Efficient Computation of Multiplicity and Directional Multiplicity of an Isolated Point

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Abstract

The dual space of a primary ideal associated to an isolated point is a topic of study which appears in several occasions in symbolic computation. In the present work we elaborate on the computation of a representation for these dual spaces. A basis for the dual space reveals the multiplicity structure of the point under study. Macaulay's algorithm is a classic algorithm for computing such a basis [18]. However it is not the most efficient algorithm due to large matrix constructions and redundant computations. There are several improvements on Macaulay's algorithm. Mourrain's integration method serves as the most advanced algorithm, constructing much smaller matrices [23]. Both algorithms are incremental, i.e., they compute a basis for the dual space degree by degree, via computing the kernel of a certain matrix at each step. Recently, an improvement on the integration method has been developed , which avoids redundancy in computations by computing both a primal and a dual basis simultaneously [19]. In this work, we generalize the latter result by computing a polynomial primal basis along with the dual basis. This reduces the size of the matrices even further. We show that a similar improvement can be applied to Macaulay's algorithm as well. We also introduce the notion of *directional multiplicity*, which has applications in studying degeneracy in many problems involving variable elimination, in particular in arrangements of planar curves.

Contents

1 Introduction

The Problem. Consider an isolated point in the variety of a given ideal and its associated primary component. The quotient of the polynomial ring modulo this associated primary ideal is a vector space, whose dimension is the multiplicity of the point. A basis for the quotient and therefore the multiplicity can be computed via Gröbner bases.

It is classically known that the dual space of the polynomial ring is isomorphic to the space of differential operators acting at the point. This space is in general infinite dimensional. However, the dual space of a primary ideal is a finite dimensional subspace and can be treated computationally. In fact, the dimension of this subspace is the multiplicity of the point. Having the dual space of this subspace, a Gröbner basis of the primary component can be obtained from it. Computing a basis for this dual space is the main topic of this work.

Considering the differential operators as polynomials, there is a bound on the degree of the monomials of such polynomials, the so called Nil-index. The existence of this bound allows us to search for a basis incrementally, i.e., degree by degree, among the monomials with degree at most Nil-index. In fact a basis can be found among the linear combinations of such monomials. Assigning symbolic coefficients for those monomials and applying some necessary and sufficient conditions that the differential operators must satisfy, we obtain a matrix whose kernel gives us the values of the coefficients.

This argument reduces the problem into the kernel computation problem in some specific matrices. Because of the structure of the matrices that are constructed at each step of the procedure, they can be very large and also there are repeated and redundant computations. The problem of making improvements via constructing smaller matrices and efficient computations is at the epicenter of this work.

It turns out that a basis for the dual space has the advantage of locality compared to a (global) Gröbner basis computation for a given input ideal. The dual space shows us the local multiplicity structure at the point of interest, which provides us with information on its intrinsic geometry. The multiplicity structure plays an essential role in several problems related to multiplicity and elimination, which can be treated via directional multiplicities that will be introduced in this work. The motivation for this work stems from our earlier investigation on using resultants in Gröbner basis computation. The idea was to project a given ideal by resultants and then use it as an element in the elimination ideal in order to facilitate computing a Gröbner basis. This problem lead us to the multiplicity problem in the elimination ideal of two affine algebraic planar curves. Directional multiplicity can be used to study the geometric properties of a point and in particular to shed light to our motivational problem.

Previous Work. Multiplicity structure of isolated points has been well studied in literature [12, 18, 30, 26, 19] and it is an active research field with recent articles on the topic, e.g. [15]. There are efficient linear algebra algorithms to compute the multiplicity structure via dual space. A historical work conducted by Macaulay [18] shows how to construct the simplest matrices in order to compute a basis for the dual space. This algorithm is still used widely and several improvements have been made that make Macaulay's algorithm faster. Wu and Zhi worked on a symbolic-numeric method for computing the primary components and the differential operators [29], which is based on an algorithm for determining the dual space that is mentioned in the book [26] by Stetter. In [30] Zeng used the ideas in Stetter's algorithm and introduced his closedness property in order to make Macaulay's matrices smaller. Mourrain gave a new algorithm based on integration in [23], which is more efficient than the algorithm of Macaulay in terms of the size of the matrices. This algorithm was improved by Mantzaflaris and Mourrain in [19] by adding a new criterion. A detailed review of the integration method and its application to root deflation methods is given in [20]

Marinari, Mora and Möller's work on dual spaces in [21, 22], includes studying the behaviour of the dual space under projection, which is the base of our result related to the use of dual elements to study the elimination ideal. A survey on dual spaces, including Marinari, Mora and Möller's main results, is given in the book by Elkadi and Mourrain [12]. Also Bates, Peterson and Sommese have worked on the multiplicity of the primary components [3]. Li and Zhi's have investigated computing the Nil-index [17]. Examining the multiplicity structure via deflation is exhibited in the work of Dayton and Zeng [11] and Leykin and Verschelde [16].

Polynomials elimination theory is an old and central topic. The two main tools in elimination theory are Gröbner Bases and resultants. Our motivational ideas for using resultants in Gröbner basis computation is described in [24], which considers the elimination problem independent of the dual computation.

Buchberger introduced and expanded the Gröbner basis concept and gave an algorithm for Gröbner bases computation in his PhD thesis [5, 6]. Gröbner bases initiated a field of study in computational commutative algebra and algebraic geometry. The applications of Gröbner bases are countless both in theoretical as well as practical problems, when dealing with algebraic systems. Apart from the application of Gröbner bases in computing the multiplicity, we will extensively use its elimination property [6] that allows computing the elimination ideals.

Resultants is a classic tool in elimination theory. It has been extensively studied by Sylvester, Bezout, Dixon, Macaulay and van der Waerden [27, 28]. A smooth introduction to resultants, including Sylvester and Macaulay resultants is given in [8] and [9]. A survey on computational methods is given in [13], and a modern view towards the topic is [14].

Contributions. The main contributions of this work are improvements to the integration method and Macaulay's algorithm. As the size of the matrices constructed in each step of the algorithms is the main obstacle in computations, we propose criteria that allow deleting some columns from the matrices in order to reduce the size of the matrices.

For the integration method, the state of the art algorithm, in Proposition 8 we give an explicit generalization of the improvement done in [19], as we detect and use a polynomial basis for the quotient rather than the monomial basis. The new primal bases are in accordance with [23, Prop. 3.7], which can be generalized for the case in question. Corollary 19 shows our criterion for deleting some columns such that the kernel of the new matrix only detects new members of a basis of the dual space, which avoids recomputing the lower degree basis elements that are obtained in the previous steps. The reduction of lower-degree elements has been employed in [19] using a different criterion; however, under certain circumstances this criterion can increase the row-size of the matrix.

For Macaulay's algorithm, we propose two criteria, each reducing the size of the matrices at each step drastically. First we show Criterion 22, which is similar to the one for the integration method that deletes some columns at each step, so that we do not recompute the previously computed basis elements. Moreover, using the properties of the dual space, we show Criterion 11 that predicts that some columns will not appear in the basis. These criteria employ the ideas of the integration method in order to reduce the size of the Macaulay matrices.

Apart from those criteria that can be used for computing the whole dual basis, we introduce the notion of directional multiplicity in Definition 5, which can give us more information than the Nil-index, a classic invariant which has been the topic of various studies in the multiplicity structure field. Our modified algorithms can be used in several cases to compute the directional multiplicity faster than the whole dual space.

An interesting interplay between the directional multiplicity and the degree of the elimination ideal is presented. As an application, in studying arrangements and topology of curves one can use directional multiplicities in order to project the extreme point of a curve.

Structure of the paper. In Section 2, we first give a short introduction to dual spaces of polynomial rings. Then we focus on the multiplicity structure of an isolated point. In this way, we introduce directional multiplicity and the extended Buchberger diagram. We show bounds on the directional multiplicities with respect to Nil-index and the intersection multiplicity. Section 3 includes our main results. After demonstrating the existing algorithms for computing the dual space, we show our improvements on those algorithms and discuss the advantages. Section 4 contains two subsections. In Subsection 4.1 we briefly show some applications of directional multiplicities in computational problems. Subsection 4.2 includes the main problems that we are considering as the future directions of research.

Notation. We introduce the following notation that will be consistently used in this paper, unless otherwise is stated. For every ideal $I \trianglelefteq \mathbb{K}[x_1, \ldots, x_n]$ and for $J \subseteq \{1, \ldots, n\}$, the elimination ideal of I with respect to J consists of those polynomials in I that contain only the variables indexed by J and is denoted by $I_J := I \cap \mathbb{K}[x_1, \ldots, \hat{x_J}, \ldots, x_n]$. The i-th elimination ideal of I is defined to be $I_{i+1,\ldots,n} := I \cap \mathbb{K}[x_{i+1}, \ldots, x_n]$.

2 Directional Multiplicity

In this section, we take a look at the dual space of an ideal. This leads us to introduce the notion of *Directional Multiplicity*. Directional multiplicities give us a lot of information about the multiplicity structure at an isolated point. Using Lemma 3 we provide a sound definition of directional multiplicity. Then, we show that directional multiplicities can be bounded and can bound some other invariants of an ideal, namely the Nil-index and the intersection multiplicity.

2.1 Preliminaries on the Dual Space of a Polynomial Ring

We present a brief review of the dual space of polynomial rings from [23]. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and consider it as a vector space over $\mathbb{K}$. Denote by $\hat{R}$ the dual of R and note that it is a (not necessarily finite dimensional) vector space. Let $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{K}^n$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and define

$$\begin{array}{rccc} \partial_\zeta^{\mathbf{a}}: & R & \longrightarrow & \mathbb{K} \\ & p & \mapsto & (d_{x_1})^{a_1} \ldots (d_{x_n})^{a_n}(p)(\zeta), \end{array} \tag{1}$$

Then $\partial_\zeta^{\mathbf{a}}$ acts on p first by differentiation and then by evaluation at the point ζ.

In [23], Proposition 2.2 states that every element of the dual of R can be written as a formal power series of linear functions. More precisely, there is an isomorphism of $\mathbb{K}$-vector spaces between $\hat{R}$ and $\mathbb{K}[[\partial_\zeta]]$ given by the following correspondence:

$$\hat{R} \ni \lambda \quad \longleftrightarrow \quad \Lambda \;=\; \sum_{\mathbf{a} \in \mathbb{N}^n} \lambda\left(\prod (x_i - \zeta_i)^{a_i}\right) \frac{1}{\prod a_i!} \partial_\zeta^{\mathbf{a}} \in \mathbb{K}[[\partial_\zeta]].$$

For the rest of this work, unless otherwise stated, we assume that $\zeta = 0$. When it is clear from the context, we will use $\partial^{\mathbf{a}}$ instead of $\partial_\zeta^{\mathbf{a}}$. Also $\mathbb{K}[[\partial_\zeta]]$ denotes the $\mathbb{K}$-vector space of power series in the variables $dx_1, \ldots, dx_n$, which are linear forms that act on R as described in Equation 1. If it is clear from the context, we will use $\mathbb{K}[[\partial]]$ instead of $\mathbb{K}[[\partial_\zeta]]$. From now on, we identify $\hat{R}$ with $\mathbb{K}[[\partial_\zeta]]$. Also we may use $\partial_\zeta^{\mathbf{a}}$ instead of $\frac{1}{\prod a_i!} \partial_\zeta^{\mathbf{a}}$ in order to simplify computations.

One can consider $\hat{R}$ as an R-module, with multiplication by ∂_ζ at i-th coordinate in $\mathbb{K}[[\partial_\zeta]]$ acting as a derivation on polynomials. The orthogonal of an ideal I of R, i.e.,

$$I^\perp \;=\; \left\{\lambda \in \hat{R} \,:\, \lambda(f) = 0 \quad \forall f \in I\right\}$$

can be seen as a linear subspace of $\mathbb{K}[[\partial_\zeta]]$, for every $\zeta \in \mathbb{K}^n$, as shown in [23], Proposition 2.6.

Among ideals in R, primary ideals corresponding to an isolated point have an important property. Unlike an arbitrary ideal, the ortoghonal of such a primary ideal I can be identified by elements of the orthogonal of I that admit only finitely many non-zero coefficients. In other words, their orthogonal contains only polynomials. In fact, not many ideals are primary ideals corresponding to isolated points. However, if the given ideal has a primary component corresponding to an isolated point, then we can forget about the other components and work only on that component. Therefore, we deal with the local properties at one single point, i.e., the point corresponding to that primary ideal. Based on the above observation, we let ζ be an isolated point of the variety of I. Then the primary decomposition of I contains a primary ideal Q_ζ whose radical is of the form $\mathfrak{m}_\zeta = \langle x_1 - \zeta_1, \ldots, x_n - \zeta_n \rangle$. If $\sqrt{I} = \mathfrak{m}_\zeta$, then we call I an $\mathfrak{m}_\zeta$-primary ideal and usually we denote it by Q_ζ. Marinari, Mora and Möller in [21] have shown that the $\mathfrak{m}_\zeta$-primary ideals are in one-to-one correspondence with the non-null vector spaces of finite dimension of $\mathbb{K}[\partial]$, which are stable under derivation. This work is attributed to Gröbner.

The following theorem and in particular its corollary are essential for the algorithms that will be presented later.

Theorem 1. *(Theorem 3.2, [23]) Let I be an ideal of R with an $\mathfrak{m}_\zeta$-primary component Q_ζ. Then*

$$\left(I^\perp \cap \mathbb{K}[\partial_\zeta]\right)^\perp = Q_\zeta \textit{ and } Q^\perp = I^\perp \cap \mathbb{K}[\partial_\zeta],$$

where $\left(I^\perp \cap \mathbb{K}[\partial_\zeta]\right)^\perp = \left\{f \in R \,:\, \lambda(f) = 0 \quad \forall \lambda \in \langle D \rangle\right\}$.

From now on, given an $\mathfrak{m}_\zeta$-primary ideal Q_ζ, D will stand for a basis for $Q_\zeta^\perp$. Therefore $\langle D \rangle = Q_\zeta^\perp = I^\perp \cap \mathbb{K}[\partial_\zeta]$.

Corollary 2. *([23]) If $I = Q_\zeta$ is an $\mathfrak{m}_\zeta$-primary ideal, then we can identify $I^\perp$ with a linear subspace of the polynomial ring $\mathbb{K}[\partial_\zeta]$.*

Therefore, we are after computing a basis for a finite-dimensional linear subspace of $\mathbb{K}[\partial_\zeta]$.

2.2 Definition and Properties

Studying dual spaces, we define the directional multiplicity and show some of its properties. We also show how directional multiplicity gives information useful for elimination.

We first prove that the set of monomials that appear in elements of $Q^{\perp}$ is exactly the set of monomials $\partial^{\mathbf{a}}$ such that $x^{\mathbf{a}} \notin Q$, where $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. Let us observe that

$$\partial^{\mathbf{a}}(x^{\mathbf{b}}) = \prod \delta_{a_i,b_j} \tag{2}$$

where $\delta_{i,j}$ is the Kronecker delta. We prove the following proposition that first appeared without a proof in [19].

Proposition 1 (Characterization of Monomials in $Q^{\perp}$)**.** *Let $Q = Q_{\zeta}$ be an $\mathfrak{m}_{\zeta}$-primary ideal. Consider $Q^{\perp}$ as a sub-vector space of $\mathbb{K}[\partial_{\zeta}]$ as above. Then*

$$\bigcup_{\Lambda \in Q^{\perp}} \operatorname{supp}(\Lambda) \;=\; \left\{ \partial^{\boldsymbol{a}} \mid x^{\boldsymbol{a}} \notin Q \right\},$$

where $\operatorname{supp}(\Lambda)$ *is the set of monomials with nonzero coefficient in* Λ.

Proof. By Theorem 1, for all f

$$f \in Q \Leftrightarrow \left(\lambda(f) = 0 \text{ for all } \lambda \in Q^{\perp}\right).$$

Now choose a basis $D \subset \mathbb{K}[\partial]$ of $Q^{\perp}$, the above implies that for all f

$$f \in Q \Leftrightarrow (\lambda(f) = 0 \text{ for all } \lambda \in D).$$

We are ready to prove the thesis:

- "$\subseteq$" If $\partial^{\mathbf{a}}$ is in $\operatorname{supp}(\Lambda)$ then the monomial $x^{\mathbf{a}}$ is not annihilated by Λ (see Equation 2), which implies $x^{\mathbf{a}} \notin Q$.
- "$\supseteq$" If $x^{\mathbf{a}} \notin Q$, then there exists $\lambda \in D$ such that $\lambda(x^{\mathbf{a}}) \neq 0$. Let $\Lambda \in \mathbb{K}[\partial]$ be the differential operator corresponding to λ, so $\Lambda(x^{\mathbf{a}}) \neq 0$. By Equation 2, we know that $m(x^{\mathbf{a}}) = 0$ for all monomials m in $\operatorname{supp}(\Lambda)$ which are different from $\partial^{\mathbf{a}}$. Hence $\partial^{\mathbf{a}}$ has to be in $\operatorname{supp}(\Lambda)$.

$\square$

Now that we have a picture of the monomials in $Q^{\perp}$, we want to know how they look like under projection. The following result shows that the objects introduced so far, behave well in the framework of elimination theory.

Proposition 2 ([12], Proposition 7.19)**.** *Let π be the linear map*

$$\begin{array}{rccc} \pi: & \mathbb{K}[[dx_1, \ldots, dx_n]] & \longrightarrow & \mathbb{K}[[dx_2, \ldots, dx_n]] \\ & \Lambda & \mapsto & \Lambda(0, dx_2, \ldots, dx_n). \end{array}$$

Also suppose that I is an ideal in R and $I_{2,\ldots,n} = I \cap \mathbb{K}[x_2, \ldots, x_n]$ is its first elimination ideal. Then we have

$$(I_{2,\ldots,n})^{\perp} \;=\; \pi\left(I^{\perp}\right).$$

We use the above proposition in order to prove the Dual Projection Lemma which shows how to get a basis of the dual space of the elimination ideal, having a basis for the dual space. Note that I in Proposition 2 can be any ideal, however the following lemma is only for the local case, i.e., when we are working on an $\mathfrak{m}_{\zeta}$-primary ideal $Q = Q_{\zeta}$.

Lemma 3 (Dual Projection Lemma)**.** *With the hypotheses of Proposition 1, suppose that $D = \{\Lambda_0, \Lambda_1, \ldots, \Lambda_{l-1}\} \subset \mathbb{K}[\partial]$ is a basis of $Q^{\perp}$. Let $Q_{2,\ldots,n} = Q \cap \mathbb{K}[x_2, \ldots, x_n]$. Then*

$$Q^{\perp}_{2,\ldots,n} = \langle \Lambda_0|_{dx_1=0}, \Lambda_1|_{dx_1=0}, \ldots, \Lambda_{l-1}|_{dx_1=0} \rangle.$$

Proof. We prove the lemma by proving two inclusions.

- $(\supseteq)$ For all $i, (1 \leqslant i \leqslant l-1)$, since $\Lambda_i \in Q^{\perp}$, therefore we have that $\Lambda_i|_{dx_1=0} \in Q^{\perp}|_{dx_1=0}$. But since by proposition 2, $Q^{\perp}|_{dx_1=0} \in Q^{\perp}_{2,\ldots,n}$, then $\Lambda_i|_{dx_1=0} \in Q^{\perp}_{2,\ldots,n}$. This means that $\langle \Lambda_0|_{dx_1=0}, \Lambda_1|_{dx_1=0} \rangle \subseteq Q^{\perp}_{2,\ldots,n}$.

($\subseteq$) Suppose that $\Lambda' \in Q^{\perp}_{2,\ldots,n}$. Since by Proposition 2, $Q^{\perp}|_{dx_1=0} \in Q^{\perp}_{2,\ldots,n}$, then $\Lambda' \in Q^{\perp}|_{dx_1=0}$. Therefore, there exists a $\Lambda \in Q^{\perp}$, such that $\Lambda' = \Lambda|_{dx_1=0}$. We know that $Q^{\perp} = \langle \Lambda_0, \Lambda_1, \ldots, \Lambda_{l-1} \rangle$. So, there exist $c_i \in \mathbb{K}$, $(1 \leqslant i \leqslant l-1)$, such that $\Lambda = \sum_{i=0}^{l-1} c_i \Lambda_i$, and therefore $\Lambda|_{dx_1=0} = \sum_{i=0}^{l-1} c_i \Lambda_i|_{dx_1=0}$, which means that $\Lambda' = \sum_{i=0}^{l-1} c_i \Lambda_i|_{dx_1=0}$. Therefore

$$\Lambda' \in \langle \Lambda_0|_{dx_1=0}, \Lambda_1|_{dx_1=0}, \ldots, \Lambda_{l-1}|_{dx_1=0} \rangle .$$

Thus, $\quad Q^{\perp}_{2,\ldots,n} \subseteq \langle \Lambda_0|_{dx_1=0}, \Lambda_1|_{dx_1=0}, \ldots, \Lambda_{l-1}|_{dx_1=0} \rangle$.

□

Corollary 4. *Let $D = \{\Lambda_0, \Lambda_1, \ldots, \Lambda_{l-1}\} \subset \mathbb{K}[\partial]$ be a basis of $Q^{\perp}$, and $Q_i = Q \cap \mathbb{K}[x_i]$, for $1 \leqslant i \leqslant n$. Denote by $\Lambda|_{dx_i \neq 0}$ the polynomial obtained by substituting $dx_j = 0$ for $1 \leqslant i \neq j \leqslant n$ in Λ. Then*

$$Q_i^{\perp} = \langle \Lambda_0|_{dx_i \neq 0}, \Lambda_1|_{dx_i \neq 0}, \ldots, \Lambda_{l-1}|_{dx_i \neq 0} \rangle .$$

Moreover, there exists $\mu_i \in \mathbb{N}$ such that

$$Q_i^{\perp} = \left\langle 1, dx_i, \ldots, dx_i^{\mu_i - 1} \right\rangle .$$

Now we have the necessary tools to define the notion of directional multiplicity.

Definition 5 (**Directional Multiplicity**). *Let ζ be an isolated point in the variety of an ideal I and Q_ζ be the corresponding $\mathfrak{m}_\zeta$-primary component. Using the notation of Corollary 4, for $1 \leqslant i \leqslant n$, we define the $i-$th directional multiplicity of ζ to be μ_i.*

In order to give an intuition of directional multiplicity, we take a look at the quotient R/Q_ζ, which we will denote by B_ζ. If we consider this quotient as a vector space, finding a basis for such a quotient was the task given to Buchberger for his PhD thesis by Gröbner, which led to the invention of Gröbner bases [5]. Let us recall that the multiplicity of ζ is defined as $dim_{\mathbb{K}}\ R/Q_\zeta$. We will denote the multiplicity by $\mu(\zeta)$ or simply by μ if ζ is clear from the context. Another notion that is highly studied in the literature and describes an intrinsic parameter of an $\mathfrak{m}_\zeta$-primary ideal is the *Nil-index*, e.g. see work in [17].

Definition 6. *The Nil-index of an $\mathfrak{m}_\zeta$-primary ideal Q_ζ is the maximum integer $\mathcal{N} \in \mathbb{N}$ such that $\mathfrak{m}_\zeta^{\mathcal{N}} \nsubseteq Q_\zeta$.*

There is a tight connection between the dual space of $\mathfrak{m}_\zeta$-primary ideals and their Nil-index.

Lemma 7. *(Lemma 3.3, [23]) The maximum degree of the elements of $I^{\perp} \cap \mathbb{K}[\partial_\zeta]$ is equal to the Nil-index of Q_ζ.*

Theorem 1 and Lemma 7 show that we can find the monomials of D by searching among those monomials of $I^{\perp}$ that have degree at most the Nil-index, i.e., there exists a degree bound over the monomials of D. These monomials are actually the monomials under the *Extended Buchberger Diagram* which is defined below.

Definition 8 (Extended Buchberger Diagram). *The Extended Buchberger Diagram of an $\mathfrak{m}_\zeta$-primary ideal Q_ζ is obtained by considering all the monomials that appear in a basis of the dual space of Q_ζ .*

We can think of the Nil-index of Q_ζ as the largest degree of the monomials under the extended Buchberger diagram. Figure 1 shows the extended Buchberger diagram and all of its monomials for Example 12.

Note that the monomials under the Buchberger diagram with respect to an ordering form a vector space basis for R/Q. They include some monomials in a basis of $Q^{\perp}$, but they do not necessarily include all the monomials in D. In particular, they may not include the highest powers of dx_i, i.e., the monomials corresponding to the directional multiplicities. However in the extended Buchberger diagram, one can see all the possible monomials in D, which are all the monomials that do now appear in Q, which include all the monomials in the Buchberger diagram of Q.

The above comments are illustrated in Figure 2. The black dots show a basis for R/Q, while the white dots are the rest of the monomials in the basis of $Q^{\perp}$, see also [20] for a similar diagram. Also Figures 3 and 4 show the quotient of the elimination ideal with respect to x and the quotient of the elimination ideal with respect to y, respectively. In Figure 3, black dots are the basis for $Q_2^{\perp}$ and the white dots are the rest of the monomials in the

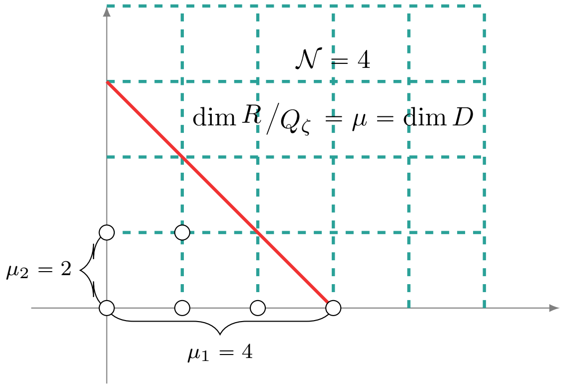


Figure 1: Extended Buchberger Diagram for Example 12

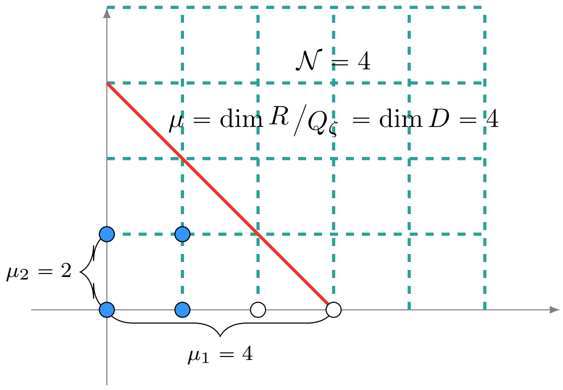


Figure 2: Extended Buchberger diagram vs a basis for B_ζ wrt a degree ordering for Example 12

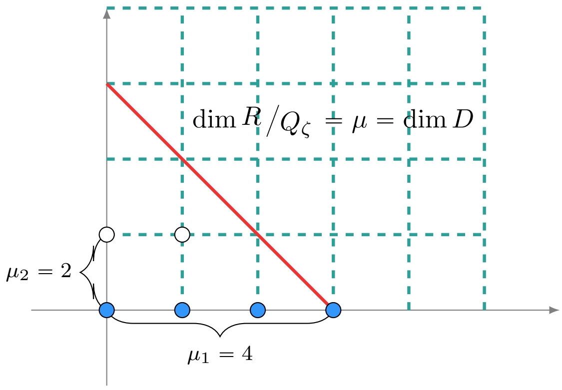


Figure 3: Extended Buchberger diagram vs directional multiplicity wrt x for Example 12

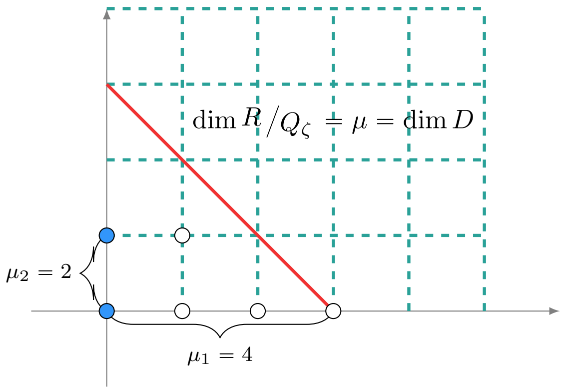


Figure 4: Extended Buchberger diagram vs directional multiplicity wrt y for Example 12

dual basis. In Figure 4, black dots are the basis for $Q_1^\perp$ and the white dots are the rest of the monomials in the dual basis.

Considering the above figures, one can see that the extended Buchberger diagram includes the Buchberger diagram with respect to every order. $\mathcal{N}$ is a bound for the degree of the members of a Gröbner basis with respect to every order. Directional multiplicity with respect to an axis is the largest intersection point of the extended Buchberger diagram with that axis. The Buchberger diagram does not necessarily have an intersection with the hyperplane $x_1 + \cdots + x_n = \mathcal{N}$, but the extended Buchberger diagram does have at least a point in common with that hyperplane.

Example 9. Let $I = \langle f_1 = x^8 + y^5, f_2 = x^7 y^4 \rangle$. The origin is a root of the system with multiplicity $\mu = 67$. We have that $\mathcal{N} = 18$, while $\mu_1 = 15, \mu_2 = 9$. The reduced Gröbner basis for I with respect to the lexicographic order ($x > y$) is $\{f_1 = x^8 + y^5, f_2 = x^7 y^4, g_y = y^9\}$, and with respect to lexicographic order ($y > x$) is $\{f_1 = y^5 + x^8, f_2 = y^4 x^7, g_x = x^{15}\}$, where g_y and g_x are the generators of the elimination ideal with respect to the lexicographic orders $x > y$ and $y > x$ respectively.

These observations give us the intuition that the directional multiplicities are at most as large as the Nil-index. Also their product gives us the volume of a cuboid which contains the Buchberger diagram. The following statements make the comments above more precise.

Remark 10. One can easily see that the Nil-index is as large as the multiplicity and also the multiplicity is bounded by the number of lattice points in the n-simplex. The simple conclusion of the definition of $\mathcal{N}$ and μ is that

$$\mathcal{N} \leqslant \mu \leqslant \text{Number of Lattice point in the n-simplex} = \binom{\mathcal{N}+n}{n}.$$

Proposition 3. *Let μ be the multiplicity of an isolated point ζ. Then*

- $\mu_i \leqslant \mu$ *for every* $1 \leqslant i \leqslant n$.
- $\mu \leqslant \prod_{1 \leqslant i \leqslant n} \mu_i$.
- $\sum_{i=1}^{n} \mu_i - n + 1 \leqslant \mu$.

Proof. For the first part, recall that $dim_{\mathbb{K}} Q_\zeta^\perp = \mu$ and that μ_i is the dimension of a vector subspace of $Q_\zeta^\perp$. Thus $\mu_i \leqslant \mu$.

For the second part, first remember that for every $1 \leqslant i \leqslant n$, μ_i is the largest degree of the elements in $Q_\zeta^\perp \cap \mathbb{K}[d_i]$. This means that $\mu_i + 1$ is the largest possible degree of x_i in R/Q. Since $\mu = dim_{\mathbb{K}} R/Q$, we conclude that $\mu \leqslant \prod_{1 \leqslant i \leqslant n} \mu_i$.

For the third statement, note that as argued above, $dx_i^{a_i} \in Q_\zeta^\perp$ if and only if $a_i < \mu_i$. This means that $x_i^{a_i} \notin Q_\zeta$ if and only if $a_i < \mu_i$. Now, for all $1 \leqslant i \leqslant n$, let $A_i := \{1, x_i, \cdots, x_i^{\mu_i - 1}\}$. Then, $\langle \bigcup A_i \rangle \subseteq R/Q_\zeta$ as vector spaces. Note that the elements of $\bigcup A_i$ are linearly independent. Then $dim \langle \bigcup A_i \rangle = \sum \mu_i - n + 1 \leqslant dim R/Q_\zeta = \mu$ and the result follows. $\square$

Proposition 4. *Let $\mathcal{N}$ be the Nil-index of Q_ζ. Then*

- $\mathcal{N} \geqslant \mu_i$ *for all* $1 \leqslant i \leqslant n$,
- $\mathcal{N} \leqslant \sum\limits_{1\leqslant i\leqslant n} \mu_i - n$

Proof. According to the definition of the Nil-index we have $\mathfrak{m}_\zeta^{\mathcal{N}} \nsubseteq Q_\zeta$ and $\mathfrak{m}_\zeta^{\mathcal{N}+1} \subseteq Q_\zeta$. Since $\mathfrak{m}_\zeta^{\mathcal{N}} = \langle x_1 - \zeta_1, \ldots, x_n - \zeta_n\rangle^{\mathcal{N}}$, therefore $(x_i - \zeta_i)^{\mathcal{N}} \notin Q_\zeta$ and $(x_i - \zeta_i)^{\mathcal{N}+1} \in Q_\zeta$. By the definition of μ_i and the Proposition 1, $dx_i^{\mu_i}(x_i - \zeta_i)^{\mathcal{N}} = 0$ and $dx_i^{\mu_i}(x_i - \zeta_i)^{\mathcal{N}-1} \neq 0$. Therefore $\mu_i \leqslant \mathcal{N}$.

For the second part, note that for all x_i, $d_{x_i}^{\mu_i - 1} \in supp(Q_\zeta^\perp)$ and $d_{x_i}^{\mu_i} \notin supp(Q_\zeta^\perp)$. Therefore by Proposition 1, $x_i^{\mu_i - 1} \notin Q_\zeta$ and $x_i^{\mu_i} \in Q_\zeta$. Consider $A = \{\mathbf{a} \in \mathbb{N}^n | \ \ |\mathbf{a}| = \sum(\mu_i - 1) + 1\}$. By the Pigeonhole principle, there exists an i, $1 \leqslant i \leqslant n$, such that $x_i^{\mu_i} | x^{\mathbf{a}}$. Therefore $x^{\mathbf{a}} \in Q_\zeta$ for all $\mathbf{a} \in A$, which implies that $\mathfrak{m}_\zeta^{|\mathbf{a}|} \subseteq Q_\zeta$ and $\mathcal{N} < |\mathbf{a}| = 1 + \sum(\mu_i - 1)$. The result follows by minimality of $\mathcal{N}$. □

Remark 11. The inequalities in the Propositions 3 and 4 are sharp. An example that shows this, is the univariate case, where $I = Q_\zeta \in \mathbb{K}[x]$. In this case the Nil-index of $I_1 = \mu_1$ is equal to its i-th directional multiplicity, which is equal to the degree of $(x - \zeta)$ in g, the monic generator of the elimination ideal. The latter doesn't happen by accident. We will discuss more about this in Section 4.2.

A geometric interpretation of the i-th directional multiplicity at an intersection point is be the number of instances of the intersection point that can be seen when we look at the intersection point in the direction parallel to the x_i axis. Some of the presented inequalities are direct consequences of the definitions, and reveal interesting properties of this new notion. In particular, knowing the directional multiplicities we can deduce information about the multiplicity or the Nil-index. Thus, the notion of directional multiplicity is, in this sense, a refinement of multiplicity and Nil-index. Moreover, in some applications, this refined information is crucial, as described in Section 4.1.

3 Algorithms for Dual Basis and Directional Multiplicity

In this section we present modifications of Macaulay's algorithm and Mourrain's integration method for computing a basis for the dual space efficiently. These algorithms give us in addition the directional multiplicities. Before presenting our modifications, we review the two approaches for computing the dual space of an $\mathfrak{m}_\zeta$-primary component of a given ideal $I = \langle f_1, \ldots, f_e\rangle \subseteq \mathbb{K}[x_1, \ldots, x_n]$. We refer the reader to [20] for a recent overview. These algorithms compute a basis D for $Q^\perp$ degree by degree. Let D_t be the subset of $\mathbb{K}[\partial_\zeta]$ that contains degree t elements of D. Then $D_0 = \langle 1\rangle$. The algorithms extend D_t into D_{t+1}, a basis for the degree $t+1$ part of $Q^\perp$, until $D_t = D_{t+1}$. Then we can conclude that $D = D_t$ and we have the basis D. We set $d_i := dx_i$ for presentation reasons in what follows.

3.1 Macaulay's Algorithm

Macaulay's algorithm [18] is the first algorithm for computing a basis for the dual space $Q^\perp$. It is based on a simple condition that the coefficients of the elements of the dual space must fulfill. Let $\Lambda = \sum\limits_{|\alpha|\leqslant\mathcal{N}} \lambda_\alpha d^\alpha$, where we use the multi-index notation with $d = d_1 d_2 \cdots d_n$. Then $\Lambda(f) = 0$ for all $f \in I$ if and only if $\Lambda(x^\beta f_i) = 0$ for all $\beta \in \mathbb{N}^n$ and $1 \leqslant i \leqslant e$. This observation, for $1 \leqslant |\beta| \leqslant \mathcal{N}$, reduces checking that $\Lambda(f) = 0$ for an infinite number of polynomials f into checking the finitely many conditions that are given in the right hand side. Namely, it suffices to impose conditions on λ_α's, the coefficients of Λ. For $1 \leqslant |\beta| \leqslant \mathcal{N}$, we obtain a system of linear homogeneous equations and construct the corresponding matrix. The rows of this matrix are labeled by $x^\beta f_i$ and the columns are labeled by d^α. Every element in the kernel of this matrix is a coefficient vector, corresponding to an element of D.

Macaulay's algorithm starts with $D_0 = \{d^0 = 1\}$. At step t, the algorithm computes the polynomials $\Lambda(x^\alpha f_i)$ for $\deg(\Lambda) \leqslant t$ and constructs the coefficient matrix. The kernel of this matrix contains coefficient vectors of elements of a basis D_t. If $D_t = D_{t-1}$, then the algorithm terminates, otherwise continues with computing D_{t+1}.

We illustrate the algorithm by two examples.

Algorithm 1: Macaulay's Algorithm

Input: A basis for an $\mathfrak{m}_\zeta$-primary ideal Q_ζ
Output: A basis for the dual of Q_ζ
def *ComputeBasis***:**
  $D_{\text{old}} = \varnothing$
  $D_{\text{new}} = \{\Lambda = d^0 = 1\}$
  while $D_{old} \neq D_{new}$**:**
    $D_{\text{old}} = D_{\text{new}}$
    Construct matrix M_{new}, the coefficient matrix of D_{new}
    $D_{\text{new}} = \text{kernel}(M_{\text{new}})$
  return D_{new}

Example 12. Let

$$\begin{aligned} f_1 &= x^2 + (y-1)^2 - 1 \\ f_2 &= y^2. \end{aligned}$$

Then for the root $(0, 0)$, we have that

$$M_1 = \begin{array}{c c} & \begin{array}{ccc} 1 & d_1 & d_2 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \end{array} & \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{3}$$

The kernel of this matrix yields the basis $D_1 = \{1, d_1\}$. In the second step, we have

$$M_2 = \begin{array}{c c} & \begin{array}{cccccc} 1 & d_1 & d_2 & d_1^2 & d_1d_2 & d_2^2 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ x_1f_1 \\ x_1f_2 \\ x_2f_1 \\ x_2f_1 \end{array} & \begin{pmatrix} 0 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{4}$$

from which we have $D_2 = \{1, d_1, 2d_1^2 + d_2\}$. The algorithm runs until step 4, during which we have a matrix of size 20×15, and

$$D_3 = D_4 = \{1, d_1, 2d_1^2 + d_2, 2d_1^3 + d_1d_2\}.$$

Thus, $\mu = 4, \mu_1 = 4$ and $\mu_2 = 2$.

Example 13. Let

$$\begin{aligned} f_1 &= y^3 \\ f_2 &= x^2y^2 \\ f_3 &= x^4 - x^3y. \end{aligned}$$

The matrices in the first, second and third steps of the algorithm are zero matrices. So we have $D_1 = \{1, d_1, d_2\}, D_2 = \{1, d_1, d_2, d_1^2, d_1d_2, d_2^2\}$ and

$$D_3 = \{1, d_1, d_2, d_1^2, d_1d_2, d_2^2, d_1^3, d_1d_2^2, d_1^2d_2\}.$$

The computation goes on till step 5, during which we have a matrix of size 45×21 whose kernel gives the dual basis

$$D_4 = D_5 = \{1, d_1, d_2, d_1^2, d_1d_2, d_2^2, d_1^3, d_1d_2^2, d_1^2d_2, d_1^4 + d_1^3d_2\}.$$

Thus, $\mu = 10, \mu_1 = 5$ and $\mu_2 = 3$.

3.2 Integration Method

Macaulay's algorithm is not efficient. In every step it builds new matrices which include previously constructed matrices, thus some computations are repeated.

In [23], Mourrain suggested another algorithm, which builds smaller matrices, further improved in [19]. We will demonstrate the improved version in this section. We first present the necessary background.

Given a basis for the vector space $B_\zeta = \mathbb{K}[x_1,\ldots,x_n]\big/Q_\zeta$, one can construct a basis D for $Q^\perp$ and vice versa. This can be deduced from the constructions in the work of Macaulay in [18]. The work of Mourrain in [23], shows the construction explicitly. Moreover, Mourrain has shown how to construct a Gröbner basis for Q_ζ having a basis for $Q^\perp$. Below we will explain the construction of D from a basis of B_ζ as in [19] in brief.

For every $\Lambda \in Q^\perp$, let $Supp(\Lambda)$ be the set of monomials that have a non-zero coefficient in Λ. Proposition 1 says that $\partial^{\mathbf{a}} \in Supp(\Lambda)$ if and only if $x^{\mathbf{a}} \notin Q_\zeta$ for $\mathbf{a} \in \mathbb{N}^n$. Let us denote by $Supp(Q^\perp)$ the union of supports of all elements of $Q^\perp$ and by s its cardinality. Then

$$Supp(Q^\perp) = \bigcup_{\Lambda\in Q^\perp} \{Supp(\Lambda)\} = \{\partial^{\mathbf{a}} | x^{\mathbf{a}} \notin Q_\zeta\}.$$

Since the degree of the monomials in $Supp(Q^\perp)$ is bounded by the Nil-index of Q_ζ, the above sets are finite. One can find a basis $B = \{x^{\beta_1},\ldots,x^{\beta_\mu}\}$ for B_ζ among the monomials in the above set. Then for every monomial $x^{\gamma_j} \in Supp(Q^\perp)$ such that $x^{\gamma_j} \notin B$ we can write

$$x^{\gamma_j} = \sum_{i=1}^{\mu} \lambda_{ij} x^{\beta_i} \mod Q_\zeta.$$

Now let

$$\Lambda_i = d^{\beta_i} + \sum_{j=1}^{s-\mu} \lambda_{ij} d^{\gamma_j}. \tag{5}$$

Then $\{\Lambda_1,\ldots,\Lambda_\mu\}$ is a basis for $Q^\perp$.

Given a basis D for $Q^\perp$, consider the matrix $M \in \mathbb{K}^{\mu\times s}$ of the coefficients of the elements of this basis. Every set of μ independent columns of M gives a basis for B_ζ. Let G be the matrix whose columns are the columns of M indexed by d^{β_i}. Then

$$G^{-1}M = \begin{array}{c} \\ \Lambda'_1 \\ \vdots \\ \Lambda'_\mu \end{array} \begin{array}{c} \begin{array}{cccccc} \beta_1 & \cdots & \beta_\mu & \gamma_1 & \cdots & \gamma_{s-\mu} \end{array} \\ \begin{pmatrix} 1 & & 0 & \lambda_{1,1} & \cdots & \lambda_{1,s-\mu} \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & \lambda_{\mu,1} & \cdots & \lambda_{\mu,s-\mu} \end{pmatrix} \end{array}, \tag{6}$$

which gives a basis of the form 5.

Having the above matrix construction, we are ready to explain Mourrain's algorithm. The algorithm is based on *integrating* elements of $Q_{t-1}^\perp$ in order to generate the elements of $Q_t^\perp$ with symbolic coefficients, and then applying necessary and sufficient conditions on the generated elements, gives a system of equations for the coefficients. Similar to Macaulay's algorithm, each vector in the kernel of the matrix determines the coefficients of an element in $Q_t^\perp$. The following definition is useful in what follows.

Definition 14. *For every $\Lambda \in \mathbb{K}[\partial]$ and $1 \leqslant i \leqslant n$, denote by $\int_i \Lambda$ the i-th integral of Λ, which is defined as follows.*

$$\int_i \Lambda = \Phi \in \mathbb{K}[\partial] \text{ such that } d_i(\Phi) = \Lambda \text{ and } \Phi(d_1,\ldots,d_{i-1},d_i=0,d_{i+1},\ldots,d_n) = 0.$$

The next theorem provides a generic, compact representation of the dual elements by exploiting the properties of derivation in the dual space.

Theorem 15 ([23, 19])**.** *Let $\{\Lambda_1,\ldots,\Lambda_m\}$ be the basis D_{t-1} with the coefficient matrix of the form 6, yielding the standard basis $B_t = \{x^{\beta_i} | 1 \leqslant i \leqslant m\}$, i.e., the elements of the basis B that are of degree up to t. An element $\Lambda \in \mathbb{K}[\partial]$ with no constant term is in D_t if and only if it is of the form*

$$\Lambda = \sum_{i=1}^{m}\sum_{k=1}^{n} \lambda_{ik} \int_k \Lambda_i(d_1,\ldots,d_k,0,\ldots,0), \tag{7}$$

where $\lambda_{ij} \in \mathbb{K}$, and the following conditions hold

1. *for all* $1 \leqslant k < l \leqslant n$,
$$\sum_{i=1}^{1} \lambda_{ik} d_l(\Lambda_i) - \sum_{i=1}^{1} \lambda_{il} d_k(\Lambda_i) = 0. \tag{8}$$

2. *for all* $1 \leqslant k \leqslant e$,
$$\Lambda(f_k) = 0 \tag{9}$$

3. *for all* $1 \leqslant i \leqslant m$,
$$\Lambda(x^{\beta_i}) = 0. \tag{10}$$

The first condition implies that the new elements Λ that have been introduced are stable by derivation. The second condition comes from the fact that Λ must be inside $Q_\zeta^\perp$. Based on Theorem 15, having $D_{t-1} = \{\Lambda_1, \ldots, \Lambda_m\}$, we have an algorithmic way to compute D_t. Consider Λ from the theorem with symbolic coefficients λ_{ik}. Plug Λ into the conditions of the theorem and obtain a system of equations. In step t the corresponding matrix will look like below.

$$M_t = \begin{array}{l} \\ \Lambda(f_1) \\ \vdots \\ \Lambda(f_e) \\ \text{Condition 8} \\ \text{Condition 10} \end{array} \begin{array}{c} \begin{array}{ccccccc} \lambda_{11}p_{11} & \ldots & \lambda_{1n}p_{1n} & \ldots & \lambda_{e1}p_{e1} & \ldots & \lambda_{en}p_{en} \end{array} \\ \left(\begin{array}{ccccccc} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{array} \right) \end{array}. \tag{11}$$

By abuse of notation and for simplifying the presentation, we use the symbolic coefficients λ_{ij} instead of the product of λ_{ij} by the polynomials $p_{ij} = \int_j \Lambda_i(d_1, \ldots, d_j, 0, \ldots, 0)$ in order to label the columns of M_t. The kernel of M_t will give us the possible values for λ_{ij}.

The first two conditions already guarantee that $\Lambda \in D_t$ [23]. However, we might have that $\Lambda \in D_{t-1}$ as well. This means that we reproduce the elements of the previous step. The third condition which has been introduced in [19], gives us a sufficient condition for having $\Lambda \in D_t \backslash D_{t-1}$. This helps with avoiding repetition of the computations that have been done in the previous steps by adding new rows to the matrix, which in some cases may lead to removing some column. It also provides a method to compute a basis D at the same time as a dual basis for B_ζ.

Algorithm 2: Integration method

Input: A basis for an $\mathfrak{m}_\zeta$-primary ideal Q_ζ
Output: A basis for $R \big/ Q_\zeta$ and a basis D for $Q_\zeta^\perp$
def *ComputeBasis*:
- $D_{\text{old}} = \varnothing$
- $D_{\text{new}} = \{\Lambda = d^0 = 1\}$
- **while** $D_{old} \neq D_{new}$:
  - $D_{\text{old}} = D_{\text{new}}$
  - $\Lambda := \sum_{i=1}^{m} \sum_{k=1}^{n} \lambda_{ik} \int_k \Lambda_i(d_1, \ldots, d_k, 0, \ldots, 0)$
  - for all $1 \leqslant k \leqslant l \leqslant n$, $\sum_{i=1}^{m} \lambda_{ik} d_l(\Lambda_i) - \sum_{i=1}^{m} \lambda_{il} d_k(\Lambda_i) = 0$
  - for all $1 \leqslant k \leqslant e$, $\Lambda(f_k) = 0$
  - for all $1 \leqslant k \leqslant m$, $\Lambda(x^{\beta_i}) = 0$
  - Construct matrix M_{new}, the coefficient matrix of Λ
  - Compute a basis K_{new} for kernel(M_{new})
  - $D_{\text{new}} = D_{\text{old}} \bigcup K_{\text{new}}$
- **return** $_{\text{new}}$

Below, we do the computations for Examples 12 and 13, first without and then with considering Condition 10.

Example 16 (Computations without Condition 10 for Example 12)**.**

$$M_1 = \begin{array}{c} \\ \Lambda(f_1) \\ \Lambda(f_2) \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \end{array}. \tag{12}$$

which is the same as the matrix in Macaulay's algorithm, and $D_1 = \{1, d_1\}$. Continuing into the second step ($\Lambda \in D_2$), we apply the first two conditions on $\Lambda = \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_1^2 + \lambda_4(d_1 d_2)$, which gives us the matrix

$$M_2 = \begin{array}{l} \\ \mathrm{C}ondition8 \\ \Lambda(f_1) \\ \Lambda(f_2) \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & d_1^2 & d_1 d_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{13}$$

which has two columns less that the second matrix of Macaulay's algorithm. We have $D_2 = \{1, 2d_1^2 + d_2, d_1\}$. The third and fourth step matrices are also smaller than the ones in Macaulay's algorithm.

Example 17 (Computations for Example 12 with Condition 10)**.** I this case the matrix of D_1 is the same, while the matrices for D_2 and D_3 are different.

$$M_1 = \begin{array}{c} \\ \Lambda(f_1) \\ \Lambda(f_2) \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \end{array}, \tag{14}$$

which is the same as the matrix in Macaulay's algorithm, and $D_1 = \{1, d_1\}$.

In step 2 we have

$$M_2 = \begin{array}{l} \\ \mathrm{C}ondition10 \\ \mathrm{C}ondition8 \\ \Lambda(f_1) \\ \Lambda(f_2) \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & d_1^2 & d_1 d_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{15}$$

Condition 10 implies that $\lambda_1 = 0$. Therefore we can remove column one from M_2.

Example 18 (Computations with and without Condition 10 for Example 13)**.** $D_0 = \{1\}$. If we do the computations without considering Condition 10, then in step 2 of the integration method, we will reach to a 3×2 zero matrix, which has one column less than the matrix in Macaulay's algorithm. The matrix in step 3 is a 3×5 zero matrix, which is much smaller than the matrix in Macaulay's method.

Re-doing the computations considering Condition 10, we get M_0 and M_1 same as above. In step 2, M_2 is a matrix of size 5×5. The two extra rows in this case comes from Condition 10. However each of the two last rows simply will have one nonzero coordinate, which implies that two of the coefficients λ_{ij} are zero. Having the value of a coefficient equal to zero means that we can remove the corresponding column from the matrix and therefore the size of the matrix will finally be 3×3, smaller than the previous one. In step 3, applying Condition 10, we will get a matrix with 4 columns instead of 9 columns in the previous case.

In the next subsection we will show modifications on the above algorithms in order to make them more efficient for computing the directional multiplicities.

3.3 Modified Algorithms for Dual Basis

In this subsection we present modifications to the integration method and Macaulay's algorithms, which make computations more efficient. In particular, we give a more efficient criterion than Condition 10 in the integration method.

We will use the following notation throughout subsection 3.3. We denote the Nil-index by $\mathcal{N}$. Let t be a fixed number between 1 and $\mathcal{N}$. We refer to the current step of the algorithm as step t. Same as previous sections, D is a basis for $Q^\perp$ and therefore $\langle D \rangle = Q^\perp$. D_t stands for the degree t part of a basis of $Q^\perp$. Obviously $\langle D_t \rangle$ is a sub-vector space of $\langle D \rangle$. If we assume that D_t is equipped with a total degree term order, e.g. degree lexicographic ordering, then the leading term of an element Λ of D_t is denoted by $\mathrm{lt}(\Lambda)$. If v is a column of a matrix M, then $M - v$ denotes the matrix obtained by deleting the column v from M.

3.3.1 Modifications on Integration Method

Let M_t denote the matrix in step t of the integration method and $\widetilde{M_t}$ denote the matrix that is constructed in step t without considering Condition 10. We assume that $D_{t-1} = \{\Lambda_1, \ldots, \Lambda_m\}$ is already computed in step $t-1$.

In the integration method, columns of M_t (similarly for $\widetilde{M_t}$) are labeled by the λ_{ij}'s appearing in Λ (see Equation 11). Fix one of the λ_{ij}'s and call it λ. We denote by v_λ the column of M_t (similarly for $\widetilde{M_t}$) that is indexed by λ. Then p_λ denotes the corresponding polynomial.

A basis D of $Q^\perp$ is in one to one correspondence with a basis K for $Ker(\widetilde{M_\mathcal{N}})$ (also D_t is in one to one correspondence with a basis K_t for $Ker(\widetilde{M_t})$). In step t, this correspondence is reduced to a correspondence between D_t and K_t, a basis of $Ker(\widetilde{M_t})$). If there exists a vector $q \in K_t$, for which the coordinate corresponding to λ in this vector is nonzero, then we say that v_λ is active in D_t. In case we explicitly know such a vector q, i.e., a particular element of the kernel corresponding to an element E of D_t, then we say that v_λ is active in E. Since, M_{t-1} is a sub-matrix of M_t and $\widetilde{M_{t-1}}$ is a sub-matrix of $\widetilde{M_t}$, if it is clear from the context, by *a column of* M_{t-1}(respectively $\widetilde{M_{t-1}}$) we will refer to the corresponding column in M_t(respectively $\widetilde{M_t}$) as well. We work on $\widetilde{M_t}$ rather than M_t in this section, although many of our arguments are correct for M_t as well.

We start with a proposition that provides us with an improvement on the integration method, related to Condition 10.

Proposition 5. *Let $\widetilde{M_t}, \widetilde{M_{t-1}}, D_t, \Lambda_i$ $(1 \leqslant i \leqslant m), \lambda, p_\lambda$ and v_λ be as above.*
Then the following hold.

1. *If v_λ is a column of $\widetilde{M_t}$, then v_λ is active in D_t if and only if v_λ can be reduced to zero by other columns of $\widetilde{M_t}$.*

2. *For all $1 \leqslant i \leqslant m$, if v_{λ_i} is active in Λ_i, K'_{t_i} is a basis for $Ker(\widetilde{M_t} - v_{\lambda_i})$ and D'_{t_i} is the set of its corresponding dual elements, then $\{\Lambda_i\} \cup D'_{t_i}$ is a basis for the degree t part of $Q^\perp$. Moreover, if v_{λ_i} is active in Λ_i, but is not active in Λ_j, $1 \leqslant j \neq i \leqslant m$, then there exists a basis D'_{t_i} such that $\Lambda_j \in D'_{t_i}, j \neq i$.*

3. *Let $K_{t_{1\ldots m}}$ be a basis for $Ker(\widetilde{M_t} - v_{\lambda_1} - \cdots - v_{\lambda_m})$ and $D_{t_{1\ldots m}}$ be the set of its corresponding dual elements. For all $1 \leqslant i \leqslant m$, if v_{λ_i} is active in Λ_i, but is not active in $1, \ldots, \Lambda_{i-1}$, then $D_{t-1} \cup D_{t_{1\ldots m}}$ is a basis for the degree t part of $Q^\perp$.*

Proof. 1. Let $v_\lambda, v_1, \ldots, v_k$ denote the columns of $\widetilde{M_t}$ and $p_\lambda, p_1, \ldots, p_k$ be the polynomials labeling the columns of $\widetilde{M_t}$. Then v_λ can be reduced to zero by $v_1, \ldots, v_k$ if and only if there exist $c_1, \ldots, c_k \in \mathbb{K}$, such that $v_\lambda = c_1 v_1 + \cdots + c_k v_k$, or equivalently $v_\lambda - c_1 v_1 - \cdots - c_k v_k = 0$. This holds if and only if $q := (1, c_1, \cdots, c_k) \in K_t$, which holds if and only if $\Lambda' := p_\lambda - c_1 p_1 - \cdots - c_k p_k \in D_t$ (Note that this is exactly the fact that Λ' in D_t corresponds to $q \in K_t$). The latter is the case if and only if v_λ is active in Λ', or equivalently v_λ is active in D_t.

2. Fix $1 \leqslant i \neq j \leqslant m$ and let q_i and q_j be the elements of K_t corresponding to Λ_i and Λ_j in D_t, respectively.
First we prove that for all $\Lambda' \in D_t$ if $\Lambda' \neq \Lambda_i$, then $\Lambda' \in \left\langle D'_{t_i} \cup \{\Lambda_i\} \right\rangle$. Let q' be the corresponding elements of Λ' in K_t. If v_{λ_i} is not active in Λ', then by part 1 it cannot be reduced to zero by the active columns in Λ'. So the column v_{λ_i} is not involved in computing Λ' via column reducing in $\widetilde{M_t}$. So Λ' can be computed via column reducing in $\widetilde{M_t} - v_\lambda$. Let q' be the corresponding element to Λ' in $Ker(\widetilde{M_t})$. Then $q' \in Ker(\widetilde{M_t} - v_\lambda)$. This means that $\Lambda' \in \left\langle D'_{t_i} \right\rangle$.
If v_{λ_i} is active in Λ', then we prove that there exists a Λ'' in D'_{t_i} such that $\Lambda' = \Lambda_i + \Lambda''$. This is because of the following. Let $q' \in K_t$ be the element corresponding to $\Lambda' \in D_t$, such that that the first coordinate of q' corresponds to v_{λ_i}. Take $q' = (1, b_1, \ldots, b_k)$. Then we have that $v_{\lambda_i} + b_1 v1 + b_2 v_2 + \cdots b_k v_k = 0$, where the columns $v_1, \ldots, v_k$ are as in the proof of part 1. Also again as in the proof of the part 1, $v_{\lambda_i} = c_1 v_1 + \cdots + c_k v_k$. Therefore $(b_1 - c_1) v_1 + \cdots + (b_k - c_k) v_k = 0$, which means that $(0, b_1 - c_1, \ldots, b_k - c_k) \in Ker(\widetilde{M_t})$, and therefore $q'' := (b_1 - c_1, \ldots, b_k - c_k) \in Ker(\widetilde{M_t} - v_\lambda)$. So one can construct a basis K'_{t_i} in such a way that $q'' \in K'_{t_i}$. Let Λ'' be the member of D'_{t_i} corresponding to q''. Then $\Lambda' = \Lambda_i + \Lambda''$.
Secondly we note that if v_{λ_i} is not active in Λ_j, for $1 \leqslant j \neq i \leqslant m$, then by the above argument, one can compute a basis K'_{t_i} (and respectively, D'_{t_i}) in such a way that $\Lambda_j \in D'_{t_i}$.
So every element of D_t can be obtained from Λ_i and an element of K'_{t_i} and therefore $\langle D_t \rangle \subseteq \left\langle \{\Lambda_i\} \cup D'_{t_i} \right\rangle$. Linear independence of the elements of $\{\Lambda_i\} \cup D'_{t_i}$ is clear, and therefore $\langle D_t \rangle = \left\langle \{\Lambda_i\} \cup D'_{t_i} \right\rangle = Q^\perp$.

3. $K_{t_{1\ldots l}}$ be a basis for $Ker(\widetilde{M_t} - v_{\lambda_1} - \cdots - v_{\lambda_l})$ and $D_{t_{1\ldots l}}$ the corresponding dual elements. Also as in the proof of the previous parts, let K_t be a basis for $Ker(\widetilde{M_t})$ and also let $q_1, \ldots, q_m \in K_t$ correspond to $\Lambda_1, \ldots, \Lambda_m$ respectively. Then from the proof of part 2 we have that $\{q_1\} \cup K_{t_1}$ is a basis for $Ker(\widetilde{M_t})$. Also by part 2 of the proposition, $q_2, \ldots, q_m \in \langle K_t \rangle$ and correspondingly $\Lambda_2, \ldots, \Lambda_m \in \langle D_{t_1} \rangle$. Now consider the matrix $\widetilde{M_t} - v_{\lambda_1}$ and the basis D_{t_1} obtained from it. Since v_{λ_2} is active in Λ_2 (which corresponds to q_2 in K_t), and it is not active in Λ_1, then we can apply part 2 of the proposition to the matrix $\widetilde{M_t} - v_{\lambda_1}$ and the basis D_{t_1} obtained by it. Then we will have that $\{q_2\} \cup K_{t_{12}}$ is a basis for $Ker(\widetilde{M_t} - v_{\lambda_1})$ and $q_3, \ldots, q_m \in \langle K_{t_{12}} \rangle$. Correspondingly, $\Lambda_3, \ldots, \Lambda_m \in \langle D_{12} \rangle$. This implies that $\{q_1, q_2\} \cup K_{t_{12}}$ is a basis for $Ker(\widetilde{M_t})$. Continuing with $v_{\lambda_i},\ i \geqslant 3$, and considering the assumption that v_{λ_i} is not active in $\Lambda_1, \ldots, \Lambda_{i-1}$, $j \neq i$, we finally get $\{q_1, \ldots, q_m\} \cup K_{t_{1\ldots m}}$ as a basis for $Ker(\widetilde{M_t})$ and correspondingly $\{\Lambda_1, \ldots, \Lambda_m\} \cup D_{1\ldots m}$ as a basis for the degree t part of $Q^{\perp}$

□

The above proposition shows that deleting some columns from $\widetilde{M_t}$ helps us to avoid re-computing the basis elements of degree at most $t-1$, which were already computed in the previous steps. Not every set of m active columns will give us degree t elements of a basis. In fact if we delete two columns that both are active in two different basis members of D_{t-1}, then we may not obtain some members of D_t, For instance Let $D_2 = \{\Lambda_1 = d_1 + d_2 + d_1^2 + d_2^2, \Lambda_2 = d_1 + d_2 + 2d_1^2 + d_1 d_2\}$ and $\Lambda' = d_1 + d_2^3 \in Ker(\widetilde{M_3})$. Then $\Lambda' \notin Ker(\widetilde{M_3} - v_{d_1} - v_{d_2})$.

Choosing the appropriate columns can be seen as a combinatorial problem. For each element of D_{t-1}, if we consider sets corresponding to the active columns in that element, then a set of columns that satisfy the assumptions of part 3 of Proposition 5 form a *System of Distinct Representatives*. However, not every set of distinct representatives gives us the appropriate columns. The above example shows this. There are combinatorial and graph theoretical equivalences for the above conditions.

In the following we show how to detect columns $v_{\lambda_1}, \ldots, v_{\lambda_m}$ that satisfy the assumption of part 3 of Proposition 5. This is done via changing the basis $\{\Lambda_1, \ldots, \Lambda_m\}$ into a new *reduced* basis $\{\Lambda_1', \ldots, \Lambda_m'\}$, in which the leading terms satisfy the assumptions of part 3 of Proposition 5.

Let $D_{t-1} = \{\Lambda_1, \ldots, \Lambda_m\}$ as above. Remember that having D_{t-1}, one can construct Matrix 6 in order to obtain a basis for the degree t part of R/Q, so that Condition 10 can be applied. Below we show constructing a similar, but smaller matrix which gives us the desired set of active columns. Same as $\widetilde{M_t}$, the columns of this matrix are labeled by the coefficients/polynomials that appear in Λ in Equation 7. Same as Matrix 6, the rows of this matrix come from $\Lambda_1, \ldots, \Lambda_m$. Let $v_{\lambda_1}, \ldots, v_{\lambda_u}$ be the columns of $\widetilde{M_t}$ such that they are active in D_{t-1}. Construct the following matrix containing the columns $v_{\lambda_1}, \ldots, v_{\lambda_u}$.

$$M' = \begin{array}{c} \\ \Lambda_1 \\ \vdots \\ \Lambda_m \end{array} \begin{array}{c} \text{Columns labeled same as } \widetilde{M_t} \\ \begin{pmatrix} & & \\ v_{\lambda_1} & \quad\cdots\quad & v_{\lambda_s} \\ & & \end{pmatrix} \end{array}, \tag{16}$$

Changing M' into a row echelon form matrix, after moving the pivot columns to the left hand, we will reach to a matrix of the following form.

$$G'^{-1}M' = \begin{array}{c} \Lambda_1' \\ \Lambda_2' \\ \vdots \\ \Lambda_m' \end{array} \begin{pmatrix} * & & * & & & & \\ 0 & \ddots & & * & & & \\ \vdots & \ddots & * & & & & \\ 0 & \cdots & 0 & * & * & \cdots & * \end{pmatrix}, \tag{17}$$

where diagonal entries are nonzero and G' is the matrix that takes care of the operations done for the column swapping and the row echelon form. Note that we will not have any zero row. This is because otherwise, if we obtain a zero row in $G'^{-1}M'$, that row is linearly dependent to the other rows. But this is in contradiction with $\Lambda_1, \ldots, \Lambda_m$ (and therefore $\Lambda_1', \ldots, \Lambda_m'$ as their linear combination) being linearly independent. Then our basis will satisfy the conditions of part 3 of Proposition 5.

Now we are ready to prove the following, which provides us with an algorithmic improvement of the integration method, more efficient than Condition 10.

Corollary 19. ***(Criterion for Deleting Active Columns)*** *Let $D_{t-1} = \{\Lambda_1, \cdots, \Lambda_m\}$, $\widetilde{M}_t$, $D_t, v_{\lambda_1}, \ldots, v_{\lambda_u}$ and $G'^{-1}M'$ be as in Equation 17, and (by abuse of notation) let $v_{\lambda_1}, \ldots, v_{\lambda_m}$ be the columns of $\widetilde{M}_t$ corresponding to the first m columns in $G'^{-1}M'$. Also let $K_{t_{1\ldots m}}$ be a basis for $Ker(\widetilde{M}_t - v_{\lambda_1} - \cdots - v_{\lambda_m})$ and $D_{t_{1\ldots m}}$ be the set of its corresponding dual elements. Then $D_{t-1} \cup D_{t_{1\ldots m}}$ is a basis for the degree t part of $Q^{\perp}$.*

Proof. We only need to prove that the columns $v_{\lambda_1}, \ldots, v_{\lambda_m}$ in $G'^{-1}M'$ satisfy the conditions of part 3 of Proposition 5. This is the case because for all $1 \leqslant i \leqslant m$, v_{λ_i} has zero in coordinates $i+1, \ldots, m$ and has non-zero coordinate i, which is the row corresponding to Λ_i. This means that for all $1 \leqslant i \leqslant m$, v_{λ_i} is not active in $\Lambda_1, \ldots, \Lambda_{i-1}$. Having the above argument, the result comes directly from Proposition 5. □

Corollary 19 provides us with an optimization in the integration method. Assume that the monomials $x^{\mathbf{a}_1}, \ldots, x^{\mathbf{a}_m}$ form a basis for the degree $t-1$ part of R/Q. If the monomial $dx^{\mathbf{a}_i}$ only appears once in Λ in Equation 7, then applying Condition 10, we have that

$$\Lambda(x^{\mathbf{a}_i}) = \lambda dx^{\mathbf{a}_i}(x^{\mathbf{a}_i}) = \lambda_i = 0.$$

This gives us an equation which adds a row to $\widetilde{M}_t$. However, instead of adding the corresponding row to $\widetilde{M}_t$, one can just plug in $\lambda_i = 0$ in the other equations obtained from Conditions 8, 9. This will remove λ_i from the other equations, or equivalently will remove the column v_{λ_i} from $\widetilde{M}_t$. If we let v_{λ_i} be the only column of M_t such that its label contains $dx^{\mathbf{a}_i}$, then v_{λ_i} is active in Λ_i and therefore according to Corollary 19, one can delete it from $\widetilde{M}_t$ in order to avoid re-computing D_{t-1}.

In what follows we show how our construction implies a basis for the quotient ring as well as a normal form algorithm and a Gröbner basis of the primary component of the isolated point in question.

Proposition 6. *Let $v_{\lambda_1}, \ldots, v_{\lambda_m}$ be the columns in the criterion for deleting active columns, i.e., Corollary 19. Also assume that $p_1, \ldots, p_m$ are the corresponding polynomials to the coefficients $\lambda_1, \ldots, \lambda_m$ in Λ in Equation 7 and let $p'_i \in \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial with the same monomials as $p_i \in \mathbb{K}[\partial]$ for $1 \leqslant i \leqslant m$. Then $\{p_1, \ldots, p_m\}$ is a basis for the degree $t-1$ part of R/Q.*

Proof. Let $l_1, \ldots, l_m$ be the leading terms of $p_1, \ldots, p_m$. Then from the discussion in the integration method, we know that $\{l_1, \ldots, l_m\}$ is a basis for the degree $t-1$ part of R/Q. Since $p_1, \ldots, p_m \in R/Q$ and also the cardinality of $\{l_1, \ldots, l_m\}$ and $\{p_1, \ldots, p_m\}$ are the same, then in order to prove that $\{p_1, \ldots, p_m\}$ is a basis for R/Q, we just need to prove that $p_1, \ldots, p_m$ are linearly independent. Without loss of generality, we can assume that l_i appears only in p_i, $1 \leqslant i \leqslant m$. Because otherwise, we can reduce $p_1, \ldots, p_m$ with respect to each other so that we obtain polynomials $p'_1, \ldots, p'_m$ such that $l_1, \ldots, l_m$ are the leading terms of $p'_1, \ldots, p'_m$ and l_i appears only in p_i, for $1 \leqslant i \leqslant m$ and also $\langle p'_1, \ldots, p'_m \rangle = \langle p_1, \ldots, p_m \rangle$. Now this shows that $p_1, \ldots, p_m$ are linearly independent, because each leading term only appears in one single polynomial and therefore no p_i can be in the span of the other p_j, $1 \leqslant j \neq i \leqslant m$. □

Let $p'_i \in \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial p_i, substituting $\partial_{x_i} = x_i$, for $1 \leqslant i \leqslant m$. Then Proposition 6 implies that the criterion for deleting active basis can be viewed as adding the equation $\Lambda(p'_i) = 0$, for $1 \leqslant i \leqslant m$. Exactly the same as Condition 10, this equation leads to adding rows to $\widetilde{M}_t$, however those rows are in the form $(0, \ldots, 0, c, 0, \ldots, 0)$, where c is a nonzero element in coordinate $i, 1 \leqslant i \leqslant m$ and therefore they result in deleting the corresponding columns. We can say even more.

Proposition 7. *Let $\{p'_1, \ldots, p'_m\} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be a (not necessarily monomial) basis for the degree t part of R/Q such that no monomial of p'_i is in Q and let $p_1, \ldots, p_m \in \mathbb{K}[\partial]$ be the polynomials p'_i, substituting $x_i = \partial_{x_i}$, for $1 \leqslant i \leqslant m$. For monomials $m_1, \ldots, m_k \notin Q$ such that $m_1, \ldots, m_k \notin Supp(p_1) \cup \ldots \cup Supp(p_m)$, write $m_j = \sum_{i=1}^{m} \lambda_{ij} p'_i$. Then $\Lambda_i = p_i + \sum_{j=1}^{k} \lambda_{ij} m_j,\ 1 \leqslant j \leqslant m$, is a basis for the degree t part of $Q^{\perp}$ and the normal form of any $g \in \mathbb{K}[x_1, \ldots, x_n]$ with respect to the basis $\{p'_1, \ldots, p'_m\}$ is*

$$NF(g) = \sum_{i=1}^{m} \Lambda_i(g) p'_i.$$

Proof. $\Lambda_1, \ldots, \Lambda_m$ are linearly independent because $p_1, \ldots, p_m$ are linearly independent in R/Q, due to the linear independence of $p'_1, \ldots, p'_m$. The latter is the case by Proposition 6. The rest of the proof is exactly the same as the proof of Lemma 2.4 in [19]. □

If $\{p'_1, \ldots, p'_m\} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is an arbitrary basis of R/Q and $\{p_1, \ldots, p_m\} \subseteq \mathbb{K}[\partial]$ are the corresponding differential polynomials, then removing the monomials that are in Q in each p'_i, we will obtain a new basis for R/Q. So this assumption in the proposition holds without loss of generality. Thus, we have the following generalization of Lemma 3.4 in [19].

Proposition 8. *Let $\{p'_1, \ldots, p'_m\} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be a basis for the degree t part of R/Q such that no monomial of p'_i is in Q. An element $\Lambda \in \mathbb{K}[\partial]$ is not zero in $Q_t^{\perp} \backslash Q_{t-1}^{\perp}$ if and only if in addition to Equations 9 and 8 it satisfies*

$$\Lambda(p_i) = 0, \;\; 1 \leqslant i \leqslant m.$$

Constructing matrices M' and $G'^{-1}M'$ in order to choose particular active columns and deleting them is a special case of the above proposition. We have the following generalization of Proposition 3.7 in [23].

Proposition 9. *Let $\prec$ be a term order and $m_j, p_i, p'_i, \Lambda_i,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant k$ be as in Proposition 8. Also let $l_i = lt(p'_i)$ and $w_1, \ldots, w_s$ be the monomials different from l_i in $p'_1, \ldots, p'_m$. Write $w_i = \sum_{j=1}^{m} \gamma_{ij} p'_j$. Consider $W = \{g_{w_i} := w_i + \sum_{j=1}^{m} \gamma_{ij} p'_j | 1 \leqslant i \leqslant s\}$, $G := \{m_j + \sum_{i=1}^{m} \lambda_{ij} p'_i | 1 \leqslant j \leqslant m\}$ and $C := \{\boldsymbol{x}^{\boldsymbol{c}} | c \in \mathbb{N}^n,\ |c| = \mathcal{N}+1\}$. Then $G \cup W \cup C$ is a Gröbner basis for Q with respect to $\prec$.*

Proof. Proof of Proposition 3.7 in [23] works here as well. We just need to note that for every $f \in Q$, $lt(f) \in \langle lt(G) \cup lt(W) \cup C \rangle$. ☐

Note that unlike Proposition 3.7 in [23] $G \cup C$ is not a Gröbner basis in this case as we don't necessarily have $\langle lt(Q) \rangle = \langle lt(G) \cup lt(W) \cup C \rangle$.

We explain the computations in step 3 of Example 3.3 in [19] using the above result. We also compare our proposition with Condition 10. This is done below in Example 20.

Example 20. Let $I = \langle f_1, f_2 \rangle \trianglelefteq \mathbb{K}[x, y]$, where

$$f_1 = x - y + x^2$$
$$f2 = x - y + y^2.$$

In step 2 of the algorithm we have that

$$\widetilde{M_2} = \begin{array}{c} \\ \text{Condition 8} \\ \Lambda(f_1) = 0 \\ \Lambda(f_2) = 0 \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & d_1^2 & d_1 d_2 + d_2^2 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \end{array},$$

from which we have $D_2 = \{\Lambda_1 = 1, \Lambda_2 = d_1 + d_2, \Lambda_3 = d_2 + d_1^2 + d_1 d_2 + d_2^2\}$. The active columns in D_2 are v_1, v_2, v_3, v_4, where v_i refers to column i and therefore matrix M' defined in 16 (ignoring $\Lambda_1 = 1$) is

$$M' = \begin{array}{c} \\ \Lambda_2 \\ \Lambda_3 \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & d_1^2 & d_1 d_2 + d_2^2 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Two instances of substituting some columns of M' and then computing its (reduced) echelon form are shown below. Matrix

$$G_1'^{-1} M' = \begin{array}{c} \\ \Lambda'_2 \\ \Lambda'_3 \end{array} \begin{array}{c} \begin{array}{cccc} d_2 & d_1^2 & d_1 & d_1 d_2 + d_2^2 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

gives columns v_2 and v_3 and matrix

$$G_2'^{-1} M' = \begin{array}{c} \\ \Lambda'_2 \\ \Lambda'_3 \end{array} \begin{array}{c} \begin{array}{cccc} d_2 & d_1 d_2 + d_2^2 & d_1 & d_1^2 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

gives columns v_2 and v_4.

For instance if we consider $G_2'^{-1}M'$, then $\Lambda_2' = d_2 + d_1 + d_1^2$ and $\Lambda_3' = d_1d_2 + d_2^2 + d_1 + d_1^2$. d_2 only appears in λ_2' and $d_1d_2 + d_2^2$ only appears in Λ_3', and therefore by deleting columns v_2 and v_4 from $\widetilde{M}_3$ we will have the following in step 3.

$$\widetilde{M}_3 - v_2 - v_4 = \begin{array}{c} \\ \text{Condition 8} \\ \text{Condition 8} \\ \Lambda(f_1) = 0 \\ \Lambda(f_2) = 0 \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_1^2 & d_1^3 - d_1^2 & d_2^3 + d_1d_2^2 + d_1^3d_2 - d_1d_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

$Ker(\widetilde{M}_3 - v_2 - v_4) = 0$ and we are done. Using any of the pairs of columns obtained via other possible matrices we would have gotten the same result.

Other observations. On one hand we show how pivoting could help and on the other hand we compare the sizes of the matrices produced by the theory presented above. Let us put an order on the monomials of D_{t-1}, e.g., degree lexicographic. Then $lt(\Lambda')$, the leading term of Λ', would be well-defined for every $\Lambda' \in D_t$. Now one can consider reducing the members of a basis of D_t with respect to each other so that $lt(\Lambda') \notin Supp(\Lambda'')$ for all $\Lambda' \neq \Lambda'' \in D_t$. We call such a basis a *reduced basis*. Then the leading term will be a monomial that uniquely appears in the reduced basis. If $\Lambda_1, \ldots, \Lambda_m$ is a basis for D_{t-1}, then removing the columns corresponding to $lt(\Lambda_1), \ldots, lt(\lambda_m)$ from $\widetilde{M}_t$ is equivalent to part 3 of Proposition 5. Using part 1 of Proposition 5, one may check whether v_λ is active in D efficiently. This must be done with precise pivoting. For that, one must start with reducing v_λ with the *appropriate* columns, without doing the column reductions for the other columns, unless it is required. In the worst case, we will need to compute the whole kernel, i.e., the whole D_t, but this is not necessarily the case all the time and therefore this can be viewed as a first potential optimization step. As a side remark, using row echelon form is also taking advantage of pivoting.

Change of the Integration Order at Each Step. We conclude by another possible optimization strategy. One can change the order of the variables at each step of the integration method in order to gain some computational advantage. Suppose that we have computed $D_{t-1} = \{\Lambda_1, \ldots, \Lambda_m\}$. Consider $n_i := \#\{dx_i^{\alpha_i} \in \bigcup_i Supp(\Lambda_i) | \alpha_i \in \mathbb{N}\}$. Now, re-order the variables in the following way: if $n_i \leqslant n_j$, then put $x_i < x_j$ (note that if the equality happens, we don't care whether x_i appears before x_j or vice versa). We call such an order a *good integrable* order. Assume that $x_{b_1} < x_{b_2} < \ldots < x_{b_n}$ is a good integrable ordering, where $b_i \in \{1, \ldots, n\}$. Now we consider $\Lambda_1, \ldots, \Lambda_m$ as polynomials in $\mathbb{K}[dx_{b_1}, \ldots, dx_{b_n}]$ and continue with the integration in the following order:

$$\Lambda = \sum_i \lambda_{i1} \int_{b_1} \Lambda_i|_{dx_{b_2} = \cdots = dx_{b_n} = 0} + \cdots + \sum_i \lambda_{in-1} \int_{b_{n-1}} \Lambda_i|_{dx_{b_2} = \cdots = dx_{b_n} = 0} + \sum_i \lambda_{in} \int_{b_n} \Lambda_i.$$

This way, we will do the least possible number of integrations. Note that the number of integrands and the number of basis elements of D_{t-1} are fixed and therefore we won't gain any advantage in terms of the size of M_t. The following example illustrates the optimization.

Example 21. Consider Example 16. In step two we have that

$$D_2 = \langle \Lambda_1 = 1, \Lambda_2 = d_1 + d_2, \Lambda_3 = -d_1 + d_1^2 + d_1d_2 + d_2^2 \rangle.$$

Then $n_1 = 3, n_2 = 2$. Therefore we change the order into $y < x$ and work on $\mathbb{K}[dy, dx]$. Then

$$\Lambda = \lambda_1 dy + \lambda_2 dx + \lambda_3 dy^2 + \lambda_4(dydx + dx^2) + \lambda_5(dy^3) + \lambda_6(dx^3 - dx^2 + dx^2dy + dxdy^2).$$

We have have only one monomial in the 5-th column of M_3, while in the original ordering, we had two:

$$\Lambda = \lambda_1 dx + \lambda_2 dy + \lambda_3 dx^2 + \lambda_4(dxdy + dy^2) + \lambda_5(dx^3 - dx^2) + \lambda_6(dy^3 + dxdy^2 + dx^2dy + dxdy).$$

Algorithm 3: Modified Integration Method

Input: A basis for an $\mathfrak{m}_\zeta$-primary ideal Q_ζ

Output: A basis for $Q_\zeta^\perp$ and directional multiplicities

def *ComputeBasis***:**

$D_{\text{old}} = \varnothing$

$D_{\text{new}} = \{\Lambda = d^0 = 1\}$

$\mu_i = 0, i = 1, \ldots, n$

while $D_{old} \neq D_{new}$**:**

$D_{\text{old}} = D_{\text{new}}$

Change the order of the variables into a good integrable order

$\Lambda := \sum_{i=1}^{s} \sum_{k=1}^{n} \lambda_{ik} \int_k \Lambda_i(d_1, \ldots, d_k, 0, \ldots, 0)$

$\forall 1 \leqslant k < l \leqslant n, \sum_{i=1}^{s} \lambda_{ik} d_l(\Lambda_i) - \sum_{i=1}^{1} \lambda_{il} d_k(\Lambda_i) = 0$

$\forall 1 \leqslant i \leqslant s, \;\; \Lambda(f_i) = 0$

Construct matrix M_{new}, the coefficient matrix of Λ

Apply Criterion 19 and choose *good* columns $v_{\lambda_1}, \ldots, v_{\lambda_m}$

$M_{\text{new}} : M_{\text{new}} - v_{\lambda_1} - \cdots - v_{\lambda_m}$

$D_{\text{new}} = D_{\text{old}} \bigcup Ker(M_{\text{new}})$

If $dx_i^{\mu_i+1} \in Supp(D_{\text{new}})$, then μ_i = highest power of dx_i in $D_{\text{new}}|_{x_i \neq 0}$

return D_{new} and μ_i

3.3.2 Modifications of Macaulay's Algorithm

For Macaulay's algorithm we use the following notation. M_t stands for the matrix in step t. Columns of M_t in Macaulay's algorithm are labeled by monomials $dx^{\mathbf{a}} \in \mathbb{K}[dx_1, \ldots, dx_n]$. Then $v_{dx^{\mathbf{a}}}$ denotes the column corresponding to $dx^{\mathbf{a}}$ in M_t. Note that $v_{dx^{\mathbf{a}}}$ is well-defined because in M_t obtained via Macaulay's algorithm, for every monomial of degree at most t, there exists a column labeled by it and vice versa. Also note that since the columns are labeled by the monomials, a column $v_{dx^{\mathbf{a}}}$ is active in a basis D of $Q^\perp$ if and only if $v_{dx^{\mathbf{a}}} \in Supp(D)$.

Below we show a modification of Proposition 5 and its corollary for Macaulay's algorithm. This enables us to make Macaulay's algorithm more efficient.

Proposition 10. *Let $M_t, M_{t-1}, D_t, \Lambda_i$ $(1 \leqslant i \leqslant m), \lambda, dx^{\mathbf{a}}$ and $v_{dx^{\mathbf{a}}}$ be as above.*
Then the following hold.

1. *If $v_{dx^{\mathbf{a}}}$ is a column of M_t, then $v_{dx^{\mathbf{a}}} \in Supp(D_t)$ if and only if $v_{dx^{\mathbf{a}}}$ can be reduced to zero by other columns of M_t.*
2. *For all $1 \leqslant i \leqslant m$, if $v_{dx_i^{\mathbf{a}}} \in Supp(\Lambda_i)$, K'_{t_i} is a basis for $Ker(\widetilde{M}_t - v_{\lambda_i})$ and D'_{t_i} is the set of its corresponding dual elements, then $\{\Lambda_i\} \cup D'_{t_i}$ is a basis for the degree t part of $Q^\perp$. Moreover, if $v_{dx^{\mathbf{a}_i}} \in Supp(\Lambda_i)$, but $v_{dx^{\mathbf{a}_i}} \notin Supp(\Lambda_j)$, $1 \leqslant j \neq i \leqslant m$, then there exists a basis D'_{t_i} such that $\Lambda_j \in D'_{t_i}, j \neq i$.*
3. *For all $1 \leqslant i \leqslant m$, if $v_{dx^{\mathbf{a}_i}} \in Supp(\Lambda_i)$, but $v_{dx^{\mathbf{a}_i}} \notin Supp(\Lambda_j)$, $1 \leqslant j \leqslant i-1$, then $D_{t-1} \cup D_{t_{1\ldots m}}$ is a basis for the degree t part of $Q^\perp$.*

Proof. Similar to the proof of Proposition 5. ☐

In order to detect columns $v_{dx^{\mathbf{a}_i}}$ that satisfy the assumptions of Proposition 10, one can simply adapt the methods mentioned for the modified integration method and equivalently form the matrices $M'', G''^{-1}M''$. Then we have the following corollary, which is the equivalent of Corollary 19 for Macaulay's algorithm.

Corollary 22. ***(Criterion for Deleting Suitable Columns in Macaulay's Matrices)*** *Let $D_{t-1} = \{\Lambda_1, \cdots, \Lambda_m\}, M_t, D_t, v_{\lambda_1}, \ldots, v_{\lambda_u}$ be as in Equation 17 and $G''^{-1}M''$ be as above and (by abuse of notation) let $v_{\lambda_1}, \ldots, v_{\lambda_m}$ be the columns of M_t corresponding to the first m columns in $G''^{-1}M''$. Also let $K_{t_{1\ldots m}}$ be a basis for $Ker(M_t - v_{\lambda_1} - \cdots - v_{\lambda_m})$ and $D_{t_{1\ldots m}}$ be the set of its corresponding dual elements. Then $D_{t-1} \cup D_{t_{1\ldots m}}$ is a basis for $Q^\perp$.*

Proof. Similar to the proof of Corollary 19. $\square$

The following provides us with more modifications on Macaulay's algorithm.

Lemma 23. *For all $1 \leqslant i \leqslant n$, $1 \leqslant m, t \leqslant N$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^n$ the following hold.*

1. *Let $dx^{\boldsymbol{a}} \in Supp(D_t)$ and $dx^{\boldsymbol{b}} | dx^{\boldsymbol{a}}$ then $dx^{\boldsymbol{b}} \in Supp(D)$. In particular, if $dx^{\boldsymbol{a}} \in Supp(D_t)$ and $dx_i^m | dx^{\boldsymbol{a}}$ then $dx_i^m, \cdots, dx_i, 1 \in Supp(D)$.*
2. *Let $dx^{\boldsymbol{b}} \notin Supp(D)$, $dx^{\boldsymbol{b}} | dx^{\boldsymbol{a}}$ and $|\boldsymbol{a}| \leqslant t$. Then $dx^{\boldsymbol{a}} \notin Supp(D_t)$. In particular, if $dx_i^m \notin Supp(D)$ then $dx_i^{m+1}, \ldots dx_i^t \notin Supp(D_t)$. Also if $dx_i^m \notin Supp(D_{t-1})$ then $dx_i^{m+1}, dx_i^{m+2}, \ldots \notin Supp(D_t)$.*

Proof. 1. For all $1 \leqslant i \leqslant n$:

$$\begin{aligned} dx^{\mathbf{a}} \in Supp(D_t) &\Leftrightarrow x^{\mathbf{a}} \notin Q_\zeta \\ x^{\mathbf{a}} \notin Q_\zeta &\Rightarrow x^{\mathbf{b}} \notin Q_\zeta \\ x^{\mathbf{b}} \notin Q_\zeta &\Leftrightarrow dx^{\mathbf{b}} \in Supp(D). \end{aligned}$$

The rest can be proved by putting $x^{\mathbf{b}} = x_i^m$.

2. Although this part can be proved directly, however, we use a simple logic argument to prove it. Consider the following notations for the three logic statements that appear in the proposition:

$$\mathtt{p} = dx^{\mathbf{a}} \in Supp(D_t), \; \mathtt{q} = dx^{\mathbf{b}} | dx^{\mathbf{a}}, \; \mathtt{r} = dx^{\mathbf{b}} \in Supp(D).$$

Then the previous part says that

$$\mathtt{p} \wedge \mathtt{q} \Rightarrow \mathtt{r}.$$

Therefore, we have the following (Note that the condition $|\mathbf{a}| \leqslant t$ is a consequence of $\mathtt{p}$):

$$\begin{aligned} (\mathtt{p} \wedge \mathtt{q} \Rightarrow \mathtt{r}) &\Leftrightarrow (\neg \mathtt{r} \Rightarrow \neg(\mathtt{p} \wedge \mathtt{q})) \\ &\Leftrightarrow (\neg \mathtt{r} \Rightarrow \neg \mathtt{p} \vee \neg \mathtt{q}) \\ &\Leftrightarrow (\neg \mathtt{r} \wedge \mathtt{q} \Rightarrow \neg \mathtt{p}), \end{aligned}$$

which means that if $dx^{\mathbf{b}} \notin Supp(D)$ and $dx^{\mathbf{b}} | dx^{\mathbf{a}}$ then $dx^{\mathbf{a}} \notin Supp(D_t)$.

$\square$

By Lemma 23, one may find some monomials in $Supp(D)$ that are of degree at most t, but not necessarily belong to $Supp(D_t)$ and therefore not necessarily they appear as monomials in the generators of D_t. Also if dx_i^m is the largest power of dx_i that appears in $Supp(D_{t-1})$ then by Lemma 23 dx_i^{m+1} is the largest possible power of dx_i that can appear in $Supp(D_t)$. Another point that we can deduce from the above proposition is that if dx_i^m is the largest power of dx_i that appears in $Supp(D_{t-1})$ and $dx_i^{m+1} \notin D_t$, then not necessarily $\mu_i = m$, because dx_i^{m+1} may appear in some other step of the algorithm and therefore, for computing μ_i, this doesn't give us a termination criterion. However, in that case there won't be anymore a leading term of the form dx_i^k, $k \in \mathbb{N}$ when we work with respect to a degree term order. Also obviously, we have that $dx_i^{\mu_i}, \ldots, dx_i \in Supp(D)$. All these monomials appear in $Supp(D_t)$ at some step of the integration method, as they only will be obtained via integrating the lower power and therefore they will appear at some step of the integration algorithm. But this doesn't imply that they necessarily appear during Macaulay's algorithm. The same applies not only for the powers of a variable x_i, but also to every monomial $dx^{\mathbf{a}} \in Supp(D_{t-1})$, i.e., $\int_i dx^{\mathbf{a}}, 1 \leqslant i \leqslant n$ is the only multiple of $dx^{\mathbf{a}}$ that can appear in $Supp(D_t)$.

Based on the above remarks, we can make the following improvement to Macaulay's algorithm.

Proposition 11 (Improvement on Macaulay's Algorithm)**.** *Let M_t be the matrix obtained via Macaulay's algorithm. Consider the set*

$$A = \left\{ \int_i dx^{\boldsymbol{a}}, 1 \leqslant i \leqslant n \; : \; dx^{\boldsymbol{a}} \in Supp(D_{t-1}) \wedge \left(\nexists dx^{\boldsymbol{b}} \in Supp(D_{t-1}) \, , \; dx^{\boldsymbol{a}} | dx^{\boldsymbol{b}} \right) \right\}. \tag{18}$$

Then $Ker(M_t - v_A) = Ker(M_t)$, where $M - v_A$ is the matrix obtained by deleting the columns corresponding to the members of A.

Proof. The proof is immediate from part 2 of Lemma 23. □

We explain the improvement by redoing the calculations for Example 12, step 3 using the above result and comparing the computations.

Example 24. Let

$$f_1 = x^2 + (y-1)^2 - 1$$
$$f_2 = y^2.$$

After doing the computations in step 2, we have $D_2 = \{1, d_1, 2d_1^2 + d_2\}$. $d_2 \in Supp(D_2)$, but $d_2^2 \notin Supp(D_2)$. So, in step 3, by the above improvement, we can remove $v_{d_1 d_2^2}$ and $v_{d_2^3}$ from M_3. Also we can remove the columns $v_1, v_{d_1}, v_{d_1^2}$ using proposition 10. So the new matrix has 5 columns, while the original matrix in Macaulay's method has 10 columns.

Algorithm 4: Modified Macaulay's Algorithm

Input: A basis for an $\mathfrak{m}_\zeta$-primary ideal Q_ζ
Output: A basis for $Q_\zeta^\perp$ and the directional multiplicities
def *ComputeBasis*:
- $D_{\text{old}} = \varnothing$
- $D_{\text{new}} = \{\Lambda = d^0 = 1\}$
- $t = 0$
- $\mu_i = 0, i = 1, \ldots, m$
- **while** $D_{old} \neq D_{new}$:
  - $D_t = D_{\text{old}}$
  - $D_{\text{old}} = D_{\text{new}}$
  - Construct matrix M_{new}, the coefficient matrix of D_{new}
  - $\forall \Lambda \in D_t$, delete a *good* active column in Λ from M_{new}
  - Compute A as in Equation 18
  - $M_{new} = M_{\text{new}} - v_A$
  - If $v_{dx_i^{\mu_i+1}} \in Ker(M_{\text{new}})$, then $\mu_i = \mu_i + 1$
  - $D_{\text{new}} = \text{kernel}(M_{\text{new}})$
  - $D_{\text{new}} = D_{\text{old}} \bigcup Ker(M_{\text{new}})$
  - $t = t + 1$
- **return** D_{new} and μ_i

Example 25. Let $I = \langle f_1, f_2, f_3 \rangle \subseteq \mathbb{K}[x, y, z]$, where

$$f_1 = 2x + 2x^2 + 2y + 2y^2 + z^2 - 1,$$
$$f_2 = (x + y - z - 1)^3 - x^3,$$
$$f_3 = (2x^3 + 2y^2 + 10z + 5z^2 + 5)^3 - 1000x^5.$$

$(0, 0, -1)$ is a root of multiplicity 18, $\mu_x = 5, \mu_y = 8, \mu_z = 8$ and $\mathcal{N} = 9$. From step 3 to step 5, the highest power of dx is 2. In step 6, the monomial dx^3 appears and in steps 7 and 8, we see the monomial dx^4. For dy and dz all the powers appear in all steps. This is a very dense system for computing μ_x and μ_y, in the sense that all the powers of dy and dz appear in all the steps. However for dx we see that we have done many redundant computations.

At the end of this section, we comment on the comparison between the size of the matrices obtained at step t of the above algorithms and their modifications, as size is a big obstacle in computations. The matrix obtained via Macaulay's algorithm has $\binom{t+n}{n}$ columns and at least the same number of rows. In the integration method, $\widetilde{M}_t$ has nm columns and $\binom{n}{2} + e$ rows. Applying Condition 10 in the integration method, one gets m extra rows, which in special cases can result in deleting at most those m rows and also at most m columns. So the size of the matrix is at least $\binom{n}{2} + e + m \times (n-1)m$. However, this is exactly the size of the matrix obtained using our modification to the integration method. Also if we let $\widehat{M}_t$ be the matrix obtained from Macaulay's algorithm applying our modifications, for every column $v_{dx^{\mathbf{a}}}$ of $\widehat{M}_t$, there exists a p_λ such that $dx^{\mathbf{a}} \in Supp(p_\lambda)$. In other words, all the monomials appearing as the columns of $\widehat{M}_t$, will appear in the columns of $\widetilde{M}_t$, but the difference is that they might be a monomial in a polynomial. This means that the number of columns of $\widetilde{M}_t$ and $\widehat{M_T}$ will be the same if every p_λ is a monomial. Thus, the columns of $\widetilde{M}_t$ are (possibly sums of) the columns of $\widehat{M}_t$. Note that many of the rows in both methods can (and in practice are) zero and can be simply deleted.

Concluding this section we provide a list of computational observations:

- Computing the directional multiplicity is basically equivalent to computing the projection of the kernel of $M_{\mathcal{N}}$. There are several classic kernel computation algorithms, e.g, *Singular Value Decomposition*. However, we are not aware of any algorithm for projection, without computing the whole kernel. Proposition 5 can be considered as a proposal for an incremental algorithm for computing kernel projection.

- Having a bound for the directional multiplicities, one can construct a single matrix and compute the dual basis using that matrix rather than running several steps. This is guaranteed by Proposition 3, part 3. The idea for constructing such a matrix is to use the resultant in order to get a bound U for directional multiplicities. Having M_U, the Macaulay matrix of size U, the kernel of M_U will give us the whole dual. Note that the main obstacles for this method are the size of M_U as well as computing the resultant. The bound U could be the Bezout bound in worst case.

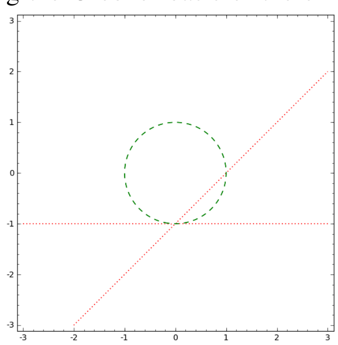
4 Applications and Future Work

4.1 Applications of Directional Multiplicity

We explore some applications of directional multiplicity. The exploration does not go into details, as the main purpose is to show the usefulness of the concept rather than present the applications themselves.

Arrangement and Topology of Planar Algebraic Curves. There are several methods in the literature for computing the arrangement and topology of a planar algebraic curve, e.g [2, 1, 7, 10, 4]. In principle, all methods use some elimination tool, e.g Gröbner basis or resultants, in order to project algebraic curves on one axis and identify the critical points (points where derivatives vanish). This is done by finding the real roots of the elimination ideal and using this information to reconstruct/identify the arrangement and topology of the curve. These approaches typically assume that no two critical points have the same projection on the axis. Our work explains what happens in that situation. In Section 4.1, we show how directional multiplicity can handle degenerate situations. Particularly, our algorithms for computing directional multiplicity with respect to an axis could be useful for computing the *multiplicity of a point in its fiber*. Devising a full algorithm for determining the topology of the algebraic curve is beyond the scope of this paper.

Geometry of the Elimination Ideal. Let $I \subseteq \mathbb{C}[x, y]$ be a zero dimensional ideal with no roots at infinity generated by two polynomials corresponding to two planar curves and $I_1 = I \cap \mathbb{C}[y] = \langle g \rangle$ be its elimination ideal. We illustrate the case of geometric degeneracy and how directional multiplicity can be used, in a concrete example. Let $f_1 = (y+1)(y-x+1)$ and $f_2 = x^2 + y^2 - 1$ as shown in the figure. The two curves intersect at two points, namely $(1, 0)$ and $(0, -1)$. Their Sylvester resultant is $2y(y+1)^3$, which implies that the projection on the y-axis of the roots $(1, 0)$ and $(0, -1)$ have multiplicity 1 and 3 respectively. On the other hand, computing the Gröbner basis of the elimination ideal in $\mathbb{C}[y]$, we obtain the unique monic generator $g = y(y+1)^2$.



f_1	$(y+1)(y-x+1)$
f_2	$x^2 + y^2 - 1$
g	$y(y+1)^2$
resultant	$2y(y+1)^3$

Observing the difference in the multiplicities of the resultant and g, the questions “when does the multiplicity drop?” and “what does the multiplicity of a factor in g mean?” arise. Using the concept of directional multiplicity, we are able to address these question in the degenerate case, as the one in the example.

The exponent of the factor of g corresponding to an intersection point is the directional multiplicity at that point. The exponent of the corresponding factor of the resultant give us the multiplicity of the intersection points. However Gröbner basis did not say much about the geometry of the intersection. Now having the concept of directional multiplicity, we can explain the generator of the elimination ideal geometrically. In general given dense polynomials $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$, let $I_1 = \langle g \rangle$ and $R_1 \ldots R_k$ be the square-free factorization of the Macaulay resultant. Then $g = R_1^{\mu_1} \ldots R_k^{\mu_k}$.

Computing Hilbert Series of Zero Dimensional Ideals. For an isolated point ζ and its corresponding $\mathfrak{m}_\zeta$-primary ideal Q_ζ, Mourrain has shown in [23] that having a base for $Q_\zeta^\perp$, one can obtain a basis for $R \big/ Q_\zeta$. Also the improvement of the integration method using Equation 10 is based on computing a dual basis along with a basis for $R \big/ Q_\zeta$. The function mapping t to the dimension of the space generated by the degree t part of this quotient is actually the Hilbert Function. Hilbert function and Hilbert series can be computed via Gröbner bases. Having the dual basis, one can compute the Hilbert function and Hilbert series. For instance, for a 0-dimensional ideal, given the set of points in the variety of the ideal, Chapter 7 of [25] shows such a method to compute the Hilbert function and series as well as the regularity. These are based on using Gröbner basis for the computations. Alternatively, one can use dual bases in order to compute these objects. In particular, directional multiplicities can be used to compute the degree of the elements of the ideal, which can be useful in computing the regularity. Finally, directional multiplicities can be used in computing the Hilbert series of the last elimination ideal.

4.2 Future Work

Directional Multiplicity with respect to an arbitrary $v \in \mathbb{R}^n$. In the definition of directional multiplicity, we have considered the n axes as the directions. One could think of defining the multiplicities in the direction of an arbitrary vector $v \in \mathbb{R}^n$. The directional multiplicities along these vectors might be useful in studying singularities of curves.

Directional Multiplicity for Sparse Systems. Let us consider the following example.

Example 26. Let $I = \left\langle f_1 = x^9 - x^6y^2, f_2 = y \right\rangle \subseteq \mathbb{K}[x, y]$. Then the origin is a root of degree 9, $\mu_1 = 9, \mu_2 = 1$. Both the integration method and Macaulay’s algorithm need to run until step 10 in order to find the dual space.

In the above example many columns (corresponding to monomials) are considered, which are equal to the zero vector. This is because the system is *sparse*. If we knew a-priori that $dx^9 \in D$, then we could have avoided the previous steps. One idea to deal with such cases is to start with the matrix M_k, where k is an upper bound for $\mathcal{N}$ and do the binary search top-down. However the only such bound that we are aware of is the Bezout bound for μ, which can be too big and hence this method is impractical. For computing μ_i, when we have a sparse system with respect to x_i, one could follow a down-top algorithm which works by a-priory adding extra columns $v_{dx_i}, \ldots, v_{dx_i^{2^t}}$ to the modified matrix M_t, where modified M_t refers to the matrix that has been obtained at step t of either modified integration method or Macaulay’s algorithm.

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