The Concept of Gröbner Reduction for Dimension in filtered free modules

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We present the concept of Gröbner reduction that is a Gröbner basis technique on filtered free modules. It allows to compute the dimension of a filtered free module viewn as a K-vector space. We apply the developed technique to the computation of a generalization of Hilbert-type dimension polynomials in $K[X]$ as well as in finitely generated difference-differential modules. The latter allows us to determine a multivariate dimension polynomial where we partition the set of derivations and the set of automorphism in a difference-differential ring in an arbitrary way.

1 Introduction

In [ZW08], the authors investigate how the theory of Gröbner bases in polynomial ideal theory can be exploited to solve problems in modules of difference-differential polynomials over the ring of difference-differential operators D. The main ingredients are the notion of reduction that is established by Gröbner bases. The developed Gröbner basis theory is used to give an answer to the question of dimension in a D-module. [KLAV98] give a detailed investigation of how to calculate difference and differential dimension polynomials, as well as generalizing the notion of dimension to difference-differential modules. [ZW08] introduced the notion of a bivariate difference-differential dimension polynomial, where they make a natural split between the set of derivations and the set of automorphisms. Levin investigated further, and partitioned the set of derivations in [Lev12, Lev13]. In this paper we show in a very general setting, that filtrations of an arbitrary associative ring D are the

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key to get full information of a set of descriptors of a module. In particular we will find out, that the partition of the basic sets is completely arbitrarily and therefore it is reasonable to construct the theory on general filtered rings. After a theoretical introduction, we apply the developed mechanism to polynomial ideals to give a general description of dimension, including Hilbert polynomials as special cases. Afterwards, we extend the existing theory of dimension in difference-differential modules to get information where we partition the set of derivations and the set of automorphisms, and compute a multivariate generalization of the bivariate dimension polynomial. This takes the work in [ZW07, ZW08, Lev12, Lev13] as possible instances of our general algorithm. An account on applications of dimension polynomials is given in [KLAV98, Chapter VII]. This paper is organized as follows: In section 2 introduce the notion of filtered free module, and introduce monomial filtered rings resp. monomial filtration on the polynomial ring and the ring of difference-differential operators. We show, how a filtration on the ring D is inherited to a free module F and how this connects to a general finitely generated module M. In section 3 this construction is used for the concept of Gröbner reduction that is used to compute the dimension in a filtered module. We show that relative reduction as introduced in [ZW08] is an instance of Gröbner reduction w.r.t. a bivariate monomial filtration. The main Theorem 1. shows how we can compute the dimension in a constructive way. In section 4 we apply the developed mechanism to polynomial ideals and introduce a general version of Hilbert polynomials. Finally we apply Gröbner reduction to the computation of multivariate difference-differential dimension polynomials in finitely generated difference-differential modules. Section 5 holds concluding remarks.

2 Filtered free modules

As a general setting throughout this report, we assume we are given a ring D with unit 1, such that D contains the field K. Expressions in D shall be of the form

$$f \in D: \ f = \sum_{\lambda \in \Lambda} a_\lambda \lambda, \qquad a_\lambda \in K, \Lambda \subseteq D,$$

the product of field elements a_λ with elements $\lambda \in \Lambda$ not necessarily commutative. The elements λ are called *monomials*. The set of *terms* of $f \in D$ is denoted by

$$T(f) := \{ \ \lambda \in \Lambda \mid \lambda \text{ appears in } f \text{ with a non-zero coefficient } \}.$$

Obviously for $f, g \in D$ we have that $T(f \pm g) \subseteq T(f) \cup T(g)$.

Next, we define a filtration on D by writing

$$(D_{\mathbf{r}}), \qquad \mathbf{r} := (r_1, \ldots, r_n) \in \mathbb{N}^n.$$

where $D_{\mathbf{r}} \subseteq D$ are additive subgroups such that

$$\begin{aligned} \cup_{\mathbf{r}\in\mathbb{N}^n} D_{\mathbf{r}} &= D \\ D_{r_1,\ldots,r_i,\ldots,r_n} &\subseteq D_{r_1,\ldots,r_i+1,\ldots,r_n} \\ D_{\mathbf{r}} D_{\mathbf{s}} &\subseteq D_{\mathbf{r}+\mathbf{s}} \\ 1 &\in D_{0,\ldots,0} =: D_0, \end{aligned}$$

holds for all $\mathbf{r}, \mathbf{s} \in \mathbb{N}^n$. The ring D together with this filtration will be called a *filtered ring.*

Definition 1 (Monomially filtered ring)**.** A filtration $(D_{\mathbf{r}})_{\mathbf{r}\in\mathbb{N}^n}$ on D is said to be *monomial* if and only if

- $D_0 = K$
- whenever f is in $D_{\mathbf{r}}$ then all terms λ that occur in f with a non-zero coefficient are also contained in $D_{\mathbf{r}}$

Hence, if

$$f = \sum_{\lambda\in\Lambda} a_\lambda \lambda \ \in D_{\mathbf{r}} \Longrightarrow \forall\ \lambda \in T(f):\ \lambda \in D_{\mathbf{r}},$$

or what is equivalent

$$f \in D_{\mathbf{r}} \Longrightarrow T(f) \subseteq D_{\mathbf{r}}.$$

We call D equipped with such a filtration a *monomially filtered ring* over the field K.

Example 1. Consider the polynomial ring $D := K[x, y, z]$ over a field K. Monomials $\lambda \in \Lambda$ are power products $x^r y^s z^t$, polynomials its K-linear combinations

$$f = \sum_{\mathbf{r}\in\mathbb{N}^3} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}} \ \in K[x, y, z], \qquad T(f) = \{\ \mathbf{x}^{\mathbf{r}} \mid a_{\mathbf{r}} \neq 0\ \}.$$

On monomials λ we define

$$|\lambda|_1 := \deg_x(\lambda), \qquad |\lambda|_2 := \deg_y(\lambda), \qquad |\lambda|_3 := \deg_z(\lambda).$$

For ring elements $f \in D$, we set

$$|f|_k := \max\{\ |\lambda|_k \mid \lambda \in T(f)\ \}, \qquad 1 \leq k \leq 3.$$

Then the filtration $D_{\mathbf{r}}$ given by

$$D_{r,s,t} := \{\ f \in D \mid |f|_1 \leq r\ \wedge\ |f|_2 \leq s\ \wedge\ |f|_3 \leq t\ \},$$

defines a monomial filtration on $K[x, y, z]$.

Example 2 (Monomial Filtration as in [ZW08])**.** Consider a field K together with a set of derivations resp. automorphisms

$$\Delta := \{\ \delta_1, \ldots, \delta_n\}, \qquad \Sigma := \{\ \sigma_1, \ldots, \sigma_m\ \}.$$

Let D be the ring of difference-differential operators over K. The monomials in Λ consist of expressions of the form

$$\lambda := \delta_1^{k_1} \ldots \delta_n^{k_n} \sigma_1^{l_1} \ldots \sigma_m^{l_m}, \qquad (k_1, \ldots, k_n) \in \mathbb{N}^n,\ (l_1, \ldots, l_m) \in \mathbb{Z}^m,$$

we set

$$|\lambda|_1 := k_1 + \ldots + k_n, \qquad |\lambda|_2 := |l_1| + \ldots + |l_m|.$$

For the operator

$$f = \sum_{\lambda \in \Lambda} a_\lambda \lambda \ \in D,$$

we define the order by

$$|f|_1 := \max\{ \ |\lambda|_1 \mid a_\lambda \neq 0 \ \}, \quad \text{and} \quad |f|_2 := \max\{ \ |\lambda|_2 \mid a_\lambda \neq 0 \ \}.$$

For $r, s \in \mathbb{N}$ let

$$D_{r,s} := \{ \ f \in D \mid |f|_1 \leq r \ \wedge \ |f|_2 \leq s \ \}.$$

Then $(D_{r,s})_{(r,s)\in\mathbb{N}^2}$ is a (bivariate) monomial filtration. We call it the *standard filtration* of D.

Let M be a finitely generated left D-module, where D is a filtered ring. A filtration of M w.r.t. the filtered ring D is a family of additive subgroups $M_{\mathbf{r}} \subseteq M$ such that for all $\mathbf{r}, \mathbf{s} \in \mathbb{N}^n$ we have that:

$$\cup_{\mathbf{r}\in\mathbb{N}^n} M_{\mathbf{r}} = M$$
$$M_{r_1,\ldots,r_i,\ldots,r_n} \subseteq M_{r_1,\ldots,r_i+1,\ldots,r_n}$$
$$D_{\mathbf{r}} M_{\mathbf{s}} \subseteq M_{\mathbf{r}+\mathbf{s}}.$$

What we call a filtered module over a filtered ring appears in the literature as multi-filtered module over a multi-filtered ring, see e.g. [Tor99].

Definition 2 (Morphism of filtered modules)**.** A D-homomorphism $\varphi : M \longrightarrow N$ between filtered D-modules is called a *morphism of filtered* D-modules or shortly a morphism if and only if

$$\varphi(M_{\mathbf{r}}) \subseteq N_{\mathbf{r}}, \qquad \mathbf{r} \in \mathbb{N}^n.$$

Obviously each $M_{\mathbf{r}}$ is a D_0-module and a morphism $\varphi : M \longrightarrow N$ induces D_0-linear maps $M_{\mathbf{r}} \longrightarrow N_{\mathbf{r}}$ for $\mathbf{r} \in \mathbb{N}^n$.

Lemma 1. *Let M, N be left D-modules, $\varphi : M \longrightarrow N$ be a morphism of modules.*

1. *If M is a filtered D-module then $\varphi(M)$ is filtered by setting $\varphi(M)_{\mathbf{r}} := \varphi(M_{\mathbf{r}})$. The map φ is then a morphism $M \longrightarrow \varphi(M)$.*
2. *If N is a filtered D-module, then M is filtered by setting $M_{\mathbf{r}} := \varphi^{-1}(N_{\mathbf{r}})$. φ is then a morphism $M \longrightarrow N$.*

Let now

$$F = D \cdot e_1 \ \oplus \ \ldots \ \oplus \ D \cdot e_m$$

be the free D-module of rank m that is generated by $E := \{ \ e_1, \ldots, e_m \ \}$. The filtration of D extends naturally to a filtration on F by

$$F_{\mathbf{r}} := D_{\mathbf{r}} \cdot e_1 \ \oplus \ \ldots \ \oplus \ D_{\mathbf{r}} \cdot e_m, \qquad \mathbf{r} \in \mathbb{N}^n.$$

By Lemma 1. each finitely generated D-module

$$M = D \cdot h_1 \ \oplus \ \ldots \ \oplus \ D \cdot h_m$$

inherits a filtration $M_{\mathbf{r}} := \pi(F_{\mathbf{r}})$ where $\pi : F \longrightarrow M$. Of course, this filtration depends on the choice of the generators $h_1, \ldots, h_m$, that is, on the epimorphism π.

Remark. The notation of $T(f)$, where $f \in D$, carries over to filtered free modules $T(h)$ where $h \in F_{\mathbf{r}}$. The monomials in a filtered free module are of the form $\lambda \cdot e_k$ where $\lambda \in \Lambda$ and $e_k \in E$.

3 Gröbner Reduction

Let the free D-module $F = D^m$ be equipped with extended filtration from D. We consider a reduction relation $\longrightarrow^{\dagger} \subseteq F \times F$, we write

$$f \longrightarrow^{\dagger} h, \qquad f, h \in F,\ f \neq h,$$

to indicate that $(f, h) \in \longrightarrow^{\dagger}$ and say that f reduces to h. The set of irreducible elements of F is denoted by

$$I^{\dagger} := \left\{ x \in F \mid \neg \left(\exists y \in F :\ x \neq y \wedge x \longrightarrow^{\dagger} y \right) \right\}.$$

Definition 3 (Gröbner reduction)**.** The relation $\longrightarrow^{\dagger}$ is called a *Gröbner reduction* for the submodule $N \subseteq F$ provided that

- $f \longrightarrow^{\dagger} h \Longrightarrow f \equiv h \pmod N$
- $\longrightarrow^{\dagger}$ is noetherian (i.e. every sequence $f_1 \longrightarrow^{\dagger} f_2 \longrightarrow^{\dagger} \ldots$ terminates)
- $I^{\dagger}$ is a monomial K-linear subspace of F, i.e. $I^{\dagger}$ is a vector space and for all $f \in F$
$$f \in I^{\dagger} \Longrightarrow T(f) \subseteq I^{\dagger}.$$
- For all $\mathbf{r} \in \mathbb{N}^n$ and for all $f, h \in F$ we have
$$f \longrightarrow^{\dagger} h \ \wedge\ f \in F_{\mathbf{r}} \Longrightarrow h \in F_{\mathbf{r}}. \tag{1}$$
- $I^{\dagger} \cap N = \{0\}$, i.e. every nonzero element in N is reducible

Remark. Condition (1) ensures that the reduction relation and the given filtration work together. We express this by saying that the reduction and the filtration are *compatible.*

Example 3 (Polynomial reduction is not a Gröbner reduction)**.** Let $D = K[x, y]$ and consider the univariate filtration

$$D_k := \{\ f \in D \mid \deg(f) \leq k\ \}.$$

Then D is monomially filtered. However, this filtration is not compatible to arbitrary term orders. If we equip D with a lexicographic order $\prec := \mathrm{lex}(x > y)$, and consider the ideal

$$N :=\ {}_D\langle x - y^2 \rangle \trianglelefteq D,$$

then the polynomial $f := x^2 \in D_2$ reduces by means of the Gröbner basis $G := \{\ g := x - y^2\ \}$ like

$$f \longrightarrow h :\Longleftrightarrow h = f - x \cdot g = x^2 - (x^2 - xy^2) = xy^2 \in D_3,$$

violating condition (1).

Notation. If $\preceq$ is a term order on ΛE, $\lambda \in \Lambda$ and $f \in F$, we will write $f \preceq \lambda$ for $\mathrm{lt}(f) \preceq \lambda$. Thus

$$f \preceq \lambda \iff \forall t \in T(f) : t \preceq \lambda.$$

Example 4 (Relative Reduction as in [ZW08])**.**
In [ZW08] the following situation is considered: Let the order relations $\prec$ and $\prec'$ be defined as follows: Given the elements

$$\lambda e_i = \delta_1^{k_1} \dots \delta_n^{k_n} \sigma_1^{l_1} \dots \sigma_m^{l_m} e_i, \qquad \mu e_j = \delta_1^{r_1} \dots \delta_n^{r_n} \sigma_1^{s_1} \dots \sigma_m^{s_m} e_j,$$

we define

$$\begin{aligned} \lambda e_i \prec \quad \mu e_j :\iff \quad & (|\lambda|_2, |\lambda|_1, e_i, k_1, \dots, k_n, |l_1|, \dots, |l_m|, l_1, \dots, l_m) \\ & <_{\text{lex}} \\ & (|\mu|_2, |\mu|_1, e_j, r_1, \dots, r_n, |s_1|, \dots, |s_m|, s_1, \dots, s_m), \end{aligned}$$

respectively

$$\begin{aligned} \lambda e_i \prec' \quad \mu e_j :\iff \quad & (|\lambda|_1, |\lambda|_2, e_i, k_1, \dots, k_n, |l_1|, \dots, |l_m|, l_1, \dots, l_m) \\ & <_{\text{lex}} \\ & (|\mu|_1, |\mu|_2, e_j, r_1, \dots, r_m, |s_1|, \dots, |s_m|, s_1, \dots, s_m). \end{aligned}$$

Let $f, h \in F$ and $g \in G$. Then

$$f \longrightarrow_g^{\text{rel}} h \ :\iff \quad f \longrightarrow_g h \quad \wedge \ \underset{\prec'}{\mathrm{lt}}(\lambda g) \preceq' \underset{\prec'}{\mathrm{lt}}(f)$$

This reduction is a special case of our concept: Let

$$D_{r,s} := \{\ u \in D \mid |u|_1 \leq r \ \wedge \ |u|_2 \leq s \ \}.$$

This gives a monomial filtration of $D = \cup_{(r,s)\in\mathbb{N}^2} D_{r,s}$. The designation

$$F_{r,s} := D_{r,s} \times \ldots \times D_{r,s}$$

establishes a monomial filtration on F w.r.t. the monomial filtration $(D_{r,s})$.

Lemma 2.

$$f \longrightarrow_g^{rel} h \quad \text{and } f \in F_{r,s} \Longrightarrow h \in F_{r,s},$$

that is, $\prec$-reduction relative to $\prec'$ is compatible with the filtration $F_{r,s}$.

Proof. Assume that $f \longrightarrow_g^{\text{rel}} h$ and $f \in F_{r,s}$. Thus, let $|f|_1 \leq r$ and $|f|_2 \leq s$ and let

$$\begin{aligned} u :=& \quad \underset{\prec}{\mathrm{lt}}(f) = \underset{\prec}{\mathrm{lt}}(\lambda g), \qquad & a_u = \underset{\prec}{\mathrm{lc}}(f), \quad a_{u'} = \underset{\prec'}{\mathrm{lc}}(f) \\ u' :=& \quad \underset{\prec'}{\mathrm{lt}}(f), & b_u = \underset{\prec}{\mathrm{lc}}(\lambda g) \end{aligned}$$

We can sort f and λg w.r.t. $\prec$ and $\prec'$ as follows:

$$\begin{aligned} f =& \quad a_u u + \varphi \quad = \quad a_{u'} u' + \varphi' \\ \lambda g =& \quad b_u u + \psi \end{aligned}$$

From the assumption of $\longrightarrow^{\text{rel}}$ reduction, we obtain that $\lambda g \preceq' u'$ and

$$h = f - \frac{\text{lc}(f)}{\text{lc}(\lambda g)}\lambda g = \varphi - \frac{a_u}{b_u}\psi.$$

Therefore

$$T(h) \subseteq T(\varphi) \cup T(\psi) = (T(f) \cup T(\lambda g))\backslash\{u\}.$$

Take $\mu \in T(h)$. If $\mu \in T(f)$ then $|\mu|_1 \leq r$ and $|\mu|_2 \leq s$. If $\mu \in T(\lambda g)$ then we have $\lambda g \preceq' u'$ and we get

$$|\mu|_1 \leq |u'|_1 \leq r.$$

Because $u = \text{lt}(\lambda g)$ we obtain $\mu \prec u$ and thus $|\mu|_2 \leq |u'|_2 \leq s$.

So in any case we obtain $|\mu|_1 \leq r$ and $|\mu|_2 \leq s$, which implies $|h|_1 \leq r \,\wedge\, |h|_2 \leq s$. Therefore $h \in F_{r,s}$, hence $\longrightarrow^{\text{rel}}$ is compatible with the filtration. □

Now consider a left module M

$$M = D \cdot h_1 + \ \dots \ + D \cdot h_m$$

of finite rank m over the monomially filtered ring D, and a free presentation

$$\{0\} \longrightarrow N \longrightarrow F \stackrel{\pi}{\longrightarrow} M \longrightarrow \{0\},$$

with $F = D^m$. Assume we are given a Gröbner reduction $\longrightarrow^{\dagger}$ for N. In particular, N is represented as a Gröbner basis for $\ker(\pi)$. Let the module M be equipped with the filtration

$$M_{\mathbf{r}} = \pi\left(F_{\mathbf{r}}\right), \qquad \mathbf{r} \in \mathbb{N}^n.$$

Let $U_{\mathbf{r}} \subseteq F_{\mathbf{r}}$ be the set of irreducible monomials in the filter space $F_{\mathbf{r}}$.

Theorem 1. *For all $\boldsymbol{r} \in \mathbb{N}^n$ the set $\pi\left(U_{\boldsymbol{r}}\right)$ comprises a K-vector space basis for the space $M_{\boldsymbol{r}}$. In particular we get*

$$\dim(M_{\boldsymbol{r}}) = |\pi(U_{\boldsymbol{r}})|, \qquad \boldsymbol{r} \in \mathbb{N}^n.$$

The map π restricted on the irreducible elements $I^{\dagger}$ is injective, hence the map π is actual a bijection making

$$\dim(M_{\boldsymbol{r}}) = |\pi(U_{\boldsymbol{r}})| = |U_{\boldsymbol{r}}|.$$

Proof. First, we note that the set of irreducible monomials $U_{\mathbf{r}}$ w.r.t. $\longrightarrow^{\dagger}$ in the filter space $F_{\mathbf{r}}$ is given by

$$U_{\mathbf{r}} = I^{\dagger} \,\cap\, \Lambda E \,\cap\, F_{\mathbf{r}},$$

their union over the n-fold integers is

$$\bigcup_{\mathbf{r}\in\mathbb{N}^n} U_{\mathbf{r}} = I^{\dagger} \,\cap\, \Lambda E \,\cap\, \left(\,\cup_{\mathbf{r}\in\mathbb{N}^n} F_{\mathbf{r}}\right) = I^{\dagger} \,\cap\, \Lambda E \,\cap\, F = I^{\dagger} \,\cap\, \Lambda E,$$

which are the irreducible monomials in F w.r.t. $\longrightarrow^{\dagger}$. To show, that π restricted on $I^{\dagger}$ is injective, consider elements $f, h \in I^{\dagger}$ such that $\pi(f) = \pi(h)$. Recall that N is a Gröbner basis of $\ker(\pi)$. We get

$$\pi(f) = \pi(h) \Longrightarrow \pi(f-h) = 0 \Longrightarrow f - h \in \ker(\pi) \Longrightarrow \ f - h \in N \,\cap\, I^{\dagger} = \{0\}.$$

The restriction of π on the set $U_{\mathbf{r}}$ is injective follows by $U_{\mathbf{r}} \subseteq I^{\dagger}$.

To prove that $\pi\left(I^{\dagger} \cap \Lambda E\right)$ is K-linearly independent. Assume

$$a_1\pi(u_1) + \ \dots \ + a_l\pi(u_l) = 0, \qquad a_i \in K, u_j \in \Lambda E.$$

This implies

$$\sum_{j=1}^{l} a_i u_i \in N \ \cap \ I^{\dagger} = 0 \Longrightarrow a_1 = \ldots = a_l = 0.$$

The set $\pi\left(U_{\mathbf{r}}\right)$ is K-linearly independent follows by $U_{\mathbf{r}} \subseteq I^{\dagger} \ \cap \ \Lambda E$.

By using the axioms from the Definition of Gröbner reduction, one can reduce any $f \in F$ until an irreducible element $h \in I^{\dagger}$ is reached. The compatibility of the reduction $\longrightarrow^{\dagger}$ with the filtration implies for all $\mathbf{r} \in \mathbb{N}^n$ that for all $f \in F_{\mathbf{r}}$ there exists some $h \in I^{\dagger} \ \cap \ F_{\mathbf{r}}$ with the property $\pi(f) = \pi(h)$.

Finally we want to prove that any element $m \in M_{\mathbf{r}}$ can be represented as K-linear combination of elements in $\pi\left(U_{\mathbf{r}}\right)$. For this element m there exists $f \in F_{\mathbf{r}}$ such that $m = \pi(f)$. Take the element $h \in I^{\dagger} \ \cap \ F_{\mathbf{r}}$, with the property $\pi(h) = \pi(f)$, whose existence was established before, and represent it as

$$h = \sum_{j=1}^{s} a_j u_j, \qquad a_j \in K, u_j \in \Lambda E.$$

Because $F_{\mathbf{r}}$ is monomial, all $u_j \in F_{\mathbf{r}}$ and since $h \in I^{\dagger}$, all terms of h must be in $I^{\dagger}$. Therefore

$$u_j \in F_{\mathbf{r}} \ \cap \ \Lambda E \ \cap \ I^{\dagger} = U_{\mathbf{r}}, \qquad \text{for all } j.$$

Consequently

$$m = \pi(h) = \sum_{j=1}^{s} a_j \pi(u_j) \quad \in \ {}_K\langle \pi\left(U_{\mathbf{r}}\right)\rangle,$$

hence, $\pi(U_{\mathbf{r}})$ is a K-basis.

□

4 Applications

4.1 Generalized Hilbert Polynomials

Theorem 2. *Let $G := \{g_1, \ldots, g_t\} \in K[X]$ be a Gröbner basis of*

$$\langle G\rangle = \mathfrak{a} = \ {}_{K[X]}\langle f_1, \ldots, f_s\rangle \ \trianglelefteq K[X] =: D,$$

w.r.t. the term order $\prec$. Define a filtration on D by

$$D_{\boldsymbol{r}} = \{ \ f \in D \mid \forall \ m \in T(f) : \quad m \preceq \boldsymbol{x}^{\boldsymbol{r}} \ \}, \qquad \boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{N}^n.$$

Then, polynomial reduction $\longrightarrow$ is a Gröbner reduction.

Example 5. Suppose, we are working over the bivariate polynomial ring $K[x,y] =: D$, and consider the ideal

$$\mathfrak{a} := {}_D\langle\, f_1 := x^4y^3,\ f_2 := x^2y^5,\ f_3 := 2x^5y^2 - 4x^3y^5\rangle \trianglelefteq D$$

The elements

$$G := \{\ g_1 := x^4y^3,\ g_2 := x^2y^5,\ g_3 := y^2x^5\ \}$$

provide a Gröbner basis for $\mathfrak{a}$ w.r.t. the graded lexicographic order $\prec := \mathrm{glex}(x > y)$. We consider the filtration

$$D_{r,s} = \{\ f \in D \mid \deg_x(f) \le r\ \wedge\ \deg_y(f) \le s\ \}.$$

Plugging in values $(r,s) \gg (0,0)$, we can count the number of irreducible monomials. This value can be interpolated as bivariate polynomial by:

$$\#\text{ of irred. monomials in } D_{r,s}:\ p_2(r,s) = 2s + 2r + 7\ \in K[r,s], \qquad (r,s) \ge (4,4).$$

From that, we see that the growth of elements is *linear* by increasing the degree in one direction. Moreover p_2 is symmetric ($p_2(r,s) = p_2(s,r)$) indicating that the growth of dimension is the same in x and y direction.

Filled circles are the irreducible elements of $\mathfrak{a}$, circles that are not filled are contained in $\mathfrak{a}$

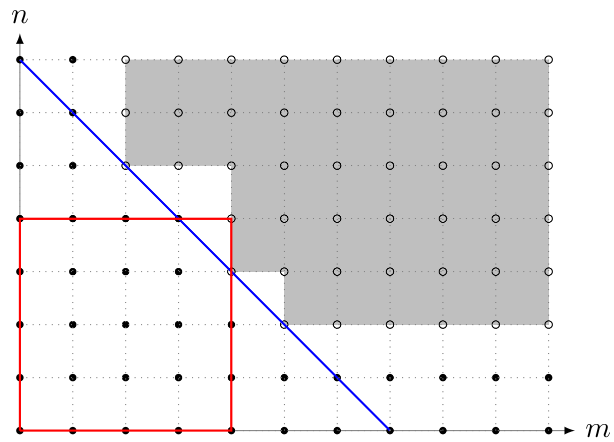

Figure 1: $(m,n) \mapsto x^m y^n$

and are reducible by polynomial reduction with elements in G. The red line indicates the index of regularity, from where on the generalized Hilbert polynomial coincides with the no. of irreducible elements. The blue line indicates the index of regularity for a total degree orderings. The number of irreducible terms in

$$D_k = \{\ f \in D \mid \deg(f) \le k\}$$

is given by

k	7	8	9	10	11	12
no. of irred. terms	33	37	41	45	49	53

For example, there are $\binom{8+2}{2} = 45$ monomials in 2 variables of total degree 8. From this 45 monomials, there are 37 monomials irreducible, leaving 8 reducible elements w.r.t. polynomial reduction with graded lexicographic order. They are given by:

$$D_8 \backslash\ I^\dagger = \{x^5y^2, x^6y^2, x^4y^3, y^3x^5, x^4y^4, x^2y^5, x^3y^5, x^2y^6\}.$$

From the 8 elements $D_8 \backslash\ I^\dagger$ there are 3 elements of degree 7, hence, in two variables, there are in total $\binom{7+2}{2} = 36$ monomials of degree 7, 3 of them reducible modulo G, giving us the value 33. The degree of the Hilbert polynomial is bounded by the number of variables, allowing us to deduce that the Hilbert polynomial is given by $p_1(k) = 4k+5$ (where $k \geq 7$). An obvious relation between $D_{r,s}$ and D_{r+s} is

$$\forall\ (r,s) \in \mathbb{N}^2 :\ D_{r,s} \subseteq D_{r+s}.$$

A less obvious connection is that for $k \geq 1$:

$$\begin{aligned} \text{irred. elements in } D_{k,k} = p_2(k,k) = &\quad 4k+7 \\ \leq &\quad 8k+5 = p_1(k+k) = \text{ irred. elements in } D_{k+k}. \end{aligned}$$

4.2 Multivariate Difference-Differential Dimension Polynomials

Suppose, we are given a Difference-Differential field $(K, +, \cdot)$ with basic sets

$$\Delta := \{\delta_1, \ldots, \delta_n\}, \qquad \Sigma := \{\sigma_1, \ldots, \sigma_m\}.$$

Let D denote the ring of difference-differential operators. Now, we make a partition of the sets of derivations Δ and the set of automorphisms Σ into p respectively q pairwise disjoint subsets,

$$\Delta := \bigcup_{i=1}^{p} \Delta_i, \qquad \Sigma := \bigcup_{j=1}^{q} \Sigma_j, \tag{2}$$

where

$$\begin{aligned} \Delta_1 &:= \{\delta_1, \ldots, \delta_{n_1}\}, \\ \Delta_k &:= \{\delta_{n_1+1+\ldots+n_{k-1}}, \ldots, \delta_{n_1+\ldots+n_k}\}, \qquad (2 \leq k \leq p), \end{aligned}$$

and $n_1 + \ldots + n_p = n$, and similar for Σ:

$$\begin{aligned} \Sigma_1 &:= \{\sigma_1, \ldots, \sigma_{m_1}\}, \\ \Sigma_k &:= \{\sigma_{m_1+1+\ldots+m_{k-1}}, \ldots, \sigma_{m_1+\ldots+m_k}\}, \qquad (2 \leq k \leq q), \end{aligned}$$

where $m_1 + \ldots + m_q = m$.

Definition 4 (Order with respect to a partition)**.**
Suppose we have given a difference-differential operator $\lambda := \delta_1^{k_1} \ldots \delta_n^{k_n} \sigma_1^{l_1} \ldots \sigma_m^{l_m} \in \Lambda$, and a partition of the sets Δ and Σ of the form (2). We define

$$|\lambda|_{\Delta_j} := \sum_{\delta_i \in \Delta_j} k_i, \qquad |\lambda|_{\Sigma_j} := \sum_{\sigma_i \in \Sigma_j} |l_i|.$$

For a difference-differential operator

$$u = \sum_{\lambda \in \Lambda} a_\lambda \lambda \quad \in D, \qquad a_\lambda \in K,$$

we define

$$|u|_\Phi := \max\{\ |\lambda|_\Phi \mid \lambda \in T(u)\ \}, \qquad \Phi \in \{\Delta_1, \ldots, \Delta_p, \Sigma_1, \ldots, \Sigma_q\}.$$

We consider D as filtered ring

$$(D_{\mathbf{r}})\,, \qquad \mathbf{r} = (r_1, \ldots, r_{p+q}) \in \mathbb{N}^{p+q},$$

where

$$D_{\mathbf{r}} := \{\ u \in D \mid 1 \le i \le p : |u|_{\Delta_i} \le r_i \text{ and } 1 \le i \le q : |u|_{\Sigma_i} \le r_{p+i}\ \} \subseteq D. \tag{3}$$

for $\mathbf{r} \in \mathbb{N}^{p+q}$.

The multivariate difference-differential polynomial of a difference-differential module $M_{\mathbf{r}}$ is a numerical polynomial. The precise statement is:

Definition 5 (Multivariate dimension polynomial)**.** Multivariate dimension polynomial
A polynomial p in $\mathbb{Q}[t_1, \ldots, t_s]$ is call s-variate numerical if and only if $p(\xi_1, \ldots, \xi_s) \in \mathbb{Z}$ for all $(\xi_1, \ldots, \xi_s) \in \mathbb{N}^s$ large enough. Let M be a difference-differential module over a difference-differential field with m derivations and n automorphisms, partitioned as given in (2). The numerical polynomial $p(t_1, \ldots, t_s)$ is called s-variate difference-differential dimension polynomial associated to M, if

1. $1 \le s \le m + n$
2. $\deg(p) \le m + n$
3. $p(r_1, \ldots, r_s) = \dim_K M_{r_1,\ldots,r_s}$ for all $(r_1, \ldots, r_s) \in \mathbb{N}^s$ large enough.

By a change of the vector space basis of polynomials of degree less equal s, to the Newton basis

$$\left\{ 1 = \binom{t}{0}, \binom{t+1}{1}, \binom{t+2}{2}, \ldots, \binom{t+s}{s} \right\} \subseteq \Pi_s := \{\ p(t) \in K[t] \mid \deg(p(t)) \le s\ \},$$

then p admits a canonical representation of the form

$$\sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_s=0}^{n_s} a_{i_1,i_2,\ldots,i_s} \binom{t_1+i_1}{i_1} \binom{t_2+i_2}{i_2} \cdots \binom{t_s+i_s}{i_s}, \qquad a_{i_1,i_2,\ldots,i_s} \in \mathbb{Z},$$

the dimension polynomial whose existence is established in [Lev12, Lev13].

Theorem 3. *Let* $(K, +, \cdot)$ *be a difference-differential field with basic sets*

$$\Delta := \{\delta_1, \dots, \delta_n\}, \qquad \Sigma := \{\sigma_1, \dots, \sigma_m\},$$

the sets partitioned as in (2)*, the ring of difference-differential operators* D *filtrated by* (3)*. Let* M *be a finitely generated difference-differential module, generated by* $\{h_1, \dots, h_s\}$*. We define* F *to be the free difference-differential module with generating set* $E := \{e_1, \dots, e_s\}$*. The epimorphism* π *is defined by:*

$$\begin{aligned} \pi :\ & F \longrightarrow M \\ & e_i \longmapsto \pi(e_i) := h_i \qquad (1 \leq i \leq s). \end{aligned}$$

For the submodule $N := \ker \pi$ *of* F*, let*

$$G := \{g_1, \dots, g_t\} \subseteq N$$

be a $\prec$*-Gröbner basis of* N *computed with a Gröbner reduction. We define*

$$\begin{aligned} U_{\boldsymbol{r}} := \{\ \lambda e \in \Lambda E \ \cap\ F_{\boldsymbol{r}} \mid \lambda \ \in\ \Lambda \quad & \text{for all } g \in G, \quad \text{for all } \mu \in \Lambda \\ & \text{such that there exists one of } \ i, j : 1 \leq i \leq p,\ 1 \leq j \leq q : \\ & \lambda e = \operatorname*{lt}_{\prec}(\mu g) \Longrightarrow (\ |\mu g|_{\Delta_i} > r_i \ \vee\ |\mu g|_{\Sigma_j} > r_{p+j}\ )\ \}. \end{aligned}$$

Then the $(p+q)$*-variate difference-differential dimension polynomial associated to* M *is the cardinality of* U*, i.e.*

$$\dim_K M_{\boldsymbol{r}} = |U_{\boldsymbol{r}}|, \qquad \boldsymbol{r} = (r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}.$$

Proof. By Theorem 1 we know that $\dim_K M_{\mathbf{r}}$ is given by $|U_{\mathbf{r}}|$, where $U_{\mathbf{r}}$ are the irreducible monomials in ΛE w.r.t. Gröbner reduction in $F_{\mathbf{r}}$. The monomial λe is reducible w.r.t. Gröbner reduction if and only if it is reducible in the classic sense and additionally for all Δ_i resp. Σ_j:

$$1 \leq i \leq p : \ |\lambda e|_{\Delta_i} \leq r_i, \qquad 1 \leq j \leq q : \ |\lambda e|_{\Sigma_j} \leq r_{p+j},$$

hence it is irreducible if for all reducible elements there exists some i or some j such that

$$1 \leq i \leq p : \ |\lambda e|_{\Delta_i} > r_i, \qquad 1 \leq j \leq q : \ |\lambda e|_{\Sigma_j} > r_{p+i}.$$

□

5 Conclusion

We have introduced the new concept of Gröbner reduction and used it to make statements about the dimension of free modules over filtered rings. The formulation is rather abstract and later on specialized to concrete instances like the ring of polynomials to generalize the notion of Hilbert polynomials with respect to a certain filtration of the underlying ring. The second major application is the computation of difference-differential dimension polynomials for modules of difference-differential operators. This extends the work appearing in [ZW08, Lev12, Lev13]. Dönch [Dön13] has shown that the computation of relative Gröbner bases might not terminate in all cases. A current research perspective is, that Gröbner reduction provides a terminating algorithm for the computation of relative Gröbner bases, and extends this concept to arbitrary n-fold filtration.

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