Birational Transformations on Algebraic Ordinary Differential Equations

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Abstract

We describe a group of birational transformations acting on the set of algebraic ordinary differential equations (AODEs) of arbitrary order n. This transformation group, by its action, partitions the set of algebraic ODEs into equivalence classes. All the elements in a given equivalence class exhibit the same behavior in terms of rational solvability. For a big family of algebraic ODEs we show how to decide whether the given equation can be transformed into an equivalent autonomous ODE.

1 Introduction

In this paper we deal with algebraic ordinary differential equations (AODEs), that is with ordinary differential equations of order n of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0 \ , \tag{1}$$

where F is a polynomial with coefficients in an algebraically closed field $\mathbb{K}$ of characteristic zero; in practice, we often take the field of complex numbers $\mathbb{C}$ as our ground field. In many problems related to AODEs, as for instance the study of existence and actual computation of rational solutions (see [FG04], [FG06], [NW10], [NW11], [NSW12]), one associates to the differential equation an algebraic variety. More precisely, one associates to (1) the hypersurface $\mathcal{V}(F)$ defined by $F(u, w, w_1, \ldots, w_n) = 0$ in the $(n+2)$-dimensional affine space $\mathbb{K}^{n+2}$, with the hope to achieve information on the solutions of (1) from the algebraic and geometric properties of $\mathcal{V}(F)$. Therefore, one distinguishes two levels of this problem: the differential and the algebraic level.

On the other hand, when working in algebraic geometry it is very usual to perform transformations that preserve the main properties and invariants of the variety, with the aim of reaching a simpler expression or a simpler geometric object from where the final conclusion might be easier to deduce. Typically, one uses birational transformations. The simplest case of birational transformations are the linear affine transformations. However, birational transformations, although preserving many important algebraic and geometric properties, in general do not preserve the differential properties. For instance, let us consider the homogeneous linear differential equation $y' = 0$. Its associated hypersurface is the plane Π_1 of equation $w_1 = 0$ in $\mathbb{K}^3$. If we consider the linear affine transformation $\{u^* = u, w^* = w, w_1^* = w_1 + w\}$, the plane Π_1 is transformed into the plane Π_2 of equation $w_1^* - w^* = 0$, corresponding to the linear differential equation $y' = y$. However, while all solutions of the first equation are rational (indeed constant), the second equation does not have any rational solution.

So the natural question is: what type of birational transformations, if any, does preserve the information (for instance, the existence of rational solutions) at the differential level? In [NSW12], we characterize the set of all linear affine transformations that preserve the rationality of the solutions of a first order ordinary differential equation. More precisely, the linear transformations are of the type $\{u^* = u, w^* = \alpha w + \beta u + \gamma, w_1^* = \alpha w_1 + \beta\}$, where $\alpha, \beta, \gamma \in \mathbb{K}$, $\alpha \neq 0$, and form a group under composition. Note that the transformation above is not of this type and that, applying one of these transformations to Π_1, one gets $w_1^* = \beta$ corresponding to $y' = \beta$, all solutions of which are rational.

In this paper, we go a step further and we characterize the set of all birational transformations that behave properly for algebraic ordinary differential equations of order n (see Theorems 2.1, 2.2), and we prove that they form a group. Moreover, we consider the action of this group on the set of all algebraic ordinary differential equations of order n and we analyze the corresponding equivalence classes. In the last part of the paper we apply this type of birational transformations for developing an algorithmic method to transform a wide class of non-autonomous equations into autonomous ones.

2 Birational transformations on integral curves

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $\mathbb{K}(x)$ be the field of rational functions in x with coefficients in $\mathbb{K}$. For any rational function $f(x)$ and positive $n \in \mathbb{N}$ the set

$$\{(x, f(x), f'(x), \ldots, f^{(n)}(x)) \mid x \in \mathbb{K}\}$$

is a rational/parametric curve in the affine space $\mathbb{K}^{n+2}$. This motivates the following definition.

Definition 2.1. For a given rational function $f(x)$ and a positive integer n the parametric space curve

$$\mathcal{C}_f^{(n)} := \{(x, f(x), f'(x), \ldots, f^{(n)}(x)) \mid x \in \mathbb{K}\} \ \subset \mathbb{K}^{n+2}$$

is called a *rational integral curve (ric) of order* n over $\mathbb{K}$ (cf. [NSW12], p. 198). f is the *defining rational function* of $\mathcal{C}_f^{(n)}$. By

$$\mathcal{RIC}^{(n)} := \{\mathcal{C}_f^{(n)} \mid f \in \mathbb{K}(x)\}$$

we denote the *set of rational integral curves of order* n over $\mathbb{K}$. •

A rational integral curve is a set of points in $\mathbb{K}^{n+2}$; but sometimes we find it convenient to consider it as a rational mapping from $\mathbb{K}$ to $\mathbb{K}^{n+2}$:

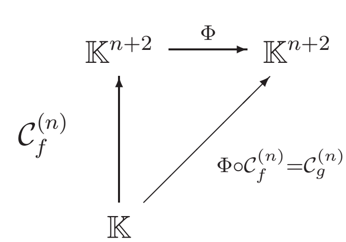
$$\begin{array}{cccc} \mathcal{C}_f^{(n)}: & \mathbb{K} & \longrightarrow & \mathbb{K}^{n+2} \\ & x & \mapsto & (x, f(x), f'(x), \ldots, f^{(n)}(x)) \end{array}.$$

We would like to find all birational transformations $\Phi : \mathbb{K}^{n+2} \to \mathbb{K}^{n+2}$ such that the induced map $\Phi^e : \mathcal{C}_f^{(n)} \mapsto \Phi \circ \mathcal{C}_f^{(n)}$ is actually a map from a non-empty subset of $\mathcal{RIC}^{(n)}$ to $\mathcal{RIC}^{(n)}$, i.e., for those $f \in \mathbb{K}(t)$ such that $\Phi^e(\mathcal{C}_f^{(n)}) = \Phi(x, f(x), f'(x), \ldots, f^{(n)}(x))$ is well defined, there exists a unique rational function $g \in \mathbb{K}(x)$ such that

$$\Phi^e(\mathcal{C}_f^{(n)}) = \mathcal{C}_g^{(n)}. \tag{2}$$

For a given birational transformation Φ on $\mathbb{K}^{n+2}$, we denote by $\mathcal{RIC}_\Phi^{(n)}$ the subset of $\mathcal{RIC}^{(n)}$ where Φ^e is defined.

The following diagram describes this situation:



Lemma 2.1. *Let* $\Phi : \mathbb{K}^{n+2} \to \mathbb{K}^{n+2}$ *be a birational map, then* $\mathcal{RIC}_\Phi^{(n)} \neq \emptyset$. *Furthermore, for almost all polynomials* $f(x)$ *of degree* n, $\mathcal{C}_f^{(n)} \in \mathcal{RIC}_\Phi^{(n)}$.

Proof. Let $G(u_1,\dots,u_{n+2})$ be the lcm of all denominators in Φ. Let

$$f = \lambda_2 + \lambda_3(x-\lambda_1) + \cdots + \frac{\lambda_{n+2}}{n!}(x-\lambda_1)^n$$

where λ_i are undetermined coefficients. First we see that $G(\mathcal{C}_f^{(n)}) \neq 0$. Indeed, if $G(\mathcal{C}_f^{(n)}) = G(x, f(x), \dots, f^{(n)}(x)) = 0$, then $G(\lambda_1, f(\lambda_1), \dots, f^{(n)}(\lambda_1)) = G(\lambda_1, \lambda_2, \dots, \lambda_{n+2}) = 0$ which is a contradiction because $G \neq 0$ and $(\lambda_1, \lambda_2, \dots, \lambda_{n+2})$ is a generic point in $\mathbb{K}^{n+2}$. Now, the set of coefficients of $G(x, f(x), \dots, f^{(n)}(x))$ w.r.t. x defines an algebraic variety V strictly included in $\mathbb{K}^{n+2}$ such that for $(\lambda_1, \lambda_2, \dots, \lambda_{n+2}) \in \mathbb{K}^{n+2} \setminus V$, the corresponding polynomial f generates a curve $\mathcal{C}_f^{(n)} \in \mathcal{RIC}_\Phi^{(n)}$. □

In the proof of the next theorem, and also throughout the paper, we will need to speak about the numerator of a rational function. Let us be precise about what we mean by that. Let a rational function $f(\overline{u}) = f_1(\overline{u})/f_2(\overline{u})$ in the (set of) variables $\overline{u}$ be represented by coprime polynomials f_1, f_2. Then we call f_1 the numerator of f, $f_1 = \mathrm{numer}(f)$. Observe that the numerator is determined up to constant multiples.
If f is a non-zero polynomial in the variable x, then by $\mathrm{LC}_x(f)$ we denote the leading coefficient of f w.r.t. x, i.e., the coefficient of $x^{\deg_x(f)}$.
Moreover, when $f(\overline{u}, \overline{v})$ is a polynomial in the (sets of) variables $\overline{u}$ and $\overline{v}$, then by the content of f w.r.t. $\overline{v}$, $\mathrm{cont}_{\overline{v}}(f)$, we denote the greatest common divisor of the coefficients of f w.r.t. $\overline{v}$. By the primitive part of f w.r.t. $\overline{v}$ we mean $\mathrm{pp}_{\overline{v}}(f) = f/\mathrm{cont}_{\overline{v}}(f)$.
For a rational function $f(u_1,\dots,u_r)$ we denote its derivative w.r.t. u_j by f_j; in particular, for the rational functions ϕ_i appearing in Theorems 2.1 and 2.2 we denote their derivative w.r.t. the j-th variable by $\phi_{i,j}$.

Theorem 2.1. *Let $\Phi : \mathbb{K}^3 \to \mathbb{K}^3$ be a birational map. If the map $\Phi^e : \mathcal{C}_f^{(1)} \mapsto \Phi \circ \mathcal{C}_f^{(1)}$ defines a map from $\mathcal{RIC}_\Phi^{(1)}$ to $\mathcal{RIC}^{(1)}$, then Φ must be of the form*

$$\Phi(u_1,u_2,u_3) = \left(u_1, \quad \frac{au_2+b}{cu_2+d}, \quad \frac{\partial}{\partial u_1}\left(\frac{au_2+b}{cu_2+d}\right) + \frac{\partial}{\partial u_2}\left(\frac{au_2+b}{cu_2+d}\right)\cdot u_3\right), \qquad (3)$$

where $a,b,c,d \in \mathbb{K}[u_1]$ such that $ad-bc \neq 0$.
Conversely, any map of the form (3) is birational and the induced map $\Phi^e : \mathcal{C}_f^{(1)} \mapsto \Phi \circ \mathcal{C}_f^{(1)}$ defines a map from $\mathcal{RIC}_\Phi^{(1)}$ to $\mathcal{RIC}^{(1)}$.

Proof. Throughout this proof, for ease of notation we will simply write $\mathcal{C}_f$ instead of $\mathcal{C}_f^{(1)}$, and $\mathcal{RIC}$ instead of $\mathcal{RIC}^{(1)}$.
Let $\Phi(\overline{u}) = (\phi_1(\overline{u}), \phi_2(\overline{u}), \phi_3(\overline{u}))$, where $\overline{u} = (u_1,u_2,u_3)$, be a birational map on $\mathbb{K}^3$. Suppose that $\phi_1(\overline{u}) = \frac{F_1(\overline{u})}{G_1(\overline{u})}$, where F_1 and G_1 are coprime polynomials in $\mathbb{K}[\overline{u}]$. F_1 and G_1 have to be such that

$$x = \frac{F_1(\mathcal{C}_f)}{G_1(\mathcal{C}_f)}$$

for $f \in \mathbb{K}(x)$ such that $\mathcal{C}_f \in \mathcal{RIC}_\Phi$. Let us consider the polynomial

$$P(\overline{u}) := u_1 G_1(\overline{u}) - F_1(\overline{u}).$$

We know that for f as above $P(x, f(x), f'(x)) = 0$. We prove that $P(\overline{u}) = 0$. Indeed, let $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{K}^3$ and consider the line $l(x) = \lambda_2 + \lambda_3(x - \lambda_1)$. Then, taking into account Lemma 2.1, for $\overline{\lambda}$ in a non-empty Zariski open subset of $\mathbb{K}^3$, $\mathcal{C}_{l(x)} \in \mathcal{RIC}_\Phi$, and so

$$P(\lambda_1, \lambda_2, \lambda_3) = P(\lambda_1, l(\lambda_1), l'(\lambda_1)) = 0.$$

Hence, $P(\overline{u}) = 0$, i.e., $\phi_1(\overline{u}) = u_1$.

Now we need to determine the form of $\phi_2(\overline{u})$ and $\phi_3(\overline{u})$ such that $\phi_2(\mathcal{C}_f)' = \phi_3(\mathcal{C}_f)$ for $f \in \mathbb{K}(x)$ such that $\mathcal{C}_f \in \mathcal{RIC}_\Phi$. This is equivalent to

$$\phi_{21}(\mathcal{C}_f) + \phi_{22}(\mathcal{C}_f)f'(x) + \phi_{23}(\mathcal{C}_f)f''(x) = \phi_3(\mathcal{C}_f) \qquad \text{for } f \in \mathbb{K}(x) \text{ such that } \mathcal{C}_f \in \mathcal{RIC}_\Phi.$$

Note that the square-free parts of the denominators of Φ_{ij} and Φ_i are equal (here we use the notation introduced just before the Theorem). Let Q be the numerator of

$$\phi_{21}(\overline{u}) + \phi_{22}(\overline{u})u_3 + \phi_{23}(\overline{u})u_4 - \phi_3(\overline{u})$$

We know that for $f \in \mathbb{K}(x)$ such that $\mathcal{C}_f \in \mathcal{RIC}_\Phi$ we have $Q(x, f(x), f'(x), f''(x)) = 0$. Let $\overline{\lambda} = (\lambda_1, \ldots, \lambda_4) \in \mathbb{K}^4$ and consider the parabola $p(x) = \lambda_2 + \lambda_3(x - \lambda_1) + \dfrac{\lambda_4}{2}(x - \lambda_1)^2$. Then, for $\overline{\lambda}$ in a non-empty Zariski open subset of $\mathbb{K}^4$, $\mathcal{C}_{p(x)} \in \mathcal{RIC}_\Phi$, and so

$$0 = Q(\lambda_1, p(\lambda_1), p'(\lambda_1), p''(\lambda_1)) = Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

Therefore, $Q(u_1, \ldots, u_4) = 0$. This implies that

$$\phi_3(\overline{u}) = \phi_{21}(\overline{u}) + \phi_{22}(\overline{u})u_3 + \phi_{23}(\overline{u})u_4.$$

Since the left hand side does not contain u_4, $\phi_{23}(\overline{u})$ must be 0. Consequently, $\phi_2(u_1, u_2, u_3)$ does not depend on the third variable. Hence, Φ has the form

$$\Phi(u_1, u_2, u_3) = (u_1,\ \phi_2(u_1, u_2),\ \phi_{21}(u_1, u_2) + \phi_{22}(u_1, u_2) \cdot u_3).$$

Now, it is enough to determine ϕ_2. We observe that ϕ_{22} is not zero, since otherwise Φ would not depend on u_3 and hence could not be birational. Let $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ be a generic point in an open subset of $\mathbb{K}^3$. Consider the system of equations

$$\begin{cases} u_1 = \lambda_1 \\ \phi_2(u_1, u_2) = \lambda_2 \\ \phi_{21}(u_1, u_2) + \phi_{22}(u_1, u_2)u_3 = \lambda_3. \end{cases} \tag{4}$$

Let $\phi_2 = \dfrac{F_2}{G_2}$ be in reduced form. Note that $F_2 \neq 0$, because Φ is birational. Consider

$$A(u_2) = F_2(\lambda_1, u_2) - \lambda_2 G_2(\lambda_1, u_2).$$

First we observe that A is of positive degree in u_2. This follows from the fact that $\dfrac{F_2}{G_2}$ does depend on u_2 (because $\phi_{22} \neq 0$). Indeed, $\deg_{u_2}(A) = 1$. If A has two different roots, the fiber will have at least 2 elements. Moreover, let us see that A does not have multiple roots for a generic $\overline{\lambda}$. The polynomial $H(u_1, u_2, w) = F_2(u_1, u_2) - wG_2(u_1, u_2)$ cannot have a multiple factor, since $\deg_w(H) = 1$ (G_2 cannot be 0) and $\gcd(F_2, G_2) = 1$. So $R(u_1, w) = \text{discriminant}_{u_2}(H) \neq 0$. If we take λ_1, λ_2 such that $R(\lambda_1, \lambda_2) \neq 0$, then A will not have multiple roots. This proves that indeed $\deg_{u_2}(A) = 1$. Therefore, $\phi_2(u_1, u_2)$ is of the form

$$\phi_2(u_1, u_2) = \frac{a(u_1)u_2 + b(u_1)}{c(u_1)u_2 + d(u_1)},$$

with $ad - bc \neq 0$.

Conversely, any map of form (3) is birational because its inverse is

$$\Phi^{-1}(u_1, u_2, u_3) = \left(u_1,\ \frac{du_2 - b}{-cu_2 + a},\ \frac{\partial}{\partial u_2}\left(\frac{du_2 - b}{-cu_2 + a}\right)u_3 + \frac{\partial}{\partial u_1}\left(\frac{du_2 - b}{-cu_2 + a}\right)\right).$$

Moreover, if $\mathcal{C}_f = (x, f(x), f'(x))$, then $\Phi(\mathcal{C}_f) = (x, g(x), g'(x))$, where $g(x) = \dfrac{a(x)f(x) + b(x)}{c(x)f(x) + d(x)}$; of course with the exceptional case in which f is a root of the denominator; namely $f = -d(x)/c(x)$. □

Now we generalize Theorem 2.1 to rational maps on higher dimensional spaces.

Theorem 2.2. *Let $\Phi = (\phi_1, \ldots, \phi_{n+2}) : \mathbb{K}^{n+2} \to \mathbb{K}^{n+2}$ be a birational map, where $n > 1$. If the map $\Phi^e : \mathcal{C}_f^{(n)} \mapsto \Phi \circ \mathcal{C}_f^{(n)}$ defines a map from $\mathcal{RIC}_\Phi^{(n)}$ to $\mathcal{RIC}^{(n)}$, then $\hat{\Phi} = (\phi_1, \ldots, \phi_{n+1}) : \mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ is birational, $\hat{\Phi}^e$ maps $\mathcal{RIC}_{\hat{\Phi}}^{(n-1)}$ to $\mathcal{RIC}^{(n-1)}$, and*

$$\phi_{n+2}(\overline{u}, u_{n+2}) \;=\; \phi_{n+1,1}(\overline{u}) + \sum_{i=2}^{n} \phi_{n+1,i}(\overline{u}) \cdot u_{i+1} + \phi_{n+1,n+1}(\overline{u}) \cdot u_{n+2}\ , \tag{5}$$

where $\overline{u} = (u_1, \ldots, u_{n+1})$.
So Φ is triangular in the sense that its i-th component depends only on the first i variables, and for $i \geq 3$ the variable u_i is introduced linearly in ϕ_i.
Conversely, any map of the form (5) extending a birational map $\hat{\Phi}$ from $\mathbb{K}^{n+1}$ to $\mathbb{K}^{n+1}$, such that $\hat{\Phi}^e$ maps $\mathcal{RIC}_{\hat{\Phi}}^{(n-1)}$ to $\mathcal{RIC}^{(n-1)}$, is birational and the induced map $\Phi^e : \mathcal{C}_f^{(n)} \mapsto \Phi \circ \mathcal{C}_f^{(n)}$ defines a map from $\mathcal{RIC}_\Phi^{(n)}$ to $\mathcal{RIC}^{(n)}$.

Proof. If $\hat{\Phi}$ would not be birational, clearly Φ could not be birational. And if $\hat{\Phi}$ would not map $\mathcal{RIC}_{\hat{\Phi}}^{(n-1)}$ to $\mathcal{RIC}^{(n-1)}$, then Φ could not map $\mathcal{RIC}_\Phi^{(n)}$ to $\mathcal{RIC}^{(n)}$.
We must have $\phi_{n+1}(\mathcal{C}_f^{(n)})' = \phi_{n+2}(\mathcal{C}_f^{(n)})$ for $f \in \mathbb{K}(x)$ such that $\mathcal{C}_f \in \mathcal{RIC}_\Phi^{(n)}$. This is equivalent to

$$\phi_{n+1,1}(\mathcal{C}_f^{(n)}) + \sum_{i=2}^{n+2} \phi_{n+1,i}(\mathcal{C}_f^{(n)}) \cdot f^{(i-1)} = \phi_{n+2}(\mathcal{C}_f^{(n)}) \quad \text{for } f \in \mathbb{K}(x) \text{ such that } \mathcal{C}_f \in \mathcal{RIC}_\Phi^{(n)}.$$

Let Q be the numerator of

$$\phi_{n+1,1}(\overline{u}) + \sum_{i=2}^{n+2} \phi_{n+1,i}(\overline{u}) \cdot u_{i+1} - \phi_{n+2}(\overline{u}, u_{n+2})$$

As in the proof on Theorem 2.1 we see that $Q(u_1, \ldots, u_{n+3}) = 0$. This implies that

$$\phi_{n+2}(\overline{u}, u_{n+2}) = \phi_{n+1,1}(\overline{u}) + \sum_{i=2}^{n+2} \phi_{n+1,i}(\overline{u}) \cdot u_{i+1}.$$

But ϕ_{n+1} does not depend on u_{n+2}, so $\phi_{n+1,n+2} = 0$. Consequently, ϕ_{n+2} is of the form (5).

Applying the process of shortening the map several times, until finally we arrive at a map on $\mathbb{K}^3$ which is covered by Theorem 2.1, we see that Φ is indeed triangular and the variables $u_3, \ldots, u_{n+2}$ are introduced linearly in the corresponding components of the map.

Conversely, any birational map $\hat{\Phi}$, such that $\hat{\Phi}^e$ maps $\mathcal{RIC}^{(n-1)}_{\hat{\Phi}^e}$ to $\mathcal{RIC}^{(n-1)}$, must be such that $\phi_{n+1,n+1} \neq 0$, since it is triangular and could not be birational otherwise. So the extension Φ of the form (5) is also triangular and birational. And Φ^e maps $\mathcal{RIC}^{(n)}_{\Phi}$ to $\mathcal{RIC}^{(n)}$.
In fact, let us determine the inverse of Φ. Φ^{-1} is also birational and $(\Phi^{-1})^e$ maps $\mathcal{RIC}^{(n)}_{(\Phi^{-1})^e}$ to $\mathcal{RIC}^{(n)}$. So it must also be of the shape (5). That means, it must be generated by a linear function in the 2nd component. If Φ is generated by the linear function

$$L(u_1, u_2) = \frac{a(u_1)u_2 + b(u_1)}{c(u_1)u_2 + d(u_1)}$$

(compare Theorem 2.1), then Φ^{-1} must be generated by the linear function

$$L^{-1}(u_1, u_2) = \frac{d(u_1)u_2 - b(u_1)}{-c(u_1)u_2 + a(u_1)} \ .$$

□

Definition 2.2. An *integral birational transformation of order* n is of the form

$$\Phi(u_1, \ldots, u_{n+2}) \ = \ \big(u_1, \phi_2(u_1, u_2), \ldots, \phi_{n+2}(u_1, \ldots, u_{n+2})\big),$$

where $\phi_2(u_1, u_2) = L(u_1, u_2)$ is an invertible linear function in $\mathbb{K}[u_1](u_2)$, i.e.,

$$L(u_1, u_2) = \frac{a(u_1)u_2 + b(u_1)}{c(u_1)u_2 + d(u_1)}, \quad \text{with} \quad a, b, c, b \in \mathbb{K}[u_1] \text{ and } ad - bc \neq 0,$$

and ϕ_r is derived from ϕ_{r-1} as in (3) and (5), for $3 \leq r \leq n+2$.
We call L the *defining function* of this integral birational transformation Φ, and we write $\Phi = \Phi_L$.
We call $\Delta = a \cdot d - b \cdot c \in \mathbb{K}[u_1]$ the *determinant* of Φ, and we write $\Delta = \Delta_\Phi$.
By

$$\mathcal{G}^{(n)} := \{\Phi_L \mid L \in \mathbb{K}[u_1](u_2) \text{ linear in } u_2 \text{ and invertible } \} \tag{6}$$

we denote the *set of integral birational transformations of order* n. •

Remark 2.1. We observe the following.

1. ¿From the proof of Theorem 2.1 and Theorem 2.2 we see that if the integral birational transformation Φ is defined by the function $L = \frac{a \cdot u_2 + b}{c \cdot u_2 + d}$, then its inverse Φ^{-1} is defined by $\frac{d \cdot u_2 - b}{-c \cdot u_2 + a}$.

2. If $\Phi \in \mathcal{G}^{(n)}$ is generated by $\frac{a \cdot u_2 + b}{c \cdot u_2 + d}$, then $\mathcal{RIC}_\Phi^{(n)} = \mathcal{RIC}^{(n)}$ if $c = 0$, and $\mathcal{RIC}_\Phi^{(n)} = \mathcal{RIC}^{(n)} \setminus \{\mathcal{C}_{-d/c}^{(n)}\}$ if $c \neq 0$.

3. The theorem may also be applied to differentiable functions instead of rational functions, in case $\mathbb{K}$ is a differential field; i.e., $\mathcal{RIC}$ may be replaced by the set

$$\{(x, f(x), f'(x), \ldots, f^{(n)}(x)) \mid f \text{ is a differentiable function in } x\}.$$

Proposition 2.1. *For a positive integer n, the set of integral birational transformations of order n, $\mathcal{G}^{(n)}$, (cf. (6)) is a subgroup (under composition) of the group of birational transformations of the space $\mathbb{K}^{n+2}$.*

Proof. Since the identity map belongs to $\mathcal{G}^{(n)}$, we have $\mathcal{G}^{(n)} \neq \emptyset$. As we have seen above (Remark 2.1), with Φ the set $\mathcal{G}^{(n)}$ also contains Φ^{-1}.
Let Φ_1, Φ_2 be in $\mathcal{G}^{(n)}$ and let $L_1(u_1, u_2), L_2(v_1, v_2)$ be their defining functions, respectively. Then the composition $\Phi_1 \circ \Phi_2$ is defined by the linear function $L_1(v_1, L_2(v_1, v_2))$, with the determinant being the product of the determinants of L_1 and L_2.
So $\mathcal{G}^{(n)}$ is a group under composition. □

Definition 2.3. We call $\mathcal{G}^{(n)}$ the *group of integral birational transformations of order n.* •

Remark 2.2. In our previous paper [NSW12], we have studied the affine case for order 1, i.e., the case in which $\Phi(u_1, u_2, u_3) := \Phi_{\alpha u_2 + \beta u_1 + \gamma} = (u_1, \alpha u_2 + \beta u_1 + \gamma, \alpha u_3 + \beta)$, where $\alpha, \beta, \gamma \in \mathbb{K}$, $\alpha \neq 0$. The set of all such affine transformations forms a subgroup of the group $\mathcal{G}^{(1)}$ of integral birational transformations on $\mathbb{K}^3$. Now, using Theorem 2.2, the linear subgroup in [NSW12] can be generalized to the n-order case. This leads to the subgroup of $\mathcal{G}^{(n)}$ $(n > 3)$ formed by all integral transformations of $\mathbb{K}^{n+2}$ of the form $\Phi(u_1, \ldots, u_{n+2}) := \Phi_{\alpha u_2 + \beta u_1 + \gamma} = (u_1, \alpha u_2 + \beta u_1 + \gamma, \alpha u_3 + \beta, \alpha u_4, \ldots, \alpha u_{n+2})$; note that $\Delta_\Phi = \alpha$.

Besides the linear subgroups mentioned in Remark 2.2, we introduce three additional subgroups. Consider the following subsets of $\mathcal{G}^{(n)}$:

$$\mathcal{G}_{inv}^{(n)} = \left\{\Phi_{u_2}, \Phi_{1/u_2}\right\}, \mathcal{G}_{mult}^{(n)} = \left\{\Phi_{a(u_1)u_2} \mid a \in \mathbb{K}(u_1), a \neq 0\right\}, \mathcal{G}_{plus}^{(n)} = \left\{\Phi_{u_2 + b(u_1)} \mid b \in \mathbb{K}(u_1)\right\}.$$

Note that $\mathcal{G}_{inv}^{(n)}$ only consists of two elements, while $\mathcal{G}_{mult}^{(n)}$ and $\mathcal{G}_{plus}^{(n)}$ are infinite sets.

Proposition 2.2. *$\mathcal{G}_{inv}^{(n)}$, $\mathcal{G}_{mult}^{(n)}$ and $\mathcal{G}_{plus}^{(n)}$ are subgroups of $\mathcal{G}^{(n)}$.*

Proof. The identity transformation is in $\mathcal{G}_{inv}^{(n)}$, $\mathcal{G}_{mult}^{(n)}$ and $\mathcal{G}_{plus}^{(n)}$.
Clearly, $\mathcal{G}_{inv}^{(n)}$ is a subgroup of $\mathcal{G}^{(n)}$, since the non-identity element in $\mathcal{G}_{inv}^{(n)}$ is its own inverse.
Let $\Phi_1, \Phi_2 \in \mathcal{G}_{mult}^{(n)}$, with defining functions $a_1(u_1)u_2$ and $a_2(u_1)u_2$, respectively. Then $\Phi_1 \circ \Phi_2^{-1}$ is in $\mathcal{G}_{mult}^{(n)}$ with defining function $\frac{a_1(u_1)}{a_2(u_1)} u_2$.

Let $\Phi_1, \Phi_2 \in \mathcal{G}^{(n)}_{plus}$, with defining functions $u_2 + b_1(u_1)$ and $u_2 + b_2(u_1)$, respectively. Then $\Phi_1 \circ \Phi_2^{-1}$ is in $\mathcal{G}^{(n)}_{plus}$ with defining function $u_2 + b_1(u_1) - b_2(u_1)$.
Therefore, $\mathcal{G}^{(n)}_{inv}$, $\mathcal{G}^{(n)}_{mult}$ and $\mathcal{G}^{(n)}_{plus}$ are subgroups of $\mathcal{G}^{(n)}$. $\square$

Proposition 2.3. *Every $\Phi \in \mathcal{G}^{(n)}$ can be decomposed into a product of elements in $\mathcal{G}^{(n)}_{inv}$, $\mathcal{G}^{(n)}_{mult}$ and $\mathcal{G}^{(n)}_{plus}$.*

Proof. Let Φ be an arbitrary element in $\mathcal{G}^{(n)}$, defined by the function

$$L(u_1, u_2) = \frac{a(u_1)u_2 + b(u_1)}{c(u_1)u_2 + d(u_1)} \ .$$

1. If $c = 0$, then we can write
$$L(u_1, u_2) = \frac{a}{d}u_2 + \frac{b}{d} \ .$$
So $\Phi = \Phi_2 \circ \Phi_1$, where $\Phi_1 \in \mathcal{G}^{(n)}_{mult}$ is defined by $\frac{a}{d}u_2$, and $\Phi_2 \in \mathcal{G}^{(n)}_{plus}$ is defined by $u_2 + \frac{b}{d}$.

2. If $c \neq 0$, then we can write
$$L(u_1, u_2) = \frac{a}{c} + \frac{bc - ad}{c^2 \left(u_2 + \frac{d}{c}\right)} \ .$$
So $\Phi = \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1$, where $\Phi_1 \in \mathcal{G}^{(n)}_{plus}$ is defined by $u_2 + \frac{d}{c}$, $\Phi_2 \in \mathcal{G}^{(n)}_{inv}$ is defined by $\frac{1}{u_2}$, $\Phi_3 \in \mathcal{G}^{(n)}_{mult}$ is defined by $\frac{bc-ad}{c^2}u_2$, and $\Phi_4 \in \mathcal{G}^{(n)}_{plus}$ is defined by $u_2 + \frac{a}{c}$.

Therefore, every element in $\mathcal{G}^{(n)}$ can be decomposed into a product of elements in those three subgroups. $\square$

3 The transformation of AODEs

In this section, we study the action of the group $\mathcal{G}^{(n)}$ on the set of all algebraic ordinary differential equations. We start with the following definition

Definition 3.1. Let $F(u, v, w_1, \ldots, w_n) \in \mathbb{K}[u, v, w_1, \ldots, w_n]$ be such that $\deg_{w_n}(F) \geq 1$. The *algebraic ordinary differential equation (AODE) of order* n (over $\mathbb{K}$) defined by F is of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0 ,$$

where y is an indeterminate over a differential extension field of $\mathbb{K}(x)$. By $'$ we denote differentiation w.r.t. x.
Let $\mathcal{AODE}^{(n)}$ be the set of all algebraic ODEs of order k over $\mathbb{K}$, where $1 \leq k \leq n$. •

If the defining polynomial of an AODE can be factored, then the set of solutions is clearly the union of the sets of solutions of the AODEs defined by the factors. So throughout this paper we will assume that the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$ is given by an irreducible polynomial F. In the following, for a given polynomial H, $\mathcal{V}(H)$ denotes the algebraic variety defined by H over $\mathbb{K}$.

Proposition 3.1. *For every $F \in \mathcal{AODE}^{(n)}$ and for every $\Phi \in \mathcal{G}^{(n)}$ there exists a non-empty Zariski dense open subset $\Omega \subset \mathcal{V}(F(u, v, w_1, \ldots, w_n)) \subset \mathbb{K}^{n+2}$ such that Φ is defined on Ω.*

Proof. The denominator of each rational component of Φ is either a constant or a power of a polynomial in $\mathbb{K}[u, v]$, and F does depend on at least one variable w_i. Therefore, since F is irreducible, no denominator of Φ vanishes on $\mathcal{V}(F)$. □

Definition 3.2. Let the irreducible polynomial F define a hypersurface $\mathcal{V}(F)$ in $\mathbb{K}^{n+2}$, and let $\Phi \in \mathcal{G}^{(n)}$ be an integral birational transformation on $\mathbb{K}^{n+2}$. Let W be the irreducible image variety of $\mathcal{V}(F)$ under Φ, i.e., the Zariski closure $\Phi(\mathcal{V}(F))^*$ of the set theoretic image. Let G be the defining polynomial of the hypersurface W; i.e. $W = \mathcal{V}(G)$. Then, we say that Φ transforms the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$ into the AODE $G(x, y, y', \ldots, y^{(n)}) = 0$. We denote this relation by $G = \Phi \cdot F$. •

So we get the following action of $\mathcal{G}^{(n)}$ on $\mathcal{AODE}^{(n)}$:

$$\begin{array}{rcl} \mathcal{G}^{(n)} \times \mathcal{AODE}^{(n)} & \longrightarrow & \mathcal{AODE}^{(n)} \\ (\Phi, F) & \longmapsto & \Phi \cdot F = F(\Phi^{-1}(x, y, y', \ldots, y^{(n)})) , \end{array}$$

This group action induces an equivalence relation in $\mathcal{AODE}^{(n)}$, say $\sim_{\mathcal{G}^{(n)}}$, and hence provides a quotient set

$$\overline{\mathcal{AODE}^{(n)}} := \mathcal{AODE}/ \sim_{\mathcal{G}^{(n)}}$$

such that if $F, G \in \mathcal{AODE}^{(n)}$ then $F \sim_{\mathcal{G}^{(n)}} G$ if and only if there exists $\Phi \in \mathcal{G}^{(n)}$ such that $\Phi \cdot F = G$. We denote the equivalence class of $F \in \mathcal{AODE}^{(n)}$ as $\overline{F}$.

Let $F \in \mathcal{AODE}^{(n)}$, $G \in \overline{F}$ and $\Phi \in \mathcal{G}^{(n)}$ such that $\Phi \cdot F = G$. Let us assume that the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$ of order n has rational solutions (a similar reasoning can be done for other types of solutions as those introduced in Remark 2.1 (3)). Because of

Remark 2.1, for every rational solution $y = f(x)$, with maybe one exception, $\mathcal{C}_f^{(n)} \in \mathcal{RIC}_\Phi^{(n)}$. Therefore, by Theorems 2.1 and 2.2, there exists a unique $g(x) \in \mathbb{K}(x)$ such that $\Phi^e(\mathcal{C}_f^{(n)}) = \mathcal{C}_g^{(n)}$. So $\Phi(x, f(x), f'(x), \ldots, f^{(n)}(x)) = (x, g(x), g'(x), \ldots, g^{(n)}(x))$. Thus,

$$G(x, g, g', \ldots, g^{(n)}) = F(\Phi^{-1}(\Phi(x, f, f', \ldots, f^{(n)}))) = F(x, f, f', \ldots, f^{(n)}) = 0.$$

We have deduced the following theorem.

Theorem 3.1. *The existence of rational solutions is an invariant property for the elements in each equivalence class of $\overline{\mathcal{AODE}^{(n)}}$. Furthermore, if $y = f(x)$ is a general rational solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ and $\Phi \cdot F = G$, then $y = g(x)$, where $\Phi^e(\mathcal{C}_f^{(n)}) = \mathcal{C}_g^{(n)}$, is a general rational solution of $G(x, y, y', \ldots, y^{(n)}) = 0$.*

Definition 3.3. Let $F(u, v, w_1, \ldots, w_n)$ be the irreducible polynomial defining the AODE

$$F(x, y, y', \ldots, y^{(n)}) = 0.$$

We say that $F(x, y, y', \ldots, y^{(n)}) = 0$ is a *parametric ordinary differential equation* (PODE) if the hypersurface defined by $F(u, v, w_1, \ldots, w_n)$ is rational; i.e. it can be rationally and properly parametrized over $\mathbb{K}$.

Let $\mathcal{PODE}^{(n)}$ be the set of all PODEs in $\mathcal{AODE}^{(n)}$. •

Remark 3.1. We observe that there exists an important difference between the cases $n = 1$ and $n > 1$. For $n = 1$ (i.e. surfaces in $\mathbb{K}^3$), Castelnouvo's Theorem (see e.g. [Sha96]) ensures that every surface rationally parametrized can be properly and rationally parametrized (note that $\mathbb{K}$ is algebraically closed of characteristic zero). However, for $n > 1$, i.e. for hypersurface in $\mathbb{K}^{n+2}$, the equivalence is not true in general. For the case $n = 1$, we refer to [NSW12].

Clearly, $\mathcal{PODE}^{(n)} \subset \mathcal{AODE}^{(n)}$. Moreover, since the elements in $\mathcal{G}^{(n)}$ are birational transformations, the rationality of the associated algebraic hypersurface is preserved when applying to the differential equation and element in $\mathcal{G}^{(n)}$. Furthermore, if $\mathcal{P}(t_1, \ldots, t_{n+1})$ is a proper rational parametrization of $F \in \mathcal{PODE}^{(n)}$ and $\Phi \in \mathcal{G}^{(n)}$ then $\Phi(\mathcal{P}(t_1, \ldots, t_{n+1}))$ is a proper rational parametrization of $\Phi \cdot F$. So, $\Phi \cdot F \in \mathcal{PODE}^{(n)}$. Therefore, $\mathcal{G}^{(n)}$ also acts on $\mathcal{PODE}^{(n)}$. Similarly, we use the notation $\overline{\mathcal{PODE}^{(n)}}$ and $\overline{F}$ for $F \in \mathcal{PODE}^{(n)}$.

Let us study the equivalence classes in $\overline{\mathcal{PODE}^{(n)}}$. From [NW10], [NW11], [HNW12] we know that every element in $\mathcal{PODE}^{(n)}$ is associated to a system of autonomous ODEs in the parameters. More precisely, let $\mathcal{P}(\bar{t}) = (\chi_1(\bar{t}), \chi_2(\bar{t}), \ldots, \chi_{n+2}(\bar{t}))$ be a proper rational parametrization of the solution hypersurface $F(u, v, w_1, \ldots, w_n) = 0$, where we assume that the Jacobian g of $(\chi_1, \chi_2, \ldots, \chi_{n+1})$ is regular; see below the role of g. Then the associated system to $F(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\mathcal{P}$ is

$$\left\{ t_1' = \frac{f_1}{g},\ t_2' = \frac{f_2}{g}, \ldots, t_{n+1}' = \frac{f_{n+1}}{g} \right\}$$

where

$$f_1 = \begin{vmatrix} 1 & \chi_{12} & \cdots & \chi_{1,n+1} \\ \chi_3 & \chi_{22} & \cdots & \chi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{n+2} & \chi_{n+1,2} & \cdots & \chi_{n+1,n+1} \end{vmatrix}, \quad f_2 = \begin{vmatrix} \chi_{11} & 1 & \cdots & \chi_{1,n+1} \\ \chi_{21} & \chi_3 & \cdots & \chi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{n+1,1} & \chi_{n+2} & \cdots & \chi_{n+1,n+1} \end{vmatrix}, \ldots,$$

$$f_{n+1} = \begin{vmatrix} \chi_{11} & \chi_{12} & \cdots & 1 \\ \chi_{21} & \chi_{22} & \cdots & \chi_3 \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{n+1,1} & \chi_{n+1,2} & \cdots & \chi_{n+2} \end{vmatrix}, \quad g = \begin{vmatrix} \chi_{11} & \chi_{12} & \cdots & \chi_{1,n+1} \\ \chi_{21} & \chi_{22} & \cdots & \chi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{n+1,1} & \chi_{n+1,2} & \cdots & \chi_{n+1,n+1} \end{vmatrix}.$$

We now prove that the associated system is invariant under this group action.

Theorem 3.2. *Let $F \in \mathcal{PODE}^{(n)}$ and $\Phi \in \mathcal{G}^{(n)}$. Let $\mathcal{P}(\bar{t}) = (\chi_1(\bar{t}), \chi_2(\bar{t}), \ldots, \chi_{n+2}(\bar{t}))$ be a proper rational parametrization of the solution surface $F(u, v, w_1, \ldots, w_n) = 0$ with $\det(J(\chi_1, \chi_2, \ldots, \chi_{n+1})) \neq 0$. Then the associated system of $F(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\mathcal{P}$ and the associated system of $(\Phi \cdot F)(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\Phi \circ \mathcal{P}$ are equal.*

Proof. The associated system of $F(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t $\mathcal{P}(\bar{t})$ is

$$\left\{ t_1' = \frac{f_1}{g},\ t_2' = \frac{f_2}{g}, \ldots, t_{n+1}' = \frac{f_{n+1}}{g} \right\}$$

where f_i and g are as above. We have

$$(\Phi \circ \mathcal{P})(\bar{t}) = \left(\chi_1,\ \phi_2(\chi_1, \chi_2),\ \phi_{21} + \phi_{22}\chi_3, \ldots, \phi_{n+1,1} + \sum_{i=2}^{n+1} \phi_{n+1,i}\chi_{i+1} \right).$$

Moreover, $(\Phi \circ \mathcal{P})(\bar{t})$ is a proper rational parametrization of the hypersurface $\Phi \cdot F = 0$. Therefore, the associated system of $(\Phi \cdot F)(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $(\Phi \circ \mathcal{P})$ is

$$\left\{ t_1' = \frac{\tilde{f}_1}{\tilde{g}},\ t_2' = \frac{\tilde{f}_2}{\tilde{g}}, \ldots,\ t_{n+1}' = \frac{\tilde{f}_{n+1}}{\tilde{g}} \right\}$$

where

$$\begin{aligned} \tilde{f}_1 = & \begin{vmatrix} 1 & \chi_{12} & \cdots & \chi_{1,n+1} \\ \phi_{21} + \phi_{22}\chi_3 & \phi_{21}\chi_{12} + \phi_{22}\chi_{22} & \cdots & \phi_{21}\chi_{1,n+1} + \phi_{22}\chi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n+1,1} + \sum_{i=2}^{n+1} \phi_{n+1,i}\chi_{i+1} & \sum_{i=1}^{n+1} \phi_{n+1,i}\chi_{i,2} & \cdots & \sum_{i=1}^{n+1} \phi_{n+1,i}\chi_{i,n+1} \end{vmatrix} \\ = & \phi_{22}\phi_{33} \ldots \phi_{n+1,n+1} f_1, \end{aligned}$$

and

$$\begin{aligned} \tilde{g} = & \begin{vmatrix} \chi_{11} & \chi_{12} & \cdots & \chi_{1,n+1} \\ \phi_{21}\chi_{11} + \phi_{22}\chi_{2,1} & \phi_{21}\chi_{12} + \phi_{22}\chi_{22} & \cdots & \phi_{21}\chi_{1,n+1} + \phi_{22}\phi_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{n+1} \phi_{n+1,i}\chi_{i,1} & \sum_{i=1}^{n+1} \phi_{n+1,i}\chi_{i,2} & \cdots & \sum_{i=1}^{n+1} \phi_{n+1,i}\chi_{i,n+1} \end{vmatrix} \\ = & \phi_{22}\phi_{33} \ldots \phi_{n+1,n+1} g. \end{aligned}$$

Similarly, we can express $\tilde{f}_i$ in terms of f_i with the same factor $\phi_{22}\phi_{33} \ldots \phi_{n+1,n+1}$. Note that

$$\phi_{22}\phi_{33} \ldots \phi_{n+1,n+1} \neq 0,$$

which implies that the jacobian condition satisfies also for $\Phi \circ \mathcal{P}$. It implies that the associated system of $F(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\mathcal{P}$ and the associated system of $(\Phi \cdot F)(x, y, y', \ldots, y^{(n)}) = 0$ w.r.t. $\Phi \circ \mathcal{P}$ are equal. $\square$

In the last part of this section, let us take a closer look at $G = \Phi \cdot F$, where $F \in \mathcal{PODE}^{(n)}$ and $\Phi \in \mathcal{G}^{(n)}$. It is clear, that G can be easily computed by means of elimination techniques as Gröbner basis, etc. Nevertheless, for further theoretical reasons, we want to have a precise description of G. For this purpose, we start with the following lemma.

Lemma 3.1. *Let Φ be a birational map from $\mathbb{K}^m$ on $\mathbb{K}^m$, and Φ^{-1} its inverse. Let $V = \mathcal{V}(P)$, with $P \in \mathbb{K}[\overline{x}]$ irreducible and non-constant such that Φ is defined on a non-empty Zariski open subset of V. Let W be the irreducible image variety of V under Φ, i.e., the Zariski closure $\Phi(V)^*$ of the set theoretic image. Then Φ^{-1} is defined on*

$$\left(\mathcal{V}(A) \setminus \mathcal{V}\left(\prod_{i=1}^{m} C_i \right) \right) ,$$

where A is the numerator of $P(\Phi^{-1}(\overline{x}))$ and C_i is the numerator of $N_i(\Phi^{-1}(\overline{x}))$, $N_i(\overline{x})$ being the denominators of the rational functions defining $\Phi(\overline{x})$. Moreover,

$$\left(\mathcal{V}(A) \setminus \mathcal{V}\left(\prod_{i=1}^{m} C_i \right) \right)^* = W \ .$$

Proof. Let Φ and Φ^{-1} be written as

$$\Phi(\overline{x}) = \left(\frac{M_1(\overline{x})}{N_1(\overline{x})}, \frac{M_2(\overline{x})}{N_2(\overline{x})}, \frac{M_3(\overline{x})}{N_3(\overline{x})} \right), \ \Phi^{-1}(\overline{x'}) = \left(\frac{M'_1(\overline{x'})}{N'_1(\overline{x'})}, \frac{M'_2(\overline{x'})}{N'_2(\overline{x'})}, \frac{M'_3(\overline{x'})}{N'_3(\overline{x'})} \right) \ .$$

We introduce the algebraic set

$$\mathcal{B} = \left\{ (\overline{x}, \overline{x'}, z) \in \mathbb{K}^m \times \mathbb{K}^m \times \mathbb{K} \ \middle| \ \begin{array}{l} P(\overline{x}) = 0 \\ x'_i N_i(\overline{x}) = M_i(\overline{x}), \ i = 1, 2, 3 \\ z K(\overline{x}, \overline{x'}) = 1 \end{array} \right\}$$

where $K(\overline{x}, \overline{x'}) = N_1(\overline{x}) N_2(\overline{x}) N_3(\overline{x}) N'_1(\overline{x'}) N'_2(\overline{x'}) N'_3(\overline{x'})$. Also, we consider the projection $\pi_{\overline{x'}} : \mathbb{K}^m \times \mathbb{K}^m \times \mathbb{K} \to \mathbb{K}^m$ such that $(\overline{x}, \overline{x'}, z) \mapsto \overline{x'}$.

Let us see that $W = \pi_{\overline{x'}}(\mathcal{B})^*$. Indeed, let Ω be the non-empty Zarsiki open subset of $\Phi(V)$ where Φ^{-1} is defined; note that, since W is irreducible because V is, then Ω is dense in W. Let $q \in \Omega$. Then there exists $p \in V$ such that $\Phi(p) = q$. Thus, $(p, q, 1/K(p, q)) \in \mathcal{B}$. So, $q \in \pi_{\overline{x'}}(\mathcal{B})$. Therefore, $\Omega \subset \pi_{\overline{x'}}(\mathcal{B})$, and hence $W = \Omega^* \subset \pi_{\overline{x'}}(\mathcal{B})^*$. Conversely, Let $q \in \pi_{\overline{x'}}(\mathcal{B})$ then there exist $p \in \mathbb{K}^n$ and $\alpha \in \mathbb{K}$ such that $(p, q, \alpha) \in \mathcal{B}$, from where one deduces that $p \in V$, $q = \Phi(p)$. So, $q \in \Phi(V)$. Therefore, $\pi_{\overline{x'}}(\mathcal{B}) \subset \Phi(V)$ and hence $\pi_{\overline{x'}}(\mathcal{B})^* \subset W$.

Finally, we prove that $\pi_{\overline{x'}}(\mathcal{B})^* = (\mathcal{V}(A) \setminus \mathcal{V}(\prod_{i=1}^{m} C_i))^*$. Indeed, let $q \in \pi_{\overline{x'}}(\mathcal{B})$. Then, there exits $p \in \mathbb{K}^m, \alpha \in \mathbb{K}$ such that $(p, q, \alpha) \in \mathcal{B}$. Thus, $p \in V$, $K(p, q) \neq 0$, from here $q = \Phi(p)$ and $\Phi^{-1}(q)$ is well defined. Moreover, $0 \neq N_i(p) = N_i(\Phi^{-1}(q))$. So, $q \notin \mathcal{V}(\prod_{i=1}^{m} C_i)$. On the other hand, $P(\Phi^{-1}(q)) = P(p) = 0$. Thus, $q \in \mathcal{V}(A)$. Therefore, taking closures, we get that $\pi_{\overline{x'}}(\mathcal{B})^* \subset (\mathcal{V}(A) \setminus \mathcal{V}(\prod_{i=1}^{m} C_i))^*$. Conversely, let B be the denominator of $P(\phi(\overline{x'}))$; observe that B is a power of the $\mathrm{lcm}(N'_1, \ldots, N'_m)$. In $\mathcal{V}(A) \setminus \mathcal{V}(\prod_{i=1}^{m} C_i)$ we consider the open set $\Sigma = \mathcal{V}(A) \setminus (\mathcal{V}(\prod_{i=1}^{m} C_i) \cup \mathcal{V}(B))$. Since $\gcd(A, B) = 1$, $\Sigma \neq \emptyset$ and

$\Sigma^* = (\mathcal{V}(A) \setminus \mathcal{V}(\prod_{i=1}^m C_i))^*$. Let $q \in \Sigma$. Since $B(q) \neq 0$, then $\Phi^{-1}(q)$ is well defined, say $p = \Phi^{-1}(q)$. So, since $q \in \mathcal{V}(A)$, $F(p) = F(\Phi^{-1}(q)) = A(q)/B(q) = 0$. Furthermore, since $q \notin \mathcal{V}(\prod_{i=1}^m C_i)$, then $0 \neq N_i(\Phi^{-1}(q)) = N_i(p)$. So, $N_i(p) \neq 0$ and $N_i'(q) \neq 0$. Therefore there exists α such that $(p, q, \alpha) \in \mathcal{B}$. Thus $Q \in \pi_{\overline{x'}}(\mathcal{B})$. Now taking closure we get the other inclusion. □

Theorem 3.3. *Let $F(u, v, w_1, \ldots, w_n)$ be the irreducible polynomial of an AODE of order n, and let $\Phi \in \mathcal{G}^{(n)}$. Then the transformed AODE is defined by the primitive part w.r.t. $\{v, w_1, \ldots, w_n\}$ of the numerator $A(u, v, w_1, \ldots, w_n)$ of the rational function $F(\Phi^{-1}(u, v, w_1, \ldots, w_n))$. Moreover, the content of A w.r.t. $\{v, w_1, \ldots, w_n\}$ is of the form Δ_Φ^r, for some non-negative integer number r.*

Proof. Let $\Phi := \Phi_L$ when $L = L_1/L_2$ and $L_1 = a(u)v + b(u), L_2 = c(u)v + d(v)$. Let $G = \Phi \cdot F$. Because of Proposition 3.1, and taking into account that $\mathcal{G}^{(n)}$ is a group, we have that $\mathcal{V}(F)$ and $\Phi(\mathcal{V}(F))^* = \mathcal{V}(G)$ satisfy the conditions of Lemma 3.1. So, we analyze for our particular case the value of C_i (with the terminology of Lemma 3.1). By Theorem 2.1 and Theorem 2.2, we know that the denominators N_i in Φ are either in $\mathbb{K}[u]$ or a power of L_2. On the other hand, $\Phi^{-1} = \Phi_{L^{-1}}$ and $L^{-1} = (vd - b)/(-cv + a)$. So, the numerator of $N_i(\Phi^{-1})$ is either in $\mathbb{K}[u]$ or Δ_Φ; so, in both cases, a polynomial in $\mathbb{K}[u]$. Therefore, G is the irreducible factor of A not depending only on u. Thus, $G = \mathrm{PP}_{\{v, w_1, \ldots, w_n\}}(A)$ and its content is Δ_Φ^r with $r \geq 0$. □

4 Application to autonomous AODEs

The AODE $F(x, y, y', \dots, y^{(n)}) = 0$ is called *autonomous* iff $F \in \mathbb{K}[y, y', \dots, y^{(n)}]$, i.e., the defining polynomial F does not depend on the first variable, namely u; or, more geometrically, if the associated hypersurface is a cylinder over the coordinate plane $u = 0$. We consider the following natural question:

> *Given* $G(x, y, y', \dots, y^{(n)}) \in \mathcal{AODE}^{(n)}$ *does there exist* $F \in \overline{G}$ *such that* $F(x, y, y', \dots, y^{(n)}) = 0$ *is autonomous? And, if so, can we compute* F*?*

This motivates the following definition.

Definition 4.1. An n-order AODE $G(x, y, y', \dots, y^{(n)}) = 0$ is said to be *quasi autonomous* iff there exists $F \in \overline{G}$ such that F is autonomous; that is, there exists an element $\Phi \in \mathcal{G}^{(n)}$ and an autonomous AODE $F(x, y, y', \dots, y^{(n)})$ such that $\Phi \cdot G(u, v, w_1, \dots, w_n) = F(v, w_1, \dots, w_n)$. •

Example 4.1. Consider the first order algebraic ODE

$$G(x, y, y') = 25x^2y'^2 - 50xyy' + 25y^2 + 12y^4 - 76xy^3 + 168x^2y^2 - 144x^3y + 32x^4 = 0.$$

Using the transformation

$$\Phi(u, v, w) = \left(u, \frac{3v - u}{-v + 2u}, \frac{5u}{(-v + 2u)^2}w + \frac{-5v}{(-v + 2u)^2}\right) \in \mathcal{G}^{(1)}$$

we get that $\Phi \cdot G$ is the primitive part w.r.t. $\{v, w\}$ of the numerator of $G(\Phi^{-1}(u, v, w))$; i.e., $\Phi \cdot G = w^2 - 4v$. Therefore, $G(x, y, y') = 0$ is transformed into the autonomous algebraic ODE $F(x, y, y') = y'^2 - 4y = 0$.

Thus, $G(x, y, y') = 0$ is a quasi autonomous AODE. In addition, we observe that G cannot be transformed into an autonomous AODE by affine transformations as considered in [NSW12].

Since Φ is birational, the rational general solution of $G(x, y, y') = 0$ is transformed into the rational general solution of $F(x, y, y') = 0$ and vice versa. It is clear that $y = (x + c)^2$ is the rational general solution of $y'^2 - 4y = 0$. Therefore,

$$y = \frac{x(2(x + c)^2 + 1)}{(x + c)^2 + 3},$$

where c is any constant, is the rational general solution of $G(x, y, y') = 0$. •

In the following, we analyze the particular case of n-order AODEs $G(x, y, y', \dots, y^{(n)}) = 0$, where $\mathrm{LC}_{w_n}(G(u, v, w_1, \dots, w_n)) \in \mathbb{K}[u, v]$.

Definition 4.2. An AODE $G(x, y, y', \dots, y^{(n)}) = 0$ is *normal* iff $\mathrm{LC}_{w_n}(G(u, v, w_1, \dots, w_n)) \in \mathbb{K}[u, v]$. •

Remark 4.1. Observe that every first order AODE is normal.

Before starting our analysis, let us study the leading coefficient, w.r.t. w_n, of $\Phi \cdot G$ where $\Phi \in \mathcal{G}^{(n)}$ and $G(u, v, w_1, \ldots, w_n)$ is an irreducible polynomial over $\mathbb{K}$ defining a normal AODE. Let us say that $\Phi^{-1} = (u, \phi_2, \ldots, \phi_{n+2}) := \Phi_{L^{-1}}$ is generated by the linear invertible rational function L^{-1}, and let Δ be the determinant of Φ^{-1} (see Def. 2.2); let $L^{-1} = \frac{L_1}{L_2}$ with $L_1 = a(u)v + b(u)$ and $L_2 = c(u)v + d(u)$. In addition let us express G as

$$G(u, v, w_1, \ldots, w_n) = \sum_{j=0}^{m} A_j(u, v, w_1, \ldots, w_{n-1}) w_n^j,$$

where $m := \deg_{w_n}(G) > 0$, $A_j \in \mathbb{K}[u, v][w_1, \ldots, w_{n-1}]$ for all $j = 0, \ldots, m-1$. Note that, by assumption, $A_m \in \mathbb{K}[u, v] \setminus \{0\}$. Then we have

$$\Phi \cdot G = G(\Phi^{-1}(u, v, w_1, \ldots, w_n)) = A_m(u, \phi_2(u, v))\phi_{n+2}^m + \sum_{j=0}^{m-1} A_j\left(u, \phi_2, \ldots, \phi_{n+1}\right)\phi_{n+2}^j.$$

We observe that (see Theorems 2.1 and 2.2) $\phi_2 \in \mathbb{K}[u, v]$ and that, for $i > 2, \phi_i \in \mathbb{K}[u, v, w_1, \ldots, w_{i-2}]$. Furthermore,

$$\begin{array}{rcll} \phi_2 & = & L^{-1} & \\ \phi_i & = & \phi_{i-1,1} + \phi_{i-1,2}w_1 + \phi_{i-1,3}w_2 + \cdots + \phi_{i-1,i-2}w_{i-3} + \phi_{i-1,i-1}w_{i-2} & \\ & = & \phi_{i-1,1} + \phi_{i-1,2}w_1 + \phi_{i-1,3}w_2 + \cdots + \phi_{i-1,i-2}w_{i-3} + \Delta w_{i-2} & \text{if } i > 2 \end{array}$$

Now, let $H(u, v, w)$ be the numerator of $G(\Phi^{-1}(u, v, w_1, \ldots, w_n))$. Note that $\deg_{w_n}(H) \leq m$. Moreover, there exists a non-negative integer ρ such that the coefficient of H w.r.t. w_n^m is

$$\Delta(u)^m L_2(u, v)^\rho \text{numer}\left(A_m\left(u, L^{-1}(u, v)\right)\right). \tag{7}$$

In this situation, we observe that $\Delta(u)^m L_2(u, v)^\rho$ is not zero, and since $A_m(u, v)$ is not zero and $(u, L^{-1}(u, v))$ defines a birational transformation from $\mathbb{K}^2$ to $\mathbb{K}^2$, then $\text{numer}\left(A_m\left(L^{-1}(u, v)\right)\right)$ is not zero either. So, using Theorem 3.3 and (7), we get the following lemma.

Lemma 4.1. *Let $G \in \mathcal{AODE}^{(n)}$ be of order n and normal. Then all elements in $\overline{G}$ are of order n and normal. Moreover, if G is expressed as above, there exist non-negative integers s and ρ such that*

$$\text{LC}_{w_n}(\Phi_L \cdot G) = \Delta(u)^s L_2(u, v)^\rho \text{numer}\left(A_m\left(u, L^{-1}(u, v)\right)\right),$$

where $\Phi_L \in \mathcal{G}^{(n)}$, and L^{-1} is expressed as $L^{-1} = \frac{L_1}{L_2}$.

Now, we come back to the original question of deciding the existence of an autonomous element in the equivalence class of an n-order normal element in $\mathcal{AODE}^{(n)}$. For this purpose, let us assume that $G \in \mathcal{AODE}^{(n)}$ is normal and quasi autonomous, and let $\Phi_L \in \mathcal{G}^{(n)}$ be such that $F = \Phi_L \cdot G$ is autonomous. Furthermore, assume that $L = L_1/L_2$ with $L_1 = a(u)v + b(u)$, $L_2 = c(u)v + d(u)$. Let F be expressed as

$$F(v, w_1, \ldots, w_n) = A_m(v)w_n^m + A_{m-1}(v, w_1, \ldots, w_{n-1})w_n^{m-1} + \cdots + A_0(v, w_1, \ldots, w_{n-1}),$$

where $m = \deg_{w_n}(F) > 0$, $A_j \in \mathbb{K}[v, w_1, \ldots, w_{n-1}]$ for $j < m$, and $A_m \in \mathbb{K}[v] \setminus \{0\}$. Since $\mathbb{K}$ is an algebraically closed field, we can distinguish the following three cases depending on A_m:

($\mathcal{A}_1$) $A_m(v) = a_{n_m} \in \mathbb{K} \setminus \{0\}$;

($\mathcal{A}_2$) $A_m(v) = a_{n_m}(v-\alpha)^{n_m}$, for some $n_m > 0$ and $\alpha \in \mathbb{K}$;

($\mathcal{A}_3$) $A_m(v) = a_{n_m}\prod_{i=1}^{n_m}(v-\alpha_i)$, for some $n_m > 1$, $\alpha_i \in \mathbb{K}$ for all $i = 1, \ldots, n_m$ and $\alpha_1 \neq \alpha_2$.

Since $F = \Phi_L \cdot G$ then $G = \Phi_{L^{-1}} \cdot F$. So, by Lemma 4.1, there exist non-negative integers s, ρ such that

$$\begin{aligned} \mathrm{LC}_{w_n}(G) = \mathrm{LC}_{w_n}(\Phi_{L^{-1}} \cdot F) &= \Delta_{\Phi_{L^{-1}}}(u)^s L_2(u,v)^\rho \mathrm{numer}\left(A_m\left(u, L(u,v)\right)\right). \\ &= a_{n_m}\Delta_{\Phi_{L^{-1}}}(u)^s L_2(u,v)^\rho \prod_{i=1}^{n_m}\left(L_1(u,v) - \alpha_i L_2(u,v)\right) \end{aligned} \tag{8}$$

understanding that, if we are in case ($\mathcal{A}_1$), then $\mathrm{LC}_{w_n}(G) = \Delta_{\Phi_{L^1}}(u)^s L_2(u,v)^\rho$.

Theorem 4.1. *Let $G \in \mathcal{AODE}^{(n)}$ be an n-order normal differential equation.*

1. *If $\mathrm{LC}_{w_n}(G)$ has a non-linear irreducible factor over $\mathbb{K}[u]$, then G is not quasi autonomous.*

2. *If a_1v+b_1 and a_2v+b_2 are two different linear factors over $\mathbb{K}[u]$ of $\mathrm{LC}_{w_n}(G)$, then the possible transformations $\Phi_L \in \mathcal{G}^{(n)}$, such that $\Phi_L \cdot G$ is autonomous, are defined by*

$$\textit{either } L^{-1} = \frac{b_1v - b_2}{-a_1v + a_2} \quad \textit{or} \quad L^{-1} = \frac{(b_2-b_1)v + b_1}{-(a_2-a_1)v - a_1}.$$

   In this case, if none of these functions, for every pair of linear factors, transforms G into an autonomous AODE, then G is not quasi autonomous.

Proof. 1. It follows from equality (8).

2. Let a_1v+b_1 and a_2v+b_2 be two different factors of $\mathrm{LC}_{w_n}(G)$ over $\mathbb{K}[u]$. We distinguish the following two possible cases.

   (a) $\begin{cases} a_1v + b_1 = cv + d \\ a_2v + b_2 = av + b - \alpha(cv+d). \end{cases}$

   We observe that, because of the plus-subgroup action (see Prop. 2.2), we can assume w.l.o.g. that $\alpha = 0$. It implies that $c = a_1, d = b_1, a = a_2, b = b_2 + \alpha b_1$. Therefore, $L = \dfrac{a_2v+b_2}{a_1v+b_1}$, and hence $L^{-1} = \dfrac{b_1v - b_2}{-a_1v+a_2}$.

   (b) $\begin{cases} a_1v + b_1 = av + b - \alpha_1(cv+d) \\ a_2v + b_2 = av + b - \alpha_2(cv+d). \end{cases}$

   As above, we can assume that one of the roots is zero; say $\alpha_1 = 0$. Then, the previous equalities implies that (note that $\alpha_2 \neq 0$)

$$L = \frac{a_1v + b_1}{\dfrac{(-a_2+a_1)v}{\alpha_2} + \dfrac{(-b_2+b_1)}{\alpha_2}}$$

Thus

$$L^{-1} = \frac{\dfrac{(b_2 - b_1)v}{\alpha_2} + b_1}{\dfrac{(-a_2 + a_1)v}{\alpha_2} - a_1}$$

Finally taking into account the mult-action subgroup, we deduce that L can be taken as

$$L^{-1} = \frac{(b_2 - b_1)v + b_1}{(-a_2 + a_1)v - a_1}$$

□

Example 4.2. We consider the AODE $G(x, y, y') = 0$ where
$G(x,y,y') = 1 + 4\,yy' - 2\,y'^2y^3 - y'^2y^2 + 2\,y'^2y - 3\,xy - 6\,y^3 + 7\,y^5 + 2\,y^4 + y'^2 - 3\,y^2 + 2\,y + 2\,y' + 2\,x - 2\,y'xy + 3\,y'^2x^4y^2 + y'^2x^3y^3 + 3\,y'^2x^5y + 3\,y'^2x^3y + y'^2xy - 12\,y'x^3y^3 - 20\,y'x^4y^2 - 12\,y'x^2y^2 - 13\,y'x^5y - 8\,y'x^3y - 10\,y'x^4y^4 - 20\,y'x^5y^3 - 20\,y'x^6y^2 - 10\,y'x^7y - 2\,y'y^5x^3 - y'y^4x^2 - 4\,y'y^3x + 2\,y'^2x^2y^3 + 6\,y'^2x^3y^2 + 6\,y'^2x^4y - 2\,y'^2xy^2 + 2\,y'^2x^2y - 20\,y'x^3y^4 - 40\,y'x^4y^3 - 40\,y'x^5y^2 + 12\,y'x^2y^3 + 4\,y'x^3y^2 - 20\,y'x^6y - 4\,y'y^5x^2 + 12\,y'y^4x - 16\,y'y^2x - 8\,y'x^2y - 10\,x^2y - 6\,x^2y^2 - 7\,x^3y - 14\,xy^2 - 4\,y'y^2 + 2\,y'x^2 + y^3x + 17\,x^3y^3 + 7\,x^4y^2 - x^5y + 35\,x^4y^4 + 40\,x^5y^3 + 25\,x^6y^2 + 8\,x^7y + y^7x^3 + 7\,y^6x^4 + 3\,y^6x^2 + 21\,y^5x^5 + 16\,y^5x^3 + 4\,y^5x + 35\,y^4x^6 + 14\,y^4x^2 + y'^2x^6 + 2\,y'^2x^4 + 2\,y'^2x^2 - 3\,y'x^6 - y'x^4 + 35\,x^7y^3 + 21\,x^8y^2 + 7\,x^9y - 2\,y'x^8 - 2\,x^3 - 2\,x^4 - x^6 + x^8 + x^{10} + x^5 + 3\,x^9 - y^7 + x^7 + 27\,y^4x + 2\,y'^2x + 4\,y'x + 40\,x^2y^3 + 28\,x^3y^2 + 9\,x^4y - 8\,y'y^3 + 35\,x^3y^4 + 45\,x^4y^3 + 29\,x^5y^2 + 9\,x^6y + 63\,x^4y^5 + 105\,x^5y^4 + 105\,x^6y^3 + 63\,x^7y^2 + 21\,x^8y + 3\,y^7x^2 + 21\,y^6x^3 - y^6x + 11\,y^5x^2 + 2\,y'^2x^5 + 2\,y'^2x^3 - 4\,y'x^7 + y'y^4 + 4\,y'y^5 - y'^2xy^3 + y'y^5x.$

The leading coeffcient of $G(u, v, w)$ w.r.t. w is

$$\mathrm{LC}_w(G) = \left(uv - v + 1 - u + u^2\right)\left(uv + 2\,v + 1 + 2\,u + u^2\right)\left(uv + u^2 + 1 + v + u\right)$$

Therefore, by Theorem 4.1, the possible elements $\Phi_L \in \mathcal{G}^{(1)}$ transforming G into an autonomous equations are such that

$$L^{-1} \in \left\{ \frac{(1 - u + u^2)\,v - 1 - 2\,u - u^2}{-(u-1)\,v + u + 2}, \frac{(1 - u + u^2)\,v - u^2 - 1 - u}{-(u-1)\,v + u + 1}, \right.$$

$$\left. \frac{(1 + 2\,u + u^2)\,v - u^2 - 1 - u}{-(u+2)\,v + u + 1}, \frac{3\,uv + 1 - u + u^2}{-3\,v + u - 1}, \frac{2\,uv + 1 - u + u^2}{-2\,v + u - 1}, \frac{uv + u^2 + 1 + u}{-v + u + 1} \right\}.$$

Indeed, when applying the 3 first we get 3 different autonomous equations and when taken the 3 last we get 3 different non-autonomous equations; all, the six, in $\overline{G}$. More precisely

- for $L^{-1} = \frac{\left(1 - u + u^2\right)v - 1 - 2\,u - u^2}{-(u-1)v + u + 2}$ we get

  $\Phi_L \cdot G(u, v, w) = 5 - 21\,vw - 162\,w^2v^2 - 81\,w^2v - 40\,v^3 - 15\,v^5 + 25\,v^4 + 42\,v^2 - 23\,v + 6\,w + 24\,wv^2 + 8\,v^6 - 2\,v^7 - 6\,wv^3 - 6\,wv^4 + 3\,wv^5,$

- for $L^{-1} = \frac{\left(1-u+u^2\right)v-u^2-1-u}{-(u-1)v+u+1}$ we get

  $\Phi_L \cdot G(u,v,w) = 1 - 3\,vw - 24\,w^2v^2 + 8\,w^2v - 15\,v^3 - 11\,v^5 + 15\,v^4 + 11\,v^2 - 5\,v + w + 2\,wv^2 + 5\,v^6 - v^7 + 2\,wv^3 - 3\,wv^4 + wv^5$

- for $L^{-1} = \frac{\left(1+2\,u+u^2\right)v-u^2-1-u}{-(u+2)v+u+1}$ we get

  $\Phi_L \cdot G(u,v,w) = 2 + 6\,vw + 3\,w^2v^2 - 2\,w^2v - 105\,v^3 - 82\,v^5 + 120\,v^4 + 55\,v^2 - 16\,v - w - 14\,wv^2 + 31\,v^6 - 5\,v^7 + 16\,wv^3 - 9\,wv^4 + 2\,wv^5$.

Therefore, G is a quasi autonomous AODE. •

Example 4.3. We consider the AODE $G(x,y,y',y'') = 0$ where
$G(x,y,y',y'') = 9\,u^5y^4y'\,y'' + 11\,u^6y^3y'\,y'' - 4\,u^6y^3y'^2y'' - 4\,u^7y^2y'^2y'' + 8\,u^4y^4y'\,y'' + 16\,u^5y^3y'\,y'' + 8\,u^6y^2y'\,y'' - 4\,u^5y^3y'^2y'' - 8\,u^6y^2y'^2y'' - 4\,u^7yy'^2y'' - 16\,u^3y^4y'\,y'' - 48\,u^4y^3y'\,y'' - 48\,u^5y^2y'\,y'' - 16\,u^6yy'\,y'' + 8\,u^4y'^2y''\,y^3 + 24\,u^5y'^2y''\,y^2 + 24\,u^6y'^2y''\,y + 3\,u^7y^2y'\,y'' + u^8yy'\,y'' + 28\,u^2y^6 + 4\,u^2y^5 + 4\,uy^6 - 8\,u^6y'^4 - 2\,u^8y''^2 - 16\,y^5u - 8\,y^4u^2 + 54\,u^3y^5 + 6\,uy^7 + 70\,y^4u^4 + 56\,y^3u^5 + 28\,y^2u^6 + 8\,yu^7 + 4\,u^6yy'^4 + u^8yy''^2 + 4\,u^5y^2y'^4 + 3\,u^7y^2y''^2 - 16\,u^2y^5y' - 16\,u^3y^4y' + 24\,u^3y^4y'^2 + 24\,u^4y^3y'^2 - 4\,u^3y^5y'' - 8\,u^4y^4y'' - 4\,u^5y^3y'' - 16\,u^4y^3y'^3 - 16\,u^5y^2y'^3 + u^5y^4y''^2 + 3\,u^6y^3y''^2 - 8\,u^4y'^4y^2 - 16\,u^5y'^4y - 12\,u^6y''^2y^2 - 8\,u^7y''^2y + 32\,y^5uy' + 64\,y^4u^2y' + 32\,y^3u^3y' - 48\,y^4u^2y'^2 - 96\,y^3u^3y'^2 - 48\,y^2u^4y'^2 + 8\,y^5u^2y'' + 24\,y^4u^3y'' + 24\,y^3u^4y'' + 8\,y^2u^5y'' + 32\,u^3y^3y'^3 + 64\,u^4y^2y'^3 + 32\,u^5yy'^3 + 8\,u^7y'^2y'' - 2\,u^4y''^2y^4 - 8\,u^5y''^2y^3 - 2\,u^7yy'^3 + 4\,u^2y^6y' - 2\,u^3y^5y'^2 + 6\,u^5y^3y'^2 + u^3y^6y'' + u^6y^3y'' - 4\,u^6y^2y'^3 - 2\,u^5y^3y' + 4\,u^6y^2y'^2 + 4\,u^6y^2y'^4 + u^8y^2y''^2 - 10\,u^3y^5y' + 24\,u^4y^4y'^2 - u^4y^5y'' - u^5y^4y'' - 18\,u^5y^3y'^3 + u^6y^4y''^2 + 2\,u^7y^3y''^2 - 8\,y^6 + y^8 + u^8$

$\deg_{w_2}(G) = 2$ and the leading coeffcient of $G(u,v,w_1,w_2)$ w.r.t. w_2 is

$$\mathrm{LC}_{w_2}(G) = u^4(uv+2v+2u)(uv-v-u)(v+u)^2.$$

Thus, $G(x,y,y'') = 0$ is normal. Moreover, by Theorem 4.1, the possible elements $\Phi_L \in \mathcal{G}^{(2)}$ transforming G into an autonomous equations are such that

$$L^{-1} \in \left\{ \frac{2\,uv+u}{-(u+2)\,v+u-1}, \frac{2\,uv-u}{-(u+2)\,v+1}, \frac{-uv-u}{-(u-1)\,v+1}, \frac{-3\,uv+2\,u}{3\,v+u+2}, \frac{-uv+2\,u}{-(-1-u)\,v+u+2}, \frac{-2\,uv+u}{-(u-2)\,v+1} \right\}.$$

Indeed, when applying the 3 first we get 3 different autonomous equations and when taken the 3 last we get 3 different non-autonomous equations; all, the six, in $\overline{G}$. More precisely

- for $L^{-1} = \frac{2\,uv+u}{-(u+2)v+u-1}$ we get

  $\Phi_L \cdot G(u,v,w_1,w_2) = v^8 - 8\,v^7 + 28\,v^6 - 56\,v^5 + 70\,v^4 - 18\,v^4w_1\,w_2 + 81\,v^3{w_2}^2 - 56\,v^3 + 45\,v^3w_1\,w_2 + 36\,v^3{w_1}^3 - 324\,v^2{w_1}^2w_2 - 54\,v^2{w_1}^3 - 162\,{w_2}^2v^2 - 27\,v^2w_1\,w_2 + 28\,v^2 + 81\,{w_2}^2v + 324\,v{w_1}^4 - 8\,v - 9\,vw_1\,w_2 + 324\,v{w_1}^2w_2 + 18\,{w_1}^3 + 1 + 9\,w_2\,w_1$,

- for $L^{-1} = \frac{2uv-u}{-(u+2)v+1}$ we get

  $\Phi_L \cdot G(u,v,w_1,w_2) = v^8 - 2v^4 w_1 w_2 + 4v^3 w_1{}^3 - 3v^3 w_2{}^2 + v^3 w_1 w_2 - 2v^2 w_1{}^3 + w_2{}^2 v^2 + 12v^2 w_1{}^2 w_2 - 12v w_1{}^4 - 4v w_1{}^2 w_2 + 4w_1{}^4$

- for $L^{-1} = \frac{-uv-u}{-(u-1)v+1}$ we get

  $\Phi_L \cdot G(u,v,w_1,w_2) = v^8 + v^4 w_1 w_2 - 2v^3 w_1{}^3 + 3v^3 w_2{}^2 + v^3 w_1 w_2 - 2v^2 w_1{}^3 + w_2{}^2 v^2 - 12v^2 w_1{}^2 w_2 + 12v w_1{}^4 - 4v w_1{}^2 w_2 + 4w_1{}^4$.

Therefore, G is a quasi autonomous AODE. •

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