Three Examples of Gröbner Reduction over Noncommutative Rings

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In this report three classes of noncommutative rings are investigated with emphasis on their properties with respect to reduction relations. The Gröbner basis concepts in these rings, being developed in the literature by several authors, are considered and it is shown that the reduction relations corresponding to these Gröbner bases obey the axioms of a general theory of Gröbner reduction.

1 Introduction

At the ISSAC 2015 conference, the authors have introduced the notion of Gröbner reduction, a general concept that covers the reduction part of several Gröbner basis techniques appearing in the literature.

In [FL15] it is proved that reduction concepts which obey the axioms of Gröbner reduction allow the derivation of the dimension polynomial of finitely generated modules.

In this paper three classes of rings are studied with emphasis on those parts of their structure which are relevant for designing a reduction concept on finitely generated free modules over them.

The rings of the first class contain differential operators of some polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$ where $\mathbf{k}$ is a field of characteristic 0 (Weyl-algebras). The rings of the second class contain

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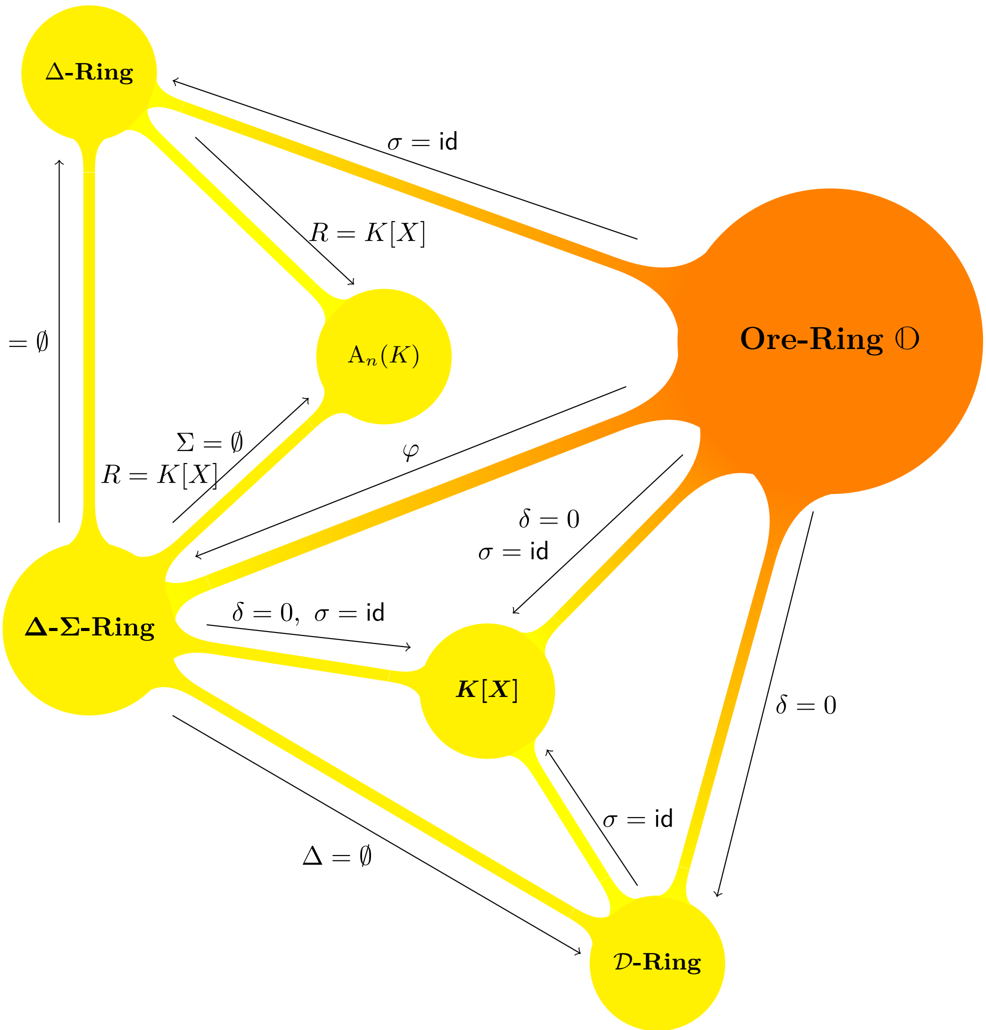
linear combinations of formal expressions equipped with a product that reflects the properties of derivations and automorphisms (difference-differential rings), while the elements of the rings of the third class are linear combinations of certain formal expressions, and the multiplication is designed to simulate skew derivations and their endomorphisms (Ore-polynomials).

The rings of all these classes are free objects in appropriate categories; the Weyl-algebras due to the behavior of partial derivatives (in characteristic 0), the other ones by design of their multiplication.

The following figure shows the relation between the three classes together with certain special types of them.



In this diagram we meet

- the ring of Ore-polynomials $\mathbb{O}$;
- the ring of difference-differential operators D;

- the ring D of differential operators with set of derivations Δ;
- the ring of difference operators $\mathcal{D}$;
- the ring of commutative polynomials $K[X]$.

Arrows indicate specialization. For example, the arrow from the node '$\Delta - \Sigma$-Ring' to the node '$K[X]$' exposes the polynomial ring $K[X]$ as a $\Delta\Sigma$-ring D by trivializing all functional ingredients of D. Similarly, the arrow from 'Ore-Ring $\mathbb{O}$' to '$\Delta - \Sigma$-Ring' accents that a $\Delta\Sigma$-ring can be obtained as a quotient of some Ore-ring.
The order of appearence of the considered ring classes in the paper follows a path of increasing generality. For each of these three types of rings the reduction concept, condensed from Gröbner basis techniques that are treated in the literature, is studied under the aspect of Gröbner reduction and the validity of axioms of Gröbner reduction is analyzed one at a time.

In order to be able to subsume all those rings under this general theory, the concept of Gröbner reduction, as introduced in [FL15], must be slightly modified. Thus, the authors introduce a new set of axioms for Gröbner reduction, weaker than the one formulated in [FL15], so that each of these reduction relations can be seen as a model of the new axioms.

Notation

$\mathbb{N}$ is the set of natural numbers ≥ 0. The symbol $\leq_\pi$ denotes the product order in $\mathbb{N}^n$, i.e.,

$$k \leq_\pi l \iff k_i \leq l_i \ (1 \leq i \leq n).$$

The length of a tupel $k \in \mathbb{N}^n$ is $|k| = k_1 + \cdots + k_n$, x^k denotes the power product $x_1^{k_1} \cdots x_n^{k_n}$. If $n = n_1 + \cdots + n_p$ is a partition of n (all $n_j > 0$) and $k = (k_1, \ldots, k_n)$ then we write x^{k^j} for the corresponding j-part of x^k, i.e.,

$$x^{k^j} = \prod_{n_1+\cdots+n_{j-1} < i \leq n_1+\cdots+n_j} x^{k_i}$$

The same notation is used in the context of other symbols (e.g., d, δ, ∂).

A 'ring' is an associative ring with unity 1. A homomorphism of rings preserves 1. A module is understood as a left module.

Throughout this paper R denotes an arbitrary (possibly noncommutative) ring containing a commutative ring K in such a way that R is a free K-module. All rings that will occur are of this type. We use the letter R in section 2 where we thematize Gröbner reduction for arbitrary such rings. As, in the text, the rings get more concrete, the letter R changes to another appropriate symbol: A will denote a Weyl-algebra, D a $\Delta\Sigma$-ring, and $\mathbb{O}$ a ring of Ore-polynomials.

For elements $a, b \in K$, if $b|a$ then we write $\frac{a}{b}$ for an element $x \in K$ with $bx = a$. Then $\frac{a}{b} \cdot b = a$, no matter which such x we had chosen. Of course, when b^{-1} exists in K then the element $\frac{a}{b}$ is unique. In any case, writing the symbol $\frac{a}{b}$, it is supposed that $b|a$.

2 Gröbner Reduction

We fix a K-basis $\Lambda \subset R$ whose elements are called *monomials*. Then the monomials of the free module $F = R^{(E)}$ on the set E are the members of the set

$$\Lambda E = \{\lambda e : (\lambda, e) \in \Lambda \times E\}.$$

Since $R^{(E)} = K^{(\Lambda E)}$ this set is a K-basis of F. We write

$$\mathrm{T}(f) = \mathrm{T}\Big(\sum_t f_t t\Big) = \{t \in \Lambda E : \ f_t \neq 0\}$$

for the support of f, i.e. the set of monomials t that appear in f with a non-zero coefficient $f_t \in K$.

There are situations where the ring K contains a field $\mathbf{k}$ that is central in R (c.f. Section 3). Then there may be two different monomial concepts:

1. $R = K^{(\Lambda_1)}$ (R is a free K-module with basis Λ_1);
2. $R = \mathbf{k}^{(\Lambda_2)}$ (R is a vector space over $\mathbf{k}$ with basis Λ_2).

In certain instances we will need the assumption that K be a field. This will be emphasized at occurence.

Definition 1. *By a p-fold filtration on R we mean a family of additive subgroups $R_r \subseteq R$, indexed by $r \in \mathbb{N}^p$, such that*

1. $R_r \cdot R_s \subseteq R_{r+s}$;
2. $r \leq_\pi s \Rightarrow R_r \subseteq R_s$;
3. $R = \bigcup_{r \in \mathbb{N}^p} R_r$;
4. $R_0 = K$.

R together with such a filtration is called a (p-fold) filtered ring*. The filtration is called* monomial*, and R a* monomially filtered ring*, when $a \in R_r \Rightarrow \mathrm{T}(a) \subseteq R_r$ for all $a \in R$ and all $r \in \mathbb{N}^p$.*

Definition 2. *Let $R = \bigcup_{r \in \mathbb{N}^p} R_r$ be a filtered ring and M an R-module. A filtration of M is a family of additive subgroups $M_r \subseteq M$ $(r \in \mathbb{N}^p)$ with the properties*

1. $R_r \cdot M_s \subseteq M_{r+s}$;
2. $r \leq_\pi s \Rightarrow M_r \subseteq M_s$;
3. $M = \bigcup_{r \in \mathbb{N}^p} M_r$.

M together with a filtration is called a filtered module *over the filtered ring R.*

A free module $F = R^{(E)}$ inherits the filtration

$$F_r = \bigoplus_{e \in E} R_r e$$

from the filtered ring R, and this filtration is monomial (w.r.t. ΛE) when the original filtration on R is monomial (w.r.t. Λ). An arbitrary R-module M turns into a filtered module via a free presentation

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

setting $M_r = \pi(F_r)$. Thus, when a set G of generators of the R-module M is determined, M obtains the filtration

$$M_r = \sum_{g \in G} R_r g.$$

Let X be a set and $\rho \subseteq X \times X$ a binary relation. We write $f \longrightarrow h$ to indicate that $(f, h) \in \rho$, and $f \longrightarrow^\star h$ when there is a chain of finite length

$$f = f_0 \longrightarrow f_1 \longrightarrow \cdots \longrightarrow f_k = h \qquad (k \in \mathbb{N}).$$

We say that f is reducible if $\exists h$ with $f \longrightarrow h$ and we write I_ρ (or I when ρ is understood) for the set of ρ-irreducible elements, that is

$$I = \{x \in X : \nexists y \in X \text{ such that } x \longrightarrow y\}.$$

A subset $Y \subseteq X$ is called ρ-stable if $y \in Y$ and $y \longrightarrow z$ implies that $z \in Y$.

In [FL15], we have formulated the following system of axioms:

Definition 3. *Let N be a submodule of a free module F over the filtered ring R. A relation $\rho \subseteq F \times F$ is called a* strong reduction *for N provided that*

I. ρ is noetherian, i.e. every sequence $f_1 \longrightarrow f_2 \longrightarrow \cdots$ terminates;

II. I is a monomial K-submodule of F, that is, I is a module and

$$\forall f \in F (f \in I \Rightarrow \mathrm{T}(f) \subseteq I);$$

III. $f \longrightarrow h \Rightarrow f \equiv h \mod N$;

IV. $N \cap I = 0$.

A strong reduction ρ for N is called a strong Gröbner reduction *if it satisfies in addition*

V. $f \in F_r \wedge f \longrightarrow h \Longrightarrow h \in F_r$, that is, each filter group F_r is ρ-stable.

We can use the concept 'strong Gröbner reduction' to determine the dimension of filter spaces of finitely generated filtered modules. We recall the main theorem, proved in [FL15].

Theorem 1. *Let K be a field, $R = K^{(\Lambda)}$ a filtered ring, and M a finitely generated R-module. Choose a free presentation*

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

and equip M with the filtration $M_r = \pi(F_r)$. Assume given a Gröbner reduction for N, let U_r be the set of irreducible monomials in the filter space F_r. Then the sets $\pi(U_r)$ provide K-vector space bases for the spaces M_r. In particular

$$\dim_K M_r = |\pi(U_r)| = |U_r| \quad (r \in \mathbb{N}^p).$$

In order to treat the rings that will appear in this paper under these aspects we need to weaken Axiom II. to

- I is a monomial subset of F.

To make things completely transparent we formulate the modified system of axioms in its entirety.

Definition 4. *Let N be a submodule of a free module F over the filtered ring R. A relation $\rho \subseteq F \times F$ is called a* weak reduction *for N provided that*

1. *ρ is noetherian;*
2. *I is a monomial subset of F, that is*
$$\forall f \in F(f \in I \Rightarrow \mathrm{T}(f) \subseteq I);$$
3. *$f \longrightarrow h \Rightarrow f \equiv h \mod N$;*
4. *$N \cap I = 0$.*

A weak reduction ρ for N is called a weak Gröbner reduction *if it satisfies in addition*

5. *$f \in F_r \wedge f \longrightarrow h \Longrightarrow h \in F_r$.*

Plainly every strong Gröbner reduction is also a weak one. Note that Axiom 4 (being present in both systems) implies that every non-zero element in N is reducible.

In the following sections we will consider several rings R and investigate reduction relations for submodules of free modules over them. While the rings are equipped with certain filtrations there is always present a well-ordering $\prec$ of the monomials ΛE that distinguishes for all $f \in F \setminus 0$ a leading term $\mathrm{LT}(f)$ and a leading coefficient $\mathrm{LC}(f)$. In each of the examples below we are now concerned with two reduction relations: Let $f, g, h \in F$, $g \neq 0$. Then we have

1. full reduction ρ

$$f \xrightarrow[g]{\rho} h \iff \exists \lambda \in \Lambda \Big(\mathrm{LT}(\lambda g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}(\lambda g)}}{\mathrm{LC}(\lambda g)} \lambda g \wedge P \Big); \qquad (1)$$

2. leading term reduction σ

$$f \xrightarrow[g]{\sigma} h \iff \exists \lambda \in \Lambda\Big(\mathrm{LT}(\lambda g) = \mathrm{LT}(f) \wedge h = f - \frac{\mathrm{LC}(f)}{\mathrm{LC}(\lambda g)}\lambda g \wedge P\Big). \tag{2}$$

The symbol 'P' denotes a predicate $P = P(f, g, \lambda, h)$ depending on the actual situation. For a set $G \subseteq F$ one has then in both cases

$$f \xrightarrow[G]{} h \iff \exists g \in G : f \xrightarrow[g]{} h\ . \tag{3}$$

These reduction concepts are the core of Gröbner bases.

Definition 5. *Consider a submodule N of a free module $F = K^{(\Lambda E)}$. Assume given a well-order $\prec$ on ΛE and a predicate $P = P(f, g, \lambda, h)$, and let ρ be the full reduction defined by these data. A subset $G \subseteq N$ is a Gröbner basis for N iff ρ is a weak reduction for N.*

Given a filtered ring and the obvious necessary data, we need to check the defining axioms in order to reveal the relation ρ as a (weak) Gröbner reduction for $N = RG$. The set G is then exposed as a Gröbner basis for N and the filter groups are ρ-stable.

So we fix a set $G \subseteq F$ and write ρ and σ for the relations $f \xrightarrow[G]{\rho} h$ and $f \xrightarrow[G]{\sigma} h$ respectively, i.e., $f \longrightarrow_\rho h$ means $f \xrightarrow[G]{\rho} h$ and similar for σ. All our examples follow the pattern along the following lines.

1. `Termination`. Fix a positive integer q and design an injection $\varphi \colon \Lambda E \longrightarrow \mathbb{N}^q$. The set $\mathbb{N}^q$ is ordered lexicographically

$$a < b \iff a_{\min\{i : a_i \neq b_i\}} < b_{\min\{i : a_i \neq b_i\}}. \tag{4}$$

   The set of monomials ΛE inherits a well order $\prec$ by means of this injection. We call such an order `induced by the injection` φ. The well order $\prec$ extends to a well order on the set of all finite subsets of ΛE (this is $\{\mathrm{T}(f) \colon f \in F\}$):

$$\mathrm{T}(f) \prec \mathrm{T}(g) \iff \max(\mathrm{T}(f) \triangle \mathrm{T}(g)) \in \mathrm{T}(g)$$

   where $\triangle$ is the symmetric difference (consider e.g. [BWK93]). It remains to check that $f \longrightarrow_\rho h$ has $\mathrm{T}(h) \prec \mathrm{T}(f)$ as a consequence: For arbitrary g, from $f \xrightarrow[g]{\rho} h$ we see that $\mathrm{LT}(\lambda g) \in \mathrm{T}(f) \setminus \mathrm{T}(h)$ - where λ is a term as mentioned in (1) - whereas for all terms t with $t \succ \mathrm{LT}(\lambda g)$ we have $h_t = f_t$. This demonstrates that $\mathrm{T}(f) \succ \mathrm{T}(h)$.

   Since $\sigma \subseteq \rho$ it is clear that $I_\rho \subseteq I_\sigma$ and σ terminates if ρ terminates. Consequently both relations terminate.

2. `The set of irreducible monomials should be a monomial subset of` F`, i.e.,` $f \in I \Rightarrow \mathrm{T}(f) \subseteq I$. The relation ρ has the property

   f is $\rho -$ reducible $\iff$ $\exists h \colon f \longrightarrow_\rho h$

$$\iff \quad \exists g \in G\, \exists \lambda \in \Lambda \Big(\mathrm{LT}(\lambda g) \in \mathrm{T}(f) \wedge P(f, g, \lambda, f - \frac{f_{\mathrm{LT}(\lambda g)}}{\mathrm{LC}(\lambda g)} \lambda g\Big).$$

In case that P does not involve h, i.e., $P = P(f, g, \lambda)$, we obtain

$$f \text{ is } \rho-\text{reducible} \iff \exists g \in G\, \exists \lambda \in \Lambda\, (\mathrm{LT}(\lambda g) \in \mathrm{T}(f) \wedge \mathrm{LC}(\lambda g) | f_{\mathrm{LT}(\lambda g)} \wedge P(f, g, \lambda)).$$

For I_ρ to be monomial it is then enough to verify the `monomial irreducibility condition`

$$\exists g \in G\, \exists \lambda \in \Lambda \Big(\mathrm{LT}(\lambda g) \in \mathrm{T}(f) \wedge \mathrm{LC}(\lambda g) \in K^\times \wedge P(\mathrm{LT}(\lambda g), g, \lambda)\Big) \Rightarrow$$
$$\exists g \in G\, \exists \lambda \in \Lambda \Big(\mathrm{LT}(\lambda g) \in \mathrm{T}(f) \wedge \mathrm{LC}(\lambda g) | f_{\mathrm{LT}(\lambda g)} \wedge P(f, g, \lambda)). \tag{5}$$

The monomial irreducibility condition for σ under the assumption $P = P(f, g, \lambda)$ is

$$\exists g \in G\, \exists \lambda \in \Lambda \Big(\mathrm{LT}(\lambda g) \in \mathrm{T}(f) \wedge \mathrm{LC}(\lambda g) \in K^\times \wedge P(\mathrm{LT}(\lambda g), g, \lambda)\Big) \Rightarrow$$
$$\exists g \in G\, \exists \lambda \in \Lambda \Big(\mathrm{LT}(\lambda g) = \mathrm{LT}(f) \wedge \mathrm{LC}(\lambda g) | \mathrm{LC}(f) \wedge P(f, g, \lambda)). \tag{6}$$

It is clear that this condition is hard to satisfy. Indeed, I_σ is not monomial in general.

3. `Compatibility of reduction with congruence modulo` $N = RG$. This is always obvious from the shape of (1) and (2).

4. $N \cap I = 0$. The validity of this condition must be guaranteed by an appropriate choice of the generator set G which is achieved by the usual Buchberger completion procedure.

5. `Each filter space should be` ρ`-stable`. In our examples we will consider univariate filtrations $(F_t)_{t \in \mathbb{N}}$ that are constructed due to the following schema:

   We start with an 'order-function' $\nu \colon \Lambda \longrightarrow \mathbb{N}$, where $\nu(\lambda)$ can be read off from $\lambda \in \Lambda$, i.e., $\nu(\lambda)$ is the sum of certain exponents that are present in λ. The function ν extends to ΛE by setting $\nu(\lambda e) = \nu(\lambda)$ ($\lambda \in \Lambda$, $e \in E$), and further to entire F (we always use the same symbol)

$$\nu(f) = \begin{cases} \max\{\nu(t) \colon t \in \mathrm{T}(f)\} & \ldots\ f \in F \setminus 0 \\ -\infty & \ldots\ f = 0. \end{cases} \tag{7}$$

   Then $\nu(f + g) \leq \max\{\nu(f), \nu(g)\}$ $(f, g \in F)$ and $\nu(c \cdot f) \leq \nu(f)$ $(f \in F,\ c \in K)$.

   The `univariate filtration induced by` ν is then

$$F_t^\nu = \{f \in F \colon \nu(f) \leq t\}\ (t \in \mathbb{N}). \tag{8}$$

   Remark that this defines implicitely the sets R_t^ν $(t \in \mathbb{N})$ since $R = R^1$.

   From the properties of ν it is plain that

- the sets F_t^ν are monomial K-modules;
- $s \le t \Rightarrow F_s^\nu \subseteq F_t^\nu$;
- $\bigcup_{t=0}^\infty F_t^\nu = F$.

It remains to check that $R_s \cdot F_t \subseteq F_{s+t}$ $(s, t \in \mathbb{N})$, which is then the only property of filtrations that depends on the actual ring structure of R.

The multivariate filtrations that we consider are constructed from univariate ones by means of intersection:

Given order functions $\nu_1, \dots, \nu_p$ and $\alpha \in \mathbb{N}^p$ we set

$$F_\alpha^{\nu_1,\dots,\nu_p} = F_{\alpha_1}^{\nu_1} \cap \dots \cap F_{\alpha_p}^{\nu_p} = \{f \in F \colon \nu_1(f) \le \alpha_1 \wedge \dots \wedge \nu_p(f) \le \alpha_p\}. \tag{9}$$

The next theorem condenses the preceeding discussion.

Theorem 2. *Let G be a subset of the free R-module $F = K^{(\Lambda E)}$. Assume that*

- *$\prec$ is a well order on ΛE;*
- *$P = P(f, g, \lambda, h)$ is a predicate $F \times G \times \Lambda \times F \longrightarrow \{0, 1\}$;*
- *ρ is the full reduction defined by $(\prec, P, G)$;*
- *$(R_t^j)_{t\in\mathbb{N}}$ is defined by an order function $\nu_j \colon \Lambda \longrightarrow \mathbb{N}$ $(1 \le j \le p)$;*
- *$R_\alpha = R_{\alpha_1}^1 \cap \dots \cap R_{\alpha_p}^p$ $(\alpha \in \mathbb{N}^p)$;*
- *$F_\alpha = \bigoplus_{e\in E} R_\alpha \cdot e$.*

Under these assumptions, if

1. *$R_s^j \cdot R_t^j \subseteq R_{s+t}^j$ $(s, t \in \mathbb{N},\ j = 1, \dots, p)$,*
2. *$P = P(f, g, \lambda)$ and the monomial irreducibility condition (5) holds,*
3. *$f \xrightarrow[g]{\rho} h \wedge f \in F_\alpha \Rightarrow h \in F_\alpha$ $(\forall g \in G)$,*
4. *$N \cap I_\rho = 0$,*

then $(F_\alpha)_{\alpha\in\mathbb{N}^p}$ is a monomial filtration on F w.r.t. the monomial filtration $(R_\alpha)_{\alpha\in\mathbb{N}^p}$ and ρ is a Gröbner reduction for RG.

3 The Weyl Algebra $\mathrm{A}_n(\mathbf{k})$

The theory of the Weyl algebra $\mathrm{A}_n(\mathbf{k})$ in n variables is the study of modules over rings of differential operators with polynomial coefficients over the field $\mathbf{k}$. In this section we develop the properties of $A_n(\mathbf{k})$ that are relevant for Gröbner reduction. In its second part we refer to work appearing in [DL12] and prove that the theory developed there is an instance of our concepts.

Let $\mathbf{k}$ be a field of characteristic 0 and let d_i denote the i-th partial derivative of the polynomial ring $\mathbf{k}[x_1, \dots, x_n]$. The Weyl algebra $A_n(\mathbf{k})$ is the $\mathbf{k}$-algebra generated by $\mathbf{k}[x_1, \dots, x_n] \cup \{d_1, \dots, d_n\}$ as a subalgebra of $\mathrm{End}_{\mathbf{k}}(\mathbf{k}[x_1, \dots, x_n])$. The multiplication in this ring obeys the rules

$$x_i x_j = x_j x_i, \qquad d_i d_j = d_j d_i, \qquad d_i x_j = x_j d_i + \delta_{i,j} \qquad (1 \leq i, j \leq n)$$

($\delta_{i,j}$ is the Kronecker symbol).

Let A denote the ring $A_n(\mathbf{k})$. We may consider A as a free $\mathbf{k}[x_1, \dots, x_n]$-module with basis $\Lambda_1 = \{d^l : l \in \mathbb{N}^n\}$. Then Λ_1 is a monoid isomorphic to $\mathbb{N}^n$ and, according to our notational conventions, K and R specialize to $K = \mathbf{k}[x_1, \dots, x_n]$ and $R = K^{(\Lambda_1)}$.

We will here stress the 2nd approach: A as a $\mathbf{k}$-vector space with distinguished set of monomials $\Lambda_2 = \{x^k d^l : (k, l) \in \mathbb{N}^n \times \mathbb{N}^n\}$. In the following we write Λ for Λ_2. Note that Λ is not closed under multiplication.

Explicitly, the product of two monomials in $A_n(\mathrm{k})$ is

$$x^k d^l \cdot x^p d^q = \sum_{v \in \mathbb{N}^n} \binom{l}{v} x^k \partial^v (x^p) d^{l+q-v} \tag{10}$$

where $\binom{l}{v} = \binom{l_1}{v_1} \cdots \binom{l_n}{v_n}$. To visualize the scope of the sum we may write

$$x^k d^l \cdot x^p d^q = \sum_{v \leq_\pi l \sqcap p} \frac{l! \cdot p!}{v! \cdot (l-v)! \cdot (p-v)!} x^{k+p-v} d^{l+q-v}$$

$l \sqcap p$ denoting the infimum of $\{l, p\}$ in $\mathbb{N}^n$.

Let A denote the Weyl algebra $A_n(\mathbf{k})$ and $F = A^{(E)}$ the free A-module with basis $E = \{e_1, \dots, e_q\}$.

Proposition 1. *$\lambda, \mu \in \Lambda$ and $t_1, t_2 \in \Lambda E$. Then*

1. $\lambda \cdot t_1 = \lambda \cdot t_2 \Rightarrow t_1 = t_2$;
2. $\lambda \cdot t_1 = \mu \cdot t_1 \Rightarrow \lambda = \mu$.

Proof. $\lambda = x^k d^l$, $t_1 = x^p d^q e_1$, $t_2 = x^r d^s e_2$.

1. If $\lambda \cdot t_1 = \lambda \cdot t_2$ then $x^k d^l \cdot x^p d^q e_1 = x^k d^l \cdot x^r d^s e_2$, therefore $e_1 = e_2$. We get

$$\sum_{u \leq_\pi l \sqcap p} b_u x^{k+p-u} d^{l+q-u} = \sum_{v \leq_\pi l \sqcap r} c_v x^{k+r-v} d^{l+s-v} \quad (b_u, c_v \in \mathbb{N}).$$

$$\exists v : x^{k+p} d^{l+q} = c_v x^{k+r-v} d^{l+s-v} \wedge \exists u : x^{k+r} d^{l+s} = b_u x^{k+p-u} d^{l+q-u}.$$

   It follows that

$$p = r - v, \; q = s - v, \; r = p - u, \; s = q - u$$

   from which we derive that $u = v = 0$. Consequently $t_1 = t_2$.

2. This is proved similarly.

□

We consider three natural order functions $\Lambda \longrightarrow \mathbb{N}$. For $\lambda = x^k d^l \in \Lambda$

$$\nu_1(\lambda) = |k| = k_1 + \cdots + k_n, \quad \nu_2(\lambda) = |l| = l_1 + \cdots + l_n, \quad \nu_0 = \nu_1 + \nu_2.$$

The extensions of the ν_j to F $(j = 0, 1, 2)$ obey the rules

$$\begin{array}{rcl} \nu_j(f+g) & \leq & \max\{\nu_j(f), \nu_j(g)\} \quad (f, g \in F) \\ \nu_j(c\,f) & = & \nu_j(f) \quad (c \in \mathbf{k} \setminus 0). \end{array} \tag{11}$$

Note that the extension of ν_0 to F is not the sum of the extensions to F from ν_1 and ν_2.

We define three well orders $\prec_x$, $\prec_d$, $\prec_0$ on ΛE:

$\prec_x$ comes from the injection

$$\Lambda E \longrightarrow \mathbb{N}^{2n+3}, \; t = x^k d^l e_i \mapsto (\nu_1(t), \nu_2(t), k_1, \ldots, k_n, l_1, \ldots, l_n, i);$$

$\prec_d$ comes from

$$\Lambda E \longrightarrow \mathbb{N}^{2n+3}, \; t = x^k d^l e_i \mapsto (\nu_2(t), \nu_1(t), l_1, \ldots, l_n, k_1, \ldots, k_n, i);$$

$\prec_0$ comes from

$$\Lambda E \longrightarrow \mathbb{N}^{2n+2}, \; t = x^k d^l e_i \mapsto (\nu_0(t), k_1, \ldots, k_n, l_1, \ldots, l_n, i).$$

The corresponding leading term and leading coefficient functions are written LT_x, LC_x, LT_d, LC_d and LT_0, LC_0 respectively.

Lemma 1. *Let $\lambda = x^k d^l \in \Lambda$, $t = x^r d^s e \in \Lambda E$, $f \in F \setminus 0$. Then*

1. $\nu_1(\lambda \cdot t) = k + r$, $\nu_2(\lambda \cdot t) = l + s$, $\nu_0(\lambda \cdot t) = k + r + l + s = \nu_1(\lambda \cdot t) + \nu_2(\lambda \cdot t)$;
2. $\mathrm{LT}_x(\lambda \cdot t) = \mathrm{LT}_d(\lambda \cdot t) = \mathrm{LT}_0(\lambda \cdot t) = x^{k+r} d^{l+s} e$;
3. $\nu_1(f) = \nu_1(\mathrm{LT}_x(f))$, $\nu_2(f) = \nu_2(\mathrm{LT}_d(f))$, $\nu_0(f) = \nu_0(\mathrm{LT}_0(f))$;
4. $\nu_j(\lambda \cdot t) = \nu_j(\lambda) + \nu_j(t)$ $(0 \leq j \leq 2)$.

Proof.

1. This is obvious from (10).

2. Also evident from (10).

3. Take $s \in \mathrm{T}(f)$. Then $s \preceq_x \mathrm{LT}_x(f)$, $s \preceq_d \mathrm{LT}_d(f)$ and $s \preceq_0 \mathrm{LT}_0(f)$. Therefore $\nu_1(s) \leq \nu_1(\mathrm{LT}_x(f))$, $\nu_2(s) \leq \nu_2(\mathrm{LT}_d(f))$ and $\nu_0(s) \leq \nu_0(\mathrm{LT}_0(f))$. Therefore

$$\begin{aligned}\nu_1(f) &= \max\{\nu_1(s) \colon s \in \mathrm{T}(f)\} = \nu_1(\mathrm{LT}_x(f));\\ \nu_2(f) &= \max\{\nu_2(s) \colon s \in \mathrm{T}(f)\} = \nu_2(\mathrm{LT}_d(f));\\ \nu_0(f) &= \max\{\nu_0(s) \colon s \in \mathrm{T}(f)\} = \nu_0(\mathrm{LT}_0(f)).\end{aligned}$$

4. This is obvious from point 1.

$\square$

Lemma 2. *Let $\lambda, \mu \in \Lambda$, $s, t \in \Lambda E$ and $j \in \{x, d, 0\}$.*

1. $\lambda \prec_j \mu \Rightarrow \mathrm{LT}_j(\lambda t) \prec_j \mathrm{LT}_j(\mu t)$;

2. $s \prec_j t \Rightarrow \mathrm{LT}_j(\lambda s) \prec_j \mathrm{LT}_j(\lambda t)$.

Proof. 1.) $\lambda = x^k d^l$, $\mu = x^r d^s$, $t = \nu e = x^\alpha d^\beta e$, $\lambda \prec_x \mu$. Then

$$\mathrm{LT}_x(\lambda t) = x^{k+\alpha} d^{l+\beta} e \text{ and } \mathrm{LT}_x(\mu t) = x^{r+\alpha} d^{s+\beta} e.$$

If $\nu_1(\lambda) < \nu_1(\mu)$ then

$$\nu_1(\mathrm{LT}_x(\lambda t)) = |k+\alpha| = |k| + |\alpha| < |r| + |\alpha| = |r+\alpha| = \nu_1(\mathrm{LT}_x(\mu t)),$$

and thus $\mathrm{LT}_x(\lambda t) \prec_x \mathrm{LT}_x(\mu t)$.

If $\nu_1(\lambda) = \nu_1(\mu)$ and $\nu_2(\lambda) < \nu_2(\mu)$ then

$$\begin{aligned}\nu_1(\mathrm{LT}_x(\lambda t)) &= |k| + |\alpha| = |r| + |\alpha| = \nu_1(\mathrm{LT}_x(\mu t))\\ \nu_2(\mathrm{LT}_x(\lambda t)) &= |l| + |\beta| < |s| + |\beta| = \nu_2(\mathrm{LT}_x(\mu t))\end{aligned}$$

which also means that $\mathrm{LT}_x(\lambda t) \prec_x \mathrm{LT}_x(\mu t)$.

If $\nu_1(\lambda) = \nu_1(\mu)$ and $\nu_2(\lambda) = \nu_2(\mu)$ and $k \neq r$ then let $j = \min\{i : k_i \neq r_i\}$. We obtain

$$(|k+\alpha|, |l+\beta|, (k+\alpha), (l+\beta), e) <_{\mathrm{lex}} (|r+\alpha|, |s+\beta|, (r+\alpha), (s+\beta), e).$$

Once again this means that $\mathrm{LT}_x(\lambda t) \prec_x \mathrm{LT}_x(\mu t)$.

If $\nu_1(\lambda) = \nu_1(\mu)$ and $\nu_2(\lambda) = \nu_2(\mu)$ and $k = r$ then l must be different from s. Let $j = \min\{i : l_i \neq s_i\}$. Then $l_j < s_j$ which results again in $\mathrm{LT}_x(\lambda t) \prec_x \mathrm{LT}_x(\mu t)$.

2.) $\lambda = x^\alpha d^\beta$, $s = x^k d^l e_1$, $t = x^p d^q e_2$, $s \prec_x t$. The proof works similarly as before. The only difference is that there is one more case: When $x^k d^l = x^p d^q$ then e_1 must be smaller than e_2 and the statement follows.

The proofs of the remainig statements are repetitions of the previous considerations by changing the subscripts of $\prec$, LT, LC and the order functions accordingly.

$\square$

Proposition 2. $\mathrm{char}(\mathbf{k}) = 0$, *$A = A_n(\mathbf{k})$, $F = A^{(E)}$ the free A-module on the set E. Let $a \in A \setminus 0$, $f \in F \setminus 0$ and $j \in \{x, d, 0\}$. Then*

$$\begin{aligned} \mathrm{LT}_j(a \cdot f) &= \mathrm{LT}_j\Big(\mathrm{LT}_j(a) \cdot \mathrm{LT}_j(f)\Big); \\ \mathrm{LC}_j(a \cdot f) &= \mathrm{LC}_j(a) \cdot \mathrm{LC}_j(f). \end{aligned} \tag{12}$$

Proof. Let $\lambda_0 = \mathrm{LT}_x(a)$, $a_0 = \mathrm{LC}_x(a)$, $t_0 = \mathrm{LT}_x(f)$, $f_0 = \mathrm{LC}_x(f)$. Thus

$$a = a_0\lambda_0 + \sum_{\lambda \prec_x \lambda_0} a_\lambda \lambda \text{ and } f = f_0 t_0 + \sum_{t \prec_x t_0} f_t t.$$

$$a \cdot f = \underbrace{a_0 f_0 \lambda_0 t_0}_{(0)} + \underbrace{\sum_{t \prec_x t_0} a_0 f_t \lambda_0 t}_{(1)} + \underbrace{\sum_{\lambda \prec_x \lambda_0} a_\lambda f_0 \lambda t_0}_{(2)} + \underbrace{\sum_{\lambda \prec_x \lambda_0} \sum_{t \prec_x t_0} a_\lambda f_t \lambda t}_{(3)}.$$

Pick out a term $\lambda_0 t$ of sum (1). Then from Lemma 2 we derive $\mathrm{LT}_x(\lambda_0 t) \prec_x \mathrm{LT}_x(\lambda_0 t_0)$. Similar things happen when choosing a term from sum (2) or (3):

$$\begin{aligned} \mathrm{LT}_x(\lambda t_0) \prec_x \mathrm{LT}_x(\lambda_0 t_0) &\text{ (term chosen from (2))}, \\ \mathrm{LT}_x(\lambda t) \prec_x \mathrm{LT}_x(\lambda_0 t) \prec_x \mathrm{LT}_x(\lambda_0 t_0) &\text{ (term chosen from (3))}, \end{aligned}$$

Let $s \in \mathrm{T}(a \cdot f)$. Then $\exists \lambda, t$ with $\lambda \in \mathrm{T}(a)$ and $t \in \mathrm{T}(f)$ and $s \in \mathrm{T}(\lambda t)$. It follows that $s \preceq_x \mathrm{LT}_x(\lambda t)$.

$$\begin{aligned} \lambda = \lambda_0 \wedge t = t_0 &\Rightarrow s \preceq_x \mathrm{LT}_x(\lambda_0 t_0); \\ \lambda = \lambda_0 \wedge t \neq t_0 &\Rightarrow s \preceq_x \mathrm{LT}_x(\lambda_0 t) \prec_x \mathrm{LT}_x(\lambda_0 t_0); \\ \lambda \neq \lambda_0 \wedge t = t_0 &\Rightarrow s \preceq_x \mathrm{LT}_x(\lambda t_0) \prec_x \mathrm{LT}_x(\lambda_0 t_0); \\ \lambda \neq \lambda_0 \wedge t \neq t_0 &\Rightarrow s \prec_x \mathrm{LT}_x(\lambda_0 t_0). \end{aligned}$$

Consequently

$$\begin{aligned} &\mathrm{LT}_x(a \cdot f) = \mathrm{LT}_x(\lambda_0 t_0) = \mathrm{LT}_{\mathrm{x}}\Big(\mathrm{LT}_x(a) \cdot \mathrm{LT}_x(f)\Big) \text{ and} \\ &\mathrm{LC}_x(a \cdot f) = a_0 f_0 = \mathrm{LC}_x(a) \cdot \mathrm{LC}_x(f). \end{aligned}$$

Again the proof of the remaining statements is a repetition by changing 'x' to 'd' or '0' respectively. $\square$

Corollary 1. *$j \in \{x, d, 0\}$. Then*

$$\lambda \in \Lambda \setminus 0 \wedge f \in F \setminus 0 \Rightarrow \mathrm{LT}_j(\lambda f) = \mathrm{LT}_j(\lambda \cdot \mathrm{LT}_j(f)).$$

Corollary 2. *Let $a \in A$, $f \in F$. Then*

$$\nu_j(a \cdot f) = \nu_j(a) + \nu_j(f) \quad (j = 0, 1, 2). \tag{13}$$

Proof. If $a = 0$ or $f = 0$ the statements are obvious. So assume $a \neq 0 \wedge f \neq 0$. Set

$$\begin{array}{ll} \lambda_1 = x^k d^l = \mathrm{LT}_x(a) & t_1 = x^\alpha d^\beta e_1 = \mathrm{LT}_x(f) \\ \lambda_2 = x^p d^q = \mathrm{LT}_d(a) & t_2 = x^\gamma d^\delta e_2 = \mathrm{LT}_d(f) \\ \lambda_3 = x^r d^s = \mathrm{LT}_0(a) & t_3 = x^\varepsilon d^\zeta e_3 = \mathrm{LT}_0(f). \end{array}$$

From Lemma 1 we get

$$\mathrm{LT}_x(\lambda_1 \cdot t_2) = x^{k+\alpha} d^{l+\beta} e_1,\ \mathrm{LT}_d(\lambda_2 \cdot t_2) = x^{p+\gamma} d^{q+\delta} e_2,\ \mathrm{LT}_0(\lambda_3 \cdot t_3) = x^{r+\varepsilon} d^{s+\zeta} e_3.$$

From (1) and (12):

$$\begin{array}{rcl} \nu_1(a \cdot f) & = & \nu_1(\mathrm{LT}_x(a \cdot f)) = \nu_1(\mathrm{LT}_x(\mathrm{LT}_x(a) \cdot \mathrm{LT}_x(f))) = \nu_1(\mathrm{LT}_x(\lambda_1 \cdot t_1)) \\ & = & \nu_1(x^{k+\alpha} d^{l+\beta} e_1) = |k+\alpha| = |k| + |\alpha| = \nu_1(\mathrm{LT}_x(a)) + \nu_1(\mathrm{LT}_x(f)) \\ & = & \nu_1(a) + \nu_1(f). \end{array}$$

$$\begin{array}{rcl} \nu_2(a \cdot f) & = & \nu_2(\mathrm{LT}_d(a \cdot f)) = \nu_2(\mathrm{LT}_d(\mathrm{LT}_d(a) \cdot \mathrm{LT}_d(f))) = \nu_2(\mathrm{LT}_d(\lambda_2 \cdot t_2)) \\ & = & \nu_2(x^{p+\gamma} d^{q+\delta} e_2) = |q+\delta| = |q| + |\delta| = \nu_2(\mathrm{LT}_d(a)) + \nu_2(\mathrm{LT}_d(f)) \\ & = & \nu_2(a) + \nu_2(f). \end{array}$$

$$\begin{array}{rcl} \nu_0(a \cdot f) & = & \nu_0(\mathrm{LT}_0(a \cdot f)) = \nu_0(\mathrm{LT}_0(\mathrm{LT}_0(a) \cdot \mathrm{LT}_0(f))) = \nu_0(\mathrm{LT}_0(\lambda_3 \cdot t_3)) \\ & = & \nu_0(x^{r+\varepsilon} d^{s+\zeta} e_3) = |r+\varepsilon+s+\zeta| = |r+s| + |\varepsilon+\zeta| = \nu_0(\mathrm{LT}_0(a)) + \nu_0(\mathrm{LT}_0(f)) \\ & = & \nu_0(a) + \nu_0(f). \end{array}$$

□

For a different proof of the statement involving ν_0 see [Cou95], chapter 2. Proposition 1 generalizes to the statement that $A_n(\mathbf{k})$ is a domain.

Corollary 3. *$a \in A$, $f \in F$. Then $a \cdot f = 0 \Rightarrow a = 0 \vee f = 0$.*

Proof. Assume $a \neq 0 \wedge f \neq 0$. Let ν denote one of ν_1, ν_2, ν_0. Then $\nu(a) \geq 0 \wedge \nu(f) \geq 0$. It follows $\nu(a \cdot f) = \nu(a) + \nu(f) \geq 0$. Consequently $a \cdot f \neq 0$. □

Definition 6. *For $r, s \in \mathbb{N}$ we set*

$$\begin{array}{rcl} F_r^0 & = & \{f \in F : \nu_0(f) \leq r\}; \\ F_r^1 & = & \{f \in F : \nu_1(f) \leq r\}; \\ F_r^2 & = & \{f \in F : \nu_2(f) \leq r\}; \\ F_{r,s} & = & F_r^1 \cap F_s^2. \end{array} \tag{14}$$

We will show that these sets define filtrations on F. Remark that we have defined implicitely $A_r^{(i)}$ and $A_{r,s}$ since we may consider A as the free module A^1.

Proposition 3.

1. *$\left(F_r^i\right)_{r \in \mathbb{N}}$ defines a univariate filtration on F $(0 \leq i \leq 2)$.*

2. *$\left(F_{r,s}\right)_{(r,s) \in \mathbb{N}^2}$ defines a bivariate filtration on F.*

Proof. From (11) it is clear that all the sets F_r^i - hence also the $F_{r,s}$ are monomial $\mathbf{k}$-vector spaces, that is, vector spaces with the property

$$f \in F_r^i \iff \mathrm{T}(f) \subseteq F_r^i \quad (i = 0, 1, 2)$$

Immediately from Corollary 2 we obtain that $A_r^i \cdot F_s^i \subseteq F_{r+s}^i$ $(i = 0, 1, 2)$. Therefore also $A_{r,s} \cdot F_{t,u} \subseteq F_{r+t,s+u}$. □

Corollary 4. $\forall\, r \in \mathbb{N} : F_r^0 \subseteq F_{r,r} \subseteq F_{2r}^0$.

Proof. By monoimiality, if $f \in F_r^0$ then $\mathrm{T}(f) \subseteq F^0$. Thus, for arbitrary $t \in \mathrm{T}(f)$, $\nu_1(t) + \nu_2(t) = \nu_0(t) \leq r$. Therefore also $\nu_1(t) \leq r$ and $\nu_2(t) \leq r$, i.e., $t \in F_r^1 \cap F_r^2 = F_{r,r}$. Thus $\mathrm{T}(f) \subseteq F_{r,r}$ and so $f \in F_{r,r}$.

Now assume that $f \in F_{r,r}$. Then $\mathrm{T}(f) \subseteq F_{r,r}$. Therefore, if $t \in \mathrm{T}(f)$ then $\nu_0(t) = \nu_1(t) + \nu_2(t) \leq r + r$. Consequently $t \in F_{2r}^0$. This shows that $f \in F_{2r}^0$. □

(x, ∂)-Gröbner Bases

Dönch and Levin [DL12] introduced the notion of an (x, ∂)-Gröbner basis for free modules over $A_n(\mathbf{k})$. We present their concepts here and prove then that the reduction relation resulting from a (x, ∂)-basis is a Gröbner reduction in our sense.

First they defined a divisibility notion in a non-standard way mimicking commutative monomials

$$x^k d^l | x^r d^s \iff (k, l) \leq_\pi (r, s).$$

This notion extends to divisibility of monomials $t_1 = x^k d^l e_1$, $t_2 = x^r d^s e_2$ of the free A-module $F = A^{(E)}$ by setting

$$t_1 | t_2 \iff x^k d^l | x^r d^s \wedge e_1 = e_2.$$

In this case the quotient $\frac{t_2}{t_1}$ is the element $x^{r-k} d^{s-l} \in \Lambda$.

Lemma 3. *Let* $t_1, t_2 \in \Lambda E$. *Then*

$$t_1 | t_2 \Rightarrow \frac{t_2}{t_1} \cdot t_1 = t_2 + \sum_j n_j s_j$$

with all $n_j \in \mathbb{N}^+$ *and* $s_j \in \Lambda E$ *such that*

$$\nu_1(s_j) < \nu_1(t_2) \wedge \nu_2(s_j) < \nu_2(t_2).$$

Proof. $t_1 = x^k d^l e$, $t_2 = x^r d^s e$, $k \leq r$, $\leq s$. Using formula (10) gives

$$\begin{aligned} \frac{t_2}{t_1} \cdot t_1 &= x^{r-k} d^{s-l} \cdot x^k d^l e = \sum_{v \leq_\pi s-l} \binom{s-l}{v} x^{r-k} \partial^v (x^k) d^{s-l+l-v} e \\ &= \Big[\binom{s-l}{0} x^{r-k} \partial^0 (x^k) d^{s-0} + \sum_{0 \neq v \leq_\pi s-l} \binom{s-l}{v} x^{r-k} \partial^v (x^k) d^{s-v} \Big] e \end{aligned}$$

$$= \quad x^r d^s e + \sum_{0\neq v \leq_\pi s-l} \binom{s-l}{v} x^{r-k} \partial^v (x^k) d^{s-v} e = t_2 + \sum_j n_j s_j.$$

Since the index v in the previous sum is in $\mathbb{N}^n \setminus 0$ the conditions on the s_j are obvious. $\square$

Lemma 4. *Let $t_1, t_2, w \in \Lambda E$. Then*

$$t_1 \prec_x t_2 \wedge t_2 | w \Rightarrow \Big(\frac{w}{t_2} t_1\Big)_w = 0.$$

Proof. Set $t_1 = x^\alpha d^\beta e_1$, $t_2 = x^\gamma d^\delta e_2$, $w = x^r d^s e_2$, $\gamma \leq_\pi r$, $\delta \leq_\pi s$. From $t_1 \prec_x t_2$ we get $\nu_1(t_1) \leq \nu_1(t_1)$ whence $|\alpha| \leq |\gamma|$. From (10) we obtain

$$\begin{aligned} \frac{w}{t_2} \cdot t_1 &= x^{r-\gamma} d^{s-\delta} \cdot x^\alpha d^\beta e_1 = \sum_{u \leq_\pi s-\delta} \binom{s-\delta}{u} x^{r-\gamma} \partial^u (x^\alpha) d^{s-\delta+\beta-u} \cdot e_1 \\ &= \sum_{u \leq_\pi s-\delta} \binom{s-\delta}{u} \frac{\alpha!}{(\alpha-u)!} x^{r-\gamma+\alpha-u} d^{s-\delta+\beta-u} \cdot e_1. \end{aligned}$$

To derive a contradiction assume that $\Big(\frac{w}{t_2} \cdot t_1\Big)_w \neq 0$. Then

$$\exists u \,\big(0 \leq_\pi u \leq_\pi s - \delta \wedge x^{r-\gamma+\alpha-u}\, d^{s-\delta+\beta-u} \cdot e_1 = x^r d^s \cdot e_2\big) \text{ that is,}$$

$e_1 = e_2 \wedge \alpha = \gamma + u \wedge \beta = \delta + u \leq_\pi s$, i.e., $u = \beta - \delta \geq_\pi 0$, $\delta \leq_\pi \beta$.
If $u >_\pi 0$ then $|\alpha| = |\gamma| + |u| > |\gamma|$, a contradiction. Therefore $u = 0$, $\alpha = \gamma \wedge \beta = \delta \wedge e_1 = e_2$, i.e., $t_1 = t_2$. This contradicts the assumption $t_1 \prec_x t_2$. Consequently $\Big(\frac{w}{t_2} t_1\Big)_w = 0$. $\square$

Let $f, g, h \in F$, $g \neq 0$. (x, ∂)-reduction defined in [DL12] amounts to the following

$$\begin{aligned} f \xrightarrow[g]{(x,\partial)} h \quad &\iff \quad \exists w \in \mathrm{T}(f) \Big(\mathrm{LT}_x(g) | w \wedge h = f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g \\ &\qquad \wedge \nu_2\big(\frac{w}{\mathrm{LT}_x(g)} \cdot \mathrm{LT}_d(g)\big) \leq \nu_2\big(\mathrm{LT}_d(f)\big)\Big). \end{aligned} \tag{15}$$

Lemma 5. *Assume that $f \xrightarrow[g]{(x,\partial)} h$ and let w be a term mentioned in (15). Then $w \notin \mathrm{T}(h)$.*

Proof. Isolating the x-leader of g gives

$$\begin{aligned} h &= f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} \Big(\mathrm{LC}_x(g) \mathrm{LT}_x(g) + \sum_{t \in \Lambda E \setminus \{\mathrm{LT}_x(g)\}} g_t t\Big) \\ &= f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} \mathrm{LC}_x(g) \mathrm{LT}_x(g) - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} \sum_{t \neq \mathrm{LT}_x(g)} g_t t \\ &= f - f_w \frac{w}{\mathrm{LT}_x(g)} \cdot \mathrm{LT}_x(g) - \sum_{t \neq \mathrm{LT}_x(g)} \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g_t t \end{aligned}$$

Aplication of Lemma 3 gives

$$h \quad = \quad f - f_w \Big(w + \sum_j n_j s_j\Big) - \sum_{t \prec_x \mathrm{LT}_x(g)} \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g_t t$$

$$= \quad f - f_w w - \sum_j n_j f_w s_j - \sum_{t \prec_x \mathrm{LT}_x(g)} \frac{f_w g_t}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} t \tag{16}$$

where all $n_j > 0$ and $\nu_1(s_j) < \nu_1(w)$, $\nu_2(s_j) < \nu_2(w)$.

Considering (16), the coefficient of w in h is

$$h_w = f_w - f_w - 0 - \sum_{t \prec_x \mathrm{LT}_x(g)} \frac{f_w g_t}{\mathrm{LC}_x(g)} \Big(\frac{w}{\mathrm{LT}_x(g)} t \Big)_w .$$

Lemma 4 now immediately provides $h_w = 0$. $\square$

It is now possible to relate (x, ∂)-reduction to Gröbner reduction.

Theorem 3. *Let P denote the predicate*

$$P(f, g, \lambda) \iff \nu_2(\lambda \cdot g) \leq \nu_2(f). \text{ Then}$$

$$f \xrightarrow[g]{(x,\partial)} h \iff \exists \lambda \in \Lambda \Big(\mathrm{LT}_x(\lambda g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}_x(\lambda g)}}{\mathrm{LC}_x(\lambda g)} \lambda g \wedge P(f, g, \lambda) \Big). \tag{17}$$

Consequently, using notation (1) from section 3, we have

$$f \xrightarrow[g]{(x,\partial)} h \iff f \xrightarrow[g]{\rho} h \, .$$

Proof. Observe that

$$P(f, g, \lambda) \iff \nu_2\big(\lambda \cdot \mathrm{LT}_d(g)\big) \leq \nu_2\big(\mathrm{LT}_d(f)\big) \text{ c.f. [DL12].}$$

Let $f \xrightarrow[g]{(x,\partial)} h$ and set $\lambda = \frac{w}{\mathrm{LT}_x(g)}$, where w is a term mentioned in (15). Write $\mathrm{LT}_x(g) = x^k d^l e$, $w = x^{k+r} d^{l+s} e$. Thus $\lambda = x^r d^s$.

$\mathrm{LT}_x(\lambda g) = \mathrm{LT}_x(\lambda \cdot \mathrm{LT}_x(g)) = \mathrm{LT}_x(x^r d^s \cdot x^k d^l e) = x^{r+k} d^{s+l} e = w$ and $\mathrm{LC}_x(\lambda g) = \mathrm{LC}_x(g)$.

It follows that

$$h = f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g = f - \frac{f_{\mathrm{LT}_x(\lambda g)}}{\mathrm{LC}_x(\lambda g)} \lambda g \text{ and } \mathrm{LT}_x(\lambda g) = w \in \mathrm{T}(f).$$

Since $f \xrightarrow[g]{(x,\partial)} h$ holds, the predicate $P(f, g, \lambda)$ is true. Consequently $f \xrightarrow[g]{\rho} h$.

Conversely, assume that $f \xrightarrow[g]{\rho} h$. Let $\lambda \in \Lambda$ be such that $\mathrm{LT}_x(\lambda g) \in \mathrm{T}(f)$ and

$$h = f - \frac{f_{\mathrm{LT}_x(\lambda g)}}{\mathrm{LC}_x(\lambda g)} \lambda g \wedge P(f, g, \lambda).$$

Set $w = \mathrm{LT}_x(\lambda g)$. Then $w \in \mathrm{T}(f)$. Write λ as $\lambda = x^u d^v$ and $\mathrm{LT}_x(g) = x^k d^l e$. Then

$$w = \mathrm{LT}_x(\lambda \cdot \mathrm{LT}_x(g)) = \mathrm{LT}_x(x^u d^v \cdot x^k d^l e) = x^{u+k} d^{v+l} e.$$

Thus $\mathrm{LT}_x(g)|w$ and $\frac{w}{\mathrm{LT}_x(g)} = x^u d^v = \lambda$. Since $\mathrm{LC}_x(\lambda g) = 1 \cdot \mathrm{LC}_x(g)$ we obtain

$$h = f - \frac{f_w}{\mathrm{LC}_x(g)} \frac{w}{\mathrm{LT}_x(g)} g \wedge \nu_2\Big(\frac{w}{\mathrm{LT}_x(g)} \cdot \mathrm{LT}_d(g)\Big) \leq \nu_2(\mathrm{LT}_d(f)).$$

Consequently $f \xrightarrow[g]{(x,\partial)} h$. □

Proposition 4. *I_ρ is monomial.*

Proof. Let $\mathrm{LT}_x(\lambda \cdot g) \in \mathrm{T}(f) \wedge P(\mathrm{LT}_x(\lambda \cdot g), g, \lambda)$. This means that

$$\nu_2(\lambda \cdot g) \leq \nu_2(\mathrm{LT}_x(\lambda \cdot g)).$$

Since $\mathrm{LT}_x(\lambda \cdot g) \in \mathrm{T}(f)$ it follows $\mathrm{LT}_x(\lambda \cdot g) \preceq_d \mathrm{LT}_d(f)$ and therefore $\nu_2(\mathrm{LT}_x(\lambda \cdot g)) \leq \nu_2(\mathrm{LT}_d(f)) = \nu_2(f)$. Thus $\nu_2(\lambda \cdot g) \leq \nu_2(f)$. Consequently $\mathrm{LT}_x(\lambda \cdot g) \in \mathrm{T}(f) \wedge P(f, g, \lambda)$. This demonstrates that the monomial irreducibility condition (5) is satisfied. □

In [DL12] the authors define Gröbner bases differently. Formulated in our notation:

Definition 7. *Let N be a submodule of $F = \mathbf{k}^{(\Lambda E)}$ and $G \subseteq N \setminus 0$. G is a (x, ∂)-Gröbner basis for N iff*

$$\forall f \in N \setminus 0 \, \exists g \in G\Big(\mathrm{LT}_x(g)|\mathrm{LT}_x(f) \wedge \nu_2(g) - \nu_2(\mathrm{LT}_x(g)) \leq \nu_2(f) - \nu_2(\mathrm{LT}_x(f)).$$

We position this notion into the frame of our concepts.

Theorem 4. *Given $G \subseteq N$, let σ denote the leading term reduction corresponding to the relation $f \xrightarrow[G]{(x,\partial)} h$, i.e.,*

$$f \xrightarrow[g]{\sigma} h \iff \exists \lambda \in \Lambda\Big(\mathrm{LT}_x(\lambda g) = \mathrm{LT}_x(f) \wedge h = f - \frac{\mathrm{LC}_x(f)}{\mathrm{LC}_x(\lambda g)} \lambda g \wedge \nu_2(\lambda \cdot g) \leq \nu_2(f)\Big). \quad (18)$$

Then G is an (x, ∂)-Gröbner basis for N iff $I_\sigma \cap N = 0$.

Proof. Let f, g be elements of $F \setminus 0$.

$$\begin{aligned} f &= f_0 t_0 + \sum_{t \prec_x t_0} f_t t = f_0' t_0' + \sum_{t \prec_d t_0'} f_t t \\ g &= g_1 t_1 + \sum_{t \prec_x t_1} g_t t = g_1' t_1' + \sum_{t \prec_d t_1'} g_t t \text{ with} \end{aligned}$$

$$\begin{array}{ll} \mathrm{LT}_x(f) = t_0 = x^{k_0} d^{l_0} e_0 & \mathrm{LT}_d(f) = t_0' = x^{k_0'} d^{l_0'} e_0' \\ \mathrm{LT}_x(g) = t_1 = x^{r_1} d^{s_1} e_1 & \mathrm{LT}_d(g) = t_1' = x^{r_1'} d^{s_1'} e_1'. \end{array}$$

Assume that G is an (x, ∂)-Gröbner basis for N and $f \in N \setminus 0$. $\exists g \in G$ such that

$$\mathrm{LT}_x(g)|\mathrm{LT}_x(f) \wedge \nu_2(g) - \nu_2(\mathrm{LT}_x(g)) \leq \nu_2(f) - \nu_2(\mathrm{LT}_x(f).$$

Using the notation from above we get

$$x^{r_1}d^{s_1}e_1|x^{k_0}d^{l_0}e_0 \wedge s'_1 - s_1 \leq l'_0 - l_0$$

hence $r_1 \leq_\pi k_0 \wedge s_1 \leq_\pi l_0 \wedge e_1 = e_0$. Set $\lambda = x^{k_0-r_1}d^{l_0-s_1}$. Then

$$\mathrm{LT}_x(\lambda \cdot g) = \mathrm{LT}_x(x^{k_0-r_1}d^{l_0-s_1} \cdot x^{r_1}d^{s_1}e_1) = x^{k_0}d^{l_0}e_0 = \mathrm{LT}_x(f).$$

$$\nu_2(\lambda \cdot g) = \nu_2(\mathrm{LT}_d(x^{k_0-r_1}d^{l_0-s_1} \cdot x^{r'_1}d^{s'_1}e'_1)) = l_0 - s_1 + s'_1 \leq l'_0 = \nu_2(f).$$

This shows that

$$f \xrightarrow[g]{\sigma} f - \tfrac{\mathrm{LC}_x(f)}{\mathrm{LC}_x(\lambda \cdot g)}\lambda \cdot g$$

that means, f is σ-reducible. Consequently, each $f \in N \setminus 0$ is σ-reducible whence $N \cap I_\sigma = 0$.

Conversely, assume that $N \cap I_\sigma = 0$ and let $f \in N \setminus 0$. Then f is σ-reducible, $\exists g \in G$ $\exists \lambda \in \Lambda$ such that

$$\mathrm{LT}_x(\lambda \cdot g) = \mathrm{LT}_x(f) \wedge h = f - \frac{\mathrm{LC}_x(f)}{\mathrm{LC}_x(\lambda \cdot g)}\lambda \cdot g \wedge \nu_2(\lambda \cdot g) \leq \nu_2(f). \tag{19}$$

Write $\lambda = x^a d^b$. From (19) we get

$$x^{k_0}d^{l_0}e_0 = \mathrm{LT}_x(f) = \mathrm{LT}_x(\lambda \cdot g) = \mathrm{LT}_x(x^a d^b \cdot x^{r_1}d^{s_1}e_1) \text{ hence}$$

$$k_0 = a + r_1 \wedge l_0 = b + s_1 \wedge e_0 = e_1 \text{ and so } r_1 \leq_\pi k_0 \wedge s_1 \leq_\pi l_0 \text{ i.e.}$$

$\mathrm{LT}_x(g)|\mathrm{LT}_x(f)$. Moreover

$$\nu_2(\lambda \cdot g) = \nu_2(\mathrm{LT}_d(\lambda \cdot g)) = \nu_2(\mathrm{LT}_d(x^a d^b \cdot x^{r'_1}d^{s'_1}e'_1)) = b + s'_1.$$

From (19) we obtain $b + s'_1 \leq l'_0$ and so

$$\underbrace{b + s_1}_{l_0} + s'_1 \leq l'_0 + s_1. \text{ Therefore}$$

$$\begin{aligned} \nu_2(g) - \nu_2(\mathrm{LT}_x(g)) &= \nu_2(x^{r'_1}d^{s'_1}e'_1) - \nu_2(x^{r_1}d^{s_1}e_1) = s'_1 - s_1 \\ &\leq l'_0 - l_0 = \nu_2(f) - \nu_2(\mathrm{LT}_x(f)). \end{aligned}$$

Therefore G is an (x, ∂)-Gröbner basis for N. □

Corollary 5. *Let ρ denote (x, ∂)-reduction (15) for N. If G is an (x, ∂)-Gröbner basis for N then ρ is a weak reduction for N. Thus, an (x, ∂)-Gröbner basis for N is a Gröbner basis for N w.r.t. ρ.*

Proof. Consider an (x, ∂)-Gröbner basis for N. Since $I_\rho \subseteq I_\sigma$ we obtain that $I_\rho \cap N = 0$. Together with Proposition 4 this says that ρ is a weak reduction for N. □

Even when G is a Gröbner basis, the corresponding reduction relation ρ is in general not a strong reduction.

Example 1. Consider $A = A_1(\mathbf{k})$, $g = xd + d^2 \in A$. Let ρ be the (x, ∂)-reduction defined by $G = \{g\}$, and $N = Ag$.

We show that $N \cap I_\rho = 0$. Let $a \in A$

$$a = a_0 x^{k_0} d^{l_0} + \sum_{\mu \prec_x x^{k_0} d^{l_0}} a_\mu \mu \quad (a_0 \neq 0). \text{ Then}$$

$$\begin{aligned} a \cdot g &= a_0 x^{k_0} d^{l_0}(xd + d^2) + \sum_{\mu \prec_x x^{k_0} d^{l_0}} a_\mu \mu(xd + d^2) \\ &= a_0 x^{k_0} d^{l_0} \cdot xd + a_0 x^{k_0} d^{l_0} \cdot d^2 + \sum_{\mu \prec_x x^{k_0} d^{l_0}} (a_\mu \mu xd + a_\mu \mu d^2). \end{aligned}$$

Set $\lambda = x^{k_0} d^{l_0}$. Then $\mathrm{LT}_x(\lambda \cdot g) = \mathrm{LT}_x(x^{k_0} d^{l_0} \cdot xd) = x^{k_0+1} d^{l_0+1}$.

$$\mathrm{LT}_x(a \cdot g) = \mathrm{LT}_x(\mathrm{LT}_x(a) \cdot \mathrm{LT}_x(g)) = \mathrm{LT}_x(x^{k_0} d^{l_0} \cdot xd) = x^{k_0+1} d^{l_0+1}$$

and thus $\mathrm{LT}_x(\lambda \cdot g) = \mathrm{LT}_x(a \cdot g) \in \mathrm{T}(a \cdot g)$.

$$\nu_2(\lambda \cdot g) = \nu_2(\mathrm{LT}_d(\lambda \cdot g)) = \nu_2(\mathrm{LT}_d(\lambda \cdot \mathrm{LT}_d(g))) = \nu_2(\mathrm{LT}_d(x^{k_0} d^{l_0} \cdot d^2)) = l_0 + 2$$

$$\nu_2(a \cdot g) = \nu_2(\mathrm{LT}_d(a \cdot g)) = \nu_2\big(\mathrm{LT}_d(\mathrm{LT}_d(a) \cdot \mathrm{LT}_d(g))\big) = \nu_2(\mathrm{LT}_d(a) \cdot d^2).$$

Now $x^{k_0} d^{l_0} \preceq_d \mathrm{LT}_d(a)$. Applying Lemma 2 gives

$$\underbrace{\mathrm{LT}_d(x^{k_0} d^{l_0} \cdot d^2)}_{x^{k_0} d^{l_0+2}} \preceq_d \mathrm{LT}_d(\mathrm{LT}_d(a) \cdot d^2). \text{ Therefore}$$

$$l_0 + 2 = \nu_2(x^{k_0} d^{l_0+2}) \leq \nu_2(\mathrm{LT}_d(\mathrm{LT}_d(a) \cdot d^2)) = \nu_2(\mathrm{LT}_d(a) \cdot d^2) \text{ and so}$$

$\nu_2(\lambda \cdot g) \leq \nu_2(a \cdot g)$. All in all

$$\exists \lambda \in \Lambda\Big(\mathrm{LT}_x(\lambda \cdot g) = \mathrm{LT}_x(a \cdot g) \wedge \nu_2(\lambda \cdot g) \leq \nu_2(a \cdot g)\Big)$$

and choosing h appropriately we see that $a \cdot g \xrightarrow[g]{\sigma} h$ whence $a \cdot g$ is σ-reducible. Therefore $N \cap I_\sigma = N \cap I_\rho = 0$. Consequently ρ is a weak reduction for $N = Ag$ and $\{g\}$ a Gröbner basis.

Now consider $f_1, f_2, g \in A$

$$f_1 = xd, \quad f_2 = d^2.$$

Then it is obvious that $f_1, f_2 \in I_\rho$. But $f_1 + f_2$ is not:

$$\mathrm{LT}_x(1 \cdot g) = xd \in \mathrm{T}(f) \wedge \nu_2(1 \cdot g) = 2 \leq \nu_2(f_1 + f_2) \text{ this means } \; f_1 + f_2 \xrightarrow[g]{\rho} 0$$

and so $f_1 + f_2 \notin I_\rho$. This shows that I_ρ is not closed under addition and therefore ρ is not a strong reduction for N.

Theorem 5. *Let $f, g, h \in F$, $g \neq 0$. Assume that $f \xrightarrow[g]{(x,\partial)} h$. Then, for arbitrary $r, s, \in \mathbb{N}$*

1. *$f \in F^1_r \Rightarrow h \in F^1_r$;*
2. *$f \in F^2_r \Rightarrow h \in F^2_r$;*
3. *$f \in F_{r,s} \Rightarrow h \in F_{r,s}$;*
4. *$f \in F^0_r \Rightarrow h \in F^0_{2r}$.*

Consequently the full reduction corresponding to an (x,∂)-Gröbner basis for a submodule $N \subseteq F$ is a weak Gröbner reduction for N w.r.t. these filtrations.

Proof. By Proposition 3 we may assume that $f \xrightarrow[g]{\rho} h$, i.e.,

$$\exists \lambda \in \Lambda\Big(\mathrm{LT}(\lambda g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}(\lambda g)}}{\mathrm{LC}(\lambda g)}\lambda g \wedge P(f, g, \lambda)\Big).$$

1. Assume that $f \in F^1_r$. Then $\nu_1(f) \leq r$ whence $\forall t \in \mathrm{T}(f) : \nu_1(t) \leq r$.
   Take $t \in \mathrm{T}(h)$. If $t \in \mathrm{T}(f)$ then $\nu_1(t) \leq r$. If $t \notin \mathrm{T}(f)$ then

   $$0 \neq h_t = -\frac{f_{\mathrm{LT}_x(\lambda g)}}{\mathrm{LC}_x(\lambda g)}(\lambda g)_t.$$

   Thus $(\lambda g)_t \neq 0$, $t \in \mathrm{T}(\lambda g)$, $t \preceq_x \mathrm{LT}_x(\lambda g) \in \mathrm{T}(f)$. Therefore $\nu_1(t) \leq \nu_1(\mathrm{LT}_x(\lambda g)) \leq r$. This means that $\mathrm{T}(h) \subseteq F^1_r$. Consequently $h \in F^1_r$.

2. $f \in F^2_r$. Then $\mathrm{T}(f) \subseteq F^2_r$. Writing out the predicate P we obtain

   $$\nu_2(\lambda \cdot \mathrm{LT}_d(g)) \leq \nu_2(\mathrm{LT}_d(f)).$$

   Take $t \in \mathrm{T}(h)$. If $t \in \mathrm{T}(f)$ then $t \in F^2_r$. If $t \notin \mathrm{T}(f)$ then, with the same argument as in the previous case, we obtain $t \in \mathrm{T}(\lambda \cdot g)$. Therefore $t \preceq_d \mathrm{LT}_d(\lambda \cdot g) = \mathrm{LT}_d(\lambda \cdot \mathrm{LT}_d(g))$. Then we derive

   $$\nu_2(t) \leq \nu_2(\mathrm{LT}_d(\lambda \cdot \mathrm{LT}_d(g)) = \nu_2(\lambda \cdot \mathrm{LT}_d(g)) \leq \nu_2(\mathrm{LT}_d(f))$$

   whence $\nu_2(t) \leq r$, that is, $t \in F^2_r$. This shows $\mathrm{T}(h) \subseteq F^2_r$. Since the filtersets are vector spaces we arrive at $h \in F^2_r$.

3. If $f \in F_{r,s}$ then $f \in F^1_r \cap F^2_s$. Therefore also $h \in F^1_r \cap F^2_s = F_{r,s}$.

4. This follows from Corollary 4 and the previous point.

□

4 The Ring of Difference-Differential Operators

Let $\delta = (\delta_1, \dots, \delta_m)$ be a tuple of derivations and $\sigma = (\sigma_1, \dots, \sigma_n)$ a tuple of automorphisms of the commutative ring K. All these maps are assumed to commute with each other. The ring D is then constructed as the free K-module on the set of formal expressions

$$\delta^k \sigma^l = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n}, \quad (k_i \in \mathbb{N}, l_i \in \mathbb{Z})$$

and a product that reflects the properties of derivations and automorphisms. We consider the elements of the set $\Lambda = \{\delta^k \sigma^l \mid (k,l) \in \mathbb{N}^m \times \mathbb{Z}^n\}$ as the distinguished monomials. Consequently elements of D are finite K-linear combinations

$$\sum_{(k,l)\in\mathbb{N}^m\times\mathbb{Z}^n} a_{k,l}\delta^k\sigma^l, \quad (a_{k,l} \in K)$$

and the product is driven by the rules

$$\delta_i \cdot c = c \cdot \delta_i + \delta_i(c) \quad \sigma_j \cdot c = \sigma_j(c)\sigma_j, \quad (c \in K).$$

We call D a difference-differential ring, or $\Delta\Sigma$-ring over K.

A left module over D is also called a *difference-differential module*, or $\Delta\Sigma$-module over K.[1] The concept covers difference modules ($\Delta = \emptyset$) as well as differential modules ($\Sigma = \emptyset$) as special instances.

Proposition 5. *Consider a field* $\mathbf{k}$ *with* $\operatorname{char}(\mathbf{k}) = 0$. *Let* $K = \mathbf{k}[x_1, \dots, x_m]$, $\Delta = \{\frac{d}{dx_1}, \dots, \frac{d}{dx_m}\}$ *and* $\Sigma = \emptyset$. *Then the resulting* $\Delta\Sigma$*-ring is the Weyl-algebra* $A_m(\mathbf{k})$.

Proof. This is due to the fact that partial derivatives have no relations among each other. Precisely: Let $\Delta^\star$ be the monoid generated (in $\operatorname{End}_{\mathbf{k}}(K)$) by Δ. Then $\Delta^\star \cong \mathbb{N}^m$ and $A_m(\mathbf{k})$ is a free K-module with basis $\Delta^\star$. $\square$

We use the notation

$$y^k = \delta^k(y) \text{ and } y_s = \sigma^s(y) \quad (k \in \mathbb{N}^m,\ s \in \mathbb{Z}^n).$$

For the free D-module $F = D^{(E)}$, the product $D \times F \longrightarrow F$ can then be written explicitly

$$\delta^k\sigma^l \cdot y\delta^r\sigma^s e = \sum_{u \le_\pi k} \binom{k}{u} y_l^{k-u} \delta^{u+r}\sigma^{l+s} e \quad (k, r \in \mathbb{N}^m;\ l, s \in \mathbb{Z}^n;\ y \in K;\ e \in E). \tag{20}$$

For $\lambda = \delta^k\sigma^l \in \Lambda$ we set

$$\nu_1(\lambda) = |k|, \quad \nu_2(\lambda) = |l|, \quad \nu_0 = \nu_1 + \nu_2. \tag{21}$$

The extensions of these functions induce the univariate filtrations

$$D_t^j = \{a \in D : \nu_j(a) \le t\} \quad (j = 0, 1, 2).$$

[1] In the literature the tuples δ and σ are denoted informally as the sets Δ and Σ, whence the name. Note though, that the mappings δ_i need not be distinct. The same is the case with the σ_j.

Proposition 6. *The family $(D^j_t)_{t\in\mathbb{N}}$ is a monomial filtration on D ($j = 0,1,2$). Therefore $(F^j_t)_{t\in\mathbb{N}}$ is a monomial filtration on F.*

Fix an enumeration of the set E and set

$$t = \delta^k \sigma^l e_i \mapsto (\nu_j(t), i, k_1, \dots, k_m, |l_1|, \dots, |l_n|, \operatorname{sgn}(l_1)+1, \dots, \operatorname{sgn}(l_n)+1) \qquad (j \in \{0,1,2\}).$$

The corresponding well-orders for monomials $s = \delta^k\sigma^l e_i$, $t = \delta^r\sigma^s e_j$ in ΛE are now

$$\begin{aligned} s \prec_j t \quad :\iff & \\ & (\nu_j(s), i, k_1, \dots, k_m, |l_1|, \dots, |l_n|, \operatorname{sgn}(l_1)+1, \dots, \operatorname{sgn}(l_n)+1) \\ & <_{\mathrm{LEX}} \\ & (\nu_j(t), j, r_1, \dots, r_m, |s_1|, \dots, |s_n|, \operatorname{sgn}(s_1)+1, \dots, \operatorname{sgn}(s_n)+1) \end{aligned}$$

Then $s \preceq_j t \Rightarrow \nu_j(s) \le \nu_j(t)$ ($j = 0,1,2$).

These orders single out $\mathrm{LT}j(f)$ and $\mathrm{LC}j(f)$ for each $f \in F \setminus 0$. According to (1) we get

$$f \xrightarrow[g]{\rho_j} h \iff \exists \lambda \in \Lambda\Big(\mathrm{LT}_j(\lambda g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}_j(f)}}{\mathrm{LC}_j(\lambda g)}\lambda g\Big)$$

and for $G \subseteq F$ ρ_j is

$$f \xrightarrow[G]{\rho_j} h \iff \exists g \in G \text{ such that } f \xrightarrow[g]{\rho_j} h\,.$$

Note that the predicate ‘P’ mentioned in (1) is empty here, that is, we may set $P = \mathrm{TRUE}$.

Proposition 7. $f \xrightarrow[G]{\rho_j} h$ *and* $f \in F^j_t \Rightarrow h \in F^j_t$.

Proof. There exists $g \in G$ and $\lambda \in \Lambda$ such that $\mathrm{LT}_j(\lambda g) \in \mathrm{T}(f)$. Therefore, from monomiality of the filtration, we get $\mathrm{LT}_j(\lambda g) \in F^j_t$. Let $b \in \mathrm{T}(\lambda g)$ be an arbitrary monomial. Then from $b \preceq_j \mathrm{LT}_j(\lambda g)$ we obtain $\nu_j(b) \le \nu_j(\mathrm{LT}_j(\lambda g) \le t$, that is, $b \in F^j_t$. Consequently $\lambda g \in F_t$, and so is $h = f - c \cdot \lambda g$. □

Together with the previous remarks, the last proposition exhibits the relations ρ_ν as Gröbner reductions.

Relative reduction in Zhou/Winkler

In [ZW07] the filtration F^0 is treated by using a variant of the term order $\prec_0$ and its corresponding reduction. In [ZW08a] the bivariate filtration $D_{r,s} = D^1_r \cap D^2_s$ occurs. For the purpose of reduction the following two term orders have been used. For monomials $u = \delta^k\sigma^l e_i$ and $v = \delta^r\sigma^s e_j$ in ΛE, set

$$\begin{aligned} u \prec_1 v \quad \iff & \\ & (\nu_2(u), \nu_1(u), i, k_1, \dots, k_m, |l_1|, \dots, |l_n|, \operatorname{sgn}(l_1)+1, \dots, \operatorname{sgn}(l_n)+1) \\ & <_{\mathrm{LEX}} \\ & (\nu_2(v), \nu_1(v), j, r_1, \dots, r_m, |s_1|, \dots, |s_n|, \operatorname{sgn}(s_1)+1, \dots, \operatorname{sgn}(s_n)+1) \end{aligned}$$

respectively

$$
\begin{aligned}
u \prec_2 v \quad &\Longleftrightarrow \\
&(\nu_1(u), \nu_2(u), i, k_1, \dots, k_m, |l_1|, \dots, |l_n|, \operatorname{sgn}(l_1)+1, \dots, \operatorname{sgn}(l_n)+1) \\
&<_{\mathrm{LEX}} \\
&(\nu_1(v), \nu_2(v), j, r_1, \dots, r_m, |s_1|, \dots, |s_n|, \operatorname{sgn}(s_1)+1, \dots, \operatorname{sgn}(s_n)+1).
\end{aligned}
$$

The appropriate reduction concept - called `relative reduction` in [ZW08a] - takes into account both of these orders. Let $f, g, h \in F$. Then $f \xrightarrow{\mathrm{rel}}_g h$ iff

$$\exists \lambda \in \Lambda\Big(\mathrm{LT}_1(\lambda g) = \mathrm{LT}_1(f) \wedge \mathrm{LT}_2(\lambda g) \preceq_2 \mathrm{LT}_2(f) \wedge h = f - \frac{\mathrm{LC}_1(f)}{\mathrm{LC}_1(\lambda g)}\lambda g\Big).$$

Here we meet leading term reduction (2) involving the predicate

$$P(f, g, \lambda) \iff \mathrm{LT}_2(\lambda g) \preceq_2 \mathrm{LT}_2(f).$$

Again, for $G \subseteq F$ relative reduction is

$$f \xrightarrow{\mathrm{rel}}_G h \iff \exists g \in G \text{ with } f \xrightarrow{\mathrm{rel}}_g h.$$

Theorem 6. *Let $F_{r,s} = \bigoplus_{e \in E} D_{r,s} e$ denote the bivariate filtration on F induced by $(D_{r,s})_{(r,s)\in\mathbb{N}^2}$. Then $f \xrightarrow{\mathrm{rel}}_g h \wedge f \in F_{r,s} \implies h \in F_{r,s}$. Consequently, the full reduction associated to $\xrightarrow{\mathrm{rel}}_G$ is a Gröbner reduction.*

A proof can be found in [FL15].

In order to solve problems arising from negative exponents, the authors of [ZW08a] introduced in [ZW06] the concept of an *orthant decomposition.* Their approach treats negative exponents by covering the set of monomials ΛE with finitely many isomorphic copies of $\mathbb{N}^m \times \mathbb{N}^n \times E$.

A detailed discussion can be found in [ZW06, ZW07, ZW08a, ZW08b, Lev12].

5 The ring of Ore polynomials

Given a K-endomorphism $\sigma\colon K \longrightarrow K$, a σ-skew derivation is an additive map $\delta\colon K \longrightarrow K$ satisfying

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b, \quad (a, b \in K).$$

An Ore-variable over K is a pair $\partial = (\sigma, \delta)$ where σ is an endomorphism and δ is a σ-skew derivation.

Let $\partial_i = (\sigma_i, \delta_i)$ be Ore-variables $(1 \le i \le n)$ such that all mappings σ_i, δ_j commute with each other. Then the Ore algebra $\mathbb{O}$ defined by $X = (\partial_1, \dots, \partial_n)$ is the free K-module on the set of formal expressions $\partial^k = \partial_1^{k_1} \cdots \partial_n^{k_n}$ with multiplication determined by the rules

$$\partial_i \cdot \partial_j = \partial_j \cdot \partial_i \text{ and } \partial_i \cdot x = \sigma_i(x)\partial_i + \delta_i(x) \quad (x \in K). \tag{22}$$

We set $\Lambda = \{\partial^k \colon k \in \mathbb{N}^n\} \cong \mathbb{N}^n$, as usual its elements are called monomials.

With the convenient notation

$$x_k^l = (\delta^l \circ \sigma^k)(x) \quad (k, l \in \mathbb{N}^n,\ x \in K) \tag{23}$$

the product in $\mathbb{O}$ may be written explicitly

$$\begin{aligned} x\partial^l \cdot y\partial^q &= \sum_{v \in \mathbb{N}^n} \binom{l}{v} x y_v^{l-v} \partial^{q+v} \\ &= \sum_{v \leq_\pi l} \binom{l}{v} x y_{l-v}^{v} \partial^{l+q-v} \end{aligned} \tag{24}$$

where $x, y \in K$ and $l, q \in \mathbb{N}^n$. In particular

$$x\partial^0 \cdot y\partial^q = xy\partial^q$$

demonstrating that K is naturally a subring of $\mathbb{O}$.

Example 2. Let δ_i be ordinary derivations $(1 \leq i \leq m)$ and σ_j automorphisms $(1 \leq j \leq n)$. Then $\mathbb{O}$ defined by the Ore-variables $\partial_i = (\delta_i, \mathrm{id})$ $(1 \leq i \leq m)$ and $\partial_j' = (0, \sigma_j)$ $(1 \leq j \leq n)$ is a difference-differential ring where the σ_j allow positive exponents only. The $\Delta\Sigma$-ring D defined by $\Delta = \{\delta_1, \ldots, \delta_m\}$ and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ as defined in the previous section can be obtained as follows:

Starting from Δ, Σ define Ore-variables

$$\begin{aligned} \partial_i &= (\delta_i, \mathrm{id}) \quad (1 \leq i \leq m) \\ \eta_j &= (0, \sigma_j) \quad (1 \leq j \leq n) \\ \phi_j &= (0, \sigma_j^{-1}) \quad (1 \leq i \leq n) \end{aligned}$$

and let $\mathbb{O}$ be the Ore-algebra defined by them. Let I be the 2-sided ideal generated by the set $\{\eta_j \cdot \phi_j - 1 \mid 1 \leq j \leq n\}$. Then $\mathbb{O}/I \cong D$.

Levin ([Lev07]) splits the set X of variables into disjoint subsets $X = X_1 \cup \cdots \cup X_p$. This gives order functions

$$\nu_j \colon \Lambda \longrightarrow \mathbb{N},\ \partial^k \mapsto \sum_{\partial_i \in X_j} k_i \quad (1 \leq j \leq p)$$

and the total degree function $\nu_0 = \nu_1 + \cdots + \nu_p$ that extend to the free module $F = \mathbb{O}^{(E)}$ where $E = \{e_1, \ldots, e_q\}$. As usual

$$\nu_j(\partial^k e) = \nu_j(\partial^k), \quad \nu_j(f) = \begin{cases} \max_{t \in \mathrm{T}(f)} \nu_j(t) & \ldots\ f \in F \setminus 0 \\ -\infty & \ldots\ f = 0 \end{cases}$$

for all $j = 0, 1, \ldots, p$.

For the remainig part of this section we assume that K be a field.

Lemma 6. *Let $x, y \in K \setminus 0$, $k, l \in \mathbb{N}^n$. Then, for $0 = 1, \ldots, p$*

$$\nu_j(x\partial^k \cdot y\partial^l e) = \nu_j(\partial^k) + \nu_j(\partial^l). \tag{25}$$

Proof. Take a term $t \in \mathrm{T}(x\partial^k \cdot y\partial^l)$. From (24) we see that $\exists v \leq_\pi k$ with $t = \partial^{l+v}e$. Thus, if $1 \leq j \leq p$ then

$$\nu_j(t) = \sum_{\partial_i \in X_j}(l_i + v_i) \leq \sum_{\partial_i \in X_j}(l_i + k_i) = \sum_{\partial_i \in X_j} l_i + \sum_{\partial_i \in X_j} k_i = \nu_j(\partial^l) + \nu_j(\partial^k) = \nu_j(xy_k\partial^{k+l}).$$

Since $xy_k \neq 0$ the assertion follows. The statement for $j = 0$ is seen by summing up all $j = 1, \ldots, p$. □

The p orders on ΛE considered in [Lev07] are defined by the p injections $\tau_j \colon \Lambda \longrightarrow \mathbb{N}^{n+p+1}$

$$\lambda = \partial^k \mapsto (\nu_j(\lambda), \nu_0(\lambda), \nu_1(\lambda), \ldots, \widehat{\nu_j(\lambda)}, \ldots, \nu_p(\lambda), k^j, k^1, \ldots, \widehat{k^j}, \ldots, k^p)$$

with notation $k = (k_1, \ldots, k_n) = (k^1, \ldots, k^p)$ where $k_i \in \mathbb{N}$, $k^j \in \mathbb{N}^{\mathrm{Card}(X_j)}$, and the extensions of τ_j to ΛE

$$\varphi_j \colon \Lambda E \longrightarrow \mathbb{N}^{n+p+2},\ t = \partial^k e_i \mapsto (\tau_j(\partial^k), i).$$

Thus, for terms $t_1, t_2 \in \Lambda E$,

$$t_1 \prec_j t_2 \iff \varphi_j(t_1) <_{\mathrm{LEX}} \varphi_j(t_2).$$

Note that $\nu_j(\partial^k) = |k^j|$ $(1 \leq j \leq p)$.

Lemma 7. *$\tau_j(\partial^{k+l}) = \tau_j(\partial^k) + \tau_j(\partial^l)$ $(1 \leq j \leq p)$.*

Proof. Using the notation from above it is plain that $(k+l)^j = k^j + l^j$. Therefore

$$\nu_j(\partial^{k+l}) = |(k+l)^j| = |k^j + l^j| = |k^j| + |l^j| = \nu_j(\partial^k) + \nu_j(\partial^l).$$

From this observation the statement is obvious. □

Leading term and leading coefficient functions are written LT_j, LC_j $(1 \leq j \leq p)$. As before it is plain that

$$\nu_j(f) = \nu_j(\mathrm{LT}_j(f))\ \forall j.$$

Proposition 8. *Let $x, y \in K \setminus 0$, $k, l \in \mathbb{N}^n$, $e \in E$. Then $\forall j = 1, \ldots, j$*

$$\mathrm{LT}_j(x\partial^k \cdot y\partial^l e) = \partial^{k+l}e.$$

Proof. Take a term $t \in \mathrm{T}(x\partial^k \cdot y\partial^l e)$ with $t \neq \partial^{k+l}e$. Then $\exists v <_\pi k$ with $t = \partial^{l+v}e$. Let $i_0 = \min\{i \colon v_i \neq k_i\}$. Then $v_{i_0} < k_{i_0}$ and $\forall i < i_0 \colon v_i = k_i$, hence also $(l+v)_{i_0} < (k+l)_{i_0}$ and $\forall i < i_0 \colon (l+v)_i = (k+l)_i$. Thus

$$i_0 = \min\{i \colon (l+v)_i \neq (k+l)_i\} \wedge (l+v)_{i_0} < (k+l)_{i_0} \wedge \forall i \colon (l+v)_i \leq (k+l)_i.$$

Therefore

$$\nu_0(t) = |l+v| = |l| + |v| < |l| + |k| = |k+l| = \nu_0(\partial^{k+l}e).$$

If $i_0 \in X_j$ then $\nu_j(t) < \nu_j(\partial^{k+l}e)$ whence $t \prec_j \partial^{k+l}e$.
If $i_0 \notin X_j$ then $\nu_j(t) \leq \nu_j(\partial^{k+l}e) \wedge \nu_0(t) < \nu_0(\partial^{k+l}e)$, and again $t \prec_j \partial^{k+l}e$. □

We may blow up the content of Lemma 7 to the following statement.

Corollary 6. *Let $a, b, x, y \in K \setminus 0$, $k, l, r \in \mathbb{N}^n$, $e_1, e_2 \in E$. Then $\forall j = 1, \ldots, p$*

$$\begin{array}{rcl} \partial^k \prec_j \partial^l & \Rightarrow & \mathrm{LT}_j(x\partial^k \cdot a\partial^r e_1) \prec_j \mathrm{LT}_j(y\partial^l \cdot b\partial^r e_2) \\ & \wedge & \mathrm{LT}_j(a\partial^r \cdot x\partial^k e_1) \prec_j \mathrm{LT}_j(b\partial^r \cdot y\partial^l e_2). \end{array}$$

Proof. We have to show that $\partial^{k+r}e_1 \prec_j \partial^{l+r}e_2$. From the hypothesis we have $\tau_j(\partial^k) <_{\mathrm{LEX}} \tau_j(\partial^l)$. Thus, using Lemma 7

$$\tau_j(\partial^{k+r}) = \tau_j(\partial^k) + \tau_j(\partial^r) <_{\mathrm{LEX}} \tau_j(\partial^l) + \tau_j(\partial^r) = \tau_j(\partial^{l+r}).$$

Therfore also $(\tau_j(\partial^{k+r}), 1) <_{\mathrm{LEX}} (\tau_j(\partial^{l+r}), 2)$. $\square$

Proposition 9. *Let $1 \leq j \leq p$. Let $a \in \mathbb{D} \setminus 0$ with $\mathrm{LT}_j(a) = \partial^{k_0}$ and $f \in F \setminus 0$. Then*

$$\begin{array}{rcl} \mathrm{LT}_j(a \cdot f) & = & \mathrm{LT}_j(a) \cdot \mathrm{LT}_j(f); \\ \mathrm{LC}_j(a \cdot f) & = & \mathrm{LC}_j(a) \cdot \sigma^{k_0}(\mathrm{LC}_j(f)). \end{array}$$

Proof. Set $a_0 = \mathrm{LC}_j(a)$, $\partial^{l_0}e_0 = \mathrm{LT}_j(f)$, $f_0 = \mathrm{LC}_j(f)$. Thus

$$a = a_0\partial^{k_0} + \sum_{\partial^k \prec_j \partial^{k_0}} a_k\partial^k \text{ and } f = f_0\partial^{l_0}e_0 + \sum_{\partial^l e \prec_j \partial^{l_0}e_0} f_{l,e}\partial^l e.$$

$$a \cdot f = \underbrace{a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0}_{(0)} + \underbrace{\sum_{\partial^l e \prec_j \partial^{l_0}e_0} a_0\partial^{k_0} \cdot f_{l,e}\partial^l e}_{(1)} + \underbrace{\sum_{\partial^k \prec_j \partial^{k_0}} a_k\partial^k \cdot f_0\partial^{l_0}e_0}_{(2)} + \underbrace{\sum_{\partial^k \prec_j \partial^{k_0}} \sum_{\partial^l e \prec_j \partial^{l_0}e_0} a_k\partial^k \cdot f_{l,e}\partial^l e}_{(3)}.$$

Pick out a summand of sum (1). If $\partial^l \prec_j \partial^{l_0}$ then $\mathrm{LT}_j(a_o\partial^{k_0} \cdot f_{l,e}\partial^l e) \prec_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0)$. If $\partial^l = \partial^{l_0}$ then $e < e_0$ and

$$\mathrm{LT}_j(a_o\partial^{k_0} \cdot f_{l,e}\partial^l e) = \partial^{k_0+l}e \prec_j \partial^{k_0+l_0}e_0 = \mathrm{LT}_j(a_k\partial^k \cdot f_0\partial^{l_0}e_0).$$

For a summand of sum (2) we obtain $\mathrm{LT}_j(a_k\partial^k \cdot f_0\partial^{l_0}e_0) \prec_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0)$.

As to sum (3), from the scope of the 1st sigma sign we derive $\mathrm{LT}_j(a_k\partial^k \cdot f_{l,e}\partial^l e) \prec_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^l e_0)$.

If $\partial^l \prec_j \partial^{l_0}$ then $\mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^l e_0) \prec_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0)$.
If $\partial^l = \partial^{l_0}$ then $\mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^l e_0) = \partial^{k_0+l} = \partial^{k_0+l_0} = \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0)$.

So, in any case, $\mathrm{LT}_j(a_k\partial^k \cdot f_{l,e}\partial^l e) \prec_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0)$.

Let $t \in \mathrm{T}(a \cdot f)$. Then t must be a term (surviving after cancellation) of one of the sum expressions (0),(1),(2),(3). Consequently $t \preceq_j \mathrm{LT}_j(a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0) = \partial^{k_0+l_0}e_0 = \partial^{k_0} \cdot \partial^{l_0}e_0 = \mathrm{LT}_j(a) \cdot \mathrm{LT}_j(f)$. Moreover we see that the expression $\partial^{k_0} \cdot \partial^{l_0}e_0$ does not cancel out. It follows that $\mathrm{LT}_j(a \cdot f) = \mathrm{LT}_j(a) \cdot \mathrm{LT}_j(f)$.

From the expansion of expression (0)

$$a_0\partial^{k_0} \cdot f_0\partial^{l_0}e_0 = \sum_{v\le\pi} \binom{k_0}{v} a_0(f_0)_v^{k_0-v}\partial^{l_0+v}e_0$$

we derive $\mathrm{LC}_j(a\cdot f) = a_0(f_0)_{k_0} = \mathrm{LC}_j(a)\cdot\mathrm{LC}_j(f)_{k_0} = \mathrm{LC}_j(a)\cdot\sigma^{k_0}(\mathrm{LC}_j(f))$. □

Corollary 7. *$\lambda = \partial^k \in \Lambda$, $f \in F\setminus 0$. Then $\forall j = 1,\dots,p$*

$$\mathrm{LT}_j(\lambda\cdot f) = \lambda\cdot\mathrm{LT}_j(f) \text{ and } \mathrm{LC}_j(\lambda\cdot f) = \sigma^k(\mathrm{LC}_j(f)).$$

Corollary 8. *$a \in \mathbb{O}$, $f \in F$. Then $\forall j = 0,\dots,p$*

$$\nu_j(a\cdot f) = \nu_j(a) + \nu_j(f).$$

Proof. If $a = 0 \vee f = 0$, the statement is true. So assume $a \neq 0 \wedge f \neq 0$. Let $1 \le j \le p$ and set $\mathrm{LT}_j(a) = \partial^{k_0}$, $\mathrm{LT}_j(f) = \partial^{l_0}e_0$. Then

$$\begin{aligned}\nu_j(a\cdot f) &= \nu_j(\mathrm{LT}_j(a\cdot f)) = \nu_j(\mathrm{LT}_j(a)\cdot\mathrm{LT}_j(f)) = \nu_j(\partial^{k_0}\cdot\partial^{l_0}e_0) = \nu_j(\partial^{k_0}) + \nu_j(\partial^{l_0})\\ &= \nu_j(\mathrm{LT}_j(a)) + \nu_j(\mathrm{LT}_j(f)) = \nu_j(a) + \nu_j(f).\end{aligned}$$

The statement for $j = 0$ follows by summation. □

Corollary 9. *$a \in \mathbb{O}$, $f \in F$. Then $a\cdot f = 0 \Rightarrow a = 0 \vee f = 0$.*
Consequently, $\mathbb{O}$ is a domain.

Proof. Let $a \neq 0 \wedge f \neq 0$ and ν one of $\nu_0, \nu_1, \dots, \nu_p$. Then $\nu(a) \ge 0 \wedge \nu(f) \ge 0$. Therefore

$$\nu(a\cdot f) = \nu(a) + \nu(f) \ge 0$$

hence $a\cdot f \neq 0$. □

The order functions ν_j propose a natural filtration concept.

$$F_t^j = \{f \in F\colon \nu_j(f) \le t\} \quad (t \in \mathbb{N},\ 0 \le j \le p). \tag{26}$$

For $\alpha \in \mathbb{N}^p$ we set

$$F_\alpha = \bigcap_{j=1}^{p} F_{a_j}^j = \{f \in F\colon \nu_1(f) \le \alpha_1 \wedge \dots \wedge \nu_p(f) \le \alpha_p\}. \tag{27}$$

Again the sets $\mathbb{O}_t^j$ and $\mathbb{O}_\alpha$ are implicitely defined ($F = \mathbb{O}^1$).

Proposition 10. *For all $1 \le j \le p$, the sets F_t^j define a univariate filtration on F w.r.t. the univariate filtration $\mathbb{O}_t^j$ in $\mathbb{O}$. Consequently F_α is a p-fold filtration w.r.t. $\mathbb{O}_\alpha$.*

Proof.

- If $f, g \in F_t^j$ then $\nu_j(f+g) \le \max\{\nu_j(f), \nu_j(g)\} \le t$, thus the sets F_t^j are abelian groups.
- $s \le t$ in $\mathbb{N}$ implies $F_s^j \subseteq F_t^j$ and $\bigcup_{t=0}^{\infty} F_t = F$.
- If $a \in \mathbb{O}_s^j \wedge f \in \mathbb{O}_t^j$ then $\nu_j(a\cdot f) = \nu_j(a) + \nu_j(f) \le s + t$, hence $\mathbb{O}_s^j \cdot F_t^j \subseteq F_{s+t}^j$.

□

Levins reduction with respect to several term orders

In [Lev07] the following theory is developed.

Definition 8. *$f, g \in F$, $g \neq 0$. Let $k, i_1, \ldots, i_r$ be distinct elements in $\{1, \ldots, p\}$, $I = \{i_1, \ldots, i_r\}$ and $L = (k, I)$. Then f is L-reduced w.r.t. g iff*

$$\neg\exists\lambda \in \Lambda\Big(\lambda \cdot \mathrm{LT}_k(g) \in \mathrm{T}(f) \wedge \forall i \in I \colon \nu_i(\lambda \cdot \mathrm{LT}_i(g)) \leq \nu_i(\mathrm{LT}_i(f))\Big) \tag{28}$$

f is L-reduced w.r.t. $G \subseteq F$ iff f is L-reduced w.r.t. g $\forall g \in G$.

The corresponding reduction concept in [Lev07] is

Definition 9. *$f, g, h \in F$, $g \neq 0$. I and $L = (k, I)$ as before. Then*

$$\begin{aligned} f \xrightarrow[g]{L} h \quad &\Longleftrightarrow \quad \exists w \in \mathrm{T}(f)\Big(\mathrm{LT}_k(g)|w \wedge h = f - \frac{f_w}{\tau_{\frac{w}{\mathrm{LT}_k(g)}}(\mathrm{LC}_k(g))} \frac{w}{\mathrm{LT}_k(g)} g \\ &\wedge \quad \forall i \in I : \nu_i\big(\frac{w}{\mathrm{LT}_k(g)} \cdot \mathrm{LT}_i(g)) \leq \nu_i\big(\mathrm{LT}_i(f)\big)\Big). \end{aligned} \tag{29}$$

Here for $\lambda \in \Lambda$ the symbol τ_λ denotes the exponent of λ - as a power of ∂ - considered as the corresponding endomorphism of K, precisely, if $\lambda = \partial^k$ and $x \in K$ then $\tau_\lambda(x) = \sigma^k(x)$.

Theorem 7. *Let $f, g, h \in F$, $g \neq 0$ and $L = (k, I)$ as before. Let P denote the predicate*

$$P(f, g, \lambda) \iff \forall i \in I : \nu_i(\lambda \cdot g) \leq \nu_i(f). \tag{30}$$

Let ρ denote the reduction relation

$$f \xrightarrow[g]{\rho} h \iff \exists\lambda \in \Lambda\Big(\mathrm{LT}_k(\lambda \cdot g) \in \mathrm{T}(f) \wedge h = f - \frac{f_{\mathrm{LT}_k(\lambda \cdot g)}}{\mathrm{LC}_k(\lambda \cdot g)} \lambda \cdot g \wedge P(f, g, \lambda)\Big). \tag{31}$$

Then

$$f \xrightarrow[g]{\rho} h \iff f \xrightarrow[g]{L} h\ .$$

Proof. Assume that $f \xrightarrow[g]{L} h$ and let $w \in \mathrm{T}(f)$ as mentioned in (29). Let $\mathrm{LT}_k(g) = \partial^l e$. Since $\mathrm{LT}_k(g)|w$ we may write $w = \partial^{l+p} e$. Set $\lambda = \frac{w}{\mathrm{LT}_k(g)} = \partial^p$. Then, by Corollary 7

$$\mathrm{LT}_k(\lambda \cdot g) = \lambda \cdot \mathrm{LT}_k(g) = \partial^p \cdot \partial^l e = \partial^{l+p} e = w.$$

In terms of the τ-notation we obtain

$$\tau_{\frac{w}{\mathrm{LT}_k(g)}}(\mathrm{LC}_k(g)) = \tau_\lambda(\mathrm{LC}_k(g)) = \sigma^p(\mathrm{LC}_k(g)) = \mathrm{LC}_k(\lambda \cdot g).$$

Consequently $\mathrm{LT}_k(\lambda \cdot g) \in \mathrm{T}(f)$ and $h = f - \frac{f_{\mathrm{LT}_k(\lambda \cdot g)}}{\mathrm{LC}_k(\lambda \cdot g)} \lambda \cdot g$. Since

$$\nu_i(\lambda \cdot g) = \nu_i(\mathrm{LT}_i(\lambda \cdot g)) = \nu_i(\lambda \cdot \mathrm{LT}_i(g)) = \nu_i\Big(\frac{w}{\mathrm{LT}_k(g)} \cdot \mathrm{LT}_i(g)\Big)$$

the formula $P(f,g,\lambda)$ is exactly the additional condition in (29), which means that $f \xrightarrow[g]{\rho} h$.

Conversely, assume that $f \xrightarrow[g]{\rho} h$. Let $\lambda = \partial^p$ as mentioned in the formula, $\mathrm{LT}_k(g) = \partial^l e$ and set $w = \mathrm{LT}_k(\lambda \cdot g) = \lambda \cdot \mathrm{LT}_k(g) = \partial^p \cdot \partial^l e = \partial^{p+l} e$. Then $\mathrm{LT}_k(g)|w$, $\frac{w}{\mathrm{LT}_k(g)} = \partial^p = \lambda$ and $\mathrm{LC}_k(\lambda \cdot g) = \sigma^p(\mathrm{LC}_k(g))$. In τ-notation:

$$\tau_{\frac{w}{\mathrm{LT}_k(g)}}(\mathrm{LC}_k(g)) = \tau_{\partial^p}(\mathrm{LC}_k(g)) = \sigma^p(\mathrm{LC}_k(g)) = \mathrm{LC}_k(\lambda \cdot g).$$

Therefore $w \in \mathrm{T}(f) \wedge \mathrm{LT}_k(g)|w$ and

$$h = f - \frac{f_w}{\tau_{\frac{w}{\mathrm{LT}_k(g)}}(\mathrm{LC}_k(g))} \frac{w}{\mathrm{LT}_k(g)} g \wedge P(f,g,\lambda)$$

and this means that $f \xrightarrow[g]{L} h$. □

Proposition 11. *P satisfies the monomial irreducibility condition (5). Therefore*

$$f \in I_\rho \Rightarrow \mathrm{T}(f) \subseteq I_\rho.$$

Proof. Assume $\exists g \in G$, $\exists \lambda \in \Lambda$ such that

$$\mathrm{LT}_k(\lambda \cdot g) \in \mathrm{T}(f) \wedge \mathrm{LC}_k(\lambda \cdot g) \in K^\times \wedge \forall i \in I : \nu_i(\lambda \cdot g) \leq \nu_i(\mathrm{LT}_k(\lambda \cdot g)).$$

Then $\nu_i(\lambda \cdot g) = \nu_i(\mathrm{LT}_k(\lambda \cdot g))$. As $\mathrm{LT}_k(\lambda \cdot g) \in \mathrm{T}(f)$ it follows $\nu_i(\mathrm{LT}_k(\lambda \cdot g)) \leq \nu_i(f)$, hence $\nu_i(\lambda \cdot g) \leq \nu_i(f)$. Consequently

$$\exists g \in G\, \exists \lambda \in \Lambda\colon \mathrm{LT}_k(\lambda \cdot g) \in \mathrm{T}(f) \wedge \mathrm{LC}_k(\lambda \cdot g) | f_{\mathrm{LT}_k(\lambda \cdot g)} \wedge \forall i \in I : \nu_i(\lambda \cdot g) \leq \nu_i(f).$$

□

6 Conclusion

We have designed the notion of a (weak) Gröbner reduction with the aim of describing several Gröbner basis concepts for finitely generated free modules over a wide class of noncommutative rings. To examine our system of axioms against known Gröbner basis techniques, we picked up three classes of such rings from the recent literature on Gröbner bases and dimension polynomials, and we have shown that the Gröbner basis notions introduced there fit into our general framework. The algorithmic generation of Gröbner bases in environments of great generality is subject to further research.

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