sqrt{33}-189}} - 1\right),$$

$$\tau_{1,3} = \frac{1}{11}\left(1 - 2\sqrt{3}\right), \qquad \tau_{1,4} = \frac{1}{11}\left(1 + 2\sqrt{3}\right),$$

where $\tau_{1,2} \approx -0.47907$. On the other hand $Z_5(0) = -1$ occurs at $\tau_{-1,2} = -\frac{1}{\sqrt{5}}, \tau_{-1,4} = 0, \tau_{-1,6} = \frac{1}{\sqrt{5}}$ and the roots $\tau_{-1,1}, \tau_{-1,3}, \tau_{-1,5}$ of $-1-z+13z^2+5z^3$ which are approximately $\tau_{-1,1} \approx -2.64701, \tau_{-1,3} \approx -0.252373, \tau_{-1,5} \approx 0.299386$,.

We are interested in those values of t, where $-1 \leq Z_5(0) \leq 1$. This happens in the intervals $T_1 = [\tau_{-1,1}, \tau_{1,1}]$, $T_2 = [\tau_{1,2}, \tau_{-1,2}]$, $T_3 = [\tau_{-1,3}, \tau_{-1,4}]$, $T_4 = [\tau_{-1,5}, \tau_{-1,6}]$. This gives a necessary condition on the interval in question. Figure 5 illustrates $Z_5(0)$.

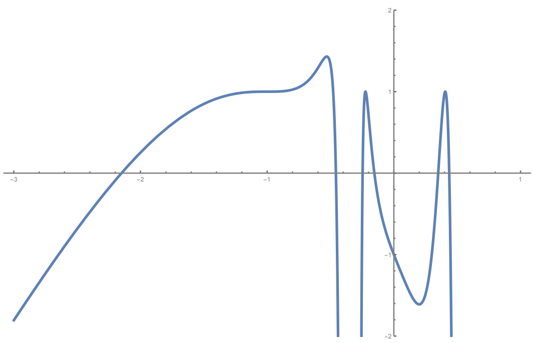

Figure 5: $Z_5(0)$ depending on t

Note, that similar considerations can be done for $x = \pm 1$. Furthermore, we can easily check, that $Z_5(\pm 1) = -1$, no matter how t is chosen. We now take a closer look to the intervals I_k. Since I_2 is the most interesting one we start with that one.

3.4.1 Interval $I_2 = [-\frac{1}{\sqrt{5}}, 0)$

By some easy investigation one finds that $Z_5(1)$ attains a local minimum for the choice $t = \frac{1}{5}(2\sqrt{5} - 5)$. Note, that α is not defined for $t = 0$. However, if we consider α to be an independent variable, we get the Chebyshev polynomial of degree 4 for $t = 0$.

In the following we show that I_2 fulfills the parameter condition. Let us consider the zeros of $Z_5'(x)$ depending on t as shown in Figure 6.

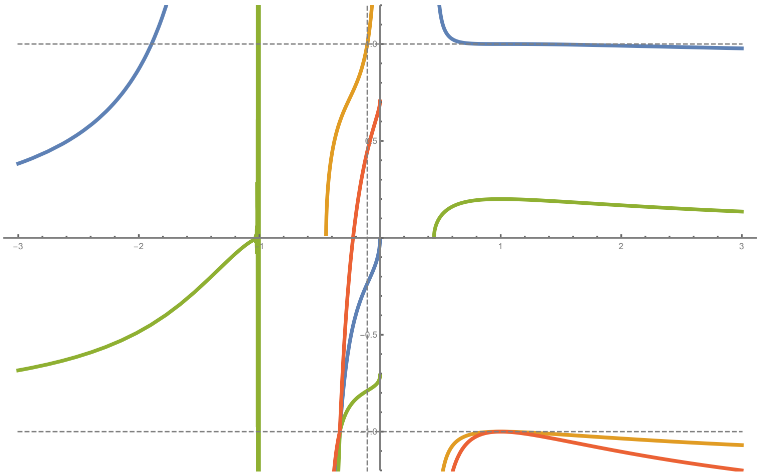

Figure 6: Solutions of $Z_5'(x) = 0$ depending on t

It is easy to compute, that Z_5' has exactly three zeros within $[-1, 1]$ when t is in the interior of I_2. We also see that they are local maxima and minima. Indeed one can computationally show, that at these maxima and minima the value ± 1 is obtained. Hence, for $\left[\frac{1}{5}(2\sqrt{5} - 5), 0\right)$ the parameter condition on Z_5 is fulfilled. In Figure 7 this is illustrated.

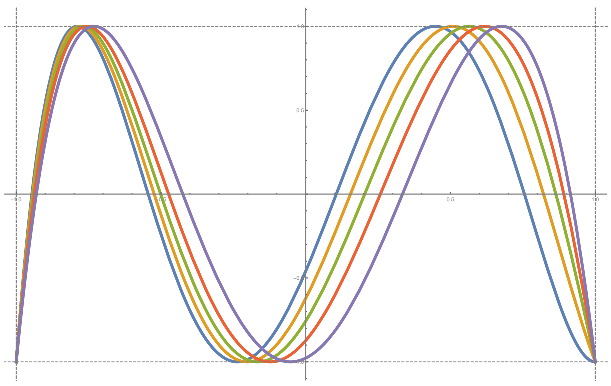

Figure 7: Z_5 for $t \in \left[\frac{1}{5}(2\sqrt{5} - 5), 0\right]$

Outside of $t \in \left[\frac{1}{5}(2\sqrt{5} - 5), 0\right]$, we get the following. If $\frac{1}{\sqrt{5}} > t > 0$ then α in Z_5 is complex and hence, Z_5 cannot be plotted in $\mathbb{R}^2$. The case $t < \frac{1}{5}(2\sqrt{5} - 5)$ is illustrated in Figure 8 in red. In this case the slope of $Z_5(1)$ is positive and hence, there is an

$\xi \in [-1, 1]$ such that $Z_5(\xi) < -1$. In blue we see the case $t = \frac{1}{5}(2\sqrt{5} - 5)$ and in green the Chebyshev polynomial of degree 4.

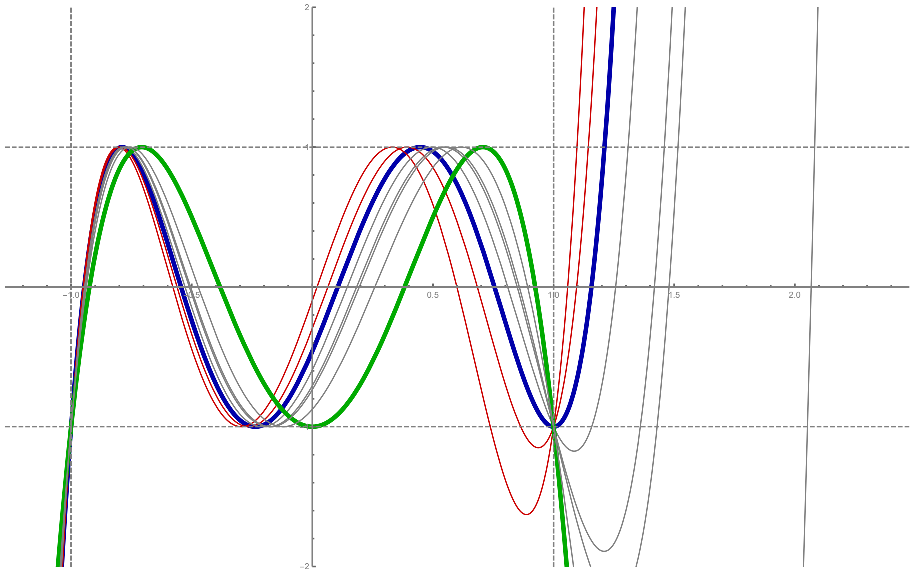

Figure 8: Z_5 for $t \in \left[\frac{1}{5}(2\sqrt{5} - 5), 0\right]$ and $t < \frac{1}{5}(2\sqrt{5} - 5)$

As a matter of the construction of the Zolotarev polynomials as a solution of a differential equation, we know (see [2]) that also $-Z_5$ is a Zolotarev polynomial. Finally, we get the following picture. In Figure 9 the gray plots come from the Z_5. The green one represents the special case of the Chebyshev polynomial of degree 4 and the blue plot is the Chebyshev polynomial of degree 5, each one together with its negative.

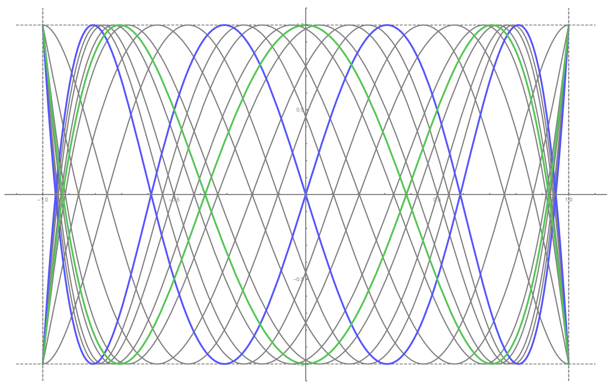

Figure 9: $\pm Z_5$ for $t \in \left[\frac{1}{5}(2\sqrt{5} - 5), 0\right]$ and Chebyshev polynomials

3.4.2 Interval $I_1 = (-\infty, -1)$

By some investigation one finds that $Z_5(1)$ attains a local minimum for the choice $t = \frac{1}{5}(-2\sqrt{5} - 5)$. However, this choice would need a different interval for x in order to reach ± 1 sufficiently often. Indeed, as we can see from direct computations illustrated in Figure 6 the polynomial has at most two extrema for $x \in [-1, 1]$ with values in $[-1, 1]$ for $t \neq \frac{1}{5}(-2\sqrt{5} - 5)$. Hence, the I_1 does not fulfill the parameter condition. Neither does any subinterval.

In Figure 10 this is illustrated where purple is $t = \frac{1}{5}(-2\sqrt{5}-5)$. The case $t < \frac{1}{5}(-2\sqrt{5}-5)$ is plotted in red. Here, the slope of $Z_5(1)$ is positive and hence, there is an $\xi \in [-1, 1]$ such that $Z_5(\xi) < -1$. In orange we show the case $t > \frac{1}{5}(-2\sqrt{5} - 5)$.

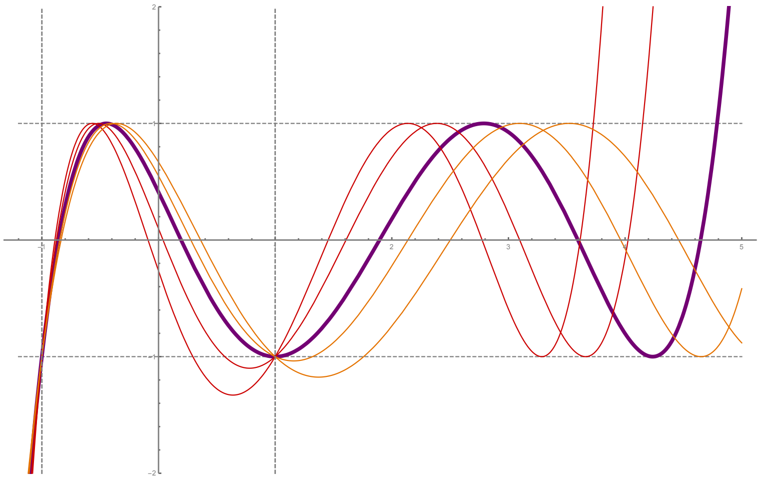

Figure 10: Z_5 for t around $\frac{1}{5}(-2\sqrt{5} - 5)$

3.4.3 Interval $I_3 = [\frac{1}{\sqrt{5}}, \infty)$

For $t = \frac{1}{\sqrt{5}}$ the polynomial Z_5 is the constant function -1. For $t = 1$ the polynomial Z_5 is not defined. From the discussion above on $Z_5(0)$ we can see that for $t > \frac{1}{\sqrt{5}}$, the polynomial Z_5 is not bounded by ± 1. Hence, I_3 does not fulfill the parameter condition.

3.5 Degree 6

As pointed out by Heinz-Joachim Rack[1] the choice $t = 1$ does fulfill the requirements, when chosen on a different interval for x. In a similar way the choice $t = -3$ works. However, for the interval in between these two choices, the polynomial Z_6 is not bounded by ± 1. This can be seen in Figure 11.

[1]Private communication, 23.06.2016

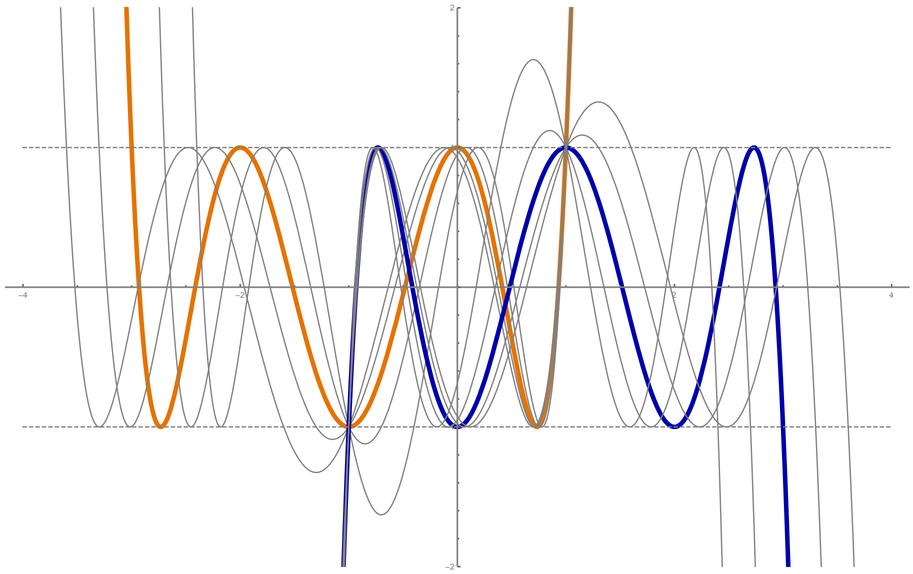

Figure 11: $\pm Z_6$ around $t = -3$ and $t = 1$

More detailed investigation of all solution of degree 6 for the Zolotarev ODE is subject to further research.

4 Conclusion

We have shown how a given parametrization of the relation curve of a Zolotarev polynomial can be used for finding higher degree Zolotarev polynomials. This describes the special structure of Z_6 and gives a reason for further investigation of this case. We have investigated the interval which fulfills the parameter condition for Z_5.

Acknowledgments

We thank Heinz-Joachim Rack for his helpful questions and comments which made us aware of the interest in the questions treated in this paper.

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