An extension of Turán's inequality for ultraspherical polynomials*

Geno Nikolov † Veronika Pillwein ‡

Abstract

Let $p_m(x) = P_m^{(\lambda)}(x)/P_m^{(\lambda)}(1)$ be the m-th ultraspherical polynomial normalized by $p_m(1) = 1$. We prove the inequality $|x|p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \geq 0$, $x \in [-1, 1]$, for $-1/2 < \lambda \leq 1/2$. Equality holds only for $x = \pm 1$ and, if n is even, for $x = 0$. Further partial results on an extension of this inequality to normalized Jacobi polynomials are given.

1 Introduction and statement of the result

Turán's inequality was formulated around 1940 for Legendre polynomials $P_n(x)$ stating that for all $n \geq 0$ and $x \in [-1, 1]$

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix} = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0,$$

where equality is only attained for $x = \pm 1$. Since its first appearance there have been several proofs provided, e.g., Szegő [22] gave four different proofs of this inequality. In the middle of the 20th century, Turán type inequalities were obtained for various classes of orthogonal polynomials such as Gegenbauer, Hermite, and Laguerre polynomials with appropriate normalization as well as refined lower and upper bounds for Turán's determinants for ultraspherical polynomials [21, 25]. The extension to the class of Jacobi polynomials was done by Gasper [8, 9] in the 1970s. In the late 1990s Szwarc [24] provided a general analysis of Turán type inequalities for sequences of orthogonal polynomials based on the coefficients of the defining three term recurrence relations. The same approach is applied in Berg and Szwarc [2] for derivation of conditions ensuring monotonicity of the normalized Turán determinants. For a historic overview and further references the reader is referred to [24] and [2].

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†Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", 5 James Bourchier Blvd., 1164 Sofia, Bulgaria, geno@fmi.uni-sofia.bg

‡Research Institute for Symbolic Computation, Johannes Kepler University, Altenberger Straße 69, A-4040 Linz, Austria, veronika.pillwein@risc.jku.at

As usual, the notation $P_m^{(\lambda)}$ stands for the m-th ultraspherical polynomial, which is orthogonal in $[-1,1]$ with respect to the weight function $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$, $\lambda > -1/2$, and is normalized by $P_m^{(\lambda)}(1) = \binom{m+2\lambda-1}{m}$. We shall need a different normalization for these polynomials, namely, we shall require that they take value 1 at $x=1$, so we set

$$p_m(x) = p_m^{(\lambda)}(x) := P_m^{(\lambda)}(x)/P_m^{(\lambda)}(1), \quad m = 0, 1, \ldots, \tag{1.1}$$

where, for the sake of brevity, the superscript $^{(\lambda)}$ will be omitted hereafter. We prove the following extension of Turán's inequality:

Theorem 1.1. *Let p_n be defined by* (1.1)*, and $\lambda \in (-1/2, 1/2]$. Then, for every $n \in \mathbb{N}$,*

$$|x| p_n^2(x) - p_{n-1}(x) p_{n+1}(x) \geq 0 \quad \textit{for every} \;\; x \in [-1,1]\,. \tag{1.2}$$

The equality in (1.2) *holds only for $x = \pm 1$ and, if n is even, for $x = 0$. Moreover,* (1.2) *fails for every $\lambda > 1/2$ and $n \in \mathbb{N}$.*

This variation of Turán's inequality was introduced by Gerhold and Kauers [11] and proven in the limit case $\lambda = 1/2$, i.e., for Legendre polynomials.

The proof of Theorem 1.1 using classical tools is given in the next section. In Section 3 we present a computer algebra approach to the proof of Theorem 1.1 indicating that statements of this form can be proven nowadays almost routinely by a computer. Both approaches have been in included in order to allow for a fair comparison of these techniques. In Section 4 we compare our lower bound for Turán's determinant for ultraspherical polynomials with the hitherto known results. In the final section we discuss an extension of Theorem 1.1 to the Jacobi case, and provide some partial results obtained with the assistance of computer algebra.

2 Classical analysis of Theorem 1.1

Assume first that $\{p_m\}$ is a general sequence of orthogonal polynomials, defined by the three term recurrence equation

$$x p_n(x) = \gamma_n p_{n+1}(x) + \alpha_n p_{n-1}(x), \;\; n = 0, 1, 2, \ldots, \tag{2.1}$$

where $p_{-1}(x) := 0$, $p_0(x) = 1$, $\alpha_0 = 0$, $\alpha_{n+1} > 0$, $\gamma_n > 0$, and $\alpha_n + \gamma_n = 1$ for every $n \in \mathbb{N}_0$. Clearly, $p_m(-1) = (-1)^m$ and $p_m(1) = 1$ for every $m \in \mathbb{N}_0$, and more generally $p_m(-x) = (-1)^m p_m(x)$. By these properties it is easy to see that $|x| p_n^2(x) - p_{n-1}(x) p_{n+1}(x)$ is an even function that vanishes at $x = \pm 1$ and, if n is even, also at $x = 0$. We therefore set

$$\widetilde{\Delta}_n(x) := x p_n^2(x) - p_{n-1}(x) p_{n+1}(x)\,, \quad n \in \mathbb{N}\,, \tag{2.2}$$

and our goal is to examine which conditions guarantee that $\widetilde{\Delta}_n(x) > 0$ for every $x \in (0,1)$. We start with some representations of $\widetilde{\Delta}_n(x)$.

Lemma 2.1. *Assume that the sequence $\{p_m\}$ satisfies the three term recurrence relation* (2.1)*. Then the following representations hold true:*

$$\gamma_n \widetilde{\Delta}_n(x) = \gamma_n x p_n^2(x) + \alpha_n p_{n-1}^2(x) - x p_{n-1}(x) p_n(x)\,, \tag{2.3}$$

$$\alpha_n \widetilde{\Delta}_n(x) = \alpha_n x p_n^2(x) + \gamma_n p_{n+1}^2(x) - x p_n(x) p_{n+1}(x)\,, \tag{2.4}$$

$$\gamma_n \widetilde{\Delta}_n(x) = (\gamma_n x - \gamma_{n-1}) p_n^2(x) + (\alpha_n - \alpha_{n-1} x) p_{n-1}^2(x) + \alpha_{n-1} \widetilde{\Delta}_{n-1}(x)\,, \tag{2.5}$$

$$\begin{aligned}\gamma_n \widetilde{\Delta}_n(x) =& x\big(\alpha_n p_{n-1}(x) - \gamma_n p_n(x)\big)\big(p_{n-1}(x) - p_n(x)\big)\\ &+ \alpha_n (1-x) p_{n-1}^2(x)\,,\end{aligned} \tag{2.6}$$

Proof. Formulae (2.3) and (2.4) follow from rewriting $\gamma_n p_{n+1}(x)$ and $\alpha_n p_{n-1}(x)$ in $\gamma_n \widetilde{\Delta}_n$ and $\alpha_n \widetilde{\Delta}_n$, respectively, using the recurrence equation (2.1). Subtracting (2.4) (with $n-1$ instead of n) from (2.3), we obtain (2.5). Formula (2.6) is deduced by multiplying $x p_{n-1}(x) p_n(x)$ in the right-hand side of (2.3) by $\gamma_n + \alpha_n$ $(=1)$, and then adding and subtracting $\alpha_n\, x\, p_{n-1}^2(x)$. □

Our next lemma shows that the inequality $\widetilde{\Delta}_n(x) > 0$ generally holds true in a subinterval of $(0,1)$.

Lemma 2.2. *Assume that the sequence $\{p_m\}$ satisfies the three term recurrence relation* (2.1)*. Then*

$$\widetilde{\Delta}_n(x) > 0 \quad \textit{for every} \quad x \in (0, 4\alpha_n\gamma_n)\,.$$

Proof. The right-hand side of (2.3) is equal to

$$\Big(\sqrt{\alpha_n} p_{n-1}(x) - \frac{x}{2\sqrt{\alpha_n}} p_n(x)\Big)^2 + \frac{x}{4\alpha_n}\big(4\alpha_n\gamma_n - x\big) p_n^2(x)\,.$$

Both summands are non-negative if $x \in (0, 4\alpha_n\gamma_n)$. Moreover, for $x \in (0, 4\alpha_n\gamma_n)$ this expression would be equal to zero only if both $p_n(x)$ and $p_{n-1}(x)$ are equal to zero, which is impossible, since the zeros of p_n and p_{n-1} interlace. □

From now on, we restrict our considerations to the case of ultraspherical polynomials, i.e., $\{p_m\} = \{p_m^{(\lambda)}\}$, as normalized by (1.1). The zeros of p_m are denoted henceforth by $x_{1,m}(\lambda) < x_{2,m}(\lambda) < \cdots < x_{m,m}(\lambda)$. We collect in the next lemma some well-known properties of ultraspherical polynomials, which will be needed for the proof of Theorem 1.1.

Lemma 2.3. *(i) $\{p_n\} = \{p_n^{(\lambda)}\}$ satisfy the recurrence relation* (2.1) *with*

$$\gamma_n = \frac{n+2\lambda}{2(n+\lambda)}, \quad \alpha_n = \frac{n}{2(n+\lambda)}\,. \tag{2.7}$$

(ii) The positive zeros of $p_n^{(\lambda)}$ are strictly monotone decreasing functions of λ in $(-1/2, \infty)$. Moreover, for every $n \ge 2$,

$$x_{n,n}(\lambda) \le \Big(\frac{(n-1)(n+2\lambda+1)}{(n+\lambda)^2 + 3\lambda + 5/4 + 3(\lambda+1/2)^2/(n-1)}\Big)^{1/2}\,. \tag{2.8}$$

(iii) The following relations hold true:

$$p_n'(x) := \frac{d}{dx}\big\{p_n^{(\lambda)}(x)\big\} = \frac{n(n+2\lambda)}{2\lambda+1}\, p_{n-1}^{(\lambda+1)}(x)\,, \tag{2.9}$$

$$p_{n-1}(x) = \frac{1}{n}(1-x^2)p_n'(x) + xp_n(x)\,. \tag{2.10}$$

The above properties of $p_n^{(\lambda)}$ are easily obtained from their analogues for $P_n^{(\lambda)}$, given, e.g., in Szegő's monograph [23]. The recurrence relation (2.1) with the coefficients γ_n and α_n given in (2.7) follows from [23, Eqn. (4.7.17)]. Formulae (2.9) and (2.10) are consequences of [23, loc. cit. (4.7.14) and (4.7.27)]. The monotone dependence of the zeros of $p_n^{(\lambda)}$ on λ follows from a well-known observation due to A. A. Markov, see e.g., [23, Theorem 6.12.1]. The upper bound (2.8) for the extreme zeros of ultraspherical polynomials is proved in [17, Lemma 3.5] (for other bounds for the extreme zeros of classical orthogonal polynomials, see, e.g., [6] and the references therein).

Set

$$z_n(\lambda) := 4\alpha_n\gamma_n = 1 - \frac{\lambda^2}{(n+\lambda)^2}\,.$$

The following is an immediate consequence of Lemma 2.2:

Corollary 2.4. *Let $\{p_m\} = \{p_m^{(\lambda)}\}$, $\lambda > -1/2$, be the sequence of ultraspherical polynomials. Then for every $n \in \mathbb{N}$,*

$$\widetilde{\Delta}_n(x) > 0\,, \quad x \in \big(0, z_n(\lambda)\big)\,.$$

In view of Corollary 2.4, (1.2) is true for $\lambda = 0$, and to prove (1.2) for $\lambda \in (-1/2, 0) \cup (0, 1/2]$, we have to show that $\widetilde{\Delta}_n(x) > 0$ when $x \in [z_n(\lambda), 1)$.

The case $n = 1$, $\lambda \in (-1/2, 1/2]$ is easily verified. Namely,

$$\widetilde{\Delta}_1(x) = x^3 - \frac{2(\lambda+1)}{2\lambda+1}x^2 + \frac{1}{2\lambda+1} = \frac{x-1}{2\lambda+1}\big((2\lambda+1)x^2 - x - 1\big)\,,$$

and the polynomial $q(x) = (2\lambda+1)x^2 - x - 1$ has a unique positive root. Since $q(1) = 2\lambda - 1 \le 0$, it follows that $q(x) < 0$, and consequently $\widetilde{\Delta}_1(x) > 0$ for every $x \in (0, 1)$.

We therefore assume in what follows that $n \ge 2$. In our proof of the inequality $\widetilde{\Delta}_n(x) > 0$, $x \in [z_n(\lambda), 1)$ we shall distinguish between the cases $\lambda \in (-1/2, 0)$ and $\lambda \in (0, 1/2)$.

Lemma 2.5. *If $\lambda \in (-1/2, 0)$, then $\widetilde{\Delta}_n(x) > 0$ for every $x \in [z_n(\lambda), 1)$.*

Proof. We use induction with respect to n. The case $n = 1$ was settled above, and we assume that, for some $n \ge 2$, $\widetilde{\Delta}_{n-1}(x) > 0$ for every $x \in [z_{n-1}(\lambda), 1)$. Since $z_{n-1}(\lambda) < z_n(\lambda)$, we have also $\widetilde{\Delta}_{n-1}(x) > 0$ for every $x \in [z_n(\lambda), 1)$.

By the interlacing property and monotonicity of the zeros of ultraspherical polynomials, we have

$$0 \le x_{n-1,n-1}(\lambda+1) \le x_{n-1,n-1}(\lambda) < x_{n,n}(\lambda) < x_{n+1,n+1}(\lambda) < 1\,, \qquad (2.11)$$

where the first two inequalities are strict unless $n = 2$, and in the latter case we have $x_{1,1}(\lambda+1) = x_{1,1}(\lambda) = 0$.

Next, we show that if $\lambda \in (-1/2, 0)$, then the largest zero of p_n', which, in view of (2.9), is $x_{n-1,n-1}(\lambda+1)$, satisfies

$$x_{n-1,n-1}(\lambda+1) < z_n(\lambda)\,. \qquad (2.12)$$

Indeed, by Lemma 2.3(ii) we readily get

$$x_{n-1,n-1}(\lambda+1) \le x_{n-1,n-1}(1/2) \le \Big(1 - \frac{5}{n^2-n+3}\Big)^{1/2} < 1 - \frac{1}{(2n-1)^2} \le z_n(\lambda)\,.$$

As is seen from (2.3) and (2.4), the inequality $\widetilde{\Delta}_n(x) > 0$ is true whenever $x > 0$ and $p_{n-1}(x)p_n(x) \le 0$ or $p_n(x)p_{n+1}(x) \le 0$, in particular, $\widetilde{\Delta}_n(x) > 0$ in the interval $[x_{n-1,n-1}(\lambda), x_{n+1,n+1}(\lambda)]$. Set

$$I_n := (x_{n-1,n-1}(\lambda+1), x_{n-1,n-1}(\lambda))\,.$$

In view of (2.11) and (2.12), the induction step from $n-1$ to n will be done if we manage to show that $\widetilde{\Delta}_n(x) > 0$ for $x \in I_n$ (this interval is void when $n = 2$) and for $x \in (x_{n+1,n+1}(\lambda), 1)$.

Assume first that $x \in (x_{n+1,n+1}(\lambda), 1)$, then $0 < p_n(x) < p_{n-1}(x)$, since the zeros of $p_n - p_{n-1}$ interlace with the zeros of p_n, and the rightmost zero of $p_n - p_{n-1}$ is at $x = 1$. Moreover, since $\alpha_n > \frac{1}{2} > \gamma_n > 0$ for $\lambda \in (-1/2, 0)$, we have $\alpha_n p_{n-1}(x) - \gamma_n p_n(x) > 0$. Then by (2.6) we conclude that

$$\gamma_n \widetilde{\Delta}_n(x) > x\big(\alpha_n p_{n-1}(x) - \gamma_n p_n(x)\big)\big(p_{n-1}(x) - p_n(x)\big) > 0\,,\ x \in (x_{n+1,n+1}(\lambda), 1)\,.$$

Now assume that $n \ge 3$ and $x \in I_n \cap [z_n(\lambda), 1)$. By (2.5) and the induction hypothesis, we have

$$\gamma_n \widetilde{\Delta}_n(x) > (\alpha_{n-1}x - \alpha_n)\Big(\frac{\gamma_n x - \gamma_{n-1}}{\alpha_{n-1}x - \alpha_n}\, p_n^2(x) - p_{n-1}^2(x)\Big)\,, \qquad (2.13)$$

and it suffices to show that the right-hand side of the inequality (2.13) is positive in $I_n \cap [z_n(\lambda), 1)$. A straightforward calculation using (2.7) shows that if $\lambda \in (-1/2, 0)$ and $x \in [z_n(\lambda), 1)$, then

$$\alpha_{n-1}x - \alpha_n \ge 4\alpha_{n-1}\alpha_n\gamma_n - \alpha_n = \alpha_n(4\alpha_{n-1}\gamma_n - 1) = -\frac{\lambda(\lambda+1)\alpha_n}{(n+\lambda)(n+\lambda-1)} > 0.$$

Therefore, the right-hand side of inequality (2.13) is positive in $I_n \cap [z_n(\lambda), 1)$ when

$$\frac{\gamma_n x - \gamma_{n-1}}{\alpha_{n-1}x - \alpha_n}\, p_n^2(x) - p_{n-1}^2(x) > 0\,, \qquad x \in I_n \cap [z_n(\lambda), 1)\,. \qquad (2.14)$$

According to (2.10) we have $p_{n-1}(x) - xp_n(x) = (1-x^2)p_n'(x)/n$, hence $p_{n-1}(x) > xp_n(x)$ for $x \in I_n$; moreover, since both $p_{n-1}(x)$ and $xp_n(x)$ are negative in I_n (see (2.11)), we get

$$x^2 p_n^2(x) > p_{n-1}^2(x)\,, \qquad x \in I_n\,. \tag{2.15}$$

We shall show that

$$\psi(x) := \frac{\gamma_n x - \gamma_{n-1}}{\alpha_{n-1}x - \alpha_n} > x^2\,, \qquad x \in [z_n(\lambda), 1)\,,$$

then obviously (2.14) is a consequence from (2.15). The function ψ is continuous in $[z_n(\lambda), 1]$; moreover, from $\alpha_n + \gamma_n = \alpha_{n-1} + \gamma_{n-1} = 1$ we find that $\psi(1) = 1$ and

$$\psi'(x) = \frac{(\alpha_n - \alpha_{n-1})(\alpha_{n-1} + \alpha_n - 1)}{(\alpha_{n-1}x - \alpha_n)^2} < 0\,, \qquad x \in [z_n(\lambda), 1)\,,$$

since $\alpha_{n-1} > \alpha_n > 1/2$. Thus, $\psi(x)$ is a decreasing function in $[z_n(\lambda), 1)$, and $\psi(x) > 1 > x^2$ therein. Consequently, (2.14) is true, and therefore $\widetilde{\Delta}_n(x) > 0$ for $x \in I_n \cap [z_n(\lambda), 1)$. The proof of Lemma 2.5 is complete. □

Next, we prove the analogue of Lemma 2.5 for the case $\lambda \in (0, 1/2]$.

Lemma 2.6. *If $\lambda \in (0, 1/2]$, then $\widetilde{\Delta}_n(x) > 0$ for every $x \in [z_n(\lambda), 1)$.*

Proof. Again, we apply induction with respect to n, and the base case $n = 1$ was already settled. As in the proof of Lemma 2.5, we assume that, for some $n \ge 2$, $\widetilde{\Delta}_{n-1}(x) > 0$ in $[z_{n-1}(\lambda, 1)$, then $\widetilde{\Delta}_{n-1}(x) > 0$ in $[z_n(\lambda), 1)$, too. To accomplish the induction step from $n-1$ to n, we observe that if $\lambda \in (0, 1/2]$, then

$$z_n(\lambda) > x_{n+1,n+1}(\lambda)\,. \tag{2.16}$$

Indeed, by Lemma 2.3(ii) we have $x_{n+1,n+1}(\lambda) < x_{n+1,n+1}(0) = \cos\frac{\pi}{2n+2}$, while $z_n(\lambda) \ge z_n(1/2) = 1 - 1/(2n+1)^2$. Then (2.16) follows from the inequality $\sin^2\frac{\pi}{4(n+1)} > \frac{1}{2(2n+1)^2}$, which is true since $\sin t > \frac{2}{\pi}t$ for $t \in (0, \pi/2)$.

In view of (2.5), the inductional hypothesis and (2.16), to prove the inequality $\widetilde{\Delta}_n(x) > 0$ for $x \in (z_n(\lambda), 1)$, it suffices to show that

$$(\gamma_n x - \gamma_{n-1})p_n^2(x) + (\alpha_n - \alpha_{n-1}x)p_{n-1}^2(x) > 0, \quad x \in [x_{n+1,n+1}(\lambda), 1)\,. \tag{2.17}$$

For $\lambda > 0$ the sequences $\{\gamma_n\}$ and $\{\alpha_n\}$ defined by (2.7) satisfy

$$\gamma_n \searrow \tfrac{1}{2} \;\text{ and }\; \alpha_n \nearrow \tfrac{1}{2} \quad \text{as} \quad n \to \infty\,.$$

Therefore, $\gamma_n x - \gamma_{n-1} \le \gamma_n - \gamma_{n-1} < 0$ and $\alpha_n - \alpha_{n-1}x \ge \alpha_n - \alpha_{n-1} > 0$. Since $p_n(x) > 0$ and $p_{n-1}(x) > 0$ for $x \in [x_{n+1,n+1}(\lambda), 1)$, the inequality (2.17) is equivalent to

$$\varphi(x) := \frac{p_{n-1}(x)}{p_n(x)} \ge \sqrt{\frac{\gamma_{n-1} - \gamma_n x}{\alpha_n - \alpha_{n-1}x}}, \qquad [x_{n+1,n+1}(\lambda), 1)\,. \tag{2.18}$$

It is well-known that $\varphi(x)$ is monotone decreasing and convex in $(x_{n,n}, \infty)$, where $x_{n,n}$ is the rightmost zero of p_n, see e.g. [23, Theorem 3.3.5] for a general result. For the sake of completeness, we propose a direct proof for the case $p_n = p_n^{(\lambda)}$. By (2.10), we have

$$\varphi(x) = \frac{p_{n-1}(x)}{p_n(x)} = x + \frac{1-x^2}{n}\,\frac{p_n'(x)}{p_n(x)} = x + \frac{1-x^2}{n}\sum_{k=1}^{n}\frac{1}{x-x_{k,n}}\,, \tag{2.19}$$

where $\{x_{k,n}\} = \{x_{k,n}(\lambda)\}$ are the zeros of p_n. Differentiating the last expression, we obtain that $\varphi'(x) < 0$ and $\varphi''(x) > 0$ for $x > x_{n,n}(\lambda)$. Indeed,

$$\begin{aligned}\varphi'(x) &= 1 - \frac{2x}{n}\sum_{k=1}^{n}\frac{1}{x-x_{k,n}} - \frac{1-x^2}{n}\sum_{k=1}^{n}\frac{1}{(x-x_{k,n})^2}\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{(x-x_{k,n})^2 - 2x(x-x_{k,n}) - 1 + x^2}{(x-x_{k,n})^2}\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{x_{k,n}^2-1}{(x-x_{k,n})^2} < 0\,,\end{aligned}$$

and

$$\varphi''(x) = \frac{2}{n}\sum_{k=1}^{n}\frac{1-x_{k,n}^2}{(x-x_{k,n})^3} > 0\,.$$

Since $\varphi(1) = 1$, it follows from the convexity of φ that $\varphi(x) > 1 + \varphi'(1)(x-1)$ in $(x_{n,n}(\lambda), 1)$. We make use of (2.19) and (2.9) to calculate $\varphi'(1)$:

$$\varphi'(1) = 1 - \frac{2}{n}\,p_n'(1) = -\frac{2n+2\lambda-1}{2\lambda+1}\,.$$

Therefore, we have

$$\varphi(x) > 1 + \frac{2n+2\lambda-1}{2\lambda+1}(1-x) \quad \text{for} \quad x \in [x_{n+1,n+1}(\lambda), 1)\,. \tag{2.20}$$

Now we estimate the right-hand side of (2.18). On using (2.7), we find

$$\frac{\gamma_{n-1}-\gamma_n x}{\alpha_n - \alpha_{n-1}x} = 1 + \frac{(1-\alpha_{n-1}-\alpha_n)(1-x)}{\alpha_n-\alpha_{n-1}x} = 1 + \frac{\lambda(2n+2\lambda-1)(1-x)}{\big(n(n+\lambda-1)-\lambda\big)(1-x)+\lambda}\,.$$

For $n \ge 1$, $\lambda > 0$ and $0 < x < 1$ we have $\big(n(n+\lambda-1)-\lambda\big)(1-x)+\lambda \ge \lambda$, therefore

$$\frac{\gamma_{n-1}-\gamma_n x}{\alpha_n-\alpha_{n-1}x} \le 1 + (2n+2\lambda-1)(1-x)\,.$$

In view of this estimate and (2.20), the inequality (2.18) will be proved if we manage to show that

$$1 + \frac{2n+2\lambda-1}{2\lambda+1}(1-x) \ge \sqrt{1+(2n+2\lambda-1)(1-x)} \quad \text{for} \quad x \in [x_{n+1,n+1}(\lambda), 1)\,.$$

After squaring both sides of this inequality, we find that a sufficient condition for its validity is $2/(2\lambda+1) \geq 1$, i.e., $\lambda \leq 1/2$. Thus, (2.17) is true, and therefore $\widetilde{\Delta}_n(x) > 0$ for $x \in [z_n(\lambda), 1)$. The proof of Lemma 2.6 is complete. $\square$

Summarizing, the inequality (2.1) follows from: 1) Corollary 2.4 for $\lambda = 0$; 2) Corollary 2.4 and Lemma 2.5 for $\lambda \in (-1/2, 0)$; 3) Corollary 2.4 and Lemma 2.6 for $\lambda \in (0, 1/2]$.

It remains to prove the last claim of Theorem 1.1, namely that if $\lambda > 1/2$, then (1.2) fails for every $n \in \mathbb{N}$. On using $p_{n-1}(1) = p_n(1) = 1$, (2.9) and (2.1), we find

$$\begin{aligned} \gamma_n \widetilde{\Delta}'_n(1) &= -\alpha_n + (2\gamma_n - 1)p'_n(1) + (2\alpha_n - 1)p'_{n-1}(1) \\ &= \frac{(2\lambda - 1)(n + 2\lambda)}{2(2\lambda + 1)(n + \lambda)}. \end{aligned}$$

If $\lambda > 1/2$, then $\widetilde{\Delta}'_n(1) > 0$, and hence $\widetilde{\Delta}_n(1 - \varepsilon) < \widetilde{\Delta}_n(1) = 0$ for a sufficiently small $\varepsilon > 0$. This completes the proof of Theorem 1.1.

3 A computer algebra approach to Theorem 1.1

Gerhold and Kauers [10] introduced a method for proving inequalities on sequences that depend on a discrete parameter and are defined by general (possibly non-linear) systems of difference equations. Many special functions are within this class, in particular classical orthogonal polynomials that can be defined by three term recurrences. In [11] they considered in particular Turán type inequalities and discovered and proved Theorem 1.1 for Legendre polynomials. Their method essentially proceeds by induction along the discrete parameter and they automatically derive a sufficient condition for the given inequality to hold. This sufficient condition consists of a system of polynomial inequalities. Note that the modified Turán inequality (1.1) is a polynomial inequality only for particular choices of n and not a polynomial inequality for symbolic n. The computer algebra algorithm that is applied to show that the system of polynomial inequalities is consistent is Cylindrical Algebraic Decomposition (CAD).

CAD was introduced in the 1970s by Collins [5] to solve the problem of real quantifier elimination. Given a quantified formula of polynomial inequalities (or, more general, rational or algebraic inequalities), a CAD computation gives an equivalent, quantifier free formula. If there are no free variables in the given expression, then this formula is one of the logical constants True or False. There are several implementations of CAD available [3, 19, 20], for this work we use the Mathematica built-in commands "CylindricalDecomposition" and "Resolve". For a recent practical overview on how to apply CAD and which problem classes are suitable as input for CAD see [15].

Let us illustrate how Lemma 2.2 can be proven using CAD. Recall that by means of identity (2.3) it had to be proven that

$$\gamma_n x p_n^2(x) + \alpha_n p_{n-1}^2(x) - x p_{n-1}(x) p_n(x) > 0, \tag{3.1}$$

for all $x \in (0, 4\alpha_n\gamma_n)$ with $\alpha_n, \gamma_n > 0$. This result follows from the more general statement, where we replace α_n by a positive real variable $a > 0$, and analogously $\gamma_n > 0$ by $c > 0$, and $p_{n-1}(x), p_n(x)$ by real variables y_{-1}, y_0. A classical quantifier elimination task for CAD is to show that

$$\forall\, a, c, x, y_{-1}, y_0 \colon \big(a > 0 \wedge c > 0 \wedge 0 < x < 4ac\big) \Rightarrow cxy_0^2 + ay_1^2 - xy_1y_0 \geq 0$$

is true (this computation can be carried out very quickly). From this more general and purely polynomial statement the non-negativity of the expression in (3.1) follows. CAD can also be used to determine for which values the strictness is violated and we find that equality is only attained for $y_{-1} = y_0 = 0$. This case, however, can be ruled out by the argument given in the proof of Lemma 2.2.

The CAD computations presented in this paper were carried out using Mathematica's built-in implementation of CAD. The computation time is negligible for all of them and could certainly be carried out with other implementations such as, e.g., [7, 3, 19, 4] as well. A reason why we stick to using the Mathematica implementation is that Kauers [12] implemented the Gerhold-Kauers method as a Mathematica package which is freely available for download. The proof of Theorem 1.1 can be carried out without knowledge of the underlying algorithm using the command "ProveInequality". The input can be formulated using standard Mathematica notation such as, e.g., `GegenbauerC` for Gegenbauer polynomials. For sake of readability below we use the traditional notation and also plug in explicitely $P_n^{(\lambda)}(1) = \frac{(2\lambda)_n}{n!}$.

In[1]:= $\mathbf{ProveInequality}[\boldsymbol{x}\big(\boldsymbol{P}_{n+1}^{(\lambda)}(\boldsymbol{x})(\boldsymbol{n}+1)!/(2\boldsymbol{\lambda})_{n+1}\big)^2$
$\quad - \boldsymbol{n}!(\boldsymbol{n}+2)!/((2\boldsymbol{\lambda})_n(2\boldsymbol{\lambda})_{n+2})\boldsymbol{P}_n^{(\lambda)}(\boldsymbol{x})\boldsymbol{P}_{n+2}^{(\lambda)}(\boldsymbol{x}) > 0,$
$\quad \mathbf{Using} \to \{-\tfrac{1}{2} < \boldsymbol{\lambda} \leq \tfrac{1}{2} \wedge 0 < \boldsymbol{x} < 1\}, \mathbf{Variable} \to \boldsymbol{n}]$

Out[1]= True

Note that the range of the continuous variables needs to be provided and that the discrete variable (along which internally the induction is performed) has to be specified. Then with a one line statement the main theorem can be proven for both positive and negative λ at once. Readers interested in the underlying mechanisms will find the intermediate steps carried out explicitly in [11] for Legendre polynomials. One drawback of the method is that the proof only delivers the final result and usually does not provide further insight. Still, not only did it provide the first computer proofs of some special function inequalities from the literature [10, 11, 13, 14], but was used to resolve some open conjectures [1, 14, 16, 18].

As mentioned earlier, the method only needs the defining recurrence of the given expressions as input. For common special functions such as Gegenbauer polynomials the defining recurrence need not be given additionally, because the program has it stored internally. In many applications it may happen that we do not have a closed form representation but only the recursive description at hand. The function call providing also the recursive definition is as follows:

In[2]:= $\mathbf{ProveInequality}[\boldsymbol{x}\boldsymbol{p}[\boldsymbol{n}+1]^2 - \boldsymbol{p}[\boldsymbol{n}]\boldsymbol{p}[\boldsymbol{n}+2]^2 > 0,$

Where $\rightarrow \{p[n+2]{=}{=}\frac{2x(\lambda+n+1)}{2\lambda+n+1}p[n+1]-\frac{(n+1)}{2\lambda+n+1}p[n], p[0]{=}{=}1, p[1]{=}{=}x\}$,
Using $\rightarrow \{-\frac{1}{2}<\lambda<\frac{1}{2}\wedge 0<x<1\}$, **Variable** $\rightarrow n$]

Out[2]= True

The function calls above very quickly deliver the desired answer, but note that CAD-computations may be costly both with respect to time and memory consumption. The run time depends on the number of variables, the degrees of the involved polynomials, as well as the number of inequalities in a doubly exponential way in the worst case. Hence, even if the computation would eventually terminate, either the computer may run out of memory before that, or the user out of patience. It is recommended, however, to be aware of the fact that these computations may take some time when using the program and not interrupt calculations after a couple of minutes (or hours). The procedure is by no means approximate and hence if the result we aim at is a proof, then it is worth investing some resources.

4 Comparison to known results

For normalized Gegenbauer polynomials $p_m(x) = P_m^{(\lambda)}(x)/P_m^{(\lambda)}(1)$ let us denote the classical Turán determinant by $\Delta_{n,\lambda}(x)$,

$$\Delta_{n,\lambda}(x) = p_n^2(x) - p_{n-1}(x)p_{n+1}(x).$$

Thiruvenkatachar and Nanjundiah [25] showed that in $(0,\infty)$ the normalized Turán determinant

$$\varphi_{n,\lambda}(x) := \frac{\Delta_{n,\lambda}(x)}{1-x^2}$$

is monotone increasing if $\lambda > 0$ and monotone decreasing if $-\frac{1}{2} < \lambda < 0$. In particular,

$$c_{n,\lambda} \le \varphi_{n,\lambda}(x) \le C_{n,\lambda}\,, \quad x \in (-1,1), \tag{4.2}$$

with sharp constants $0 < c_{n,\lambda} < C_{n,\lambda}$, given by $c_{n,\lambda} = p_n^2(0) - p_{n-1}(0)p_{n+1}(0)$ and $C_{n,\lambda} = (2\lambda+1)^{-1}$, if $\lambda > 0$, and with interchanged $c_{n,\lambda}$ and $C_{n,\lambda}$, if $-1/2 < \lambda < 0$. Theorem 1.1 asserts that if $\lambda \in (-\frac{1}{2}, \frac{1}{2}]$, then for $x \in (-1,1)$,

$$\varphi_{n,\lambda}(x) \ge \frac{p_n(x)^2}{1+|x|} := g_{n,\lambda}(x)\,. \tag{4.3}$$

A result of a similar nature due to Szász [21] asserts that for every $\lambda \in (0,1)$,

$$\varphi_{n,\lambda}(x) \ge \frac{\lambda(1-p_n^2(x))}{(n+\lambda-1)(n+2\lambda)(1-x^2)} =: h_{n,\lambda}(x)\,. \tag{4.4}$$

In view of (4.2), (4.3) and (4.4), it is of interest to compare the normalized Turán determinant $\varphi_{n,\lambda}(x)$ with its lower bounds $g_{n,\lambda}(x)$, $h_{n,\lambda}(x)$ in the case $0 < \lambda \le 1/2$, and with $g_{n,\lambda}(x)$ in the case $-1/2 < \lambda < 0$.

It turns out that for $0 < \lambda \le 1/2$ the inequality $g_{n,\lambda}(x) \ge c_{n,\lambda}$ holds only near the endpoints of $[-1,1]$, i.e., our pointwise lower bound $g_{n,\lambda}(x)$ for

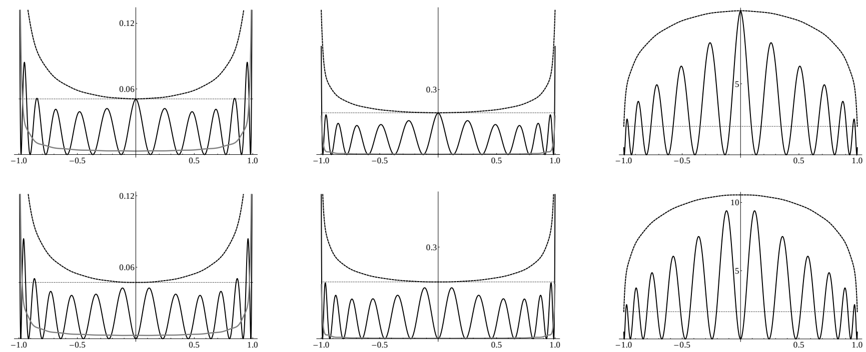

Figure 1: Graphs of $g_{n,\lambda}(x)$ (black), $h_{n,\lambda}(x)$ (gray), $\phi_{n,\lambda}(x)$ (dashed), and $c_{n,\lambda}$ (dotted) for $n = 12$ (top line) and $n = 13$ (bottom line), and (from left to right) $\lambda = \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$.

the normalized Turán determinant $\varphi_{n,\lambda}(x)$ improves upon the "uniform" lower bound $c_{n,\lambda}$ only on a subset of $[-1,1]$ of a small measure. However, for most $x \in (-1,1)$ our pointwise estimate is better than the Szász one.

The situation changes when λ is negative. Namely, in that case the inequality $g_{n,\lambda}(x) < c_{n,\lambda}$ holds only in some small neighborhoods of the zeros of p_n. Thus, in the case $-1/2 < \lambda < 0$, for most $x \in (-1,1)$ Theorem 1.1 furnishes a better pointwise bound than the "uniform" bound $c_{n,\lambda}$. Also, the local maxima of $g_{n,\lambda}(x)$ imitate the shape of $\varphi_{n,\lambda}(x)$. See Fig. 1.

5 The non-symmetric case

In view of Gasper's result [8, 9], it seems reasonable to look for extension of Theorem 1 to the normalized Jacobi polynomials. Even though the most general result can not be proven using the Gerhold-Kauers method yet (at least not within a reasonable amount of time), several interesting observations can be made with the assistance of computer algebra.

Recall that $P_m^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, is the m-th Jacobi polynomial, orthogonal with respect to the weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ in $[-1,1]$, and normalized by $P_m^{(\alpha,\beta)}(1) = \binom{m+\alpha}{m}$. As in the classical study of Turán determinants for Jacobi polynomials by Gasper [8, 9] we shall need a different normalization for these polynomials, namely, we shall require that they take value 1 at $x = 1$. Hence we set, oppressing the dependency on α and β for sake of brevity,

$$q_m(x) = P_m^{(\alpha,\beta)}(x)/P_m^{(\alpha,\beta)}(1), \quad m = 0, 1, 2, \ldots, \tag{5.5}$$

and study the following extension of Turán's determinant,

$$\widehat{\Delta}_m(x) = |x|q_m^2(x) - q_{m-1}(x)q_{m+1}(x), \quad x \in [-1,1], \quad -1 < \alpha \le \beta \le 0.$$

Note that by the standard relation between Jacobi and Gegenbauer polynomials the range of α and β includes the previously discussed case. In the general case of $\alpha \neq \beta$ we still have $\widehat{\Delta}_m(1) = 0$, but the polynomials $\widehat{\Delta}_m(x)$ are no longer symmetric with respect to the origin.

Assume first that $\{q_m\}$ is a general sequence of orthogonal polynomials, defined by the three term recurrence equation

$$xq_n(x) = \gamma_n q_{n+1}(x) + \beta_n q_n(x) + \alpha_n q_{n-1}(x), \quad n = 0, 1, 2, \dots, \tag{5.6}$$

where $q_{-1}(x) = 0$ and $q_0(x) = 1$, $\alpha_0 = 0$, and $\alpha_{n+1} > 0, \gamma_n > 0$ and $\alpha_n + \beta_n + \gamma_n = 1$ for every $n \in \mathbb{N}_0$. In a complete analogy to Lemma 2.1 we can derive the following identities.

Lemma 5.1. *Assume that the sequence $\{q_m\}$ satisfies the three term recurrence relation* (5.6)*. Then the following representations hold true:*

$$\gamma_n \widehat{\Delta}_n(x) = |x| \gamma_n q_n(x)^2 + \alpha_n q_{n-1}(x)^2 - (x - \beta_n) q_{n-1}(x) q_n(x)\,, \tag{5.7}$$

$$\alpha_n \widehat{\Delta}_n(x) = \alpha_n |x| q_n^2(x) + \gamma_n q_{n+1}^2(x) - (x - \beta_n) q_n(x) q_{n+1}(x)\,, \tag{5.8}$$

$$\begin{aligned} \gamma_n \widehat{\Delta}_n(x) = (\gamma_n |x| - \gamma_{n-1}) q_n^2(x) + (\alpha_n - \alpha_{n-1}|x|) q_{n-1}^2(x) \\ + (\beta_n - \beta_{n-1}) q_{n-1}(x) q_n(x) + \alpha_{n-1} \widehat{\Delta}_{n-1}(x)\,. \end{aligned} \tag{5.9}$$

Also, a version of Lemma 2.2 for general orthogonal polynomial sequences (5.6) can be proven, where we distinguish between the cases $x \geq 0$ and $x < 0$.

Lemma 5.2. *Assume the sequence $\{q_m\}$ satisfies the three term recurrence relation* (5.6) *with $\beta_n = 1 - \alpha_n - \gamma_n$.*

If $0 \leq \alpha_n \leq 1$ and $0 \leq \gamma_n \leq 1$ and $x \in [\xi_1, \xi_2] \subseteq [0, 1]$, where

$$\xi_i = \beta_n + 2\alpha_n\gamma_n + (-1)^i 2\sqrt{\alpha_n(1-\alpha_n)\gamma_n(1-\gamma_n)}, \quad i = 1, 2,$$

then $\widehat{\Delta}_n(x) \geq 0$.

If $0 \leq \alpha_n \leq 1$ and $\frac{1-\alpha_n}{1+\alpha_n} \leq \gamma_n \leq 1$ and $x \in [\zeta_1, \zeta_2] \subseteq [-1, 0]$, where

$$\zeta_i = \beta_n - 2\alpha_n\gamma_n + (-1)^i 2\sqrt{\alpha_n\gamma_n(\alpha_n\gamma_n + \alpha_n + \gamma_n - 1)}, \quad i = 1, 2,$$

then $\widehat{\Delta}_n(x) \geq 0$.

Proof. For the proof we also distinguish between the cases $x \geq 0$ and $x < 0$ and write identity (5.7) without absolute value. A CAD computation quickly confirms that

$$\begin{aligned} \forall\, y_0, y_{-1}, a, c, x \in \mathbb{R}\colon\ (0 \leq c \leq 1 \wedge 0 \leq a \leq 1 \wedge \xi_1 \leq x \leq \xi_2) \\ \Longrightarrow cxy_0^2 + ay_{-1}^2 - (x - (1-a-c)) y_{-1} y_0 \geq 0 \end{aligned}$$

is true. This general result in combination with the assumptions stated in the lemma and identity (5.7) yields $\widehat{\Delta}_n(x) \geq 0$ for non-negative x. The analogous formula yielding positivity of the modified Turán determinant for negative x is

$$\begin{aligned} \forall\, y_0, y_{-1}, a, c, x \in \mathbb{R}\colon\ \left(\tfrac{1-a}{1+a} \leq c \leq 1 \wedge 0 \leq a \leq 1 \wedge \zeta_1 < x < \zeta_2\right) \\ \Longrightarrow -cxy_0^2 + ay_{-1}^2 - (x - (1-a-c)) y_{-1} y_0 \geq 0. \end{aligned}$$

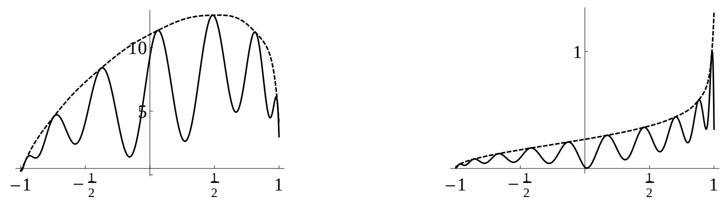

Figure 2: Graphs of $\widehat{\Delta}_n(x)/(1-x)$ (solid line) and $\Delta_n(x)/(1-x)$ (dashed) for $n=7, \alpha=-\frac{3}{4}, \beta=-\frac{1}{4}$(left) and $n=10, \alpha=-\frac{1}{4}, \beta=0$ (right)

$\square$

For the normalized Jacobi polynomials the recurrence coefficients are given by

$$\alpha_n = \frac{2n(\beta+n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)},$$
$$\gamma_n = \frac{2(\alpha+n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)},$$

and $\beta_n = 1-\alpha_n-\gamma_n$. With a CAD computation it is easily verified that the assumptions of Lemma 5.2 are satisfied if $-1<\alpha\le\beta\le 0$ and $n\ge 2$ (the cases $n=0,1$ can easily be verified independently). For fixed α and β, the bounds derived in Lemma 5.2 when specialized to the Jacobi recurrence coefficients have the following limits

$$\lim_{n\to\infty}\zeta_1=-1, \quad \lim_{n\to\infty}\zeta_2=0, \quad \lim_{n\to\infty}\xi_1=0, \quad \text{and} \quad \lim_{n\to\infty}\xi_1=1.$$

Note that if we restrict ourselves to the ultraspherical case $\alpha=\beta=\lambda-1/2$, then $\xi_1=\zeta_2=0$ and $\xi_2=-\zeta_1$ coincides with $z_n(\lambda)$. For particular choices of α,β with $-1<\alpha<\beta\le 0$ (not including the already proven ultraspherical case) and for concrete degrees n, it is easy to verify using CAD that $\widehat{\Delta}_n(x)\ge 0$. The calculations proving the inequality for more than 1500 random samples of α,β with degrees n ranging up to 15 on the whole interval $[-1,1]$ take less than half an hour. Still, we did not succeed in proving the inequality with symbolic parameters. Using the well-known identities [23]

$$\begin{aligned}&\tfrac{d}{dx}P_n^{(\alpha,\beta)}(x)=\tfrac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x),\\ &P_n^{(\alpha,\beta)}(-x)=(-1)^nP_n^{(\beta,\alpha)}(-x), \quad \text{and} \quad P_n^{(\alpha,\beta)}(1)=\tfrac{(\alpha+1)_n}{n!},\end{aligned}$$

we can obtain some partial results by investigating the special cases $x=\pm1$. For $x>0$ we have

$$\widehat{\Delta}_n(x)=xq_n(x)^2-q_{n-1}(x)q_{n+1}(x),$$

and using the identities above we find that $\widehat{\Delta}_n'(1)=\frac{\alpha}{\alpha+1}$. Recall that $\widehat{\Delta}_n(1)=0$ for all $n\ge1$ and since $\frac{\alpha}{\alpha+1}>0$ for positive α, we have that $\widehat{\Delta}_n(1-0)<0$ if $\alpha>0$. Hence, the upper bound 0 on α and β is sharp and can not be extended.

For $x = -1$ we have

$$\widehat{\Delta}_n(-1) = \left(\frac{(\beta+1)_n}{(\alpha+1)_n}\right)^2 - \frac{(\beta+1)_{n-1}}{(\alpha+1)_{n-1}}\frac{(\beta+1)_{n+1}}{(\alpha+1)_{n+1}}.$$

Using SumCracker we obtain quickly that $\widehat{\Delta}_n(-1) > 0$ for $n \geq 1$ and $\alpha \neq \beta$:

In[3]:= **ProveInequality**[$\left(\frac{(\beta+1)_{n+1}}{(\alpha+1)_{n+1}}\right)^2 - \frac{(\beta+1)_n}{(\alpha+1)_n}\frac{(\beta+1)_{n+2}}{(\alpha+1)_{n+2}} > 0$,
Using $\to \{-1 < \alpha < \beta \leq 0\}$, **Variable** $\to n$]

Out[3]= True

The inequality above could also be easily verified by hand. Note that since $\alpha, \beta > -1$ the arguments of the rising factorials are positive and so is $(\alpha+1)_{n+1}/(\beta+1)_n(\alpha+1)_{n+2}/(\beta+1)_{n+1}$. Multiplying the inequality by this term the left hand side simplifies to $\beta - \alpha$, which is positive in the given range.

In order to be able to apply a similar argument as for $x = 1$ we multiply $\widehat{\Delta}_n(x)$ with the factor $1 + x$ that is non-negative for $x \in [-1, 0]$ and consider

$$f_n(x) = (1+x)\big(-xq_n(x)^2 - q_{n-1}(x)q_{n+1}(x)\big) = (1+x)\widehat{\Delta}_n(x), \quad x \in [-1, 0].$$

Now $f_n(-1) = 0$ and $f_n(x) \geq 0$ if and only if $\widehat{\Delta}_n(x) \geq 0$ (in $[-1, 0]$). Furthermore, we have

$$f_n'(x) = \widehat{\Delta}_n(x) + (1+x)\widehat{\Delta}_n'(x),$$

and thus $f_n'(-1) = \widehat{\Delta}_n(-1)$, which is positive as we showed earlier. From these observations it follows that there exists $\delta > 0$ such that $\widehat{\Delta}_n(x) > 0$ in $(-1, -1+\delta) \cup (1-\delta, 1)$.

In conclusion we formulate a conjecture, which, if true, would yield a refinement of the result of Gasper [9].

Conjecture 5.3. *For all $n \geq 0$ and $-1 < \alpha \leq \beta \leq 0$ and all $x \in [-1, 1]$,*

$$\widehat{\Delta}_m(x) = |x|q_m^2(x) - q_{m-1}(x)q_{m+1}(x) \geq 0.$$

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