AN ALGORITHM TO PROVE ALGEBRAIC RELATIONS INVOLVING ETA QUOTIENTS

CRISTIAN-SILVIU RADU

Abstract. In this paper we present an algorithm which can prove algebraic relations involving η-quotients, where η is the Dedekind eta function.

1. The Problem

Let N be a positive integer throughout this paper. We denote by $R(N)$ the set of integer sequences $r = (r_\delta)_{\delta|N}$ indexed by the positive divisors δ of N; $\tilde{r} = (\tilde{r}_\delta)_{\delta|N}$ is defined by $\tilde{r}_\delta := r_{N/\delta}$. For $r \in R(N)$ we define an associated η-quotient as

$$f(r)(\tau) := \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}, \quad \tau \in \mathbb{H},$$

where $\eta(\tau) := e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty}(1-q^n)$, $q = q(\tau) := e^{2\pi i \tau}$, is the Dedekind eta function and $\mathbb{H} := \{x \in \mathbb{C} : \text{Im}(x) > 0\}$.

The input to our algorithm is $n \in \mathbb{N}$, $r^{(j)} \in R(N)$ and $a_j \in \mathbb{Q}$ for $j = 1, \dots, n$; the output is true or false depending whether

$$\sum_{1\leq j\leq n} a_j f(r^{(j)})(\tau) \equiv 0, \tag{1}$$

is true or false [1]. The new contribution of this paper is that we reduce the proving of the identity (1), to the proving of a finite number of identities of the type (1) under additional constraints; in particular, in each such identity the terms are modular functions for the group $\Gamma_0(N)$.

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[1]Using "$\equiv$" is short hand for meaning equality for all $\tau \in \mathbb{H}$.

2. The First Problem Reduction

Recall that

$$\eta(-1/\tau) \equiv (-i\tau)^{1/2}\eta(\tau). \tag{2}$$

Applying $\tau \mapsto -1/(N\tau)$ to both sides of the identity (1) we obtain by (2) ,

$$\sum_{1\leq j\leq n} a_j \prod_{\delta|N} (-i/\delta)^{\frac{r_\delta}{2}} \times \tau^{\frac{\sum_{\delta|N} r_\delta^{(j)}}{2}} f(\tilde{r}^{(j)})(\tau) \equiv 0.$$

We may rewrite this sum as

$$\sum_{k=m_1}^{m_2} \tau^{k/2} \sum_{\substack{1\leq j\leq n \\ \sum_{\delta|N} r_\delta^{(j)} = \frac{k}{2}}} a_j \prod_{\delta|N} (-i/\delta)^{\frac{r_\delta}{2}} f(\tilde{r}^{(j)})(\tau) \equiv 0 \tag{3}$$

for some $m_1, m_2 \in \mathbb{Z}$ with $m_1 \leq m_2$.

Lemma 2.1. *Let n be a positive integer and $f_k : \mathbb{H} \to \mathbb{C}$ such that $f_k(\tau + 24) \equiv f_k(\tau)$ for $k = 0, \ldots, n$. Then*

$$\sum_{k=0}^{n} \tau^{k/2} f_k(\tau) \equiv 0 \tag{4}$$

iff $f_k(\tau) \equiv 0$ for $k = 1, \ldots, n$.

Proof. Applying $\tau \mapsto \tau + 24$ to both sides of (4) m times we obtain

$$\sum_{k=0}^{n} (\tau + 24m)^{k/2} f_k(\tau) \equiv 0.$$

Therefore

$$\sum_{k=0}^{n} (\tau + 24m)^{k/2} f_k(\tau) \equiv 0, \quad m = 0, \ldots n$$

which we may write in matrix form:

$$\begin{pmatrix} 1 & \tau^{1/2} & \tau & \ldots & \tau^{n/2} \\ 1 & (\tau+24)^{1/2} & \tau+24 & \ldots & (\tau+24)^{n/2} \\ 1 & (\tau+48)^{1/2} & \tau+48 & \ldots & (\tau+48)^{n/2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (\tau+24n)^{1/2} & \tau+24n & \ldots & (\tau+24n)^{n/2} \end{pmatrix} \begin{pmatrix} f_0(\tau) \\ f_1(\tau) \\ f_2(\tau) \\ \vdots \\ f_n(\tau) \end{pmatrix} \equiv 0.$$

This matrix is a Vandermonde-matrix with determinant

$$\prod_{0\leq i<j\leq n} (((\tau + 24j)^{1/2} - (\tau + 24i)^{1/2}).$$

Hence for all $\tau \in \mathbb{H}$ this matrix is invertible. Multiplying both sides by the inverse we obtain $f_k(\tau) \equiv 0$ for $k = 0, \dots, n$. $\square$

For $k \in \mathbb{Z}$ we define

$$S(k) := \Big\{ r \in R(N) : 2 \sum_{\delta | N} r_\delta = k \Big\}.$$

Since $\eta(\tau + 24) \equiv \eta(\tau)$ we have $f(r)(\tau + 24) \equiv f(r)(\tau)$ for all $r \in R(N)$. Multiplying both sides of (3) by $\tau^{-m_1/2}$ we obtain:

$$\sum_{k=0}^{m_2 - m_1} \tau^{\frac{k}{2}} \sum_{\substack{1 \le j \le n \\ r^{(j)} \in S(k+m_1)}} a_j \prod_{\delta | N} (-i/\delta)^{\frac{r_\delta}{2}} f(\tilde{r}^{(j)})(\tau) \equiv 0.$$

Now we apply Lemma 2.1 to conclude that

$$\sum_{\substack{1 \le j \le n \\ r^{(j)} \in S(k)}} a_j \prod_{\delta | N} (-i/\delta)^{\frac{r_\delta}{2}} f(\tilde{r}^{(j)})(\tau) \equiv 0$$

for all $k \in \{m_1, \dots, m_2\}$. Multiplying with $\tau^{k/2}$ and applying again the involution $\tau \mapsto -1/(N\tau)$ to both sides of the last equation we obtain

$$\sum_{\substack{1 \le j \le n \\ r^{(j)} \in S(k)}} a_j f(r^{(j)})(\tau) \equiv 0 \tag{5}$$

for all $k \in \{m_1, \dots, m_2\}$. Summarizing, we have shown that to prove (1) is equivalent to prove (5) for all

$$k \in \Big\{ \min_{1 \le j \le n} \sum_{\delta | N} r_\delta^{(j)}, \dots, \max_{1 \le j \le n} \sum_{\delta | N} r_\delta^{(j)} \Big\} \tag{6}$$

Therefore without loss of generality we concern ourselves with proving identities of the type (5) for all k as in (6). Hence we can from now on restrict the input to our algorithm to be of the type (5).

If for a given k there is no j with $r^{(j)} \in S(k)$, then (5) is trivially 0 and there is nothing to do or there exists $m_k \in \{1, \dots, n\}$ such that $r^{(m_k)} \in S(k)$ and we divide (5) by $f(r^{(m_k)})(\tau)$ and obtain

$$\sum_{\substack{1 \le j \le n \\ s^{(j)} \in S(0)}} a_j f(s^{(j)})(\tau) \equiv 0$$

where $s^{(j)} := r^{(j)} - r^{(m_k)}$. We call the above identity an identity of weight zero.

The structure of this paper is as follows. In Section 3 we split weight zeros identities into further smaller identities which we call “almost modular identities”. In Section 4 we split almost modular identities into further smaller identities which we call “modular identities”. In Section 5 we give an algorithm for proving modular identities and conclude with a simple example.

3. Weight Zero Identities

The input to our algorithm is $n \in \mathbb{N}$, $r^{(j)} \in R(N)$ with $r^{(j)} \in S(0)$ and $a_j \in \mathbb{Q}$ for $j = 1, \dots, n$; the output is true or false depending whether

$$\sum_{1 \le j \le n} a_j f(r^{(j)})(\tau) \equiv 0, \tag{7}$$

is true or false. For $k \in \{0, \dots, 23\}$ we define

$$S_1(k) := \{r \in S(0) : \sum_{\delta | N} \delta r_\delta \equiv k \pmod{24}\}.$$

Note that if $\tau \mapsto \tau + 1$ then $\eta(\tau) \mapsto e^{\frac{\pi i}{12}}\eta(\tau)$ and $f(r)(\tau) \mapsto e^{\pi i \frac{\sum_{\delta|N} \delta r_\delta}{12}} f(r)(\tau)$. Hence applying $\tau \mapsto \tau + 1$ to (7) gives

$$\sum_{1 \le j \le n} a_j e^{\pi i \frac{\sum_{\delta|N} \delta r_\delta^{(j)}}{12}} f(r^{(j)})(\tau) \equiv 0.$$

which is equivalent to

$$\sum_{k=0}^{23} e^{\frac{\pi i k}{12}} \sum_{\substack{1 \le j \le n \\ r^{(j)} \in S_1(k)}} a_j f(r^{(j)})(\tau) \equiv 0.$$

Applying $\tau \mapsto \tau + 1$ to the above equation m times we obtain

$$\sum_{k=0}^{23} e^{\frac{\pi i k m}{12}} \sum_{\substack{1 \le j \le n \\ r^{(j)} \in S_1(k)}} a_j f(r^{(j)})(\tau) \equiv 0.$$

Writing

$$F_k(\tau) :\equiv \sum_{\substack{1 \le j \le n \\ r^{(j)} \in S_1(k)}} a_j f(r^{(j)})(\tau)$$

we have

$$\sum_{k=0}^{23} e^{\frac{\pi i k m}{12}} F_k(\tau) \equiv 0$$

for $m = 0, \ldots, 23$ which in matrix form may be written as

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ e^{\frac{2\cdot 0\pi i}{24}} & e^{\frac{2\cdot 1\pi i}{24}} & e^{\frac{2\cdot 2\pi i}{24}} & \cdots & e^{\frac{2\cdot 23\pi i}{24}} \\ e^{\frac{4\cdot 0\pi i}{24}} & e^{\frac{4\cdot 1\pi i}{24}} & e^{\frac{4\cdot 2\pi i}{24}} & \cdots & e^{\frac{4\cdot 23\pi i}{24}} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ e^{\frac{46\cdot 0\pi i}{24}} & e^{\frac{46\cdot 1\pi i}{24}} & e^{\frac{46\cdot 2\pi i}{24}} & \cdots & e^{\frac{46\cdot 23\pi i}{24}} \end{pmatrix} \begin{pmatrix} F_0(\tau) \\ F_1(\tau) \\ F_2(\tau) \\ \vdots \\ F_{23}(\tau) \end{pmatrix} \equiv 0.$$

This is the transpose of a Vandermonde matrix with nonzero determinant independent of τ. Therefore $F_k(\tau) \equiv 0$ for $k = 0, \ldots, 23$ which is equivalent to

$$\sum_{\substack{1\le j\le n \\ r^{(j)}\in S_1(k)}} a_j f(r^{(j)})(\tau) \equiv 0 \tag{8}$$

for $k = 0, \ldots, 23$. We apply $\tau \mapsto -1/(N\tau)$ to (8) and obtain

$$\sum_{\substack{1\le j\le n \\ r^{(j)}\in S_1(k)}} \tilde{a}_j f(\tilde{r}^{(j)})(\tau) \equiv 0 \tag{9}$$

where

$$\tilde{a}_j := a_j \prod_{\delta|N} (-i/\delta)^{\frac{r_\delta^{(j)}}{2}}.$$

For $k, \ell \in \{0, \ldots, 23\}$ we define

$$S_2(k,\ell) := \{r \in S_1(k) : \sum_{\delta|N} \delta \tilde{r}_\delta \equiv \ell \pmod{24}\}.$$

We apply the same reasoning as above to (9) and conclude that (9) is equivalent to

$$\sum_{\substack{1\le j\le n \\ r^{(j)}\in S_2(k,\ell)}} \tilde{a}_j f(\tilde{r}^{(j)})(\tau) \equiv 0 \tag{10}$$

for $\ell = 0, \ldots, 23$. Applying again the involution $\tau \mapsto -1/(N\tau)$ to (10) gives

$$\sum_{\substack{1\le j\le n \\ r^{(j)}\in S_2(k,\ell)}} a_j f(r^{(j)})(\tau) \equiv 0. \tag{11}$$

Summarizing, we have proven that one can prove a weight zero identity (7) to be true or false if we can prove an identity of type (11) to be true or false. Dividing identity (11) by any nonzero term $f(r^{(d)})(\tau)$ we obtain the identity:

$$\sum_{\substack{1\le j\le n \\ s^{(j)}\in S_2(0,0)}} a_j f(s^{(j)})(\tau) \equiv 0 \tag{12}$$

where $s^{(j)} := r^{(j)} - r^{(d)}$ and $\sum_{\delta|N} s_\delta^{(j)} = 0$, recalling the assumption on the input for (7).

We call identities of the type (12) almost modular identities.

4. Almost Modular Identities

In view of (12), the input to our algorithm is $n \in \mathbb{N}$, $r^{(j)} \in R(N)$ with

$$r^{(j)} \in S_2(0,0)$$

and $a_j \in \mathbb{Q}$ for $j = 1, \dots, n$; the output is true or false depending whether

$$\sum_{1 \le j \le n} a_j f(r^{(j)})(\tau) \equiv 0, \tag{13}$$

is true or false. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, (the group of 2×2 matrices over the integers with determinant equal to one). If $a, c > 0$ and $\gcd(a, 6) = 1$, Newman [4] proved

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) \equiv \left(\frac{c}{a}\right) e^{-\frac{\pi i a}{12}(c-b-3)} (-i(c\tau + d))^{1/2} \eta(\tau),$$

where $\left(\frac{c}{a}\right)$ is the Legendre-Jacobi symbol. If, in addition, we assume that $c \equiv 0 \pmod{N}$ we obtain

$$\begin{aligned}
f(r^{(j)})\Big(\frac{a\tau + b}{c\tau + d}\Big) &\equiv f(r^{(j)})\Big(\frac{a(\delta\tau) + \delta b}{\frac{c}{\delta}(\delta\tau) + d}\Big) \\
&\equiv \prod_{\delta|N} \Big(\frac{c/\delta}{a}\Big)^{r_\delta^{(j)}} e^{-\frac{\pi i a}{12}(\sum_{\delta|N} c r_\delta^{(j)}/\delta - b\sum_{\delta|N} \delta r_\delta^{(j)} - 3\sum_{\delta|N} r_\delta^{(j)})} f(r^{(j)})(\tau) \\
&\equiv \prod_{\delta|N} \Big(\frac{\delta c}{a}\Big)^{r_\delta^{(j)}} e^{-\frac{\pi i a}{12}(c/N \sum_{\delta|N} \delta \tilde{r}_\delta^{(j)} - b\sum_{\delta|N} \delta r_\delta^{(j)} - 3\sum_{\delta|N} r_\delta^{(j)})} f(r^{(j)})(\tau) \\
&\equiv \Big(\frac{\prod_{\delta|N} \delta^{|r_\delta^{(j)}|}}{a}\Big) f(r^{(j)})(\tau),
\end{aligned}$$

for $j = 1, \dots, n$. Let $p_0, p_1, \dots, p_n$ be the primes dividing N. For $\overline{e} = (e_0, \dots, e_n) \in \{0,1\}^{n+1}$ we define

$$S_3(\overline{e}) := \{r \in S_2(0,0) : \prod_{\delta|N} \delta^{|r_\delta^{(j)}|}/(p_0^{e_0} \cdots p_n^{e_n}) \text{ is a square. }\}.$$

We may write (13) as

$$\sum_{1\le j\le n} a_j f(r^{(j)})(\tau) \equiv \sum_{\overline{e}\in\{0,1\}^{n+1}} F(\overline{e})(\tau) \equiv 0,$$

where

$$F(\overline{e})(\tau) :\equiv \sum_{\substack{1\le j\le n\\ r^{(j)}\in S_3(\overline{e})}} a_j f(r^{(j)})(\tau).$$

Lemma 4.1. *Let $P_1,\dots,P_k$ be pairwise different odd primes, then for every $\mu_0,\mu_1,\dots,\mu_k\in\{-1,1\}$ there exist an $a\in\mathbb{N}$, $\gcd(a,6)=1$ such that $\left(\frac{P_i}{a}\right)=\mu_i$ for $i=1,\dots,k$ and $\left(\frac{2}{a}\right)=\mu_0$.*

Proof. By Chinese remaindering we can solve the system

$$\begin{array}{rcll} a & \equiv & v_0 & \pmod 8\\ a & \equiv & v_1 & \pmod{P_1}\\ \vdots & \vdots & \vdots & \\ a & \equiv & v_k & \pmod{P_k}. \end{array}$$

Here the v_i are such that $\left(\frac{v_i}{P_i}\right)=\mu_i$ for $i=1,\dots,k$ and $v_0=1$ if $\mu_0=1$ and $v_0=5$ if $\mu_0=-1$. In this case $\left(\frac{P_i}{a}\right)=(-1)^{\frac{P_i-1}{2}\frac{a-1}{2}}\left(\frac{a}{P_i}\right)=\mu_i$ and $\left(\frac{2}{a}\right)=\mu_0$. $\square$

Let $(m_0,\dots,m_n)\in\{1,-1\}^{n+1}$ be fixed. Without loss of generality assume for the given primes that $p_0<\dots<p_n$. If $p_0=2$ apply Lemma 4.1 with $k=n$, $P_i=p_i$ for $i=1,\dots,k$ and $\mu_i=m_i$ for $i=0,\dots,k$. If $p_0\neq 2$ then apply Lemma 4.1 with $k=n+1$, $P_i=p_{i-1}$, $i=1,\dots,k$ and $\mu_i=m_{i-1}$ for $i=1,\dots,k$, then the $a=a(m_0,\dots,m_n)\in\mathbb{N}$ given by the lemma is such that $\left(\frac{p_i}{a}\right)=m_i$ for $i=0,\dots,n$. Let b,c,d with $N|c$ and $c>0$ be such that $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$ (note that $\gcd(a,6N)=1$ because of $\left(\frac{p_i}{a}\right)\neq 0$). Then applying $\tau\mapsto\frac{a\tau+b}{c\tau+d}$ to the identity (13) we obtain:

$$\sum_{\overline{e}\in\{0,1\}^{n+1}} \overline{m}^{\overline{e}}\cdot F(\overline{e})(\tau)\equiv 0,$$

where for $\overline{x}\in\{0,1\}^{n+1}$ and $\overline{y}\in\{-1,1\}^{n+1}$ we define

$$\overline{y}^{\overline{x}} := y_0^{x_0}\dots y_n^{x_n}.$$

Hence for each $\overline{m}\in\{-1,1\}^{n+1}$ we obtain a new identity. This gives in total 2^{n+1} identities. Let $\overline{m_i}=(m_{0,i},\dots,m_{n,i})\in\{-1,1\}^{n+1}$ for $i=1,\dots,2^{n+1}$ be all the elements of $\{-1,1\}^{n+1}$ and $\overline{e_i}=(e_{0,i},\dots,e_{n,i})\in\{0,1\}$ for $i=1,\dots,2^{n+1}$ be all the elements of $\{0,1\}^{n+1}$. Then we may write the $\nu := 2^{n+1}$ identities in matrix form as follows

$$\begin{pmatrix} m_{0,1}^{e_{0,1}}\cdots m_{n,1}^{e_{n,1}} & m_{0,1}^{e_{0,2}}\cdots m_{n,1}^{e_{n,2}} & \dots & m_{0,1}^{e_{0,\nu}}\cdots m_{n,1}^{e_{n,\nu}} \\ m_{0,2}^{e_{0,1}}\cdots m_{n,2}^{e_{n,1}} & m_{0,2}^{e_{0,2}}\cdots m_{n,2}^{e_{n,2}} & \dots & m_{0,2}^{e_{0,\nu}}\cdots m_{n,2}^{e_{n,\nu}} \\ \vdots & \vdots & \ddots & \vdots \\ m_{0,\nu}^{e_{0,1}}\cdots m_{n,\nu}^{e_{n,1}} & m_{0,\nu}^{e_{0,2}}\cdots m_{n,\nu}^{e_{n,2}} & \dots & m_{0,\nu}^{e_{0,\mu}}\cdots m_{n,\nu}^{e_{n,\nu}}. \end{pmatrix} \begin{pmatrix} F(\overline{e_1})(\tau) \\ F(\overline{e_2})(\tau) \\ \vdots \\ F(\overline{e_\nu})(\tau) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

In the $\nu \times \nu$ matrix, which we call M, the scalar product between row i and row j equals to

$$\prod_{s=0}^{n}(1 + m_{s,i}m_{s,j}).$$

Therefore $MM^T = 2^{n+1}I$ where I is the identity matrix. In particular, M is a nonsingular matrix. Therefore

$$\sum_{\substack{1\leq j\leq n \\ r^{(j)}\in S_3(\overline{e_i})}} a_j f(r^{(j)})(\tau) \equiv F(\overline{e_i})(\tau) \equiv 0$$

for $i = 1, \dots, \nu$. Dividing out the whole identity with some nonzero term we obtain an identity of the form

$$\sum_{\substack{1\leq j\leq n \\ r^{(j)}\in S_3(\overline{e})}} a_j f(s^{(j)})(\tau) \equiv 0. \tag{14}$$

where $s^{(j)} := r^{(j)} - r^{(d)}$ for $j = 1, \dots, n$ and $r^{(d)} \in S_3(\overline{e})$ is chosen such that $a_d \neq 0$. Note that $\prod_{\delta|N} \delta^{|s_\delta^{(j)}|}$ is a square.

We call a reduced identity like (14) a modular identity which, summarizing, is an identity of the form

$$\sum_{1\leq j\leq n} a_j f(r^{(j)})(\tau) \equiv 0$$

with $a_j \in \mathbb{Q}$ and $r^{(j)} \in R(N)$ for $j \in \{1, \dots, n\}$ with the properties:

$$\sum_{\delta|N} r_\delta^{(j)} = 0, \tag{15}$$

$$\sum_{\delta|N} \delta r_\delta^{(j)} \equiv 0 \pmod{24}, \tag{16}$$

$$\sum_{\delta|N} \delta \tilde{r}_\delta^{(j)} \equiv 0 \pmod{24}, \tag{17}$$

$$\prod_{\delta|N} \delta^{|r_\delta^{(j)}|} = x_j^2 \text{ , for some } x_j \in \mathbb{Z}. \tag{18}$$

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5. Modular Identities

In this section we explain how modular identities are proven algorithmically. In order to do this we use the fact that each term in a modular identity falls into a class of holomorphic functions called modular functions. Modular functions are mapped isomorphically to meromorphic functions on a compact Riemann surface. The reason we mention this is that one can decide algorithmically if a meromorphic function on a compact Riemann surface is zero or not. Furthermore, we present a classical lemma (Lemma 5.3) that has been used by authors without proof, for example [2, p. 4827], and therefore we decided to prove it here.

Let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Newman [4] discovered the following theorem:

Theorem 5.1. *Let $r \in R(N)$, then*

$$\sum_{\delta|N} r_\delta = 0,$$

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24},$$

$$\sum_{\delta|N} \delta \tilde{r}_\delta \equiv 0 \pmod{24},$$

$$\prod_{\delta|N} \delta^{|r_\delta|} = x^2 \text{ , for some } x \in \mathbb{Z}.$$

iff

$$f(r)\Big(\frac{a\tau + b}{c\tau + d}\Big) \equiv f(r)(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

Recall that $\mathbb{H} := \{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > 0\}$. For any $r \in R(N)$, $f(r)$ is a meromorphic function on $\mathbb{H}$. By Newman's theorem the eta quotients which appear as terms in a modular identity satisfy additionally

$$f(r)\Big(\frac{a\tau+b}{c\tau+d}\Big) \equiv f(r)(\tau) \tag{19}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. We will explain now how we can prove identities involving such terms.

Following [5, p. 526], we use that holomorphic functions h on $\mathbb{H}$, with the additional property

$$h\Big(\frac{u\tau+v}{t\tau+w}\Big) \equiv h(\tau) \tag{20}$$

for all $\begin{pmatrix} u & v \\ t & w \end{pmatrix} \in \Gamma_0(N)$, have for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ a Laurent expansion in powers of $e^{2\pi i n(\gamma^{-1}\tau)/w_\gamma}$ where

$$w_\gamma := \min\Big\{h \in \mathbb{N}^* : \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma_0(N)\gamma\Big\}.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we define $\gamma\tau := \frac{a\tau+b}{c\tau+d}$ for $\tau \in \mathbb{H}$, $\gamma\infty := \frac{a}{c}$ and for $x/y \in \mathbb{Q}$ we define

$$\gamma(x/y) := \begin{cases} \infty, & \text{if } c(x/y)+d = 0, \\ \frac{a(x/y)+b}{c(x/y)+d}, & \text{otherwise.} \end{cases}$$

In this way $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$.

Since the function $f(r)$ has the property (20) because of (19) it follows that it has such a Laurent expansion for each γ. In addition, by Lemma 5.2 below it follows that for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ this Laurent expansion has finite principal part, namely:

$$f(r)(\tau) \equiv \sum_{n=d_\gamma}^{\infty} c_n(\gamma) e^{2\pi i n(\gamma^{-1}\tau)/w_\gamma}.$$

As in [5, p. 526] for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we define $\mathrm{ord}^{\gamma}_{a/c}(f(r))$ to be the smallest integer n for which $c_n(\gamma) \neq 0$. Note that $\gamma\infty = \frac{a}{c}$, and it is not difficult to check that for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma_1\infty = \gamma_2\infty = \frac{a}{c}$ we have

$$\mathrm{ord}^{\gamma_1}_{a/c}(f) = \mathrm{ord}^{\gamma_2}_{a/c}(f).$$

Hence we can define

$$\mathrm{ord}_{a/c}(f(r)) := \mathrm{ord}^{\gamma}_{a/c}(f(r)),$$

and when $a = 1$, $c = 0$ one should interpret $a/c = \infty$.

The value of $\mathrm{ord}_{a/c}(f(r))$ at $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$ can be computed by the following lemma due to Ligozat [1]:

Lemma 5.2 (Ligozat)**.** *Let $r \in R(N)$. Then*

$$\mathrm{ord}_{a/c}(f(r)) = \frac{N}{24c \cdot \gcd(c, N/c)} \sum_{\delta | N} \frac{\gcd(\delta, c)^2 r_\delta}{\delta}.$$

So our functions $f(r)$, besides having the property (19) and being holomorphic on $\mathbb{H}$, also have the property that for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ have a Laurent expansion in powers of $e^{2\pi i n(\gamma^{-1}\tau)/w_\gamma}$ with finite principal part. We call such functions modular functions (on $\Gamma_0(N)$). Denote by $X_0(N)$ the set of orbits of the action of $\Gamma_0(N)$ on $\mathbb{H}^*$. We denote the orbit of $\tau \in \mathbb{H}^*$ by $[\tau] \in X_0(N)$. We can then view a modular function f naturally as a function $\tilde{f}$ on $X_0(N)$ by defining $\tilde{f}([\tau]) := f(\tau)$ for $\tau \in \mathbb{H}$. The definition of $\tilde{f}$ at the points

$$C_0(N) := \{[\tau] : \tau \in \mathbb{Q} \cup \{\infty\}\}$$

needs to be considered separately, see [5, p. 532]. Next, the space $X_0(N)$ is next transformed into a compact topological space, by making $\mathbb{H}^*$ a topological space and giving $X_0(N)$ the quotient topology. Finally one transforms $X_0(N)$ into a compact Riemann surface. What is important is that the function $\widetilde{f(r)}$ becomes a meromorphic function on $X_0(N)$ which is holomorphic at all points from

$$U_0(N) := \{[\tau] : \tau \in \mathbb{H}\}.$$

Furthermore to each meromorphic function $\tilde{f}$ on a compact Riemann surface one can assign an order to $\tilde{f}$ at each point $[\tau] \in X_0(N)$ and we denote this by $\mathrm{ord}_{[\tau]}(\tilde{f})$. It turns out that $\mathrm{ord}_{[\tau]}(\tilde{f}) = \mathrm{ord}_\tau(f)$ for every $\tau \in \mathbb{Q} \cup \{\infty\}$.

The reason we want to view a modular function f as meromorphic function $\tilde{f}$ on a compact Riemann surface is that we can then use an important theorem that applies to nonzero meromorphic functions on a compact Riemann surface. Namely, if $\tilde{f} \neq 0$ is a meromorphic function on a compact Riemann surface then the number of poles of $\tilde{f}$ equal to the number of zeros of $\tilde{f}$, more precisely, for our case this means $\sum_{[\tau] \in X_0(N)} \mathrm{ord}_{[\tau]}(\tilde{f}) = 0$, see [3, Prop. 4.12]. Note that $X_0(N)$ is the disjoint union of $U_0(N)$ and $C_0(N)$, and as we mentioned above $\mathrm{ord}_{[\tau]}(\tilde{f}) \geq 0$

for $[\tau] \in U_0(N)$. Therefore

$$\begin{aligned} 0 &= \sum_{[\tau]\in X_0(N)} \mathrm{ord}_{[\tau]}(\tilde{f}) = \sum_{[\tau]\in U_0(N)} \mathrm{ord}_{[\tau]}(\tilde{f}) + \sum_{[\tau]\in C_0(N)} \mathrm{ord}_{[\tau]}(\tilde{f}) \\ &\geq \sum_{[\tau]\in C_0(N)} \mathrm{ord}_{[\tau]}(\tilde{f}). \end{aligned}$$

Note that this translates into

$$\sum_{\tau\in S} \mathrm{ord}_{[\tau]}(\tilde{f}) \leq 0 \tag{21}$$

where S is a complete set of representatives of $C_0(N)$, that is $C_0(N) = \{[\tau] : \tau \in S\}$ such that for every $x_1, x_2 \in S$ we have $[x_1] \neq [x_2]$.

Such a complete set of representatives S can be computed by using the following lemma.

Lemma 5.3. *Let $S \subseteq \mathbb{Q}$ be defined by $S := \cup_{d|N} S_d$ where S_d is the unique subset of $\{a/d : a \in \{1,\dots,d\}, \gcd(a,d) = 1\}$ with the property that for every $x \in \{1,\dots,\gcd(d,N/d)\}$ with $\gcd(x,\gcd(d,N/d)) = 1$ there exists an unique $a/d \in S_d$ such that $a \equiv x \pmod{\gcd(d,N/d)}$. Then S is a complete set of representatives of $C_0(N)$.*

Proof. We split the proof into three smaller parts.

(A). For $i = 1,2$, let $a_i, c_i \in \mathbb{Z}$ with $\gcd(a_i,c_i) = 1$. Then there exists $\gamma \in \Gamma_0(N)$ such that $\gamma\frac{a_1}{c_1} = \frac{a_2}{c_2}$ iff there exist $b_i, d_i \in \mathbb{Z}$ with $a_i d_i - b_i c_i = 1$ such that $d_1c_2 - d_2c_1 \equiv 0 \pmod{\gcd(N, c_1c_2)}$.

Proof of (A): Assume that there exists $\gamma \in \Gamma_0(N)$ such that $\gamma\frac{a_1}{c_1} = \frac{a_2}{c_2}$. By the extended Euclidean algorithm there exist b_i, d_i be such that $a_id_i - b_ic_i = 1$. Set $\gamma_i := \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. Then $\gamma_i\infty = \frac{a_i}{c_i}$ which implies that $\gamma\gamma_1\infty = \gamma_2\infty$ and $\gamma_2^{-1}\gamma\gamma_1\infty = \infty$. Consequently, $\gamma_2^{-1}\gamma\gamma_1 = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ for some $h \in \mathbb{Z}$. Multiplying γ_2 to the left and γ_1^{-1} to the right we obtain

$$\gamma = \begin{pmatrix} * & * \\ d_1c_2 - d_2c_1 + hc_1c_2 & * \end{pmatrix}.$$

In particular, since $\gamma \in \Gamma_0(N)$ it follows that

$$d_1c_2 - d_2c_1 + hc_1c_2 \equiv 0 \pmod{N},$$

which implies that $d_1c_2 - d_2c_1 \equiv 0 \pmod{\gcd(N, c_1c_2)}$.

Now assume that there exist $c_i, d_i \in \mathbb{Z}$ such that $a_i d_i - b_i c_i = 1$ and $d_1 c_2 - d_2 c_1 \equiv 0 \pmod{\gcd(N, c_1 c_2)}$. Then for some $k \in \mathbb{Z}$ we have $d_1c_2 - d_2c_1 - k\gcd(N, c_1c_2) = 0$, by the extended Euclidean algorithm there exist $u, v \in \mathbb{Z}$ such that $uc_1c_2 + vN = \gcd(N, c_1c_2)$ and consequently $d_1c_2 - d_2c_1 - kuc_1c_2 = kvN$. Set $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, then

$$\gamma := \gamma_2 \begin{pmatrix} 1 & ku \\ 0 & 1 \end{pmatrix} \gamma_1^{-1} = \begin{pmatrix} * & * \\ d_1c_2 - d_2c_1 - kuc_1c_2 & * \end{pmatrix}.$$

Hence $\gamma \in \Gamma_0(N)$ and one verifies $\gamma \frac{a_1}{c_1} = \frac{a_2}{c_2}$.

(B). For all $\frac{a_1}{c_1} \in \mathbb{Q} \cup \{\infty\}$ there exist $u \in \mathbb{S}$ and $\gamma \in \Gamma_0(N)$ such that $\gamma \frac{a_1}{c_1} = u$.

Note: Here we interpret $\infty = \frac{1}{0}$.

Proof of (B): Let $b_1, d_1 \in \mathbb{Z}$ be such that $a_1d_1 - b_1c_1 = 1$. Set $c_2 := \gcd(c_1, N)$ and choose $a_2 \in \mathbb{Z}$ defined uniquely by the property $a_2 \equiv a_1 \frac{c_1}{c_2} \pmod{\gcd(N/c_2, c_2)}$ and $a_2/c_2 \in S$. Let b_2, d_2 be integers such that $a_2d_2 - b_2c_2 = 1$. Then

$$a_1\frac{c_1}{c_2} - a_2 \equiv 0 \pmod{\gcd(N/c_2, c_2)} \Rightarrow d_2\frac{c_1}{c_2} - d_1 \equiv 0 \pmod{\gcd(N/c_2, c_2)}$$
$$\Rightarrow d_2\frac{c_1}{c_2} - d_1 \equiv 0 \pmod{\gcd(N/c_2, c_1)} \Rightarrow d_2c_1 - d_1c_2 \equiv 0 \pmod{\gcd(N, c_1c_2)}.$$

This by (A) implies that there exist $\gamma \in \Gamma_0(N)$ such that $\gamma \frac{a_1}{c_1} = \frac{a_2}{c_2}$.

(C). Let $\frac{a_1}{c_1}, \frac{a_2}{c_2} \in S$. If there is $\gamma \in \Gamma_0(N)$ such that $\gamma \frac{a_1}{c_1} = \frac{a_2}{c_2}$, then $\frac{a_1}{c_1} = \frac{a_2}{c_2}$.

Proof of (C): Assume that there exists $\gamma \in \Gamma_0(N)$ such that $\gamma \frac{a_1}{c_1} = \frac{a_2}{c_2}$, then by (A) there exist $b_i, d_i \in \mathbb{Z}$ with $a_id_i - b_ic_i = 1$ such that $d_2c_1 - c_1d_2 \equiv 0 \pmod{\gcd(N, c_1c_2)}$. Since $c_1, c_2 | N$, we have $c_1 | c_2$ and $c_2 | c_1$, and thus $c_1 = c_2 := c$. This implies $c(d_2 - d_1) \equiv 0 \pmod{\gcd(N, c^2)}$ which is equivalent to $d_2 - d_1 \equiv 0 \pmod{\gcd(N/c, c)}$, which is equivalent to $a_2 \equiv a_1 \pmod{\gcd(N/c, c)}$ and by the definition of S we have $a_1 = a_2$. □

Example: We want to prove the modular identity:

$$1 - \frac{\eta(28\tau)\eta(7\tau)^2\eta(4\tau)\eta(\tau)^2}{\eta(14\tau)^3\eta(2\tau)^3} - 2\frac{\eta(28\tau)^2\eta(7\tau)\eta(4\tau)^2\eta(\tau)}{\eta(14\tau)^3\eta(2\tau)^3} \equiv 0. \tag{22}$$

This may be rewritten as:

$$1 - f(r^{(1)})(\tau) - 2f(r^{(2)})(\tau) \equiv 0$$

where $r^{(1)}, r^{(2)} \in R(28)$ are defined by

$$(r_1^{(1)}, r_2^{(1)}, r_4^{(1)}, r_7^{(1)}, r_{14}^{(1)}, r_{28}^{(1)}) := (2, -3, 1, 2, -3, 1)$$

and

$$(r_1^{(2)}, r_2^{(2)}, r_4^{(2)}, r_7^{(2)}, r_{14}^{(2)}, r_{28}^{(2)}) := (1, -3, 2, 1, -3, 2).$$

Note that $r^{(1)}$ and $r^{(2)}$ satisfy (15)-(18) for $N = 28$. Next note that $f(\tilde{r^{(1)}})$ and $f(\tilde{r^{(2)}})$ are meromorphic functions on $X_0(28)$. We have by Lemma 5.3 that

$$\{[1], [1/2], [1/4], [1/7], [1/14], [1/28]\} = C_0(28).$$

By Ligozat's theorem:

$$\begin{aligned}
\mathrm{ord}_{[1]}(f(\tilde{r^{(1)}})) &= 1\\
\mathrm{ord}_{[1/2]}(f(\tilde{r^{(1)}})) &= -1\\
\mathrm{ord}_{[1/4]}(f(\tilde{r^{(1)}})) &= 0\\
\mathrm{ord}_{[1/7]}(f(\tilde{r^{(1)}})) &= 1\\
\mathrm{ord}_{[1/14]}(f(\tilde{r^{(1)}})) &= -1
\end{aligned}$$

$$\begin{aligned}
\mathrm{ord}_{[1]}(f(\tilde{r^{(2)}})) &= 0\\
\mathrm{ord}_{[1/2]}(f(\tilde{r^{(2)}})) &= -1\\
\mathrm{ord}_{[1/4]}(f(\tilde{r^{(2)}})) &= 1\\
\mathrm{ord}_{[1/7]}(f(\tilde{r^{(2)}})) &= 0\\
\mathrm{ord}_{[1/14]}(f(\tilde{r^{(2)}})) &= -1
\end{aligned}$$

We define

$$F(\tau) := 1 - f(r^{(1)})(\tau) - 2f(r^{(2)})(\tau).$$

Hence we have

$$\begin{aligned}
&\sum_{[\tau]\in C_0(28)} \mathrm{ord}_{[\tau]}(\tilde{F})\\
&= \mathrm{ord}_{[1]}(\tilde{F}) + \mathrm{ord}_{[1/2]}(\tilde{F}) + \mathrm{ord}_{[1/4]}(\tilde{F}) + \mathrm{ord}_{[1/7]}(\tilde{F}) + \mathrm{ord}_{[1/14]}(\tilde{F}) + \mathrm{ord}_{[1/28]}(\tilde{F})\\
&\geq 0 - 1 + 0 + 0 - 1 + \mathrm{ord}_{[1/28]}(\tilde{F}).
\end{aligned}$$

In order to bound the order of $\tilde{F}$ at the point $[1/28] = [\infty]$ we compute the q-expansion of

$$F(\tau) = 0 + 0q + 0q^2 + \ldots.$$

Therefore $\mathrm{ord}_{[1/28]} \tilde{F} \geq 3$, that is $\tilde{F}$ has least a triple zero at $[1/28]$. In particular

$$\sum_{[\tau]\in C_0(28)} \mathrm{ord}_{[\tau]}(\tilde{F}) \geq -2 + 3 = 1. \tag{23}$$

Hence $\tilde{F} = 0$ because if $\tilde{F} \neq 0$ then (21) would apply which says $\sum_{[\tau]\in C_0(28)} \mathrm{ord}_{[\tau]}(\tilde{F}) \leq 0$ and this is a contradiction to (23). It follows that $\tilde{F} = 0$ and hence $F = 0$ and we have proven the identity (22).

5.1. **The Algorithm in a Nutshell.** The strategy in the above example can be applied to any modular identity $F = 0$, where the notion of modular identity is defined at the end of Section 4. First assume that $F \neq 0$. Take each term $f(r^{(i)})$ appearing in F and compute its order at each point $[\tau_j] \in C_0(N) - [\infty]$, then

$$\mathrm{ord}_{[\tau_j]}(\tilde{F}) \geq o_j := \min\{\mathrm{ord}_{[\tau_j]}(f(\widetilde{r^{(i)}})) : i \in \{1, \dots, n\}\}.$$

This implies that

$$\sum_{[\tau]\in C_0(N)} \mathrm{ord}_{[\tau]}(\tilde{F}) \geq o_1 + \dots + o_{|C_0(N)|-1} + \mathrm{ord}_{[\infty]}(\tilde{F}).$$

To obtain a contradiction to (21) we need to prove that

$$\mathrm{ord}_{[\infty]}(\tilde{F}) \geq -(o_1 + \dots + o_{|C_0(N)|-1}) + 1. \tag{24}$$

This is done by looking at the expansion of F in powers of q, if F is indeed zero then each computed coefficient in the expansion of F has to be zero. If some coefficient of F is not zero, then clearly $F \neq 0$ and we are done disproving the identity $F = 0$. Hence in case $F = 0$ we must have

$$F(\tau) = 0 + 0q + \dots + 0q^{-(o_1+\dots+o_{|C_0(N)|-1})-1} + \dots$$

which by (24) implies $\sum_{[\tau]\in C_0(N)} \mathrm{ord}_{[\tau]}(\tilde{F}) \geq 1$ contradicting (21), and therefore our assumption $F \neq 0$ is false.

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Research Institute for Symbolic Computation (RISC), Johannes Kepler University, A-4040 Linz, Austria