A procedure for solving autonomous AODEs

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In this report we present a procedure for solving first order autonomous algebraic ordinary differential equations by means of algebraic curves and parametrizations. In particular we look at solutions expressible by radicals and show the connection to existing theory on rational solutions.

1 Introduction

Consider the field of rational functions $\mathbb{K}(x)$ for a field $\mathbb{K}$ and let $\frac{d}{dx}$ be the usual derivative. Then $\mathbb{K}(x)$ is a differential field. We call $\mathbb{K}(x)\{y\}$ the ring of differential polynomials. Its elements are polynomials in y and the derivatives of y, i. e. $\mathbb{K}(x)\{y\} = \mathbb{K}(x)[y, y', y'', \ldots]$. An algebraic ordinary differential equation (AODE) is one of the form $F(x, y, y', \ldots, y^{(n)}) = 0$ where $F \in \mathbb{K}(x)\{y\}$ and F is also a polynomial in x. The AODE is called autonomous if $F \in \mathbb{K}\{y\}$, i. e. if the coefficients of F do not depend on the variable of differentiation x. For a given AODE we are interested in deciding whether it has rational or radical solutions and, in the affirmative case, determining all of them.

In order to define the notion of a general solution we go a little more into detail. Let Σ be a prime differential ideal in $\mathbb{K}(x)\{y\}$. Then we call η a generic zero of Σ if for any

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differential polynomial P we have $P(\eta) = 0 \Leftrightarrow P \in \Sigma$. Such an η exists in a suitable extension field.

Let F be an irreducible differential polynomial of order n. Then $\{F\}$, the radical differential ideal generated by F, can be decomposed into two parts. There is one component where the separant $\frac{\partial F}{\partial y^{(n)}}$ also vanishes. This part represents the singular solutions. The component we are interested in is the one where the separant does not vanish. It is a prime differential ideal $\Sigma_F := \{F\} : \langle \frac{\partial F}{\partial y^{(n)}} \rangle$ and represents the general component (see for instance Ritt [16]). A generic zero of Σ_F is called general solution of $F = 0$. We say it is a rational general solution if it is of the form $y = \frac{a_k x^k + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$, where the a_i and b_i are algebraic over $\mathbb{K}$.

Here we consider only first order AODEs. For solving differential equations $G(x, y, y') = 0$ or $F(y, y') = 0$ we will look at the corresponding surface $G(x, y, z) = 0$ or curve $F(y, z) = 0$ respectively where we replace the derivative of y by a transcendental variable z.

An algebraic curve $\mathcal{C}$ is a one-dimensional algebraic variety, i. e. a zero set of a square-free bivariate polynomial $f \in \mathbb{K}[x, y]$, $\mathcal{C} = \{(a, b) \in \mathbb{A}^2 \,|\, f(a, b) = 0\}$. We call the polynomial f the defining polynomial. An important aspect of algebraic curves is their parametrizability. Consider an irreducible plane algebraic curve defined by an irreducible polynomial f. A tuple of rational functions $\mathcal{P}(t) = (r(t), s(t))$ is called a rational parametrization of the curve if $f(r(t), s(t)) = 0$ and not both $r(t)$ and $s(t)$ are constant. A parametrization can be considered as a map $\mathcal{P}(t) : \mathbb{A} \to \mathcal{C}$. By abuse of notation we also call this map a parametrization. Later we will see other kinds of parametrizations. We call a parametrization $\mathcal{P}(t)$ proper if it is a birational map or in other words if for almost every point (x, y) on the curve we find exactly one t such that $\mathcal{P}(t) = (x, y)$. Parametrizations of higher dimensional algebraic varieties are defined in a similar way.

In this report we use parametrizations of curves for solving AODEs. Hubert [8] already studies solutions of AODEs of the form $F(x, y, y') = 0$. She gives a method for finding a basis of the general solution of the equation by computing a Gröbner basis of the prime differential ideal of the general component. The solutions, however, are given implicitly. Later Feng and Gao [2, 3] start using parametrizations for solving first order autonomous AODEs, i. e. $F(y, y') = 0$. They provide an algorithm to actually solve such AODEs with coefficients in $\mathbb{Q}$ by using rational parametrizations of the algebraic curve $F(x, y) = 0$. The key fact they are proving is that any rational solution of the AODE gives a proper parametrization of the corresponding algebraic curve. For this they use a degree bound derived in Sendra and Winkler [21]. On the other hand, if a proper parametrization of the algebraic curve fulfills some requirements Feng and Gao can generate a rational solution of the AODE. From the rational solution it is then possible to create a rational general solution by shifting the variable by a constant. Finally Feng and Gao derive an algorithm to compute a rational solution of an AODE, if it exists.

By using this approach we can take advantage of the well known theory of algebraic curves and rational parametrizations (see for instance [22, 23]).

Recently Ngô and Winkler [12, 14, 13] worked on generalizations of what Feng and Gao started. They considered non-autonomous AODEs $F(x, y, y') = 0$. Instead of algebraic curves as in the autonomous case, algebraic surfaces play a role there. For an algorithm to find rational parametrizations of surfaces see [17]. Using a rational parametrization of the surface Ngô and Winkler derive a special kind of system of differential equations. For these so called associated systems solution methods exist. From a rational solution of such a system they find a rational general solution of the original differential equation.

Aroca, Cano, Feng and Gao give in [1] a necessary and sufficient condition for an autonomous AODE to have an algebraic solution. They also provide a polynomial time algorithm to find such a solution if it exists. This solution, however, is implicit whereas we are interested in explicit solutions.

Furthermore Huang, Ngô and Winkler continue their work in considering higher order equations. A first result can be found in [7]. A generalization of Ngô and Winkler [14] to trivariate systems of ODEs can be found in [5, 6]. Ngô, Sendra and Winkler [11, 10] also considered classification of AODEs, i. e. they look for transformations of AODEs which keep the associated system invariant.

In this work we stick to the case of first order autonomous AODES but try to extend the results to radical solutions using radical parametrizations (see Section 2.2). We do so by investigating a procedure which has the rational solutions as a special case (see Section 2.1). As shown in [1, 2] it is enough to look for a single non-trivial solution, for if $y(x)$ is a solution, so is $y(x + c)$ for a constant c and the latter is also a general solution. In Section 2.3 we give examples of non-radical solutions that can be found by the given procedure. Finally in Section 2.4 we look for advantages of the procedure and compare it to existing algorithms.

2 A procedure for solving first order autonomous AODEs

Let $F(y, y') = 0$ be an autonomous algebraic ordinary differential equation (AODE). We consider the corresponding algebraic curve $F(y, z) = 0$. Then obviously $P_y := (y(t), y'(t))$ (for a solution y of the AODE) is a parametrization of F.

Now we take an arbitrary parametrization $P(t) = (r(t), s(t))$, i. e. functions r and s not both constant such that $F(r(t), s(t)) = 0$. We define $A_P(t) := \frac{s(t)}{r'(t)}$. If it is clear which parametrization is considered, we write A. Suppose we are given any parametrization of F and we want to find a solution of the AODE. We assume the parametrization is of the form $P_g(t) = (r(t), s(t)) = (y(g(t)), y'(g(t)))$ where g and y are unknown. In the rational case we know that each parametrization can be obtained by any other by application of a rational function. If we can find g and especially its inverse function we also find y. For our given P_g we can compute A_{P_g} and furthermore we have

$$A_{P_g}(t) = \frac{y'(g(t))}{\frac{d}{dt}(y(g(t)))} = \frac{y'(g(t))}{g'(t)y'(g(t))} = \frac{1}{g'(t)}.$$

Hence, by reformulation we get an expression for the unknown g':

$$g'(t) = \frac{1}{A_{P_g}(t)}.$$

Using integration and inverse functions the procedure continues as follows

$$\begin{aligned} g(t) &= \int g'(t)dt = \int \frac{1}{A_{P_g}(t)} dt, \\ y(x) &= r(g^{-1}(x)). \end{aligned}$$

Kamke [9] already mentions this procedure where he restricts to continuously differential functions r and s which satisfy $F(r(t), s(t)) = 0$. However, he does not mention where to get these functions from.

In general g is not a bijective function. Hence, when we talk about an inverse function we actually mean one branch of a multivalued inverse. Each branch inverse will give us a solution to the differential equation.

We might add any constant c to the solution of the indefinite integral. Assume $g(t)$ is a solution of the integral and g^{-1} its inverse. Then $\bar{g}(t) = g(t) + c$ is as well a solution and $\bar{g}^{-1}(t) = g^{-1}(t - c)$. We know that if $y(x)$ is a solution of the AODE, so is $y(x + c)$. Hence, we may postpone the introduction of c to the end of the procedure.

The procedure finds a solution if we can compute the integral and the inverse function. On the other hand it does not give us any clue on the existence of a solution in case either part does not work. Neither do we know whether we found all solutions.

2.1 Rational solutions

Feng and Gao [2] found an algorithm for computing all rational general solutions of an autonomous first order AODE. They use the fact that $(y(x), y'(x))$ is a proper rational parametrization. The main part of their algorithm says that there is a rational general solution if and only if for any proper rational parametrization $P(t) = (r(t), s(t))$ we have that $A_P(t) = q \in \mathbb{Q}$ or $A_P(t) = a(t - b)^2$ with $a, b \in \mathbb{Q}$. The solutions therefore are $r(q(x + c))$ or $r(b - \frac{1}{a(x+c)})$ respectively.

We will show now that the algorithm accords with our procedure. Assume we are given an AODE with a proper parametrization $P = (r(t), s(t))$. Assume further that $A_P(t) = q \in \mathbb{Q}$ or $A_P(t) = a(t - b)^2$. Then we get from the procedure

$$\begin{aligned} A_P(t) &= q, & A_P(t) &= a(t-b)^2, \\ g'(t) &= \frac{1}{q}, & g'(t) &= \frac{1}{a(t-b)^2}, \\ g(t) &= \frac{t}{q} + c, & g(t) &= -\frac{1}{a(t-b)} + c, \\ g^{-1}(t) &= q(t-c), & g^{-1}(t) &= \frac{-1 + ab(t-c)}{a(t-c)}. \end{aligned}$$

We see that $r(g^{-1}(t))$ is exactly what Feng and Gao found aside from the sign of c. As mentioned above, our procedure does not give an answer to whether the AODE has a rational solution in case A is not of this special form. It might, however, find non rational solutions for some AODEs. Nevertheless, Feng and Gao [2] already proved that there is a rational general solution if and only if A is of the special form mentioned above and all rational general solution can be found by the algorithm.

2.2 Radical solutions

Now we extend our set of possible parametrizations and also the set, in which we are looking for solutions, to the functions including radical expressions.

The research area of radical parametrizations is rather new. Sendra and Sevilla [19] recently published a paper on parametrizations of curves using radical expressions. In this paper Sendra and Sevilla define the notion of radical parametrization and they provide algorithms to find such parametrizations in certain cases which include but are not restricted to curves of genus less or equal 4. Every rational parametrization will be a radical one but obviously not the other way round. Further considerations of radical parametrizations can be found in Schicho and Sevilla [18] and Harrison [4]. There is also a paper on radical parametrization of surfaces by Sendra and Sevilla [20]. Nevertheless, for the beginning we will restrict to the case of first order autonomous equations and hence to algebraic curves.

Definition 2.1.
Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A field extension $\mathbb{K} \subseteq \mathbb{L}$ is called a radical field extension *iff $\mathbb{L}$ is the splitting field of a polynomial of the form $x^k - a \in \mathbb{K}[x]$, where k is a positive integer and $a \neq 0$. A* tower of radical field extensions *of $\mathbb{K}$ is a finite sequence of fields*

$$\mathbb{K} = \mathbb{K}_0 \subseteq \mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \ldots \subseteq \mathbb{K}_m$$

such that for all $i \in \{1, \ldots, m\}$, the extension $\mathbb{K}_{i-1} \subseteq \mathbb{K}_i$ is radical.
A field $\mathbb{E}$ is a radical extension field *of $\mathbb{K}$ iff there is a tower of radical field extensions of $\mathbb{K}$ with $\mathbb{E}$ as its last element.*
A polynomial $h(x) \in \mathbb{K}[x]$ is solvable by radicals *over $\mathbb{K}$ iff there is a radical extension field of $\mathbb{K}$ containing the splitting field of h.*

Let now $\mathcal{C}$ be an affine plane curve over $\mathbb{K}$ defined by an irreducible polynomial $f(x, y)$. According to [19], $\mathcal{C}$ is parametrizable by radicals *iff there is a radical extension field $\mathbb{E}$ of $\mathbb{K}(t)$ and a pair $(r(t), s(t)) \in \mathbb{E}^2 \backslash \mathbb{K}^2$ such that $f(r(t), s(t)) = 0$. Then the pair $(r(t), s(t))$ is called a* radical parametrization *of the curve $\mathcal{C}$.*

We call a function $f(x)$ over $\mathbb{K}$ a radical function *if there is a radical extension field of $\mathbb{K}(x)$ containing $f(x)$. Hence, a* radical solution *of an AODE is a solution that is a radical function. A* radical general solution *is a general solution which is radical.*

Computing radical parametrizations as in [19] goes back to solving algebraic equations of degree less or equal four. Depending on the degree we might therefore get more than one solution to such an equation. Each solution yields one branch of a parametrization. Therefore, we use the notation $a^{\frac{1}{n}}$ for any n-th root of a.

We will now see that the procedure mentioned above yields information about solvability in some cases.

Theorem 2.2.
Let $P(t) = (r(t), s(t))$ be a radical parametrization of the curve $F(y, z) = 0$. Assume $A_P(t) = a(b + t)^n$ for some $n \in \mathbb{Q}\backslash\{1\}$.
Then $r(h(t))$, with $h(t) = -b + (-(n-1)a(t+c))^{\frac{1}{1-n}}$, is a general radical solution of the AODE $F(y, y') = 0$.

Proof. From the procedure we get

$$
\begin{aligned}
g'(t) &= \frac{1}{A_P(t)} = \frac{1}{a(b+t)^n},\\
g(t) &= \int g'(t)dt = \frac{(b+t)^{1-n}}{a(1-n)},\\
g^{-1}(t) &= -b + (-(n-1)at)^{\frac{1}{1-n}}.
\end{aligned}
$$

Then $y(x) = r(g^{-1}(x))$ is a solution of F. Let $h(x) = g^{-1}(x + c)$ for some constant c. Then $r(h(x))$ is a general solution of F. $\square$

The algorithm of Feng and Gao is therefore a special case of this one with $n = 0$ or $n = 2$ and a rational parametrization. In exactly these two cases g^{-1} is a rational function. Furthermore, Feng and Gao [2] showed that all rational solutions can be found like this, assuming the usage of a rational parametrization. The existence of a rational parametrization is of course necessary to find a rational solution. However, in the procedure we might use a radical parametrization of the same curve which is not rational and we can still find a rational solution.

In Theorem 2.2 $n = 1$ is excluded because in this case the function g contains a logarithm and its inverse an exponential term.

Example 2.3.
The equation $y^5 - y'^2 = 0$ gives rise to the radical parametrization $\left(\frac{1}{t}, -\frac{1}{t^{5/2}}\right)$ with corresponding $A(t) = \frac{1}{\sqrt{t}}$. We can compute $g(t) = \frac{2t^{3/2}}{3}$ and $g^{-1}(t) = \left(\frac{3}{2}\right)^{2/3} t^{2/3}$. Hence, $\frac{\left(\frac{2}{3}\right)^{2/3}}{(x+c)^{2/3}}$ is a solution of the AODE.

As a corollary of Theorem 2.2 we get the following statement for AODEs where the parametrization yields another form of $A(t)$.

Corollary 2.4.
Assume we have a radical parametrization $P(t) = (r(t), s(t))$ *of an autonomous curve* $F(y, z) = 0$ *and assume* $A(t) = \frac{a(b+t^k)^n}{kt^{k-1}}$ *with* $k \in \mathbb{Q}$. *Then the AODE has a radical solution.*

Proof. By transforming the parametrization by $f(t) = t^{1/k}$ to the radical parametrization $\bar{P}(t) = (r(f(t)), s(f(t)))$ we compute

$$A_{\bar{P}}(t) = \frac{s(f(t))}{\frac{\partial}{\partial t}(r(f(t)))} = \frac{A(f(t))}{f'(t)} = \frac{a(b + f(t)^k)^n}{kf(t)^{k-1}f'(t)} = \frac{a(b + t^{\frac{k}{k}})^n}{kt^{\frac{k-1}{k}}\frac{1}{k}t^{\frac{1-k}{k}}} = a(b + t)^n,$$

which can be solved by Theorem 2.2. □

In contrast to the rational case there are more possible forms for A now. In the following we will see another rather simple form of A which might occur. Here we do not know immediately whether or not the procedure will lead to a solution.

Theorem 2.5.
Let $P(t) = (r(t), s(t))$ *be a radical parametrization of the curve* $F(y, z) = 0$. *Assume* $A(t) = \frac{at^n}{b+t^m}$ *for some* $a, b \in \mathbb{Q}$ *and* $m, n \in \mathbb{Q}$ *with* $m \neq n-1$ *and* $n \neq 1$. *Then the AODE* $F(y, y') = 0$ *has a radical solution if the equation*

$$b(m - n + 1)h(t)^{1-n} - (n - 1)h(t)^{m-n+1} + (n - 1)(m - n + 1)at = 0 \tag{1}$$

has a non-zero radical solution for $h = h(t)$. *A general solution of the AODE is then* $r(h(x + c))$.

Proof. The procedure yields

$$g'(t) = \frac{1}{A_P(t)} = \frac{b + t^m}{at^n},$$
$$g(t) = \int g'(t)dt = \frac{1}{a}t^{1-n}\left(\frac{b}{1-n} + \frac{t^m}{1 + m - n}\right).$$

The inverse of g is only computable if the equation

$$\frac{1}{a}h(t)^{1-n}\left(\frac{b}{1-n} + \frac{h(t)^m}{1 + m - n}\right) = t$$

can be solved for $h(t)$. By a reformulation and the assumptions for m and n this is equivalent to (1). □

In case we have $n = 1$ or $m = n - 1$ the integral is a function containing a logarithm and the inverse function yields an expression containing the Lambert W-Function.

Example 2.6.
The equation $-y^5 - y' + y^8 y' = 0$ *gives rise to the radical parametrization* $P(t) = \left(\frac{1}{t}, \frac{t^3}{1-t^8}\right)$ *with corresponding* $A(t) = \frac{t^5}{-1+t^8}$. *Then equation* (1) *has a solution, e. g.* $-\left(2t - \sqrt{-1+4t^2}\right)^{1/4}$. *Hence,* $-\left(2(x+c) - \sqrt{(-1+4(x+c)^2}\right)^{-1/4}$ *is a solution of the AODE.*

It remains to show when equation (1) is solvable (i. e. when $g(t)$ in the proof of Theorem 2.5 has an inverse which is expressible by radicals). The following theorem due to Ritt [15] will help us to do so.

Theorem 2.7.
A polynomial g *has an inverse expressible by radicals if and only if it can be decomposed in*

- *linear polynomials,*
- *power polynomials* x^n *for* $n \in \mathbb{N}$*,*
- *Chebyshev polynomials and*
- *degree* 4 *polynomials.*

Certainly also polynomials of degree 2 and 3 are invertible by radicals but it can be shown, that Theorem 2.7 applies. We will show now that a certain polynomial is not decomposable into non-linear factors.

Theorem 2.8.
Let $g(t) = C_1 t^\alpha + C_2 t^\beta \in \mathbb{K}[t]$ *where* $\mathbb{K}$ *is a field of characteristic zero,* $C_1, C_2 \in \mathbb{K}\backslash\{0\}$, $\alpha, \beta \in \mathbb{N}$, $\gcd(\alpha,\beta) = 1$ *and* $\beta > \alpha > 0$ *and* $\beta > 4$. *Then* g *cannot be decomposed into two polynomials* $f(h(x))$ *each of degree higher than one.*

Proof. Assume $g(t) = f(h(t))$ with $f = \sum_{i=0}^{n} a_i x^i$ and $h = \sum_{k=0}^{m} b_k x^k$ and $a_n \neq 0$, $b_m \neq 0$, $m, n > 1$. In case $b_0 \neq 0$ it follows that $g(t) = \bar{f}(\bar{h}(t))$ where $\bar{f}(t) = f(b_0 + t)$ and $\bar{h}(t) = h(t) - b_0$. Hence, without loss of generality we can assume that $b_0 = 0$ and therefore also $a_0 = 0$.

Let now $\tau \in \{1, \ldots, m\}$ such that $b_\tau \neq 0$ and $b_l = 0$ for all $l \in \{1, \ldots, \tau - 1\}$. Similarly let $\pi \in \{1, \ldots, n\}$ such that $a_\pi \neq 0$ and $a_l = 0$ for all $l \in \{1, \ldots, \pi - 1\}$.

This implies that $\mathrm{coef}_l(g) = 0$ for all $l \in \{1, \ldots, \tau\pi - 1\}$ and $\mathrm{coef}_{\tau\pi}(g) = a_\pi b_\tau^\pi \neq 0$. Hence, $\alpha = \tau\pi$. Furthermore, we know that

$$0 = \mathrm{coef}_{m(n-1)+l}(g) = c_{n-1} a_n b_m^{n-1} b_l + \sum_{k=0}^{n-2} c_k a_n b_m^k B_k$$

for all $l \in \{\tau, \ldots, m-1\}$, where c_k are non-zero constants and

$$B_k = \sum_{\substack{\varepsilon_{l+1}+\ldots+\varepsilon_{m-1}=n-k \\ (l+1)\varepsilon_{l+1}+\ldots+(m-1)\varepsilon_{m-1}=m(n-1)+l-km}} b_{l+1}^{\varepsilon_{l+1}} \cdot \ldots \cdot b_{m-1}^{\varepsilon_{m-1}}.$$

This yields, that $0 = \mathrm{coef}_{mn-1}(g) = c_{n-1} a_n b_m^{n-1} b_{m-1}$, hence, $b_{m-1} = 0$. By induction it follows that $b_l = 0$ for all $l \in \{\tau, \ldots, m-1\}$ which contradicts $b_\tau \neq 0$.

Therefore, $\tau = m$ which makes $\{\tau, \ldots, m-1\}$ empty. But then we have $m \mid \alpha$ and $m \mid \beta$ which contradicts $\gcd(\alpha, \beta) = 1$ since $m \neq 1$. □

Let us now consider the function

$$g(t) = \frac{1}{a} t^{1-n} \left(\frac{b}{1-n} + \frac{t^m}{1+m-n} \right)$$

from the proof of Theorem 2.5. Assume that $1 - n = \frac{z_1}{d_1}$ and $m - n + 1 = \frac{z_2}{d_2}$ with $z_1, z_2 \in \mathbb{Z}$, $d_1, d_2 \in \mathbb{N}$ such that $\gcd(z_1, d_1) = \gcd(z_2, d_2) = 1$. Then $g(t) = \bar{g}(h(t))$ where $h(t) = t^{\frac{d}{d_1 d_2}}$ and

$$\bar{g}(t) = \frac{1}{a} t^{\bar{n}} \left(\frac{b}{1-n} + \frac{t^{\bar{m}-\bar{n}}}{1+m-n} \right) \tag{2}$$

with $\bar{n} = \frac{(1-n)d_1 d_2}{d}$, $\bar{m} = \frac{(m-n+1)d_1 d_2}{d}$ and $d = \gcd(z_1 d_2, z_2 d_1)$. Hence, $\bar{m}, \bar{n} \in \mathbb{Z}$ with $\gcd(\bar{m}, \bar{n}) = 1$. The function h has an inverse expressible by radicals. If now g has an inverse expressible by radicals than so does $\bar{g}$, i.e. $\bar{g}^{-1} = h \circ g^{-1}$. If $\bar{g}$ has an inverse expressible by radicals, so does g. In case $m - n + 1, 1 - n \in \mathbb{N}$ also $\bar{m}, \bar{n} \in \mathbb{N}$. On the other hand if $n - m - 1, n - 1 \in \mathbb{N}$ we get a polynomial by a decomposition with a factor $f(t) = t^{-1}$.

If not both $\bar{n}$ and $\bar{m}$ are positive and not both are negative but $|\bar{m}| + |\bar{n}| \leqslant 4$, computing the inverse function of $\bar{g}$ is the same as solving an equation of degree less or equal 4, which can be done by radicals.

The above discussion and Theorem 2.7 and 2.8 imply

Corollary 2.9.
The function $g(t) = \frac{1}{a} t^{1-n} \left(\frac{b}{1-n} + \frac{t^m}{1+m-n} \right)$ *from the proof of Theorem 2.5 has an inverse expressible by radicals if (using the notation from above)*

- $b = 0$ *or*
- $\bar{m}, \bar{n} \in \mathbb{N}$ *and* $\max(|\bar{m}|, |\bar{n}|) \leqslant 4$.
- $-\bar{m}, -\bar{n} \in \mathbb{N}$ *and* $\max(|\bar{m}|, |\bar{n}|) \leqslant 4$.
- $-\bar{m}, \bar{n} \in \mathbb{N}$ *and* $|\bar{m}| + |\bar{n}| \leqslant 4$.
- $\bar{m}, -\bar{n} \in \mathbb{N}$ *and* $|\bar{m}| + |\bar{n}| \leqslant 4$.

It has no inverse expressible by radicals in the cases

- $\bar{m}, \bar{n} \in \mathbb{N}$ *and* $\max(\bar{m}, \bar{n}) > 4$,
- $-\bar{m}, -\bar{n} \in \mathbb{N}$ *and* $\max(|\bar{m}|, |\bar{n}|) > 4$.

Hence, in some cases we are able to decide the solvability of an AODE with properties as in Theorem 2.5. Nevertheless, the procedure is not complete, since even Corollary 2.9 does not cover all possible cases for m and n.

2.3 Non-Radical solutions

The procedure is not restricted to the radical case but might also solve some AODEs with non-radical solutions.

Example 2.10.
Consider the equation $y^3+y^2+y'^2=0$. *The corresponding curve has the parametrization* $P(t)=(-1-t^2, t(-1-t^2))$. *We get* $A(t)=\frac{1}{2}(1+t^2)$ *and hence,* $g(t)=\int \frac{1}{A(t)}dt = 2\arctan(t)$. *The inverse function is* $g^{-1}(t)=\tan(\frac{t}{2})$ *and hence,* $y(x)=-1-\tan(\frac{x+c}{2})^2$ *is a solution.*

Beside trigonometric solutions we might also find exponential solutions.

Example 2.11.
Consider the AODE $y^2+y'^2+2yy'+y=0$. *We get the rational parametrization* $(-\frac{1}{(1+t)^2}, -\frac{t}{(1+t)^2})$. *With* $A(t)=-\frac{1}{2}t(1+t)$ *we compute* $g(t)=-2\log(t)+2\log(1+t)$ *and hence* $g^{-1}(t)=\frac{1}{-1+\mathrm{e}^{t/2}}$, *which leads to the solution* $-\mathrm{e}^{-(x+c)}(-1+\mathrm{e}^{(x+c)/2})^2$.

We see that it is not even necessary to use radical parametrizations in order to find non-radical solutions.

2.4 Comparison

In many books on differential equations we can find a method for transforming an autonomous ODE of any order $F(y, y', \ldots, y^{(n)})=0$ to an equation of lower order by substituting $u(y)=y'$ (see for instance [24, 9]). For the case of first order ODEs this method yields a solution. It turns out that this method is somehow related to our procedure. The method does the following.

- Substitute $u(y)=y'$.
- Solve $F(y, u(y))=0$ for $u(y)$.
- Solve $\int \frac{1}{u(y)}dy = x$ for y.

This can be interpreted in terms of parametrizations and our general procedure as follows

- Compute a parametrization of the form $P=(t, s(t))$,
- $g'(t)=\frac{1}{A_P(t)}=\frac{1}{s(t)}$,
- $g(t)=\int \frac{1}{s(t)}dt$.
- Compute the inverse h of g, i. e. $g(h(t))=t$,
- $y(t)=h(t)$.

We will now give some arguments concerning the possibilities and benefits of the general procedure. Since in the procedure any radical parametrization can be used we might take advantage of picking a good one as we will see in the following example.

Example 2.12.
We consider the AODE $y'^6 + 49yy'^2 - 7$ *and find a parametrization of the form* $(t, s(t))$*:*

$$\left(t, \frac{\sqrt{\left(756 + 84\sqrt{28812t^3 + 81}\right)^{2/3} - 588t}}{\sqrt{6}\left(756 + 84\sqrt{28812t^3 + 81}\right)^{1/6}}\right).$$

Neither Mathematica 8 *nor* Maple 16 *can solve the corresponding integral explicitly and hence, the procedure stops. Neither of them is capable of solving the differential equation in explicit form by the built in functions for solving ODEs. Nevertheless, we can try our procedure using other parametrizations. An obvious one to try next is*

$$(r(t), s(t)) = \left(-\frac{-7 + t^6}{49t^2}, t\right).$$

It turns out that here we get $g(t) = \frac{2}{21t^3} - \frac{4t^3}{147}$*. Its inverse can be computed* $g^{-1}(t) = \frac{1}{2}\left(-147t - \sqrt{7}\sqrt{32 + 3087t^2}\right)^{1/3}$*. Applying* $g^{-1}(x + c)$ *to* $r(t)$ *we get the solution*

$$y(x) = -\frac{4\left(-7 + \frac{1}{64}\left(-147(c+x) - \sqrt{7}\sqrt{32 + 3087(c+x)^2}\right)^2\right)}{49\left(-147(c+x) - \sqrt{7}\sqrt{32 + 3087(c+x)^2}\right)^{2/3}}.$$

The procedure might find a radical solution of an AODE by using a rational parametrization as we have seen in Example 2.6 and 2.12. As long as we are looking for rational solutions only, the corresponding curve has to have genus zero. Now we can also solve some examples where the genus of the corresponding curve is higher than zero and hence there is no rational parametrization. The AODE in Example 2.13 below corresponds to a curve with genus 1.

Example 2.13.
Consider the AODE $-y^3 - 4y^5 + 4y^7 - 2y' - 8y^2y' + 8y^4y' + 8yy'^2 = 0$*. We compute a parametrization and get*

$$\left(\frac{1}{t}, \frac{-4 + 4t^2 + t^4}{t\left(4t^2 - 4t^4 - t^6 - \sqrt{-16t^4 + 16t^8 + 8t^{10} + t^{12}}\right)}\right)$$

as one of the branches. The procedure yields

$$\begin{aligned}
A(t) &= -\frac{t\left(-4 + 4t^2 + t^4\right)}{4t^2 - 4t^4 - t^6 - \sqrt{-16t^4 + 16t^8 + 8t^{10} + t^{12}}},\\
g(t) &= \frac{2t^4 + t^6 + \sqrt{t^4\left(2 + t^2\right)^2\left(-4 + 4t^2 + t^4\right)}}{4t^2 + 2t^4},\\
g^{-1}(t) &= -\frac{\sqrt{1 + t^2}}{\sqrt{1 + t}},\\
y(x) &= -\frac{\sqrt{1 + c + x}}{\sqrt{1 + (c + x)^2}}.
\end{aligned}$$

Again Mathematica 8 *cannot compute a solution in reasonable time and* Maple 16 *only computes constant and implicit ones.*

3 Conclusion

We have introduced a procedure for solving autonomous first order ordinary differential equations. The procedure works in a given class of expressions if we can compute a certain integral and an inverse function in this class. In case of looking for rational solutions it does exactly what was known before. Furthermore, we have found some cases in which we find radical solutions. However, these cases are not yet complete and hence, this part is subject to further investigation. In case the procedure works, we have one or more solutions of the AODE. So far we do not know, whether we have all. Neither do we know anything about solvability of the AODE if the procedure does not work. We have seen that the choice of parametrization makes a difference. The influence of the choice of parametrization on the solvability of the integration and inversion problem is also a topic for further investigation.

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