Bounds for D-Finite Closure Properties

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ABSTRACT

We provide bounds on the size of operators obtained by algorithms for executing D-finite closure properties. For operators of small order, we give bounds on the degree and on the height (bit-size). For higher order operators, we give degree bounds that are parameterized with respect to the order and reflect the phenomenon that higher order operators may have lower degrees (order-degree curves).

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algorithms*

General Terms

Algorithms

Keywords

Ore Operators, Holonomic Functions

1. INTRODUCTION

A common way of representing special functions in computer algebra systems is via functional equations of which they are a solution, or equivalently, by linear operators which map the function under consideration to zero. Functions admitting such a representation are called *D-finite*. Arithmetic on D-finite functions translates into arithmetic of operators. For such computations it is common that the output may be much larger than the input. But how large? This is the question we wish to discuss in this paper.

Estimates on the output size are interesting because they enter in a crucial way into the complexity analysis for the corresponding operations, and because algorithms based on evaluation/interpolation depend on an a-priori knowledge of the size of the result. Bounds on the bit size are also needed for the design of "two-line algorithms" in the sense of [15]. For these reasons, there has been some activity concerning bounds in recent years, especially for estimating the sizes of operators arising from creative telescoping [11, 1, 5, 4, 8], i.e., algorithms for definite summation and integration.

The focus in the present paper is on *closure properties.* Closure properties refer to the fact that when f and g are D-finite, then so are $f+g$ and fg and various other derived functions. We say that the class of D-finite functions is closed under these operations. Algorithms for "executing closure properties" belong to the standard repertoire of computer algebra since the 1990s [12, 10]. Our goal is to estimate the size of operators annihilating $f+g$ or fg depending on assumptions on the sizes of operators annihilating f and g.

It is easy to get good bounds on the *order* of the output of closure property algorithms. Such bounds are well known [13, 14, 7]. We add here bounds on the *degree* of the polynomial coefficients of the output operators, and also on their *height,* which measures the size of the coefficients in the polynomial coefficients. We also give degree bounds that are parameterized by the order and reflect the phenomenon that the degree decreases as the order grows. Although all these results are in principle obtained by the same reasoning as the classical bounds on the order, actually computing them is somewhat more laborious. We therefore believe that it is worthwhile working them out once and for all and having them available in the literature for reference.

1.1 Notation

Let R be an integral domain. We consider the Ore algebra $\mathbb{A} = R[x][\partial]$ with the commutation rule

$$\partial p = \sigma(p)\partial + \delta(p) \quad (p \in R[x])$$

where $\sigma\colon R[x] \to R[x]$ is a homomorphism and $\delta\colon R[x] \to R[x]$ is a σ-derivation. For definitions of these notions and further basic facts about Ore algebras, see [3]. Two important examples of Ore algebras are the algebra of linear differential operators (where $\sigma = \mathrm{id}$ and $\delta = \frac{d}{dx}$) and the algebra of linear recurrence operators (where $\sigma(x) = x + 1$, $\sigma|_R = \mathrm{id}$ and $\delta = 0$).

Elements of Ore algebras are called operators. We can let them act on $R[x]$-modules $\mathcal{F}$ of "functions" in such a way that $p \cdot f = pf$ for all $p \in R[x]$ and $f \in \mathcal{F}$ and $(L+M)\cdot f = (L\cdot f) + (M \cdot f)$ and $(LM)\cdot f = L\cdot(M\cdot f)$ for all $L, M \in \mathbb{A}$ and all $f \in \mathcal{F}$. A function $f \in \mathcal{F}$ is then called D-finite (with respect to the action of $\mathbb{A}$ on $\mathcal{F}$) if there exists $L \in \mathbb{A} \setminus \{0\}$ with $L \cdot f = 0$.

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Operators $L \in \mathbb{A}$ have the form

$$L = \ell_0 + \ell_1 \partial + \cdots + \ell_r \partial^r$$

with $\ell_0, \ldots, \ell_r \in R[x]$. When $\ell_r \neq 0$, we call $\operatorname{ord}(P) := r$ the *order* of the operator L. The *degree* of L is defined as the maximum degree of its polynomial coefficients: $\deg(L) := \max_{i=0}^r \deg(\ell_i)$.

We assume that for the ground ring R a size function $\operatorname{ht}\colon R \to \mathbb{R}$ is given with the properties $\operatorname{ht}(0) = 0$, $\operatorname{ht}(a) \geq 0$, $\operatorname{ht}(a) = \operatorname{ht}(-a)$, for all $a \in R$, $\operatorname{ht}(ab) \leq \operatorname{ht}(a) + \operatorname{ht}(b)$ for all $a, b \in R$, and

$$\operatorname{ht}\Big(\sum_{i=1}^n a_i\Big) \leq \operatorname{ht}(n-1) + \max_{i=1}^n \operatorname{ht}(a_i) \tag{1}$$

for any $a_1, \ldots, a_n \in R$. For example, when $R = \mathbb{Z}$, we can take $\operatorname{ht}(a) = \log(1 + |a|)$, and when $R = K[t]$, we can take $\operatorname{ht}(a) = 1 + \deg(a)$ (using $\deg(0) := -1$). The *height* of a polynomial $p = c_0 + c_1 x + \cdots + c_d x^d \in R[x]$ is defined as $\operatorname{ht}(p) := \max_{i=0}^d \operatorname{ht}(c_i)$. Note that we have

$$\operatorname{ht}(pq) \leq \operatorname{ht}(\min\{\deg(p), \deg(q)\}) + \operatorname{ht}(p) + \operatorname{ht}(q)$$

for all $p, q \in R[x]$ (but of course $\operatorname{ht}(1p) = \operatorname{ht}(p)$). Observe that the height of a polynomial depends on the basis of $R[x]$ and that we use the standard basis $1, x, x^2, \ldots$ in our definition. The *height* of an operator $L = \ell_0 + \ell_1 \partial + \cdots + \ell_r \partial^r$ is defined as $\operatorname{ht}(L) := \max_{i=0}^r \operatorname{ht}(\ell_i)$.

We also need to know how σ and δ change the degree and the height of elements of $R[x]$. In order to avoid unnecessary notational and computational overhead, let us assume throughout that $\deg(\sigma(p)) \leq \deg(p)$ and $\deg(\delta(p)) \leq \deg(p)$ for all $p \in R[x]$. This covers most algebras arising in applications. For the height, we assume that a function $c\colon \mathbb{R}^2 \to \mathbb{R}$ is given such that for all $p, q \in R[x]$ with $\deg(p), \deg(q) \leq d$ and $\operatorname{ht}(p), \operatorname{ht}(q) \leq h$ we have $\operatorname{ht}(\pm\sigma(p) + \delta(q)) \leq c(d, h)$. Note that this definition implies $\operatorname{ht}(\partial L) \leq c(\deg(L), \operatorname{ht}(L))$ for every $L \in R[x][\partial]$. We assume that c is nonnegative, in both arguments non-decreasing, and satisfies a triangle inequality with respect to the second argument. For example, for the algebra of differential operators we can take $c(d, h) = \operatorname{ht}(1) + \operatorname{ht}(d) + h$, and a possible choice for the algebra of recurrence operators is $c(d, h) = d \operatorname{ht}(2) + h$.

We will need to iterate the function c in the second argument, and we will write the composed functions using the following notation:

$$c^{(n)}(d, h) := c(d, c^{(n-1)}(d, h)), \quad c^{(0)}(d, h) := h$$

We assume that this function is also non-decreasing with respect to n. With this notation we then have $\operatorname{ht}(\partial^n L) \leq c^{(n)}(\deg(L), \operatorname{ht}(L))$, and more generally, using also height properties stated earlier,

$$\begin{aligned}\operatorname{ht}(ML) \leq{}& \operatorname{ht}(\operatorname{ord}(M)) + \operatorname{ht}(\min\{\deg(M), \deg(L)\}) \\ &+ \operatorname{ht}(M) + c^{(\operatorname{ord}(M))}(\deg(L), \operatorname{ht}(L))\end{aligned} \tag{2}$$

for any two operators $L, M \in R[x][\partial]$. It is also not difficult to see that when $p \in R[x]$ and $n \in \mathbb{N}$, then for $p^{[n]} := p\sigma(p) \cdots \sigma^{n-1}(p)$ we have

$$\operatorname{ht}(p^{[n]}) \leq (n-1) \operatorname{ht}(\deg(p)) + n\, c^{(n-1)}\big(\deg(p), \operatorname{ht}(p)\big). \tag{3}$$

1.2 Argument Structure

If the function f_1 is annihilated by an operator L_1 and the function f_2 is annihilated by another operator L_2, and if L is an operator such that $L = M_1 L_1 = M_2 L_2$ for two other operators M_1, M_2, then L annihilates the function $f_1 + f_2$. It is easy to see that such an operator L always exists. For, suppose $L_1 = \ell_{1,0} + \cdots + \ell_{1,r} \partial^r$ and $L_2 = \ell_{2,0} + \cdots + \ell_{2,s} \partial^s$ are given. Make an ansatz $M_1 = m_{1,0} + \cdots + m_{1,s} \partial^s$, $M_2 = m_{2,0} + \cdots + m_{2,r} \partial^r$ with undetermined coefficients $m_{i,j}$ for two left multipliers. Compute the coefficients of the operator $M_1 L_1 - M_2 L_2$. They will be linear combinations of the undetermined $m_{i,j}$ with coefficients in $R[x]$. Equating coefficients of ∂^k in $M_1 L_1 - M_2 L_2$ to zero gives a linear system over $R[x]$ with $(s+1) + (r+1)$ variables but only $(s+r)+1$ equations. This system must have a nontrivial solution.

All the following arguments will be based on this idea: make an ansatz with undetermined coefficients, compare coefficients, observe that there are more variables than equations, conclude that there must be a solution. The technical difficulty consists in deriving reasonably good estimates for the degrees and the heights of the entries in the linear system. We then use the following lemma to turn them into estimates on the size of the solution vectors.

Lemma 1. *Let $A = ((a_{i,j})) \in R[x]^{n \times m}$ be a matrix with $\deg(a_{i,j}) \leq d$ and $\operatorname{ht}(a_{i,j}) \leq h$ for all i, j. Assume that $n < m$ so that the matrix has a nontrivial nullspace. Then there exists a vector $v = (v_1, \ldots, v_m) \in \ker A \subseteq R[x]^m \setminus \{0\}$ with $\deg(v_i) \leq nd$ and $\operatorname{ht}(v_i) \leq \operatorname{ht}(n!) + (n-1)\operatorname{ht}(d) + nh$ for all $i = 1, \ldots, m$.*

Proof. Let k be the rank of A when viewed as matrix over $\operatorname{Quot}(R[x])$. By choosing a maximal linearly independent set of rows from A, we may assume that $A \in R[x]^{k \times m}$. By permuting the columns if necessary, we may further assume that $A = (A_1, A_2)$ for some $A_1 \in R[x]^{k \times k}$ and $A_2 \in R[x]^{k \times (m-k)}$ with $\det(A_1) \neq 0$. By Cramer's rule, the vector $(v_1, \ldots, v_m)$ with $v_{k+1} = -\det(A_1)$, $v_i = 0$ ($i = k+2, \ldots, m$), and $v_i = \det(A_{1|i})$ ($i = 1, \ldots, k$) where $A_{1|i}$ is the matrix obtained from A_1 by replacing the ith column by the first column of A_2 belongs to $\ker A$. From the determinant formula

$$\det(A_1) = \sum_{\pi \in S_k} \operatorname{sgn}(\pi) \prod_{i=1}^k a_{i,\pi(i)}$$

it follows that $\deg(\det(A_1)) \leq kd$ and

$$\operatorname{ht}(\det(A_1)) \leq \operatorname{ht}(k!) + (k-1)\operatorname{ht}(d) + kh.$$

The same bounds apply for all the determinants $\det(A_{1|i})$ and hence for all coordinates v_i of the solution vector. Since $k \leq n$, the claim follows. ∎

2. COMMON LEFT MULTIPLES ("PLUS")

For the differential case, the computation of common left multiples was studied in detail by Bostan et al. for ISSAC 2012 [2]. Their Theorem 6 says that if L is the least common left multiple of differential operators $L_1, \ldots, L_n$, then $\operatorname{ord}(L) \leq r := \sum_{k=1}^n \operatorname{ord}(L_k)$ and

$$\deg(L) \leq (n(r+1) - r) \max_{k=1}^n \deg(L_k).$$

Without insisting in $\operatorname{ord}(L)$ being minimal, we reprove this result for arbitrary Ore algebras and supplement it with a bound on the height (Section 2.1). We then give a bound on the degree of common multiples of non-minimal order and show that the degree decreases as the order grows (Section 2.2).

2.1 Operators of Small Order

By a common left multiple of "small order", we mean a left multiple of L_k whose order is at most the sum of the orders of the L_k. The actual order of the *least* common left multiple may be smaller than this, for instance if some of the L_k have a non-trivial common right divisor. For investigating the size of common left multiples of small order, we compare coefficients of ∂^i and consider linear systems with coefficients in $R[x]$.

Theorem 2. *Let $L_1, \ldots, L_n \in R[x][\partial]$, suppose $\deg(L_k) \leq d$ and $\operatorname{ht}(L_k) \leq h$ for $k = 1, \ldots, n$. Then there is a common left multiple $L \in R[x][\partial]$ of $L_1, \ldots, L_n$ with*

$$\begin{aligned} \operatorname{ord}(L) &\leq r := \sum_{k=1}^{n} \operatorname{ord}(L_k) \\ \deg(L) &\leq (n(r+1) - r)d \\ \operatorname{ht}(L) &\leq \operatorname{ht}(r) + \operatorname{ht}((n(r+1) - r - 1)!) \\ &\quad + (n(r+1) - r - 1)\operatorname{ht}(d) \\ &\quad + (n(r+1) - r)c^{(r)}(d, h) \end{aligned}$$

Proof. Make an ansatz for n operators $M_k = m_{k,0} + m_{k,1}\partial + \cdots + m_{k,r-\operatorname{ord}(L_k)}\partial^{r-\operatorname{ord}(L_k)}$ with undetermined coefficients $m_{k,i}$ ($k = 1, \ldots, n$; $i = 0, \ldots, r - \operatorname{ord}(L_k)$). We wish to determine the $m_{k,i} \in R[x]$ such that

$$M_1 L_1 = M_2 L_2 = \cdots = M_n L_n \;(= L)$$

by comparing coefficients with respect to ∂ and solving the resulting linear system. Each $M_k L_k$ is an operator of order r whose coefficients are $R[x]$-linear combinations of the undetermined $m_{k,i}$ with coefficients that are bounded in degree by d and in height by $c^{(r)}(d, h)$. Coefficient comparison therefore leads to a system of linear equations with $\sum_{k=1}^{n}(r - \operatorname{ord}(L_k) + 1) = nr - \sum_{k=1}^{n} \operatorname{ord}(L_k) + n = n(r+1) - r$ variables and $(n-1)(r+1) = n(r+1) - r - 1$ equations, which according to Lemma 1 has a solution vector with coordinates v_i with $\deg(v_i) \leq (n(r+1) - r - 1)d$ and $\operatorname{ht}(v_i) \leq \operatorname{ht}((n(r+1) - r - 1)!) + (n(r+1) - r - 2)\operatorname{ht}(d) + (n(r+1) - r - 1)c^{(r)}(d, h)$. If M_1 is an operator with coefficients of this shape, we get for $L = M_1 L_1$ the size estimates as stated in the theorem by (2). ∎

Experiments indicate that the bounds on order and degree are tight for random operators. The bound on the height seems to be off by a constant factor.

Experiment 3. *Consider the algebra $\mathbb{Z}[x][\partial]$ with $\sigma(x) = x + 1$ and $\delta = 0$, set $\operatorname{ht}(a) = \log(1 + |a|)$ for $a \in \mathbb{Z}$, and define $c(d, h) = d\operatorname{ht}(2) + h$. Instead of the recursive definition $c^{(r)}(d, h)$ we use $c^{(r)}(d, h) = d\operatorname{ht}(r+1) + h$, which is justified because $\delta = 0$ and $\operatorname{ht}(\sigma^r(p)) \leq \deg(p)\operatorname{ht}(r + 1) + \operatorname{ht}(p)$ for every $p \in \mathbb{Z}[x]$.*

For two randomly chosen operators $L_1, L_2 \in \mathbb{Z}[x][\partial]$ of order, degree, and height s ($s = 2, 4, 8, 16, 32$) we found that the order and degree of their least common left multiple match exactly the bounds stated in the theorem. The bound stated for the height seems to overshoot by a constant factor only. The data is given in the first two rows of the following table. In the third and fourth row we give the corresponding data for random operators in $R[x][\partial]$ with $R = \mathbb{Z}_{1091}[t]$ and $\operatorname{ht}(a) = \deg_t(a)$. In this case, we can take $c(d, h) = h$ and find that the bound of Theorem 2 is tight.

s	*2*	*4*	*8*	*16*	*32*
height bound	*46.8*	*163.2*	*635.7*	*2646.3*	*11403.3*
actual height	*17.3*	*76.7*	*347.6*	*1615.9*	*7575.4*
height bound	*12*	*40*	*144*	*544*	*2112*
actual height	*12*	*40*	*144*	*544*	*2112*

2.2 Order-Degree Curve

The next result says that there exist higher order common left multiples of lower degree. Also this was already observed by Bostan et al. [2], who in their Section 6 show that the total arithmetic size (order times degree) of higher order common multiples may be asymptotically smaller than the arithmetic size of the least common left multiples. We state this result more explicitly as a formula for an *order-degree curve,* a hyperbola which constitutes a degree bound d in dependence of the order r of the multiple. More results on order-degree curves can be found in [5, 4, 6].

Technically, the result is again obtained by making an ansatz and comparing coefficients, but this time, coefficients with respect to $x^j\partial^i$ are compared, and the resulting linear system has coefficients in R rather than in $R[x]$. According to our experience, non-minimal order operators of low degree have unreasonably large height, which is why in practice they are used only in domains where the height is bounded, such as finite fields. We have therefore not derived height bounds for these operators. A result on the height of non-minimal operators arising in creative telescoping can be found in [8].

Theorem 4. *Let $L_1, \ldots, L_n \in R[x][\partial]$ with $r_i = \operatorname{ord}(L_i)$ and $d_i = \deg(L_i)$ for all i. Let*

$$r \geq \sum_{k=1}^{n} r_k \;\text{ and }\; d \geq \frac{(r+1)\sum_{k=1}^{n} d_k - \sum_{k=1}^{n} r_k d_k}{r + 1 - \sum_{k=1}^{n} r_k}.$$

Then there exists a common left multiple $L \neq 0$ of $L_1, \ldots, L_n$ with $\operatorname{ord}(L) \leq r$ and $\deg(L) \leq d$.

Proof. For $r, d \geq 0$, make an ansatz for n operators

$$M_k = \sum_{i=0}^{r-r_k} \sum_{j=0}^{d-d_k} m_{i,j,k}\, x^j \partial^i$$

with undetermined coefficients $m_{i,j,k}$. We wish to determine the $m_{i,j,k} \in R$ such that $M_1 L_1 = \cdots = M_n L_n$. Then $M_k L_k$ is a common left multiple of $L_1, \ldots, L_n$ of order at most r and degree at most d. Coefficient comparison in the ansatz gives a linear system over R with

$$\begin{aligned} &\sum_{k=1}^{n} \sum_{i=0}^{r-r_k} \sum_{j=0}^{d-d_k} 1 \\ &= n(r+1)(d+1) - (r+1)\sum_{k=1}^{n} d_k - (d+1)\sum_{k=1}^{n} r_k + \sum_{k=1}^{n} r_k d_k \end{aligned}$$

variables and $(n-1)(r+1)(d+1)$ equations. It has a solution when

$$(r+1)(d+1) - (r+1)\sum_{k=1}^{n} d_k - (d+1)\sum_{k=1}^{n} r_k + \sum_{k=1}^{n} r_k d_k > 0.$$

For r and d satisfying the constraints in the theorem, this inequality is true. ■

Experiment 5. *For three operators L_1, L_2, L_3 of order 5 and degree 5, the theorem says that they admit a common left multiple L of order r and degree d for every $r \geq 15$ and $d \geq \frac{15(r-4)}{r-14}$. When we took three such operators at random from the algebra $\mathbb{Z}[x][\partial]$ with $\sigma(x) = x + 1$ and $\delta = 0$, we found the degrees of their left multiple to match this bound exactly. We also found that the leading coefficient of their least common left multiple L had removable factor of degree 150, so that the order-degree curve from Theorem 4 matches the order-degree curve given in Theorem 9 in [6].*

3. POLYNOMIALS ("TIMES")

If two functions f_1 and f_2 are annihilated by operators L_1, L_2, respectively, then a common left multiple L of L_1, L_2 annihilates the sum $f_1 + f_2$. We now turn to operators L which annihilate the product $f_1 f_2$, more generally, some function f that depends polynomially on given functions $f_1, \dots, f_n$ and their derivatives (or shifts). Before we can do this, we need to specify how operators should act on products of functions.

3.1 Actions on Polynomial Rings

Consider the ring extension

$$\mathbf{R} = R[x][y_{i,j} : i = 1, \dots, n, j \geq 0].$$

We want the Ore algebra $R[x][\partial]$ to act on $\mathbf{R}$ in such a way that $p \cdot P = pP$ and $\partial \cdot (pP) = \sigma(p)(\partial \cdot P) + \delta(p)P$ and $\partial \cdot (P + Q) = (\partial \cdot P) + (\partial \cdot Q)$ for all $p \in R[x], P, Q \in \mathbf{R}$, and $\partial \cdot y_{i,j} = y_{i,j+1}$ for all $i \in \mathbb{N}$. The polynomial variables $y_{i,j}$ are meant to represent the functions $\partial^j \cdot f_i$. For the product, we require that there are $\alpha, \beta, \gamma \in \{0, 1, -1\}$ such that for all $P, Q \in \mathbf{R}$ we have

$$\partial \cdot (PQ) = \alpha\, (\partial \cdot P)(\partial \cdot Q) + \beta\, ((\partial \cdot P)Q + P(\partial \cdot Q)) + \gamma\, PQ. \quad (4)$$

To fix the action, it then remains to specify how ∂ acts on $R[x]$. Two canonical options are $\partial \cdot p = \sigma(p)$ and $\partial \cdot p = \delta(p)$.

In the first case, i.e., when "$\partial = \sigma$", we have

$$\begin{aligned}\sigma(p) = \partial \cdot p = \partial \cdot (p1) &= \sigma(p)(\partial \cdot 1) + \delta(p)\\ &= \sigma(p)\sigma(1) + \delta(p) = \sigma(p) + \delta(p),\end{aligned}$$

so this option is only available when $\delta = 0$, and then, since

$$\partial \cdot (pq) = \sigma(p)(\partial \cdot q) + 0 = (\partial \cdot p)(\partial \cdot q)$$

for all $p, q \in R[x] \subseteq \mathbf{R}$ we must have $\alpha = 1, \beta = \gamma = 0$ for the multiplication rule.

There is more diversity when "$\partial = \delta$". For example, in the differential case ($\sigma = \mathrm{id}$, $\delta = \frac{d}{dx}$), we have $\alpha = 0, \beta = 1, \gamma = 0$, and for difference operators ($\delta = \Delta = \sigma - \mathrm{id}$) we have $\alpha = 1, \beta = 1, \gamma = 0$.

Observe that the action of $R[x][\partial]$ on $\mathbf{R}$ is an extension of the action of $R[x][\partial]$ on $R[x]$.

Every $P \in \mathbf{R}$ is a polynomial in the variables $y_{i,j}$ with coefficients that are polynomials in x over R. We write $\mathrm{ht}(P)$ for the maximum of the heights of all the elements of R appearing in coefficients of the polynomial, $\deg(P)$ for the degree of P with respect to x, and $\mathrm{Deg}(P) = (D_1, \dots, D_n)$ where D_i is the total degree of P when viewed as polynomial in the variables $y_{i,0}, y_{i,1}, y_{i,2}, \dots$. For such degree vectors, we write $(D_1, \dots, D_n) \leq (E_1, \dots, E_n)$ if $D_i \leq E_i$ for all i. Addition and maxima of such vectors is meant componentwise. We write $\mathrm{Ord}(P) = (S_1, \dots, S_n)$ if $S_i \in \mathbb{N}$ is the largest index such that the variable y_{i,S_i} appears in P.

A polynomial P with $\mathrm{Deg}(P) = (D_1, \dots, D_n)$ is called homogeneous if it is homogeneous with respect to each group $y_{i,0}, y_{i,1}, \dots$ of variables, i.e., if for every monomial $\prod_{i,j} y_{i,j}^{e_{i,j}}$ in P and every $i = 1, \dots, n$ we have $\sum_j e_{i,j} = D_i$.

Lemma 6. *1. For homogeneous polynomials $P, Q \in \mathbf{R}$ with $\mathrm{Ord}(P) = (S_1, \dots, S_n)$, $\mathrm{Deg}(P) = (D_1, \dots, D_n)$, $\mathrm{Ord}(Q) = (T_1, \dots, T_n)$, $\mathrm{Deg}(Q) = (E_1, \dots, E_n)$, we have*

$$\begin{aligned}\mathrm{Ord}(PQ) &\leq \max\{\mathrm{Ord}(P), \mathrm{Ord}(Q)\}\\ \mathrm{Deg}(PQ) &\leq \mathrm{Deg}(P) + \mathrm{Deg}(Q)\\ \deg(PQ) &\leq \deg(P) + \deg(Q)\\ \mathrm{ht}(PQ) &\leq \min\Big\{\sum_{i=1}^n \mathrm{ht}\big(\tbinom{D_i+S_i}{D_i}\big), \sum_{i=1}^n \mathrm{ht}\big(\tbinom{E_i+T_i}{E_i}\big)\Big\}\\ &\quad + \mathrm{ht}(\min\{\deg(P), \deg(Q)\})\\ &\quad + \mathrm{ht}(P) + \mathrm{ht}(Q)\end{aligned}$$

The first term in the expression for $\mathrm{ht}(PQ)$ can be dropped if P or Q have just one monomial, in particular, when P or Q are in $R[x]$.

2. For $k \in \mathbb{N}$ and a polynomial $P \in \mathbf{R}$ with $\mathrm{Deg}(P) = (D_1, \dots, D_n) \neq (0, \dots, 0)$ we have

$$\begin{aligned}\mathrm{Ord}(\partial^k \cdot P) &\leq \mathrm{Ord}(P) + (k, k, \dots, k)\\ \mathrm{Deg}(\partial^k \cdot P) &\leq \mathrm{Deg}(P)\\ \deg(\partial^k \cdot P) &\leq \deg(P)\\ \mathrm{ht}(\partial^k \cdot P) &\leq k\, \mathrm{ht}(4) \sum_{i=1}^n D_i + c^{(k)}(\deg(P), \mathrm{ht}(P))\end{aligned}$$

Proof. 1. The claims on orders and degrees are clear. For the claim on the height, observe that the coefficient of every monomial in PQ is a sum over products pq, where p is a coefficient of P and q a coefficient of Q. We have

$$\begin{aligned}\mathrm{ht}(pq) &\leq \mathrm{ht}(\min\{\deg(p), \deg(q)\}) + \mathrm{ht}(p) + \mathrm{ht}(q)\\ &\leq \mathrm{ht}(\min\{\deg(P), \deg(Q)\}) + \mathrm{ht}(P) + \mathrm{ht}(Q).\end{aligned}$$

When p or q have just one monomial, this completes the proof. Otherwise, the number of summands pq in such a sum is bounded by the number of terms in P and by the number of terms in Q. The claim follows because a homogeneous polynomial of degree D_i in $S_i + 1$ variables has at most $\binom{D_i+S_i+1-1}{D_i}$ terms.

2. It suffices to consider the case $k = 1$. The general case follows by repeating the argument k times. The claims on orders and degrees follow directly from the product rule for the action of ∂ on $\mathbf{R}$ and the assumption that σ and δ do not increase degree.

For the bound on the height, write $P = \sum_\ell p_\ell \tau_\ell$ for some $p_\ell \in R[x]$ and distinct monomials $\tau_\ell = \prod_{i,j} y_{i,j}^{e_{i,j}}$. Then $\partial \cdot P = \sum_\ell (\sigma(p_\ell)(\partial \cdot \tau_\ell) + \delta(p_\ell)\tau_\ell)$ can be written as a sum $\sum_m q_m \sigma_m$ where the σ_m are distinct monomials and the q_m are sums of several polynomials $\sigma(p_\ell)$ or $-\sigma(p_\ell)$, and possibly one polynomial $\delta(p_\ell)$. Each of these polynomials has height at most $c(\deg P, \mathrm{ht}\, P)$. We show that these sums have at most 4^D summands, where $D = D_1+\cdots+D_n$. Then the claim follows from (1) and $\operatorname{ht}(4^D) \leq \operatorname{ht}(4)D$. For one part, the number of summands is caused by the fact that for two fixed monomials σ and τ, the application of ∂ to τ may create the monomial σ more than once. For the other part, a fixed term σ may turn up for several terms τ. We need to discuss both effects.

For the first effect, for any two monomials σ, τ let $a_{\sigma,\tau}$ be the number of times the monomial σ appears in $\partial\cdot\tau$, and set $a_{\sigma,\tau} := 0$ if σ or τ is not a monomial. We show by induction on D that $a_{\sigma,\tau} \leq 2^D-1$. For $D = 1$ we have $\tau = y_{i,j}$ for some i, j, so $\partial \cdot \tau = y_{i,j+1}$, so $a_{\sigma,\tau} = [[\sigma = y_{i,j+1}]] \leq 1 = 2^1 - 1$, where $[[\cdot]]$ denotes the Iverson bracket. Now assume the bound is true for $D - 1 \geq 1$. Writing $\tau = \tilde{\tau} y_{i,j}$, the product rule (4) gives

$$\partial \cdot (\tilde{\tau} y_{i,j}) = \alpha(\partial \cdot \tilde{\tau}) y_{i,j+1} + \beta(\partial \cdot \tilde{\tau}) y_{i,j} + \beta\tilde{\tau} y_{i,j+1} + \gamma\tilde{\tau} y_{i,j}.$$

It follows that

$$a_{\sigma,\tau} \leq \underbrace{\underbrace{\underbrace{a_{\sigma/y_{i,j+1},\tilde{\tau}}}_{\leq 2^{D-1}-1} + \underbrace{a_{\sigma/y_{i,j},\tilde{\tau}}}_{\leq 2^{D-1}-1}}_{\leq 2^D-2} + \underbrace{[[\sigma{=}\tilde{\tau} y_{i,j}]] + [[\sigma{=}\tilde{\tau} y_{i,j+1}]]}_{\leq 1}}_{\leq 2^D-1},$$

as claimed.

For the second effect, the total number of contributions to a coefficient q_m in $\partial \cdot P$ is bounded by $\sum_\tau a_{\sigma_m,\tau} \leq \sum_\tau (2^D - 1)$. For the summation range, it suffices to let τ run over at most 2^D "neighbouring" terms of σ_m, for if $\sigma_m = y_{i_1,j_1} y_{i_2,j_2} \cdots y_{i_D,j_D}$, then the only terms τ for which $\partial \cdot \tau$ may involve σ_m are those of the form

$$y_{i_1,j_1-e_1} y_{i_2,j_2-e_2} \cdots y_{i_D,j_D-e_D}$$

with $(e_1, \ldots, e_D) \in \{0,1\}^D$. These are 2^D many. ∎

3.2 Normal Forms

If the functions $f_1, \ldots, f_n \in \mathcal{F}$ are solutions of the operators $L_1, \ldots, L_n$ then every function

$$f = P(f_1, \ldots, f_n, \ \ldots\ldots, \partial^m \cdot f_1, \ldots, \partial^m \cdot f_n)$$

where P is a multivariate polynomial is again D-finite. To see this, it suffices to observe that D-finiteness is preserved under addition, multiplication, and application of ∂, because the expression for f can be decomposed into a finite number of these operations. For computing an annihilating operator for f, it suffices to have algorithms for performing these closure properties and apply them repeatedly. For obtaining a bound on the order of an annihilating operator for f, it suffices to have such bounds for these operations. However, it turns out that the bounds obtained in this way overshoot significantly, and the corresponding algorithm has a horrible performance.

It is much better to consider an algorithm that computes an annihilating operator for f directly from the polynomial P, and this is what we will do next. Observe that the relations $L_i \cdot f_i = 0$ can be used to rewrite f as another polynomial V in the functions $\partial^j \cdot f_i$ with $j < \operatorname{ord}(L_i)$ only. In the following lemma, we analyze how the size of V depends on the size of P.

Lemma 7. *Let $L_1, \ldots, L_n \in R[x][\partial]$, $r_i = \operatorname{ord}(L_i)$, $p_i = \operatorname{lc}(L_i)$ $(i = 1, \ldots, n)$ and consider the ideal*

$$\begin{aligned}\mathfrak{a} = \big\langle & L_1 \cdot y_{1,0},\ \partial L_1 \cdot y_{1,0},\ \partial^2 L_1 \cdot y_{1,0}, \ldots \\ & L_2 \cdot y_{2,0},\ \partial L_2 \cdot y_{2,0},\ \partial^2 L_2 \cdot y_{2,0}, \ldots \\ & \ldots \\ & L_n \cdot y_{n,0},\ \partial L_n \cdot y_{n,0},\ \partial^2 L_n \cdot y_{n,0}, \ldots \big\rangle \subseteq \mathbf{R}.\end{aligned}$$

For every $m \in \mathbb{N}$ and every homogeneous polynomial $P \in \mathbf{R}$ with $\operatorname{Deg}(P) = (D_1, \ldots, D_n)$ and $\operatorname{Ord}(P) < (r_1+m, \ldots, r_n+m)$ there exists a homogeneous polynomial $V \in \mathbf{R}$ with

$$\Big(\prod_{i=1}^n (p_i^{D_i})^{[m]}\Big) P \equiv V \bmod \mathfrak{a}$$

and

$$\begin{aligned}&\operatorname{Ord}(V) < (r_1, \ldots, r_n)\\ &\operatorname{Deg}(V) \leq (D_1, \ldots, D_n)\\ &\deg(V) \leq \deg(P) + m\sum_{i=1}^n D_i \deg(L_i)\\ &\operatorname{ht}(V) \leq \operatorname{ht}(P) + m\sum_{i=1}^n \Big(\operatorname{ht}(D_i+1) + D_i \operatorname{ht}(r_i+m)\\ &\qquad + D_i \operatorname{ht}(\deg(L_i)) + D_i\, c^{(m)}(\deg(L_i), \operatorname{ht}(L_i))\Big).\end{aligned}$$

Proof. Induction on m. For $m = 0$ there is nothing to show (take $V = P$). Suppose the claim is true for $m - 1$. Write

$$P = \sum_{j_1=0}^{D_1} \cdots \sum_{j_n=0}^{D_n} P_{j_1,\ldots,j_n} \prod_{i=1}^n y_{i,r_i+m-1}^{j_i}$$

for some $P_{j_1,\ldots,j_n}$ with $\operatorname{Ord}(P_{j_1,\ldots,j_n}) < (r_1+m-1, \ldots, r_n+m-1)$ and $\operatorname{Deg}(P_{j_1,\ldots,j_n}) \leq (D_1 - j_1, \ldots, D_n - j_n)$. Then

$$\begin{aligned}&\Big(\prod_{i=1}^n \sigma^{m-1}(p_i^{D_i})\Big) P\\ &= \sum_{j_1=0}^{D_1} \cdots \sum_{j_n=0}^{D_n} \tilde{P}_{j_1,\ldots,j_n} \prod_{i=1}^n \big(\sigma^{m-1}(p_i) y_{i,r_i+m-1}\big)^{j_i}\\ &\equiv \underbrace{\sum_{j_1=0}^{D_1} \cdots \sum_{j_n=0}^{D_n} \tilde{P}_{j_1,\ldots,j_n} \prod_{i=1}^n \tilde{Q}_i^{j_i}}_{=:\tilde{P}} \quad \bmod \mathfrak{a},\end{aligned}$$

where

$$\begin{aligned}\tilde{P}_{j_1,\ldots,j_n} &= P_{j_1,\ldots,j_n} \prod_{i=1}^n \sigma^{m-1}(p_i)^{D_i-j_i}\\ \tilde{Q}_i &= \sigma^{m-1}(p_i) y_{i,r_i+m-1} - \big(\partial^{m-1} L_i \cdot y_{i,0}\big).\end{aligned}$$

First, because of $\operatorname{Ord}(\tilde{P}_{j_1,\ldots,j_n}), \operatorname{Ord}(\tilde{Q}_i^{j_i}) < (r_1+m-1, \ldots, r_n+m-1)$ we have $\operatorname{Ord}(\tilde{P}) < (r_1+m-1, \ldots, r_n+m-1)$. Second, because of $\operatorname{Deg}(\tilde{P}_{j_1,\ldots,j_n}) = \operatorname{Deg}(P_{j_1,\ldots,j_n}) \leq (D_1 - j_1, \ldots, D_n - j_n)$ and $\operatorname{Deg}(\prod_{i=1}^n \tilde{Q}_i^{j_i}) \leq (j_1, \ldots, j_n)$ we have $\operatorname{Deg}(\tilde{P}) \leq (D_1, \ldots, D_n)$. Third, because of

$$\begin{aligned}&\deg(\tilde{P}_{j_1,\ldots,j_n}) \leq \deg(P) + \sum_{i=1}^n (D_i - j_i) \deg(L_i),\\ &\deg\Big(\prod_{i=1}^n \tilde{Q}_i^{j_i}\Big) \leq \sum_{i=1}^n j_i \deg(L_i)\end{aligned}$$

we have $\deg(\tilde{P}) \le \deg(P) + \sum_{i=1}^n D_i \deg(L_i)$. Fourth, because of these degree estimates and

$$\operatorname{ht}(\tilde{P}_{j_1,\dots,j_n}) \le \operatorname{ht}(P) + \sum_{i=1}^n (D_i - j_i)\Big(\operatorname{ht}(\deg(L_i)) + c^{(m)}(\deg(L_i), \operatorname{ht}(L_i))\Big)$$

and $\operatorname{ht}(\tilde{Q}_i) \le c^{(m)}(\deg(L_i), \operatorname{ht}(L_i))$, we have, by $\sum_{i=1}^n j_i$ fold application of Lemma 6.(1),

$$\operatorname{ht}\Big(\tilde{P}_{j_1,\dots,j_n} \prod_{i=1}^n \tilde{Q}_i^{j_i}\Big) \le \operatorname{ht}(P) + \sum_{i=1}^n \Big(j_i \operatorname{ht}(r_i + m) + D_i \operatorname{ht}(\deg(L_i)) + D_i c^{(m)}(\deg(L_i), \operatorname{ht}(L_i))\Big)$$

and therefore, because $\tilde{P}$ is a sum of at most $\prod_{i=1}^n (D_i + 1)$ such terms,

$$\operatorname{ht}(\tilde{P}) \le \operatorname{ht}(P) + \sum_{i=1}^n \Big(\operatorname{ht}(D_i + 1) + D_i \operatorname{ht}(r_i + m) + D_i \operatorname{ht}(\deg(L_i)) + D_i\, c^{(m)}(\deg(L_i), \operatorname{ht}(L_i))\Big).$$

By induction hypothesis, there exists V such that

$$\Big(\prod_{i=1}^n (p_i^{D_i})^{[m-1]}\Big) \tilde{P} \equiv V \bmod \mathfrak{a}$$

with $\operatorname{Ord}(\tilde{V}) < (r_1, \dots, r_n)$, $\operatorname{Deg}(\tilde{V}) \le (D_1, \dots, D_n)$,

$$\begin{aligned}\deg(V) &\le \deg(\tilde{P}) + (m-1) \sum_{i=1}^n D_i \deg(L_i) \\ &\le \deg(P) + m \sum_{i=1}^n D_i \deg(L_i)\end{aligned}$$

$$\begin{aligned}\operatorname{ht}(V) &\le \operatorname{ht}(\tilde{P}) + (m-1) \sum_{i=1}^n \Big(\operatorname{ht}(D_i+1) + D_i \operatorname{ht}(r_i+m-1) \\ &\quad + D_i \operatorname{ht}(\deg(L_i)) + D_i\, c^{(m-1)}(\deg(L_i), \operatorname{ht}(L_i))\Big) \\ &\le \operatorname{ht}(P) + m \sum_{i=1}^n \Big(\operatorname{ht}(D_i + 1) + D_i \operatorname{ht}(r_i + m) \\ &\quad + D_i \operatorname{ht}(\deg(L_i)) + D_i\, c^{(m)}(\deg(L_i), \operatorname{ht}(L_i))\Big).\end{aligned}$$

Finally, because of

$$\begin{aligned}\Big(\prod_{i=1}^n (p_i^{D_i})^{[m]}\Big) P &= \Big(\prod_{i=1}^n (p_i^{D_i})^{[m-1]}\Big) \Big(\prod_{i=1}^n \sigma^{m-1}(p_i^{D_i})\Big) P \\ &\equiv \Big(\prod_{i=1}^n (p_i^{D_i})^{[m-1]}\Big) \tilde{P} \equiv V \bmod \mathfrak{a},\end{aligned}$$

the polynomial V has all the required properties. ∎

3.3 Small Orders

We are now ready to state the main result, which bounds the size of an operator which annihilates a function given as a polynomial of $f_1, \dots, f_n$ and their derivatives or shifts.

We consider only homogeneous polynomials. If a function f is expressed in terms of $f_1, \dots, f_n$ via an inhomogeneous polynomial P, we can write $P = P_1 + P_2 + \cdots + P_s$ where each P_i is homogeneous, then apply the theorem to the P_i separately, and then combine the resulting bounds using Theorem 2 to obtain a bound for P. This is fair because it seems that the overestimation explained at the beginning of the previous section only happens when homogeneous components are not handled as a whole but subdivided further into sums of even smaller polynomials.

Theorem 8. *Let $L_1, \dots, L_n \in R[x][\partial]$, $r_i = \operatorname{ord}(L_i)$ ($i = 1, \dots, n$). Let $\mathfrak{a} \subseteq \mathbf{R}$ be as in Lemma 7. Let $P \in \mathbf{R}$ be a homogeneous polynomial with $\operatorname{Deg}(P) = (D_1, \dots, D_n)$ and $\operatorname{Ord}(P) < (r_1, \dots, r_n)$. Then there exists an operator $L \in R[x][\partial] \setminus \{0\}$ and a polynomial $p \in R[x] \setminus \{0\}$ with $pL \cdot P \in \mathfrak{a}$ and*

$$\begin{aligned}\operatorname{ord}(L) &\le m := \prod_{i=1}^n \tbinom{D_i + r_i - 1}{D_i} \\ \deg(L) &\le m \deg(P) + m^2 \sum_{i=1}^n D_i \deg(L_i) \\ \operatorname{ht}(L) &\le \operatorname{ht}(m!) + m\, c^{(m)}(\deg(P), \operatorname{ht}(P))) \\ &\quad + (m-1) \operatorname{ht}\Big(\deg(P) + m \sum_{i=1}^n D_i \deg(L_i)\Big) \\ &\quad + m^2 \sum_{i=1}^n \Big(\operatorname{ht}(4) D_i + \operatorname{ht}(D_i + 1) + D_i \operatorname{ht}(r_i + m) \\ &\quad + \operatorname{ht}(\deg(L_i)) + c^{(m)}(\deg(L_i), \operatorname{ht}(L_i))\Big).\end{aligned}$$

Proof. Let $p_i = \operatorname{lc}(L_i)$ and $p = \prod_{i=1}^n (p_i^{D_i})^{[m]}$. We show that there exist $\ell_0, \dots, \ell_m \in R[x]$, not all zero, such that

$$p \sum_{k=0}^m \ell_k (\partial^k \cdot P) \in \mathfrak{a}. \tag{5}$$

Consider the polynomials

$$P_k = \Big(\prod_{i=1}^n (\sigma^k(p_i)^{D_i})^{[m-k]}\Big)(\partial^k \cdot P)$$

for $k = 0, \dots, m$. Bounds for $\partial^k \cdot P$ can be obtained from Lemma 6.(2). Applying Lemma 6.(1) with $\prod_{i=1}^n \sigma^{k+j}(p_i)^{D_i}$ as P and $\big(\prod_{i=1}^n (\sigma^k(p_i)^{D_i})^{[j]}\big)(\partial^k \cdot P)$ as Q, for $j = 0, \dots, m - k - 1$ (so that there are altogether $m - k$ applications of the Lemma), we obtain

$$\begin{aligned}\operatorname{Ord}(P_k) &< (r_1 + k, \dots, r_n + k) \\ \operatorname{Deg}(P_k) &= (D_1, \dots, D_n) \\ \deg(P_k) &\le \deg(P) + (m-k) \sum_{i=1}^n D_i \deg(L_i) \\ \operatorname{ht}(P_k) &\le k \operatorname{ht}(4) \sum_{i=1}^n D_i + c^{(m)}(\deg(P), \operatorname{ht}(P)) \\ &\quad + (m-k) \sum_{i=1}^n \Big(\operatorname{ht}(\deg(L_i)) + c^{(m)}(\deg(L_i), \operatorname{ht}(L_i))\Big)\end{aligned}$$

for all $k = 0, \dots, m$, where we have used $c^{(k)}(\cdot, \cdot) \le c^{(m)}(\cdot, \cdot)$, $\deg(p_i) \le \deg(L_i)$, and $\operatorname{ht}(p_i) \le \operatorname{ht}(L_i)$ to bring the expression for the height into the form stated here.

Using Lemma 7 and the above bounds for P_k, we find for each $k \le m$ a V_k with

$$\Big(\prod_{i=1}^n (p_i^{D_i})^{[m]}\Big)(\partial^k \cdot P) = \Big(\prod_{i=1}^n (p_i^{D_i})^{[k]}\Big) P_k \equiv V_k \bmod \mathfrak{a}$$

and $\operatorname{Ord}(V_k) < (r_1, \dots, r_n)$, $\operatorname{Deg}(V_k) = (D_1, \dots, D_n)$,

$$\deg(V_k) \le \deg(P) + m \sum_{i=1}^{n} D_i \deg(L_i)$$

$$\begin{aligned}\operatorname{ht}(V_k) &\le k \operatorname{ht}(4) \sum_{i=1}^{n} D_i + c^{(m)}(\deg(P), \operatorname{ht}(P)) \\ &\quad + (m-k) \sum_{i=1}^{n} \Big(\operatorname{ht}(\deg(L_i)) + c^{(m)}(\deg(L_i), \operatorname{ht}(L_i)) \Big) \\ &\quad + k \sum_{i=1}^{n} \Big(\operatorname{ht}(D_i + 1) + D_i \operatorname{ht}(r_i + k) + D_i \operatorname{ht}(\deg(L_i)) \\ &\qquad + D_i c^{(k)}(\deg(L_i), \operatorname{ht}(L_i)) \Big) \\ &\le m \sum_{i=1}^{n} \Big(\operatorname{ht}(4) D_i + \operatorname{ht}(D_i + 1) + D_i \operatorname{ht}(r_i + m) \\ &\qquad + \operatorname{ht}(\deg(L_i)) + c^{(m)}(\deg(L_i), \operatorname{ht}(L_i)) \Big) \\ &\quad + c^{(m)}(\deg(P), \operatorname{ht}(P)).\end{aligned}$$

In the ansatz $\sum_{k=0}^{m} \ell_k V_k \stackrel{!}{=} 0$ with undetermined coefficients $\ell_0, \dots, \ell_m$, compare coefficients with respect to terms $\prod_{i,j} y_{i,j}^{e_{i,j}}$. This gives a linear system over $R[x]$ with $m+1$ variables, $\prod_{i=1}^{n} \binom{D_i + r_i - 1}{D_i} = m$ equations, and with coefficients of degree at most $\deg(P) + m \sum_{i=1}^{n} D_i \deg(L_i)$ and height at most

$$\begin{aligned} m \sum_{i=1}^{n} \Big(&\operatorname{ht}(4) D_i + \operatorname{ht}(D_i + 1) + D_i \operatorname{ht}(r_i + m) + \operatorname{ht}(\deg(L_i)) \\ &+ c^{(m)}(\deg(L_i), \operatorname{ht}(L_i)) \Big) + c^{(m)}(\deg(P), \operatorname{ht}(P)).\end{aligned}$$

By Lemma 1, the theorem follows. ∎

In its full generality, the theorem is a bit bulky. For convenient reference, and as example applications, we rephrase it for three important special cases. The first concerns simple products of the form $f_1 f_2$ and powers f^k, the second is what is called "D-finite Ore action" in Koutschan's package [9], and the third is the Wronskian. Observe that the bound for the order of the symmetric power is lower than the bound that would follow by applying the bound for the symmetric product $k-1$ times.

Corollary 9. ***(Symmetric Product and Power)***

1. *Let $L_1, L_2 \in R[x][\partial]$ and let $f_1, f_2 \in \mathcal{F}$ be solutions of L_1, L_2, respectively. Let $r_1 = \operatorname{ord}(L_1)$ and $r_2 = \operatorname{ord}(L_2)$ and let d, h be such that $\deg(L_1), \deg(L_2) \le d$ and $\operatorname{ht}(L_1), \operatorname{ht}(L_2) \le h$. Then there exists an operator $M \in R[x][\partial]$ with $M \cdot (f_1 f_2) = 0$ and*

$$\operatorname{ord}(M) \le r_1 r_2, \qquad \deg(M) \le 2 d r_1^2 r_2^2,$$
$$\begin{aligned}\operatorname{ht}(M) &\le \operatorname{ht}((r_1 r_2)!) + (r_1 r_2 - 1) \operatorname{ht}(2 r_1 r_2 d) + r_1 r_2 \operatorname{ht}(1) \\ &\quad + 2 r_1^2 r_2^2 \big(2 \operatorname{ht}(4) + 3 \operatorname{ht}(r_1 r_2) + \operatorname{ht}(d) + c^{(r_1 r_2)}(d, h) \big)\end{aligned}$$

2. *Let $L \in R[x][\partial]$, $r = \operatorname{ord}(L)$, $d = \deg(L)$, $h = \operatorname{ht}(L)$, and let $f \in \mathcal{F}$ be a solution of L. Let $k \in \mathbb{N}$. Then there exists an operator $M \in R[x][\partial]$ with $M \cdot (f^k) = 0$ and*

$$\operatorname{ord}(M) \le \tbinom{k+r}{k} =: m, \qquad \deg(M) \le k d m^2,$$
$$\begin{aligned}\operatorname{ht}(M) &\le \operatorname{ht}(m!) + m \operatorname{ht}(1) + (m-1) \operatorname{ht}(mkd) \\ &\quad + m^2 \big(k \operatorname{ht}(4) + \operatorname{ht}(k+1) + k \operatorname{ht}(r+m) \\ &\quad + \operatorname{ht}(d) + c^{(m)}(d, h) \big)\end{aligned}$$

Proof. For part 1, apply the theorem with $n = 2$ and $P = y_{1,0} y_{2,0}$. Note that $\operatorname{Ord}(P) = (0,0) < (r_1, r_2)$, $\operatorname{Deg}(P) = (1,1)$, $\deg(P) = 0$, and $\operatorname{ht}(P) = \operatorname{ht}(1)$. For part 2, take $n = 1$, $P = y_{1,0}^k$. Note that $\operatorname{Ord}(P) = 0 < r$, $\operatorname{Deg}(P) = k$, $\deg(P) = 0$, and $\operatorname{ht}(P) = \operatorname{ht}(1)$. ∎

Corollary 10. ***(Associates)*** *Let $L \in R[x][\partial]$ and let $f \in \mathcal{F}$ be a solution of L. Let $A \in R[x][\partial]$ be another operator with $\operatorname{ord}(A) < \operatorname{ord}(L) := r$. Then $A \cdot f$ is annihilated by an operator M with*

$$\operatorname{ord}(M) \le r, \qquad \deg(M) \le r \deg(A) + r^2 \deg(L),$$
$$\begin{aligned}\operatorname{ht}(M) &\le \operatorname{ht}(r!) + r\, c^{(r)}(\deg(A), \operatorname{ht}(A)) \\ &\quad + (r-1) \operatorname{ht}(\deg(A) + r \deg(L)) \\ &\quad + r^2 \big(4 \operatorname{ht}(2) + \operatorname{ht}(r) + \operatorname{ht}(\deg(L)) + c^{(r)}(\deg(L), \operatorname{ht}(L)) \big)\end{aligned}$$

Proof. Apply Theorem 8 with $n = 1$ and $P = A \cdot y_{1,0}$. Note that $\operatorname{Ord}(P) < r - 1$, $\operatorname{Deg}(P) = D_1 = 1$, $\deg(P) = \deg(A)$, and $\operatorname{ht}(P) = \operatorname{ht}(A)$. In the expression for the height, we used $\operatorname{ht}(4) + \operatorname{ht}(1+1) + \operatorname{ht}(r + r - 1) \le 3 \operatorname{ht}(2) + \operatorname{ht}(2r) \le 4 \operatorname{ht}(2) + \operatorname{ht}(r)$. ∎

Corollary 11. ***(Wronskian)*** *Let $L_1, \dots, L_r \in R[x][\partial]$ be operators of order r, degree d and height h. Let $f_1, \dots, f_r \in \mathcal{F}$ be solutions of $L_1, \dots, L_r$, respectively, and consider*

$$w := \begin{vmatrix} f_1 & f_2 & \cdots & f_r \\ \partial \cdot f_1 & \partial \cdot f_2 & \cdots & \partial \cdot f_r \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{r-1} \cdot f_1 & \partial^{r-1} \cdot f_2 & \cdots & \partial^{r-1} \cdot f_r \end{vmatrix}.$$

Then there exists an operator $M \in R[x][\partial]$ with $M \cdot w = 0$ and

$$\operatorname{ord}(M) \le r^r =: m, \qquad \deg(M) \le m^2 r^2 d,$$
$$\begin{aligned}\operatorname{ht}(M) &\le \operatorname{ht}(m!) + m \operatorname{ht}(1) + (m-1) \operatorname{ht}(m r^2 d) \\ &\quad + m^2 r \big((r+1)(\operatorname{ht}(4) + \operatorname{ht}(r)) + \operatorname{ht}(d) + c^{(m)}(d, h) \big).\end{aligned}$$

Proof. Apply Theorem 8 with $n = r$, $P \in \mathbf{R}$ the polynomial obtained by replacing f_i by $y_{i,0}$ in the expression given for w. Note that $\operatorname{Ord}(P) < (r, \dots, r)$, $\operatorname{Deg}(P) = (1, \dots, 1)$, $\deg(P) = 0$, and $\operatorname{ht}(P) = \operatorname{ht}(1)$. ∎

Experiment 12. *To check the bounds of Theorem 8 for plausibility, we have computed the symmetric product $L = L_1 \otimes L_2$ for two random operators $L_1, L_2 \in \mathbb{Z}[x][\partial]$ of order and degree and height bounded by s, for $s = 2, 3, 4, 5$. It turned out that the order of L meets the bound stated in the theorem. The bounds for degree and height are not as tight, but the data suggests that they are only off by some constant factor. The results are given in the table below.*

s	*2*	*3*	*4*	*5*
degree bound	*64*	*486*	*2048*	*6250*
actual degree	*16*	*90*	*320*	*850*
height bound	*471.5*	*3495.*	*14677.*	*44980.2*
actual height	*23.29*	*185.12*	*865.95*	*2693.30*

3.4 Order-Degree Curve

Finally, the following result provides an order-degree curve for operators which annihilates a function that is given as a polynomial of $f_1, \ldots, f_n$ and their derivatives/shifts. Once more, the technical difference in the argument is that coefficient comparison is done with respect to the variables $y_{i,j}$ as well as x, giving a linear system over R rather than over $R[x]$.

Theorem 13. *Let $L_1, \ldots, L_n \in R[x][\partial]$, $r_i = \operatorname{ord}(L_i)$, $d_i = \deg(L_i)$. Let $\mathfrak{a} \subseteq \mathbf{R}$ be as in Lemma 7. Let $P \in \mathbf{R}$ be a homogeneous polynomial with $\operatorname{Ord}(P) < (r_1, \ldots, r_n)$ and $\operatorname{Deg}(P) = (D_1, \ldots, D_n)$. Let*

$$r \geq m := \prod_{i=1}^{n} \binom{D_i + r_i - 1}{D_i} \text{ and } d \geq \frac{r\,m \sum_{i=1}^{n} D_i d_i + m \deg(P)}{r + 1 - m}.$$

Then there exists an operator $L \in R[x][\partial] \setminus \{0\}$ and a polynomial $p \in R[x] \setminus \{0\}$ with $pL \cdot P \in \mathfrak{a}$ and $\operatorname{ord}(L) \leq r$ and $\deg(L) \leq d$.

Proof. For $k = 0, \ldots, r$, let V_k be as in the proof of Theorem 8 but with r in place of m so that $\operatorname{Ord}(V_k) < (r_1, \ldots, r_n)$ and $\deg(V_k) \leq \deg(P) + r \sum_{i=1}^{n} D_i d_i$. Make an ansatz $L = \sum_{i=0}^{r} \sum_{j=0}^{d} \ell_{i,j} x^j \partial^i$ for an operator of order r and degree d. We wish to determine the $\ell_{i,j}$ such that

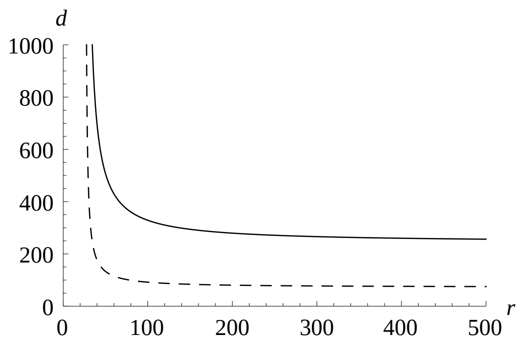
$$\sum_{i=0}^{r} \sum_{j=0}^{d} \ell_{i,j} x^j V_i = 0.$$

Coefficient comparison gives a linear system over R with $(r+1)(d+1)$ variables and

$$\max_{k=1}^{n} (d + 1 + \deg(V_k)) m = m\left(d + 1 + \deg(P) + r \sum_{i=1}^{n} D_i d_i\right)$$

equations. For r and d as in the theorem, there are more variables than equations, and therefore a nontrivial solution. ∎

Experiment 14. *From the algebra $\mathbb{Z}[x][\partial]$ with $\sigma(x) = x+1$ and $\delta = 0$ we picked three random operators L_1, L_2, L_3 of order, degree, and height 3, and we computed operators L annihilating the Wronskian w associated to these operators (cf. Cor. 11 above). In the following figure we compare the degree bound obtained by last year's result [6] from the minimal order operator L (dotted) to the a-priori degree bound of Theorem 13 (solid). That the new bound overshoots is the price we have to pay for the feature that this bound can be calculated without knowing L.*



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