MK-fuzzy automata*

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Abstract

We introduce MK-fuzzy automata over a bimonoid K which is related to the fuzzification of the McCarthy-Kleene logic. Our automata are inspired by, and intend to contribute to, practical applications being in development in the LogicGuard project [15, 16]. We investigate closure properties of the class of MK-fuzzy languages accepted by MK-fuzzy automata as well as by their deterministic counterparts.

1 Introduction

Fuzzy automata constitute a special model of weighted automata but historically have been defined and studied separately, mostly inspired by the fuzzy logic theory. The original fuzzy automaton model was assigned, to words, values from the lattice $[0, 1]$ with the usual max and min operations. Later on, fuzzy automata were investigated also over more general structures like for instance lattices, residuated lattices, and l-monoids. Several real world applications are modelled by fuzzy automata. We refer the reader to [13] for fuzzy automata theory and applications, to [14] for a generalization of them and their connection to weighted automata, and to [1] for fuzzy semirings related to automata. For weighted automata theory the interesting reader should consult for instance [6, 7, 9].

On the other hand, McCarthy-Kleene logic (MK-logic for short) (cf. for instance [2, 3, 4]), a combination of the logics of McCarthy and Kleene, is a four-valued logic and it has been proved reasonable for the LogicGuard project which pursues research on security systems [11, 15, 16]. On this strand, and for the development of the fuzzification of the MK-logic and relative models, we introduce MK-fuzzy automata, and this paper is a first attempt to study these models.

Our MK-fuzzy automata assign, to words, values from the bimonoid

$$K = \{(t, f, u, e) \in [0, 1]^4 \mid t + f + u + e = 1\}$$

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where its operations, called MK-disjunction and MK-conjunction, are inspired by the fuzzification of the MK-logic. Formal series with values in K are called MK-fuzzy languages.

Classical operations in formal series over semirings cannot be defined in a unique way over bimonoids due to the lack of commutativity and distributivity properties. Notable examples are the Cauchy product and the star operation. On the other hand, in weighted automata theory over semirings (cf. for instance [6, 7]) or even more general structures with a multiplicative zero, like valuation monoids [8], the initial and terminal distributions are defined for all the states of the automaton, and the mapping assigning weights to the transitions is defined for all the transitions of the automaton. Whenever a state is assigned with the zero value by the initial (resp. the terminal) distribution, then the occurrence of this state at the beginning (resp. at the end) of a path assigns the zero weight to the whole path. Similarly, if a transition with zero weight occurs in a path, then the path gets the zero weight. If the weight structure is weaker than a semiring, for instance a bimonoid like in our case, then the lack of commutativity, distributivity, and multiplicative zero properties has a serious impact at the automata models considered over such a weight structure. For instance the value assigned by the automaton to a word cannot be defined in a unique way. Due to these difficulties, and since not any interesting bimonoid structure was known so far, there is a lack of work on weighted automata over bimonoids. According to our best knowledge the most relative works deal with automata and transducers over strong bimonoids where the first operation is commutative and there is a multiplicative zero [5, 10, 12].

For our MK-fuzzy automata, where a multiplicative zero is missing from the bimonoid K, we consider a set of initial states, a set of transitions, and a set of final states and define on these sets the initial distribution, the mapping assigning truth values to the transitions of the automaton, and the terminal distribution, respectively. Our model is nondeterministic. Since the MK-disjunction is not commutative, we require the state set of the MK-fuzzy automaton to be totally ordered. Then the paths of the automaton over any word, can be ordered according to lexicographic order, and hence we can define the value of K assigned by the MK-fuzzy automaton to the given word.

We show that the class of recognizable MK-fuzzy languages accepted by MK-fuzzy automata, is closed under MK-disjunction and inverse strict alphabetic homomorphisms. We introduce also the deterministic version of our automata and show that the class of MK-fuzzy languages accepted by these models, called deterministic recognizable, is closed under scalar MK-disjunctions. Furthermore, the Cauchy product of two deterministic recognizable MK-fuzzy languages is a recognizable MK-fuzzy language. Due to the structure of the bimonoid K, we can define several notions of supports of MK-fuzzy languages. We show that the strong support, related to the first component of the elements in K, of a deterministic recognizable MK-fuzzy language is a recognizable language.

2 Preliminaries

Let A be an alphabet, i.e., a finite nonempty set. As usually, we denote by A^* the set of all finite words over A and define $A^+ = A^* \setminus \{\varepsilon\}$, where ε is the empty word. The length of a word w, i.e., the number of the letters of w is denoted as usual by $|w|$.

Throughout the paper A will denote an alphabet.

A *bimonoid* $(K, +, \cdot, 0, 1)$ (cf. [10]) consists of a set K, two binary operations $+$ and $\cdot$ and two constant elements 0 and 1 such that $(K, +, 0)$ and $(K, \cdot, 1)$ are monoids. If the monoid $(K, +, 0)$ is commutative and 0 acts as a multiplicative zero, i.e., $k \cdot 0 = 0 \cdot k = 0$ for every $k \in K$, then the bimonoid is called *strong*. The bimonoid is denoted simply by K if the operations and the constant elements are understood. A semiring is a strong bimonoid with the property that the multiplication distributes over addition.

A bimonoid K is called zero-sum free if $k + k' = 0$ implies $k = k' = 0$, and it is called zero-divisor free if $k \cdot k' = 0$ implies $k = 0$ or $k' = 0$, for every $k, k' \in K$.

In this paper we deal with a new type of fuzzy sets. More precisely, the values belong to a subset K of the Cartesian product $[0, 1]^4 = [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ which is defined as follows:

$$K = \{(t, f, u, e) \in [0, 1]^4 \mid t + f + u + e = 1\}.$$

Due to practical applications, where our theory is motivated by (cf. [11]), we refer to the four components of the elements of K to as the *true*, *false*, *unknown*, and *error* value, respectively. We shall denote the elements of K with bold symbols and we shall call them the *truth values* of our fuzzy sets. For $\mathbf{k} = (t, f, u, e) \in K$ we shall write sometimes $x(\mathbf{k})$ for $x \in \{t, f, u, e\}$, to denote the x value of $\mathbf{k}$.

For every $\mathbf{k_1} = (t_1, f_1, u_1, e_1), \mathbf{k_2} = (t_2, f_2, u_2, e_2) \in K$ we let $\mathbf{k_3} = \mathbf{k_1} \vee \mathbf{k_2}$ and $\mathbf{k_4} = \mathbf{k_1} \wedge \mathbf{k_2}$ where $\mathbf{k_3} = (t_3, f_3, u_3, e_3)$ and $\mathbf{k_4} = (t_4, f_4, u_4, e_4)$ are defined by the relations

- $t_3 = t_1 + (f_1 + u_1)t_2$
- $f_3 = f_1 f_2$
- $u_3 = f_1 u_2 + u_1(f_2 + u_2)$
- $e_4 = e_1 + (f_1 + u_1)e_2$

and

- $t_4 = t_1 t_2$
- $f_4 = f_1 + (t_1 + u_1)f_2$
- $u_4 = t_1 u_2 + u_1(t_2 + u_2)$
- $e_4 = e_1 + (t_1 + u_1)e_2$.

It is not difficult to see that $\mathbf{k_3}, \mathbf{k_4} \in K$, therefore $\vee$ and $\wedge$ are well-defined operations on K. Indeed, let us present the proof for $\mathbf{k_4}$; the proof for $\mathbf{k_3}$ is similar. By standard computations we get $0 \leq t_4, f_4, u_4, e_4 \leq 1$. Furthermore, we calculate

$$\begin{aligned} t_4 + f_4 + u_4 + e_4 &= t_1 t_2 + f_1 + (t_1 + u_1)f_2 + t_1 u_2 + u_1(t_2 + u_2) + e_1 + (t_1 + u_1)e_2 \\ &= t_1 t_2 + f_1 + t_1 f_2 + u_1 f_2 + t_1 u_2 + u_1 t_2 + u_1 u_2 + e_1 + t_1 e_2 + u_1 e_2 \\ &= t_1(t_2 + f_2 + u_2 + e_2) + f_1 + u_1(f_2 + t_2 + u_2 + e_2) + e_1 \\ &= t_1 + f_1 + u_1 + e_2 = 1 \end{aligned}$$

as wanted.

We call $\vee$ the *MK-disjunction* (*disjunction* for simplicity) and $\wedge$ the *MK-conjunction* (*conjunction* for simplicity).

Proposition 1 *The disjunction and conjunction operations on K are associative with unit elements $\mathbf{0} = (0, 1, 0, 0)$ and $\mathbf{1} = (1, 0, 0, 0)$, respectively.*

Proof. Let $\mathbf{k_1} = (t_1, f_1, u_1, e_1), \mathbf{k_2} = (t_2, f_2, u_2, e_2), \mathbf{k_3} = (t_3, f_3, u_3, e_3) \in K$. We show firstly, the associativity property for the disjunction operation. For this we let $(\mathbf{k_1} \vee \mathbf{k_2}) \vee \mathbf{k_3} = (t, f, u, e)$ and $\mathbf{k_1} \vee (\mathbf{k_2} \vee \mathbf{k_3}) = (t', f', u', e')$. By definition, we have

$\mathbf{k_1} \vee \mathbf{k_2} = (t_1 + (f_1 + u_1)t_2, f_1 f_2, f_1 u_2 + u_1(f_2 + u_2), e_1 + (f_1 + u_1)e_2)$

and

$\mathbf{k_2} \vee \mathbf{k_3} = (t_2 + (f_2 + u_2)t_3, f_2 f_3, f_2 u_3 + u_2(f_3 + u_3), e_2 + (f_2 + u_2)e_3)$.

Furthermore, we get

$$\begin{aligned} t &= t_1 + (f_1 + u_1)t_2 + (f_1 f_2 + f_1 u_2 + u_1(f_2 + u_2))t_3 \\ &= t_1 + (f_1 + u_1)t_2 + f_1 f_2 t_3 + f_1 u_2 t_3 + u_1 f_2 t_3 + u_1 u_2 t_3 \\ &= t_1 + (f_1 + u_1)t_2 + (f_1 + u_1) f_2 t_3 + (f_1 + u_1) u_2 t_3 \\ &= t_1 + (f_1 + u_1)(t_2 + (f_2 + u_2)t_3) \\ &= t', \end{aligned}$$

$$\begin{aligned} f &= (f_1 f_2) f_3 \\ &= f_2(f_2 f_3) \\ &= f', \end{aligned}$$

$$\begin{aligned} u &= f_1 f_2 u_3 + (f_1 u_2 + u_1(f_2 + u_2))(f_3 + u_3) \\ &= f_1 f_2 u_3 + f_1 u_2 f_3 + f_1 u_2 u_3 + u_1 f_2 f_3 + u_1 f_2 u_3 + u_1 u_2 f_3 + u_1 u_2 u_3 \\ &= f_1(f_2 u_3 + u_2(f_3 + u_3)) + u_1(f_2 f_3 + f_2 u_3 + u_2(f_3 + u_3)) \\ &= u', \end{aligned}$$

$$\begin{aligned} e &= e_1 + (f_1 + u_1)e_2 + (f_1 f_2 + f_1 u_2 + u_1(f_2 + u_2))e_3 \\ &= e_1 + (f_1 + u_1)e_2 + f_1 f_2 e_3 + f_1 u_2 e_3 + u_1 f_2 e_3 + u_1 u_2 e_3 \\ &= e_1 + (f_1 + u_1)e_2 + (f_1 + u_1) f_2 e_3 + (f_1 + u_1) u_2 e_3 \\ &= e_1 + (f_1 + u_1)(e_2 + (f_2 + u_2)e_3) \\ &= e' \end{aligned}$$

which implies that $(\mathbf{k_1} \vee \mathbf{k_2}) \vee \mathbf{k_3} = \mathbf{k_1} \vee (\mathbf{k_2} \vee \mathbf{k_3})$.

Next we proceed with the associativity of conjunction. For this we let $(\mathbf{k_1} \wedge \mathbf{k_2}) \wedge \mathbf{k_3} = (\tilde{t}, \tilde{f}, \tilde{u}, \tilde{e})$ and $\mathbf{k_1} \wedge (\mathbf{k_2} \wedge \mathbf{k_3}) = (\tilde{t}', \tilde{f}', \tilde{u}', \tilde{e}')$. By definition, we have

$\mathbf{k_1} \wedge \mathbf{k_2} = (t_1 t_2, f_1 + (t_1 + u_1) f_2, t_1 u_2 + u_1(t_2 + u_2), e_1 + (t_1 + u_1)e_2)$

and

$\mathbf{k_2} \wedge \mathbf{k_3} = (t_2 t_3, f_2 + (t_2 + u_2) f_3, t_2 u_3 + u_2(t_3 + u_3), e_2 + (t_2 + u_2)e_3)$.

Then we get

$$\begin{aligned}\tilde{t} &= (t_1t_2)t_3\\ &= t_1(t_2t_3)\\ &= \tilde{t}',\end{aligned}$$

$$\begin{aligned}\tilde{f} &= f_1 + (t_1+u_1)f_2 + (t_1t_2 + t_1u_2 + u_1(t_2+u_2))f_3\\ &= f_1 + (t_1+u_1)f_2 + t_1t_2f_3 + t_1u_2f_3 + u_1t_2f_3 + u_1u_2f_3\\ &= f_1 + (t_1+u_1)f_2 + (t_1+u_1)t_2f_3 + (t_1+u_1)u_2f_3\\ &= f_1 + (t_1+u_1)f_2 + (t_1+u_1)(t_2+u_2)f_3\\ &= f_1 + (t_1+u_1)(f_2 + (t_2+u_2)f_3)\\ &= \tilde{f}',\end{aligned}$$

$$\begin{aligned}\tilde{u} &= t_1t_2u_3 + (t_1u_2 + u_1(t_2+u_2))(t_3+u_3)\\ &= t_1t_2u_3 + t_1u_2t_3 + t_1u_2u_3 + u_1t_2t_3 + u_1t_2u_3 + u_1u_2t_3 + u_1u_2u_3\\ &= t_1(t_2u_3 + u_2t_3 + u_2u_3) + u_1(t_2t_3 + t_2u_3 + u_2t_3 + u_2u_3)\\ &= t_1(t_2u_3 + u_2(t_3+u_3)) + u_1(t_2t_3 + t_2u_3 + u_2(t_3+u_3))\\ &= \tilde{u}',\end{aligned}$$

$$\begin{aligned}\tilde{e} &= e_1 + (t_1+u_1)e_2 + (t_1t_2 + t_1u_2 + u_1(t_2+u_2))e_3\\ &= e_1 + t_1e_2 + u_1e_2 + t_1t_2e_3 + t_1u_2e_3 + u_1t_2e_3 + u_1u_2e_3\\ &= e_1 + (t_1+u_1)e_2 + (t_1+u_1)t_2e_3 + (t_1+u_1)u_2e_3\\ &= e_1 + (t_1+u_1)(e_2 + t_2e_3 + u_2e_3)\\ &= e_1 + (t_1+u_1)(e_2 + (t_2+u_2)e_3)\\ &= \tilde{e}'\end{aligned}$$

and hence $(\mathbf{k_1} \wedge \mathbf{k_2}) \wedge \mathbf{k_3} = \mathbf{k_1} \wedge (\mathbf{k_2} \wedge \mathbf{k_3})$, as required.

We show now that $\mathbf{0}, \mathbf{1}$ are the unit elements of disjunction and conjunction, respectively. Indeed, we have $\mathbf{k_1} \vee \mathbf{0} = (t_1 + (f_1+u_1)0, f_11, f_10 + u_1(1+0), e_1 + (f_1+u_1)0) = (t_1, f_1, u_1, e_1)$ and $\mathbf{0} \vee \mathbf{k_1} = (0 + (1+0)t_1, 1f_1, 1u_1 + 0(f_1+u_1), 0 + (1+0)e_1) = (t_1, f_1, u_1, e_1)$. Finally, $\mathbf{k_1} \wedge \mathbf{1} = (t_11, f_1 + (t_1+u_1)0, t_10 + u_1(1+0), e_1 + (t_1+u_1)0) = (t_1, f_1, u_1, e_1)$ and $\mathbf{1} \wedge \mathbf{k_1} = (1t_1, 0 + (1+0)f_1, 1u_1 + 0(t_1+u_1), 0 + (1+0)e_1) = (t_1, f_1, u_1, e_1)$, and we are done. ■

By Proposition 1, we immediately get the next corollary.

Corollary 2 *The structure $(K, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a bimonoid.*

Nevertheless, by the following proposition we conclude that the bimonoid $(K, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is not strong.

Proposition 3 *Both the disjunction and conjunction operations on K are not commutative and idempotent. Furthermore, for every $\mathbf{k} = (t, f, u, e) \in K$ we get $\mathbf{0} \wedge \mathbf{k} = \mathbf{0}$ and $\mathbf{k} \wedge \mathbf{0} = (0, t+f+u, 0, e)$.*

Proof. Consider the elements $\mathbf{k} = (0.3, 0.2, 0.4, 0.1), \mathbf{k}' = (0.9, 0.05, 0.03, 0.02)$ of K. Then we get $\mathbf{k} \vee \mathbf{k}' = (0.84, 0.01, 0.038, 0.112)$, $\mathbf{k}' \vee \mathbf{k} = (0.924, 0.01, 0.038, 0.028)$, $\mathbf{k} \wedge \mathbf{k}' = (0.27, 0.235, 0.381, 0.114)$, $\mathbf{k}' \wedge \mathbf{k} = (0.27, 0.236, 0.381, 0.113)$, $\mathbf{k} \vee \mathbf{k} = (0.48, 0.04, 0.32, 0.16)$, and $\mathbf{k} \wedge \mathbf{k} = (0.09, 0.34, 0.4, 0.17)$.
The remaining part of our proposition is proved by a standard calculation. ■

Proposition 4 *Both the disjunction and conjunction on K do not distribute to each other.*

Proof. Let $\mathbf{k} = (t, f, u, e)$ an arbitrary element in K. Then we can easily show that $\mathbf{k} \wedge (\mathbf{0} \vee \mathbf{1}) \neq (\mathbf{k} \wedge \mathbf{0}) \vee (\mathbf{k} \wedge \mathbf{1})$ and $(\mathbf{1} \vee \mathbf{k}) \wedge \mathbf{0} \neq (\mathbf{1} \wedge \mathbf{0}) \vee (\mathbf{k} \wedge \mathbf{0})$, which imply that conjunction is neither left nor right distributive to disjunction. Similarly, we get $\mathbf{k} \vee (\mathbf{0} \wedge \mathbf{1}) \neq (\mathbf{k} \vee \mathbf{0}) \wedge (\mathbf{k} \vee \mathbf{1})$ and $(\mathbf{0} \wedge \mathbf{k}) \vee \mathbf{1} \neq (\mathbf{0} \vee \mathbf{1}) \wedge (\mathbf{k} \vee \mathbf{1})$, i.e., disjunction is neither left nor right distributive to conjunction. ■

Proposition 5 *The bimonoid K is zero-sum free and zero-divisor free.*

Proof. We show firstly that K is zero-sum free. For this let $\mathbf{k_1} = (t_1, f_1, u_1, e_1), \mathbf{k_2} = (t_2, f_2, u_2, e_2) \in K$ and assume that $\mathbf{k}_1 \vee \mathbf{k}_2 = \mathbf{0}$. Hence, we get

$$(t_1 + (f_1 + u_1)t_2,\ f_1 f_2,\ f_1 u_2 + u_1(f_2 + u_2),\ e_1 + (f_1 + u_1)e_2) = (0, 1, 0, 0),$$

i.e.,

- $t_1 + (f_1 + u_1)t_2 = 0,$
- $f_1 f_2 = 1,$
- $f_1 u_2 + u_1(f_2 + u_2) = 0,$
- $e_1 + (f_1 + u_1)e_2 = 0.$

Since $0 \leq f_1, f_2 \leq 1$ the second equality implies that $f_1 = f_2 = 1$, which in turn means that $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{0}$, as required.

Next assume that $\mathbf{k}_1 \wedge \mathbf{k}_2 = \mathbf{0}$, i.e.,

$$(t_1 t_2,\ f_1 + (t_1 + u_1)f_2,\ t_1 u_2 + u_1(t_2 + u_2),\ e_1 + (t_1 + u_1)e_2) = (0, 1, 0, 0)$$

and hence,

- $t_1 t_2 = 0,$
- $f_1 + (t_1 + u_1)f_2 = 1,$
- $t_1 u_2 + u_1(t_2 + u_2) = 0,$
- $e_1 + (t_1 + u_1)e_2 = 0.$

The first equality implies that $t_1 = 0$ or $t_2 = 0$.

i) Let $t_1 = 0$. Then, we get

  - $f_1 + u_1 f_2 = 1,$
  - $u_1 t_2 = u_1 u_2 = 0,$
  - $e_1 = u_1 e_2 = 0.$

By the second relation we get $u_1 = 0$ or $u_2 = 0$. If $u_1 = 0$ since also $e_1 = 0$, by our assumption we get $f_1 = 1$, i.e., $\mathbf{k}_1 = \mathbf{0}$.
Assume that $u_1 \neq 0$. Then by the second and third relations we get respectively, $t_2 = u_2 = 0$ and $e_2 = 0$, which implies $f_2 = 1$, i.e., $\mathbf{k}_2 = \mathbf{0}$.

ii) Let $t_1 \neq 0$ and $t_2 = 0$. Then we have

- $f_1 + (t_1 + u_1)f_2 = 1$,
- $t_1 u_2 = u_1 u_2 = 0$,
- $e_1 = t_1 e_2 = u_1 e_2 = 0$.

By the second equality we get $u_2 = 0$ and by the third one $e_2 = 0$. Taking into account our assumption, we conclude that $f_2 = 1$, and hence $\mathbf{k}_2 = \mathbf{0}$.

Therefore, in every case $\mathbf{k}_1 = \mathbf{0}$ or $\mathbf{k}_2 = \mathbf{0}$, and our proof is completed. ■

Proposition 6 *For every* $\mathbf{k} \in K$ *we have* $\mathbf{e} \vee \mathbf{k} = \mathbf{e}$ *and* $\mathbf{e} \wedge \mathbf{k} = \mathbf{e}$ *where* $\mathbf{e} = (0, 0, 0, 1)$. *Furthermore, we have*

$$\mathbf{k_1} \vee \mathbf{k_2} = \mathbf{e} \implies \mathbf{k_1} = \mathbf{e} \text{ or } \mathbf{k_2} = \mathbf{e}$$

and

$$\mathbf{k_1} \wedge \mathbf{k_2} = \mathbf{e} \implies \mathbf{k_1} = \mathbf{e} \text{ or } \mathbf{k_2} = \mathbf{e}$$

for every $\mathbf{k_1}, \mathbf{k_2} \in K$.

Proof. Our first claim is trivially proved by the definition of disjunction and conjunction operations. Next, let $\mathbf{k_1} = (t_1, f_1, u_1, e_1), \mathbf{k_2} = (t_2, f_2, u_2, e_2) \in K$ and assume firstly that $\mathbf{k_1} \vee \mathbf{k_2} = \mathbf{e}$. Then we get

- $t_1 + (f_1 + u_1)t_2 = 0$,
- $f_1 f_2 = 0$,
- $f_1 u_2 + u_1(f_2 + u_2) = 0$,
- $e_1 + (f_1 + u_1)e_2 = 1$.

Therefore
$t_1 + (f_1 + u_1)t_2 + f_1 f_2 + f_1 u_2 + u_1(f_2 + u_2) = 0$,
i.e.,
$t_1 + (f_1 + u_1)(t_2 + f_2 + u_2) = 0$
which implies
$t_1 = 0$ and $(f_1 + u_1 = 0$ or $t_2 + f_2 + u_2 = 0)$,
hence, $e_1 = 1$ or $e_2 = 1$, as required.

Finally, let $\mathbf{k_1} \wedge \mathbf{k_2} = \mathbf{e}$. Then we get

- $t_1 t_2 = 0$,
- $f_1 + (t_1 + u_1)f_2 = 0$,
- $t_1 u_2 + u_1(t_2 + u_2) = 0$,

- $e_1 + (t_1 + u_1)e_2 = 1$.

Thus

$$t_1 t_2 + f_1 + (t_1 + u_1) f_2 + t_1 u_2 + u_1 (t_2 + u_2) = 0,$$

i.e.,

$$f_1 + (t_1 + u_1)(t_2 + f_2 + u_2) = 0$$

which implies

$$f_1 = 0 \text{ and } (t_1 + u_1 = 0 \text{ or } t_2 + f_2 + u_2 = 0),$$

hence,

$$e_1 = 1 \text{ or } e_2 = 1,$$

and our proof is completed. ■

An *MK-fuzzy language over A and K* is a mapping $s : A^* \to K$. The *strong support of* s is the language $\mathrm{stgsupp}(s) = \{w \in A^* \mid t(s(w)) \neq 0\}$. For every $w \in A^*$ the MK-fuzzy language $\overline{w}$ is determined by $\overline{w}(u) = \mathbf{1}$ if $u = w$, and $\overline{w}(u) = \mathbf{0}$ otherwise. The *constant* MK-fuzzy language $\widetilde{\mathbf{k}}$ ($\mathbf{k} \in K$) is defined, for every $w \in A^*$, by $\widetilde{\mathbf{k}}(w) = \mathbf{k}$. We shall denote by $K\langle\langle A^* \rangle\rangle$ the class of all MK-fuzzy languages over A and K.

Let $s, r \in K\langle\langle A^* \rangle\rangle$ and $\mathbf{k} \in K$. The *MK-disjunction* (or simply *disjunction*) $s \vee r$, the *MK-conjunction* (or simply *conjunction*) $s \wedge r$, and the *scalar MK-conjunctions* (simply *scalar conjunctions*) $\mathbf{k} \wedge s$ and $s \wedge \mathbf{k}$ are defined as follows: $s \vee r(w) = s(w) \vee r(w)$, $s \wedge r(w) = s(w) \wedge r(w)$, and $(\mathbf{k} \wedge s)(w) = \mathbf{k} \wedge s(w)$, $(s \wedge \mathbf{k})(w) = s(w) \wedge \mathbf{k}$ for every $w \in A^*$. Since the disjunction and conjunction operations among MK-fuzzy languages are defined elementwise, we can easily show that the properties of the structure $\left(K\langle\langle A^* \rangle\rangle, \vee, \wedge, \widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}\right)$ are inherited by the properties of the structure $(K, \vee, \wedge, \mathbf{0}, \mathbf{1})$, hence $\left(K\langle\langle A^* \rangle\rangle, \vee, \wedge, \widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}\right)$ is a bimonoid.

Next, we define the *Cauchy product rs of* $r, s \in K\langle\langle A^* \rangle\rangle$ in the following way. For every $w = a_0 \dots a_{n-1} \in A^*$ with $a_0, \dots, a_{n-1} \in A$ we let

$$rs(w) = (r(\varepsilon) \wedge s(a_0 \dots a_{n-1})) \vee (r(a_0) \wedge s(a_1 \dots a_{n-1})) \vee \dots \vee (r(a_0 \dots a_{n-1}) \wedge s(\varepsilon)).$$

Since disjunction and conjunction are not commutative, and they do not distribute to each other, the Cauchy product is not associative as we show in the next proposition.

Proposition 7 *The Cauchy product operation is not associative.*

Proof. Let $\mathbf{k} = (t, f, u, e) \in K$. By Proposition 3 we get $\mathbf{k} \wedge \mathbf{0} = (0, t + f + u, 0, e)$ and we can easily see that the false value of the element in K resulting by applying n-times the disjunction operation on $\mathbf{k} \wedge \mathbf{0}$ with itself, is $(t + f + u)^{n+1}$. Consider the constant MK-fuzzy languages $\widetilde{\mathbf{k}}, \widetilde{\mathbf{0}}$, and $\widetilde{\mathbf{1}}$, and the word $w = a_0 a_1 \in A^*$. Then we have

$$\begin{aligned}
\widetilde{\mathbf{k}}(\widetilde{\mathbf{0}}\widetilde{\mathbf{1}})(w) &= \left(\widetilde{\mathbf{k}}(\varepsilon) \wedge \widetilde{\mathbf{0}}\widetilde{\mathbf{1}}(a_0 a_1)\right) \vee \left(\widetilde{\mathbf{k}}(a_0) \wedge \widetilde{\mathbf{0}}\widetilde{\mathbf{1}}(a_1)\right) \vee \left(\widetilde{\mathbf{k}}(a_0 a_1) \wedge \widetilde{\mathbf{0}}\widetilde{\mathbf{1}}(\varepsilon)\right) \\
&= \left(\widetilde{\mathbf{k}}(\varepsilon) \wedge \left(\left(\widetilde{\mathbf{0}}(\varepsilon) \wedge \widetilde{\mathbf{1}}(a_0 a_1)\right) \vee \left(\widetilde{\mathbf{0}}(a_0) \wedge \widetilde{\mathbf{1}}(a_1)\right) \vee \left(\widetilde{\mathbf{0}}(a_0 a_1) \wedge \widetilde{\mathbf{1}}(\varepsilon)\right)\right)\right) \\
&\quad \vee \left(\widetilde{\mathbf{k}}(a_0) \wedge \left(\left(\widetilde{\mathbf{0}}(\varepsilon) \wedge \widetilde{\mathbf{1}}(a_1)\right) \vee \left(\widetilde{\mathbf{0}}(a_1) \wedge \widetilde{\mathbf{1}}(\varepsilon)\right)\right)\right) \vee \left(\widetilde{\mathbf{k}}(a_0 a_1) \wedge \left(\widetilde{\mathbf{0}}(\varepsilon) \wedge \widetilde{\mathbf{1}}(\varepsilon)\right)\right) \\
&= (\mathbf{k} \wedge (\mathbf{0} \vee \mathbf{0} \vee \mathbf{0})) \vee (\mathbf{k} \wedge (\mathbf{0} \vee \mathbf{0})) \vee (\mathbf{k} \wedge \mathbf{0}) \\
&= (\mathbf{k} \wedge \mathbf{0}) \vee (\mathbf{k} \wedge \mathbf{0}) \vee (\mathbf{k} \wedge \mathbf{0}) \\
&= \left(\dots, (t + f + u)^3, \dots, \dots\right).
\end{aligned}$$

On the other hand, we get

$$
\begin{aligned}
(\widetilde{\mathbf{k}}\widetilde{\mathbf{0}})\widetilde{\mathbf{1}}(w) &= \left(\widetilde{\mathbf{k}}\widetilde{\mathbf{0}}(\varepsilon) \wedge \widetilde{\mathbf{1}}(a_0a_1)\right) \vee \left(\widetilde{\mathbf{k}}\widetilde{\mathbf{0}}(a_0) \wedge \widetilde{\mathbf{1}}(a_1)\right) \vee \left(\widetilde{\mathbf{k}}\widetilde{\mathbf{0}}(a_0a_1) \wedge \widetilde{\mathbf{1}}(\varepsilon)\right) \\
&= \left(\left(\widetilde{\mathbf{k}}(\varepsilon) \wedge \widetilde{\mathbf{0}}(\varepsilon)\right) \wedge \widetilde{\mathbf{1}}(a_0a_1)\right) \vee \left(\left(\left(\widetilde{\mathbf{k}}(\varepsilon) \wedge \widetilde{\mathbf{0}}(a_0)\right) \vee \left(\widetilde{\mathbf{k}}(a_0) \wedge \widetilde{\mathbf{0}}(\varepsilon)\right)\right) \wedge \widetilde{\mathbf{1}}(a_1)\right) \\
&\quad \vee \left(\left(\left(\widetilde{\mathbf{k}}(\varepsilon) \wedge \widetilde{\mathbf{0}}(a_0a_1)\right) \vee \left(\widetilde{\mathbf{k}}(a_0) \wedge \widetilde{\mathbf{0}}(a_1)\right) \vee \left(\widetilde{\mathbf{k}}(a_0a_1) \wedge \widetilde{\mathbf{0}}(\varepsilon)\right)\right) \wedge \widetilde{\mathbf{1}}(\varepsilon)\right) \\
&= (\mathbf{k} \wedge \mathbf{0}) \vee ((\mathbf{k} \wedge \mathbf{0}) \vee (\mathbf{k} \wedge \mathbf{0})) \vee ((\mathbf{k} \wedge \mathbf{0}) \vee (\mathbf{k} \wedge \mathbf{0}) \vee (\mathbf{k} \wedge \mathbf{0})) \\
&= \left(\ldots, (t+f+u)^6, \ldots, \ldots\right).
\end{aligned}
$$

Hence, we get $\widetilde{\mathbf{k}}(\widetilde{\mathbf{0}}\widetilde{\mathbf{1}})(w) \neq (\widetilde{\mathbf{k}}\widetilde{\mathbf{0}})\widetilde{\mathbf{1}}(w)$ which implies that $\widetilde{\mathbf{k}}(\widetilde{\mathbf{0}}\widetilde{\mathbf{1}}) \neq (\widetilde{\mathbf{k}}\widetilde{\mathbf{0}})\widetilde{\mathbf{1}}$, and our proof is completed. ■

The last proposition shows that the nth-iteration of an MK-fuzzy language cannot be defined in a unique way. We define it as follows. Let $r \in K\langle\langle A^*\rangle\rangle$. The *$n$th-iteration of r* is the MK-fuzzy language r^n which is determined for every $n \geq 0$ by

$r^0 = \overline{\varepsilon}$, and

$r^{n+1} = r^n r$ for $n \geq 0$.

By the above definition, for $n = 0$ we derive $r^1 = r$. The *star r^* of r* is defined for every $w \in A^*$ by

$r^*(w) = r^0(w) \vee r^1(w) \vee \ldots \vee r^{|w|}(w)$.

Let B be another alphabet and $h : A^* \to B^*$ be a strict alphabetic homomorphism, i.e., $h(a) \in B$ for every $a \in A$. If $r \in K\langle\langle B^*\rangle\rangle$, then the MK-fuzzy language $h^{-1}(r) \in K\langle\langle A^*\rangle\rangle$ is determined by $h^{-1}(r)(w) = r(h(w))$ for every $w \in A^*$.

3 MK-fuzzy automata

In this section we introduce the model of MK-fuzzy automata over A and K and investigate closure properties of the class of their behaviors. Due to the lack of commutativity, distributivity, and multiplicative zero properties of K, we assume that our automata are equipped with a set of initial states, a set of transitions, and a set of final states and define on these sets the initial distribution, the mapping assigning truth values to the transitions of the automaton, and the terminal distribution, respectively. Furthermore, we require that the state set of the automaton is totally ordered. We show that the class of MK-fuzzy languages accepted by MK-fuzzy automata is closed under MK-disjunction and inverse strict alphabetic homomorphisms. We consider also the deterministic counterpart of our model and show that the class of MK-fuzzy languages accepted by deterministic MK-fuzzy automata is closed under scalar MK-disjunctions. We prove that the Cauchy product of two deterministic recognizable MK-fuzzy languages is a recognizable MK-fuzzy language. For this we need to introduce the notion of an unambiguous MK-fuzzy automaton. Furthermore, we show that the strong support of a deterministic recognizable MK-fuzzy language is a recognizable language.

Definition 8 *An* MK-fuzzy automaton over A *and* K *is a seven-tuple* $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ *where* Q *is the* finite state set *which is assumed to be totally ordered,* I *is the* set of initial states, $T \subseteq Q \times A \times Q$ *is the* set of transitions, F *is the* set of final states, $in : I \to K$ *is the* initial distribution*, $wt: T \to K$ is a mapping assigning* truth values *to the transitions of the automaton, and $ter: F \to K$ is the* final distribution*.*

Let $w = a_0 \ldots a_{n-1}$ be a word over A with $a_0, \ldots, a_{n-1} \in A$. A *path* $P_w^{(\mathcal{A})}$ (or simply P_w if the automaton is understood) *of $\mathcal{A}$ over w* is a sequence of transitions $P_w^{(\mathcal{A})} := ((q_i, a_i, q_{i+1}))_{0 \le i \le n-1}$, $(q_i, a_i, q_{i+1}) \in T$ for every $0 \le i \le n-1$, with $q_0 \in I$ and $q_n \in F$. The *weight of* $P_w^{(\mathcal{A})}$ is the truth value

$$weight\left(P_w^{(\mathcal{A})}\right) = in(q_0) \wedge \bigwedge_{0 \le i \le n-1} wt\left((q_i, a_i, q_{i+1})\right) \wedge ter(q_n).$$

The set of paths of $\mathcal{A}$ over w can be totally ordered as follows. For two paths $P_w = ((q_i, a_i, q_{i+1}))_{0 \le i \le n-1}$ and $P'_w = \left((q'_i, a_i, q'_{i+1})\right)_{0 \le i \le n-1}$ we let

$$P_w \le P'_w \quad \text{iff} \quad q_0 \ldots q_{n-1} \le_{lex} q'_0 \ldots q'_{n-1}.$$

The *behavior of* $\mathcal{A}$ is the MK-fuzzy language $\|\mathcal{A}\| : A^* \to K$ and it is defined in the following way. Let $w \in A^+$ and $\{P_{w,1}, \ldots, P_{w,m}\}$ be the set of all paths of $\mathcal{A}$ over w. Furthermore, assume that $P_{w,1} \le \ldots \le P_{w,m}$. Then, we set

$$\|\mathcal{A}\|(w) = weight(P_{w,1}) \vee \ldots \vee weight(P_{w,m}).$$

If there are no paths of $\mathcal{A}$ over w, then we let $\|\mathcal{A}\|(w) = \mathbf{0}$. If $w = \varepsilon$, then

$$\|\mathcal{A}\|(\varepsilon) = (in(q_{i_1}) \wedge ter(q_{i_1})) \vee \ldots \vee (in(q_{i_m}) \wedge ter(q_{i_m}))$$

where $I \cap F = \{q_{i_1}, \ldots, q_{i_m}\}$ and $q_{i_1} \le \ldots \le q_{i_m}$. This implies that if $I \cap F = \emptyset$, then $\|\mathcal{A}\|(\varepsilon) = \mathbf{0}$. An MK-fuzzy language $s : A^* \to K$ is called *recognizable* if there is an MK-fuzzy automaton $\mathcal{A}$ over A and K such that $s = \|\mathcal{A}\|$. We denote by $Rec(K, A)$ the class of all recognizable MK-fuzzy languages over A and K.

Example 9 *Let $w \in A^*$. We show that $\overline{w} \in Rec(K, A)$. Indeed, let first $w = a_0 \ldots a_{n-1} \in A^+$. We consider the MK-fuzzy automaton $\mathcal{A}_w = (Q, I, T, F, in, wt, ter)$ with*

- $Q = \{q_0, \ldots, q_n\}$,
- $I = \{q_0\}$,
- $T = \{(q_i, a_i, q_{i+1}) \mid 0 \le i \le n-1\}$,
- $F = \{q_n\}$,
- $in(q_0) = \mathbf{1}$,
- $wt((q_i, a_i, q_{i+1})) = \mathbf{1}$ *for every* $0 \le i \le n-1$, *and*
- $ter(q_n) = \mathbf{1}$.

Then it can be easily seen that there is a unique path $P_w = ((q_i, a_i, q_{i+1}))_{0 \le i \le n-1}$ of $\mathcal{A}_w$ over w with $weight(P_w) = \mathbf{1}$, whereas for every other word $u \ne w$ there is not any path of $\mathcal{A}_w$ over u, i.e., $\|\mathcal{A}_w\|(u) = \mathbf{0}$. Hence, we obtain $\|\mathcal{A}_w\| = \overline{w}$.

If $w = \varepsilon$, then we consider the MK-fuzzy automaton $\mathcal{A}_\varepsilon = \{\{q\}, \{q\}, \emptyset, \{q\}, in, wt, ter\}$ with $in(q) = ter(q) = \mathbf{1}$. Then $\|\mathcal{A}_\varepsilon\|(\varepsilon) = \mathbf{1}$ and $\|\mathcal{A}_\varepsilon\|(u) = \mathbf{0}$ for every $u \ne \varepsilon$, i.e., $\|\mathcal{A}_\varepsilon\| = \overline{\varepsilon}$.

Theorem 10 *The class $Rec(K,A)$ is closed under disjunction.*

Proof. Let $\mathcal{A}_1 = (Q_1, I_1, T_1, F_1, in_1, wt_1, ter_1)$, $\mathcal{A}_2 = (Q_2, I_2, T_2, F_2, in_2, wt_2, ter_2)$ be two MK-fuzzy automata over A and K. Without any loss, we assume that $Q_1 \cap Q_2 = \emptyset$. We define a total order on $Q_1 \cup Q_2$ by preserving the orders of Q_1 and Q_2 and letting $\max Q_1 \leq \min Q_2$. We consider the MK-fuzzy automaton $\mathcal{A} = (Q, T, I, F, in, wt, ter)$ with $Q = Q_1 \cup Q_2$, $I = I_1 \cup I_2$, $T = T_1 \cup T_2$, $F = F_1 \cup F_2$, and in, wt, ter are defined respectively by

- $in(q) = \begin{cases} in_1(q) & \text{if } q \in I_1 \\ in_2(q) & \text{if } q \in I_2 \end{cases}$

  for every $q \in I$,

- $wt((q,a,q')) = \begin{cases} wt_1((q,a,q')) & \text{if } (q,a,q') \in T_1 \\ wt_2((q,a,q')) & \text{if } (q,a,q') \in T_2 \end{cases}$

  for every $(q,a,q') \in T$, and

- $ter(q) = \begin{cases} ter_1(q) & \text{if } q \in F_1 \\ ter_2(q) & \text{if } q \in F_2 \end{cases}$

  for every $q \in F$.

Consider a word $w = a_0 \dots a_{n-1} \in A^*$ and a path $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ of $\mathcal{A}$ over w. By definition of T, the transitions of $P_w^{(\mathcal{A})}$ belong either to T_1 or to T_2, hence $P_w^{(\mathcal{A})}$ is either a path of $\mathcal{A}_1$ over w or a path of $\mathcal{A}_2$ over w. Conversely, every path of $\mathcal{A}_1$ (resp. of $\mathcal{A}_2$) over w is also a path of $\mathcal{A}$ over w. Taking into account the order of Q, we get

$$\|\mathcal{A}\|(w) = \|\mathcal{A}_1\|(w) \vee \|\mathcal{A}_2\|(w).$$

Hence, $\|\mathcal{A}\| = \|\mathcal{A}_1\| \vee \|\mathcal{A}_2\|$ which implies that $\|\mathcal{A}_1\| \vee \|\mathcal{A}_2\| \in Rec(K,A)$, as required. ■

Theorem 11 *Let $h : A^* \to B^*$ be a strict alphabetic homomorphism. Then $s \in Rec(K,B)$ implies $h^{-1}(s) \in Rec(K,A)$.*

Proof. Let $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ be an MK-fuzzy automaton over B and K accepting s. We consider the MK-fuzzy automaton $\mathcal{A}' = (Q, I, T', F, in, wt', ter)$ over A and K, where $T' = \{(q,a,q') \in Q \times A \times Q \mid (q, h(a), q') \in T\}$ and the mapping $wt' : T' \to K$ is defined by $wt'((q,a,q')) = wt((q,h(a),q'))$ for every $(q,a,q') \in T'$.
Let $w = a_0 \dots a_{n-1} \in A^*$ and $P_{h(w)}^{(\mathcal{A})} = ((q_i, h(a_i), q_{i+1}))_{0 \leq i \leq n-1}$ be a path of $\mathcal{A}$ over $h(w)$. By construction of $\mathcal{A}'$, there is a path $P_w^{(\mathcal{A}')} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ of $\mathcal{A}$ over w, and vice -versa. Trivially $weight\left(P_w^{(\mathcal{A}')}\right) = weight\left(P_{h(w)}^{(\mathcal{A})}\right)$. Furthermore, if $P_{h(w),1}^{(\mathcal{A})}, \dots, P_{h(w),m}^{(\mathcal{A})}$ are all the paths of $\mathcal{A}$ over $h(w)$ and

$$P_{h(w),1}^{(\mathcal{A})} \leq \dots \leq P_{h(w),m}^{(\mathcal{A})},$$

then for the corresponding paths $P_{w,1}^{(\mathcal{A}')}, \dots, P_{w,m}^{(\mathcal{A}')}$ of $\mathcal{A}'$ over w, we get

$$P_{w,1}^{(\mathcal{A}')} \leq \dots \leq P_{w,m}^{(\mathcal{A}')}.$$

We conclude that $\|\mathcal{A}'\|(w) = \|\mathcal{A}\|(h(w))$, i.e., $\|\mathcal{A}'\|(w) = h^{-1}(s)(w)$ for every word $w \in A^*$. Hence, $\mathcal{A}' = h^{-1}(s)$ and our proof is completed. ■

In the sequel, we deal with the deterministic counterpart of our models and investigate properties of the class of MK-fuzzy languages accepted by such automata. An MK-fuzzy automaton $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ over A and K is called *deterministic* if $I = \{q_0\}$ and for every $q \in Q, a \in A$ there is at most one $q' \in Q$ such that $(q, a, q') \in T$. Then for every word $w \in A^*$ there is at most one path P_w of $\mathcal{A}$ over w, which in turn implies that we can relax the order relation of Q. Nevertheless, in the sequel, sometimes we will need the state set of a deterministic MK-fuzzy automaton to be ordered. A deterministic MK-fuzzy automaton $\mathcal{A}$ is simply written as $\mathcal{A} = (Q, q_0, T, F, in, wt, ter)$. An MK-fuzzy language $s \in K\langle\langle A^*\rangle\rangle$ is called *deterministic recognizable* if there is a deterministic MK-fuzzy automaton $\mathcal{A}$ over A and K such that $s = \|\mathcal{A}\|$. We denote by $DRec(K, A)$ the class of all deterministic recognizable MK-fuzzy languages over A and K.

An MK-fuzzy automaton $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ is called *unambiguous* if for every word $w \in A^*$ there is at most one path P_w of $\mathcal{A}$ over A. Clearly, every deterministic MK-fuzzy automaton is unambiguous as well, but the converse is not always true.

Theorem 12 *Let $s \in DRec(K, A)$ and $\mathbf{k} \in K$. Then $\mathbf{k} \wedge s, s \wedge \mathbf{k} \in DRec(K, A)$.*

Proof. Let $\mathcal{A} = (Q, q_0, T, F, in, wt, ter)$ be a deterministic MK-fuzzy automaton accepting s. We consider the deterministic MK-fuzzy automata $\mathcal{A}' = (Q, q_0, T, F, in', wt, ter)$ and $\mathcal{A}'' = (Q, q_0, T, F, in, wt, ter'')$ with $in'(q_0) = \mathbf{k} \wedge in(q_0)$, and $ter''(q) = ter(q) \wedge \mathbf{k}$ for every $q \in F$, respectively. Then, it is a routine matter to formally show that $\|\mathcal{A}'\| = \mathbf{k} \wedge s$ and $\|\mathcal{A}''\| = s \wedge \mathbf{k}$. ■

Corollary 13 *Let $w \in A^*$ and $\mathbf{k} \in K$. Then $\mathbf{k} \wedge \overline{w}, \overline{w} \wedge \mathbf{k} \in DRec(K, A)$.*

Proof. The MK-fuzzy automaton $\mathcal{A}_w$, for every $w \in A^*$, accepting $\overline{w}$ (cf. Example 9) is deterministic, hence we obtain our result by Theorem 12. ■

In the sequel, we investigate the closure of the class of deterministic recognizable MK-fuzzy languages under Cauchy product. More precisely, we show that the Cauchy product of two deterministic recognizable MK-fuzzy languages is a recognizable MK-fuzzy language. For this, we will need the notion of a normalized MK-fuzzy automaton and some preliminary results which present their own interest.

Definition 14 *An MK-fuzzy automaton $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ is called* normalized *if $I = \{q_{in}\}$, $q_{in} \notin F$, $in(q_{in}) = \mathbf{1}$, $ter(q) = \mathbf{1}$ for every $q \in F$, $(q, a, q_{in}) \notin T$ for every $q \in Q, a \in A$, and $(q, a, q') \notin T$ for every $q \in F, a \in A$, and $q' \in Q$.*

By the above definition, if $\mathcal{A}$ is a normalized MK-fuzzy automaton, then $\|\mathcal{A}\|(\varepsilon) = \mathbf{0}$. A normalized MK-fuzzy automaton $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ will be simply denoted by $\mathcal{A} = (Q, q_{in}, T, F, wt)$.

Proposition 15 *For every deterministic MK-fuzzy automaton $\mathcal{A} = (Q, q_0, T, F, in, wt, ter)$ we can effectively construct a normalized unambiguous MK-fuzzy automaton $\mathcal{A}'$ such that $\|\mathcal{A}'\|(w) = \|\mathcal{A}\|(w)$ for every $w \in A^+$, and $\|\mathcal{A}'\|(\varepsilon) = \mathbf{0}$.*

Proof. We consider a new state q_{in} not belonging to Q and a copy $\overline{F} = \{\overline{q} \mid q \in F\}$ of F. We define a total order on $\overline{F}$ by $\overline{q} \leq \overline{q'}$ iff $q \leq q'$ for every $q, q' \in F$. We let $Q' = Q \cup \{q_{in}\} \cup \overline{F}$ and extend the order of Q on Q' as follows: $q_{in} \leq \min Q$ and $\max Q \leq \min \overline{F}$. Let now $\mathcal{A}' = (Q', q_{in}, T', \overline{F}, wt')$ be the normalized MK-fuzzy automaton with

- $T' = T \cup \{(q_{in}, a, p) \mid (q_0, a, p) \in T\} \cup \{(q, a, \overline{p}) \mid (q, a, p) \in T, p \in F\}$
  $\cup \{(q_{in}, a, \overline{p}) \mid (q_0, a, p) \in T, p \in F\}$, and

- $wt'((q, a, q')) = \begin{cases} wt((q, a, q')) & \text{if } (q, a, q') \in T \\ in(q_0) \wedge wt((q_0, a, q')) & \text{if } q = q_{in} \\ wt((q, a, p)) \wedge ter(p) & \text{if } q' = \overline{p} \in \overline{F} \\ in(q_0) \wedge wt((q_0, a, p)) \wedge ter(p) & \text{if } q = q_{in}, q' = \overline{p} \in \overline{F} \end{cases}$,

for every $(q, a, q') \in T'$.

Let $w = a_0 \dots a_{n-1} \in A^+$. If there is path

$$P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$$

of $\mathcal{A}$ over w, then by construction of $\mathcal{A}'$, there is a path

$$P_w^{(\mathcal{A}')} = (q_{in}, a_0, q_1) \, ((q_i, a_i, q_{i+1}))_{1 \leq i \leq n-2} \, (q_{n-1}, a_{n-1}, \overline{q_n})$$

of $\mathcal{A}'$ over w, and vice-versa. Moreover, we get

$$\begin{aligned} & weight\left(P_w^{(\mathcal{A}')}\right) \\ & = wt'((q_{in}, a_0, q_1)) \wedge \bigwedge_{1 \leq i \leq n-2} wt'((q_i, a_i, q_{i+1})) \wedge wt'((q_{n-1}, a_{n-1}, \overline{q_n})) \\ & = in(q_0) \wedge wt((q_0, a_0, q_1)) \wedge \bigwedge_{1 \leq i \leq n-2} wt((q_i, a_i, q_{i+1})) \wedge wt((q_{n-1}, a_{n-1}, q_n)) \wedge ter(q_n) \\ & = weight\left(P_w^{(\mathcal{A})}\right). \end{aligned}$$

Since the path $P_w^{(\mathcal{A})}$ is unique, whenever it exists, the path $P_w^{(\mathcal{A}')}$ is unique, hence the MK-fuzzy automaton $\mathcal{A}'$ is unambiguous, and $\|\mathcal{A}'\|(w) = \|\mathcal{A}\|(w)$. Finally, $\|\mathcal{A}'\|(\varepsilon) = \mathbf{0}$ since $q_{in} \notin \overline{F}$, and this concludes our proof. ■

Lemma 16 *Let $s \in K\langle\langle A^* \rangle\rangle$ and $\mathbf{k} \in K$. If s is accepted by a normalized unambiguous MK-fuzzy automaton, then $s \wedge \mathbf{k}$ is accepted also by a normalized unambiguous MK-fuzzy automaton.*

Proof. Let $\mathcal{A} = (Q, q_{in}, T, F, wt)$ be a normalized unambiguous MK-fuzzy automaton accepting s. We consider the MK-fuzzy automaton $\mathcal{A}' = (Q, q_{in}, T, F, wt')$ with

$$wt'((q, a, q')) = \begin{cases} wt((q, a, q')) \wedge \mathbf{k} & \text{if } q' \in F \\ wt((q, a, q')) & \text{otherwise} \end{cases},$$

for every $(q, a, q') \in T$.

By construction, the automaton $\mathcal{A}'$ is normalized. Let now $w \in A^+$ and assume that there is a path $P_w^{(\mathcal{A}')}$ of $\mathcal{A}'$ over w. Then $P_w^{(\mathcal{A}')}$ is also a path of $\mathcal{A}$ over w. Since $\mathcal{A}$ is unambiguous, we conclude that $P_w^{(\mathcal{A}')}$ is unique which implies that the normalized MK-fuzzy automaton $\mathcal{A}'$ is also unambiguous. Furthermore, by a standard computation we get $weight\left(P_w^{(\mathcal{A}')}\right) = weight\left(P_w^{(\mathcal{A})}\right) \wedge \mathbf{k}$, i.e., $\|\mathcal{A}'\|(w) = \|\mathcal{A}\|(w) \wedge \mathbf{k}$. On the other hand, $\|\mathcal{A}\|(\varepsilon) = \mathbf{0}$ and $\|\mathcal{A}'\|(\varepsilon) = \mathbf{0}$, since both $\mathcal{A}$ and $\mathcal{A}'$ are normalized, hence $\|\mathcal{A}'\|(\varepsilon) = \|\mathcal{A}'\|(\varepsilon) \wedge \mathbf{k}$. Therefore, we conclude $\|\mathcal{A}'\| = s \wedge \mathbf{k}$, as required. ■

Theorem 17 *Let $r, s \in DRec(K, A)$. Then $rs \in Rec(K, A)$.*

Proof. Since $r, s \in DRec(K, A)$, there are deterministic MK-fuzzy automata accepting them. Then, by Proposition 15, we can effectively construct normalized unambiguous MK-fuzzy automata $\mathcal{A}_1 = \left(Q_1, q_{in}^{(1)}, T_1, F_1, wt_1\right)$ and $\mathcal{A}_2 = \left(Q_2, q_{in}^{(2)}, T_2, F_2, wt_2\right)$ such that $\|\mathcal{A}_1\|(w) = r(w)$ and $\|\mathcal{A}_2\|(w) = s(w)$ for every $w \in A^+$. Without any loss we assume that $Q_1 \cap Q_2 = \emptyset$, otherwise we apply a renaming. We consider the MK-fuzzy automaton $\mathcal{A} = \left(Q, \{q_{in}^{(1)}\}, T, F_2, in, wt, ter\right)$ with

- $Q = (Q_1 \setminus F_1) \cup Q_2$,
- $T = \left\{\left(q^{(1)}, a, p^{(1)}\right) \in T_1 \mid p^{(1)} \notin F_1\right\} \cup T_2 \cup \left\{\left(q^{(1)}, a, q_{in}^{(2)}\right) \mid \text{ there exists } p^{(1)} \in F_1 \text{ such that } \left(q^{(1)}, a, p^{(1)}\right) \in T_1\right\}$,
- $in\left(q_{in}^{(1)}\right) = \mathbf{1}$,
- $wt((q,a,p)) = \begin{cases} wt_1((q,a,p)) & \text{if } (q,a,p) \in T_1 \\ wt_2((q,a,p)) & \text{if } (q,a,p) \in T_2 \\ wt_1((q,a,p^{(1)})) & \text{if } q \in Q_1 \setminus F_1, p = q_{in}^{(2)}, p^{(1)} \in F_1, \text{ and } \left(q, a, p^{(1)}\right) \in T_1 \end{cases}$

  for every $(q, q, p) \in T$, and
- $ter(q) = \mathbf{1}$ for every $q \in F_2$.

We note that in case $p = q_{in}^{(2)}$ the value $wt((q, a, p))$ is well-defined. Indeed, since the original MK-fuzzy automaton accepting r is deterministic, by construction of $\mathcal{A}_1$ we get that there is at most one $p^{(1)} \in F_1$ such that $(q, a, p^{(1)}) \in T_1$.

We define a total order on Q by preserving the orders of Q_1 and Q_2 and letting $\max Q_2 \leq \min Q_1$. Let $w = a_0 \dots a_{n-1} \in A^+$ and $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ be a path of $\mathcal{A}$ over w. Then $q_0 = q_{in}^{(1)}$ and $q_n \in F_2$ which, by construction of T, means that $n > 1$ and there is an index $0 < j < n$ such that $q_j = q_{in}^{(2)}$, $q_1, \dots, q_{j-1} \in Q_1 \setminus F_1$, and $q_{j+1}, \dots, q_{n-1} \in Q_2$. This in turn implies that there is a path $P_{a_0 \dots a_{j-1}}^{(\mathcal{A}_1)} = \left(q_{in}^{(1)}, a_0, q_1\right)((q_i, a_i, q_{i+1}))_{1 \leq i \leq j-2}\left(q_{j-1}, a_{j-1}, p^{(1)}\right)$ of $\mathcal{A}_1$ over $a_0 \dots a_{j-1}$, with $p^{(1)} \in F_1$, and a path $P_{a_j \dots a_{n-1}}^{(\mathcal{A}_2)} = \left(q_{in}^{(2)}, a_j, q_{j+1}\right)((q_i, a_i, q_{i+1}))_{j+1 \leq i \leq n-1}$ of $\mathcal{A}_2$ over $a_j \dots a_{n-1}$. Since the MK-fuzzy automata $\mathcal{A}_1$ and $\mathcal{A}_2$ are unambiguous, these paths are unique. Furthermore, we get

$$\begin{aligned} weight\left(P_w^{(\mathcal{A})}\right) &= \bigwedge_{0\le i\le n-1} wt((q_i,a_i,q_{i+1})) \\ &= wt_1\left(\left(q_{in}^{(1)},a_0,q_1\right)\right)\wedge\ldots\wedge wt_1\left(\left(q_{j-1},a_{j-1},p^{(1)}\right)\right) \\ &\qquad \wedge wt_2\left(\left(q_{in}^{(2)},a_j,q_{j+1}\right)\right)\wedge\ldots\wedge wt_2\left((q_{n-1},a_{n-1},q_n)\right) \\ &= weight\left(P_{a_0\ldots a_{j-1}}^{(\mathcal{A}_1)}\right)\wedge weight\left(P_{a_j\ldots a_{n-1}}^{(\mathcal{A}_2)}\right) \\ &= \|\mathcal{A}_1\|(a_0\ldots a_{j-1})\wedge\|\mathcal{A}_2(a_j\ldots a_{n-1}) \end{aligned}$$

where the last equality holds by the uniqueness of the paths $P_{a_0\ldots a_{j-1}}^{(\mathcal{A}_1)}$ and $P_{a_j\ldots a_{n-1}}^{(\mathcal{A}_2)}$.

Conversely, keeping the above notations, if there is an index $0\le j\le n-1$ such that there are (unique) paths $P_{a_0\ldots a_{j-1}}^{(\mathcal{A}_1)}$ of $\mathcal{A}_1$ over $a_0\ldots a_{j-1}$ and $P_{a_j\ldots a_{n-1}}^{(\mathcal{A}_2)}$ of $\mathcal{A}_2$ over $a_j\ldots a_{n-1}$, then $\|\mathcal{A}_1\|(a_0\ldots a_{j-1}) = weight\left(P_{a_0\ldots a_{j-1}}^{(\mathcal{A}_1)}\right)$ and $\|\mathcal{A}_2\|(a_j\ldots a_{n-1}) = weight\left(P_{a_j\ldots a_{n-1}}^{(\mathcal{A}_2)}\right)$. Then, by construction of $\mathcal{A}$, there is a path $P_w^{(\mathcal{A})}$ of $\mathcal{A}$ over w with

$$weight\left(P_w^{(\mathcal{A})}\right) = weight\left(P_{a_0\ldots a_{j-1}}^{(\mathcal{A}_1)}\right)\wedge weight\left(P_{a_j\ldots a_{n-1}}^{(\mathcal{A}_2)}\right).$$

We conclude that

$$\|\mathcal{A}\|(w) = (r(a_0)\wedge s(a_1\ldots a_{n-1}))\vee\ldots\vee(r(a_0\ldots a_{n-2})\wedge s(a_{n-1}))$$

for every $w=a_0\ldots a_{n-1}\in A^+$.

Now, by Theorem 12, the series $r(\varepsilon)\wedge s$ is deterministic recognizable, hence by Proposition 15 there is a normalized unambiguous MK-fuzzy automaton $\mathcal{A}_3$ such that $\|\mathcal{A}_3\|(w) = (r(\varepsilon)\wedge s)(w)$ for every $w\in A^+$, and $\|\mathcal{A}_3\|(\varepsilon) = \mathbf{0}$. Furthermore, by Corollary 13 and Lemma 16 respectively, the MK-fuzzy languages $\overline{\varepsilon}\wedge r(\varepsilon)\wedge s(\varepsilon)$ and $\|\mathcal{A}_1\|\wedge s(\varepsilon)$ are recognizable. Therefore, since

$$rs = (\overline{\varepsilon}\wedge r(\varepsilon)\wedge s(\varepsilon))\vee\|\mathcal{A}_3\|\vee\|\mathcal{A}\|\vee(\|\mathcal{A}_1\|\wedge s(\varepsilon)),$$

we conclude our proof by Theorem 10. ■

Proposition 18 *Let $s\in DRec(K,A)$. Then the strong support of s is a recognizable language.*

Proof. Let $\mathcal{A}=(Q,q_0,T,F,in,wt,ter)$ be a deterministic MK-fuzzy automaton over A and K accepting s. Assume firstly that $t(in(q_0))=0$. Then for every word $w\in A^*$, if there is a path $P_w^{(\mathcal{A})}$ (which is unique) of $\mathcal{A}$ over w, we get $t\left(weight\left(P_w^{(\mathcal{A})}\right)\right)=0$. This implies that $\mathrm{stgsupp}(s)=\emptyset$ which is recognizable. Next let $t(in(q_0))\neq 0$. We consider the finite automaton $\mathcal{A}'=(Q,A,q_0,T',F')$ with $T'=\{(q,a,q')\in T\mid t(wt((q,a,q')))\neq 0\}$ and $F'=\{q\in F\mid t(ter(q))\neq 0\}$. By definition the automaton $\mathcal{A}'$ is deterministic. Let $w=a_0\ldots a_{n-1}\in A^+$ being accepted by $\mathcal{A}'$. Hence, there is a unique successful path $P_w^{(\mathcal{A}')} = ((q_i,a_i,q_{i+1}))_{0\le i\le n-1}$ of $\mathcal{A}'$ over w. By construction of $\mathcal{A}'$, we get that $P_w^{(\mathcal{A}')}$ is also a path of $\mathcal{A}$ over w. Moreover it holds

$$t\left(weight\left(P_w^{(\mathcal{A}')}\right)\right) = t(in(q_0)) \cdot \prod_{0 \leq i \leq n-1} t(wt((q_i, a_i, q_{i+1}))) \cdot t(ter(q_n))$$

and by our assumption we get $t\left(weight\left(P_w^{(\mathcal{A}')}\right)\right) \neq 0$ which in turn implies that $w \in \mathrm{stgsupp}(s)$. If $\varepsilon \in L(\mathcal{A}')$, then $q_0 \in F'$, and hence $t(in(q_0)) \cdot t(ter(q_0)) \neq 0$, i.e., $\varepsilon \in \mathrm{stgsupp}(s)$.

Conversely assume that $w = a_0 \dots a_{n-1} \in A^+$ is in $\mathrm{stgsupp}(s)$. Then, there is a unique path $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ of $\mathcal{A}$ over w with $t\left(weight\left(P_w^{(\mathcal{A})}\right)\right) \neq 0$. By construction of the finite automaton $\mathcal{A}'$, we get that $P_w^{(\mathcal{A})}$ is a successful path of $\mathcal{A}'$ over w, and thus $w \in L(\mathcal{A}')$. If $\varepsilon \in \mathrm{stgsupp}(s)$, then $t(ter(q_0)) \neq 0$, hence $q_0 \in F'$, i.e., $\varepsilon \in L(\mathcal{A}')$, and our proof is completed. ■

4 Conclusion

We introduced the bimonoid K related to the fuzzification of MK-logic, and investigated MK-fuzzy automata over K. Our models inspired by the real practical applications being in development within the project LogicGuard [15, 16]. We proved properties of the class of MK-fuzzy languages accepted by MK-fuzzy automata as well as by their deterministic counterpart. Several problems remain open and they are under investigation. For instance, whether the closure of the class of recognizable MK-fuzzy languages is closed under MK-conjunction, Cauchy product and star operation, as well as whether the class of deterministic recognizable MK-fuzzy languages is closed under MK-disjunction and conjunction, Cauchy product, and star operation. Furthermore, due to the four-valued elements of K, there are several notions of supports and it is greatly desirable for applications to check which of them constitute recognizable languages. We intend also to study MK-fuzzy automata over infinite words as well as over nonlinear structures like trees.

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