Symbolic Summation in Difference Rings and Applications

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Symbolic summation started with Abramov [1] for rational expressions and has been pushed forward by Gosper [7], Zeilberger [30], Petkovšek [15] and Paule [14] to tackle sums for hypergeometric expressions. In the last decade the class of input sums has been extended significantly and covers, for instance, hypergeometric multi-sums [29, 4], holonomic sequences [6, 13], unspecified sequences [11], radical expressions [12] or Stirling numbers [10].

We will focus on a new difference ring approach. The foundation was led by Karr's summation algorithm [8, 9], which enables one to rephrase indefinite nested sums and products in the setting of difference fields.

Definition. Let $\mathbb{F}$ be a field with characteristic 0 and let σ be a field automorphism of $\mathbb{F}$. Then $(\mathbb{F},\sigma)$ is called a *difference field*; the *constant field* of $\mathbb{F}$ is defined by $\mathbb{K}=\text{const}(\mathbb{F},\sigma)=\{f\in\mathbb{F}\mid\sigma(f)=f\}$. A difference field $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ is called a $\Pi\Sigma^*$*-field* if

$$\mathbb{K}=\mathbb{F}_0\leq\mathbb{F}_1\leq\dots\leq\mathbb{F}_e=\mathbb{F}$$

is a tower of field extensions where for all $1\leq i\leq e$ each $\mathbb{F}_i=\mathbb{F}_{i-1}(t_i)$ is a transcendental field extension of $\mathbb{F}_{i-1}$ and for σ one of the following holds[1]:

- $\frac{\sigma(t_i)}{t_i}\in(\mathbb{F}_{i-1})^*$ *(Π-field extension)*;
- $\sigma(t_i)-t_i\in\mathbb{F}_{i-1}$ *(Σ^*-field extension)*.

Many new ideas have been incorporated into a strong summation theory, see e.g., [5, 17, 5, 16, 20, 18, 19, 22, 23, 25] which led to new algorithms for the summation paradigms of telescoping, creative telescoping and recurrence solving [21]. However, this elegant difference field approach has one central drawback. Alternating signs cannot be represented in such a field: zero-divisors like

$$(1+(-1)^k)(1-(-1)^k)=0$$

are introduced which can be formulated only within a ring. We will present a class of difference rings in which one can represent algorithmically indefinite nested sums and products together with the alternating sign, and more generally products over primitive roots of unity [26].

Definition. Let $\mathbb{A}$ be a ring with characteristic 0 and let σ be a ring automorphism of $\mathbb{A}$. Then $(\mathbb{A},\sigma)$ is called a *difference ring*; the *constant ring* of $\mathbb{A}$ is defined by $\text{const}(\mathbb{A},\sigma)=\{f\in\mathbb{A}\mid\sigma(f)=f\}$. A difference ring $(\mathbb{A},\sigma)$ is called an $R\Pi\Sigma^*$*-extension* of a difference field $(\mathbb{F},\sigma)$ if the constants remain unchanged, i.e., $\text{const}(\mathbb{A},\sigma)=\text{const}(\mathbb{F},\sigma)$ and if

$$\mathbb{F}=\mathbb{A}_0\leq\mathbb{A}_1\leq\dots\leq\mathbb{A}_e=\mathbb{A}$$

is a tower of ring extensions where for all $1\leq i\leq e$ one of the following holds:

- $\mathbb{A}_i=\mathbb{A}_{i-1}[t_i]$ is a ring extension subject to the relation $t_i^n=1$ for some $n>1$ where $\frac{\sigma(t_i)}{t_i}\in(\mathbb{A}_{i-1})^*$ is a primitive nth root of unity *(R-extension)*;
- $\mathbb{A}_i=\mathbb{A}_{i-1}[t_i,t_i^{-1}]$ is a Laurent polynomial ring extension with $\frac{\sigma(t_i)}{t_i}\in(\mathbb{A}_{i-1})^*$ *(Π-extension)*;
- $\mathbb{A}_i=\mathbb{A}_{i-1}[t_i]$ is a polynomial ring extension with $\sigma(t_i)-t_i\in\mathbb{A}_{i-1}$ *(Σ^*-extension)*.

Representation of nested sums in $R\Pi\Sigma^*$-rings

We will restrict ourselves to basic $R\Pi\Sigma^*$-extensions [27]: for R-extensions we require that $\frac{\sigma(t_i)}{t_i}\in\text{const}(\mathbb{F},\sigma)^*$ holds and for Π-extensions we assume that $\frac{\sigma(t_i)}{t_i}\in\mathbb{F}^*$ holds. Together with an appropriate difference field $(\mathbb{F},\sigma)$ one can represent in this setting expressions (and in particular sequences produced by these expressions) in terms of indefinite nested sums defined over hypergeometric, q-hypergeometric or q-mixed hypergeometric products [27]. In the following we consider the hypergeometric case only. Let $f(k)$ be an expression in a variable k that evaluates at non-negative integers (from a certain point on) to elements of a field $\mathbb{K}$. Then $f(k)$ is called a *nested sum expression over hypergeometric products* w.r.t $\mathbb{K}$ and k if it is composed recursively by

- elements from the rational function field $\mathbb{K}(k)$;
- hypergeometric products of the form $\prod_{j=l}^k h(j)$ with $l\in\mathbb{N}=\{0,1,\dots\}$ and a rational function $h(j)\in\mathbb{K}(j)^*$, where $h(\nu)$ has no pole and is non-zero for all $\nu\in\mathbb{N}$ with $\nu\geq l$;
- the three operations $+$, $-$ and $\cdot$;

[1] For a ring or field $\mathbb{A}$ we denote by $\mathbb{A}^*$ the set of units.

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- sums of the form $\sum_{j=l}^{k} h(j)$ with $l \in \mathbb{N}$ and with $h(j)$ being a nested sum expression over hypergeometric products w.r.t. $\mathbb{K}$ and j and being free of k; here l is chosen big enough such that $h(j)|_{j\mapsto\nu}$ does not introduce poles for all $\nu \geq l$. $\sum_{j=l}^{k} h(j)$ is also called a nested sum over hypergeometric products w.r.t. $\mathbb{K}$ and k.

Example. Consider the sum

$$S(k) = \sum_{j=2}^{k} F(j) = \sum_{j=2}^{k} \frac{(-1)^j - \sum_{i=1}^{j} \frac{(-1)^i}{i}}{(-1+j)j}$$

which belongs to the class of nested sum expressions over hypergeometric products w.r.t. $\mathbb{Q}$ and k. As a consequence it follows that any shifted version of the objects within $S(k)$ can be expressed again by the non-shifted versions: we have

$$(-1)^{k+1} = -(-1)^k, \tag{1}$$

$$\sum_{i=1}^{k+1} \frac{(-1)^i}{i} = \sum_{i=1}^{k} \frac{(-1)^i}{i} + \frac{-(-1)^k}{k+1} \tag{2}$$

and $S(k+1) = S(k) + F(k+1)$ with

$$F(k+1) = -\frac{1}{(k+1)^2}(-1)^k - \frac{1}{k(k+1)} \sum_{i=1}^{k} \frac{(-1)^i}{i}; \tag{3}$$

note that the equalities on the expression level are justified since both sides agree for any evaluation $k \mapsto \nu$ with $\nu \in \mathbb{N}$. In the following we will construct a difference ring composed by a tower of $R\Pi\Sigma^*$-extensions in which the objects $(-1)^k$, $\sum_{i=1}^{k} \frac{(-1)^i}{i}$ and $S(k)$ with their explicitly given shift-behaviors are represented accordingly. Starting with the difference field $(\mathbb{Q}(x), \sigma)$ with $\sigma(x) = x+1$ and constant field $\mathbb{Q}$ (which is a $\Pi\Sigma^*$-field) we proceed as follows:

(1) We take the ring extension $\mathbb{Q}(x)[y]$ subject to the relation $y^2 = 1$. Further, we extend σ to a ring automorphism of $\mathbb{Q}(x)[y]$ with $\sigma(y) = -y$. In this way, we can model $(-1)^k$ with $((-1)^k)^2 = 1$ and (1) by y. By [27, Prop 2.20] we conclude that $\text{const}(\mathbb{Q}(x)[y], \sigma) = \text{const}(\mathbb{Q}(x), \sigma)$ holds, i.e., that $(\mathbb{Q}(x)[y], \sigma)$ forms an $R\Pi\Sigma^*$-extension of $(\mathbb{Q}(x), \sigma)$.

(2) Next, we want to model $\sum_{i=1}^{k} \frac{(-1)^i}{i}$ with (2). First, we check if this is possible in our given difference ring. To be more precise, we test if there is a $g \in \mathbb{Q}(x)[y]$ with $\sigma(g) = g + \frac{-y}{x+1}$. Our telescoping algorithms [26] (see also below) show that such a solution is not possible. Therefore we take the polynomial ring extension $\mathbb{Q}(x)[y][s]$ and extend σ to a ring automorphism of $\mathbb{Q}(x)[y][s]$ with $\sigma(s) = s + \frac{-y}{x+1}$. Using [26, Theorem 2.12] the non-existence of a solution $g \in \mathbb{Q}(x)[y]$ of $\sigma(g) = g + \frac{-y}{x+1}$ implies that

$$\text{const}(\mathbb{Q}(x)[y][s], \sigma) = \text{const}(\mathbb{Q}(x)[y], \sigma) = \text{const}(\mathbb{Q}(x), \sigma)$$

holds, Hence the constructed difference ring $(\mathbb{Q}(x)[y][s], \sigma)$ forms an $R\Pi\Sigma^*$-extension of $(\mathbb{Q}(x), \sigma)$ in which we can represent the sum $\sum_{i=1}^{k} \frac{(-1)^i}{i}$ by s.

(3) Finally, we want to model $S(k) = \sum_{j=2}^{k} F(j)$ with (3). As above, we check if this is possible in the given difference ring $(\mathbb{Q}(x)[y][s], \sigma)$. By our earlier construction we can rephrase $F(k+1)$ with $f = -\frac{yx+(x+1)s}{x(x+1)^2}$. Thus we check if there exists a $g \in \mathbb{Q}(x)[y][s]$ such that $\sigma(g) = g + f$ holds. This time our telescoping algorithms produce $g = \frac{y}{x}$. We conclude that for $G(k) = \frac{1}{k} \sum_{i=1}^{k} \frac{(-1)^i}{i} + c$ with $c \in \mathbb{Q}$ we obtain $G(\nu+1) = G(\nu) + F(\nu+1)$ for all $\nu \in \mathbb{N}$. Choosing $c = 1$ we get $G(1) = S(1) = 0$. Together with $S(\nu+1) = S(\nu) + F(\nu+1)$ for all $\nu \in \mathbb{N}$ we get

$$S(k) = \frac{1}{k} \sum_{i=1}^{k} \frac{(-1)^i}{i} + 1. \tag{4}$$

This means that both sides agree whenever we evaluate k at positive integers. Hence $S(k)$ can be represented by $g+1 = \frac{s}{x} + 1$ in $(\mathbb{Q}(x)[y][s], \sigma)$. Internally, we used the map

$$\text{expr}_k(a) = a|_{x\mapsto k, y\mapsto(-1)^k, s\mapsto\sum_{i=1}^{k}\frac{(-1)^i}{i}}$$

for $a \in \mathbb{Q}(x)[y][s]$ that links our difference ring elements to our nested sum expressions. E.g., $\text{expr}_k(f)$ and $\text{expr}_k(g+1)$ are precisely the right hand sides of (3) and (4), respectively.

In general, given an $R\Pi\Sigma^*$-extension over a $\Pi\Sigma^*$-field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x+1$ such a map expr_k is canonically induced by the action of σ and is uniquely determined up to choice of the lower summation/multiplication bounds and of additive constants (for sums) or multiplicative constants (for products) from $\mathbb{K}$. We are now ready to introduce the following problem which has been solved in [27, Subsec. 7.1].

Problem RNS: Representation of Nested Sums. Given a nested sum expression over hypergeometric products $A(k)$, find a nested sum expression over hypergeometric products $B(k)$ and a δ with the following properties:

(1) $A(\nu) = B(\nu)$ for all $\nu \in \mathbb{N}$ with $\nu \geq \delta$;

(2) the nested sums and hypergeometric products in $B(k)$ (except products of the form α^k with α being a root of unity) are algebraically independent among each other.

Our solution. We start with the $\Pi\Sigma^*$-field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x+1$. Then we compute a basic $R\Pi\Sigma^*$-extension $(\mathbb{K}(x)[y][p_1, p_1^{-1}] \dots [p_r, p_r^{-1}][s_1] \dots [s_e], \sigma)$ of $(\mathbb{K}(x), \sigma)$ with an element $a \in \mathbb{A}$ and a $\delta \in \mathbb{N}$ such that property (1) of Problem RNS holds with $B(k) := \text{expr}_k(a)$; for details see the above example (for the product case see [24]). By the explicitly given map expr_k it follows that $B(k)$ is given in terms of one root of unity product $R(k) = \text{expr}_k(y) = \alpha^k$ (note that one R-extension is sufficient by [27, Lemma 2.22]), hypergeometric products $P_i(k) := \text{expr}_k(p_i)$ with $1 \leq i \leq r$ and nested sums $S_i(k) = \text{expr}_k(s_i)$ with $1 \leq i \leq e$.

Now consider the ring of sequences $\mathbb{K}^{\mathbb{N}}$ with component-wise addition and multiplication, and define $\text{ev}\colon \mathbb{K}(x) \to \mathbb{K}^{\mathbb{N}}$ by

$$\text{ev}(\tfrac{p}{q}, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0 \end{cases}$$

where $p, q \in \mathbb{K}[x]$, $q \neq 0$ are co-prime. Then $\tau\colon \mathbb{K}(x) \to \mathbb{K}^{\mathbb{N}}$ with $\tau(a) = \langle \text{ev}(a, k) \rangle_{k\geq0}$ for $a \in \mathbb{K}(x)$ establishes a ring monomorphism. Thus the field $\mathbb{K}(x)$ and the field of rational sequences $F := \tau(\mathbb{K}(x))$ are isomorphic. Further, difference ring theory[2] [27] (see also [28]) implies that

$$S[\langle P_1(\nu)\rangle_{\nu\geq0}, \langle \tfrac{1}{P_1(\nu)}\rangle_{\nu\geq0}] \dots [\langle P_r(\nu)\rangle_{\nu\geq0}, \langle \tfrac{1}{P_r(\nu)}\rangle_{\nu\geq0}] [\langle S_1(\nu)\rangle_{\nu\geq0}] \dots [\langle S_e(\nu)\rangle_{\nu\geq0}]$$

forms a (Laurent) polynomial ring extension over the ring

$$S = F[\langle R(\nu)\rangle_{\nu\geq0}] = \tau(\mathbb{K}(x))[\langle \alpha^\nu \rangle_{\nu\geq0}]$$

of rational sequences adjoined with $\langle \alpha^\nu \rangle_{\nu\geq0}$. In conclusion, the arising sums and products in $B(k)$ (except $R(k) = \alpha^k$) are algebraically independent among each other. □

[2] Internally, one works within the difference ring of sequences: there two sequences are considered to be equal if they agree from a certain point on.

Among many interesting features we emphasize that our solution of Problem RNS contains the zero-recognition problem: the input $A(k)$ evaluates to zero from a certain point on if and only if the output $B(k)$ is the zero-expression.

Summation paradigms in difference rings

As illustrated in the above example, our solution of Problem RNS relies on algorithms that decide if a telescoping equation has a solution within the already constructed difference ring. More generally, we can use efficient algorithms that solve the so-called parameterized telescoping problem.

Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$, and take the elements $f_0, \dots, f_d \in \mathbb{A}$. Then we are interested in finding all solutions $g \in \mathbb{A}$ and $c_0, \dots, c_d \in \mathbb{K}$ such that

$$\sigma(g) - g = c_0\, f_0 + \dots + c_d\, f_d \tag{5}$$

holds. Since the set of solutions

$$\{(c_0, \dots, c_d, g) \in \mathbb{K}^{d+1} \times \mathbb{A} \mid \text{ (5) holds}\} \tag{6}$$

forms a $\mathbb{K}$-vector space of dimension $\leq d+2$, we can formulate this task as follows.

Problem PT: Parameterized Telescoping.
Given a difference ring $(\mathbb{A}, \sigma)$ with constant field $\mathbb{K}$ and given $f_0, \dots, f_d \in \mathbb{A}$; find a basis of (6).

General algorithms for this task are worked out in [26, 27] relying on algorithms from [8]. In particular, we can solve this problem if we take a $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ over a constant field $\mathbb{K}$ and choose any basic $R\Pi\Sigma^*$-extension $(\mathbb{A}, \sigma)$ on top of $(\mathbb{F}, \sigma)$; by definition we have $\mathbb{K} = \text{const}(\mathbb{F}, \sigma) = \text{const}(\mathbb{A}, \sigma)$.

The special case $d = 0$ of Problem PT is nothing else than the telescoping problem in the setting of difference rings and is applied iteratively to solve Problem RNS. At this point we emphasize that refined telescoping algorithms can be exploited which leads to improved versions of Problem RNS: e.g., the sums in $B(k)$ are minimal nested [17, 19, 22, 25].

Creative telescoping. Problem PT and our algorithms cover as special case Zeilberger's celebrated summation paradigm of creative telescoping. In its simplest form it can be formulated as follows.
Given $d \in \mathbb{N}$ and a nested sum expression $F(n, k)$ over hypergeometric products w.r.t. $\mathbb{K}$ and k where $F(n, k)$ depends on an extra parameter n; here $\mathbb{K} = \mathbb{K}'(n)$ is supposed to be a rational function field over a subfield $\mathbb{K}'$.
Find constants $c_0, \dots, c_d$, free of k, and a nested sum expression $G(n, k)$ w.r.t. $\mathbb{K}$ and k, which depends only on the sums and products given in $F(n, k)$, such that the creative telescoping equation

$$G(n, \nu+1) - G(n, \nu) = c_0\, F(n, \nu) + \dots + c_d F(n+d, \nu) \tag{7}$$

holds for all evaluations $\nu \in \mathbb{N}$ with $\nu \geq \lambda$ for some $\lambda \in \mathbb{N}$.
We can tackle this problem as follows: We take the difference field $(\mathbb{K}(x), \sigma)$ with $\sigma(x) = x + 1$ and construct an $R\Pi\Sigma^*$-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $f_0, \dots, f_d \in \mathbb{A}$ and a $\delta \in \mathbb{N}$ such that for $0 \leq i \leq d$ the evaluations of $\text{expr}_k(f_i)$ and $F(n+i, k)$ with $k \mapsto \nu$ agree for all non-negative integers $\nu \geq \delta$ (compare Problem RNS). If we fail to find a solution $c_0, \dots, c_d \in \mathbb{K}$ (not all c_i being non-zero) and $g \in \mathbb{A}$ for (5), then we conclude that the desired solution of (7) does not exist; in this case we can either increase d or we can use our enhanced algorithms [17, 19, 22, 25] for Problem PT. Otherwise, if we find such a solution, we obtain the desired $G(n, k) := \text{expr}_k(g)$ and a $\lambda \in \mathbb{N}$ with $\lambda \geq \delta$ such that (7) holds all $\nu \in \mathbb{N}$ with $\nu \geq \lambda$.
Given such a solution, we can sum (7) over k and get

$$G(n, a+1) - G(n, 1) = c_0\, S(n, a) + \dots + c_d\, S(n+d, a)$$

for the sum $S(n, a) = \sum_{k=\lambda}^{a} F(n, k)$. This means that both sides of the obtained recurrence in terms of nested sums over hypergeometric products agree for any evaluation $a \mapsto \nu$ with $\nu \in \mathbb{N}$ and $\nu \geq \lambda$. Further, we can specialize a, e.g., to n which yields a recurrence for the definite sum $\sum_{k=\lambda}^{n} F(n, k)$.

Recurrence solving. Finally, we can look for solutions of a given recurrence relation

$$a_0(k) S(k) + a_1(k) S(k+1) + \dots + a_d(k) S(k+d) = b(k) \tag{8}$$

where $a_i(k)$ are polynomials in $\mathbb{K}[k]$ and $b(k)$ is a nested sum expression over hypergeometric products w.r.t. $\mathbb{K}$ and k. More precisely, we find the so-called d'Alembertian solutions [2, 3]. This means that we find all solutions of (8) that are expressible in terms of nested sums over hypergeometric products. Internally, the expression $b(k)$ is rephrased to a difference ring composed by a tower of $R\Pi\Sigma^*$-extensions and variants of the algorithms from [15, 5, 16, 20, 18] are used to compute these solutions. We remark that the produced solutions are highly nested. As a consequence, the simplification of these solutions with our telescoping algorithms for Problem RNS is a crucial and highly challenging task.

Applications

Combining this toolbox of computing recurrences and solving them in terms of nested sums over hypergeometric, q-hypergeometric or q-mixed hypergeometric products, we obtain an efficient summation machinery that has been built into the Mathematica package Sigma [21].

In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

For instance, after entering the definite sum

In[2]:= **mySum = SigmaSum[(1 − (−1)^k)SigmaBinomial[n, k] SigmaSum[SigmaPower[−1, i]/i, {i, 1, k}], {k, 0, n}]**

Out[2]= $\sum_{k=0}^{n} (1 - (-1)^k) \binom{n}{k} \sum_{i=1}^{k} \frac{(-1)^i}{i}$

one can calculate a linear recurrence by the function call

In[3]:= **rec = GenerateRecurrence[mySum]**

Out[3]= $-4n\text{SUM}[n] + 4(n+1)\text{SUM}[n+1] + (-n-2)\text{SUM}[n+2] = \frac{-2n}{n+1}$

Internally, one activates the parameterized telescoping algorithms mentioned above. Next, we solve the computed recurrence in terms of nested sums over hypergeometric products by executing the function call

In[4]:= **recSol = SolveRecurrence[rec[[1]], SUM[n]]**

Out[4]= $\{\{0, -2^n\}\{0, -\frac{2^n}{n}\}, \{1, \frac{1}{n} - 2^n \sum_{i=1}^{n} \frac{2^{-i}}{i}\}\}$

This means that $h_1(n) = -2^n$ and $h_2(n) = -\frac{2^n}{n}$ are two linearly independent solutions of the homogeneous version of the input recurrence and $p(n) = \frac{1}{n} - 2^n \sum_{i=1}^{n} \frac{2^{-i}}{i}$ is a particular solution of the recurrence itself. In a nutshell, we obtain the full solution set

$$\{c_1\, h_1(n) + c_2\, h(n) + p(n) \mid c_1, c_2 \in \mathbb{Q}\}. \tag{9}$$

In particular, the arising sums and products are algebraically independent among each other. Finally, we determine c_1, c_2 such that (9) equals to $S(n)$ for $n = 1, 2$:

In[5]:= **sol = FindLinearCombination[recSol, mySum, n, 2]**

Out[5]= $-2^{n}\sum_{i=1}^{n}\frac{2^{-i}}{i}+\frac{1-2^{n}}{n}$

Since this expression and $S(n)$ are both a solution of the same recurrence of order 2, they must agree for all $n \in \mathbb{N}$.

In this talk we exploit this summation machinery of `Sigma` and demonstrate challenging problems coming from combinatorics and particle physics; in the latter case see, e.g., [27] and references therein.

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