# Denominator Bounds for Higher Order Systems of Linear Recurrence Equations

— Extended Abstract —

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## 1 Introduction

Let $(K, \sigma)$ be a difference field. We define the set of constants by $\operatorname{const} K = \{c \in K \mid \sigma(c) = c\}$. A $\Pi\Sigma^*$-*extension* of $K$ is a field of rational functions $K(t)$ over $K$ together with an extension of $\sigma$ to $K(t)$ given by either $\sigma(t) = at$ ($\Pi$ case) or $\sigma(t) = t + b$ ($\Sigma^*$ case) for some non-zero $a$ or $b \in K$ such that $\operatorname{const} K(t) = \operatorname{const} K$ holds. See, for example, [8, 9] for more details on $\Pi\Sigma^*$-extensions.

In this work, we consider coupled systems of recurrence equations of the form

$$A_s \sigma^s(y) + \ldots + A_1 \sigma(y) + A_0 y = b \tag{1}$$

where $A_0, \ldots, A_s \in K(t)^{m\times n}$ are matrices and $b \in K(t)^m$ is a vector. Our goal is to find rational solutions, that is, all $y \in K(t)^n$ which satisfy the system. A first step is to find a nonzero polynomial $d \in K[t]$ such that $dy$ has only polynomial entries for all possible solutions $y$. This polynomial is known as *denominator bound* or *universal denominator*.

Most existing algorithms as for instance [3, 1] work by translating the higher order system to a first order system. We only know of one method, [4], dealing directly with higher order systems. Our algorithm is similar to that later work; however, we expand it in several points: 1. Most importantly, we address the problem for general $\Pi\Sigma^*$ extensions instead of concentrating on the case $\operatorname{const} K = K$ and the shift operator $t \mapsto t + 1$. 2. In addition our method does not require the system matrices to be square or their rows to be linearly independent.

## 2 Results

The special case of a scalar recurrence ($m = 1$) for a $\Pi\Sigma^*$-extension $(K(t), \sigma)$ of $(K, \sigma)$ has been treated in [6, 10]. The derived algorithm generalises Abramov's denominator bounding algorithm [2] that has been introduced for the rational case, i.e., for the situation that $\operatorname{const}(K) = K$ and $\sigma(t) = t + 1$. Exploiting the observation that Abramov's algorithm can be formulated in a straightforward fashion [7] enables us to tackle the denominator bounding problem for the general case $m \geq 1$ in a given $\Pi\Sigma^*$-extension.

More precisely, in this poster we present a way to derive the denominator bound for the "aperiodic" part directly from the highest and lowest coefficient matrix $A_\ell$ and $A_0$ for the case that both matrices are regular. It is convenient to consider two cases.

For the $\Sigma^*$-case we show that for any solution $y = \frac{p}{d} \in K(t)^m$ ($p \in K[t]^m \setminus \{0\}$, $d \in K[t] \setminus \{0\}$ and $\gcd(p, d) = 1$) of the system of (1) the denominator $d$ fulfils

$$d \;\Big|\; \gcd\Big(\prod_{j=0}^{D} \sigma^{-\ell-j}(\tilde{m}), \prod_{j=0}^{D} \sigma^{j}(\tilde{p})\Big) \tag{2}$$

where $\tilde{m}$ is the denominator of $A_\ell^{-1}$, $\tilde{p}$ is the denominator of $A_0^{-1}$ and $D$ is the dispersion of $\sigma^{-\ell}(\tilde{m})$ and $\tilde{p}$. Here the dispersion of $a, b \in K[t]$ is defined by

$$\operatorname{disp}(a, b) = \max\{n > 0 \mid \gcd(a, \sigma^n(b)) \neq 1\}$$

with the convention that $\max \varnothing = -1$; and the denominator of a matrix means the least common multiple of the denominators of its entries. It is important to note that for any $a, b \in K[t] \setminus \{0\}$ the dispersion $\operatorname{disp}(a, b)$ is finite; for details see [6, 9]. This implies in particular that the products in our formula (2) are well defined.

For the $\Pi$-case the situation is slightly more complicated. For any $a, b \in K[t] \setminus \{0\}$ the dispersion $\operatorname{disp}(a, b)$ is finite if and only if $t \nmid a$ or $t \nmid b$; see again [6, 9]. Using this extra insight, we show that for any solution $y = \frac{p}{d\,t^r} \in K(t)^m$ ($r \geq 0$, $p \in K[t]^m \setminus \{0\}$, $d \in K[t] \setminus \{0\}$ with $t \nmid d$ and $\gcd(p, d\,t^r) = 1$) of the system (1) the factor $d$ of the denominator fulfils (2) where $\tilde{m}$ and $\tilde{p}$ are defined as above but where the possibly occurring factors $t$ are removed. In the $\Pi$ case, what is left of a polynomial after removing factors of $t$ is usually known as its aperiodoc part within the $\Pi\Sigma^*$ setting; that is, we are computing the so-called aperiodic denominator bound here. Again this construction implies that $D = \operatorname{disp}(\sigma^{-\ell}(\tilde{m}), \tilde{p})$ in (2) is finite.

Moreover, we provide a discussion on how to deal with systems where the leading or trailing coefficient matrices are singular. Here, we have to preprocess the system using so-called row and column reduction—see, for example, [5]. In brief, this method considers the system matrix $A = A_\ell \sigma^\ell + \ldots + A_1 \sigma + A_0$ as a matrix over the operator ring $K(t)[\sigma]$ and uses elementary row or column transformations in order to make the leading matrix $A_\ell$ regular. A slight modification of the method can be used to work on the trailing matrix $A_0$ as well. Thus we obtain two equivalent systems from which we get a similar denominator bound to (2).

For the $\Pi$-case we also provide some preliminary results on the missing "periodic" denominator bound $t^r$ for some $r \geq 0$. Here we focus on the special case that $\operatorname{const} K = K$.

# References


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