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Standard Bases in Finitely Generated Difference-Skew-Differential Modules and Their Application to Dimension Polynomials

DISSERTATION

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Kurzfassung / Abstract

Kurzfassung

Diese Doktorarbeit behandelt verschiedene Arten von Standardbasen in endlich erzeugten Moduln über dem Ring der Differenzen-Schiefdifferentialoperatoren, deren Berechnung und Anwendungen auf die Berechnung multivariater Dimensions(quasi)polynome. Sie besteht aus zwei Teilen. Der erste behandelt Standardbasen in Moduln über dem Ring der Differenzen-Schiefdifferentialoperatoren. Der zweite Teil behandelt mit solchen Moduln assoziierte uni- und multivariate Dimensionsquasipolynome.

Wir beginnen damit, die Begriffe von Schiefdifferential-, Differenzen- und Differenzen-Schiefdifferentialoperatoren in Erinnerung zu rufen. Schiefdifferentialoperatoren sind eine Verallgemeinerung von kommutativen Polynomen, Differentialoperatoren und Differenzenoperatoren. Zur numerischen Lösung linearer partieller Differentialgleichungen betrachtet man oft das zugehörige Differenzenschema, welches durch inverse Differenzenoperatoren beschrieben werden kann. Indem wir die Begriffe der Schiefdifferential- und Differenzenoperatoren kombinieren, betrachten wir Differenzen-Schiefdifferentialoperatoren. Wir präsentieren Matrixdarstellungen verallgemeinerter Termordnungen. Dann führen wir den Begriff der gewichtet relativen Gröbnerbasen in endlich erzeugten Moduln von Differenzen-Schiefdifferentialoperatoren ein. Diese stellen eine Verallgemeinerung von Gröbnerbasen, relativen Gröbnerbasen und Gröbnerbasen bezüglich mehrerer Ordnungen dar. Wir geben eine Methode zu deren Berechnung an. Des Weiteren geben wir eine Charakterisierung gewichtet relativer Gröbnerbasen an, welche den Anstoß zu Komplexitätsüberlegungen für die erwähnte Methode liefert. Ein Teilresultat für kommutative Polynome wurde bei ACA 2011 [Dön11] präsentiert. Wir verallgemeinern den Begriff der Randbasen zu endlich erzeugten Moduln von Differenzen-Schiefdifferentialoperatoren und stellen eine Beziehung zwischen Rand- und Gröbnerbasen in diesem Umfeld her. Durch die Betrachtung von Multiplikationsendomorphismen leiten wir, in Analogie zu S-Polynomen, Kriterien ab, um zu überprüfen, ob eine Randvorbasis bereits eine Randbasis ist. Algorithmen zur Berechnung von Randbasen von nulldimensionalen Moduln sind im Anhang enthalten.

Wir führen auch den Begriff gewichteter Filtrierungen von Moduln über Differenzen-Schiefdifferentialoperatorringen ein und generalisieren die klassische Theorie der Dimensionspolynome assoziiert mit exzellenten Filtrierungen zu exzellenten gewichteten Filtrierungen. Wir beweisen die Existenz von mit solchen exzellenten gewichteten Filtrierungen assoziierten Quasipolynomen. Mit Hilfe des Moduls der Differentiale erweitern wir dieses Resultat zu Differentialkörpererweiterungen. Eine andere Erweiterung unseres Resultates betrifft gewichtete Multifiltrierungen und multivariate Dimensionsfunktionen. Schlussendlich geben wir mehrere Beispiele für Dimensions(quasi)polynome bekannter Differential- und Differenzengleichungssysteme aus der mathematischen Physik an.

Abstract

This thesis treats different kinds of standard bases in finitely generated modules over the ring of difference-skew-differential operators, their computation and their application to the computation of multivariate dimension (quasi-)polynomials. It consists of two parts. The first deals with standard bases in modules over the ring of difference-skew-differential operators. The second part deals with uni- and multivariate dimension quasipolynomials associated with such modules.

We start by recalling the notions of skew-differential, difference, and difference-skew-differential operators. Skew-differential operators are a generalization of commutative polynomials, differential operators, and difference operators. For the numeric solution of linear differential equations one often considers the associated difference scheme which can be described in terms of inversive difference operators. Combining the notions of skew-differential operators and dif-

ference operators we consider difference-skew-differential operators. We present matrix representations for generalized term orders. Then we introduce the notion of weight relative Gröbner bases in finitely generated modules of difference-skew-differential operators generalizing the notions of Gröbner bases, relative Gröbner bases and Gröbner bases with respect to several orderings. We provide a method for their computation. Furthermore we give a characterization of weight relative Gröbner bases. This naturally gives rise to complexity considerations for the aforementioned method. A partial result for commutative polynomials has been presented at ACA 2011 [Dön11]. We go on by generalizing the notion of border bases to finitely generated modules of difference-skew-differential operators. We establish a connection between border and Gröbner bases in this setting. Considering multiplication endomorphisms we also derive some S-polynomial-like criteria for a border prebasis to be a border basis. Algorithms for the computation of border bases of zero-dimensional modules are included in the appendix.

We also introduce the notion of weighted filtrations of modules over rings of difference-skew-differential operators and generalize the classical theory of dimension polynomials associated with excellent filtrations to excellent weighted filtrations. We prove the existence of dimension quasipolynomials associated with such excellent weighted filtrations. Considering the module of differentials we can extend this result to differential field extensions. Another extension of our results regards weighted multifiltrations and multivariate dimension functions. Finally we provide several examples for dimension (quasi)polynomials of well-known systems of differential and difference equations from mathematical physics.

Curriculum vitæ

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Published and submitted papers

1. C. Dönch, *Bivariate difference-differential dimension polynomials and their computation in Maple*, Tech. Report 09-19, RISC Report Series, University of Linz, Austria, December 2009
2. C. Dönch, F. Winkler, *Bivariate Difference-Differential Dimension Polynomials and Their Computation in Maple*, In: Proceedings of the 8th International Conference on Applied Informatics, Attila Egri-Nagy, Emőd Kovács, Gergely Kovásznai, Gábor Kusper, Tibor Tómács (ed.), 211-218, 2010
3. C. Dönch, *Characterization of Relative Gröbner Bases*, Journal of Symbolic Computation, submitted
4. C. Dönch, A.B. Levin, *Computation of the Strength of PDEs of Mathematical Physics and their Difference Approximations*, `http://arxiv.org/abs/1205.6762`, 2012

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Eidesstattliche Erklärung / Affidavit

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Die vorliegende Dissertation ist mit dem elektronisch übermittelten Textdokument identisch. [JKU10]

Christian Dönch

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Also I would like to thank my colleague Johannes Middeke. Although formally Alexander has been my second advisor at least after Johannes finished his PhD studies I humorously dubbed him as such. Not only did he explain Gröbner bases and Buchberger's algorithm to me when I came to RISC but also took great many efforts in answering multitudinous questions on algebra in such a way that I being not much of an algebraist was able to understand his explanations. Whenever I asked him for advice he immediately took the time to answer my questions and he has always been an inexhaustible source of helpful comments and suggestions. To a big part it is his merit that I found the atmosphere at RISC so supportive and stimulating.

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Contents

Chapter 1

Introduction

1.1 Introduction

1.1.1 Overview

In this thesis we study different kinds of standard bases in finitely generated modules over rings of difference-skew-differential operators and their application to the computation of difference-skew-differential dimension polynomials taking into account different weights associated with the different operators involved.

Taking a look at the standard literature for differential and difference dimension polynomials – most notably [KLMP99] – there is a distinction between differential, difference, and inversive difference operators which naturally arises from the different properties of these classes of operators. Working with them however it turns out immediately that differential and difference operators are not that different at all. A unified theory for differential and difference operators can be found in the theory of Ore polynomials or skew-polynomials first appearing in [Ore33]. Throughout this thesis we will use the term *skew-differential operators*.

Inversive difference operators are distinguished from difference operators because in the former case the terms involved form a group whereas in the later case they form a monoid. Several approaches have been developed to deal with this issue [PZ96, PU99, ZW06, ZW08a, ZW08b, LW11]. Unless otherwise noted, throughout this thesis difference operators will always be considered as a special case of skew-differential operators. Therefore, from now on, whenever we use the term "difference operator" we mean an inversive difference operator.

1.1.2 Outline

In the beginning of the second chapter we recall the notions of skew-differential, difference, and difference-skew-differential operators.

In Section 2.2 we recall the notion of and provide a matrix representation for generalized term orders.

In Section 2.3 we unify the theories of relative Gröbner bases as developed in [ZW08a] and Gröbner bases with respect to several orderings as developed in [Lev07a, Lev08] and introduce the notion of weight relative Gröbner bases in finitely generated modules over rings of difference-skew-differential operators. We state a method for the computation of weight relative Gröbner bases and prove its correctness. We provide a characterization of weight relative Gröbner bases which gives rise to considerations leading to the result that in some situations no finite weight relative Gröbner basis exists. This is illustrated by an extended example regarding relative Gröbner bases of polynomial ideals. It turns out that if one drops the "relative"part then for every finitely generated difference-skew-differential module and suitable choice of weights a finite weight Gröbner basis always exists. In this case the aforementioned method gives rise to an algorithm for its computation.

In Section 2.4 we extend the notion of border bases from polynomial ideals to finitely generated modules over rings of difference-skew-differential operators. We show that there exists an intrinsic connection between border bases and Gröbner bases modules over the ring of difference-skew-differential operators. Considering multiplication endomorphisms we derive S-polynomial-like criteria for a border-prebasis to be a border basis. Algorithms for the computation of border bases for zero-dimensional modules over the ring of difference-skew-differential operators are provided in Appendix 3.

The third chapter deals with uni- and multivariate dimension quasipolynomials. Section 3.1 is devoted to the existence and computation of uni- and multivariate dimension quasipolynomials where the different skew-differential and difference operators are associated with certain weights. We introduce the notion of weighted filtrations of modules over filtered rings of difference-skew-differential operators and apply the theory of weight relative Gröbner bases in order to prove the existence of uni- and multivariate dimension quasipolynomials associated with such modules. The provided proves are constructive and give rise to an algorithm for the computation of said quasipolynomials. In Section 3.2 the relation between Einstein's notion of the "strength" of a system of differential equations governing a physical field and differential dimension polynomials is pointed out. We provide several examples of differential and difference dimension (quasi)polynomials of well known systems from mathematical physics.

Our main original contributions are:

(i) introducing the notion of weight relative Gröbner bases in finitely generated modules of difference-skew-differential operators and clarifying their relation to Gröbner bases, relative Gröbner bases, and Gröbner bases with respect to several orderings in the sense of Levin,

(ii) extending the notion of border bases to finitely generated modules of difference-skew-differential operators,

(iii) introducing the notion of (excellent) weighted filtrations,

(iv) proving the existence of weighted difference-skew-differential dimension quasipolynomials and establishing the general form of multivariate difference-skew-differential dimension functions.

Chapter 2

Standard bases for difference-skew-differential operators

2.1 Difference-skew-differential operators

The following definition generalizes the definitions of differential, difference, and difference-differential operators as provided in [KLMP99].

2.1.1 Definition of difference-skew-differential operators

Throughout this thesis the symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}_+, \mathbb{Q}_0$ and $\mathbb{R}$ denote the sets of nonnegative integers, integers, rational numbers, positive rational numbers, nonnegative rational numbers, and real numbers, respectively. We assume all rings to have a unit element, every subring of a ring contains the ring's unit element. Ring homomorphisms are considered to be unitary, i.e., mapping unit element to unit element. By the module over a ring R we always mean a unitary left R-module.

Definition 2.1.1.

(i) (a) *Let R be a ring, and τ an endomorphisms on R. A function δ on R is called* τ-derivation *or* skew-derivation with respect to τ *if and only if for all $a, b \in R$ we have*

$$\begin{aligned} \delta(a+b) &= \delta(a) + \delta(b), \text{ and} \\ \delta(ab) &= \delta(a)b + \tau(a)\delta(b). \end{aligned}$$

(b) *Let R be a commutative ring (respectively a field),$\{\tau_1, \ldots, \tau_m\}$ a set of mutually commuting injective endomorphisms on R, and $\Delta = \{\delta_1, \ldots, \delta_m\}$ a set of mutually commuting skew-derivations on R such that for all $i = 1, \ldots, m$ the skew-derivation δ_i is a skew-derivation with respect to τ_i. Then R is called a* skew-differential ring *or* Δ-ring *(respectively a* skew-differential field *or* Δ-field*) with basic set of skew-derivations Δ.*

(c) *Let R be a Δ-ring (respectively Δ-field) and S a subring (respectively subfield) of R that is closed with respect to the action of any operator from Δ. Then S is a Δ-ring (respectively Δ-field) which will be called* Δ-subring *(respectively* Δ-subfield*) of R and R is called* Δ-ring!extension *(respectively* Δ-field extension*) of S.*

(d) *By $[\Delta]$ we denote the commutative monoid generated by Δ, i.e.,*

$$\{\delta^k \mid k \in \mathbb{N}^m\},$$

where we use multi-index notation, i.e., $\delta^k = \delta_1^{k_1} \cdots \delta_m^{k_m}$ where $k = (k_1, \ldots, k_m)$. Elements of $[\Delta]$ are called skew-differential (Δ-) terms.

(e) *The free R-module generated by $[\Delta]$ will be denoted by $R[\Delta]$. Hence elements of $R[\Delta]$ are of the form $\sum_{\lambda \in [\Delta]} a_\lambda \lambda$ with $a_\lambda \in R$ and only finitely many a_λ are not vanishing. $R[\Delta]$ can be equipped with a natural ring structure with the commutation rules*

   i. *$\lambda\mu = \mu\lambda$ for all $\lambda, \mu \in [\Delta]$, and*

   ii. *$\delta_i r = \tau_i(r)\delta_i + \delta_i(r)$ for all $1 \le i \le m, r \in R$.*

   The obtained ring is called the ring of skew-differential (Δ-) operators *over R.*

(f) *The order of any $\lambda = \delta^k = \delta_1^{k_1} \cdots \delta_m^{k_m} \in [\Delta]$ is given by* $\operatorname{ord} \lambda = k_1 + \ldots + k_m$ *and the order of $f = \sum_{\lambda \in [\Delta]} a_\lambda \lambda \in R[\Delta]$ is given by*

$$\operatorname{ord} f = \max\{\operatorname{ord} \lambda \mid a_\lambda \neq 0\}.$$

(g) *A left module over the ring $R[\Delta]$ is called an $R[\Delta]$-module. If $G \subseteq R[\Delta]$ then by ${}_{R[\Delta]}\langle G \rangle$ we denote the $R[\Delta]$-module generated by G. If the ring $R[\Delta]$ is clear from the context we write $\langle G \rangle$ instead of ${}_{R[\Delta]}\langle G \rangle$.*

(ii) (a) *A commutative ring (respectively a field) R together with a finite set $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ of mutually commuting automorphisms of R is called a* difference ring *or* Σ-ring *(respectively* difference field *or* Σ-field*) with basic set Σ.*

(b) *Let R be a Σ-ring (respectively Σ-field) and S a subring (respectively subfield) of R that is closed with respect to the action of any operator from Σ. Then S is a Σ-ring (respectively Σ-field) which will be called* Σ-subring *(respectively* Σ-subfield*) of R and R is called* Σ-ring extension *(respectively* Σ-field extension*) of S.*

(c) *By Σ^* we denote the set $\{\sigma_1, \sigma_1^{-1}, \ldots, \sigma_n, \sigma_n^{-1}\}$ and by $[\Sigma^*]$ we denote the free commutative group generated by Σ, i.e.,*

$$[\Sigma^*] = \{\sigma^l \mid l \in \mathbb{Z}^n\}.$$

Elements of $[\Sigma^*]$ are called difference (Σ-) terms*.*

(d) *The free left R-module generated by $[\Sigma^*]$ we will denote by $R[\Sigma^*]$. Hence, elements of $R[\Sigma^*]$ are of the form $\sum_{\lambda \in [\Sigma^*]} a_\lambda \lambda$ with $a_\lambda \in R$ for any $\lambda \in [\Sigma^*]$ and only finitely many a_λ are not vanishing. $R[\Sigma^*]$ can be equipped with a natural ring structure with the commutation rules*

i. *$\lambda\mu = \mu\lambda$ for all $\lambda, \mu \in [\Sigma^*]$,*
ii. *$\sigma_i r = \sigma_i(r)\sigma_i$ for all $1 \leq i \leq n, r \in R$, and*
iii. *$\sigma_i^{-1} r = \sigma_i^{-1}(r)\sigma_i^{-1}$ for all $1 \leq i \leq n, r \in R$.*

The obtained ring is called the ring of difference (Σ-) operators *over the ring R.*

(e) *The order of any $\lambda = \sigma^l = \sigma_1^{l_1} \cdots \sigma_m^{l_m} \in [\Delta]$ is given by* $\operatorname{ord} \lambda = |l_1| + \ldots + |l_m|$ *and the order of $0 \neq f = \sum_{\lambda \in [\Sigma^*]} a_\lambda \lambda \in R[\Sigma^*]$ is given by*

$$\operatorname{ord} f = \max\{\operatorname{ord} \lambda \mid a_\lambda \neq 0\}.$$

(f) *A left module of the ring $R[\Sigma^*]$ is called $R[\Sigma^*]$-module. If $G \subseteq R[\Sigma^*]$ then by ${}_{R[\Sigma^*]}\langle G \rangle$ we denote the $R[\Sigma^*]$-module generated by G. If the ring $R[\Sigma^*]$ is clear from the context we write $\langle G \rangle$ instead of ${}_{R[\Sigma^*]}\langle G \rangle$.*

(iii) (a) *Let R be a commutative ring (respectively a field), $T = \{\tau_1, \ldots, \tau_m\}$ a set of mutually commuting injective endomorphisms on R, $\Delta = \{\delta_1, \ldots, \delta_m\}$ a set of skew-derivations such that for $i = 1, \ldots, m$ the skew derivation δ_i is a skew-derivation with respect to τ_i, and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ a set of automorphisms on R with $\lambda \circ \mu = \mu \circ \lambda$ for any $\lambda, \mu \in T \cup \Delta \cup \Sigma$. Then R is called a* difference-skew-differential ring *or* Δ-Σ-ring *(respectively* difference-skew-differential field *or* Δ-Σ-field*) with basic set of skew-derivations Δ and basic set of automorphisms Σ.*

(b) *Let R be a Δ-Σ-ring (respectively Δ-Σ-field) and S a subring (respectively subfield) of R that is closed with respect to the action of any operator from $\Delta \cup \Sigma$. Then S is a Δ-Σ-ring (respectively Δ-Σ-field) which will be called* Δ-Σ-subring *(respectively* Δ-Σ-subfield*) of R and R is called* Δ-Σ-ring extension *(respectively* Δ-Σ-field extension*) of S.*

(c) *By $[\Delta, \Sigma^*]$ we denote the set*

$$\{\delta^k \sigma^l \mid k \in \mathbb{N}^m, l \in \mathbb{Z}^n\},$$

where we use multi-index notation. Elements of $[\Delta, \Sigma^*]$ are called difference-skew-differential (Δ-Σ-) terms*.*

(d) *The free R-module generated by $[\Delta, \Sigma^*]$ we will denote by $R[\Delta, \Sigma^*]$. Hence, elements of $R[\Delta, \Sigma^*]$ are of the form $\sum_{\lambda \in [\Delta, \Sigma^*]} a_\lambda \lambda$ with $a_\lambda \in R$ and only finitely many a_λ are not vanishing. $R[\Delta, \Sigma^*]$ can be equipped with a natural ring structure with the commutation rules*

i. *$\lambda\mu = \mu\lambda$ for all $\lambda, \mu \in [\Delta, \Sigma^*]$,*
ii. *$\delta_i r = \tau_i(r)\delta + \delta_i(r)$ for all $1 \leq i \leq m, r \in R$,*

iii. $\sigma_i r = \sigma_i(r)\sigma_i$ *for all* $1 \leq j \leq n, r \in R$, *and*

iv. $\sigma_i^{-1} r = \sigma_i^{-1}(r)\sigma_i^{-1}$ *for all* $1 \leq j \leq n, r \in R$.

The obtained ring is called the ring of difference-skew-differential (Δ-Σ-) operators *over* R.

(e) The order of any $\lambda = \delta^k \sigma^l = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \in [\Delta, \Sigma^*]$ *is given by* $\operatorname{ord} \lambda = k_1 + \ldots + k_m + |l_1| + \ldots + |l_n|$ *and the order of* $0 \neq f = \sum_{\lambda \in [\Delta, \Sigma^*]} a_\lambda \lambda \in R[\Delta, \Sigma^*]$ *is given by*

$$\operatorname{ord} f = \max\{\operatorname{ord} \lambda \mid a_\lambda \neq 0\}.$$

(f) A left module of the ring $R[\Delta, \Sigma^*]$ *is called* $R[\Delta, \Sigma^*]$*-module. If* $G \subseteq R[\Delta, \Sigma^*]$ *then by* ${}_{R[\Delta,\Sigma^*]}\langle G \rangle$ *we denote the* $R[\Delta, \Sigma^*]$*-module generated by* G. *If the ring* $R[\Delta, \Sigma^*]$ *is clear from the context we write* $\langle G \rangle$ *instead of* ${}_{R[\Delta,\Sigma^*]}\langle G \rangle$.

□

Obviously, (i) and (ii) can be considered as special cases of (iii) in Definition 2.1.1 providing clarifications for the case that $n = 0$ or $m = 0$, respectively. From now on whenever we consider a difference-skew-differential ring (or field) with basic set of skew-derivations Δ, and basic set of automorphisms Σ we also allow $\Delta = \emptyset$ or $\Sigma = \emptyset$.

For reasons of convenience throughout this thesis we will often use multi-index notation meaning that if we consider a difference-skew-differential term $\delta^u \sigma^v$ then $u = (u_1, \ldots, u_m) \in \mathbb{N}^m, v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ and $\delta^u \sigma^v = \delta_1^{u_1} \cdots \delta_m^{u_m} \sigma_1^{v_1} \cdots \sigma_n^{v_n}$.

There are several popular approaches to difference-skew-differential operators. We hope that the kind reader will find at least one of the two motivations we present satisfactionable.

2.1.2 Difference-skew-differential operators arising from physical applications

Linear partial differential operators arise naturally in physics, chemistry, biology, and many other sciences. For example, Gauss's flux theorem relates the divergence $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ of an electric field E to the distribution of electric charges ρ and the electric constant ϵ_0 via

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}.$$

Differences come into play when one discretizes a (partial) differential equation in order to solve it. That means, replacing every occurrence of

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

in a given differential equation by

$$\frac{f(x+h_0) - f(x)}{h_0}$$

for fixed h_0 and solving the resulting difference equation.

A second possibility for differences to arise is by considering systems involving time delays. Then naturally expressions of the form

$$f(t+1) - f(t)$$

have to be treated.

2.1.3 Ore polynomials

The theory of Ore polynomials originates from the works of Øystein Ore [Ore32a, Ore32b, Ore33] examining polynomials satisfying a certain commutativity condition. Consider a ring R together with maps $d, s : R \to R$ such that

(i) s is an injective ring homomorphism, and

(ii) d is a homomorphism of abelian groups satisfying for any $r_1, r_2 \in R$ the equation

$$d(r_1 r_2) = s(r_1)d(r_2) + d(r_1)r_2.$$

By $[\partial]$ we denote the commutative monoid generated by an element ∂, i.e.,

$$[\partial] := \{\partial^k \mid k \in \mathbb{N}\}.$$

Then the set $R[\partial]$ can be equipped with a ring structure by the commutation rule $\partial r = s(r)\partial + d(r)$ for all $r \in R$.

Definition 2.1.2. *The ring $R[\partial]$ together with this commutation relation is called* ring of Ore polynomials over R with respect to s and d *and is denoted by $R[\partial; s, d]$.*

Example 2.1.3. *Choosing $s = 1$ implies $d(r_1 r_2) = r_1 d(r_2) + d(r_1)r_2$, i.e., in this case d is a derivation on R. Furthermore we have*

$$\partial r_1 = r_1 \partial + d(r_1).$$

So $R[\partial]$ is the ring of differential operators over R (in this case one can consider ∂ as a symbol for the operator $r \mapsto d(r)$).

Example 2.1.4. *Choosing $d = 0$ implies $\partial r = s(r)\partial$, i.e., ∂ acts on R as an operator associated with the endomorphism s. If R is a ring (or a field) of functions over a field K, then s is typically a mapping of the form $f(x) \mapsto f(x+h)$, where $f(x) \in R, h \in K$. Therefore the name "difference operator".*

Keeping in mind that d and s correspond to derivations and shifts, respectively, it turns out that the choices $s = 1$ and $d = 0$, respectively, are trivial.

2.1.4 The module of differentials

In this subsection we present the module of differentials as introduced in [KLMP99].

Definition 2.1.5. *Let R be a ring, M an R-module, and $D : R \to M$ an additive map such that for any $r_1, r_2 \in R$ we have*

$$D(r_1 r_2) = D(r_1)r_2 + r_1 D(r_2).$$

Then D is called a derivation *from R to M. The set of all derivations from R to M is denoted by* $\mathrm{Der}(R, M)$.

In [Bou70, Chapter V, §9, Proposition 4] the following proposition is provided.

Proposition 2.1.6. *Let Ω be a field, E a subfield of Ω, and F a separable algebraic field extension of E contained in Ω. Then every derivation D of E into Ω can be uniquely extended to a derivation $\bar{D}$ of F into Ω.*

Let $F \subseteq G$ be fields. The set $\mathrm{Der}_F\, G$ is defined by

$$\mathrm{Der}_F\, G := \{\delta \in \mathrm{Der}(G, G) \mid \forall_{f \in F} \delta(f) = 0\}.$$

Then $\mathrm{Der}_F\, G$ is a G-vector space. Let $(\mathrm{Der}_F\, G)^*$ denote its dual space which again is a G-vector space. For every element $g \in G$ let $dg \in (\mathrm{Der}_F\, G)^*$ such that for all $\delta \in \mathrm{Der}_F\, G$ we have

$$(dg)(\delta) = \delta(g).$$

Then $g \mapsto dg$ is F-linear and

$$\begin{aligned} d(g_1 g_2)(\delta) &= \delta(g_1 g_2) \\ &= \delta(g_1)g_2 + g_1\delta(g_2) \\ &= ((dg_1)g_2 + g_1(dg_2))(\delta). \end{aligned}$$

Hence, $g \mapsto dg$ is a derivative.

Definition 2.1.7. *The G-vector space generated by $\{dg \mid g \in G\}$ will be denoted by $\Omega_F(G)$. It is called the* module of differentials *associated with the field extension $G \supseteq F$.*

If $B \subseteq G$, then the intersection of all subfields of G containing F and B is denoted by $F(B)$ and is called the *field extension of F generated by B*. If B is finite then $F(B)$ is called a *finitely generated field extension of F*.

In [KLMP99] the following proposition is provided.

Proposition 2.1.8. *Let F be a field and $G = F(g_1, \ldots, g_k)$ a finitely generated field extension of F. Then $\Omega_F(G)$ is a finite dimensional G-vector space with generators $dg_1, \ldots, dg_k$.*

2.2 Generalized term orders

Let K be a difference-skew-differential field of characteristic 0, $\{\tau_1, \ldots, \tau_m\}$ a set of mutually commuting injective endomorphisms on K, $\{\delta_1, \ldots, \delta_m\}$ a basic set of skew-derivations such that for $i = 1, \ldots, m$ the skew-derivation δ_i is a skew-derivation with respect to τ_i, respectively, and $\{\sigma_1, \ldots, \sigma_n\}$ a basic set of automorphisms. Since the set of difference-skew-differential terms $[\Delta, \Sigma^*]$ is isomorphic to $\mathbb{N}^m \times \mathbb{Z}^n$ for reasons of convenience throughout this section we will consider $\mathbb{N}^m \times \mathbb{Z}^n$ instead of $[\Delta, \Sigma^*]$.

2.2.1 Orthant decompositions

Zhou and Winkler [ZW06, ZW08a, ZW08b] suggested to decompose the set $\mathbb{N}^m \times \mathbb{Z}^n$ so that every component of such a decomposition is isomorphic to the $m+n$-fold nonnegative integers providing a possibility for a natural extension of admissible orders as used in the theory of Gröbner bases. These so-called generalized term orders were first introduced by Pauer and Zampieri on sets of monomials in a polynomial ring for modelling problems in system theory [PZ96].

Definition 2.2.1. *Let $\mathbb{Z}^n = \bigcup_{k=1}^p \mathbb{Z}_k^{(n)}$ such that for all $1 \leq k \leq p$ we have*

(i) $0 \in \mathbb{Z}_k^{(n)}$ and apart from 0 the set $\mathbb{Z}_k^{(n)}$ contains no two inverse elements,

(ii) $\mathbb{Z}_k^{(n)}$ is isomorphic to $\mathbb{N}^n$ as a semigroup, and

(iii) $\mathbb{Z}_k^{(n)}$ generates $\mathbb{Z}^n$ as a group.

Then $\{\mathbb{Z}_k^{(n)} \mid 1 \leq k \leq p\}$ is called an orthant decomposition *of $\mathbb{Z}^n$ and each of its components $\mathbb{Z}_k^{(n)}$ is called an* orthant *of the orthant decomposition $\{\mathbb{Z}_k^{(n)} \mid 1 \leq k \leq p\}$. Furthermore we call $\{\mathbb{N}^m \times \mathbb{Z}_k^{(n)} \mid 1 \leq k \leq p\}$ an orthant decomposition of $\mathbb{N}^m \times \mathbb{Z}^n$ and each of its components $\mathbb{N}^m \times \mathbb{Z}^{(n)}$ is called an orthant of the orthant decomposition $\{\mathbb{N}^m \times \mathbb{Z}_k^{(n)} \mid 1 \leq k \leq p\}$.* □

Every orthant decomposition of $\mathbb{N}^m \times \mathbb{Z}^n$ induces a decomposition of $[\Delta, \Sigma^*]$ which we call orthant decomposition of $[\Delta, \Sigma^*]$.

For the following examples of orthant decompositions see also [ZW06].

Example 2.2.2. *The standard example of an orthant decomposition consists of* $\{\mathbb{Z}_k^{(n)} \mid 1 \leq k \leq 2^n\}$ *being the set of all distinct cartesian products of* n *sets each of which is either* $\mathbb{N}$ *or* $-\mathbb{N}$*, i.e.,* $\mathbb{Z}_k^{(n)}$ *is generated as a semigroup by*

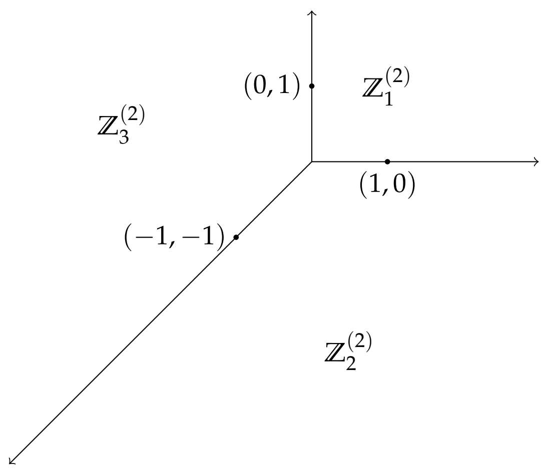
$$\{(c_1, 0, \ldots, 0), (0, c_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, c_n)\},$$

where for $i = 1, \ldots, n$ *either* $c_i = 1$ *or* $c_i = -1$*. This orthant decomposition of* $\mathbb{Z}^n$ *is called the* canonical orthant decomposition. □

Example 2.2.3. *Let* $n = 2$ *and define an orthant decomposition* $\{\mathbb{Z}_k^{(2)} \mid 1 \leq k \leq 3\}$ *of* $\mathbb{Z}^2$ *by*

$$\begin{aligned} \mathbb{Z}_1^{(2)} &:= \{(k_1, k_2) \mid k_1, k_2 \in \mathbb{N}\} \\ \mathbb{Z}_2^{(2)} &:= \{(k_1 - k_2, -k_2) \mid k_1, k_2 \in \mathbb{N}\} \\ \mathbb{Z}_3^{(2)} &:= \{(-k_1, k_2 - k_1) \mid k_1, k_2 \in \mathbb{N}\}, \end{aligned}$$

i.e., $\mathbb{Z}_1^{(2)}, \mathbb{Z}_2^{(2)}$ *and* $\mathbb{Z}_3^{(2)}$ *are generated by* $\{(1,0),(0,1)\}, \{(1,0),(-1,-1)\}$ *and* $\{(0,1),(-1,-1)\}$*, respectively.*



□

Consider an orthant decomposition $\Xi = \{\mathbb{Z}_k^{(n)} \mid 1 \leq k \leq p\}$ of $\mathbb{Z}^n$. Since for $1 \leq k \leq p$ the orthant $\mathbb{Z}_k^{(n)}$ has n generators the set of all generators of the orthant decomposition Ξ is finite, say $\{\xi_1, \ldots, \xi_r\}$ – there exist $1 \leq k_1, \ldots, k_n \leq r$ such that $\mathbb{Z}_k^{(n)}$ is generated by $\xi_{k_1}, \ldots, \xi_{k_n}$. Then we say that $\xi_1, \ldots, \xi_r$ are the generators of the orthant decomposition Ξ. If $\Xi' = \{\mathbb{N}^m \times \mathbb{Z}_k^{(n)} \mid 1 \leq k \leq p\}$ is an according orthant decomposition of $\mathbb{N}^m \times \mathbb{Z}^n$ then we still call $\{\xi_1, \ldots, \xi_r\}$ the set of generators of the orthant decomposition Ξ' of $\mathbb{N}^m \times \mathbb{Z}^n$.

2.2.2 Definition of generalized term orders

Definition 2.2.4. *Let* Ξ *be an orthant decomposition of* $\mathbb{N}^m \times \mathbb{Z}^n$*,* $E = \{e_1, \ldots, e_q\}$ *a finite set and let* $\prec$ *be a total order on* $\mathbb{N}^m \times \mathbb{Z}^n \times E$ *such that for all* $1 \leq i, j \leq q, \lambda, \eta\mu \in \mathbb{N}^m \times \mathbb{Z}^n$ *we have*

(i) $(0, e_i) \prec (\lambda, e_i)$ *, and*

(ii) $(\lambda, e_i) \prec (\eta, e_j)$ *implies* $(\lambda + \mu, e_i) \prec (\eta + \mu, e_j)$ *if* η *and* μ *belong to the same orthant.*

Then $\prec$ *is called a* generalized term order *on* $\mathbb{N}^m \times \mathbb{Z}^n \times E$ *with respect to the orthant decomposition* Ξ. *If* $\Sigma = \emptyset$ *then* $\prec$ *is also referred to as* admissible order.

If E is a finite set of generators of a free difference-skew-differential module we identify $\mathbb{N}^m \times \mathbb{Z}^n \times E$ with $[\Delta, \Sigma^*]E$. Then every generalized term order (respectively admissible order) $\prec$ on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ induces an order on $[\Delta, \Sigma^*]E$ which we call a generalized term order (respectively admissible order) on $[\Delta, \Sigma^*]E$.

If no confussion is possible we simply say that $\prec$ is a generalized term order or admissible order, respectively.

Obviously, if $\prec$ is a generalized term order with respect to the orthant decomposition $\Xi = \{\Xi_k \mid 1 \leq k \leq p\}$ and there exist $1 \leq k_1, k_2 \leq p$ with $k_1 \neq k_2$ and $\Xi_{k_1} \subseteq \Xi_{k_2}$ then $\prec$ is also a generalized term order with respect to the orthant decompositions $\{\Xi_k \mid 1 \leq k \leq p, k \neq k_1\}$. Therefore from now on whenever we consider a generalized term order $\prec$ with respect to the orthant decomposition Ξ we assume that Ξ does not contain any orthant which is entirely contained in another orthant of the orthant decomposition Ξ.

For a better understanding of the relation between orthant decompositions and generalized term orders consider an orthant decomposition $\Xi = \{\Xi_k \mid 1 \leq k \leq p\}$ of $\mathbb{Z}^n$ such that the intersection of two orthants, say Ξ_1 and Ξ_2 generates $\mathbb{Z}^n$ as a group. Let E be finite and $\prec$ a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ with respect to $\{\mathbb{N}^m \times \Xi_k \mid 1 \leq k \leq p\}, \lambda_1, \eta_1 \in \mathbb{N}^m \times \Xi_1, e, e' \in E$ with $(\lambda_1, e) \prec (\eta_1, e')$ and $\mu \in \mathbb{Z}^{m+n}$ such that $\lambda_1 + \mu = \eta_1$. Then for $\lambda_2, \eta_2 \in \mathbb{N}^m \times \Xi_2$ with $\lambda_2 + \mu = \eta_2$ there exist $\mu_1, \mu_2 \in \mathbb{N}^m \times (\Xi_1 \cap \Xi_2)$ such that $\lambda_1 + \mu_1 = \lambda_2 + \mu_2 \in \mathbb{N}^m \times (\Xi_1 \cap \Xi_2)$ and $\eta_1 + \mu_1 = \eta_2 + \mu_2 \in \mathbb{N}^m \times (\Xi_1 \cap \Xi_2)$. Hence, $(\lambda_2 + \mu_2, e) = (\lambda_1 + \mu_1, e) \prec (\eta_1 + \mu_1, e') = (\eta_2 + \mu_2, e')$ and $(\lambda_2, e) \prec (\eta_2, e')$. From these considerations we obtain:

Lemma 2.2.5. *Let* E *be finite and* $\Xi = \{\Xi_k \mid 1 \leq k \leq p\}$ *an orthant decomposition of* $\mathbb{Z}^n$ *such that* $\Xi_1 \cap \Xi_2$ *generates* $\mathbb{Z}^n$ *as a group.*

(i) If there exist $\eta \in \Xi_1, \lambda \in \Xi_2$ *with* $\eta\lambda = 1$ *then there cannot exist any generalized term order on* $\mathbb{N}^m \times \mathbb{Z}^n \times E$ *with respect to* $\{\mathbb{N}^m \times \Xi_k \mid 1 \leq k \leq p\}$*, and*

(ii) if there exists a generalized term order $\prec$ *on* $\mathbb{N}^m \times \mathbb{Z}^n \times E$ *with respect to* $\{\mathbb{N}^m \times \Xi_k \mid 1 \leq k \leq p\}$ *then it also is a generalized term order with respect to the orthant decomposition* $\{\mathbb{N}^m \times (\Xi_1 \cup \Xi_2), \mathbb{N}^m \times \Xi_3, \ldots, \mathbb{N}^m \times \Xi_p\}$.

From now on unless otherwise noted we always assume that if we consider a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ with respect to the orthant decomposition $\Xi = \{\mathbb{N}^m \times \Xi_k \mid 1 \leq k \leq p\}$ of $\mathbb{N}^m \times \mathbb{Z}^n$ then there exist no $1 \leq k_1, k_2 \leq p$ with $k_1 \neq k_2$ and $\Xi_{k_1} \cap \Xi_{k_2}$ generating $\mathbb{Z}^n$ as a group.

2.2.3 Characterization of admissible orders

At EUROCAL'85 Robbiano presented a classification of admissible orders over the polynomial ring [Rob85]. A classification of monomial orders for free modules over polynomial rings was provided by Rust and Reid [RR97], and independently Horn [Hor98]. Already earlier partial classifications were obtained by e.g., Carrà-Ferro and Sit [CS94], Caboara and Silvestri [CS99].

First we recall Robbiano's theorem classifying admissible orders on $\mathbb{N}^m$.

Theorem 2.2.6. *Let* $k \in \{1, \ldots, m\}, u_1, \ldots, u_k \in \mathbb{R}^m$ *and let* $U \in \mathbb{R}^{k \times m}$ *be the matrix with rows* $u_1, \ldots, u_k$. *By* d_i *we denote the dimension of the* $\mathbb{Q}$*-vector space spanned by the entries of* u_i. *Suppose* $u_1, \ldots, u_k$ *are such that*

(i) $d_1 + \ldots + d_k = m$,

(ii) for $i = 1, \ldots, k$ *we have* $\|u_i\| = 1$, *and*

(iii) for $i = 2, \ldots, k$ the vector u_i is an element of the real completion of the rational subspace orthogonal to the real space spanned by $u_1, \ldots, u_{i-1}$.

Then every linear order $\prec$ on $\mathbb{Q}^m$ corresponds one-to-one with U by

$$a \prec b :\Longleftrightarrow Ua <_{\text{lex}} Ub,$$

where $a, b \in \mathbb{Q}^m$ and $<_{\text{lex}}$ denotes the lexicographic order (cf. [Win96, Ex. 8.2.1 a]). The order $\prec$ restricts to an admissible order on $\mathbb{N}^m$ if and only if the topmost non-vanishing entry in every column of U is positive.

For $a = (a_1, \ldots, a_c)$ with $c \geq d$ let $\mathrm{Pr}_d(a) := (a_1, \ldots, a_d)$.

For $E = \{e_1, \ldots, e_q\}$ the classification of admissible orders on $\mathbb{N}^m \times E$ as provided by Rust and Reid [RR97] is given in the following theorem.

Theorem 2.2.7. *Let $k_1, \ldots, k_q \in \{1, \ldots, m\}$. For $s = 1, \ldots, q$ let $\gamma_s \in \mathbb{R}^{k_s}$ and $U_s \in \mathbb{R}^{k_s \times m}$ a matrix corresponding to an admissible order on $\mathbb{N}^m$ as in Theorem 2.2.6. By m_{ij} we denote the largest non-vanishing integer for which U_i and U_j have the first m_{ij} rows in common. Let $E = \{e_1, \ldots, e_q\}$ and let $T = (t_{ij}) \in \mathbb{N}^{q \times q}$. Let α be an element of the symmetric group on $\{1, \ldots, q\}$ such that for all $1 \leq i, j, k \leq q$ we have*

(i) $0 \leq t_{ij} \leq m_{ij}$,

(ii) $t_{ii} = m_{ii} = n_i$,

(iii) $t_{ij} = t_{ji}$,

(iv) $t_{ik} \geq \min\{t_{ij}, t_{jk}\}$, and

(v) whenever $t_{ik} > \max\{t_{ij}, t_{jk}\}$ and $\alpha(i) < \alpha(j)$ then $\alpha(k) < \alpha(j)$.

Then for $a, b \in \mathbb{N}^m, 1 \leq i, j \leq q$ an admissible order $\prec$ on $\mathbb{N}^m \times E$ is defined by

$$(a, e_i) \prec (b, e_j) :\Longleftrightarrow (\mathrm{Pr}_{t_{ij}}(U_i a + \gamma_i), \alpha(i)) <_{\text{lex}} (\mathrm{Pr}_{t_{ij}}(Ub + \gamma_j), \alpha(j)). \tag{2.1}$$

Conversely, any admissible order on $\mathbb{N}^m \times E$ can be represented as in (2.1) *by matrices $U_1, \ldots, U_q$, vectors $\gamma_1, \ldots, \gamma_q$, a matrix $T \in \mathbb{N}^{q \times q}$ and an element α of the symmetric group on $\{1, \ldots, q\}$.*

2.2.4 Representation of generalized term orders

Consider a generalized term order $\prec$ with respect to the orthant decomposition $\{\mathbb{N}^m \times \mathbb{Z}_k^{(n)} \mid 1 \leq k \leq p\}$ of $\mathbb{N}^m \times \mathbb{Z}^n$ with generators $\xi_1 \prec \ldots \prec \xi_r$. Let $k_1, \ldots, k_n \in \{1, \ldots, r\}$ pairwise distinct such that the orthant $\mathbb{N}^m \times \mathbb{Z}_k^{(n)}$ is generated by $\xi_{k_1} \prec \ldots \prec \xi_{k_n}$. We assume that for $1 \leq k, l \leq p$ with $k \neq l$ there exists $s_{kl} \in \{1, \ldots, n\}$ such that

(i) for all $1 \leq i < s_{kl}$ we have $k_i = l_i$, and

(ii) $k_{s_{kl}} < l_{s_{kl}}$.

For $b \in \mathbb{Z}^n$ let

$$k_b := \min\{k \mid 1 \leq k \leq p, b \in \mathbb{Z}_k^{(n)}\}.$$

By the above assumptions on orthant decompositions there exist unique $\beta_1, \ldots, \beta_r \in \mathbb{N}$ with

(i) $b = \sum_{i=1}^{r} \beta_i \xi_i$,

(ii) $\beta_i \geq 0$ for all i with ξ_i being a generator of $\mathbb{Z}_{k_b}^{(n)}$, and

(iii) $\beta_i = 0$ for all i with ξ_i not being a generator of $\mathbb{Z}^{(n)}_{k_b}$.

We define $\phi : \mathbb{Z}^n \to \mathbb{N}^r$ by

$$\phi(b) := (\beta_1, \ldots, \beta_r). \tag{2.2}$$

Lemma 2.2.8. *Let $E = \{e_1, \ldots, e_q\}$ and $\prec$ a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ with respect to the orthant decomposition $\Xi = \{\mathbb{N}^m \times \mathbb{Z}^{(n)}_k \mid 1 \leq k \leq p\}$ where the orthants and their generators meet the assumptions given in the definition of ϕ above. Then there exist*

(i) $k_1, \ldots, k_q \in \{1, \ldots, m+r\}$ such that for every $s = 1, \ldots, q$ there exist $\gamma_s \in \mathbb{R}^{k_s}$ and a matrix $U_s \in \mathbb{R}^{k_s \times (m+r)}$ corresponding to an admissible order on $\mathbb{N}^{m+r}$ as in Theorem 2.2.6,

(ii) $T = (t_{ij}) \in \mathbb{N}^{q \times q}$, and

(iii) an element α of the symmetric group on $\{1, \ldots, q\}$

such that for all $1 \leq i, j, k \leq q$ we have

(i) $0 \leq t_{ij} \leq m_{ij}$,

(ii) $t_{ii} = m_{ii} = n_i$,

(iii) $t_{ij} = t_{ji}$,

(iv) $t_{ik} \geq \min\{t_{ij}, t_{jk}\}$, and

(v) whenever $t_{ik} > \max\{t_{ij}, t_{jk}\}$ and $\alpha(i) < \alpha(j)$ then $\alpha(k) < \alpha(j)$.

Then for $(a_1, b_1), (a_2, b_2) \in \mathbb{N}^m \times \mathbb{Z}^n, 1 \leq i, j \leq q$ we have

$$\begin{aligned}&(a_1, b_1, e_i) \prec (a_2, b_2, e_j) \iff \\ &\quad (\mathrm{Pr}_{t_{ij}}(U_i(a_1, \phi(b_1)) + \gamma_i), \alpha(i)) <_{\mathrm{lex}} (\mathrm{Pr}_{t_{ij}}(U_j(a_2, \phi(b_2)) + \gamma_j), \alpha(j)).\end{aligned} \tag{2.3}$$

Proof. Let $\xi_1, \ldots, \xi_r$ be the generators of the orthant decomposition Ξ and for $k = 1, \ldots, p$ let $k_1, \ldots, k_n$ be such that Ξ_k is generated by $\xi_{k_1}, \ldots, \xi_{k_n}$. Note that ϕ as defined in (2.2) is injective. We define $\psi : \mathbb{N}^r \to Z^n$ by

$$(\beta_1, \ldots, \beta_r) \mapsto \sum_{i=1}^{r} \beta_i \xi_i.$$

Hence, for $i = 1, 2$ and $\beta^{(i)} = (\beta^{(i)}_1, \ldots, \beta^{(i)}_r) \in \mathbb{N}^r$ we have

$$\psi(\beta^{(1)}) + \psi(\beta^{(2)}) = \psi(\beta^{(1)} + \beta^{(2)}).$$

Let $\beta^{(1)}, \beta^{(2)} \in \phi(\mathbb{Z}^n)$ with $\beta^{(1)} + \beta^{(2)} \in \phi(\mathbb{Z}^n)$. Since $\psi(\beta^{(1)}) \in \mathbb{Z}^n$ there exist $0 < t \in \mathbb{N}$ and $j_1, \ldots, j_t \in \{1, \ldots, p\}$ such that

$$\psi(\beta^{(1)}) \in \mathbb{Z}^{(n)}_{j_1} \cap \ldots \cap \mathbb{Z}^{(n)}_{j_t}.$$

If there exists $1 \leq i \leq r$ such that $\beta^{(1)}_i > 0$ and there exists a unique $k \in \{1, \ldots, p\}$ with $i \in \{k_1, \ldots, k_n\}$ then $\beta^{(1)} + \beta^{(2)}, \beta^{(2)} \in \phi(\mathbb{Z}^n)$ implies $\beta^{(1)} + \beta^{(2)}, \beta^{(2)} \in \phi(\mathbb{Z}^{(n)}_k)$ and $\psi(\beta^{(1)} + \beta^{(2)}), \psi(\beta^{(2)}) \in \mathbb{Z}^{(n)}_k$.

If there exists $v \in \{1, \ldots, p\}$ and $k^{(1)}, \ldots, k^{(v)}$ such that for all $i \in \{1, \ldots, r\}$ with $\beta^{(1)}_i > 0$ we have $i \in \{k^{(1)}_1, \ldots, k^{(1)}_n\} \cap \ldots \cap \{k^{(v)}_1, \ldots, k^{(v)}_n\}$ then there exists $k \in \{k^{(1)}, \ldots, k^{(v)}\}$ (not necessarily unique) such that $\beta^{(1)} + \beta^{(2)}, \beta^{(2)} \in \phi(\mathbb{Z}^{(n)}_k)$.

So now let $a_1, a_2, a_3 \in \mathbb{N}^m, \beta^{(1)}, \beta^{(2)}, \beta^{(3)} \in \phi(\mathbb{Z}^n), i_1, i_2 \in \{1, \dots, q\}$ such that $\beta^{(1)} + \beta^{(3)}, \beta^{(2)} + \beta^{(3)} \in \phi(\mathbb{Z}^n)$ and

$$(a_1, \psi(\beta^{(1)}), e_{i_1}) \prec (a_2, \psi(\beta^{(2)}), e_{i_2}).$$

Then there exists $k \in \{1, \dots, p\}$ such that $\psi(\beta^{(3)}) \in \mathbb{Z}_k^{(n)}$ and then also $\psi(\beta^{(2)}), \psi(\beta^{(2)} + \beta^{(3)}) \in \mathbb{Z}_k^{(n)}$. Hence, we have

$$(a_1 + a_3, \psi(\beta^{(1)}) + \psi(\beta^{(3)}), e_{i_1}) \prec (a_2 + a_3, \psi(\beta^{(2)}) + \psi(\beta^{(3)}), e_{i_2}).$$

Since $\prec$ is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ this yields

$$\begin{aligned} (a_1 + a_3, \psi(\beta^{(1)} + \beta^{(3)}), e_{i_1}) &= (a_1 + a_3, \psi(\beta^{(1)}) + \psi(\beta^{(3)}), e_{i_1}) \\ &\prec (a_2 + a_3, \psi(\beta^{(2)}) + \psi(\beta^{(3)}), e_{i_2}) \\ &= (a_2 + a_3, \psi(\beta^{(2)} + \beta^{(3)}), e_{i_2}). \end{aligned}$$

We define a partial order $\prec'$ on $\mathbb{N}^m \times \mathbb{N}^r \times E$ by the following rules (for $a = (a_1, \dots, a_m) \in \mathbb{N}^m, b = (b_1, \dots, b_r) \in \mathbb{N}^r$ we identify $(a, b) \in \mathbb{N}^m \times \mathbb{N}^r$ with $(a_1, \dots, a_m, b_1, \dots, b_r) \in \mathbb{N}^{m+r}$):

(i) If $a_1, a_2 \in \mathbb{N}^m, \beta^{(1)}, \beta^{(2)} \in \phi(\mathbb{Z}^n), i_1, i_2 \in \{1, \dots, q\}$ and

$$(a_1, \psi(\beta^{(1)}), e_{i_1}) \prec (a_2, \psi(\beta^{(2)}), e_{i_2})$$

then

$$(a_1, \beta^{(1)}, e_{i_1}) \prec' (a_2, \beta^{(2)}, e_{i_2}),$$

(ii) $(a, 0, e) \prec' (a, b, e)$ for all $a \in \mathbb{N}^m, b \in \mathbb{N}^r, e \in E$,

(iii) $(a, 0, e) \prec' (a + b, 0, e)$ for all $a, b \in \mathbb{N}^m, e \in E$.

Then $\mathbb{N}^m \times \mathbb{N}^r \times E$ together with $\prec'$ is a well-founded set. Hence, there exists a well-ordering $\prec''$ on $\mathbb{N}^m \times \mathbb{N}^r \times E$ extending $\prec'$ (see [Har05]). Since $\mathbb{N}^m \times \mathbb{N}^r \times E$ satisfies the cancellation law (cf. [KR00, Defg. 1.3.3.] it follows that $\prec''$ is an admissible order (see [KR00]). Applying Theorem 2.2.7 proves the claim. □

Example 2.2.9. *Let* $m = 0, n = 2, E = \{e\}$ *– and let* $\zeta_1 = (0, -1), \zeta_2 = (0, 1), \zeta_3 = (-1, 0), \zeta_4 = (1, 0)$. *Let* $\{\mathbb{Z}_j^{(2)} \mid j = 1, \dots, 4\}$ *be such that*

$$\mathbb{Z}_j^{(2)} \text{ is generated by } \begin{cases} \{\zeta_1, \zeta_2\} & \text{if } j = 1, \\ \{\zeta_1, \zeta_3\} & \text{if } j = 2, \\ \{\zeta_2, \zeta_4\} & \text{if } j = 3, \\ \{\zeta_3, \zeta_4\} & \text{if } j = 4. \end{cases}$$

We identify $\mathbb{Z}^2 \times E$ *with* $\mathbb{Z}^2$ *and define the generalized term order* $\prec$ *on* $\mathbb{Z}^2$ *by*

$$\begin{aligned} &(a_1, a_2) \prec (b_1, b_2) :\Longleftrightarrow \\ &\quad (|a_1| + |a_2|, |a_1|, a_1, a_2) \\ &\quad <_{\text{lex}} (|b_1| + |b_2|, |b_1|, b_1, b_2). \end{aligned}$$

Then $\phi : \mathbb{Z}^2 \to \mathbb{N}^4$ *is given by*

$$\phi(a_1, a_2) = \left(\frac{a_2 - |a_2|}{2}, \frac{a_2 + |a_2|}{2}, \frac{a_1 - |a_1|}{2}, \frac{a_1 + |a_1|}{2} \right)$$

and $\psi : \mathbb{N}^4 \to \mathbb{Z}^2$ *is given by*

$$\psi(\alpha_1, \ldots, \alpha_4) = (\alpha_4 - \alpha_3, \alpha_2 - \alpha_1).$$

Let

$$U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then M *corresponds to an admissible order* $\prec'$ *on* $\mathbb{N}^4$ *by*

$$\alpha \prec' \beta :\Longleftrightarrow M\alpha <_{\text{lex}} M\beta,$$

for $\alpha, \beta \in \mathbb{N}^4$ *and to the generalized term order* $\prec$ *by*

$$a \prec b \Longleftrightarrow M\phi(a) <_{\text{lex}} M\phi(b),$$

for $a, b \in \mathbb{Z}^2$. □

Remark 2.2.10. *From Lemma 2.2.8 it follows that the set of generalized term orders on* $\mathbb{N}^m \times \mathbb{Z}^n \times E$ *with respect to any orthant decomposition of* $\mathbb{N}^m \times \mathbb{Z}^n$ *is countable. Since the set of orthant decompositions is countable, too, we conclude that the set of generalized term orders on* $\mathbb{N}^m \times \mathbb{Z}^n \times E$ *is countable.*

2.3 Generalized Gröbner bases

The notion of Gröbner bases is well-recognized as an effective tool for the algorithmic treatment of polynomial algebra. It was first introduced by Buchberger in his Ph.D thesis [Buc65] although not called so at that time. Gröbner bases for difference-differential operators were introduced by Zhou and Winkler [ZW06, ZW08b]. Levin [Lev07a] considers the problem of computing multivariate dimension polynomials associated with modules over rings of Ore polynomials by means of so-called Gröbner bases with respect to several orderings. Zhou and Winkler consider bivariate dimension polynomials and introduce the notion of relative Gröbner bases as means of solution [ZW08a]. Pauer and Unterkircher considered Gröbner bases in Laurent polynomial rings and their applications to difference operators [PU99].

In this section for the sake of unified notation and to avoid having several definitions of different kinds of standard bases we combine the notions of relative Gröbner bases and Gröbner bases with respect to several orderings introducing weight relative Gröbner bases and provide their characterization. We will also provide methods of computation of such bases.

Unless otherwise noted, throughout this section let K be a difference-skew-differential field, $\{\tau_1, \ldots, \tau_m\}$ a set of mutually commuting injective endomorphisms on K, $\Delta = \{\delta_1, \ldots, \delta_m\}$ a basic set of skew-derivations such that for $i = 1, \ldots, m$ the skew-derivation δ_i is a skew-derivation with respect to τ_i, respectively, and $\{\sigma_1, \ldots, \sigma_n\}$ a basic set of automorphisms. By E we always denote the finite set $\{e_1, \ldots, e_q\}$ of free generators of a free difference-skew-differential module.

2.3.1 Orders with respect to orthant decompositions

Definition 2.3.1. *Let* $\prec$ *be a generalized term order on* $[\Delta, \Sigma^*]E$ *as in Definition 2.2.4. Then for every* $f = \sum_{\lambda \in [\Delta, \Sigma^*]E} a_\lambda \lambda$, *by* $\mathrm{lt}_\prec(f) := \max_\prec\{\lambda \mid a_\lambda \neq 0\}, \mathrm{lc}_\prec(f) := a_{\mathrm{lt}_\prec(f)}$, *and* $\mathrm{in}_\prec(f) := \mathrm{lc}_\prec(f)\,\mathrm{lt}_\prec(f)$ *we denote the* leading term, leading coefficient, and initial of f w.r.t. $\prec$, *resp. If no confusion is possible we write* lt, lc, *and* in *instead of* $\mathrm{lt}_\prec, \mathrm{lc}_\prec$, *and* $\mathrm{in}_\prec$, *resp.*

One of the most used characterizations of Gröbner bases is provided in terms of the following definition [AL94].

Definition 2.3.2. *Let $K, [X]$ and $K[X]$ denote a field of characteristic 0, the commutative semigroup generated by a finite set $X = \{x_1, \dots, x_m\}$, and the commutative polynomial ring with indeterminates $x_1, \dots, x_m$ over K, respectively. A subset G of an ideal I of the ring $K[X]$ is called a* Gröbner basis of I *with respect to the admissible order $\prec$ on $[X]$ iff for every $f \in I \setminus \{0\}$ there exists $g \in G$ such that $\mathrm{lt}(g)$ divides $\mathrm{lt}(f)$.*

For the more general case of a difference-skew-differential module this definition is not restrictive enough for the reason that in the polynomial ring $K[X]$ for any polynomial $f \in K[X] \setminus \{0\}$, term $\mu \in [X]$ and admissible order $\prec$ on $[X]$ we have

$$\mathrm{lt}(\mu f) = \mu \,\mathrm{lt}(f).$$

If K is a difference-skew-differential field with basic set of skew-derivations Δ and basic set of automorphisms Σ instead then in general for $f \in K[\Delta, \Sigma^*]E, \mu \in [\Delta, \Sigma^*]$ and a generalized term order $\prec$ on $[\Delta, \Sigma^*]$ we have

$$\mathrm{lt}(\mu f) \neq \mu \,\mathrm{lt}(f).$$

For $f = \sum_{\lambda \in [\Delta, \Sigma^*]} a_\lambda \lambda$ the set $\{\lambda \mid a_\lambda \neq 0\}$ is called the *support* of f and is denoted by $\mathrm{supp}(f)$. For $g = \sum_{\mu \in [\Delta, \Sigma^*], e \in E} b_{\mu e} \mu e$ the set $\{\mu e \mid b_{\mu e} \neq 0\}$ is called the *support* of g and is denoted by $\mathrm{supp}(g)$.

Lemma 2.3.3. [ZW08a, Lem. 3.2 and 3.3] *Let $\prec$ be a generalized term order on $[\Delta, \Sigma^*]$ with respect to the orthant decomposition $\{[\Delta, \Sigma^*]_k \mid 1 \leq k \leq p\}$, $f = \sum_{\lambda \in [\Delta, \Sigma^*]E} a_\lambda \lambda \in K[\Delta, \Sigma^*]E \setminus \{0\}$, with only finitely many $a_\lambda \in K$ not vanishing.*

(i) For $\mu \in [\Delta, \Sigma^*]$ we have $\mathrm{lt}(\mu f) = \max_\prec \{\mu a_\lambda \mid a_\lambda \neq 0\}$. In particular $\mathrm{lt}(\mu f) = \mu \lambda$ for a unique $\lambda \in \mathrm{supp}(f)$.

(ii) If for some $k \in \{1, \dots, p\}$ we have $\mathrm{lt}(f) \in [\Delta, \Sigma^*]_k E$ then $\mathrm{lt}(\mu f) = \mu \,\mathrm{lt}(f) \in [\Delta, \Sigma^*]_k E$ for any $\mu \in [\Delta, \Sigma^*]_k$.

(iii) For each $k \in \{1, \dots, p\}$ there exists some $\mu \in [\Delta, \Sigma^*]$ and a unique term λ_k of f such that

$$\mathrm{lt}(\mu f) = \mu \lambda_k \in [\Delta, \Sigma^*]_k E,$$

i.e., if for some $\mu_1, \mu_2 \in [\Delta, \Sigma^*]$ we have $\mathrm{lt}(\mu_1 f) = \mu_1 \lambda_{k_1} \in [\Delta, \Sigma^*]_k E$ and $\mathrm{lt}(\mu_2 f) = \mu_2 \lambda_{k_2} \in [\Delta, \Sigma^*]_k E$ then $\lambda_{k_1} = \lambda_{k_2}$. The term λ_k will then be denoted by $\mathrm{lt}_{k;\prec}(f)$ or $\mathrm{lt}_k(f)$ if no confusion is possible.

For the very same reason also the definition of S-polynomials becomes more complicated for a pair of difference-skew-differential operators. We follow the approach outlined in [ZW08a].

Definition 2.3.4. *Let $f, g \in K[\Delta, \Sigma^*]E \setminus \{0\}$ and let $\prec$ be a generalized term order on $[\Delta, \Sigma^*]E$. Let $\{[\Delta, \Sigma^*]_k \mid 1 \leq k \leq p\}$ be an orthant decomposition of $[\Delta, \Sigma^*]$. For every orthant $[\Delta, \Sigma^*]_k$ let $V_\prec(k, f, g)$ be a finite system of generators of the $K[\Delta, \Sigma^*]_k$-module*

$$\begin{aligned} &{}_{K[\Delta, \Sigma^*]_k}\langle \mathrm{lt}_\prec(\lambda f) \in [\Delta, \Sigma^*]_k E \mid \lambda \in [\Delta, \Sigma^*] \rangle \\ \cap \; &{}_{K[\Delta, \Sigma^*]_k}\langle \mathrm{lt}_\prec(\eta g) \in [\Delta, \Sigma^*]_k E \mid \eta \in [\Delta, \Sigma^*] \rangle. \end{aligned}$$

For every $k \in \{1, \dots, p\}, v \in V_\prec(k, f, g)$ the operator

$$S_\prec(k, f, g, v) := \frac{v}{\mathrm{lt}_{k;\prec}(f)} \frac{f}{\mathrm{lc}_{k;\prec}(f)} - \frac{v}{\mathrm{lt}_{k;\prec}(g)} \frac{g}{\mathrm{lc}_{k;\prec}(g)}$$

is called an S-polynomial *of f and g with respect to k, $\prec$, and v. If no confussion is possible we will write $S(k, f, g, v)$ instead of $S_\prec(k, f, g, v)$.* □

We generalize the notions of Gröbner bases with respect to several orderings and relative Gröbner bases as introduced in [Lev07a] and [ZW08a], respectively.

Definition 2.3.5. *We consider the vector* $\alpha = (\alpha_1, \ldots, \alpha_{m+n}) \in \mathbb{Q}_0^{m+n}$. *Then for a difference-skew-differential term* $\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \in [\Delta, \Sigma^*]$ *the* α-order ord_α *of* λ *is defined by*

$$\mathrm{ord}_\alpha(\lambda) := \alpha_1 k_1 + \cdots + \alpha_m k_m + \alpha_{m+1}|l_1| + \cdots + \alpha_{m+n}|l_n|,$$

and for any $f = \sum_{\lambda \in [\Delta,\Sigma^*]} a_\lambda \lambda \in K[\Delta, \Sigma^*] \setminus \{0\}$ *the* α*-order of* f *is defined by*

$$\mathrm{ord}_\alpha(f) := \max\{\mathrm{ord}_\alpha(\lambda) \mid a_\lambda \neq 0\}.$$

If E *is a finite set generating a free difference-skew-differential module then for any* $f = \sum_{e \in E} f_e e \in K[\Delta, \Sigma^*]E \setminus \{0\}$ *with* $f_e \in K[\Delta, \Sigma^*]$ *for all* $e \in E$ *the* α*-order of* f *is defined by*

$$\mathrm{ord}_\alpha(f) := \max\{\mathrm{ord}_\alpha(f_e) \mid f_e \neq 0\}.$$

We refine our notation taking into account also orthant decompositions in the following way.

Definition 2.3.6. *Let* Ξ *be an orthant decomposition of* $[\Delta, \Sigma^*]$ *with generators* $\xi_1, \ldots, \xi_r$. *Let* $\alpha = (\alpha_1, \ldots, \alpha_{r+m}) \in \mathbb{Q}_0^{r+m}$. *We say that* α *is a* weight vector *associated with the orthant decomposition* Ξ. *Then for a difference-skew-differential term* $\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \in [\Delta, \Sigma^*]$ *the* α-Ξ-order $\mathrm{ord}_{\alpha,\Xi}$ *of* λ *is defined by*

$$\begin{aligned}\mathrm{ord}_{\alpha,\Xi}(\lambda) \quad := \quad \min\{&\alpha_1 k_1 + \cdots + \alpha_m k_m + \alpha_{m+1} l_1 + \cdots + \alpha_{r+m} l_r \mid \\ &\delta_1^{k_1} \cdots \delta_m^{k_m} \xi_1^{l_1} \cdots \xi_r^{l_r} = \lambda \text{ and all } \xi_i \text{ with} \\ &l_i \neq 0 \text{ are generators of the same orthant}\},\end{aligned}$$

and for any $f = \sum_{\lambda \in [\Delta,\Sigma^*]} a_\lambda \lambda \in K[\Delta, \Sigma^*] \setminus \{0\}$ *the* α-Ξ*-order of* f *is defined by*

$$\mathrm{ord}_{\alpha,\Xi}(f) := \max\{\mathrm{ord}_{\alpha,\Xi}(\lambda) \mid a_\lambda \neq 0\}.$$

If E *is a finite set generating a free difference-skew-differential module then for any* $f = \sum_{e \in E} f_e e \in K[\Delta, \Sigma^*]E \setminus \{0\}$ *with* $f_e \in K[\Delta, \Sigma^*]$ *for all* $e \in E$ *the* α-Ξ*-order of* f *is defined by*

$$\mathrm{ord}_{\alpha,\Xi}(f) := \max\{\mathrm{ord}_{\alpha,\Xi}(f_e) \mid f_e \neq 0\}.$$

Let $t \in \mathbb{N}, \mathcal{T} \in \mathbb{Q}_0^{(m+r) \times t}$ a matrix with columns $\mathcal{T}_1, \ldots, \mathcal{T}_t$ which we consider as weight vectors associated with the orthant decomposition Ξ generated by $\xi_1, \ldots, \xi_r$ and $\lambda = \delta^u \sigma^v e \in [\Delta, \Sigma^*]E$. If $\prec$ is a generalized term order on $[\Delta, \Sigma^*]E$ satisfying for some $j \in \{1, \ldots, t\}$ and for all $\lambda, \mu \in [\Delta, \Sigma^*]E$ the condition

$$\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda) < \mathrm{ord}_{\mathcal{T}_j,\Xi}(\mu) \Longrightarrow \lambda \prec_j \mu$$

then we say that $\prec$ respects $\mathcal{T}_j$. If $\prec_1, \ldots, \prec_t$ are generalized term orders on $[\Delta, \Sigma^*]E$ such that for any $j \in \{1, \ldots, t\}$ the order $\prec_j$ respects $\mathcal{T}^{(j)}$ then we say that $\prec_1, \ldots, \prec_t$ respects $\mathcal{T}$. We call $\mathcal{T}$ a t-weight matrix or simply weight matrix if t is clear from the context. From now on unless otherwise noted whenever we consider a weight matrix $\mathcal{T}$ we mean a t-weight matrix where t could be possibly vanishing.

2.3.2 Reduction

We introduce a suitable reduction relation generalizing relative reduction as introduced by Zhou and Winkler [ZW08a] and reduction with respect to several orderings as introduced by Levin [Lev07a].

Definition 2.3.7. *Let $f, g \in K[\Delta, \Sigma^*]E \setminus \{0\}$, $\prec$ a generalized term order, $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ a weight matrix, and $\mathcal{R}$ a finite set of generalized term orders. If there exists $\lambda \in [\Delta, \Sigma^*]$ such that for all $j \in \{1, \ldots, t\}$ and $\prec' \in R$ we have*

(i) $\mathrm{lt}_\prec(\lambda g) = \mathrm{lt}_\prec(f)$,

(ii) $\mathrm{lt}_{\prec'}(\lambda g) \preceq' \mathrm{lt}_{\prec'}(f)$, *and*

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda g) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f)$,

then we say that f is $\prec$-reducible to $f - \mathrm{lc}_\prec(f)\lambda\frac{g}{\mathrm{lc}_\prec(g)}$ modulo g relative to $\mathcal{R}$ respecting $\mathcal{T}$.

Let $G \subseteq K[\Delta, \Sigma^*]E \setminus \{0\}$. If there exist $f_0, \ldots, f_{s-1} \in K[\Delta, \Sigma^*]E \setminus \{0\}$, $f_s \in K[\Delta, \Sigma^*]E$, $g_0, \ldots, g_s \in G$ such that for all $i = 1, \ldots, s$ the difference-skew-differential operator f_{i-1} is $\prec$-reducible to f_i modulo g_{i-1} relative to $\mathcal{R}$ respecting $\mathcal{T}$ then we say that f_0 is $\prec$-reducible to f_s modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$.

Algorithm 2.3.6 describes the reduction process for finite G and $\mathcal{R}$:

Algorithm 2.3.6 `reduce`

IN: $f \in K[\Delta, \Sigma^*]E \setminus \{0\}$, finite $G \subseteq K[\Delta, \Sigma^*]E \setminus \{0\}$, a generalized term order $\prec$, a weight matrix $\mathcal{T}$, and a finite set of generalized term orders $\mathcal{R}$

OUT: h such that f is $\prec$-reducible to h modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$ and h is not $\prec$-reducible modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$.

$h := f$

while there exist $g \in G, \lambda \in [\Delta, \Sigma^*]$ such that h is $\prec$-reducible modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$ and $\mathrm{lt}_\prec(\lambda g) = \mathrm{lt}_\prec(h)$ **do**

  $h := h - \mathrm{lc}_\prec(f)\lambda\frac{g}{\mathrm{lc}_\prec(g)}$

end while

return h;

Theorem 2.3.7. *Algorithm 2.3.6 is correct and terminates.*

Proof. The correctness of Algorithm 2.3.6 is an immediate consequence of Definition 2.3.7.

For termination we observe that by Lemma 2.2.8 there exists $r \in \mathbb{N}$ such that the generalized term order $\prec$ induces an admissible order on $\mathbb{N}^{m+r} \times E$. Now let $f = f_0, f_1, f_2, \ldots \in K[\Delta, \Sigma^*]E$ be the intermediate reduction results appearing during the execution of algorithm 2.3.6 and denote the elements in $\mathbb{N}^{m+r} \times E$ corresponding to $\mathrm{lt}_\prec(f_0), \mathrm{lt}_\prec(f_1), \mathrm{lt}_\prec(f_2), \ldots$ by $\bar{f}_0, \bar{f}_1, \bar{f}_2, \ldots$. Since for every $e \in E$ the ring $\mathbb{N}^{m+r} \times \{e\}$ is noetherian the set $\{\bar{f}_i \mid i = 0, 1, 2, \ldots\} \cap \mathbb{N}^{m+r} \times \{e\}$ contains a minimal element with respect to $\prec$ and therefore is finite. Since E is finite we conclude that $\bar{f}_0, \bar{f}_1, \bar{f}_2, \ldots$ and hence also $f_0, f_1, f_2, \ldots$ must be finite. □

Remark 2.3.8. *Let $f \in K[\Delta, \Sigma^*]E \setminus \{0\}$, $G \subseteq K[\Delta, \Sigma^*]E \setminus \{0\}$ finite, $\prec$ a generalized term order, $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ a weight matrix, $\mathcal{R}$ a set of generalized term orders and $h \in K[\Delta, \Sigma^*]E$ such that f is $\prec$-reducible modulo G to h relative to $\mathcal{R}$ respecting $\mathcal{T}$. From Theorem 2.3.7 it follows that then there exist $g_1, \ldots, g_s \in G$, $h_1, \ldots, h_s \in K[\Delta, \Sigma^*]E$ such that for all $\prec' \in \mathcal{R} \cup \{\prec\}$, $j \in \{1, \ldots, t\}$ we have*

(i) $f = \sum_{i=1}^s h_i g_i + h$,

(ii) $\mathrm{lt}_{\prec'} \preceq' \mathrm{lt}_{\prec'}(h_i g_i)$, *and*

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(h_i g_i) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f)$.

□

2.3.3 Definition of weight relative Gröbner bases

Definition 2.3.7 gives rise to the following definition of weight relative Gröbner bases.

Definition 2.3.9. *Let $\prec$ be a generalized term order, $\mathcal{T} \in \mathbb{Q}_0^{(m+r)\times t}$ a weight matrix , and $\mathcal{R}$ a set of generalized term orders. Let $M \subseteq K[\Delta, \Sigma^*]$ be a difference-skew-differential module and let $G \subseteq M \setminus \{0\}$. If every $f \in M \setminus \{0\}$ is $\prec$-reducible modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$ then G is called a* $\prec$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$. *If no confusion is possible we will say that G is a weight relative Gröbner basis.*

Remark 2.3.10. *(i) Definition 2.3.9 includes the definitions of Gröbner bases ($t = 0, \mathcal{R} = \emptyset$, [ZW06, Def. 3.5.]), relative Gröbner bases ($t = 0$, [ZW08a, Def. 3.3.], if G happens to be finite), and Gröbner bases with respect to several orderingsdecomp ($\mathcal{R} = \emptyset$, $\mathcal{T}$ choosen appropriately, [Lev07a, Def. 3.3.]). If $t = 0$ and $\mathcal{R} = \emptyset$ then we also call G a* Gröbner basis with respect to $\prec$.

(ii) Let $\mathcal{T} \in \mathbb{Q}_0^{(m+r)\times(t)}$ be a weight matrix with columns $\mathcal{T}_1, \ldots, \mathcal{T}_t$, and $\mathcal{T}' \in \mathbb{Q}_0^{(m+r)\times(t-1)}$ the weight matrix with columns $\mathcal{T}_1, \ldots, \mathcal{T}_{t-1}$. Furthermore let $\prec_t$ be a generalized term order respecting $\mathcal{T}_t$. If G is a $\prec$-Gröbner basis relative to $\mathcal{R} \cup \{\prec_t\}$ respecting $\mathcal{T}'$ then it is also a $\prec$-Gröbner basis relative to $\mathcal{R}$ respecting $\mathcal{T}$.

We provide a generalization of [ZW08a, Prop. 3.1.]. The proof is a direct consequence of Definition 2.3.9.

Proposition 2.3.11. *Let $\prec$ be a generalized term order, $\mathcal{T} \in \mathbb{Q}_0^{(m+r)\times t}$, $\mathcal{R}$ a set of generalized term orders, $M \subseteq K[\Delta, \Sigma^*]E$ a difference-skew-differential module and $G \subseteq M \setminus \{0\}$. TFAE*

(i) G is a weight relative Gröbner basis,

(ii) $f \in M$ if and only if $f = 0$ or f is $\prec$-reducible to 0 modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$,

(iii) every $f \in M \setminus \{0\}$ is $\prec$-reducible modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$.

Corollary 2.3.12. *Let $\prec$ be a generalized term order, $\mathcal{T} \in \mathbb{Q}_0^{(m+r)\times t}$, $\mathcal{R}$ a set of generalized term orders, $M \subseteq K[\Delta, \Sigma^*]E$ a difference-skew-differential module, G a weight relative Gröbner basis and $f \in K[\Delta, \Sigma^*]E$. Then there exists a unique $h \in K[\Delta, \Sigma^*]E$ such that f is $\prec$-reducible to h modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$ and h is not $\prec$-reducible modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$.*

Definition 2.3.13. *The unique element h whose existence is established by Corollary 2.3.12 is called the* normal form of f modulo G.

2.3.4 Computation of weight relative Gröbner bases

The following lemma is a somewhat enhanced version of [AL94, L. 1.7.5.] and [Lev07a, Prop. 3.9.].

Lemma 2.3.14. *Let $f, g_1, \ldots, g_s \in K[\Delta, \Sigma^*]E$, $\prec$ a generalized term order with respect to the orthant decomposition $\Xi = \{\Xi_k \mid 1 \le k \le p\}$, $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ a weight matrix, $\mathcal{R}$ a set of generalized term orders, $c_1, \ldots, c_s \in K, \lambda_1, \ldots, \lambda_s \in [\Delta, \Sigma^*]$ and $u \in [\Delta, \Sigma^*]E$ such that for all $\prec' \in \mathcal{R}$, $j \in \{1, \ldots, t\}$ and for some $k \in \{1, \ldots, p\}$ we have*

(i) $f = \sum_{i=1}^{s} c_i \lambda_i g_i$,

(ii) $\mathrm{lt}_\prec(f) \prec \mathrm{lt}_\prec(\lambda_1 g_1) = \ldots = \mathrm{lt}_\prec(\lambda_s g_s) = u \in [\Delta, \Sigma^*]_k E$,

(iii) $\mathrm{lt}_{\prec'}(\lambda_i g_i) \preceq' \mathrm{lt}_{\prec'}(f)$, and

(iv) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda_i g_i) \le \mathrm{ord}_{\mathcal{T}_j,\Xi}(f)$.

Then for $1 \leq s_1, s_2 \leq s$ *there exist* $v_{s_1,s_2} \in V_{\prec}(k, g_{s_1}, g_{s_2}), c_{s_1,s_2} \in K$ *such that for* $\theta_{s_1,s_2} := \frac{u}{v_{s_1,s_2}}$ *and for all* $\prec' \in \mathcal{R}, j \in \{1, \ldots, t\}$ *we have*

(i) $f = \sum_{s_1,s_2=1}^{s} c_{s_1,s_2}\theta_{s_1,s_2} S_{\prec}(k, g_{s_1}, g_{s_2}, v_{s_1,s_2})$,

(ii) $\mathrm{lt}_{\prec}(\theta_{s_1,s_2} S_{\prec}(k, g_{s_1}, g_{s_2}, v_{s_1,s_2})) \prec u$,

(iii) $\mathrm{lt}_{\prec'}(\theta_{s_1,s_2} S_{\prec}(k, g_{s_1}, g_{s_2}, v_{s_1,s_2})) \preceq' \mathrm{lt}_{\prec'}(f)$, *and*

(iv) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\theta_{s_1,s_2} S_{\prec}(k, g_{s_1}, g_{s_2}, v_{s_1,s_2})) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f)$.

Proof. W.l.o.g. $\mathrm{lc}_{k;\prec}(g_i) = 1$ for $i = 1, \ldots, s$. Then $\mathrm{lt}_{\prec}(f) \prec u$ and $c_1, \ldots, c_s \in K$ imply $c_1 + \cdots + c_s = 0$. By Definition 2.3.4 for $v_{s_1,s_2} \in V_{\prec}(k, g_{s_1}, g_{s_2})$ we have

$$S(k, g_{s_1}, g_{s_2}, v_{s_1,s_2}) = \frac{v_{s_1,s_2}}{\mathrm{lt}_{k;\prec}(g_{s_1})} g_{s_1} - \frac{v_{s_1,s_2}}{\mathrm{lt}_{k;\prec}(g_{s_2})} g_{s_2},$$

and for $i = 2, \ldots, s-1$ we have

$$\frac{u}{\mathrm{lt}_{k;\prec}(g_i)} g_i = \frac{u}{v_{i-1,i}} \frac{v_{i-1,i}}{\mathrm{lt}_{k;\prec}(g_i)} g_i = \frac{u}{v_{i,i+1}} \frac{v_{i,i+1}}{\mathrm{lt}_{k;\prec}(g_i)} g_i.$$

Using this and

$$\begin{aligned} \frac{u}{\mathrm{lt}_{k;\prec}(g_1)} g_1 &= \frac{u}{v_{1,2}} \frac{v_{1,2}}{\mathrm{lt}_{k;\prec}(g_1)} g_1, \\ \frac{u}{\mathrm{lt}_{k;\prec}(g_s)} g_s &= \frac{u}{v_{s-1,s}} \frac{v_{s-1,s}}{\mathrm{lt}_{k;\prec}(g_s)} g_s \end{aligned}$$

we obtain

$$\begin{aligned} f &= c_1\lambda_1 g_1 + \cdots c_s \lambda_s f_s \\ &= c_1 \frac{u}{\mathrm{lt}_{k;\prec}(g_1)} g_1 + \cdots + c_s \frac{u}{\mathrm{lt}_{k;\prec}(g_s)} g_s \\ &= c_1\theta_{1,2} \left(\frac{v_{1,2}}{\mathrm{lt}_{k;\prec}(g_1)} g_1 - \frac{v_{1,2}}{\mathrm{lt}_{k;\prec}(g_2)} g_2 \right) \\ &\quad + (c_1 + c_2)\theta_{2,3} \left(\frac{v_{2,3}}{\mathrm{lt}_{k;\prec}(g_2)} g_2 - \frac{v_{2,3}}{\mathrm{lt}_{k;\prec}(g_3)} g_3 \right) + \cdots \\ &\quad + (c_1 + \cdots + c_{s-1})\theta_{s-1,s} \left(\frac{v_{s-1,s}}{\mathrm{lt}_{k;\prec}(g_{s-1})} g_{s-1} - \frac{v_{s-1,s}}{\mathrm{lt}_{k;\prec}(g_s)} g_s \right) \\ &\quad + (c_1 + \cdots + c_s) \frac{u}{\mathrm{lt}_{k;\prec}(g_s)} \\ &= c_1\theta_{1,2} S_{\prec}(k, g_1, g_2, v_{1,2}) + (c_1 + c_2)\theta_{2,3} S_{\prec}(k, g_2, g_3, v_{2,3}) + \cdots \\ &\quad + (c_1 + \cdots + c_{s-1})\theta_{s-1,s} S_{\prec}(k, g_{s-1}, g_s, v_{s-1,s}). \end{aligned}$$

and for all $\prec' \in \mathcal{R}, j \in \{1, \ldots, t\}, i = 2, \ldots, s$ we have

(i) $\mathrm{lt}_{\prec}(\theta_{i-1,i} S_{\prec}(k, g_{i-1}, g_i, v_{i-1,i})) \prec u$,

(ii) $\mathrm{lt}_{\prec'}(\theta_{i-1,i} S_{\prec}(k, g_{i-1}, g_i, v_{i-1,i})) \preceq' \mathrm{lt}_{\prec'}(f)$, and

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\theta_{i-1,i} S_{\prec}(k, g_{i-1}, g_i, v_{i-1,i})) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f)$.

□

For Gröbner bases there are several equivalent characterizations which play a key role e.g. in basis transformation algorithms or algorithms for fast computation of Gröbner bases. Since for weight relative Gröbner bases the computational cost is even higher we would like to have more useful characterizations for them. The following result extends a well known result (see e.g. [AL94, Theorem 1.6.2.] for Gröbner bases to weight relative Gröbner bases.

Lemma 2.3.15. *Let $M \trianglelefteq K[\Delta, \Sigma^*]E$ be a difference-skew-differential module and let $G \subseteq M \setminus \{0\}$ be finite. Let $\prec$ be a generalized term order, $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ a weight matrix and $\mathcal{R}$ a set of generalized term orders. The following are equivalent:*

(i) For every $f \in M \setminus \{0\}$ there exist $h_1, \ldots, h_s \in K[\Delta, \Sigma^*]E, g_1, \ldots, g_s \in G$ such that

(a) $f = \sum_{i=1}^s h_i g_i$,

(b) for $i = 1, \ldots, s$ and every $\prec' \in \mathcal{R} \cup \{\prec\}$ we have

$$\mathrm{lt}_{\prec'}(h_i g_i) \preceq' \mathrm{lt}_{\prec'}(f),$$

and

(c) for $i = 1, \ldots, s$ and every $j \in \{1, \ldots, t\}$ we have

$$\mathrm{ord}_{\mathcal{T}_j, \Xi}(h_i g_i) \leq \mathrm{ord}_{\mathcal{T}_j, \Xi}(f).$$

(ii) G is a $\prec$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$.

Proof. "(i)$\Longrightarrow$(ii)": Assume that for every $f \in M \setminus \{0\}$ there exist $h_1, \ldots, h_s \in K[\Delta, \Sigma^*]E$ such that (a), (b), and (c) hold. Then there exists $i_0 \in \{1, \ldots, s\}$ such that $\mathrm{lt}_\prec(h_{i_0} g_{i_0}) = \mathrm{lt}_\prec(f)$. Hence, there exists a term λ in h_{i_0} with

$$\mathrm{lt}_\prec(\lambda g_{i_0}) = \mathrm{lt}_\prec(f).$$

On the other hand for every $\prec' \in \mathcal{R}$ we have

$$\mathrm{lt}_{\prec'}(\lambda g_{i_0}) \preceq' \mathrm{lt}_{\prec'}(h_{i_0} g_{i_0}) \preceq' \mathrm{lt}_{\prec'}(f)$$

and for every $j = 1, \ldots, t$ we have

$$\mathrm{ord}_{\mathcal{T}_j, \Xi}(\lambda g_{i_0}) \leq \mathrm{ord}_{\mathcal{T}_j, \Xi}(h_{i_0} g_{i_0}) \leq \mathrm{ord}_{\mathcal{T}_j, \Xi}(f)$$

and we conclude that f is $\prec$-reducible modulo g_{i_0} relative to $\mathcal{R}$ respecting $\mathcal{T}$. By Proposition 2.3.11 G is a $\prec$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$.
"(ii)$\Longrightarrow$(i)": Assume that G is a $\prec$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$. By Lemma 2.3.11 every $f \in M \setminus \{0\}$ is $\prec$-reducible to 0 modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$. Then by Definition 2.3.7 there exist $s \in \mathbb{N}$, $f_1, \ldots, f_s \in K[\Delta, \Sigma^*]E$ with $f_1 = f$ and $g_1, \ldots, g_s \in G, j_1, \ldots, j_{s'} \in \{1, \ldots, s\}, \lambda_1, \ldots, \lambda_{s'} \in [\Delta, \Sigma^*]$ such that

(i) $0 = f_1 - \sum_{i=1}^{s'} \mathrm{lc}_\prec(f_i) \lambda_i \frac{g_{j_i}}{\mathrm{lc}_\prec(g_{j_i})}$,

(ii) $\mathrm{lt}_{\prec'}(\lambda_i g_{j_i}) \preceq' \mathrm{lt}_{\prec'}(f_1)$ for all $i \in \{1, \ldots, s'\}, \prec' \in \mathcal{R}$, and

(iii) $\mathrm{ord}_{\mathcal{T}_j, \Xi}(\lambda_i g_{j_i}) \leq \mathrm{ord}_{\mathcal{T}_j, \Xi}(f_0)$ for all $i \in \{1, \ldots, s'\}, j \in \{1, \ldots, t\}$.

Collecting coefficients' belonging to the same element of G it is clear that there exist $h_1, \ldots, h_{s'} \in K[\Delta, \Sigma^*]$ such that (a), (b), and (c) hold. □

The following theorem generalizing [Lev07a, Thm. 3.10.] and [ZW08a, Thm. 3.3.] establishes an S-polynomial criterion giving rise to a method for computing weight relative Gröbner bases.

Theorem 2.3.16. *Let $\prec, \prec_0$ be generalized term orders, $\Xi = \{\Xi_k \mid 1 \leq k \leq p\}$ the orthant decomposition with respect to which $\prec_0$ is defined, $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ a weight matrix with columns $\mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(t)}$, $\mathcal{R}$ a set of generalized term orders, $M \subseteq K[\Delta, \Sigma^*]E$ a difference-skew-differential module and $G = \{g_1, \ldots, g_s\} \subseteq M \setminus \{0\}$ a $\prec$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$. If for all $k = 1, \ldots, p, g, g' \in G$ and for all $v \in V_{\prec_0}(k, g, g')$ the S-polynomial $S_{\prec_0}(k, g, g', v)$ is $\prec_0$-reducible to 0 modulo G relative to $\mathcal{R} \cup \{\prec\}$ respecting $\mathcal{T}$, then G is a $\prec_0$-Gröbner basis relative to $\mathcal{R} \cup \{\prec\}$ respecting $\mathcal{T}$.*

So, if we assume that there exist generalized term orders $\prec'_1, \ldots, \prec'_s$ such that $\mathcal{R} = \{\prec'_1, \ldots, \prec'_s\}$ and that there exist generalized term orders $\prec_1, \ldots, \prec_t$ respecting $\mathcal{T}$ then a $\prec$-Gröbner basis relative to $\mathcal{R}$ respecting $\mathcal{T}$ can be computed iteratively by first computing a $\prec_t$-Gröbner basis, then – using Theorem 2.3.16 – a $\prec_{t-1}$-Gröbner basis respecting $\mathcal{T}^{(t)}$, and so on until we obtain a $\prec$-Gröbner basis relative to $\mathcal{R}$ respecting $\mathcal{T}$.

Proof. Let $f \in M \setminus \{0\}$ and $\prec_1, \ldots, \prec_t$ respecting $\mathcal{T}$. We have to show that there exist $\lambda \in [\Delta, \Sigma^*], g \in G$ such that for all $j \in \{1, \ldots, t\}$ and $\prec' \in R \cup \{\prec\}$ we have

(i) $\mathrm{lt}_{\prec_0}(\lambda g) = \mathrm{lt}_{\prec_0}(f)$,

(ii) $\mathrm{lt}_{\prec'}(\lambda g) \preceq' \mathrm{lt}_{\prec'}(f)$, and

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda g) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f)$.

By assumption there exist $h_1, \ldots, h_s \in K[\Delta, \Sigma^*]E$ such that for all $\prec' \in \mathcal{R} \cup \{\prec\}, j \in \{1, \ldots, t\}$ we have

(i) $f = \sum_{i=1}^{s} h_i g_i + h$,

(ii) $\mathrm{lt}_{\prec'} \preceq' \mathrm{lt}_{\prec'}(h_i g_i)$, and

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(h_i g_i) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f)$.

Let $u := \max_{\prec_0}\{\mathrm{lt}_{\prec_0}(h_i g_i) \mid i = 1, \ldots, s\}$ and let k be such that $u \in \Xi_k E$. W.l.o.g. we assume that $h_1, \ldots, h_s$ are choosen such that u is minimal with respect to $\prec_0$. For any $i \in \{1, \ldots, s\}$ and $\lambda \in \mathrm{supp}(h_i)$ we have

$$\mathrm{lt}_{\prec_0}(\lambda g_i) \preceq_0 \mathrm{lt}_{\prec_0}(f).$$

If for some $i \in \{1, \ldots, s\}$ we have $\mathrm{lt}_{\prec_0}(f) = u = \mathrm{lt}_{\prec_0}(h_i g_i)$ then there exists $\lambda \in \mathrm{supp}(h_i)$ such that for all $j \in \{1, \ldots, t\}$ and $\prec' \in R \cup \{\prec\}$ we have

(i) $\mathrm{lt}_{\prec_0}(\lambda g_i) = \mathrm{lt}_{\prec_0}(f)$,

(ii) $\mathrm{lt}_{\prec'}(\lambda g_i) \preceq' \mathrm{lt}_{\prec'}(f)$, and

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda g_i) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f)$.

Hence, f would be $\prec_0$-reducible modulo G relative to $\mathcal{R} \cup \{\prec\}$ respecting $\mathcal{T}$. So assume $\mathrm{lt}_{\prec_0}(f) \prec_0 u$ and let

$$B := \{i \mid \mathrm{lt}_{\prec_0}(h_i g_i) = u\}.$$

Then for every $i \in B$ there exists a unique difference-skew-differential term $\lambda_i \in \mathrm{supp}(h_i)$ such that for every term $\eta \in \mathrm{supp}(h_i) \setminus \{\lambda_i\}$ we have

$$u = \mathrm{lt}_{\prec_0}(\lambda_i g_i) \succ_0 \mathrm{lt}_{\prec_0}(\eta g_i).$$

If we denote the coefficient of λ_i in h_i by c_i then

$$\begin{aligned} f &= \sum_{i \in B} h_i g_i + \sum_{i \notin B} h_i g_i \\ &= \sum_{i \in B} c_i \lambda_i g_i + \sum_{i \in B} (h_i - c_i \lambda_i) g_i + \sum_{i \notin B} h_i g_i, \end{aligned}$$

and all terms appearing in the last two sums are $\prec_0 u$. For $i \in B$ by $\mu_i \in \operatorname{supp}(g_i)$ we denote the difference-skew-differential term satisfying $u = \operatorname{lt}_{\prec_0}(\lambda_i g_i) = \lambda_i \mu_i$. If we denote the coefficient of μ_i in g_i by d_i then we obtain

$$\begin{aligned} \sum_{i \in B} c_i \lambda_i &= \sum_{i \in B} c_i \lambda_i d_i \mu_i + \sum_{i \in B} c_i \lambda_i d_i \left(\frac{g_i}{d_i} - \mu_i \right) \\ &= \left(\sum_{i \in B} c_i d_i' \right) u + \sum_{i \in B} c_i d_i' \lambda_i \left(\frac{g_i}{d_i} - \mu_i \right), \end{aligned}$$

for some $d_i' \in K$ and all terms appearing in the last sum are $\prec_0 u$. From $\operatorname{lt}_{\prec_0}(f) \prec_0 u$ we obtain $\sum_{i \in B} c_i d_i' = 0$. By Lemma 2.3.14 for $1 \leq s_1, s_2 \leq s$ there exist $b_{s_1 s_2}$ such that we can write

$$\sum_{i \in B} c_i d_i' \lambda_i \frac{g_i}{d_i} = \sum_{1 \leq s_1, s_2 \leq s} b_{s_1 s_2} \left(\lambda_{s_1} \frac{g_{s_1}}{d_{s_1}} - \lambda_{s_2} \frac{g_{s_2}}{d_{s_2}} \right).$$

Since $\lambda_{s_1} \mu_{s_1} = \lambda_{s_2} \mu_{s_2} = u \in \Xi_k E$ we obtain $\mu_{s_1} = \operatorname{lt}_{k;\prec_0}(g_{s_1}), \mu_{s_2} = \operatorname{lt}_{k;\prec_0}(g_{s_2}), d_{s_1} = \operatorname{lc}_{k;\prec_0}(g_{s_1})$, $d_{s_2} = \operatorname{lc}_{k;\prec_0}(g_{s_2}), \lambda_{s_1} = \frac{u}{\operatorname{lt}_{k;\prec_0}(g_{s_1})}, \lambda_{s_2} = \frac{u}{\operatorname{lt}_{k;\prec_0}(g_{s_2})}$ and $\lambda_{s_1} \frac{g_{s_1}}{d_{s_1}} - \lambda_{s_2} \frac{g_{s_2}}{d_{s_2}} = \frac{u}{\operatorname{lt}_{k;\prec_0}(g_{s_1})} \frac{g_{s_1}}{\operatorname{lc}_{k;\prec_0}(g_{s_1})} - \frac{u}{\operatorname{lt}_{k;\prec_0}(g_{s_2})} \frac{g_{s_2}}{\operatorname{lc}_{k;\prec_0}(g_{s_2})}$ with

$$\operatorname{lt}_{k;\prec_0} \left(\lambda_{s_1} \frac{g_{s_1}}{d_{s_1}} - \lambda_{s_2} \frac{g_{s_2}}{d_{s_2}} \right) \prec_0 u.$$

Since for all $v \in V(k, g_{s_1}, g_{s_2})$ the S-polynomial $S(k, g_{s_1}, g_{s_2}, v)$ is $\prec_0$-reducible to 0 modulo G relative to $\mathcal{R} \cup \{\prec\}$ respecting $\mathcal{T}$ and for some $v \in V(k, g_{s_1}, g_{s_2})$ we have

$$\lambda_{s_1} \frac{g_{s_1}}{d_{s_1}} - \lambda_{s_2} \frac{g_{s_2}}{d_{s_2}} = \frac{u}{\operatorname{lcm}(\operatorname{lt}_{k;\prec_0}(g_{s_1}), \operatorname{lt}_{k;\prec_0}(g_{s_2}))} S(k, g_{s_1}, g_{s_2}, v)$$

by Lemma 2.3.15 we can write

$$\lambda_{s_1} \frac{g_{s_1}}{d_{s_1}} - \lambda_{s_2} \frac{g_{s_2}}{d_{s_2}} = \sum_{i=1}^{s} p_i g_i$$

with

(i) $\operatorname{lt}_{\prec_0}(p_i g_i) \prec_0 u$,

(ii) $\operatorname{lt}_{\prec'}(p_i g_i) \preceq' \max_{\prec'}\{\operatorname{lt}_{\prec'}(\lambda_{s_1} g_{s_1}), \operatorname{lt}_{\prec'}(\lambda_{s_2} g_{s_2})\} \preceq' \operatorname{lt}_{\prec'}(f)$ for all $\prec' \in \mathcal{R} \cup \{\prec\}$, and

(iii) $\operatorname{ord}_{\mathcal{T}_j, \Xi}(p_i g_i) \leq \max_{\prec_j}\{\operatorname{ord}_{\mathcal{T}_j, \Xi}(\lambda_{s_1} g_{s_1}), \operatorname{ord}_{\mathcal{T}_j, \Xi}(\lambda_{s_2} g_{s_2})\} \leq \operatorname{ord}_{\mathcal{T}_j, \Xi}(f)$ for all $j \in \{1, \ldots, t\}$.

Hence, there exist $h_1', \ldots, h_s'$ such that

(i) $f = \sum_{i=1}^{s} h_i' g_i$,

(ii) $\max_{\prec_0}\{\operatorname{lt}_{\prec_0}(h_i' g_i) \mid i = 1, \ldots, s\} \prec_0 u$,

(iii) for all $\prec' \in \mathcal{R} \cup \{\prec\}$ we have $\operatorname{lt}_{\prec'}(h_i' g_i) \preceq' \operatorname{lt}_{\prec'}(f)$, and

(iv) for all $j = 1, \ldots, t$ we have $\operatorname{ord}_{\mathcal{T}_j, \Xi}(h_i' g_i) \leq \operatorname{ord}_{\mathcal{T}_j, \Xi}(f)$.

This is a contradiction to our assumption on the minimality of u which proves the claim. □

Using Theorem 2.3.16 we obtain a method for computing weight relative Gröbner bases.

The correctness of Method 2.3.17 is an immediate consequence of Theorem 2.3.16. As far as termination is concerned we will later state an example in which Method 2.3.17 does not terminate.

Method 2.3.17 `basis`

IN: finite $G \subseteq K[\Delta, \Sigma^*]E \setminus \{0\}$, a generalized term order $\prec$ with respect to the orthant decomposition $\Xi = \{\Xi_k \mid 1 \leq k \leq p\}$, a weight matrix $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ with columns $\mathcal{T}_1, \ldots, \mathcal{T}_t$ where t is possibly vanishing, and a finite set of generalized term orders $\mathcal{R}$ being possibly empty

OUT: a $\prec$-Gröbner basis $\tilde{G}$ of the difference-skew-differential module generated by G relative to $\mathcal{R}$ respecting $\mathcal{T}$.

$\tilde{G} := G$
if $t > 0$ **then**
  choose a generalized term order $\prec_t$ respecting $\mathcal{T}_t$
  let $\mathcal{T}'$ be the weight matrix with columns $\mathcal{T}_1, \ldots, \mathcal{T}_{t-1}$
  $\tilde{G} :=$ `basis`$(\tilde{G}, \prec_t, \mathcal{T}', \mathcal{R})$
else if $\mathcal{R} \neq \emptyset$ **then**
  choose $\prec' \in \mathcal{R}$
  $\tilde{G} :=$ `basis`$(\tilde{G}, \prec', \mathcal{T}, \mathcal{R} \setminus \{\prec'\})$
end if
while there exist $g, g' \in \tilde{G}, k \in \{1, \ldots, p\}$ and $v \in V_\prec(k, g, g')$ such that we have `reduce`$(S_\prec(k, g, g', v), \tilde{G}, \prec, \mathcal{T}, \mathcal{R}) \neq 0$ **do**
  $\tilde{G} := \tilde{G} \cup \{$`reduce`$(S_\prec(k, g, g', v), \tilde{G}, \prec, \mathcal{T}, \mathcal{R})\}$
end while
return $\tilde{G}$

2.3.5 Symmetry

Interestingly enough there also exists some symmetry property for weight relative Gröbner bases.

Lemma 2.3.18. *Let $\prec$ be a generalized term order, $\mathcal{T} \in \mathbb{Q}_0^{(m+r)\times t}$, $\mathcal{R}$ a set of generalized term orders, $M \subseteq K[\Delta, \Sigma^*]E$ a difference-skew-differential module and $G \subseteq M \setminus \{0\}$ a $\prec$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$. Let $\prec^{(1)} \in \mathcal{R}$ and $\mathcal{R}_1 \subseteq \mathcal{R}$. Then*

(i) G is a $\prec^{(1)}$-Gröbner basis of M relative to $\{\prec\} \cup \mathcal{R} \setminus \{\prec^{(1)}\}$ respecting $\mathcal{T}$, and

(ii) G is a $\prec$-Gröbner basis of M relative to $\mathcal{R}_1$ respecting $\mathcal{T}$.

In particular, G is a Gröbner basis of M with respect to $\prec$ and every $\prec^{(1)} \in \mathcal{R}$.

Proof. Assume that G is a $\prec$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$ and let $f \in M \setminus \{0\}$. Then by Proposition 2.3.11 and Definition 2.3.7 there exist $s \in \mathbb{N}, f_0, f_1, \ldots, f_{s-1} \in M \setminus \{0\}, f_s = 0, g_0, \ldots, g_{s-1} \in G$ such that $f = f_0$ and for $i = 0, \ldots, s-1$ the difference-skew-differential operator f_i is $\prec$-reducible to f_{i+1} modulo g_i relative to $\mathcal{R}$ respecting $\mathcal{T}$, i.e., there exist $\lambda_0, \ldots, \lambda_{s-1} \in [\Delta, \Sigma^*]$ such that for $i = 0, \ldots, s-1$, $\prec' \in \mathcal{R}$ and $j = 1, \ldots, t$ we have

(i) $\mathrm{lt}_\prec(\lambda_i g_i) = \mathrm{lt}_\prec(f_i)$,

(ii) $\mathrm{lt}_{\prec'}(\lambda_i g_i) \preceq' \mathrm{lt}_{\prec'}(f_i)$, and

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda_i g_i) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f_i)$.

Hence, there exists $i_0 \in \{1, \ldots, s-1\}$ such that for $\prec'' \in \mathcal{R} \setminus \{\prec^{(1)}\}$ and $j = 1, \ldots, t$ we have

(i) $\mathrm{lt}_{\prec^{(1)}}(\lambda_{i_0} g_{i_0}) = \mathrm{lt}_{\prec^{(1)}}(f_{i_0}) = \mathrm{lt}_{\prec^{(1)}}(f_0)$,

(ii) $\mathrm{lt}_\prec(\lambda_{i_0} g_{i_0}) = \mathrm{lt}_\prec(f_{i_0}) \preceq \mathrm{lt}_\prec(f_0)$,

(iii) $\mathrm{lt}_{\prec'}(\lambda_{i_0} g_{i_0}) \preceq' \mathrm{lt}_{\prec'}(f_{i_0}) \preceq' \mathrm{lt}_{\prec'}(f_0)$, and

(iv) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda_{i_0} g_{i_0}) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f_{i_0}) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f_0)$.

So f_0 is $\prec^{(1)}$-reducible modulo G relative to $\{\prec\} \cup \mathcal{R} \setminus \{\prec^{(1)}\}$ respecting $\mathcal{T}$, i.e., G is a $\prec^{(1)}$-Gröbner basis of M relative to $\{\prec\} \cup \mathcal{R} \setminus \{\prec^{(1)}\}$ respecting $\mathcal{T}$ so (i) holds. From Definition 2.3.7 it follows that if $f \in M \setminus \{0\}$ is $\prec$-reducible modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$ then it is also $\prec$-reducible modulo G relative to $\mathcal{R}_1 \subseteq \mathcal{R}$ respecting $\mathcal{T}$. $\square$

2.3.6 Characterization of weight relative Gröbner bases

The following theorem (see e.g. [Win96, Theorem 8.3.4.]) provides a well-known characterization of Gröbner bases. We are going to develope a similar characterization for weight relative Gröbner bases.

Theorem 2.3.19. *Let K be a field, $\prec$ an admissible order on $[X]$, $I \trianglelefteq K[X]$ an ideal and $G \subseteq I$ finite. Then G is a Gröbner basis for I if and only if $\langle \mathrm{in}_\prec(I) \rangle = \langle \mathrm{in}_\prec(G) \rangle$.*

Along these lines Levin [Lev07a] provides a characterization for Gröbner bases with respect to several orderings, i.e., a characterization for weight relative Gröbner bases with $\mathcal{R} = \emptyset$. We use a slightly more complicated approach for the general case.

Let $\prec$ be a generalized term order, $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ a weight matrix, and $\mathcal{R} = \{\prec'_l \mid 1 \le l \le L\}$ a finite set of generalized term orders.

For $l \in L$ by Lemma 2.2.8 there exist a map $\phi^{(l)}$, matrices $U_1^{(l)}, \ldots, U_q^{(l)}, T^{(l)} = (t_{ij}^{(l)})$, vectors $\gamma_1^{(l)}, \ldots, \gamma_q^{(l)}$ and an element $\alpha^{(l)}$ of the symmetric group on $\{1, \ldots, q\}$ such that for (a_1, b_1, e_i), $(a_2, b_2, e_j) \in \mathbb{N}^m \times \mathbb{Z}^n \times E$ using the notation of Lemma 2.2.8 we have

$$\begin{aligned}\delta^{a_1}\sigma^{b_1}e_i \prec'_l \delta^{a_2}\sigma^{b_2}e_j \quad \Longleftrightarrow \quad & (\mathrm{Pr}_{t_{ij}^{(l)}}(U_i^{(l)}(a_1, \phi^{(l)}(b_1)) + \gamma_i^{(l)}), \alpha^{(l)}(i)) \\ & <_{\mathrm{lex}} (\mathrm{Pr}_{t_{ij}^{(l)}}(U_j^{(l)}(a_2, \phi^{(l)}(b_2)) + \gamma_j^{(l)}), \alpha^{(l)}(j)).\end{aligned}$$

For every difference differential operator $f \in K[\Delta, \Sigma^*]E$ let $(a_l^f)_{l=1}^L \in (\mathbb{N}^m)^L$, $(b_l^f)_{l=1}^L \in (\mathbb{Z}^n)^L$, $(j_{fl})_{l=1}^L \in \{1, \ldots, q\}^L$ such that for $l = 1, \ldots, L$ we have

$$\mathrm{lt}_{\prec'_l}(f) = \delta^{a_l^f}\sigma^{b_l^f}e_{j_{fl}}.$$

Introduce new symbols $y_1, \ldots, y_L, z_1, \ldots, z_t$. Then for every $g \in K[\Delta, \Sigma^*]E \setminus \{0\}$ let the map τ_g be defined by

$$\begin{aligned}&\tau_g : K[\Delta, \Sigma^*]E \setminus \{0\} \ni f \mapsto \qquad (2.4)\\ &\mathrm{lt}_\prec(f) \cdot \left(\prod_{l=1}^{L} y_l^{\left(\mathrm{Pr}_{t_{j_{fl},j_{gl}}^{(l)}}\left(U_{j_{fl}}^{(l)}\left(a_l^f, \phi^{(l)}\left(b_l^f\right)\right) + \gamma_{j_{fl}}^{(l)}\right), \alpha^{(l)}(j_{fl})\right)} \right) \cdot \prod_{j=1}^{t} z_j^{\mathrm{ord}_{\mathcal{T}_j, \Xi}(f)}.\end{aligned}$$

If

(i) $\mathrm{lt}_\prec(f) = \mathrm{lt}_\prec(g)$,

(ii) for every $l \in \{1, \ldots, L\}$ we have

$$\begin{aligned}&\left(\mathrm{Pr}_{t_{j_{fl},j_{gl}}^{(l)}}\left(U_{j_{gl}}^{(l)}\left(a_l^g, \phi^{(l)}\left(b_l^g\right)\right) + \gamma_{j_{gl}}^{(l)}\right), \alpha^{(l)}(j_{gl})\right)\\ &\quad <_{\mathrm{lex}} \left(\mathrm{Pr}_{t_{j_{fl},j_{gl}}^{(l)}}\left(U_{j_{fl}}^{(l)}\left(a_l^f, \phi^{(l)}\left(b_l^f\right)\right) + \gamma_{j_{fl}}^{(l)}\right), \alpha^{(l)}(j_{fl})\right),\end{aligned}$$

and

(iii) $\prod_{j=1}^{t} z_j^{\mathrm{ord}_{\mathcal{T}_j,\Xi}(g)}$ divides $\prod_{j=1}^{t} z_j^{\mathrm{ord}_{\mathcal{T}_j,\Xi}(f)}$ in the ordinary sense,

then we say that $\tau_f(g)$ divides $\tau_g(f)$ and write $\tau_f(g)|\tau_g(f)$.

Lemma 2.3.20. *Let $f, g \in K[\Delta, \Sigma^*]E \setminus \{0\}$, $\prec$ a generalized term order on $[\Delta, \Sigma^*]E$, $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ a weight matrix, $\mathcal{R} = \{\prec'_l \mid 1 \leq l \leq L\}$ a finite set of generalized term orders, and τ as in* (2.4). *f is $\prec$-reducible modulo g relative to $\mathcal{R}$ respecting $\mathcal{T}$ if and only if there exists $\lambda \in [\Delta, \Sigma^*]$ such that $\tau_f(\lambda g)$ divides $\tau_{\lambda g}(f)$.*

Proof. By Definition 2.3.7 f is $\prec$-reducible modulo g relative to $\mathcal{R}$ respecting $\mathcal{T}$ if and only if there exists $\lambda \in [\Delta, \Sigma^*]$ such that for all $l \in \{1, \ldots, L\}$ and all $j \in \{1, \ldots, t\}$ we have

(i) $\mathrm{lt}_\prec(\lambda g) = \mathrm{lt}_\prec(f)$,

(ii) $\mathrm{lt}_{\prec'_l}(\lambda g) \preceq'_l \mathrm{lt}_{\prec'_l}(f)$, and

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda g) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f)$.

By Lemma 2.2.8 this is equivalent to $\tau_f(\lambda g)|\tau_{\lambda g}(f)$. □

By Proposition 2.3.11 we obtain the following

Corollary 2.3.21. *Let $M \subseteq K[\Delta, \Sigma^*]E$ be a difference-skew-differential module and $G \subseteq M \setminus \{0\}$. TFAE*

(i) G is a $\prec$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$, and

(ii) for every $f \in M \setminus \{0\}$ there exist $g \in G, \lambda \in [\Delta, \Sigma^*]$ with $\tau_f(\lambda g)|\tau_{\lambda g}(f)$.

Obviously, the most important part of this characterization is how to deal with the set $\mathcal{R} = \{\prec'_l \mid 1 \leq l \leq L\}$ of generalized term orders. The fact that E contains several elements only complicates the representations of the generalized orders contained in $\mathcal{R}$ and the representation of a generalized term order is very similar to that of an admissible order.

2.3.7 Extended example: characterization of relative Gröbner bases for modules over rings of differential operators

Let $n = 0, E = \{1\}$. Consider $\mathcal{R} = \{\prec\}$ and $t = 0$, i.e., we are characterizing relative Gröbner bases for differential modules. By Theorem 2.2.6 there exist $s_\prec \in \{1, \ldots, m\}$ and $U_\prec \in \mathbb{R}^{s_\prec \times m}$ such that

$$\begin{array}{rcl} \alpha_\prec : [\Delta] & \to & \mathbb{R}^{s_\prec} \\ \delta^a & \mapsto & U_\prec \cdot a \end{array}$$

is an injective homomorphism of monoids. Note that for $\lambda, \mu \in [\Delta]$ we have

$$\alpha(\lambda\mu) = \alpha(\lambda) + \alpha(\mu).$$

Let

$$V := \{a_1 U \cdot e_1 + \cdots + a_m U \cdot e_m \mid \forall_{i \in \{1,\ldots,m\}} a_i \in \mathbb{Z}, e_i \text{ the } i\text{-th unit vector in } \mathbb{N}^m\}.$$

Let us consider a new symbol z and let

$$\begin{array}{rcl} [\Delta, z]_U & := & \{\delta^k z^{U \cdot k} \mid k \in \mathbb{N}^m\} \\ \overline{[\Delta, z]_U} & := & \{\delta^k z^v \mid k \in \mathbb{N}^m,\ v \in V,\ 0 \leq_{\mathrm{lex}} v - U \cdot k\}. \end{array}$$

For $v_1, v_2 \in V$ define $z^{v_1} z^{v_2} = z^{v_1+v_2}$ and $z^v \delta^k = \delta^k z^v$. Then $[\Delta, z]_U$ and $\overline{[\Delta, z]_U}$ can be considered as multiplicative monoids.

Since E consists of a single element for any $f, g \in K[\Delta, \Sigma^*]E$ the maps τ_f and τ_g coincide. Therefore we will omit the index and write τ. Define $\tau : K[\Delta] \to \overline{[\Delta, z]_U}$ by

$$\tau(f) := \mathrm{lt}_{\prec}(f) z^{\alpha(\mathrm{lt}_{\prec'}(f))}.$$

We get the following lemma.

Lemma 2.3.22. *Let $G = \{g_1, \dots, g_r\} \subseteq K[\Delta]$ be finite, $M := {}_{K[\Delta]}\langle G \rangle$ and let $\prec, \prec'$ be two admissible orders on $[\Delta]$. TFAE*

(i) G is a $\prec$-Gröbner basis of M relative to $\prec'$,

(ii) $\tau(M) \subseteq \overline{[\Delta, z]_U} \tau(G)$,

(iii) $\overline{[\Delta, z]_U} \tau(M) = \overline{[\Delta, z]_U} \tau(G)$,

(iv) ${}_{K\overline{[\Delta,z]_U}}\langle \tau(M) \rangle = {}_{K\overline{[\Delta,z]_U}}\langle \tau(G) \rangle$.

Proof. "(i)$\Longrightarrow$(ii)": Let G be a $\prec$-Gröbner basis of M relative to $\prec'$. Then every $f \in M \setminus \{0\}$ is $\prec$-reducible relative to $\prec'$, i.e., there exist $\lambda \in [\Delta]$, $g \in G$ such that

(i) $\mathrm{lt}_{\prec}(\lambda g) = \mathrm{lt}_{\prec}(f)$, and

(ii) $\mathrm{lt}_{\prec'}(\lambda g) \preceq' \mathrm{lt}_{\prec'}(f)$.

Hence, $\alpha(\lambda) + \alpha(\mathrm{lt}_{\prec'}(g)) \leq_{\mathrm{lex}} \alpha(\mathrm{lt}_{\prec'}(f))$ and we obtain

$$\lambda z^{\alpha(\mathrm{lt}_{\prec'}(f)) - \alpha(\mathrm{lt}_{\prec'}(g))} \in \overline{[\Delta, z]_U}.$$

On the other hand

$$\begin{aligned} \lambda z^{\alpha(\mathrm{lt}_{\prec'}(f)) - \alpha(\mathrm{lt}_{\prec'}(g))} \tau(g) &= \lambda z^{\alpha(\mathrm{lt}_{\prec'}(f)) - \alpha(\mathrm{lt}_{\prec'}(g))} \mathrm{lt}_{\prec}(g) z^{\alpha(\mathrm{lt}_{\prec'}(g))} \\ &= \mathrm{lt}_{\prec}(f) z^{\alpha(\mathrm{lt}_{\prec'}(f))} \\ &= \tau(f) \end{aligned}$$

and we conclude $\tau(M) \subseteq \overline{[\Delta, z]_U} \tau(G)$.
"(ii)$\Longrightarrow$(iii)": From $\tau(M) \subseteq \overline{[\Delta, z]_U} \tau(G)$ we immediately obtain $\overline{[\Delta, z]_U} \tau(M) \subseteq \overline{[\Delta, z]_U} \tau(G)$. On the other hand from $M \supseteq G$ we get $\overline{[\Delta, z]_U} \tau(M) \supseteq \overline{[\Delta, z]_U} \tau(G)$ and conclude $\overline{[\Delta, z]_U} \tau(M) = \overline{[\Delta, z]_U} \tau(G)$.
"(iii)$\Longrightarrow$(iv)": From $\overline{[\Delta, z]_U} \tau(M) = \overline{[\Delta, z]_U} \tau(G)$ we obtain

$$\begin{aligned} {}_{K\overline{[\Delta,z]_U}}\langle \tau(M) \rangle &= {}_{K\overline{[\Delta,z]_U}}\langle \overline{[\Delta, z]_U} \tau(M) \rangle \\ &= {}_{K\overline{[\Delta,z]_U}}\langle \overline{[\Delta, z]_U} \tau(G) \rangle \\ &= {}_{K\overline{[\Delta,z]_U}}\langle \tau(G) \rangle. \end{aligned}$$

"(iv)$\Longrightarrow$(i)": Suppose ${}_{K\overline{[\Delta,z]_U}}\langle \tau(M) \rangle = {}_{K\overline{[\Delta,z]_U}}\langle \tau(G) \rangle$ and let $f \in M \setminus \{0\}$. Then $\tau(f) \in \tau(M)$ and there exists $h_1, \dots, h_r \in K\overline{[\Delta, z]_U}$ such that

$$\tau(f) = \sum_{i=1}^{r} h_i \tau(g_i).$$

In fact, since $\tau(f)$ and all the $\tau(g)$ are monomials there exists a particular $g \in G$ such that $\tau(f) = h_g \tau(g)$ for some monomial $h_g \in \overline{[\Delta, z]_U}$. Then $h_g = \lambda z^v$ for some $\lambda = \delta^k z^{U \cdot k} \in [\Delta, z]_U$ and $0 \leq_{\text{lex}} v \in V$. Hence, $\text{lt}_\prec(f) = \delta^k \,\text{lt}_\prec(g)$ and

$$\begin{aligned} z^{\alpha(\text{lt}_{\prec'}(f))} &= z^v z^{U \cdot k} z^{\alpha(\text{lt}_{\prec'}(g))} \\ &= z^v z^{\alpha(\delta^k) + \alpha(\text{lt}_{\prec'}(g))} \\ &= z^v z^{\alpha(\text{lt}_{\prec'}(\delta^k g))}, \end{aligned}$$

i.e., $\alpha(\text{lt}_{\prec'}(\delta^k g)) \leq_{\text{lex}} \alpha(\text{lt}_{\prec'}(f))$ which implies $\text{lt}_{\prec'}(\delta^k g) \preceq' \text{lt}_{\prec'}(f)$. We conclude that f is $\prec$-reducible modulo G relative to $\prec'$ which is equivalent to G being a $\prec$-Gröbner basis relative to $\prec'$. □

Since $K\overline{[\Delta, z]_U}$ is not necessarily Noetherian it is by no means obvious that for every module $M \subseteq K[\Delta]$ there exists a finite basis of ${}_{K\overline{[\Delta,z]_U}}\langle \tau(M) \rangle$.

Consider the particular case $m = 3$, i.e., $\Delta = \{\delta_1, \delta_2, \delta_3\}$ and let $\prec = \text{lex}(\delta_3 > \delta_1 > \delta_2)$ and $\prec' = \text{grevlex}(\delta_3, \delta_2, \delta_1)$ on $[\Delta]$, i.e., they are given by

$$\delta_1^{a_1} \delta_2^{a_2} \delta_3^{a_3} \prec \delta_1^{b_1} \delta_2^{b_2} \delta_3^{b_3} :\Longleftrightarrow (a_3, a_1, a_2) <_{\text{lex}} (b_3, b_1, b_2),$$

and

$$\begin{aligned} &\delta_1^{a_1} \delta_2^{a_2} \delta_3^{a_3} \prec' \delta_1^{b_1} \delta_2^{b_2} \delta_3^{b_3} :\Longleftrightarrow \\ &\qquad (a_1 + a_2 + a_3, -a_1, -a_2) <_{\text{lex}} (b_1 + b_2 + b_3, -b_1, -b_2). \end{aligned}$$

The leading terms with respect to $\prec$ and $\prec'$ will be underlined and dotted underlined, respectively. For $i \in \mathbb{N}$ let

$$\begin{aligned} G_i &= \{f_0 := \underset{\cdots}{\delta_1^3 \delta_2^2} + \underline{\delta_1^4 \delta_2}, f_1 := \underset{\cdots}{\delta_2^3 \delta_3^2} + \underline{\delta_1 \delta_2^2 \delta_3^2}\} \\ &\quad \cup \{g_j := \underset{\cdots\cdots}{\delta_1^{3+4j}} \delta_2 \delta_3 + \underline{\delta_2^{2+4j} \delta_3^2} \mid j = 0, \ldots, i\}. \end{aligned}$$

It can be easily verified (e.g. using Maple) that G_0 is a Gröbner basis of $M := \langle f_0, g_0 \rangle$ with respect to $\prec'$. We use the method provided by [ZW08a] (which is a special case of Method 2.3.17, p.25) for computing a $\prec$-Gröbner basis of M relative to $\prec'$. For every $i \in \mathbb{N}$ the S-polynomial of f_0 and g_i with respect to $\prec$ is given by

$$S_\prec(f_0, g_i) = \delta_2^{1+4i} \delta_3^2 f_0 - \delta_1^4 g_i = \underline{\delta_1^3 \delta_2^{3+4i} \delta_3^2} - \underset{\cdots\cdots}{\delta_1^{7+4i} \delta_2 \delta_3}.$$

Then for every $0 \leq j \leq i$ we have

(i) $\text{lt}_\prec(\delta_1^3 \delta_2^{1+4(i-j)} g_j) = \text{lt}_\prec(\underset{\cdots\cdots\cdots}{\delta_1^{6+4j} \delta_2^{2+4(i-j)} \delta_3} + \underline{\delta_1^3 \delta_2^{3+4i} \delta_3^2}) = \text{lt}_\prec(S_\prec(f_0, g_i))$ and

(ii) $\text{lt}_{\prec'}(S_\prec(f_0, g_i)) = \delta_1^{7+4i} \delta_2 \delta_3 \prec' \delta_1^{6+4j} \delta_2^{2+4(i-j)} \delta_3 = \text{lt}_{\prec'}(\delta_1^3 \delta_2^{1+4(i-j)} g_j)$.

Hence, $S_\prec(f_0, g_i)$ is not $\prec$-reducible modulo $\{g_0, \ldots, g_i\}$ relative to $\prec'$. Furthermore it is not $\prec$-reducible modulo f_0. Nevertheless we have

(i) $\text{lt}_\prec(\delta_1^2 \delta_2^{1+4i} f_1) = \text{lt}_\prec(\underset{\cdots\cdots}{\delta_1^2 \delta_2^{4+4i} \delta_3^2} + \underline{\delta_1^3 \delta_2^{3+4i} \delta_3^2}) = \text{lt}_\prec(S_\prec(f_0, g_i))$ and

(ii) $\text{lt}_{\prec'}(\delta_1^2 \delta_2^{1+4i} f_1) = \delta_1^2 \delta_2^{4+4i} \delta_3^2 \prec' \delta_1^{7+4i} \delta_2 \delta_3 = \text{lt}_{\prec'}(S_\prec(f_0, g_i))$.

Hence, $S_{\prec}(f_0, g_i))$ is $\prec$-reducible modulo f_1 relative to $\prec'$ to

$$S(f_0, g_i) - \delta_1^2\delta_2^{1+4i} f_1 = -\delta_1^{7+4i}\delta_2\delta_3 - \underline{\delta_1^2\delta_2^{4+4i}\delta_3^2} =: h_1.$$

It is immediate that h_1 is not $\prec$-reducible modulo f_0. Furthermore for every $0 \leq j \leq i$ we have

(i) $\mathrm{lt}_{\prec}(\delta_1^2\delta_2^{2+4(i-j)} g_j) = \delta_1^2\delta_2^{4+4i}\delta_3^2 = \mathrm{lt}_{\prec}(h_1)$ and

(ii) $\mathrm{lt}_{\prec'}(h_1) = \delta_1^{7+4i}\delta_2\delta_3 \prec' \delta_1^{5+4j}\delta_2^{3+4(i-j)}\delta_3 = \mathrm{lt}_{\prec'}(\delta_1^2\delta_2^{2+4(i-j)} g_j)$.

So h_1 is not $\prec$-reducible modulo $\{g_0, \ldots, g_i\}$ relative to $\prec'$. Nevertheless we have

(i) $\mathrm{lt}_{\prec}(\delta_1\delta_2^{2+4i} f_1) = \mathrm{lt}_{\prec}(\delta_1\delta_2^{5+4i}\delta_3^2 + \underline{\delta_1^2\delta_2^{4+4i}\delta_3^2}) = \mathrm{lt}_{\prec}(h_1)$ and

(ii) $\mathrm{lt}_{\prec'}(\delta_1\delta_2^{2+4i} f_1) = \delta_1\delta_2^{5+4i}\delta_3^2 \prec' \delta_1^{7+4i}\delta_2\delta_3 = \mathrm{lt}_{\prec'}(h_1)$.

Hence, h_1 is $\prec$-reducible modulo f_1 relative to $\prec'$ to

$$h_1 + \delta_1\delta_2^{2+4i} f_1 = -\delta_1^{7+4i}\delta_2\delta_3 + \underline{\delta_1\delta_2^{5+4i}\delta_3^2} =: h_2.$$

It is immediate that h_2 is not $\prec$-reducible modulo f_0. Furthermore for every $0 \leq j \leq i$ we have

(i) $\mathrm{lt}_{\prec}(\delta_1\delta_2^{3+4(i-j)} g_j) = \delta_1\delta_2^{5+4i}\delta_3^2 = \mathrm{lt}_{\prec}(h_2)$ and

(ii) $\mathrm{lt}_{\prec'}(h_2) = \delta_1^{7+4i}\delta_2\delta_3 \prec' \delta_1^{4+4j}\delta_2^{4+4(i-j)}\delta_3 = \mathrm{lt}_{\prec'}(\delta_1\delta_2^{3+4(i-j)} g_j)$.

So h_2 is not $\prec$-reducible modulo $\{g_0, \ldots, g_i\}$ relative to $\prec'$. Nevertheless we have

(i) $\mathrm{lt}_{\prec}(\delta_2^{3+4i} f_1) = \mathrm{lt}_{\prec}(\delta_2^{6+4i}\delta_3^2 + \underline{\delta_1\delta_2^{5+4i}\delta_3^2}) = \mathrm{lt}_{\prec}(h_2)$ and

(ii) $\mathrm{lt}_{\prec'}(\delta_2^{3+4i} f_1) = \delta_2^{6+4i}\delta_3^2 \prec' \delta_1^{7+4i}\delta_2\delta_3 = \mathrm{lt}_{\prec'}(h_2)$.

Hence, h_2 is $\prec$-reducible modulo f_1 relative to $\prec'$ to

$$h_2 - \delta_2^{3+4i} f_1 = -\delta_1^{7+4i}\delta_2\delta_3 - \underline{\delta_2^{6+4i}\delta_3^2} = -g_{i+1}.$$

It is immediate that g_{i+1} is not $\prec$-reducible modulo $\{f_0, f_1\}$. Furthermore for every $0 \leq j \leq i$ we have

(i) $\mathrm{lt}_{\prec}(\delta_2^{4+4(i-j)} g_j) = \delta_2^{6+4i}\delta_3^2 = \mathrm{lt}_{\prec}(g_{i+1})$ and

(ii) $\mathrm{lt}_{\prec'}(g_{i+1}) = \delta_1^{7+4i}\delta_2\delta_3 \prec' \delta_1^{3+4j}\delta_2^{5+4(i-j)}\delta_3 = \mathrm{lt}_{\prec'}(\delta_2^{4+4(i-j)} g_j)$.

So g_{i+1} is not $\prec$-reducible modulo $\{g_0, \ldots, g_i\}$ relative to $\prec'$.

Hence, the method provided by [ZW08a] will not terminate on this example. Nevertheless it could still be the case that there exists a $\prec$-Gröbner basis of M relative to $\prec'$ but the provided method cannot compute it.

Now suppose there exists an element $h \in M \setminus \{0\}$ such that infinitely many g_i are $\prec$-reducible modulo h relative to $\prec'$. Then $\mathrm{lt}_{\prec}(h) = \delta_2^{a_1}\delta_3^{a_2}$ and $\mathrm{lt}_{\prec'}(h) = \delta_1^{b_1}\delta_2^{b_2}\delta_3^{b_3}$, i.e.,

$$\begin{pmatrix} a_1 + a_2 \\ 0 \\ 1 \end{pmatrix} \leq_{\mathrm{lex}} \begin{pmatrix} b_1 + b_2 + b_3 \\ -b_1 \\ -b_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_3 \\ b_1 \\ b_2 \end{pmatrix} \leq_{\mathrm{lex}} \begin{pmatrix} a_2 \\ 0 \\ a_1 \end{pmatrix}.$$

Since infinitly many g_i are $\prec$-reducible modulo h relative to $\prec'$ we also have

$$\begin{pmatrix} b_1 + b_2 + 2 + 4i - a_1 + b_3 + 2 - a_2 \\ -b_1 \\ -b_2 - 2 - 4i + a_1 \end{pmatrix} \leq_{\mathrm{lex}} \begin{pmatrix} 5 + 4i \\ -3 - 4i \\ -1 \end{pmatrix}$$

for infinitly many $i \in \mathbb{N}$. We have to distinguish several cases:

$a_1 + a_2 < b_1 + b_2 + b_3$: Then $4 + 4i < b_1 + b_2 + b_3 - a_1 - a_2 + 4 + 4i \leq 5 + 4i$, i.e., $b_1 + b_2 + b_3 - a_1 - a_2 = 1$. Then we have $-b_1 \leq -3 - 4_i$, i.e., $b_1 \geq 3 + 4i$. This can obviously not happen for infinitely many $i \in \mathbb{N}$.

$a_1 + a_2 = b_1 + b_2 + b_3$: We distinguish further

$0 < -b_1$: Then $b_1 < 0$ which is not possible.

$0 = -b_1$: Then $-1 \leq -b_2$, i.e., $b_2 \leq 1$.

$b_2 = 0$: Then $b_3 = a_1 + a_2$ and $b_3 \leq a_2$.

$b_3 < a_2$: Then $a_1 < 0$ which is not possible.

$b_3 = a_2$: Then $a_1 = 0$, i.e., $h = c\delta_3^{a_2}+$ lower terms with respect to $\prec$ and $\prec'$ for some $c \in K \setminus \{0\}$. Since G_0 is a $\prec'$-Gröbner basis of M we see that h is not $\prec'$-reducible, i.e., $h \notin M$.

$b_2 = 1$: Then $b_3 = a_1 + a_2 - 1$ and $b_3 \leq a_2$.

$b_3 < a_2$: Then $b_3 = a_2 - 1$ and $a_1 = 0$, i.e., $h = c_1\delta_3^{b_3+1} + c_2\delta_2\delta_3^{b_3}+$ lower terms with respect to $\prec$ and $\prec'$ for some $c_1, c_2 \in K \setminus \{0\}$. Again since G_0 is a $\prec'$-Gröbner basis of M we see that h is not $\prec'$-reducible, i.e., $h \notin M$.

$b_3 = a_2$: Then $1 = b_2 \leq a_1 = 1$, i.e., $h = c\delta_2\delta_3^{b_3}+$ lower terms with respect to $\prec$ and $\prec'$ for some $c \in K \setminus \{0\}$. Again since G_0 is a $\prec'$-Gröbner basis of M we see that h is not $\prec'$-reducible, i.e., $h \notin M$

We conclude that there cannot exist any $\prec$-Gröbner basis of I relative to $\prec'$. □

2.3.8 Change of orders

Although Example 2.3.7 reveals a mayor drawback of the concept of weight relative Gröbner bases we will still stick to this general setting.

It is quite obvious that the computation of a weight relative Gröbner basis is very laborious. Therefore we provide the following result providing sufficient conditions for a change of orders. It is similar to the corresponding result in the theory of Gröbner walks [CKM97].

Lemma 2.3.23. *Let $\prec_1, \prec_2$ be generalized term orders, $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ a weight matrix, $\mathcal{R}$ a set of generalized term orders, $M \subseteq K[\Delta, \Sigma^*]E$ a difference-skew-differential module and $G \subseteq M \setminus \{0\}$ a $\prec_1$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$ such that for all $g \in G, \lambda \in [\Delta, \Sigma^*]$ we have*

$$\mathrm{lt}_{\prec_1}(\lambda g) = \mathrm{lt}_{\prec_2}(\lambda g).$$

Then G is a $\prec_2$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$.

Proof. Let $f_0 \in M \setminus \{0\}$. Then f_0 is $\prec_1$-reducible modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$. Hence, there exist $\lambda_0, \ldots, \lambda_s \in [\Delta, \Sigma^*], g_0, \ldots, g_s \in G, f_1, \ldots, f_s \in K[\Delta, \Sigma^*]$ such that for all $i \in \{0, \ldots, s-1\}, \prec' \in \mathcal{R}, j \in \{1, \ldots, t\}$ we have

(i) $\mathrm{lt}_{\prec_1}(\lambda_i g_i) = \mathrm{lt}_{\prec_1}(f_i)$,

(ii) $\mathrm{lt}_{\prec'}(\lambda_i g_i) \preceq' \mathrm{lt}_{\prec'}(f_i)$,

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda_i g_i) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f_i)$, and

(iv) f_{i-1} is $\prec_1$-reducible to f_i modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$.

We have to distinguish the following two cases:

$\mathrm{lt}_{\prec_1}(f_0) = \mathrm{lt}_{\prec_2}(f_0)$: Because of $\mathrm{lt}_{\prec_1}(\lambda_0 g_0) = \mathrm{lt}_{\prec_2}(\lambda_0 g_0)$ we get that f_0 is $\prec_2$-reducible modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$. Hence, G is a $\prec_2$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$.

$\mathrm{lt}_{\prec_1}(f_0) \prec_2 \mathrm{lt}_{\prec_2}(f_0)$: We obtain f_1 by removing $\mathrm{lt}_{\prec_1}(f_0)$ from f_0 and replacing it with a linear combination of finitely many strictly smaller terms than $\mathrm{lt}_{\prec_1}(f_0)$ with respect to $\prec_1$ and $\prec_2$ which are not bigger than $\mathrm{lt}_{\prec'}(f_0)$ for any $\prec' \in \mathcal{R}$ and which do not have a higher order than f_0 with respect to $\mathcal{T}_j$ for any $j \in \{1, \ldots, t\}$. Since G is a $\prec_1$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$ and $f_0 \in M$ after finitely many reduction steps – say s' – we obtain $f_{s'}$ such that $\mathrm{lt}_{\prec_1}(f_{s'}) = \mathrm{lt}_{\prec_2}(f_{s'})$ and there exist $\lambda_{s'} \in [\Delta, \Sigma^*], g_{s'} \in G$ such that for all $\prec' \in \mathcal{R}$ and $j \in \{1, \ldots, t\}$ we have

(i) $\mathrm{lt}_{\prec_2}(\lambda_{s'} g_{s'}) = \mathrm{lt}_{\prec_1}(\lambda_{s'} g_{s'}) = \mathrm{lt}_{\prec_1}(f_{s'}) = \mathrm{lt}_{\prec_2}(f_{s'})$,

(ii) $\mathrm{lt}_{\prec'}(\lambda_{s'} g_{s'}) \preceq' \mathrm{lt}_{\prec'}(f_{s'}) \preceq' \mathrm{lt}_{\prec'}(f_0)$, and

(iii) $\mathrm{ord}_{\mathcal{T}_j,\Xi}(\lambda_{s'} g_{s'}) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f_{s'}) \leq \mathrm{ord}_{\mathcal{T}_j,\Xi}(f_0)$.

Hence, f_0 is $\prec_2$-reducible modulo G relative to $\mathcal{R}$ respecting $\mathcal{T}$, i.e., G is a $\prec_2$-Gröbner basis of M relative to $\mathcal{R}$ respecting $\mathcal{T}$.

□

Using Lemma 2.3.18 we obtain the following

Corollary 2.3.24. *Let $\prec_1, \prec_2$ be generalized term orders, $\mathcal{R}, \mathcal{R}_1, \mathcal{R}_2$ sets of generalized term orders, $\mathcal{T} \in \mathbb{Q}_0^{(r+m)\times t}$ a weight matrix, $M \subseteq K[\Delta, \Sigma^*]E$ a difference-skew-differential module and $G \subseteq M \setminus \{0\}$ a $\prec_1$-Gröbner basis of M relative to $\mathcal{R} \cup \mathcal{R}_1$ respecting $\mathcal{T}$ such that for all $\prec_1' \in \mathcal{R}_1 \cup \{\prec_1\}$, $\prec_2' \in \mathcal{R}_2 \cup \{\prec_2\}$, $g \in G$, $\lambda \in [\Delta, \Sigma^*]$ we have*

$$\mathrm{lt}_{\prec_1'}(\lambda g) = \mathrm{lt}_{\prec_2'}(\lambda g).$$

Then G is a $\prec_2$-Gröbner basis of M relative to $\mathcal{R} \cup \mathcal{R}_2$ respecting $\mathcal{T}_2$.

2.4 Border bases for difference-skew-differential operators respecting orthant decompositions

In numerical polynomial algebra border bases have the advantage that they are numerically more stable than Gröbner bases (see [Ste04]). Due to their nature they are mostly used for solving zero-dimensional polynomial systems of equations (see [AS88, Möl93, Mou99]). In this chapter we extend the notion of border bases to difference-skew-differential modules and give some results which resemble the well known ones by Kehrein, Kreuzer and Robbiano [KK05],[KKR05],[KK06]. Most of the proofs carry over and we only have to take care of some technical details.

We use the notion of orthant decompositions to define border bases for difference-skew-differential modules and establish a connection to Gröbner bases for difference-skew-differential operators resembling the well-known result for polynomial ideals (see [KKR05, Proposition 4.4.9.]).

We give the definition of a difference-skew-differential order module with respect to an orthant decomposition. Based on this we define the index of a difference-skew-differential term with respect to a difference-skew-differential order module and prove some of its fundamental properties. We introduce the notion of difference-skew-differential border prebases and of reduction of difference-skew-differential operators with respect to a given border prebasis. The goal of making reduction canonical leads naturally to the concept of difference-skew-differential border bases. We establish a connection between difference-skew-differential border bases and difference-skew-differential Gröbner bases with respect to the same orthant decomposition. A normal form function for difference-skew-differential operators is introduced and its relation to difference-skew-differential border bases is demonstrated. Furthermore we consider relations between multiplication endomorphisms and difference-skew-differential border bases. For polynomial ideals this leads to characterizing border bases by the property that for all pairs of neighboring border basis elements the corresponding S-polynomial is reducible to 0 [KK06]. For difference-skew-differential border bases it turns out that also certain multiples of border basis elements have to be reducible to 0. For these basis elements we introduce the notion of i-individuals.

Unless otherwise noted, throughout this section let K be a difference-skew-differential field with $\{\tau_1, \ldots, \tau_m\}$ a set of mutually commuting endomorphisms on K, $\Delta = \{\delta_1, \ldots, \delta_m\}$ a basic set of skew-derivations such that for $i = 1, \ldots, m$ the skew-derivation δ_i is a skew-derivation with respect to τ_i, respectively, and $\{\sigma_1, \ldots, \sigma_n\}$ a basic set of automorphisms. By E we always denote the finite set $\{e_1, \ldots, e_q\}$ of free generators of a free difference-skew-differential module.

2.4.1 Order and index

Definition 2.4.1. *Let $\Xi = \{[\Delta, \Sigma^*]_k \mid 1 \leq k \leq p\}$ be an orthant decomposition with generators $\xi_1, \ldots, \xi_r$. For $\lambda \in [\Delta, \Sigma^*]$ we define the* order *of μ with respect to Ξ (or the* Ξ-order *of μ) by*

$$\begin{aligned}\mathrm{ord}_\Xi(\mu) \quad := \quad \min\{\alpha_1 + \ldots + \alpha_m + \beta_1 + \ldots + \beta_r \mid \\ \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_r \in \mathbb{N}, \\ \delta_1^{\alpha_1} \cdots \delta_m^{\alpha_m} \xi_1^{\beta_1} \cdots \xi_r^{\beta_r} = \mu\},\end{aligned}$$

and extend this definition to $[\Delta, \Sigma^*]E$ by $\mathrm{ord}_\Xi(\mu e) := \mathrm{ord}_\Xi(\mu)$ for any $e \in E$. For $p = p_1\lambda_1 + \ldots + p_s\lambda_s \in K[\Delta, \Sigma^*]E$ with $p_1, \ldots, p_s \in K \setminus \{0\}$ and $\lambda_1, \ldots, \lambda_s \in [\Delta, \Sigma^*]E$ mutually distinct let the order *of p with respect to Ξ be given by*

$$\mathrm{ord}_\Xi(p) := \max_{1 \leq i \leq s}\{\mathrm{ord}_\Xi(\lambda_i)\}.$$

If no confusion is possible we write ord *instead of* ord_Ξ.

Similar to [KKR05, Def. 4.3.1] for polynomial ideals we introduce the notion of a difference-skew-differential order module.

Definition 2.4.2. *Let $\Xi = \{[\Delta, \Sigma^*]_k \mid 1 \leq k \leq p\}$ be an orthant decomposition of $[\Delta, \Sigma^*]$. A set $\emptyset \neq \mathcal{O} \subseteq [\Delta, \Sigma^*]E$ is called* difference-skew-differential order module with respect to Ξ *(or* Ξ-difference-skew-differential order module*) if and only if for any $\lambda \in \mathcal{O}, 1 \leq k \leq p$ and $e \in E$ such that $\lambda \in [\Delta, \Sigma^*]_k e$ and for all $\eta, \mu \in [\Delta, \Sigma]_k$ such that $\eta\mu e = \lambda$ we have $\eta e, \mu e \in \mathcal{O}$.*

Consider an orthant decomposition $\Xi = \{[\Delta, \Sigma^*]_k \mid 1 \leq k \leq p\}$ with generators $\xi_1, \ldots, \xi_r$. For any nonempty set $V \subseteq [\Delta, \Sigma^*]E$ and any difference-skew-differential term $\lambda \in [\Delta, \Sigma^*]$ let $\lambda V := \{\lambda v \mid v \in V\}$, and

$$\begin{aligned}E(V) \quad := \quad V \cup \{\mu e' \mid e' \in E, \mathrm{ord}_\Xi(\mu) = \min_{v \in V}\{\mathrm{ord}_\Xi(v)\}, \\ \text{and there exists } e \in E \text{ with } \mu e \in V\}.\end{aligned}$$

Let $V^{[0]} := V$ and define for any $d > 0$ recursively

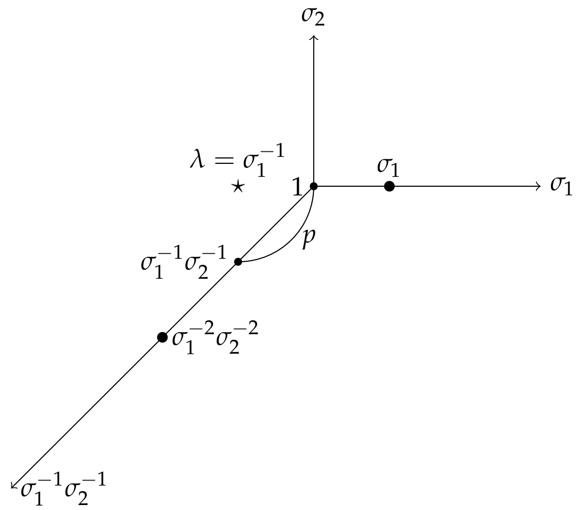
$$V^{[d]} := E(V^{[d-1]}) \cup \delta_1 V^{[d-1]} \cup \ldots \cup \delta_m V^{[d-1]} \cup \xi_1 V^{[d-1]} \cup \ldots \cup \xi_r V^{[d-1]}.$$

For $p = p_1\lambda_1 + \ldots + p_s\lambda_s \in K[\Delta, \Sigma^*]E$ with $p_1, \ldots, p_s \in K \setminus \{0\}$ and $\lambda_1, \ldots, \lambda_s \in \bigcup_{d \geq 0} V^{[d]}$ mutually distinct let the *index* of p with respect to V be given by (compare [KKR05, Def. 4.3.5])

$$\mathrm{ind}_V(p) := \max_{1 \leq i \leq s} \min\{d \mid \lambda_i \in V^{[d]}\}.$$

Note that if V is a difference-skew-differential order module then ind_V is defined for all $p \in K[\Delta, \Sigma^*]E$. If no confusion is possible we write ind instead of ind_V.

Example 2.4.3. *Let $m = 0, n = 2, E = \{1\}$ and Ξ the orthant decomposition of Example 2.2.3 having the generators $\xi_1 = \sigma_1, \xi_2 = \sigma_2$ and $\xi_3 = \sigma_1^{-1}\sigma_2^{-1}$. Let $V := \{\sigma_1^{-2}\sigma_2^{-2}, \sigma_1\}$, $p := 1 + 3\sigma_1^{-1}\sigma_2^{-1}$ and $\lambda := \sigma_1^{-1}$. Then $\mathrm{ord}_\Xi(\lambda) = \mathrm{ord}_\Xi(\xi_2\xi_3) = 2$.*



Furthermore we have $V^{[1]} = V \cup \{\sigma_1^2, \sigma_1\sigma_2, \sigma_2^{-1}, \sigma_1^{-1}\sigma_2^{-2}, \sigma_1^{-2}\sigma_2^{-1}, \sigma_1^{-3}\sigma_2^{-3}\}$ and $V^{[2]} = V^{[1]} \cup \{\sigma_1^3, \sigma_1^2\sigma_2, \sigma_1\sigma_2^{-1}, \sigma_1\sigma_2^2, 1, \sigma_1\sigma_2^{-1}, \sigma_1^{-1}\sigma_2^{-1}, \sigma_1^{-2}\sigma_2^{-3}, \sigma_1^{-2}, \sigma_1^{-3}\sigma_2^{-2}, \sigma_1^{-4}\sigma_2^{-4}\}$. Hence, $\mathrm{ind}_V(p) = 2$. □

If $V \subseteq K[\Delta, \Sigma^*]E$ then by (V) we denote the K-vector space spanned by the elements of V and by $\langle V \rangle$ we denote the

2.4.2 Border and border closure

Then we can define border and border closure of difference-skew-differential order modules similar to [KKR05, Def. 4.3.2].

Definition 2.4.4. *Let Ξ be an orthant decomposition, $\mathcal{O}$ a Ξ-difference-skew-differential order module and $k \geq 2$. Then $\partial\mathcal{O} := \mathcal{O}^{[1]} \setminus \mathcal{O}$ is called the* Ξ-border *of $\mathcal{O}$. By convention $\partial^0\mathcal{O} := \mathcal{O}$. The set*

$$\partial^k\mathcal{O} := \mathcal{O}^{[k]} \setminus \mathcal{O}^{[k-1]}$$

is called the k-th Ξ-border *of $\mathcal{O}$ and $\mathcal{O}^{[k]}$ is called the* k-th Ξ-border closure *of $\mathcal{O}$.*

Kehrein, Kreuzer and Robbiano [KKR05, Prop. 4.3.4] proved some properties of order ideals, their borders and border closures for polynomial rings. We provide according results for the ring of difference-skew-differential operators.

Lemma 2.4.5. *Let Ξ be an orthant decomposition, $\mathcal{O} \subseteq [\Delta, \Sigma^*]E$ a Ξ-difference-skew-differential order module and $1 \leq k \in \mathbb{N}$. Then*

(i) We have a disjoint union $\mathcal{O}^{[k]} = \bigcup_{i=0}^k \partial^i\mathcal{O}$,

(ii) $\partial^{k+1}\mathcal{O} = \{\lambda \cdot o \mid \lambda \in [\Delta, \Sigma^*],\ \mathrm{ord}_\Xi(\lambda) = k,\ o \in \mathcal{O}^{[1]}\} \setminus \{\lambda \cdot o \mid \lambda \in [\Delta, \Sigma^*],\ \mathrm{ord}_\Xi(\lambda) < k,\ o \in \mathcal{O}^{[1]}\}$,

(iii) we have a disjoint union $[\Delta, \Sigma^*]E = \bigcup_{i\in\mathbb{N}} \partial^i\mathcal{O}$,

(iv) for every $\lambda \in \partial^k\mathcal{O}$ with $k \geq 1$ there exist $\lambda' \in \mathcal{O}^{[1]}$, $\lambda'' \in [\Delta, \Sigma^*]$ with $\mathrm{ord}_\Xi(\lambda'') = k - 1$ such that $\lambda = \lambda'\lambda''$.

Proof. (i) By definition we have

$$\mathcal{O}^{[1]} = E(\mathcal{O}) \cup \{\lambda \cdot o \mid \lambda \in [\Delta, \Sigma^*],\ \mathrm{ord}_\Xi(\lambda) = 1,\ o \in \mathcal{O}\}.$$

By induction we obtain

$$\begin{aligned} \mathcal{O}^{[k+1]} &= \mathcal{O}^{[k]} \cup \{\lambda \cdot o \mid \lambda \in [\Delta, \Sigma^*],\ \mathrm{ord}_\Xi(\lambda) = 1,\ o \in \mathcal{O}^{[k]}\} \\ &= \mathcal{O}^{[k]} \cup \{\lambda \cdot o \mid \lambda \in [\Delta, \Sigma^*],\ \mathrm{ord}_\Xi(\lambda) = k,\ o \in \mathcal{O}^{[1]}\}. \end{aligned}$$

(ii) Follows immediately from Definition 2.4.4 of $\partial^{k+1}\mathcal{O}$.

(iii) The inclusion $[\Delta, \Sigma^*]E \supseteq \bigcup_{i \in \mathbb{N}} \partial^i \mathcal{O}$ is clear. The opposite inclusion follows from Definition 2.4.4 of $\partial\mathcal{O}$ and the fact that for every $\lambda \in [\Delta, \Sigma^*]$ there exist $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_r \in \mathbb{N}$ with $\lambda = \delta_1^{\alpha_1} \cdots \delta_m^{\alpha_m} \zeta_1^{\beta_1} \cdots \zeta_r^{\beta_r}$. Hence, every $\lambda \in [\Delta, \Sigma^*]E$ is an element of $\partial^{i_\lambda}\mathcal{O}$ for some $i_\lambda \in \mathbb{N}$. By (i) and Definition 2.4.4 the union is disjoint.

(iv) Follows from the second claim.

□

With this lemma at hand we can prove the following properties for the index (see also [KKR05, Prop. 4.3.6]).

Lemma 2.4.6. *Let Ξ be an orthant decomposition and $\mathcal{O} \subseteq [\Delta, \Sigma^*]E$ a Ξ-difference-skew-differential order module.*

(i) For $\lambda \in [\Delta, \Sigma^*]E \setminus \mathcal{O}$ we have

$$\mathrm{ind}_\mathcal{O}(\lambda) = \min\{i \mid \lambda = \lambda'\lambda'',\ \mathrm{ord}_\Xi(\lambda') = i - 1,\ \lambda' \in [\Delta, \Sigma^*],\ \lambda'' \in \mathcal{O}^{[1]}\},$$

(ii) $\lambda \in [\Delta, \Sigma^*], \mu \in [\Delta, \Sigma^*]E \Longrightarrow \mathrm{ind}_\mathcal{O}(\lambda\mu) \leq \mathrm{ord}_\Xi(\lambda) + \mathrm{ind}_\mathcal{O}(\mu)$,

(iii) For $f, g \in K[\Delta, \Sigma^*]E \setminus \{0\}$ with $f + g \neq 0$ we have

$$\mathrm{ind}_\mathcal{O}(f + g) \leq \max\{\mathrm{ind}_\mathcal{O}(f), \mathrm{ind}_\mathcal{O}(g)\},$$

(iv) For $f \in K[\Delta, \Sigma^*] \setminus \{0\}, g \in K[\Delta, \Sigma^*]E \setminus \{0\}$ we have

$$\mathrm{ind}_\mathcal{O}(fg) \leq \mathrm{ord}_\Xi(f) + \mathrm{ind}_\mathcal{O}(g).$$

Proof. (i) The case $\mathrm{ind}_\mathcal{O}(\lambda) = 0$ is clear. For $\mathrm{ind}_\mathcal{O}(\lambda) \geq 1$ the claim follows from Lemma 2.4.5(iv) and the definition of the index.

(ii) Follows from (i).

(iii) Follows from $\mathrm{supp}(f + g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$.

(iv) It is clear that $\mathrm{supp}(fg) \subseteq \{\lambda\mu \mid \lambda \in \mathrm{supp}(f),\ \mu \in \mathrm{supp}(g)\}$. Applying (ii) proves the claim.

□

2.4.3 Border prebases and border division

For the definition of a border prebasis in polynomial rings we refer to [KKR05, Def. 4.3.8].

Observe that $[\Delta, \Sigma^*]E$ is countable. Hence, for any orthant decomposition Ξ any Ξ-difference-skew-differential order module and its Ξ-border are countable.

Definition 2.4.7. *Let Ξ be an orthant decomposition, $I, J \subseteq \mathbb{N}$ index sets and $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module with border $\partial\mathcal{O} = \{b_j \mid j \in J\}$. Let $G = \{g_j \mid j \in J\} \subseteq K[\Delta, \Sigma^*]E$ be such that for each $j \in J$ we have*

$$g_j = b_j - \sum_{i \in I} \alpha_{ij} t_i,$$

with only finitely many $\alpha_i \in K$ not vanishing. Then G is called $\mathcal{O}$-border prebasis with respect to Ξ *(or* $\mathcal{O}$-border prebasis *or* difference-skew-differential border prebasis *if no confusion is possible).*

From now on unless otherwise stated we assume that for any orthant decomposition Ξ and any Ξ-difference-skew-differential order module $\mathcal{O}$ we have index sets $I, J \subseteq \mathbb{N}$ such that we can write $\mathcal{O} = \{t_i \mid i \in I\}$ and $\partial\mathcal{O} = \{b_j \mid j \in J\}$.

Given an orthant decomposition Ξ, a Ξ-difference-skew-differential order module $\mathcal{O}$, an $\mathcal{O}$-border prebasis and a difference-skew-differential operator $f \in K[\Delta, \Sigma^*]E$ consider the following algorithm 2.4.18 (see also [KKR05, Prop. 4.3.10]):

Algorithm 2.4.18 Border division algorithm

IN: Ξ an orthant decomposition, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module, $\partial\mathcal{O} = \{b_j \mid j \in J\}$ its border, $\{g_j \mid j \in J\}$ an $\mathcal{O}$-border prebasis, $f \in K[\Delta, \Sigma^*]E$.
OUT: $(c_i)_{i \in I}$, $(f_j)_{j \in J}$ such that $f = \sum_{i \in I} c_i t_i + \sum_{j \in J} f_j g_j$ and $\operatorname{ord}_\Xi(f_j) \leq \operatorname{ind}_\mathcal{O}(f) - 1$ for all $j \in J$ with $f_j g_j \neq 0$.

$(c_i)_{i \in I} := (0)_{i \in I}$; $(f_j)_{j \in J} := (0)_{j \in J}$; $h := f$;
while $\operatorname{ind}_\mathcal{O}(h) > 0$ **do**
  write $h = a_1 h_1 + \ldots + a_s h_s$ such that $a_1, \ldots, a_s \in K \setminus \{0\}$ and $h_1, \ldots, h_s \in [\Delta, \Sigma^*]E$ mutually distinct satisfy $\operatorname{ind}_\mathcal{O}(h_1) = \operatorname{ind}_\mathcal{O}(h)$. Choose $j_0 \in J$ such that there exists $\lambda \in [\Delta, \Sigma^*]$ with $\operatorname{ord}_\Xi(\lambda) = \operatorname{ind}_\mathcal{O}(h) - 1$ and $h_1 = \lambda b_{j_0}$;
  $h = h - a_1 \lambda g_{j_0}$;
  $f_{j_0} = f_{j_0} + a_1 \lambda$;
end while
choose $(c_i)_{i \in I}$ such that $h = \sum_{i \in I} c_i t_i$; $h = 0$;
return $(c_i)_{i \in I}$, $(f_j)_{j \in J}$;

Theorem 2.4.8. *Algorithm 2.4.18 is correct and terminates.*

Proof. First we show that the while loop can be executed and terminates. By Lemma 2.4.5 there exist $\mu \in [\Delta, \Sigma^*]$ with $\operatorname{ord}_\Xi(\mu) = \operatorname{ind}(h_1) - 1$ and $j_0 \in J$ such that $h_1 = \mu b_{j_0}$ and there exists no such factorization with a term μ of smaller order with respect to Ξ.

For termination we consider the subtraction

$$h - a_1 \lambda g_{j_0} = a_1 h_1 + \ldots + a_s h_s - a_1 \lambda b_{j_0} + a_1 \lambda \sum_{i \in I} \alpha_{i, j_0} t_i,$$

where $a_1 h_1 = a_1 \lambda b_{j_0}$. Lemma 2.4.6 shows that in h a term of index $\operatorname{ind}(h)$ is replaced by a finite number of terms in $\mathcal{O}^{[\operatorname{ind}(h)-1]}$. Since the index of a given term is finite and there are only finitely many terms in h having the same index as h the loop terminates.

For correctness consider the equation

$$f = h + \sum_{i \in I} c_i t_i + \sum_{j \in J} f_j g_j.$$

It is certainely satisfied before we enter the while loop. The f_js are only changed inside the loop where the subtraction $h - a_1\lambda g_{j_0}$ makes up with the addition $(f_{j_0} + a_1\lambda)g_{j_0}$. The coefficients c_i are changed immediately after termination of the while loop in the step where they are choosen such that h is written as $\sum_{i\in I} c_i t_i$. Hence, the stated representation is computed by the algorithm. Furthermore the output does not depend on the particular choice of h_1 since h_1 is replaced by terms of strictly smaller index, i.e., the "reductions" of several terms of the same index do not interfere with each other. $\square$

Note 2.4.9. *Although the output of Algorithm 2.4.18 does not depend on the actual choice of h_1 in the while loop it still depends on the choice of j_0.*

Let Ξ be an orthant decomposition, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module, $G = \{g_j \mid j \in J\}$ an $\mathcal{O}$-border prebasis and $f \in K[\Delta, \Sigma^*]E$ a difference-skew-differential operator. Then the set

$$\operatorname{rem}_{\mathcal{O},G}(f) = \left\{ \sum_{i\in I} c_i t_i \;\middle|\; (c_i)_{i\in I},\ (f_j)_{j\in J} \text{ is a possible output of Algorithm 2.4.18 applied to } \Xi, \mathcal{O}, G \text{ and } f \right\}$$

is called the *set of $\mathcal{O}$-remainders of f* (see also [KKR05, Def. 4.3.12]). If the context is clear we also sometimes write $\operatorname{rem}(f)$ instead of $\operatorname{rem}_{\mathcal{O},G}(f)$.

We obtain the following corollary (see also [KKR05, Cor. 4.3.4]).

Corollary 2.4.10. *Let $\mathcal{O} = \{t_i \mid i \in I\}$ be a difference-skew-differential order module and $G = \{g_j \mid j \in J\}$ an $\mathcal{O}$-border prebasis. Then the residue classes of the terms in $\mathcal{O}$ generate $K[\Delta, \Sigma^*]E/\langle G\rangle$ as a K-vector space.*

2.4.4 Definition of border bases

Note 2.4.11. *Although the residue classes of the terms $\{t_i \mid i \in I\}$ generate $K[\Delta, \Sigma^*]E/\langle G\rangle$ they do not necessarily form a basis.*

Example 2.4.12. *Let $m = 0, n = 2, E = \{1\}$ and Ξ be the orthant decomosition from Example 2.2.3. Let $\mathcal{O} := \{1\}$. Then $\partial\mathcal{O} = \{\sigma_1, \sigma_2, \sigma_1^{-1}\sigma_2^{-1}\}$. Let $a_1, a_2 \in K$ and $G := \{\sigma_1 - a_1, \sigma_2 - a_2, \sigma_1^{-1}\sigma_2^{-1} - 1\}$. Applying Algorithm 2.4.18 yields $\operatorname{rem}_{\mathcal{O},G}(\sigma_1\sigma_2) = \{\sigma_2(a_1)a_2, \sigma_1(a_2)a_1\}$ and hence, the residue class of 1 does not necessarily form a K-vector space basis of $K[\Sigma^*]/\langle G\rangle$.*

This is addressed by the following definition which extends [KKR05, Def. 4.4.1] to our setting.

Definition 2.4.13. *Let Ξ be an orthant decomposition, $\mathcal{M}$ a difference-skew-differential module and $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module. Let $G = \{g_j \mid j \in J\} \subseteq \mathcal{M}$ be an $\mathcal{O}$-*border basis *of $\mathcal{M}$ if the residue classes of the terms in $\mathcal{O}$ form a K-vector space basis of $K[\Delta, \Sigma^*]E/\mathcal{M}$. If no ambiguities are possible we simply speak of a* difference-skew-differential border basis.

As for polynomial ideals also for difference-skew-differential modules one can show that an $\mathcal{O}$-border basis of $\mathcal{M}$ actually generates $\mathcal{M}$ (see also [KKR05, Prop. 4.4.2]).

Lemma 2.4.14. *Let Ξ be an orthant decomposition, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module, $\mathcal{M}$ a difference-skew-differential module and $G = \{g_j \mid j \in J\}$ an $\mathcal{O}$-border basis of $\mathcal{M}$. Then $\mathcal{M} = \langle G\rangle$.*

Proof. Let $f \in \mathcal{M}$. Applying Algorithm 2.4.18 to $\Xi, \mathcal{O}, \partial\mathcal{O}, G$ and f yields $(c_i)_{i\in I}$, $(f_j)_{j\in J}$ such that

$$f = \sum_{i\in I} c_i t_i + \sum_{j\in J} f_j g_j.$$

Hence,

$$0 = f = \sum_{i\in I} c_i t_i \qquad \text{mod } \mathcal{M}.$$

Since G is an $\mathcal{O}$-border basis the elements of $\mathcal{O}$ form a K-vector space basis of $K[\Delta, \Sigma^*]E/\mathcal{M}$. We conclude that for all $i \in I$ we have $c_i = 0$ which implies

$$f = \sum_{j\in J} f_j g_j.$$

Hence, $\mathcal{M} \subseteq \langle G \rangle$. The opposite inclusion is obvious and the claim follows. □

We will use the following lemma providing conditions for an $\mathcal{O}$-border prebasis to be an $\mathcal{O}$-border basis (see [KKR05, Rem. 4.4.3]).

Lemma 2.4.15. *Let Ξ be an orthant decomposition, $\mathcal{O}$ a Ξ-difference-skew-differential order module, G an $\mathcal{O}$-border prebasis and $\mathcal{M} = \langle G \rangle$. The following are equivalent:*

(i) G is an $\mathcal{O}$-border basis of $\mathcal{M}$,

(ii) $\mathcal{M} \cap (\mathcal{O}) = \{0\}$,

(iii) $K[\Delta, \Sigma^*]E = \mathcal{M} \oplus (\mathcal{O})$.

Proof. By Corollary 2.4.10 we have that the residue classes of the terms in $\mathcal{O}$ generate $K[\Delta, \Sigma^*]E/\mathcal{M}$. Since G being an $\mathcal{O}$-border basis is equivalent to these residue classes being linearly independent the claim follows. □

With this lemma we can prove the following theorem (see also [KK05, Prop. 9]).

Theorem 2.4.16. *Let Ξ be an orthant decomposition, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module and $G = \{g_j \mid j \in J\}$ an $\mathcal{O}$-border prebasis. Then the following are equivalent:*

(i) G is an $\mathcal{O}$ border basis of $\langle G \rangle$.

(ii) For every $f \in \langle G \rangle \setminus \{0\}$ there exists $(f_j)_{j\in J} \subseteq K[\Delta, \Sigma^*]$ such that $f = \sum_{j\in J} f_j g_j$ and for all f_j with $f_j g_j \neq 0$ we have $\operatorname{ord}_\Xi(f_j) \leq \operatorname{ind}(f) - 1$.

(iii) For every $f \in \langle G \rangle \setminus \{0\}$ there exists $(f_j)_{j\in J} \subseteq K[\Delta, \Sigma^*]$ such that $f = \sum_{j\in J} f_j g_j$ and

$$\max\{\operatorname{ord}_\Xi(f_j) \mid j \in J,\ f_j g_j \neq 0\} = \operatorname{ind}(f) - 1.$$

Proof. "(i)$\Longrightarrow$(ii)": Let $f \in \langle G \rangle \setminus \{0\}$ then Algorithm 2.4.18 applied to $\Xi, \mathcal{O}, G$ and f returns $(c_i)_{i\in I} \subseteq K$ and $(f_j)_{j\in J} \subseteq K[\Delta, \Sigma^*]$ such that

$$f = \sum_{i\in I} c_i t_i + \sum_{j\in J} f_j g_j$$

and $\operatorname{ord}_\Xi(f_j) \leq \operatorname{ind}(f) - 1$ for all $j \in J$. We have $\sum_{i\in I} c_i t_i = 0$ modulo $\langle G \rangle$ and since the residue classes of the terms in $\mathcal{O}$ form a basis of $K[\Delta, \Sigma^*]E/\langle G \rangle$ we have $c_i = 0$ for all $i \in I$.
"(ii)$\Longrightarrow$(iii)": Lemma 2.4.6 implies that for $\operatorname{ord}_\Xi(f_j) < \operatorname{ind}(f) - 1$ we have

$$\begin{aligned} \operatorname{ind}(f_j g_j) &\leq \operatorname{ord}_\Xi(f_j) + \operatorname{ind}(g_j) \\ &= \operatorname{ord}_\Xi(f_j) + 1 \\ &< \operatorname{ind}(f). \end{aligned}$$

Again by Lemma 2.4.6 there exists $j_0 \in J$ such that $\operatorname{ind}(f) = \operatorname{ind}(f_{j_0} g_{j_0}) \leq \operatorname{ord}_\Xi(f_{j_0}) + 1$, i.e. $\operatorname{ord}_\Xi(f_{j_0}) = \operatorname{ind}(f) - 1$.
"(iii)$\Longrightarrow$(i)": Let $(c_i)_{i\in I} \subseteq K$ be such that $f := \sum_{i\in I} c_i t_i \in \langle G\rangle \setminus \{0\}$. Then there exist $(f_j)_{j\in J} \subseteq K[\Delta, \Sigma^*]$ such that $f = \sum_{j\in J} f_j g_j$. Since $f \neq 0$ we have $\max\{\operatorname{ord}_\Xi(f_j) \mid j \in J,\ f_j g_j \neq 0\} \geq 0$. On the other hand $\operatorname{ind}(f) - 1 = -1$ which contradicts (iii). Hence $\langle G\rangle \cap (\mathcal{O}) = \{0\}$ and Lemma 2.4.15 implies that G is an $\mathcal{O}$-border basis. □

2.4.5 Border form module

For the definition of border form and border form ideal in polynomial rings see [KK05, Def. 10].

Definition 2.4.17. *Let $0 \neq f = \sum_{\lambda\in[\Delta,\Sigma^*]E} a_\lambda \lambda$, with only finitely many $a_\lambda \in K$ different from 0. Let $\mathcal{O}$ be a difference-skew-differential order module. The operator $\mathrm{BF}_\mathcal{O}(f) = \sum_{\operatorname{ind}(\lambda)=\operatorname{ind}(f)} a_\lambda \lambda$ is called the* $\mathcal{O}$-border form *of f. We define $\mathrm{BF}_\mathcal{O}(0) := 0$. For a difference-skew-differential module $\mathcal{M} \subseteq K[\Delta, \Sigma^*]E$ the module $\mathrm{BF}_\mathcal{O}(\mathcal{M}) := \langle \mathrm{BF}_\mathcal{O}(f) \mid f \in \mathcal{M}\rangle$ is called the* $\mathcal{O}$-border form module *of $\mathcal{M}$. If no confusion is possible we write* BF *instead of* $\mathrm{BF}_\mathcal{O}$.

The following lemma immediately carries over from the polynomial setting (see [KK05, Prop. 11]).

Lemma 2.4.18. *Let Ξ be an orthant decomposition, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module and $G = \{g_j \mid j \in J\}$ an $\mathcal{O}$-border prebasis. The following are equivalent:*

(i) G is an $\mathcal{O}$-border basis of $\langle G\rangle$.

(ii) For $f \in \langle G\rangle \setminus \{0\}$ we have $\operatorname{supp}(\mathrm{BF}(f)) \subseteq [\Delta, \Sigma^*]E \setminus \mathcal{O}$.

Proof. "(i)$\Longrightarrow$(ii)": Let $f \in \langle G\rangle \setminus \{0\}$ and suppose there exists $i_0 \in I$ such that $t_{i_0} \in \operatorname{supp}(\mathrm{BF}(f))$. Then $\operatorname{supp}(f) \subseteq \mathcal{O}$, i.e., there exist $(c_i)_{i\in I} \subseteq K$ such that $f = \sum_{i\in I} c_i t_i$. Since G is an $\mathcal{O}$ border basis we obtain $c_i = 0$ for all $i \in I$ which contradicts the assumtion $f \neq 0$.
"(ii)$\Longrightarrow$(i)": Let $(c_i)_{i\in I} \subseteq K$ be such that $f := \sum_{i\in I} c_i t_i \in \langle G\rangle$. Then $\operatorname{supp}(\mathrm{BF}(f)) \subseteq [\Sigma] \setminus \mathcal{O}$ implies $c_i = 0$. Hence, $\langle G\rangle \cap (\mathcal{O}) = \{0\}$ and Lemma 2.4.15 implies that G is an $\mathcal{O}$-border basis of $\langle G\rangle$. □

2.4.6 Border bases and Gröbner bases

For the existence of $\mathcal{O}$-border bases we have the following theorem (see also [KKR05, Thm. 4.4.4]).

Theorem 2.4.19. *Let Ξ be an orthant decomposition, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module and $\mathcal{M} \trianglelefteq K[\Delta, \Sigma^*]E$ a difference-skew-differential module such that the residue classes of the terms in $\mathcal{O}$ form a K-vector space basis of $K[\Delta, \Sigma^*]E/\mathcal{M}$.*

(i) There exists a unique $\mathcal{O}$-border basis G of $\mathcal{M}$.

(ii) Let $F \subseteq \mathcal{M}$ be an $\mathcal{O}$-border prebasis. Then F is the $\mathcal{O}$-border basis of $\mathcal{M}$.

Proof. (i) Let $\partial\mathcal{O} = \{b_j \mid j \in J\}$. Then the residue class of every b_{j_0} in $K[\Delta, \Sigma^*]E/\mathcal{M}$ is linearly dependent on the residue classes of the terms in $\mathcal{O}$. Hence, for $i \in I$ there exist $\alpha_{i,j_0} \in K$ such that

$$g_{j_0} := b_{j_0} - \sum_{i\in I} \alpha_{i,j_0} t_i \in \mathcal{M}.$$

Then $G := \{g_j \mid j \in J\}$ is an $\mathcal{O}$-border prebasis and hence also an $\mathcal{O}$-border basis of $\mathcal{M}$. Then Lemma 2.4.15 implies $\mathcal{M} \cap (\mathcal{O}) = \{0\}$. Let $F = \{f_j \mid j \in J\}$ be another $\mathcal{O}$-border basis of $\mathcal{M}$ different from G. Then there exists $j_0 \in J$ such that $g_{j_0} - f_{j_0} \in \mathcal{M} \setminus \{0\}$ and $\operatorname{supp}(g_{j_0} - f_{j_0}) \subseteq \mathcal{O}$ contradicting $\mathcal{M} \cap (\mathcal{O}) = \{0\}$.

(ii) By Definition 2.4.13 the set F is an $\mathcal{O}$-border basis of $\mathcal{M}$. Applying (i) proves the claim. $\square$

The following theorem is a generalization of Macaulay's basis theorem (see e.g. [KR00, Thm. 1.5.7]) to modules of difference-skew-differential operators.

Macaulay's basis theorem for difference-skew-differential modules 2.4.20. *Let $\prec$ be a generalized term order on $[\Delta, \Sigma^*]E$ and let $\mathcal{M}$ be a difference-skew-differential module. The residue classes of the elements of $[\Delta, \Sigma^*]E \setminus \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\}$ form a basis of the K-vector space $K[\Delta, \Sigma^*]E/\mathcal{M}$.*

Proof. Let $\mathcal{O}_\prec(\mathcal{M}) := [\Delta, \Sigma^*]E \setminus \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\}$. First we prove $\sum_{b\in\mathcal{O}_\prec(\mathcal{M})} Kb + \mathcal{M} = K[\Delta, \Sigma^*]E$. In contradiction suppose

$$\sum_{b\in\mathcal{O}_\prec(\mathcal{M})} Kb + \mathcal{M} \subset K[\Delta, \Sigma^*]E,$$

i.e., there exists $0 \neq m \in K[\Delta, \Sigma^*]E \setminus (\sum_{b\in\mathcal{O}_\prec(\mathcal{M})} Kb + \mathcal{M})$ such that $\mathrm{lt}_\prec(m) \preceq \mathrm{lt}_\prec(\tilde{m})$ for all $\tilde{m} \in K[\Delta, \Sigma^*]E \setminus (\sum_{b\in\mathcal{O}_\prec(\mathcal{M})} Kb + \mathcal{M})$. If $\mathrm{lt}_\prec(m) \in \mathcal{O}_\prec(\mathcal{M})$ then

$$m - \mathrm{lc}_\prec(m)\,\mathrm{lt}_\prec(m) \in K[\Delta, \Sigma^*]E \setminus \left(\sum_{b\in\mathcal{O}_\prec(\mathcal{M})} Kb + \mathcal{M}\right)$$

having a strictly smaller leading term than m which contradicts our assumption on m. Hence, $\mathrm{lt}_\prec(m) \in \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\}$, i.e., there exists $m' \in \mathcal{M}$ such that $\mathrm{lt}_\prec(m) = \mathrm{lt}_\prec(m')$. Then

$$m - \frac{\mathrm{lc}_\prec(m)}{\mathrm{lc}_\prec(m')} m' \in K[\Delta, \Sigma^*]E \setminus \left(\sum_{b\in\mathcal{O}_\prec(\mathcal{M})} Kb + \mathcal{M}\right)$$

having a strictly smaller leading term then m which contradicts our assumption on m. We conlude that the residue classes of the elements of $\mathcal{O}_\prec(\mathcal{M})$ generate $K[\Delta, \Sigma^*]E/\mathcal{M}$.

Suppose now that there exist $c_1, \ldots, c_s \in K \setminus \{0\}$, $m_1, \ldots, m_s \in \mathcal{O}_\prec(\mathcal{M})$ such that

$$m = \sum_{i=1}^{s} c_i m_i \in \mathcal{M}.$$

Then $\mathrm{lt}_\prec(m) \in \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\}$ since $m \in \mathcal{M}$. On the other hand $\mathrm{lt}_\prec(m) \in \mathrm{supp}(m) \subseteq \{m_1, \ldots, m_s\} \subseteq \mathcal{O}_\prec(\mathcal{M})$. Hence

$$\mathrm{lt}_\prec(m) \in \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\} \cap \mathcal{O}_\prec(\mathcal{M}) = \emptyset.$$

We conlude that the elements of $\mathcal{O}_\prec(\mathcal{M})$ form a K-vector space basis of $K[\Delta, \Sigma^*]E/\mathcal{M}$. $\square$

Let $\Xi = \{[\Delta, \Sigma^*]_k \mid 1 \leq k \leq p\}$ be an orthant decomposition and $\mathcal{O}$ a Ξ-difference-skew-differential order module. Then for any $k \in \{1, \ldots, p\}$ we have that $K[\Delta, \Sigma^*]_k E \setminus \mathcal{O}$ is a monomial $K[\Delta, \Sigma^*]_k E$-module, i.e., there exists a generating set consisting only of monomials. By $\mathcal{C}(\mathcal{O})_k \subseteq [\Delta, \Sigma^*]_k E$ we denote its minimal generating set. Let $\mathcal{C}(\mathcal{O}) := \bigcup_{k=1}^{p} \mathcal{C}(\mathcal{O})_k$. Furthermore for any generalized term order $\prec$ and difference-skew-differential module $\mathcal{M}$ the set $\mathcal{O}_\prec(\mathcal{M}) = [\Delta, \Sigma^*]E \setminus \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\}$ is a Ξ-difference-skew-differential order module.

Using Macaulay's basis theorem for difference-skew-differential operators and the notion of Gröbner bases for difference-skew-differential – which by Remark 2.3.10 is a special case of weight relative Gröbner bases – we obtain the following result stating that every difference-skew-differential module possesses an $\mathcal{O}$-border basis for some Ξ-difference-skew-differential order module $\mathcal{O}$ if there exists a generalized term order with respect to the orthant decomposition Ξ (compare [KKR05, Prop. 4.4.6]).

Lemma 2.4.21. *Let $\Xi = \{[\Delta, \Sigma^*]_k \mid 1 \leq k \leq p\}$ be an orthant decomposition, $\prec$ a generalized term order on $[\Delta, \Sigma^*]E$ with respect to Ξ and let $\mathcal{M}$ be a difference-skew-differential module. Let $\mathcal{O}_\prec(\mathcal{M}) := [\Delta, \Sigma^*]E \setminus \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\}$ and let $\partial\mathcal{O}_\prec(\mathcal{M}) = \{b_j \mid j \in J\}$. Then there exists a unique $\mathcal{O}_\prec(\mathcal{M})$-border basis $G = \{g_j \mid j \in J\}$ of $\mathcal{M}$. Let J' be an index set such that $\mathcal{C}(\mathcal{O}_\prec(\mathcal{M})) = \{b_j \mid j \in J'\}$. Then the set $G' := \{g_j \mid j \in J'\}$ is a $\prec$-Gröbner basis of $\mathcal{M}$.*

Proof. By Macaulay's basis theorem for difference-skew-differential modules 2.4.20 the residue classes of the terms in $\mathcal{O}_\prec(\mathcal{M})$ form a basis of the K-vector space $K[\Delta, \Sigma^*]E/\mathcal{M}$. Hence, by Theorem 2.4.19 there exists a unique $\mathcal{O}_\prec(\mathcal{M})$-border basis $G = \{g_j \mid j \in J\}$ of $\mathcal{M}$.

By Definition 2.3.9 a set $F \subseteq \mathcal{M}$ forms a $\prec$-Gröbner basis of $\mathcal{M}$ if and only if every element $f \in \mathcal{M} \setminus \{0\}$ is $\prec$-reducible to 0 modulo F. By Proposition 2.3.11 this is equivalent to every element $f \in \mathcal{M} \setminus \{0\}$ being $\prec$-reducible modulo F at all. So let $f_0 \in \mathcal{M} \setminus \{0\}$. Then $\mathrm{lt}_\prec(f_0) \in [\Delta, \Sigma^*]_k E \setminus \mathcal{O}_\prec(\mathcal{M})$ for some $k \in \{1, \ldots, p\}$. Hence, there exists $j_0 \in J'$ with $b_{j_0} \in [\Delta, \Sigma^*]_k E$ such that

$$\mathrm{lt}_\prec(f_0) \in {}_{K[\Delta,\Sigma^*]_k}\langle b_{j_0} \rangle,$$

where ${}_{K[\Delta,\Sigma^*]_k}\langle b_{j_0} \rangle$ denotes the $K[\Delta, \Sigma^*]_k$ module generated by b_{j_0}. That means, there exists some $\lambda \in [\Delta, \Sigma^*]_k$ such that $\mathrm{lt}_\prec(f_0) = \lambda b_{j_0}$. On the other hand

$$\begin{aligned} \mathrm{supp}(g_{j_0} - b_{j_0}) &\subseteq \mathcal{O}_\prec(\mathcal{M}) \\ &= [\Delta, \Sigma^*]E \setminus \{\mathrm{lt}(f) \mid f \in \mathcal{M} \setminus \{0\}\} \end{aligned}$$

and $g_{j_0} \in \mathcal{M}$. Hence, $\mathrm{lt}(g_{j_0}) = b_{j_0}$. By the definition of generalized term orders we obtain $\mathrm{lt}(\lambda g_{j_0}) = \lambda b_{j_0}$ and

$$\mathrm{lt}_\prec(f_0 - \lambda g_{j_0}) \prec \mathrm{lt}_\prec(f_0),$$

i.e., f_0 is $\prec$-reducible modulo G'. □

We even can establish the following theorem (see also [KKR05, Prop. 4.4.9]).

Theorem 2.4.22. *Let Ξ be an orthant decomposition, $\prec$ a generalized term order with respect to Ξ, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module with $\partial\mathcal{O} = \{b_j \mid j \in J\}$ and $\mathcal{M}$ a difference-skew-differential module such that the residue classes of the terms in $\mathcal{O}$ form a basis of the K-vector space $K[\Delta, \Sigma^*]E/\mathcal{M}$. Let $G = \{g_j \mid j \in J\}$ be the $\mathcal{O}$-border basis of $\mathcal{M}$. Let J' be an index set such that $\mathcal{C}(\mathcal{O}) = \{b_j \mid j \in J'\}$ and let $G' = \{g_j \mid j \in J'\}$. TFAE*

(i) $\mathcal{O} = \mathcal{O}_\prec(\mathcal{M})$,

(ii) for $j \in J$ *we have* $\mathrm{lt}_\prec(g_j) = b_j$,

(iii) for $j \in J'$ *we have* $\mathrm{lt}_\prec(g_j) = b_j$.

If these conditions are satisfied then G' is a $\prec$-Gröbner basis of $\mathcal{M}$.

Proof. "(i) $\Longrightarrow$(ii)": By the definition of an $\mathcal{O}$-border prebasis for $j \in J$ there exists $(\alpha_{ij})_{i \in I} \subseteq K$ such that $g_j = b_j - \sum_{i \in I} \alpha_{ij} t_i$ and only finitely many α_{ij} are different from 0. Since $\mathcal{O} = \mathcal{O}_\prec(\mathcal{M})$ it is obvious that none of the t_i can be the leading term of any element of $\mathcal{M}$. By Lemma 2.4.14 we have $g_j \in \mathcal{M}$. We obtain $\mathrm{lt}_\prec(g_j) \in \mathrm{supp}(g_j) \setminus \mathcal{O} = \{b_j\}$. In addition by Lemma 2.4.21 we get that G' is a $\prec$-Gröbner basis of $\mathcal{M}$.
"(ii)$\Longrightarrow$(iii)": Follows from $J' \subseteq J$.
"(iii)$\Longrightarrow$(i)": Assume that for all $j \in J'$ we have $\mathrm{lt}_\prec(g_j) = b_j$. Since

$$\bigcup_{k=1}^{p} [\Delta, \Sigma^*]_k \left(\{\mathrm{lt}_\prec(g_j) \mid j \in J'\} \cap [\Delta, \Sigma^*]_k E\right) \subseteq \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\}$$

we obtain

$$\begin{aligned}\mathcal{O} &= [\Delta,\Sigma^*]E \setminus \bigcup_{k=1}^{p} [\Delta,\Sigma^*]_k \left(\{\mathrm{lt}_\prec(g_j) \mid j \in J'\} \cap [\Delta,\Sigma^*]_k E\right) \\ &\supseteq [\Delta,\Sigma^*]E \setminus \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\} \\ &= \mathcal{O}_\prec(\mathcal{M}).\end{aligned}$$

By Macaulay's basis theorem 2.4.20 the residue classes of the terms in $\mathcal{O}_\prec(\mathcal{M})$ also form a basis of the K-vector space $K[\Delta,\Sigma^*]E/\mathcal{M}$. We conclude

$$\mathcal{O} = \mathcal{O}_\prec(\mathcal{M}).$$

□

Example 2.4.23. *Let $E = \{1\}, m = 0, n = 2, \mathcal{O} := \{1, \sigma_1, \sigma_2^{-1}, \sigma_1^{-1}\sigma_2^{-1}\}$ and*

$$F := \left\{\sigma_2 - \sigma_1 - \sigma_2^{-1}, \sigma_1^2 - 1 + \frac{1}{2}\sigma_1^{-1}\sigma_2^{-1}, \sigma_1^{-2}\sigma_2^{-2} - 2\right\}.$$

Let $\phi : [\Sigma^*] \to \mathbb{R}^4$ be given by

$$\phi(\sigma_1^{k_1}\sigma_2^{k_2}) := \begin{cases} (k_1 + 3k_2, k_1, k_2, 0) & \text{if } k_1, k_2 \geq 0, \\ (k_1 - 2k_2, k_1 - k_2, 0, -k_2) & \text{if } k_2 < 0 \text{ and } k_1 \geq k_2, \\ (3k_2 - 4k_1, 0, k_2 - k_1, -k_1) & \text{if } k_1 < 0 \text{ and } k_2 > k_1. \end{cases}$$

Let $\lambda, \mu \in [\Sigma^*]$ and let the generalized term order $\prec$ be defined by

$$\lambda \prec \mu :\Longleftrightarrow \phi(\lambda) <_{\mathrm{lex}} \phi(\mu).$$

Using Algorithm 2.3.17, p.25 (or, e.g., [ZW06, Thm 3.3]) it is easy although tedious to verify that F is a Gröbner basis of $\mathcal{M} := \langle F \rangle$ with respect to $\prec$. From this we obtain $\mathcal{O}_\prec(\mathcal{M}) = \mathcal{O}$ and $\mathcal{C}(\mathcal{O}) = \{\sigma_2, \sigma_1^2, \sigma_1^{-2}\sigma_2^{-2}\}$. Hence, completing F to an $\mathcal{O}$-border prebasis with elements from $\mathcal{M}$ will yield an $\mathcal{O}$-border basis of $\mathcal{M}$. The unique $\mathcal{O}$-border basis of $\mathcal{M}$ is given by

$$\begin{aligned} F \quad \cup \quad & \left\{\sigma_1^{-1}\sigma_2^{-2} - 2\sigma_2, \sigma_1\sigma_2 - \frac{1}{2}\sigma_1^{-1}\sigma_2^{-1}, \right. \\ & \left. \sigma_1^{-1} - 2\sigma_1 - \sigma_2^{-1}, \sigma_1\sigma_2^{-1} + 1 - \sigma_1^{-1}\sigma_2^{-1}\right\}. \end{aligned}$$

□

2.4.7 Normal forms

In Gröbner basis theory the representative of a residue class in $K[\Delta,\Sigma^*]E/\mathcal{M}$ is defined as the normal form of an operator f which is defined as the unique – with respect to the chosen admissible order – remainder of f under division by an according Gröbner basis of $\mathcal{M}$. In fact, the defining property of Gröbner bases is that the remainder under division by the Gröbner basis is unique. For border division we have the following lemma (see also [KKR05, Prop. 4.4.11]).

Lemma 2.4.24. *Let Ξ be an orthant decomposition, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module with $\partial\mathcal{O} = \{b_j \mid j \in J\}$. Let $\mathcal{M}$ be a difference-skew-differential module and $G = \{g_j \mid j \in J\}$ an $\mathcal{O}$-border basis of $\mathcal{M}$. Then for any $f \in K[\Delta,\Sigma^*]E$ the set of $\mathcal{O}$-remainders $\mathrm{rem}_{\mathcal{O},G}(f)$ of f contains only one element.*

Proof. Let $f \in K[\Delta, \Sigma^*]E$ and assume that there are $(c_i)_{i\in I}, (\tilde{c}_i)_{i\in I} \subseteq K$, $(f_j)_{j\in J}, (\tilde{f}_j)_{j\in J} \subseteq K[\Delta, \Sigma^*]$ such that

$$\begin{aligned} f &= \sum_{i\in I} c_i t_i + \sum_{j\in J} f_j g_j \\ &= \sum_{i\in I} \tilde{c}_i t_i + \sum_{j\in J} \tilde{f}_j g_j. \end{aligned}$$

Then

$$\sum_{i\in I} (c_i - \tilde{c}_i) t_i \in (\mathcal{O}) \cap \mathcal{M}.$$

Since G is an $\mathcal{O}$-border basis of $\mathcal{M}$ we have $(\mathcal{O}) \cap \mathcal{M} = \{0\}$. So the set of $\mathcal{O}$-remainders of f contains only one element. □

Hence, we can define a normal form (compare [KKR05, Def. 4.4.12]).

Definition 2.4.25. *Let Ξ be an orthant decomposition, $\mathcal{O}$ a Ξ-difference-skew-differential order module, $\mathcal{M}$ a difference-skew-differential module and G an $\mathcal{O}$-border basis of $\mathcal{M}$. Then for any $f \in K[\Delta, \Sigma^*]E$ define the* normal form *of f with respect to $\mathcal{O}$ and $\mathcal{M}$ to be the unique element $\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f) \in (\mathcal{O})$ such that*

$$\{\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f)\} = \mathrm{rem}_{\mathcal{O},G}(f).$$

If $\mathcal{O}$ and $\mathcal{M}$ are clear from the context we simply write $\bar{f}$ instead of $\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f)$.

This leads to the following lemma (compare [KK05, Prop. 14]).

Lemma 2.4.26. *Let Ξ be an orthant decomposition, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module, $\mathcal{M}$ a difference-skew-differential module and $G = \{g_j \mid j \in J\} \subseteq \mathcal{M}$ an $\mathcal{O}$-border prebasis. The following are equivalent:*

(i) G is an $\mathcal{O}$-border basis of $\mathcal{M}$.

(ii) For $f \in K[\Delta, \Sigma^*]E$ we have $\mathrm{rem}_{\mathcal{O},G}(f) = \{0\}$ if and only if $f \in \mathcal{M}$.

Proof. "(i)$\Longrightarrow$(ii)": Let $f \in K[\Delta, \Sigma^*]E$ such that $\mathrm{rem}_{\mathcal{O},G}(f) = \{0\}$. By the definition of the set of $\mathcal{O}$-remainders there exist $(f_j)_{j\in J} \subseteq K[\Delta, \Sigma^*]$ such that $f = \sum_{j\in J} f_j g_j$, i.e., $f \in \langle G \rangle$. Conversely, let $f \in \mathcal{M}$. Applying Algorithm 2.4.18 to $\Xi, \mathcal{O}, G$ and f returns $(c_i)_{i\in I} \subseteq K$, $(f_j)_{j\in J} \subseteq K[\Delta, \Sigma^*]$ such that $f = \sum_{i\in I} c_i t_i + \sum_{j\in J} f_j g_j$, i.e., $\sum_{i\in I} c_i t_i \in \mathrm{rem}_{\mathcal{O},G}(f)$. Since $f \in \mathcal{M}$ we have $\mathrm{rem}_{\mathcal{O},G}(f) \subseteq \mathcal{M}$. G being an $\mathcal{O}$-border basis implies $\mathrm{rem}_{\mathcal{O},G}(f) \subseteq \mathcal{M} \cap (\mathcal{O}) = \{0\}$, i.e., $\mathrm{rem}_{\mathcal{O},G}(f) = \{0\}$.
"(ii)$\Longrightarrow$(i)": Let $(c_i)_{i\in I} \subseteq K$ be such that $f := \sum_{i\in I} c_i t_i \in \mathcal{M} \setminus \{0\}$. Applying Algorithm 2.4.18 to $\Xi, \mathcal{O}, G$ and f returns $(c_i)_{i\in I}$ and $(0)_{j\in J}$ such that $\sum_{i\in I} c_i t_i \in \mathrm{rem}_{\mathcal{O},G}(f) = \{0\}$ contradicting $f \neq 0$. Hence, $\mathcal{M} \cap (\mathcal{O}) = \{0\}$ and Lemma 2.4.6 implies that G is the $\mathcal{O}$-border basis of $\mathcal{M}$. □

We obtain the following relation between normal forms and Gröbner bases (compare [KKR05, Def. 4.4.13]).

Lemma 2.4.27. *Let Ξ be an orthant decomposition, $\mathcal{O}$ a Ξ-difference-skew-differential order module and $\mathcal{M}$ a difference-skew-differential module possessing an $\mathcal{O}$-border basis.*

(i) If there exists a generalized term order $\prec$ such that $\mathcal{O} = \mathcal{O}_\prec(\mathcal{M})$ then for all $f \in K[\Delta, \Sigma^*]E$ we have that $\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f) \in f + \mathcal{M}$ is $\prec$-irreducible modulo any Gröbner basis of $\mathcal{M}$ with respect to $\prec$.

(ii) For $f \in K[\Delta, \Sigma^*]E$ we have $\mathrm{NF}_{\mathcal{O},\mathcal{M}}(\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f)) = \mathrm{NF}_{\mathcal{O},\mathcal{M}}(f)$.

(iii) For $f_1, f_2 \in K[\Delta, \Sigma^*]E$ we have

$$\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f_1 - f_2) = \mathrm{NF}_{\mathcal{O},\mathcal{M}}(f_1) - \mathrm{NF}_{\mathcal{O},\mathcal{M}}(f_2)$$

and

$$\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f_1 f_2) = \mathrm{NF}_{\mathcal{O},\mathcal{M}}(\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f_1)\,\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f_2)).$$

Proof. Since $\mathcal{M}$ possesses an $\mathcal{O}$-border basis for every $f \in K[\Delta, \Sigma^*]E$ there exists $f_0 \in K[\Delta, \Sigma^*]E$ such that

$$(f + \mathcal{M}) \cap (\mathcal{O}) = \{f_0\}.$$

Hence, there exists a uniquely determined operator in $f + \mathcal{M}$ whose support is contained in $\mathcal{O}$ which proves (ii) and (iii). Since $\mathrm{supp}(\mathrm{NF}_{\mathcal{O},\mathcal{M}}(f)) \subseteq \mathcal{O}$ we have $f_0 = \mathrm{NF}_{\mathcal{O},\mathcal{M}}(f)$ and f_0 is $\prec$-irreducible modulo any $g \in \mathcal{M} \setminus \{0\}$ since $\mathcal{O} \cap \{\mathrm{lt}_\prec(g) \mid g \in \mathcal{M} \setminus \{0\}\} = \emptyset$, i.e., (i) holds. □

2.4.8 Multiplication endomorphisms

Effective computation with residue classes basically boils down to figuring out how to perform multiplications. For zero-dimensional modules multiplication of residue classes can be described by multiplication tables which give rise to multipication endomorphisms. If the module in concern is not zero-dimensional then consequently the associated multiplication table is not finite but still we can consider multiplication endomorphisms in this case. By this considerations Kehrein and Kreuzer [KK05] derived an S-polynomial criterion for border bases of polynomial ideals. Due to the group structure of $[\Sigma^*]$ contained in $[\Delta, \Sigma^*]$ for border bases of difference-skew-differential modules the according result becomes slightly more complicated.

Let Ξ be an orthant decomposition with generators $\xi_1, \ldots, \xi_r$, $\mathcal{O} = \{t_i \mid i \in I\}$ a Ξ-difference-skew-differential order module with border $\partial\mathcal{O} = \{b_j \mid j \in J\}$ and $G = \{g_j = b_j - \sum_{i \in I} \alpha_{ij} t_i \mid j \in J\}$ an $\mathcal{O}$-border prebasis. We identify $\{f \in K[\Delta, \Sigma^*]E \mid \mathrm{supp}(f) \subseteq \mathcal{O}^{[1]}\}$ with $(\mathcal{O}^{[1]})$ and define a K-linear projection $N : (\mathcal{O}^{[1]}) \to (\mathcal{O})$ by

$$N : t_i \mapsto t_i, \qquad \text{and} \qquad N : b_j \mapsto \sum_{i \in I} \alpha_{ij} t_i.$$

For reasons of convenience define $\xi_{r+1} := \delta_1, \ldots, \xi_{r+m} := \delta_m$ and for $1 \leq k \leq r + m$ let $M_k : (\mathcal{O}) \to (\mathcal{O})$ be given by

$$p \mapsto M_k(p) := N(\xi_k p). \tag{2.5}$$

For every $\lambda \in [\Delta, \Sigma^*]E \setminus E$ let

$$\begin{aligned} C_\lambda := \{(l, \mu) \in \{1, \ldots, r+m\} \times [\Delta, \Sigma^*]E \quad | \quad & \xi_l \mu = \lambda,\ \mathrm{ord}_\Xi(\mu) = \mathrm{ord}_\Xi(\lambda) - 1, \\ & \mathrm{ind}_\mathcal{O}(\mu) \leq \max\{\mathrm{ind}_\mathcal{O}(\lambda) - 1, 0\}\}. \end{aligned}$$

If $M_1, \ldots, M_{r+m}$ are mutually commuting let $M = (M_1, \ldots, M_{r+m})$ and for $e \in E, \lambda \in [\Delta, \Sigma^*]E \setminus E$ define recursively

$$\begin{aligned} e[M] &:= N(e), \\ \lambda[M] &:= \frac{1}{|C_\lambda|} \sum_{(l,\mu) \in C_\lambda} M_l(\mu[M]). \end{aligned}$$

For $p = \sum_{q=1}^{s} \alpha_q \lambda_q \in K[\Delta, \Sigma^*]E$ with $\alpha_1, \ldots, \alpha_s \in K$, $\lambda_1, \ldots, \lambda_s \in [\Delta, \Sigma^*]E$ define $p(M) := \sum_{q=1}^{s} \alpha_q \lambda_q[M]$.

Example 2.4.28. *Let m, n, Ξ, E and $\mathcal{O} = \{1, \sigma_1, \sigma_2^{-1}, \sigma_1^{-1}\sigma_2^{-1}\}$ be as in Example 2.4.23 and let $G = \{g_1, \ldots, g_7\}$ be given by*

$$\begin{aligned}
g_1 &:= \sigma_2 - \sigma_1 - \sigma_2^{-1}, \\
g_2 &:= \sigma_1^2 - 1 + \frac{1}{2}\sigma_1^{-1}\sigma_2^{-1}, \\
g_3 &:= \sigma_1^{-1} - 2\sigma_1 - \sigma_2^{-1}, \\
g_4 &:= \sigma_1\sigma_2 - \frac{1}{2}\sigma_1^{-1}\sigma_2^{-1}, \\
g_5 &:= \sigma_1\sigma_2^{-1} - \sigma_1^{-1}\sigma_2^{-1} + 1, \\
g_6 &:= \sigma_1^{-1}\sigma_2^{-2} - 2\sigma_2, \\
g_7 &:= \sigma_1^{-2}\sigma_2^{-2} - 2,
\end{aligned}$$

i.e., $N : (\mathcal{O}^{[1]}) \to (\mathcal{O})$ is given by

$$\begin{aligned}
\sigma_2 &\mapsto \sigma_1 + \sigma_2^{-1}, \\
\sigma_1^2 &\mapsto 1 - \frac{1}{2}\sigma_1^{-1}\sigma_2^{-1}, \\
\sigma_1^{-1} &\mapsto 2\sigma_1 + \sigma_2^{-1}, \\
\sigma_1\sigma_2 &\mapsto \frac{1}{2}\sigma_1^{-1}\sigma_2^{-1}, \\
\sigma_1\sigma_2^{-1} &\mapsto \sigma_1^{-1}\sigma_2^{-1} - 1, \\
\sigma_1^{-1}\sigma_2^{-2} &\mapsto 2\sigma_2, \\
\sigma_1^{-2}\sigma_2^{-2} &\mapsto 2.
\end{aligned}$$

The generators of Ξ are $\xi_1 = \sigma_1, \xi_2 = \sigma_2$ and $\xi_3 = \sigma_1^{-1}\sigma_2^{-1}$. Then for $\alpha \in K$ the maps M_1, M_2 and $M_3 : (\mathcal{O}) \to (\mathcal{O})$ are given by

$$\begin{aligned}
&M_1(\alpha_1 + \alpha_2\sigma_1 + \alpha_3\sigma_2^{-1} + \alpha_4\sigma_1^{-1}\sigma_2^{-1}) = \\
&\qquad (s_1(\alpha_2) - s_1(\alpha_3)) + s_1(\alpha_1)\sigma_1 + s_1(\alpha_4)\sigma_2^{-1} + \left(s_1(\alpha_3) - \frac{s_1(\alpha_2)}{2}\right)\sigma_1^{-1}\sigma_2^{-1}, \\
&M_2(\alpha_1 + \alpha_2\sigma_1 + \alpha_3\sigma_2^{-1} + \alpha_4\sigma_1^{-1}\sigma_2^{-1}) = \\
&\qquad s_2(\alpha_3) + (s_2(\alpha_1) + 2s_2(\alpha_4))\sigma_1 + (s_2(\alpha_1) + s_2(\alpha_4))\sigma_2^{-1} + \frac{s_2(\alpha_2)}{2}\sigma_1^{-1}\sigma_2^{-1} \\
&M_3(\alpha_1 + \alpha_2\sigma_1 + \alpha_3\sigma_2^{-1} + \alpha_4\sigma_1^{-1}\sigma_2^{-1}) = \\
&\qquad 2s_1^{-1}(s_2^{-1}(\alpha_4)) + 2s_1^{-1}(s_2^{-1}(\alpha_3))\sigma_1 + s_1^{-1}(s_2^{-1}(\alpha_2))\sigma_2^{-1} \\
&\qquad + s_1^{-1}(s_2^{-1}(\alpha_1))\sigma_1^{-1}\sigma_2^{-1}.
\end{aligned}$$

It is easy to check that M_1, M_2 and M_3 are mutually commuting. Let $\alpha \in K$ and $p := \sigma_1^2 - \alpha\sigma_1\sigma_2^{-1}$. First we compute $\sigma_1^2[M]$. From $C_{\sigma_1^2} = \{(1, \sigma_1)\}$ we get $\sigma_1^2[M] = M_1(\sigma_1[M])$ and from $C_{\sigma_1} = \{(1, 1)\}$ we obtain

$$\sigma_1^2[M] = M_1(M_1(1)) = M_1^2(1) = 1 - \frac{1}{2}\sigma_1^{-1}\sigma_2^{-1}.$$

Next we compute $\sigma_1\sigma_2^{-1}[M]$. From $C_{\sigma_1\sigma_2^{-1}} = \{(1, \sigma_2^{-1})\}$ we get $\sigma_1\sigma_2^{-1}[M] = M_1(\sigma_2^{-1}[M])$ and from $C_{\sigma_2^{-1}} = \{(1, \sigma_1^{-1}\sigma_2^{-1}), (3, \sigma_1)\}$ we obtain

$$\sigma_1\sigma_2^{-1}[M] = M_1\left(\frac{1}{2}(M_1(\sigma_1^{-1}\sigma_2^{-1}[M]) + M_3(\sigma_1[M]))\right).$$

Since $C_{\sigma_1^{-1}\sigma_2^{-1}} = \{(3,1)\}$ *we have*

$$\sigma_1\sigma_2^{-1}[M] = M_1\left(\frac{1}{2}(M_1(M_3(1)) + M_3(M_1(1)))\right) = M_1^2 M_3(1) = \sigma_1^{-1}\sigma_2^{-1} - 1.$$

We conlude

$$p(M) = 1 - \alpha - \left(\frac{1}{2} + \alpha\right)\sigma_1^{-1}\sigma_2^{-1}.$$

□

Then we have the following lemma (compare [Mou99, Prop. 3.2]).

Lemma 2.4.29. *Let* $M_1, \dots, M_{r+m}$ *be mutually commuting. Then for any* $p \in K[\Delta, \Sigma^*]E$ *with* $\operatorname{supp}(p) \subseteq \mathcal{O}^{[1]}$ *we have*

$$p(M) = N(p).$$

Proof. Since the maps $p \mapsto p(M)$ and $p \mapsto N(p)$ are K-linear it suffices to assume p to be a difference-skew-differential term. If $p = e$ for some $e \in E$ then $p(M) = M^0 e[M] = N(e)$, i.e., for operators of Ξ-order 0 the claim holds. Assume now that it holds for all monomials $\tilde{p} \in \mathcal{O}$ with $\operatorname{ord}_\Xi(\tilde{p}) < d$. Let p be in $\mathcal{O}^{[1]}$ with $\operatorname{ord}_\Xi(p) = d$. Then for all $(l, \mu) \in C_p$ we have $\operatorname{ord}_\Xi(\mu) < d$, $\mu \in \mathcal{O}$ and

$$\begin{aligned} p(M) &= \frac{1}{|C_p|}\sum_{(l,\mu)\in C_p} M_l(\mu[M]) \\ &= \frac{1}{|C_p|}\sum_{(l,\mu)\in C_p} M_l(N(\mu)) \\ &= \frac{1}{|C_p|}\sum_{(l,\mu)\in C_p} N(\xi_l N(\mu)) \\ &= \frac{1}{|C_p|}\sum_{(l,\mu)\in C_p} N(\xi_l \mu) \\ &= N(p). \end{aligned}$$

□

Example 2.4.30. *Let* $m, n, \Xi, E, \mathcal{O}, G, M_1, M_2, M_3, N$ *and* p *be as in Example 2.4.28. Then we have*

$$p(M) = 1 + \alpha - \left(\frac{1}{2} + \alpha\right)\sigma_1^{-1}\sigma_2^{-1}.$$

On the other hand we have

$$\begin{aligned} N(p) &= 1 - \frac{1}{2}\sigma_1^{-1}\sigma_2^{-1} - \alpha(\sigma_1^{-1}\sigma_2^{-1} - 1) \\ &= 1 + \alpha - \left(\frac{1}{2} + \alpha\right)\sigma_1^{-1}\sigma_2^{-1}. \end{aligned}$$

□

Then [Mou99, Prop. 3.3] carries over to our setting.

Lemma 2.4.31. *Let* $M_1, \dots, M_{r+m}$ *be mutually commuting. Then*

$$\begin{aligned} \langle \operatorname{Ker}(N) \rangle &= (p - p(M) \mid p \in K[\Delta, \Sigma^*]E) \\ &= \{p \in K[\Delta, \Sigma^*]E \mid p(M) = 0\}. \end{aligned}$$

Proof. Let
$$\Upsilon := (\, p - p(M) \mid p \in K[\Delta, \Sigma^*]E)$$
and
$$\Upsilon' := \{\, p \in K[\Delta, \Sigma^*]E \mid p(M) = 0\}\,,$$
i.e., we have to show
$$\langle \mathrm{Ker}(N) \rangle = \Upsilon = \Upsilon'.$$
For $p \in K[\Delta, \Sigma^*]E$ we have $\mathrm{supp}(p(M)) \in \mathcal{O}$ and Lemma 2.4.29 together with the definition of N yields
$$\begin{aligned} (p - p(M))(M) &= p(M) - p(M)(M) \\ &= p(M) - p(M) \\ &= 0, \end{aligned}$$
i.e., $\Upsilon \subseteq \Upsilon'$. Conversely, for $p \in \Upsilon'$ with $p(M) = 0$ we have
$$p = p - p(M) \in \Upsilon,$$
i.e., $\Upsilon = \Upsilon'$. The kernel of N is generated by the elements $p - N(p)$, where $p \in K[\Delta, \Sigma^*]E$ with $\mathrm{supp}(p) \subseteq \mathcal{O}^{[1]}$. Hence, by Lemma 2.4.29 the module $\langle \mathrm{Ker}(N) \rangle$ is generated by the elements $p - p(M)$, where $\mathrm{supp}(p) \subseteq \mathcal{O}^{[1]}$ and we get $\langle \mathrm{Ker}(N) \rangle \subseteq \Upsilon$. Obviously, for any $e \in E$ we have $\xi^0 e - \xi^0 e[M] \in \Upsilon$ and $\xi^0 e - \xi^0 e[M] \in \langle \mathrm{Ker}(N) \rangle$. Assume now that for all $\mu \in [\Delta, \Sigma^*]E$ with $\mathrm{ord}_\Xi(\mu) < d$ we have
$$\mu - \mu[M] \in \langle \mathrm{Ker}(N) \rangle$$
and consider $\lambda \in [\Delta, \Sigma^*]E$ with $\mathrm{ord}_\Xi(\lambda) = d$. Then
$$\begin{aligned} &\lambda - \lambda[M] \\ &= \frac{1}{|C_\lambda|} \sum_{(l,\mu) \in C_\lambda} \Big[\xi_l(\mu - \mu[M]) + \xi_l \mu[M] - M_l(\mu[M]) \Big]. \end{aligned}$$
On the other hand $\mathrm{supp}(\xi_l \mu[M]) \subseteq \mathcal{O}^{[1]}$ and by assumption
$$\begin{aligned} \xi_l \mu[M] - M_l(\mu[M]) &= \xi_l \mu[M] - N(\xi_l \mu[M]) \\ &\in \langle \mathrm{Ker}(N) \rangle. \end{aligned}$$
Hence,
$$\lambda - \lambda[M] \in \langle \mathrm{Ker}(N) \rangle$$
and we conclude $\Upsilon = \langle \mathrm{Ker}(N) \rangle$. □

These two lemmata together yield the following theorem which extends [Mou99, Thm. 3.1] and [KKR05, Thm. 4.4.17] to our setting.

Theorem 2.4.32. *The maps $M_1, \ldots, M_{r+m}$ are mutually commuting if and only if the set $G = \{g_j \mid j \in J\}$ is an $\mathcal{O}$-border basis of $\langle G \rangle$.*

Proof. "$\Longleftarrow$": Let G be an $\mathcal{O}$-border basis of $\langle G \rangle$. Then Lemma 2.4.15 implies $K[\Delta, \Sigma^*]E = (\mathcal{O}) \oplus \langle G \rangle$. By the definition of N we have
$$\mathrm{Ker}(N) = \left\{ \sum_{j \in J} a_j g_j \;\middle|\; \forall_{j \in J} a_j \in K \right\},$$

i.e., $K[\Delta,\Sigma^*]E = (\mathcal{O}) \oplus \langle \mathrm{Ker}(N)\rangle$. So we have $\mathcal{O} \cap \langle \mathrm{Ker}(N)\rangle = \{0\}$. Then for any $p \in (\mathcal{O})$ we get $M_l(p) = \xi_l p \mod \langle \mathrm{Ker}(N)\rangle$. Hence,

$$\begin{aligned}(M_{l_1} \circ M_{l_2} - M_{l_2} \circ M_{l_1})(p) &= (\xi_{l_1}\xi_{l_2} - \xi_{l_2}\xi_{l_1})p \\ &= 0 \mod \langle \mathrm{Ker}(N)\rangle.\end{aligned}$$

On the other hand

$$(M_{l_1} \circ M_{l_2} - M_{l_2} \circ M_{l_1})(p) \in (\mathcal{O}).$$

We conclude

$$(M_{l_1} \circ M_{l_2} - M_{l_2} \circ M_{l_1})(p) = 0,$$

i.e.,$(M_{l_1} \circ M_{l_2} - M_{l_2} \circ M_{l_1})(p) = 0$ for all p. Hence, M_{l_1} and M_{l_2} commute.
"$\Longrightarrow$": The map $K[\Delta,\Sigma^*]E \ni p \mapsto p(M) \in (\mathcal{O})$ is surjectiv. Lemma 2.4.29 implies that restricted to $(\mathcal{O})$ it is the identity map. Because of Lemma 2.4.31 its kernel coincides with the ideal $\langle \mathrm{Ker}(N)\rangle = \langle G\rangle$. So $(\mathcal{O}) \simeq K[\Delta,\Sigma^*]E/\langle G\rangle$ and $K[\Delta,\Sigma^*]E = (\mathcal{O}) \oplus \langle G\rangle$. By Lemma 2.4.15 we conclude that G is an $\mathcal{O}$-border basis of $\langle G\rangle$. □

Example 2.4.33. *Let $m, n, \Xi, E, \mathcal{O}, G, M_1, M_2, M_3$ be as in Example 2.4.28. Since we already saw that M_1, M_2, M_3 are mutually commuting we conclude by Theorem 2.4.32 that G is the $\mathcal{O}$-border basis of $\langle G\rangle$.* □

2.4.9 Commutativity condition

As for polynomial ideals (see [KK05]) also for modules of difference-skew-differential operators it pays off to take a closer look at the commutativity condition derived in the previous subsection. We will consider an element t_{i_1} of the order module $\mathcal{O}$ and multiplications by ξ_{l_1} and ξ_{l_2}. Then we have to distinguish several cases depending on whether $\xi_{l_1}t_{i_1}, \xi_{l_2}t_{i_1}$ and $\xi_{l_1}\xi_{l_2}t_{i_1}$ are elements of $\mathcal{O}$ or $\partial\mathcal{O}$. In the polynomial situation there are four cases to distinguish. Two carry directly over to our setting. For the remaining two we have to take into account noncommutativity in form of the action of skew-derivations and automorphisms on the coefficients. Dealing with difference-skew-differential operators there is a fifth case to consider due to $[\Sigma^*]$ being a group contained in $[\Delta,\Sigma^*]$.

In this subsection on several occasions we will make use of the Kronecker delta. It will be denoted by δ with two subindices and therefore cannot be confussed with a δ denoting a skew-derivative because those only carry one subindex.

Case 1: $\xi_{l_1}\xi_{l_2}t_{i_1}, \xi_{l_1}t_{i_1}, \xi_{l_2}t_{i_1} \in \mathcal{O}$. Then there exist $i_2, i_3, i_4 \in I$ such that $\xi_{l_1}t_{i_1} = t_{i_2}$, $\xi_{l_2}t_{i_1} = t_{i_3}$ and $\xi_{l_1}\xi_{l_2}t_{i_1} = t_{i_4}$. Then

$$M_{l_1}M_{l_2}t_{i_1} = M_{l_1}t_{i_3} = t_{i_4} = M_{l_2}t_{i_2} = M_{l_2}M_{l_1}t_{i_1}$$

and commutativity holds by definition of the maps M_{l_1}, M_{l_2}.

Case 2: $\xi_{l_1}\xi_{l_2}t_{i_1} \in \partial\mathcal{O}$, $\xi_{l_1}t_{i_1}$, $\xi_{l_2}t_{i_1} \in \mathcal{O}$. Then there exist $i_2, i_3 \in I$, $j_1 \in J$ such that $\xi_{l_1}t_{i_1} = t_{i_2}$, $\xi_{l_2}t_{i_1} = t_{i_3}$ and $\xi_{l_1}\xi_{l_2}t_{i_1} = b_{j_1}$. Then

$$M_{l_1}M_{l_2}t_{i_1} = M_{l_1}t_{i_3} = \sum_{i\in I}\alpha_{i,j_1}t_i = M_{l_2}t_{i_2} = M_{l_2}M_{l_1}t_{i_1}$$

and commutativity holds by the definition of M_{l_1}, M_{l_2}.

Case 3: $\xi_{l_1}t_{i_1} \in \mathcal{O}$, $\xi_{l_2}t_{i_1}, \xi_{l_1}\xi_{l_2}t_{i_1} \in \partial\mathcal{O}$. Then there exist $i_2 \in I$, $j_1, j_2 \in J$ such that $\xi_{l_1}t_{i_1} = t_{i_2}$, $\xi_{l_2}t_{i_1} = b_{j_1}$ and $\xi_{l_1}\xi_{l_2}t_{i_1} = b_{j_2}$. We get

$$M_{l_1}M_{l_2}t_{i_1} = M_{l_1}\sum_{i\in I}\alpha_{i,j_1}t_i$$

and

$$M_{l_2}M_{l_1}t_{i_1} = M_{l_2}t_{i_2} = \sum_{i\in I}\alpha_{i,j_2}t_i.$$

If $l_1 \in \{1,\dots,r\}$ then because of the action of the automorphism ξ_{l_1} the commutativity condition becomes

$$\begin{aligned}&(\alpha_{k,j_2})_{k\in I}\\ &= \left(\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}}\delta_{k,\psi(i)}\xi_{l_1}(\alpha_{i,j_1}) + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde{\psi}(i)}}\xi_{l_1}(\alpha_{i,j_1})\alpha_{k,\tilde{\psi}(i)}\right)_{k\in I},\end{aligned}$$

where $\psi: I\to I,\ \tilde{\psi}: I\to J$ are such that $\xi_{l_1}t_i = t_{\psi(i)}$ and $\xi_{l_1}t_i = b_{\tilde{\psi}(i)}$, the first sum runs over all $i\in I$ such that $\xi_{l_1}t_i\in\mathcal{O}$ (it only consists of one summand for which $\xi_{l_1}t_i = t_k$) and the second sum runs over all $i\in I$ such that $\xi_{l_1}t_i\in\partial\mathcal{O}$.

If $l_1\in\{r+1,\dots,r+m\}$ then because of the action of the derivation $\xi_{l_1} = \delta_{l_1-r}$ the commutativity condition becomes

$$\begin{aligned}&(\alpha_{k,j_2})_{k\in I}\\ &= \left(\delta_{l_1-r}(\alpha_{k,j_1}) + \sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}}\delta_{k,\psi(i)}\alpha_{i,j_1} + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde{\psi}(i)}}\alpha_{i,j_1}\alpha_{k,\tilde{\psi}(i)}\right)_{k\in I},\end{aligned}$$

where $\psi: I\to I,\ \tilde{\psi}: I\to J$ are such that $\xi_{l_1}t_i = t_{\psi(i)}$ and $\xi_{l_1}t_i = b_{\tilde{\psi}(i)}$, the first sum runs over all $i\in I$ such that $\xi_{l_1}t_i\in\mathcal{O}$ (it only consists of one summand for which $\xi_{l_1}t_i = t_k$) and the second sum runs over all $i\in I$ such that $\xi_{l_1}t_i\in\partial\mathcal{O}$.

Case 4: $\xi_{l_1}t_{i_1}, \xi_{l_2}t_{i_1}\in\partial\mathcal{O}$. Then there exist $j_1, j_2\in J$ such that $\xi_{l_1}t_{i_1} = b_{j_1},\ \xi_{l_2}t_{i_1} = b_{j_2}$. We get

$$M_{l_1}M_{l_2}t_{i_1} = M_{l_1}\sum_{i\in I}\alpha_{i,j_2}t_i$$

and

$$M_{l_2}M_{l_1}t_{i_1} = M_{l_2}\sum_{i\in I}\alpha_{i,j_1}t_i.$$

If $l_1, l_2\in\{1,\dots,r\}$ then because of the actions of the automorphisms ξ_{l_1} and ξ_{l_2} the commutativity condition becomes

$$\begin{aligned}&\left(\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}}\delta_{k,\psi(i)}\xi_{l_1}(\alpha_{i,j_2}) + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde{\psi}(i)}}\xi_{l_1}(\alpha_{i,j_2})\alpha_{k,\tilde{\psi}(i)}\right)_{k\in I}\\ &= \left(\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_2}t_i=t_{\rho(i)}}\delta_{k,\rho(i)}\xi_{l_2}(\alpha_{i,j_1}) + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_2}t_i=b_{\tilde{\rho}(i)}}\xi_{l_2}(\alpha_{i,j_1})\alpha_{k,\tilde{\rho}(i)}\right)_{k\in I},\end{aligned}$$

where $\psi,\rho: I\to I,\ \tilde{\psi},\tilde{\rho}: I\to J$ are such that $\xi_{l_1}t_i = t_{\psi(i)}, \xi_{l_2}t_i = t_{\rho(i)}$ and $\xi_{l_1}t_i = b_{\tilde{\psi}(i)}, \xi_{l_2}t_i = b_{\tilde{\rho}(i)}$, the first sum runs over all $i\in I$ such that $\xi_{l_1}t_i\in\mathcal{O}$, the second sum runs over all $i\in I$ such that $\xi_{l_1}t_i\in\partial\mathcal{O}$, the third sum runs over all $i\in I$ such that $\xi_{l_2}t_i\in\mathcal{O}$ and the fourth sum runs over all $i\in I$ such that $\xi_{l_2}t_i\in\partial\mathcal{O}$.

If $l_1\in\{1,\dots,r\}, l_2\in\{r+1,\dots,r+m\}$ then because of the actions of the automorphism ξ_{l_1} and the derivation $\xi_{l_2} = \delta_{l_2-r}$ the commutativity condition becomes

$$\left(\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}}\delta_{k,\psi(i)}\xi_{l_1}(\alpha_{i,j_2}) + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde{\psi}(i)}}\xi_{l_1}(\alpha_{i,j_2})\alpha_{k,\tilde{\psi}(i)}\right)_{k\in I}$$

$$= \left(\delta_{l_2-r}(\alpha_{k,j_1}) + \sum_{i\in I,\ \mathcal{O}\ni \xi_{l_2} t_i = t_{\rho(i)}} \delta_{k,\rho(i)}\alpha_{i,j_1} + \sum_{i\in I,\ \partial\mathcal{O}\ni \xi_{l_2} t_i = b_{\tilde\rho(i)}} \alpha_{i,j_1}\alpha_{k,\tilde\rho(i)} \right)_{k\in I},$$

where $\psi,\rho : I \to I$, $\tilde\psi,\tilde\rho : I \to J$ are such that $\xi_{l_1} t_i = t_{\psi(i)}, \xi_{l_2} t_i = t_{\rho(i)}$ and $\xi_{l_1} t_i = b_{\tilde\psi(i)}, \xi_{l_2} t_i = b_{\tilde\rho(i)}$, the first sum runs over all $i \in I$ such that $\xi_{l_1} t_i \in \mathcal{O}$, the second sum runs over all $i \in I$ such that $\xi_{l_1} t_i \in \partial\mathcal{O}$, the third sum runs over all $i \in I$ such that $\xi_{l_2} t_i \in \mathcal{O}$ and the fourth sum runs over all $i \in I$ such that $\xi_{l_2} t_i \in \partial\mathcal{O}$.

If $l_1, l_2 \in \{r+1,\dots,r+m\}$ then because of the action of the derivations $\xi_{l_1} = \delta_{l_1-r}$ and $\xi_{l_2} = \delta_{l_2-r}$ the commutativity condition becomes

$$\left(\delta_{l_1-r}(\alpha_{k,j_2}) + \sum_{i\in I,\ \mathcal{O}\ni \xi_{l_1} t_i = t_{\psi(i)}} \delta_{k,\psi(i)}\alpha_{i,j_2} + \sum_{i\in I,\ \partial\mathcal{O}\ni \xi_{l_1} t_i = b_{\tilde\psi(i)}} \alpha_{i,j_2}\alpha_{k,\tilde\psi(i)} \right)_{k\in I}$$
$$= \left(\delta_{l_2-r}(\alpha_{k,j_1}) + \sum_{i\in I,\ \mathcal{O}\ni \xi_{l_2} t_i = t_{\rho(i)}} \delta_{k,\rho(i)}\alpha_{i,j_1} + \sum_{i\in I,\ \partial\mathcal{O}\ni \xi_{l_2} t_i = b_{\tilde\rho(i)}} \alpha_{i,j_1}\alpha_{k,\tilde\rho(i)} \right)_{k\in I},$$

where $\psi,\rho : I \to I$, $\tilde\psi,\tilde\rho : I \to J$ are such that $\xi_{l_1} t_i = t_{\psi(i)}, \xi_{l_2} t_i = t_{\rho(i)}$ and $\xi_{l_1} t_i = b_{\tilde\psi(i)}, \xi_{l_2} t_i = b_{\tilde\rho(i)}$, the first sum runs over all $i \in I$ such that $\xi_{l_1} t_i \in \mathcal{O}$, the second sum runs over all $i \in I$ such that $\xi_{l_1} t_i \in \partial\mathcal{O}$, the third sum runs over all $i \in I$ such that $\xi_{l_2} t_i \in \mathcal{O}$ and the fourth sum runs over all $i \in I$ such that $\xi_{l_2} t_i \in \partial\mathcal{O}$.

Case 5: $\xi_{l_1}\xi_{l_2} t_{i_1}, \xi_{l_2} t_{i_1} \in \mathcal{O},\ \xi_{l_1} t_{i_1} \in \partial\mathcal{O}$. Then there exist $i_2, i_3 \in I,\ j_1 \in J$ such that $\xi_{l_2} t_{i_1} = t_{i_2},\ \xi_{l_1}\xi_{l_2} t_{i_1} = t_{i_3}$ and $\xi_{l_1} t_{i_1} = b_{j_1}$. We get

$$M_{l_1} M_{l_2} t_{i_1} = M_{l_1} t_{i_2} = t_{i_3}$$

and

$$M_{l_2} M_{l_1} t_{i_1} = M_{l_2} \sum_{i\in I} \alpha_{i,j_1} t_i.$$

Obviously, $l_2 \in \{1,\dots,r\}$. Hence, because of the action of the automorphism $\xi_{l_2} := s^{k_{l_2}}$ associated to $\xi_{l_2} = \sigma^{k_{l_2}}$, resp., the commutativity condition becomes

$$(\delta_{i_3 k})_{k\in I}$$
$$= \left(\sum_{i\in I,\ \mathcal{O}\ni \xi_{l_2} t_i = t_{\rho(i)}} \delta_{k,\rho(i)} \xi_{l_2}(\alpha_{i,j_1}) + \sum_{i\in I,\ \partial\mathcal{O}\ni \xi_{l_2} t_i = b_{\tilde\rho(i)}} \xi_{l_2}(\alpha_{i,j_1})\alpha_{k,\tilde\rho(i)} \right)_{k\in I},$$

where $\rho : I \to I, \tilde\rho : I \to J$ are such that $\xi_{l_2} t_i = t_{\rho(i)}$ and $\xi_{l_2} t_i = b_{\tilde\rho(i)}$, the first sum runs over all $i \in I$ such that $\xi_{l_2} t_i \in \mathcal{O}$ and the second sum runs over all $i \in I$ such that $\xi_{l_2} t_i \in \partial\mathcal{O}$.

2.4.10 S- and T-polynomials

Let us take a closer look at the third, fourth and fifth case above to see whether the commutativity conditions formulated there can be represented more intuitively. It turns out that for the third and fourth case the results from [KK05] carry over.

First consider the relation $\xi_{l_1} b_{j_1} = b_{j_2}$. If $l_1 \in \{1,\dots,r\}$ then for the corresponding combination of border operators we have

$$\begin{aligned} g_{j_2} - \xi_{l_1} g_{j_1} &= \left(b_{j_2} - \sum_{i\in I} \alpha_{i,j_2} t_i \right) - \xi_{l_1} \left(b_{j_1} - \sum_{i\in I} \alpha_{i,j_1} t_i \right) \\ &= -\sum_{i\in I} \alpha_{i,j_2} t_i + \sum_{i\in I} \xi_{l_1}(\alpha_{i,j_1}) \xi_{l_1} t_i \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i\in I}\alpha_{i,j_2}t_i + \sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}} \xi_{l_1}(\alpha_{i,j_1})t_{\psi(i)} \\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}} \xi_{l_1}(\alpha_{i,j_1})b_{\tilde\psi(i)} \\
&= -\sum_{i\in I}\alpha_{i,j_2}t_i + \sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}} \xi_{l_1}(\alpha_{i,j_1})t_{\psi(i)} \\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}} \xi_{l_1}(\alpha_{i,j_1})g_{\tilde\psi(i)} \\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}} \left(\xi_{l_1}(\alpha_{i,j_1})\sum_{k\in I}\alpha_{k,\tilde\psi(i)}t_k\right).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&-\sum_{i\in I}\alpha_{i,j_2}t_i + \sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}} \xi_{l_1}(\alpha_{i,j_1})t_{\psi(i)} \\
&+ \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}} \left(\xi_{l_1}(\alpha_{i,j_1})\sum_{k\in I}\alpha_{k,\tilde\psi(i)}t_k\right) \quad = \quad 0 \mod \langle G\rangle. \qquad (2.6)
\end{aligned}
$$

If $l_1 \in \{r+1,\dots,r+m\}$ then for the corresponding combination of border operators we have

$$
\begin{aligned}
g_{j_2} - \xi_{l_1}g_{j_1} &= \left(b_{j_2} - \sum_{i\in I}\alpha_{i,j_2}t_i\right) - \delta_{l_1-r}\left(b_{j_1} - \sum_{i\in I}\alpha_{i,j_1}t_i\right) \\
&= -\sum_{i\in I}\alpha_{i,j_2}t_i + \sum_{i\in I}\alpha_{i,j_1}\delta_{l_1-r}t_i + \sum_{i\in I}\delta_{l_1-r}(\alpha_{i,j_1})t_i \\
&= -\sum_{i\in I}\alpha_{i,j_2}t_i + \sum_{i\in I,\ \mathcal{O}\ni\delta_{l_1-r}t_i=t_{\psi(i)}} \alpha_{i,j_1}t_{\psi(i)} \\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_1-r}t_i=b_{\tilde\psi(i)}} \alpha_{i,j_1}b_{\tilde\psi(i)} + \sum_{i\in I}\delta_{l_1-r}(\alpha_{i,j_1})t_i \\
&= -\sum_{i\in I}\alpha_{i,j_2}t_i + \sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}} \alpha_{i,j_1}t_{\psi(i)} \\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}} \alpha_{i,j_1}g_{\tilde\psi(i)} + \sum_{i\in I}\delta_{l_1-r}(\alpha_{i,j_1})t_i \\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}} \left(\alpha_{i,j_1}\sum_{k\in I}\alpha_{k,\tilde\psi(i)}t_k\right).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&-\sum_{i\in I}\alpha_{i,j_2}t_i + \sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}} \alpha_{i,j_1}t_{\psi(i)} + \sum_{i\in I}\delta_{l_1-r}(\alpha_{i,j_1})t_i \qquad (2.7) \\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}} \left(\alpha_{i,j_1}\sum_{k\in I}\alpha_{k,\tilde\psi(i)}t_k\right) \quad = \quad 0 \mod \langle G\rangle.
\end{aligned}
$$

If G is a border basis then for any $l_1 \in \{1,\dots,r+m\}$ and each $i \in I$ the coefficient of t_i in the left hand sides of (2.6) and (2.8) above must vanish. This in turn is equivalent to the commutativity condition obtained in case 3 above.

Now consider the relation $\xi_{l_1}b_{j_2} = \xi_{l_2}b_{j_1}$. If $l_1, l_2 \in \{1,\dots,r\}$ then for the corresponding combination of border operators we have

$$\xi_{l_1}g_{j_2} - \xi_{l_2}g_{j_1}$$

$$
\begin{aligned}
&= \xi_{l_1}\left(b_{j_2} - \sum_{i\in I}\alpha_{i,j_2}t_i\right) - \xi_{l_2}\left(b_{j_1} - \sum_{i\in I}\alpha_{i,j_1}t_i\right)\\
&= -\sum_{i\in I}\xi_{l_1}(\alpha_{i,j_2})\xi_{l_1}t_i + \sum_{i\in I}\xi_{l_2}(\alpha_{i,j_1})\xi_{l_2}t_i\\
&= -\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}}\xi_{l_1}(\alpha_{i,j_2})t_{\psi(i)} - \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}}\xi_{l_1}(\alpha_{i,j_2})b_{\tilde\psi(i)}\\
&\quad + \sum_{i\in I,\ \mathcal{O}\ni\xi_{l_2}t_i=t_{\rho(i)}}\xi_{l_2}(\alpha_{i,j_1})t_{\rho(i)} + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_2}t_i=b_{\tilde\rho(i)}}\xi_{l_2}(\alpha_{i,j_1})b_{\tilde\rho(i)}\\
&= -\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}}\xi_{l_1}(\alpha_{i,j_2})t_{\psi(i)} - \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}}\xi_{l_1}(\alpha_{i,j_2})g_{\tilde\psi(i)}\\
&\quad - \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}}\left(\xi_{l_1}(\alpha_{i,j_2})\sum_{k\in I}\alpha_{k,\tilde\psi(i)}t_k\right)\\
&\quad + \sum_{i\in I,\ \mathcal{O}\ni\xi_{l_2}t_i=t_{\rho(i)}}\xi_{l_2}(\alpha_{i,j_1})t_{\rho(i)} + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_2}t_i=b_{\tilde\rho(i)}}\xi_{l_2}(\alpha_{i,j_1})g_{\tilde\rho(i)}\\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_2}t_i=b_{\tilde\rho(i)}}\left(\xi_{l_2}(\alpha_{i,j_1})\sum_{k\in I}\alpha_{k,\tilde\rho(i)}t_k\right).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&-\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}}\xi_{l_1}(\alpha_{i,j_2})t_{\psi(i)} - \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}}\left(\xi_{l_1}(\alpha_{i,j_2})\sum_{k\in I}\alpha_{k,\tilde\psi(i)}t_k\right)\\
&+\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_2}t_i=t_{\rho(i)}}\xi_{l_2}(\alpha_{i,j_1})t_{\rho(i)} + \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_2}t_i=b_{\tilde\rho(i)}}\left(\xi_{l_2}(\alpha_{i,j_1})\sum_{k\in I}\alpha_{k,\tilde\rho(i)}t_k\right)\\
&\qquad = 0 \mod \langle G\rangle. \qquad (2.8)
\end{aligned}
$$

If $l_1 \in \{1,\dots,r\}, l_2 \in \{r+1,\dots,r+m\}$ then for the corresponding combination of border operators we have

$$
\begin{aligned}
&\xi_{l_1}g_{j_2} - \xi_{l_2}g_{j_1}\\
&= \xi_{l_1}\left(b_{j_2} - \sum_{i\in I}\alpha_{i,j_2}t_i\right) - \delta_{l_2-r}\left(b_{j_1} - \sum_{i\in I}\alpha_{i,j_1}t_i\right)\\
&= -\sum_{i\in I}\xi_{l_1}(\alpha_{i,j_2})\xi_{l_1}t_i + \sum_{i\in I}\alpha_{i,j_1}\delta_{l_2-r}t_i + \sum_{i\in I}\delta_{l_2-r}(\alpha_{i,j_1})t_i\\
&= -\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}}\xi_{l_1}(\alpha_{i,j_2})t_{\psi(i)} - \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}}\xi_{l_1}(\alpha_{i,j_2})b_{\tilde\psi(i)}\\
&\quad + \sum_{i\in I,\ \mathcal{O}\ni\delta_{l_2-r}t_i=t_{\rho(i)}}\alpha_{i,j_1}t_{\rho(i)} + \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_2-r}t_i=b_{\tilde\rho(i)}}\alpha_{i,j_1}b_{\tilde\rho(i)} + \sum_{i\in I}\delta_{l_2-r}(\alpha_{i,j_1})t_i\\
&= -\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}}\xi_{l_1}(\alpha_{i,j_2})t_{\psi(i)} - \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}}\xi_{l_1}(\alpha_{i,j_2})g_{\tilde\psi(i)}\\
&\quad - \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde\psi(i)}}\left(\xi_{l_1}(\alpha_{i,j_2})\sum_{k\in I}\alpha_{k,\tilde\psi(i)}t_k\right)\\
&\quad + \sum_{i\in I,\ \mathcal{O}\ni\delta_{l_2-r}t_i=t_{\rho(i)}}\alpha_{i,j_1}t_{\rho(i)} + \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_2-r}t_i=b_{\tilde\rho(i)}}\alpha_{i,j_1}g_{\tilde\rho(i)}\\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_2-r}t_i=b_{\tilde\rho(i)}}\left(\alpha_{i,j_1}\sum_{k\in I}\alpha_{k,\tilde\rho(i)}t_k\right) + \sum_{i\in I}\delta_{l_2-r}(\alpha_{i,j_1})t_i.
\end{aligned}
$$

Hence, we obtain

$$\begin{aligned}
-\sum_{i\in I,\ \mathcal{O}\ni\xi_{l_1}t_i=t_{\psi(i)}} \xi_{l_1}(\alpha_{i,j_2})t_{\psi(i)} - \sum_{i\in I,\ \partial\mathcal{O}\ni\xi_{l_1}t_i=b_{\tilde{\psi}(i)}} \left(\xi_{l_1}(\alpha_{i,j_2})\sum_{k\in I}\alpha_{k,\tilde{\psi}(i)}t_k\right)& \\
+\sum_{i\in I,\ \mathcal{O}\ni\delta_{l_2-r}t_i=t_{\rho(i)}} \alpha_{i,j_1}t_{\rho(i)} + \sum_{i\in I}\delta_{l_2-r}(\alpha_{i,j_1})t_i& \\
+\sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_2-r}t_i=b_{\tilde{\rho}(i)}} \left(\alpha_{i,j_1}\sum_{k\in I}\alpha_{k,\tilde{\rho}(i)}t_k\right)& \\
=0 \mod \langle G\rangle.& \qquad (2.9)
\end{aligned}$$

If $l_1, l_2 \in \{r+1,\ldots,r+m\}$ then for the corresponding combination of border operators we have

$$\begin{aligned}
&\xi_{l_1}g_{j_2} - \xi_{l_2}g_{j_1} \\
&= \delta_{l_1-r}\left(b_{j_2} - \sum_{i\in I}\alpha_{i,j_2}t_i\right) - \delta_{l_2-r}\left(b_{j_1} - \sum_{i\in I}\alpha_{i,j_1}t_i\right) \\
&= -\sum_{i\in I}\alpha_{i,j_2}\delta_{l_1-r}t_i - \sum_{i\in I}\delta_{l_1-r}(\alpha_{i,j_2})t_i + \sum_{i\in I}\alpha_{i,j_1}\delta_{l_2-r}t_i + \sum_{i\in I}\delta_{l_2-r}(\alpha_{i,j_1})t_i \\
&= -\sum_{i\in I,\ \mathcal{O}\ni\delta_{l_1-r}t_i=t_{\psi(i)}} \alpha_{i,j_2}t_{\psi(i)} - \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_1-r}t_i=b_{\tilde{\psi}(i)}} \alpha_{i,j_2}b_{\tilde{\psi}(i)} - \sum_{i\in I}\delta_{l_1-r}(\alpha_{i,j_2})t_i \\
&\quad + \sum_{i\in I,\ \mathcal{O}\ni\delta_{l_2-r}t_i=t_{\rho(i)}} \alpha_{i,j_1}t_{\rho(i)} + \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_2-r}t_i=b_{\tilde{\rho}(i)}} \alpha_{i,j_1}b_{\tilde{\rho}(i)} + \sum_{i\in I}\delta_{l_2-r}(\alpha_{i,j_1})t_i \\
&= -\sum_{i\in I,\ \mathcal{O}\ni\delta_{l_1-r}t_i=t_{\psi(i)}} \alpha_{i,j_2}t_{\psi(i)} - \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_1-r}t_i=b_{\tilde{\psi}(i)}} \alpha_{i,j_2}g_{\tilde{\psi}(i)} \\
&\quad - \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_1-r}t_i=b_{\tilde{\psi}(i)}} \left(\alpha_{i,j_2}\sum_{k\in I}\alpha_{k,\tilde{\psi}(i)}t_k\right) - \sum_{i\in I}\delta_{l_1-r}(\alpha_{i,j_2})t_i \\
&\quad + \sum_{i\in I,\ \mathcal{O}\ni\delta_{l_2-r}t_i=t_{\rho(i)}} \alpha_{i,j_1}t_{\rho(i)} + \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_2-r}t_i=b_{\tilde{\rho}(i)}} \alpha_{i,j_1}g_{\tilde{\rho}(i)} \\
&\quad + \sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_2-r}t_i=b_{\tilde{\rho}(i)}} \left(\alpha_{i,j_1}\sum_{k\in I}\alpha_{k,\tilde{\rho}(i)}t_k\right) + \sum_{i\in I}\delta_{l_2-r}(\alpha_{i,j_1})t_i.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
-\sum_{i\in I,\ \mathcal{O}\ni\delta_{l_1-r}t_i=t_{\psi(i)}} \alpha_{i,j_2}t_{\psi(i)} - \sum_{i\in I}\delta_{l_1-r}(\alpha_{i,j_2})t_i& \\
-\sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_1-r}t_i=b_{\tilde{\psi}(i)}} \left(\alpha_{i,j_2}\sum_{k\in I}\alpha_{k,\tilde{\psi}(i)}t_k\right)& \\
+\sum_{i\in I,\ \mathcal{O}\ni\delta_{l_2-r}t_i=t_{\rho(i)}} \alpha_{i,j_1}t_{\rho(i)} + \sum_{i\in I}\delta_{l_2-r}(\alpha_{i,j_1})t_i& \\
+\sum_{i\in I,\ \partial\mathcal{O}\ni\delta_{l_2-r}t_i=b_{\tilde{\rho}(i)}} \left(\alpha_{i,j_1}\sum_{k\in I}\alpha_{k,\tilde{\rho}(i)}t_k\right)& \\
=0 \mod \langle G\rangle.& \qquad (2.10)
\end{aligned}$$

Again, if G is a border basis then for each $i \in I$ the coefficient of t_i in the left hand sides of (2.8),(2.9) and (2.10) above must vanish. This in turn is equivalent to the commutativity condition obtained in case 4 above.

Last consider the relation $\xi_{l_2} b_{j_1} = t_{i_3}$. Then $l_2 \in \{1, \dots, r\}$ and we have

$$\begin{aligned}
\xi_{l_2} g_{j_1} &= \xi_{l_2} \left(b_{j_1} - \sum_{i \in I} \alpha_{i,j_1} t_i \right) \\
&= t_{i_3} - \sum_{i \in I} \xi_{l_2}(\alpha_{i,j_1}) \xi_{l_2} t_i \\
&= t_{i_3} - \sum_{i \in I,\ \mathcal{O} \ni \xi_{l_2} t_i = t_{\rho(i)}} \xi_{l_2}(\alpha_{i,j_1}) t_{\rho(i)} - \sum_{i \in I,\ \partial\mathcal{O} \ni \xi_{l_2} t_i = b_{\tilde{\rho}(i)}} \xi_{l_2}(\alpha_{i,j_1}) b_{\tilde{\rho}(i)} \\
&= t_{i_3} - \sum_{i \in I,\ \mathcal{O} \ni \xi_{l_2} t_i = t_{\rho(i)}} \xi_{l_2}(\alpha_{i,j_1}) t_{\rho(i)} - \sum_{i \in I,\ \partial\mathcal{O} \ni \xi_{l_2} t_i = b_{\tilde{\rho}(i)}} \xi_{l_2}(\alpha_{i,j_1}) g_{\tilde{\rho}(i)} \\
&\quad - \sum_{i \in I,\ \partial\mathcal{O} \ni \xi_{l_2} t_i = b_{\tilde{\rho}(i)}} \left(\xi_{l_2}(\alpha_{i,j_1}) \sum_{k \in I} \alpha_{k,\tilde{\rho}(i)} t_k \right)
\end{aligned}$$

Hence, we obtain

$$t_{i_3} - \sum_{i \in I,\ \mathcal{O} \ni \xi_{l_2} t_i = t_{\rho(i)}} \xi_{l_2}(\alpha_{i,j_1}) t_{\rho(i)} - \sum_{i \in I,\ \partial\mathcal{O} \ni \xi_{l_2} t_i = b_{\tilde{\rho}(i)}} \left(\xi_{l_2}(\alpha_{i,j_1}) \sum_{k \in I} \alpha_{k,\tilde{\rho}(i)} t_k \right) = 0 \mod \langle G \rangle.$$

Again if G is a border basis then for each $i \in I$ the coefficient of t_i in the left hand side above must vanish. This in turn is equivalent to the commutativity condition obtained in case 5 above.

On the other hand if these three commutativity conditions are satisfied then we know from the considerations above that the operators $M_1, \dots, M_{r+m}$ are mutually commuting and Theorem 2.4.32 implies that the $\mathcal{O}$-border prebasis G is in fact the $\mathcal{O}$-border basis of $\langle G \rangle$.

This motivates the following definition (see also [KK05, Def. 17]).

Definition 2.4.34. *Let $j_1 \neq j_2 \in J$.*

(i) The border terms b_{j_1}, b_{j_2} are called i_1-next-door neighbors *if there exist $i_1, i_2 \in \{1, \dots, r+m\}$, $i \in I$ such that*

  (a) $\xi_{i_1} b_{j_1} = b_{j_2}$,

  (b) $\xi_{i_2} t_i = b_{j_1}$, and

  (c) $\xi_{i_1} t_i \in \mathcal{O}$.

(ii) The border terms b_{j_1}, b_{j_2} are called i_1, i_2-across-the-street neighbors *if there exist $i_1, i_2 \in \{1, \dots, r+m\}$, $i \in I$ such that*

  (a) $\xi_{i_1} b_{j_1} = \xi_{i_2} b_{j_2}$,

  (b) $\xi_{i_1} t_i = b_{j_2}$, and

  (c) $\xi_{i_2} t_i = b_{j_1}$.

(iii) The border terms b_{j_1}, b_{j_2} are called neighbors *if they are next-door neighbors or across-the-street neighbors.*

(iv) The border term b_j is called i_1-individual *if there exist $i_1 \in \{1, \dots, r\}$, $i_2 \in \{1, \dots, r+m\}$, $i \in I$ such that*

  (a) $\xi_{i_1} b_j \in \mathcal{O}$,

  (b) $\xi_{i_2} t_i = b_j$ and

  (c) $\xi_{i_1} t_i \in \mathcal{O}$.

Remark 2.4.35. *It is possible that b_{j_1} and b_{j_2} are next-door and across-the-street neighbors at the same time. Consider, e.g., $m = 0, n = 2, E = \{1\}$ and let the orthant decomposition Ξ have the generators $\xi_1 = \sigma_1, \xi_2 = \sigma_1\sigma_2, \xi_3 = \sigma_2, \xi_4 = \sigma_1^{-1}, \xi_5 = \sigma_2^{-1}$ where for $i = 1, \ldots, 4$ the orthant Ξ_i is generated by ξ_i, ξ_{i+1} and Ξ_5 is generated by ξ_5, ξ_1. Let $\mathcal{O} := \{1, \sigma_2\}$. Then $b_1 := \sigma_1$ and $b_2 := \sigma_1\sigma_2$ are elements of $\partial\mathcal{O}$. On one hand b_1 and b_2 are 3-next-door neighbors and on the other hand they are 2, 1-across-the-street neighbors.*

In contrast to Gröbner bases for border bases it only makes sense to define S-polynomials for operators which are associated to neighbors (see also [KK06, Def. 2]). In addition we also have to deal with operators associated to i-individuals.

Definition 2.4.36. *Let $\mathcal{O}$ be a difference-skew-differential order module $G = \{g_j \mid j \in J\}$ an $\mathcal{O}$-border prebasis and let $i, i_1, i_2 \in I, j_1, j_2 \in J$ such that b_{j_1}, b_{j_2} are*

(i) i-next-door neighbors with $\xi_i b_{j_1} = b_{j_2}$. Then we define the i-th S-polynomial *of g_{j_1} and g_{j_2} by*

$$S_i(g_{j_1}, g_{j_2}) := \xi_i g_{j_1} - g_{j_2}.$$

(ii) i_1, i_2-across-the-street neighbors with $\xi_{i_1} b_{j_1} = \xi_{i_2} b_{j_2}$. Then we define the i_1, i_2-th S-polynomial *of g_{j_1} and g_{j_2} by*

$$S_{i_1,i_2}(g_{j_1}, g_{j_2}) := \xi_{i_1} g_{j_1} - \xi_{i_2} g_{j_2}.$$

Let $i \in \{1, \ldots, r\}$, $j \in J$ be such that the border term b_j is i-individual. Then we define the i-th T-polynomial *of g_j by*

$$T_i(g_j) := \xi_i g_j.$$

Remark 2.4.37. *It follows from Remark 2.4.35 that two $\mathcal{O}$-border prebasis elements can have several S-polynomials. If they have a unique S-polynomial we sometimes omit the index specifying the kind of S-polynomial.*

From the considerations above we obtain the following theorem resembling [KK05, Prop. 18] for our setting.

Theorem 2.4.38. *Let $\mathcal{O}$ be a difference-skew-differential order module and $G = \{g_j \mid j \in J\}$ an $\mathcal{O}$-border prebasis. Let $\mathcal{M} = \langle G \rangle$. TFAE*

(i) G is an $\mathcal{O}$-border basis of $\mathcal{M}$.

(ii) For $j_1, j_2 \in J$ and $\lambda, \mu \in [\Delta, \Sigma^*]$ such that $\lambda b_{j_1} = \mu b_{j_2}$ we have

$$0 \in \mathrm{rem}_{\mathcal{O},G}(\lambda g_{j_1} - \mu g_{j_2}),$$

and for $j \in J$ and $\lambda \in [\Delta, \Sigma^*]$ we have

$$0 \in \mathrm{rem}_{\mathcal{O},G}(\lambda g_j).$$

(iii) For all $j_1, j_2 \in J$ and $i \in \{1, \ldots, r + m\}$ such that b_{j_1} and b_{j_2} are i-next-door neighbors we have

$$0 \in \mathrm{rem}_{\mathcal{O},G}(S_i(g_{j_1}, g_{j_2})),$$

for all $j_1, j_2 \in J$ and $i_1, i_2 \in \{1, \ldots, r + m\}$ such that b_{j_1} and b_{j_2} are i_1, i_2-across-the-street-neighbors we have

$$0 \in \mathrm{rem}_{\mathcal{O},G}(S_{i_1,i_2}(g_{j_1}, g_{j_2})),$$

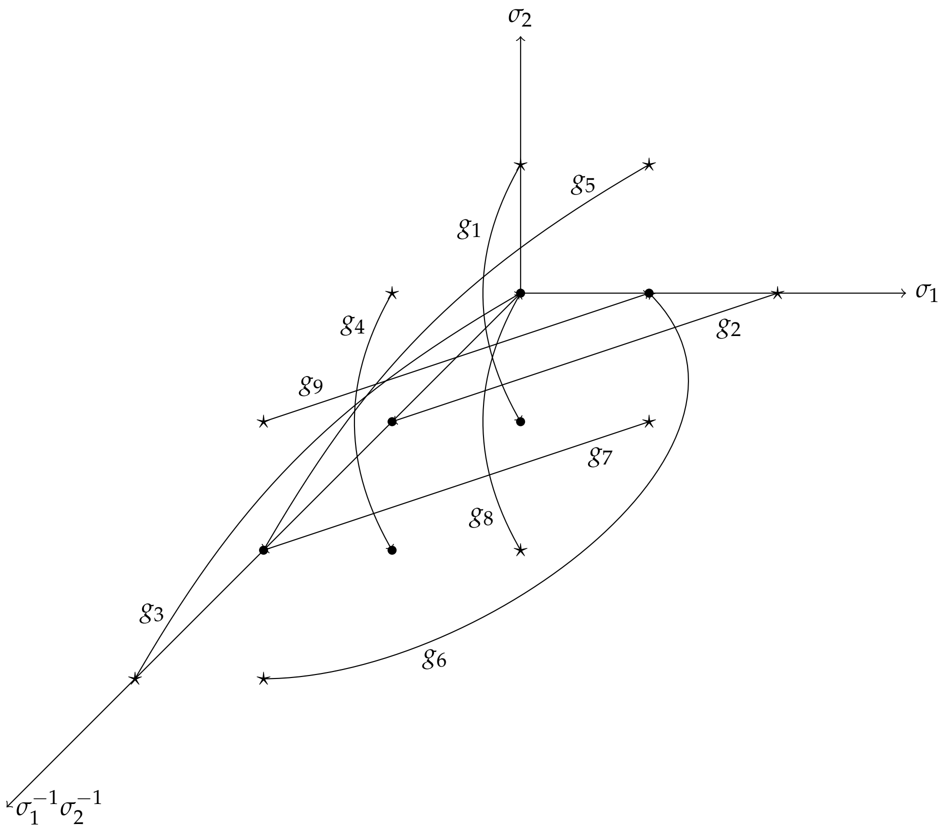
and for all $j \in J$ and $i \in \{1, \ldots, r\}$ such that b_j is i-individual we have

$$0 \in \mathrm{rem}_{\mathcal{O},G}(T_i(g_j)).$$

Proof. "(i)$\Longrightarrow$(ii)": For $\lambda, \mu \in [\Delta, \Sigma^*]$ we have $\lambda g_{j_1} - \mu g_{j_2} \in \mathcal{M}$. If G is the $\mathcal{O}$-border basis of $\langle G \rangle$ then by Lemma 2.4.26 we obtain $\mathrm{rem}_{\mathcal{O},G}(\lambda g_{j_1} - \mu g_{j_2}) = \{0\}$ and $\mathrm{rem}_{\mathcal{O},G}(\lambda g_j) = \{0\}$.
"(ii)$\Longrightarrow$(iii)": Obvious.
"(iii)$\Longrightarrow$(i)": The above calculations show that from (iii) we get that the operators $M_1, \dots, M_r$ are mutually commuting. By Theorem 2.4.32 this is equivalent to G being the $\mathcal{O}$-border basis of $\langle G \rangle$. □

Example 2.4.39. *Let $m = 0, n = 2, E = \{1\}$ and let $\Xi = \{\Xi_k \mid 1 \le k \le 3\}$, where Ξ_1 is generated by σ_1, σ_2, Ξ_2 is generated by $\sigma_1, \sigma_1^{-1}\sigma_2^{-1}$, and Ξ_3 is generated by $\sigma_2, \sigma_1^{-1}\sigma_2^{-1}$ (compare Example 2.2.3). We set $\xi_1 := \sigma_1, \xi_2 := \sigma_2$ and $\xi_3 := \sigma_1^{-1}\sigma_2^{-1}$. Let $\mathcal{O} := \{1, \sigma_1, \sigma_2^{-1}, \sigma_1^{-1}\sigma_2^{-1}, \sigma_1^{-1}\sigma_2^{-1}, \sigma_1^{-2}\sigma_2^{-1}\}$. Then $\partial\mathcal{O} = \{\sigma_2, \sigma_1^2, \sigma_1^{-3}\sigma_2^{-3}, \sigma_1^{-1}, \sigma_1\sigma_2, \sigma_1^{-2}\sigma_2^{-3}, \sigma_1\sigma_2^{-1}, \sigma_2^{-2}, \sigma_1^{-2}\sigma_2^{-1}\}$. Let $\alpha \in K$ be such that $\sigma_2(\alpha) = \alpha$ and let $\mathcal{M} := \langle G \rangle$ where $G = \{g_1, \dots, g_9\}$ is given by*

$$
\begin{array}{ll}
g_1 := \sigma_2 - \sigma_2^{-1}, & g_2 := \sigma_1^2 - \frac{1}{\alpha}\sigma_1^{-1}\sigma_2^{-1}, \\
g_3 := \sigma_1^{-3}\sigma_2^{-3} - s_1^{-2}(\alpha), & g_4 := \sigma_1^{-1} - \sigma_1^{-1}\sigma_2^{-2}, \\
g_5 := \sigma_1\sigma_2 - \frac{1}{s_1^{-1}(\alpha)}\sigma_1^{-2}\sigma_2^{-2}, & g_6 := \sigma_1^{-2}\sigma_2^{-3} - s_1^{-1}(\alpha)\sigma_1, \\
g_7 := \sigma_1\sigma_2^{-1} - \frac{1}{s_1^{-1}(\alpha)}\sigma_1^{-2}\sigma_2^{-2}, & g_8 := \sigma_2^{-2} - 1, \\
g_9 := \sigma_1^{-2}\sigma_2^{-1} - s_1^{-1}(\alpha)\sigma_1. &
\end{array}
$$



The set of next-door neighbors consists of the pairs $\{(b_1, b_5), (b_3, b_6), (b_1, b_4), (b_4, b_9), (b_2, b_7), (b_7, b_8)\}$, the set of across-the-street neighbors consists of the pairs $\{(b_2, b_5), (b_6, b_8), (b_3, b_9)\}$ and the set of individuals consists of $\{b_4, b_5, b_6, b_9\}$. We have

$$
\mathrm{rem}_{\mathcal{O},G}(S_1(g_1, g_5)) \quad = \quad \mathrm{rem}_{\mathcal{O},G}\left(-\sigma_1\sigma_2^{-1} + \frac{1}{s_1^{-1}(\alpha)}\sigma_1^{-2}\sigma_2^{-2}\right)
$$

$$
\begin{aligned}
&= \operatorname{rem}_{\mathcal{O},G}(-g_7) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(S_1(g_3,g_6)) &= \operatorname{rem}_{\mathcal{O},G}(-s_1^{-1}(\alpha)\sigma_1 + s_1^{-1}(\alpha)\sigma_1) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(S_3(g_1,g_4)) &= \operatorname{rem}_{\mathcal{O},G}(-\sigma_1^{-1}\sigma_2^{-2} + \sigma_1^{-1}\sigma_2^{-2}) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(S_3(g_4,g_9)) &= \operatorname{rem}_{\mathcal{O},G}(-\sigma_1^{-2}\sigma_2^{-3} + s_1^{-1}(\alpha)\sigma_1) \\
&= \operatorname{rem}_{\mathcal{O},G}(-g_6) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(S_3(g_2,g_7)) &= \operatorname{rem}_{\mathcal{O},G}\left(-\frac{1}{s_1^{-1}(\alpha)}\sigma_1^{-2}\sigma_2^{-2} + \frac{1}{s_1^{-1}(\alpha)}\sigma_1^{-2}\sigma_2^{-2}\right) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(S_3(g_7,g_8)) &= \operatorname{rem}_{\mathcal{O},G}\left(-\frac{1}{s_1^{-2}(\alpha)}\sigma_1^{-3}\sigma_2^{-3} + 1\right) \\
&= \operatorname{rem}_{\mathcal{O},G}\left(-\frac{1}{s_1^{-2}(\alpha)}g_3\right) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(S_{2,1}(g_2,g_5)) &= \operatorname{rem}_{\mathcal{O},G}\left(-\frac{1}{\alpha}\sigma_1^{-1} + \frac{1}{\alpha}\sigma_1^{-1}\sigma_2^{-2}\right) \\
&= \operatorname{rem}_{\mathcal{O},G}\left(-\frac{1}{\alpha}g_4\right) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(S_{1,3}(g_6,g_8)) &= \operatorname{rem}_{\mathcal{O},G}(-\alpha\sigma_1^2 + \sigma_1^{-1}\sigma_2^{-1}) \\
&= \operatorname{rem}_{\mathcal{O},G}(-\alpha g_2) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(S_{2,3}(g_3,g_9)) &= \operatorname{rem}_{\mathcal{O},G}(-s_1^{-2}(\alpha)\sigma_2 + s_1^{-2}(\alpha)\sigma_2^{-1}) \\
&= \operatorname{rem}_{\mathcal{O},G}(-s_1^{-2}(\alpha)g_1) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(T_1(g_4)) &= \operatorname{rem}_{\mathcal{O},G}(1 - \sigma_2^{-2}) \\
&= \operatorname{rem}_{\mathcal{O},G}(-g_8) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(T_3(g_5)) &= \operatorname{rem}_{\mathcal{O},G}\left(1 - \frac{1}{s_1^{-2}(\alpha)}\sigma_1^{-3}\sigma_2^{-3}\right) \\
&= \operatorname{rem}_{\mathcal{O},G}\left(-\frac{1}{s_1^{-2}(\alpha)}g_3\right) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(T_2(g_6)) &= \operatorname{rem}_{\mathcal{O},G}(\sigma_1^{-2}\sigma_2^{-2} - s_1^{-1}(\alpha)\sigma_1\sigma_2) \\
&= \operatorname{rem}_{\mathcal{O},G}(-s_1^{-1}(\alpha)g_5) \\
&= \{0\}, \\
\operatorname{rem}_{\mathcal{O},G}(T_1(g_9)) &= \operatorname{rem}_{\mathcal{O},G}(\sigma_1^{-1}\sigma_2^{-1} - \alpha\sigma_1^2) \\
&= \operatorname{rem}_{\mathcal{O},G}(-\alpha g_2)
\end{aligned}
$$

$$= \quad \{0\}.$$

Using Theorem 2.4.38 we conclude that G is the $\mathcal{O}$-border basis of $\mathcal{M}$.

Let the generalized term order $\prec$ be as in Example 2.4.23. Since $\mathcal{C}(\mathcal{O}) = \{b_1, b_2, b_3\}$ and for $j = 1, 2, 3$ we have $\mathrm{lt}_{\prec}(g_j) = b_j$ it follows from Theorem 2.4.22 that $\{g_1, g_2, g_3\}$ is a $\prec$-Gröbner basis of $\mathcal{M}$. $\square$

Chapter 3

Difference-skew-differential dimension polynomials and Einstein's strength of systems of difference-differential quations

As pointed out, e.g., in [Eis95, Bre98] Hilbert polynomials occupy a key position in algebraic geometry, combinatorics, and commutative algebra. The prefered method of their algorithmic computation for filtered and graded modules is provided by the theory of Gröbner bases – see, e.g., [CLO92]. Kolchin [Kol64] introduced the differential dimension polynomial as the equivalent of the Hilbert polynomial in differential algebra. For a given system of differential equations the associated differential dimension polynomial describes the number of arbitrary constants in the system's general solution.

As pointed out by Levin [Lev07a] the importance of differential dimension polynomials rests on three pilars.

(i) Mikhalev and Pankratev [MP80] showed that for a system of linear differential equations the associated differential dimension polynomial expresses the system's "strength" in the sense of Einstein [Ein53].

(ii) A differential dimension polynomial carries invariants characterizing a difference-differential field extension independent of the choosen representation [Joh69a, JS78, Kol73, Sit78, MP80].

(iii) Dimension polynomials of prime differential ideals are useful tools in dimension theory of differential rings [Joh69b, KLMP99].

The theory of Gröbner bases in modules over rings of differential operators was developed by Mikhalev and Pankratev [MP80, MP89], Oaku and Shimoyama [OS94], Insa and Pauer [IP98]. Characteristic set methods for the computation of differential dimension polynomials arising from the proof of Kolchin's theorem [Kol73] were developed by Mikhalev and Pankratev [MP80]. A third method of computation of a differential dimension polynomial associated with a differential field extension uses the Hilbert polynomial of the associated module of Kähler differentials which Johnson proved to coincide with the differential dimension polynomial in concern [Joh69a, Joh69c]. Mikhalev and Pankratev computed Einstein's strength for several systems of differential equations from mathematical physics, including – amongst others – the Wave equation, Maxwell's equations, and Dirac equations.

Difference equations naturally arise in numerical solution methods for differential equations. Consequently there exists a theory of difference dimension polynomials, too. They were introduced by Levin [Lev78, Lev80, Lev82, Lev85a] for difference field extensions and modules over rings of difference operators.

Considering (a system of) partial differential equations it is natural to try isolating one of the involved derivations and apply a difference scheme for it leading to a system of difference-differential equations. Another reason to consider such systems is provided by considering differential equations involving, e.g., time delays giving rise to shifts and hence involving differences. The computation of differential, and difference dimension polynomials using Gröbner basis techniques in modules over rings of differential, and difference operators, respectively, is explained in [KLMP99].

Again it was Levin [Lev85b, Lev87] who combined the notions of differential and difference dimension polynomials and considered difference-differential dimension polynomials. Results for difference-differential modules and field extensions are provided in [LM88, LM91]. The algorithmic computation of difference-differential dimension polynomials by Gröbner bases in modules over rings of difference-differential operators is developed in [Lev00, ZW06, ZW08b].

Another direction of research on dimension polynomials associated with systems of partial difference-differential equations emanates from grouping the involved derivations and automorphisms in different groups and considering the degrees of freedom of the system with respect to these groups. Levin [Lev07a, Lev07b, Lev07c, Lev08] as well as Zhou and Winkler [ZW08a] provided algorithms based on Gröbner basis techniques for their computation.

Considering derivatives with weights Shananin was able to prove several interesting analytic results [Sha00a, Sha00b, Sha02, Sha09].

3.1 Uni- and multivariate difference-skew-differential dimension polynomials

In this section we will introduce the notion of weighted filtrations and consider dimension functions for excellently weighted filtered modules over rings of difference-skew-differential operators. This leads to a generalization of Kolchin's result on differential dimension polynomials.

Unless otherwise noted, throughout this section let K be a difference-skew-differential field, $\{\tau_1, \ldots, \tau_m\}$ a set of mutually commuting injective endomorphisms on K, $\Delta = \{\delta_1, \ldots, \delta_m\}$ a basic set of skew-derivations such that for $i = 1, \ldots, m$ the skew-derivation δ_i is a skew-derivation with respect to τ_i, respectively, and $\{\sigma_1, \ldots, \sigma_n\}$ a basic set of automorphisms. By E we always denote the finite set $\{e_1, \ldots, e_q\}$ of free generators of a free difference-skew-differential module.

3.1.1 Weighted filtrations

For any finitely generated difference-skew-differential module $\mathcal{M}$ with generators $m_1, \ldots, m_q$, a weight vector $\alpha \in \mathbb{Q}_0^{m+n}$, and $k \in \mathbb{Z}$ define the *α-filtration* $(\mathcal{M}_{\alpha,k})_{k\in\mathbb{Z}}$ of $\mathcal{M}$ by

$$\mathcal{M}_{\alpha,k} := \left\{ \sum_{j=1}^{q} p_j m_j \;\middle|\; \forall_{1\leq j\leq q} p_j \in K[\Delta, \Sigma^*],\ \mathrm{ord}_\alpha(p_j) \leq k \right\},$$

and let

$$\begin{aligned} \bar{\psi}_\alpha : \mathbb{Z} &\to \mathbb{N}, \\ k &\mapsto \dim_K(\mathcal{M}_{\alpha,k}). \end{aligned}$$

If no confusion is possible we will write $\mathcal{M}_k$ and $\bar{\psi}$ instead of $\mathcal{M}_{\alpha,k}$ and $\bar{\psi}_\alpha$, respectively. To the present author Franz Winkler raised the question whether $\bar{\psi}_\alpha$ can be computed by Gröbner basis methods and independently Alexander Levin brought up the question of its general form.

We rephrase the problem in the following more general way. Let Ξ be an orthant decomposition of $[\Delta, \Sigma^*]$ with generators $\xi_1, \ldots, \xi_r$. For any finitely generated difference-skew-differential module $\mathcal{M}$ with generators $m_1, \ldots, m_q$, a weight vector $\alpha \in \mathbb{Q}_0^{m+r}$, and $k \in \mathbb{Z}$ define the *α-Ξ-filtration* $(\mathcal{M}_{\alpha,\Xi,k})_{k\in\mathbb{Z}}$ of $\mathcal{M}$

$$\mathcal{M}_{\alpha,\Xi,k} := \left\{ \sum_{j=1}^{q} p_j m_j \;\middle|\; \forall_{1\leq j\leq q} p_j \in K[\Delta, \Sigma^*],\ \mathrm{ord}_{\alpha,\Xi}(p_j) \leq k \right\},$$

and let

$$\begin{aligned} \bar{\psi}_{\alpha,\Xi} : \mathbb{Z} &\to \mathbb{N}, \\ k &\mapsto \dim_K(\mathcal{M}_{\alpha,\Xi,k}). \end{aligned}$$

If α is clear from the context we will write $\mathcal{M}_{\Xi,k}$ and $\bar{\psi}_\Xi$ instead of $\mathcal{M}_{\alpha,\Xi,k}$ and $\bar{\psi}_{\alpha,\Xi}$, respectively.

Recall that a *filtered ring* is a ring R together with an ascending chain $(R_k)_{k\in\mathbb{Z}}$ of additive subgroups of R such that $1 \in R_0$ and for all $k, l \in \mathbb{Z}$ we have $R_k R_l \subseteq R_{k+l}$. The family $(R_k)_{k\in\mathbb{Z}}$ is called (ascending) *filtration* of R and its elements are called *components* of the filtration.

We introduce the notion of weighted filtrations as follows.

Definition 3.1.1. *Let Ξ be an orthant decomposition of $R = K[\Delta, \Sigma^*]$ with generators $\xi_1, \ldots, \xi_r$, and consider a weight vector $\alpha \in \mathbb{Q}_0^{m+r}$. A left R-module $\mathcal{M}$ is called a (left)* filtered R-module *if there exists an ascending chain $(\mathcal{M}_k)_{k\in\mathbb{Z}}$ of additive subgroups of $\mathcal{M}$ such that for all $k, l \in \mathbb{Z}$ we have $\{f \in R \mid \mathrm{ord}_{\alpha,\Xi}(f) \leq k\}\mathcal{M}_l \subseteq \mathcal{M}_{k+l}$. The family $(\mathcal{M}_k)_{k\in\mathbb{Z}}$ is called* weighted (or α-) filtration *of $\mathcal{M}$ and its elements are called* components *of the filtration. If* *(i) for every $k \in \mathbb{Z}$ the component $\mathcal{M}_k$ is finitely generated as a K-vector space, and*

(ii) there exist $0 < s \in \mathbb{N}$ and $l_0 \in \mathbb{Z}$ such that for all $l_0 \leq l \in \mathbb{Z}$ we have

$$\mathcal{M}_l = \left\{ f \in R \;\middle|\; \mathrm{ord}_{\alpha,\Xi}(f) \leq \left\lfloor \frac{l - l_0}{s} \right\rfloor s \right\} \mathcal{M}_{l_0 + (l - l_0 \mod s)}.$$

then the weighted filtration $(\mathcal{M}_k)_{k \in \mathbb{Z}}$ is called excellent *and s is called the* period *of $(\mathcal{M}_l)_{l \in \mathbb{Z}}$.*

Example 3.1.2. *Let Ξ be an orthant decomposition of $[\Delta, \Sigma^*]$ with generators $\xi_1, \dots, \xi_r$, and $\alpha \in \mathbb{Q}_+^{m+r}$. Define the* least common multiple lcm *of α by*

$$\begin{aligned} \mathrm{lcm}(\alpha) \quad := \quad & \min\{p \mid \text{there exist } a_1, \dots, a_{r+m} \in \mathbb{N} \text{ such that} \\ & a_1\alpha_1 = \cdots = a_{r+m}\alpha_{r+m} = p \in \mathbb{N}\}, \end{aligned}$$

and define the filtration $(K[\Delta, \Sigma^*]_k)_{k \in \mathbb{Z}}$ by

$$K[\Delta, \Sigma^*]_k := \{f \in K[\Delta, \Sigma^*] \mid \mathrm{ord}_{\alpha,\Xi}(f) \leq k\}.$$

Let $\mathcal{M}$ be a finitely generated difference-skew-differential module. Then $(M_{\alpha,\Xi,k})_{k \in \mathbb{Z}}$ is a weighted filtration of $\mathcal{M}$, every component $\mathcal{M}_{\alpha,\Xi,k}$ is finitely generated as a K-vector space, and for any $q_1 \in \mathbb{Z}, q_2 \in \{0, \dots, \mathrm{lcm}(\alpha) - 1\}$ satisfying

$$q_1 = q_2 \mod \mathrm{lcm}(\alpha)$$

we have

$$\mathcal{M}_{\alpha,\Xi,q_1} = K[\Delta, \Sigma^*]_{q_1 - q_2} \mathcal{M}_{\alpha,\Xi,q_2},$$

i.e., the weighted filtration $(M_{\alpha,\Xi,k})_{k \in \mathbb{Z}}$ is excellent.

3.1.2 Univariate difference-skew-differential dimension polynomials

We recall some basic facts about quasipolynomials [Sta97]. Remember that a function $f : \mathbb{N} \to \mathbb{N}$ is called *quasipolynomial* if there exists $s \in \mathbb{N}$ and polynomials $p_0, \dots, p_{s-1}$ such that for all $n \in \mathbb{N}$ with $n \equiv i \mod s$ we have $f(n) = p_i(n)$. The polynomials $p_0, \dots, p_{s-1}$ are called the *constituents* and s is called the *period* of f. Equivalently f can be written as $f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n)$ for periodic functions $c_d, \dots, c_0$ with integral period. If $c_d \not\equiv 0$ then the degree of f is defined to be d. If $f \equiv 0$ then $\deg(f) := -\infty$.

Obviously, any finite sum of quasipolynomials is again a quasipolynomial.

Remark 3.1.3. *It is easy to see that for any quasipolynomial f with period s and any $0 < m, n \in \mathbb{N}$ also the function $g : \mathbb{N} \to \mathbb{N}$ defined by $g : t \mapsto f(m + tn)$ is a quasipolynomial in t with period $\leq \mathrm{lcm}(s, n)$.*

Theorem 3.1.4. *Let $\mathcal{M}$ be a finitely generated difference-skew-differential module with generators $m_1, \dots, m_q$, Ξ an orthant decomposition of $[\Delta, \Sigma^*]$ with generators $\xi_1, \dots, \xi_r$, $E = \{e_1, \dots, e_q\}$ a finite set of generators of a free difference-skew-differential module, $\alpha \in \mathbb{Q}_0^{m+r}$ a weight vector and $\prec$ a generalized term order on $[\Delta, \Sigma^*]E$ respecting α. By $\pi : K[\Delta, \Sigma^*]E \to \mathcal{M}$ we denote the difference-skew-differential epimorphism given by $\pi(e_i) = m_i$ for $i = 1, \dots, q$. Let G be a $\prec$-Gröbner basis of the $K[\Delta, \Sigma^*]E$-submodule $\ker(\pi)$ and for all $k \in \mathbb{N}$ define*

$$U_k := \{\lambda \in [\Delta, \Sigma^*]E \mid \mathrm{ord}_{\alpha,\Xi}(\lambda) \leq k, \; \nexists_{\mu \in [\Delta,\Sigma^*], g \in G} \, \mathrm{lt}_\prec(\mu g) = \lambda\}.$$

Then for all $k \in \mathbb{N}$ we have $|U_k| = \dim_K(\mathcal{M}_{\alpha,\Xi,k})$.

Proof. We show that $\pi(U_k)$ is a basis of the K-vector space $M_{\alpha,\Xi,k}$. Consider $\lambda m_i \notin \pi(U_k)$ with $i \in \{1,\dots,q\}, \lambda \in [\Delta,\Sigma^*]$ and $\mathrm{ord}_{\alpha,\Xi}(\lambda) \leq k$. First we show that λm_i can be written as a finite K-linear combination of elements of $\pi(U_k)$. Obviously this holds true for $\lambda = 1$. So assume that it holds for all terms ηm_j with $\pi(\eta m_j) = \eta e_j \prec \lambda e_i$. From $\lambda m_i \notin \pi(U_k)$ we obtain $\lambda e_i \notin U_k$, i.e., there exist $\mu \in [\Delta,\Sigma^*], g \in G$ with $\lambda e_i = \mathrm{lt}(\mu g)$ and $\mathrm{ord}_{\alpha,\Xi}(\mu g) \leq k$. Hence,

$$\mu g = a\lambda e_i + \sum_{\mathrm{ord}_{\alpha,\Xi}(\eta)\leq k, j\in\{1,\dots,q\}} a_{\eta,j}\eta e_j,$$

where $a \neq 0$ and $a_{\eta,j} \neq 0$ for only finitely many η, j. Obviously, $\eta e_j \prec \lambda e_i = \mathrm{lt}(\mu g)$ and since $\prec$ respects α we obtain $\mathrm{ord}_{\alpha,\Xi}(\eta) \leq k$. From $G \subseteq N = \ker(\pi)$ we get $\pi(g) = 0$ which implies

$$\begin{aligned} 0 &= \mu\pi(g) \\ &= \pi(\mu g) \\ &= a\pi(\lambda e_i) + \sum_{\mathrm{ord}_{\alpha,\Xi}(\eta)\leq k, j\in\{1,\dots,q\}} a_{\eta,j}\pi(\eta e_j) \\ &= a\lambda m_i + \sum_{\mathrm{ord}_{\alpha,\Xi}(\eta)\leq k, j\in\{1,\dots,q\}} a_{\eta,j}\eta m_j. \end{aligned}$$

Hence, λm_i is a finite K-linear combination of elements of the form ηm_j with $\mathrm{ord}_{\alpha,\Xi}(\eta) \leq k$ and $\eta e_j \prec \lambda e_i$. By induction we conclude that there exist $b_{\eta,j} \in K$ such that

$$\lambda m_i = \sum_{\mathrm{ord}_{\alpha,\Xi}(\eta)\leq k, j\in\{1,\dots,q\}} b_{\eta,j}\eta m_j$$

with $\eta m_j \in \pi(U_k)$ for all η, j such that $b_{\eta,j} \neq 0$.

Regarding K-linear independence assume that there exist $a_1,\dots,a_v \in K, u_1,\dots,u_v \in U_k$ with $\sum_{i=1}^{v} a_i\pi(u_i) = 0$. Then $f = \sum_{i=1}^{v} a_i u_i \in N$ and from

$$u_i \notin \{\mathrm{lt}(\mu g) \mid \mathrm{ord}_{\alpha,\Xi}(\mu) \leq k,\ g \in G\}$$

we get

$$\mathrm{lt}(f) \notin \{\mathrm{lt}(\mu g) \mid \mathrm{ord}_{\alpha,\Xi}(\mu) \leq k,\ g \in G\}.$$

So $f \in N$ is $\prec$-reduced modulo G, i.e., $f = 0$. This implies $a_1 = \cdots = a_v = 0$ which means that $\pi(U_k)$ is K-linearly independent, i.e., it is a basis of $\mathcal{M}_{\alpha,\Xi,k}$. Since π is a bijection on U_k we conclude $|U_k| = \dim_K(\mathcal{M}_{\alpha,\Xi,k})$. □

Using Gröbner basis techniques it is easy to compute the cardinality of U_k for a fixed k. The theory of Ehrhart polynomials enables us to compute the cardinality of U_k and hence also the K-dimension of $\mathcal{M}_k$ efficiently.

For the definition and some fundamental properties of Ehrhart quasipolynomials see also [Sta97]. By a convex polytope P in $\mathbb{R}^v$ or a convex $\mathbb{R}^v$-polytope we mean the convex hull of a finite set of points in $\mathbb{R}^v$. For some $d \in \{0,\dots,v\}$ the affine span of P is a d-dimensional affine subspace of $\mathbb{R}^v$. A point $a \in P$ is a vertex of P if it is not an element of the interior of any line segment contained in P. If V denotes the set of vertices of P then V is finite and P is the convex hull of V. P is called *integer polytope* if all vertices of P have integer coordinates and it is called *rational polytope* if all vertices of P have rational coordinates.

Let P be a convex polytope in $\mathbb{R}^v$. For any $u \in \mathbb{N}$ by uP we denote the polytope obtained by expanding P by a factor of u in each dimension. Ehrhart [Ehr62] proved the following theorem.

Theorem 3.1.5. *Let P be a convex integer polytope in $\mathbb{R}^v$ and for $0 < t \in \mathbb{N}$ let $f(t)$ denote the number of points contained in $\mathbb{Z}^v \cap tP$. Then f is a polynomial in t and* $\deg(f) \leq v$.

In [Sta97] the following theorem is proven.

Theorem 3.1.6. *Let $\tilde{P}$ be a convex rational polytope in $\mathbb{R}^v$ and for $0 < t \in \mathbb{N}$ let $\tilde{f}(t)$ denote the number of points contained in $\mathbb{Z}^v \cap P$. Then $\tilde{f}$ is a quasipolynomial in t.*

Definition 3.1.7. *The polynomial f, and quasipolynomial $\tilde{f}$ whose existence is established in Theorem 3.1.5, and Theorem 3.1.6 is called* Ehrhart polynomial of *P, and* Ehrhart quasipolynomial of *$\tilde{P}$, respectively.*

We will make use of the following lemma.

Lemma 3.1.8. *Let K be a difference-skew-differential field with basic sets $\Delta = \{\delta_1, \ldots, \delta_m\}$ and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ of skew-derivations and automorphisms, respectively. Let Ξ be an orthant decomposition of $[\Delta, \Sigma^*]$, $E = \{e_1, \ldots, e_q\}$ a finite set of generators of a free difference-skew-differential module and $G \subseteq K[\Delta, \Sigma^*]E$ finite. Furthermore let $\alpha \in \mathbb{Q}_+^{m+r}$ be a weight vector and $\prec$ a generalized term order respecting α. For $k \in \mathbb{N}$ let $U_k \subseteq [\Delta, \Sigma^*]E$ be given by*

$$U_k := \{\lambda \in [\Delta, \Sigma^*]E \mid \operatorname{ord}_{\alpha,\Xi}(\lambda) \leq k,\ \nexists_{\mu \in [\Delta,\Sigma^*], g \in G} \operatorname{lt}_{\prec}(\mu g) = \lambda\}.$$

Then there exists a quasipolynomial f such that for all $k \in \mathbb{N}$ sufficiently large we have $f(k) = |U_k|$.

Proof. Let Ξ be the orthant decomposition $\{\Xi_l \mid 1 \leq l \leq p\}$ of $[\Delta, \Sigma^*]$. By Exercise 10.2) of [BK94] the intersection of two finitely generated subsemigroups of a commutative semigroup is finitely generated. It follows that for any $\emptyset \neq S \subseteq \{1, \ldots, p\}$ the intersection $\Xi_S := \bigcap_{s \in S} \Xi_s$ is finitely generated, say by elements $\xi_{S,1}, \ldots, \xi_{S,v_S}$ for some $v_S \in \mathbb{N}$ (note that $m \leq v_S$ for all $\emptyset \neq S \subseteq \{1, \ldots, p\}$). Assume that $\operatorname{ord}_{\alpha,\Xi}(\xi_{S,1}) \leq \ldots \leq \operatorname{ord}_{\alpha,\Xi}(\xi_{S,v_S})$. Naturally, because $\xi_{S,1}, \ldots, \xi_{S,v_S}$ are a generating set for Ξ_S but not necessarily a basis, the elements $\xi_{S,1}, \ldots, \xi_{S,v_S}$ satisfy relations of the form

$$\xi_{S,1}^{k_1} \cdots \xi_{S,v_S}^{k_{v_S}} - \xi_{S,1}^{l_1} \cdots \xi_{S,v_S}^{l_{v_S}} = 0.$$

Let $X_S := \{x_1, \ldots, x_{v_s}\}$ and consider the polynomial ring $K[X_S]$, let $M = \{m_1, \ldots, m_q\}$ be generators of a free $K[X_S]$-module and let $\pi : K[x_1, \ldots, x_{v_s}]M \to K[\xi_{S,1}, \ldots, \xi_{S,v_S}]E$ be the natural epimorphism $\forall_{i \in \{1,\ldots,v_S\}, j \in \{1,\ldots,q\}} x_i m_j \mapsto \xi_{S,i} e_j$. Let $\tilde{G}_S$ be a Gröbner basis of $\ker(\pi)$ with respect to an admissible order $\prec_S$ satisfying for all $\lambda = x_1^{t_1} \cdots x_{v_S}^{t_{v_S}}, \mu = x_1^{l_1} \cdots x_{v_S}^{l_{v_S}}, j \in \{1, \ldots, q\}$ the condition

$$\sum_{i=1}^{v_S} t_i \operatorname{ord}_{\alpha,\Xi}(\xi_{S,i}) < \sum_{i=1}^{v_s} l_i \operatorname{ord}_{\alpha,\Xi}(\xi_{S,i}) \Longrightarrow \lambda m_j \prec_S \mu m_j,$$

and M_S the set of all elements of $[X_S]M$ which are irreducible modulo $\tilde{G}_S$. For any $\lambda \in [X_S]M$ by $\operatorname{NF}_{\prec_S}(\lambda)$ we denote the normal form of λ under reduction modulo $\tilde{G}_S$ with respect to $\prec_S$. Then there exists a natural isomorpism $\phi_S : \Xi_S E \to M_S$ given by $\phi_S : \Xi_S E \ni \xi_{S,1}^{t_1} \cdots \xi_{S,v_S}^{t_{v_S}} e_j \mapsto \operatorname{NF}_{\prec_S}(x_1^{t_1} \cdots x_{v_S}^{t_{v_s}} m_j) \in M_S$.

The ring $K[X_S]$ is Noetherian which impies that the (left) $K[X_S]$-submodule $M_S \subseteq K[X_S]M$ generated by the set

$$\left\{ x_1^{t_1} \cdots x_{v_S}^{t_{v_S}} m_j \;\middle|\; \exists_{\mu \in [\Delta,\Sigma^*], g \in G} \xi_{S,1}^{t_1} \cdots \xi_{S,v_S}^{t_{v_S}} e_j = \operatorname{lt}_{\prec}(\mu g) \in K\Xi_S E \right\}$$

is finitely generated. Let G_S be a Gröbner basis of $\ker(\pi) \cup M_S$ with respect to $\prec_S$. Note that $\operatorname{ord}_{\alpha,\Xi}(\xi_{S,i}) \in \mathbb{Q}_+$ for all $\emptyset \neq S \subseteq \{1, \ldots, p\}, i \in \{1, \ldots, v_S\}$. Now for any $\mu = x_1^{t_1} \cdots x_{v_S}^{t_{v_S}} \in [X_S]$ let $m_{\mu,1}, m_{\mu,2} \in \mathbb{N}$ with $\gcd(m_{\mu,1}, m_{\mu,2}) = 1$ and $\operatorname{ord}_{\alpha,\Xi}(\xi_{S,1}^{t_1} \cdots \xi_{S,v_S}^{t_{v_S}}) \equiv m_{\mu,1}/m_{\mu,2} \mod 1$. For $k \in \mathbb{N}$ let $P_{\mu,S}(k)$ denote the rational $\mathbb{R}^{v_S}$-polytope with vertices $(t_1, \ldots, t_{v_S}), (t_1 + (\frac{m_{\mu,2}-m_{\mu,1}}{m_{\mu,2}} + k) \operatorname{ord}_{\alpha,\Xi}(\xi_{S,1})^{-1}, t_2, \ldots, t_{v_S}), \ldots, (t_1, \ldots, t_{v_S-1}, k_{v_S} + (\frac{m_{\mu,2}-m_{\mu,1}}{m_{\mu,2}} + k) \operatorname{ord}_{\alpha,\Xi}(\xi_{S,v_S})^{-1})$. Let $\tilde{P}_{\mu,S}$ be the rational $\mathbb{R}^{v_S}$ polytope with vertices $(0, \ldots, 0), ((m_{\mu,2} \operatorname{ord}_{\alpha,\Xi}(\xi_{S,1}))^{-1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, (m_{\mu,2} \operatorname{ord}_{\alpha,\Xi}(\xi_{S,v_S}))^{-1})$. Then $\tilde{P}_{\mu,S}$ satisfies

$$(m_{\mu,2} - m_{\mu,1} + k m_{\mu,2}) \tilde{P}_{\mu,S} = P_{\mu,S}(k) - (t_1, \ldots, t_{v_S}). \tag{3.1}$$

By Theorem 3.1.6 there exists a quasipolynomial $\tilde{f}_{\mu,S}$ such that for all $0 < k \in \mathbb{N}$ we have

$$\tilde{f}_{\mu,S}(k) = |k\tilde{P}_{\mu,S} \cap \mathbb{Z}^{v_S}|,$$

and by Remark 3.1.3 it follows from (3.1) that there exists a quasipolynomial $f_{\mu,S}$ such that for all $k \in \mathbb{N}$ with $k > \mathrm{ord}_{\alpha,\Xi}(\mu)$ we have

$$f_{\mu,S}(k) = |P_{\mu,S}(\lfloor k - \mathrm{ord}_{\alpha,\Xi}(\mu) \rfloor) \cap \mathbb{Z}^{v_S}|.$$

For ease of notation we use the convention $f_{0,S} := 0$. Since G_S is finite there exist $w_S \in \mathbb{N}, g_{S,1}, \ldots, g_{S,w_S} \in K[X_S]M$ such that we can write $G_S = \{g_{S,1}, \ldots, g_{S,w_S}\}$. For $1 \leq w \leq w_S$ and $1 \leq v_1 < \cdots < v_w \leq w_S$ by $u_{v_1,\ldots,v_w}$ we denote the least common multiple of $\{\mathrm{lt}_{\prec_S}(g_{S,v_i}) \mid i = 1, \ldots, w\}$. By the Principle of Inclusion-Exclusion [Sta97, Ch. 2.1] we obtain that there exists a quasipolynomial f_S given by

$$f_S(k) = \sum_{j=1}^{q} f_{e_j,S}(k) + \sum_{w=1}^{w_S} \sum_{1 \leq v_1 < \cdots < v_w \leq w_S} (-1)^w f_{u_{v_1,\ldots,v_w},S}(k),$$

such that for all $k \in \mathbb{N}$ sufficiently large we have

$$f_S(k) = \left| P_{1,S}(k) \setminus \bigcup_{i \in \{1,\ldots,w_S\}} P_{\mathrm{lt}_{\prec_S}(g_{S,i}),S}(k) \right|.$$

On the other hand we have

$$\begin{aligned} & P_{1,S}(k) \setminus \bigcup_{i \in \{1,\ldots,w_S\}} P_{\mathrm{lt}_{\prec_S}(g_{S,i}),S}(k) \\ = \quad & \phi\left(\{\lambda \in \Xi_S E \mid \mathrm{ord}_{\alpha,\Xi}(\lambda) \leq k, \nexists_{\mu \in [\Delta,\Sigma^*], g \in G} \mathrm{lt}_{\prec}(\mu g) = \lambda\}\right). \end{aligned}$$

Since ϕ_S is an isomorphism we obtain for all $k \in \mathbb{N}$ sufficiently large

$$f_S(k) = \left|\{\lambda \in \Xi_S E \mid \mathrm{ord}_{\alpha,\Xi}(\lambda) \leq k, \nexists_{\mu \in [\Delta,\Sigma^*], g \in G} \mathrm{lt}_{\prec}(\mu g) = \lambda\}\right|.$$

Therefore there exists a quasipolynomial f given by

$$f(k) = \sum_{\emptyset \neq S \subseteq \{1,\ldots,p\}} (-1)^{|S|-1} f_S(t),$$

satisfying for all $k \in \mathbb{N}$ sufficiently large

$$f(k) = \left|\{\lambda \in K[\Delta,\Sigma^*]E \mid \mathrm{ord}_{\alpha,\Xi}(\lambda) \leq k, \nexists_{\mu \in [\Delta,\Sigma^*], g \in G} \mathrm{lt}_{\prec}(\mu g) = \lambda\}\right|.$$

□

From Theorem 3.1.4 and Lemma 3.1.8 we obtain the following corollary extending [KLMP99, Thm. 6.7.3.] to the described setting.

Corollary 3.1.9. *Let K be a difference-skew-differential field with basic sets $\Delta = \{\delta_1, \ldots, \delta_m\}$ and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ of derivations and automorphisms, respectively. Let Ξ be an orthant decomposition, α a weight vector, and $\mathcal{M}$ a finitely generated $K[\Delta,\Sigma^*]$-module with excellent weighted filtration $(\mathcal{M}_{\alpha,\Xi,k})_{k\in\mathbb{Z}}$. Then there exists a quasipolynomial $\psi_{\alpha,\Xi}$ satisfying for all sufficiently large $k \in \mathbb{N}$ the equation*

$$\psi_{\alpha,\Xi}(k) = \dim_K(\mathcal{M}_{\alpha,\Xi,k}).$$

Definition 3.1.10. *The quasipolynomial $\psi_{\alpha,\Xi}$ whose existence has been established in Corollary 3.1.9 is called* difference-skew-differential dimension quasipolynomial *associated with the excellent weighted filtration $(\mathcal{M}_{\alpha,\Xi,k})_{k\in\mathbb{Z}}$ or* α-Ξ-difference-skew-differential dimension quasipolynomial *(or* α-dimension quasipolynomial *if Ξ is clear from the context) associated with $\mathcal{M}$.*

Definition 3.1.11. *If Ξ is the canonical orthant decomposition and $\alpha = (1, \ldots, 1) \in \mathbb{Q}_+^{m+2^n}$ then the degree of $\psi_{\alpha,\Xi}$ is called the* dimension *of the difference-skew-differential module $\mathcal{M}$.*

3.1.3 Weighted differential dimension polynomials of differential field extensions

It has been pointed out to the author by Alexander Levin that Corollary 3.1.9 allows us to generalize Kolchin's result on differential dimension polynomials [Kol73, Chapter II, Theorem 6]. In order to express his (and also our) result we extend the notion of filtration of a differential field extension (cf. [KLMP99, Definition 5.2.1.]) to our setting introducing weights.

Definition 3.1.12. *Let G be a differential field with basic set $\Delta = \{\delta_1, \ldots, \delta_m\}$ of derivations, F a Δ-subfield of G, and $\alpha \in \mathbb{Q}_+^m$ a weight vector. An α-*filtration *or* weighted filtration *of G over F is an ascending sequence $(G_k)_{k\in\mathbb{Z}}$ of (nondifferential) subfields of G such that*

(i) for all $k < 0$ we have $G_k = F$,

(ii) for all $\lambda \in [\Delta], g \in G_k$ with $\mathrm{ord}_\alpha(\lambda) \leq 1$ we have $\lambda g \in G_{k+1}$, and

(iii) $\bigcup_{k\in\mathbb{Z}} G_k = G$.

If in addition

(iv) for all $k \in \mathbb{Z}$ the field G_k is finitely generated over F, and

(v) there exist $0 < s \in \mathbb{N}$ and $k_0 \in \mathbb{Z}$ such that for all $k_0 < k \in \mathbb{Z}$ we have

$$G_k = G_{k_0+(k-k_0 \mod s)}\left(K[\Delta]_{\left\lfloor \frac{k-k_0}{s} \right\rfloor} G_{k_0+(k-k_0 \mod s)}\right),$$

then the weighted filtration $(G_k)_{k\in\mathbb{Z}}$ is called excellent.

Consider an excellent α-filtration $(G_k)_{k\in\mathbb{Z}}$ of a Δ-extension G of F. For $k \in \mathbb{Z}$ let Ω_k denote the G-subspace of the module of differentials $\Omega_{G/F}$ generated by $\{dg \mid g \in G_k\}$ (cf. Subsection 2.1.4 for the definition of dg). Then $(\Omega_k)_{k\in\mathbb{Z}}$ is an excellent α-filtration of $\Omega_{G/F}$.

Recall that for a field extension G of a field F the *transcendence degree* of G over F is defined as the maximal number of elements of G which are algebraically independent over F and is denoted by $\mathrm{trdeg}_F\, G$.

Theorem 3.1.13. *Let F be a differential field with basic set $\Delta = \{\delta_1, \ldots, \delta_m\}$, $G = F(g_1, \ldots, g_q)$ a finitely generated Δ-extension of F, and $\alpha \in \mathbb{Q}_+^m$ a weight vector. Let the filtration $(G_k)_{k\in}$ be given by*

$$G_k = F(\{\lambda g_i \mid \mathrm{ord}_\alpha(\lambda) \leq k, i = 1, \ldots, q\}).$$

Then there exists a quasipolynomial χ such that for all $k \in \mathbb{N}$ sufficiently large we have $\chi(k) = \mathrm{trdeg}_F\, G_k$ and χ coincides with the α-dimension quasipolynomial of $\Omega_{G/F}$.

Proof. By Corollary 3.1.9 there exists a quasipolynomial χ such that for all $k \in \mathbb{N}$ sufficiently large we have $\chi(k) = \dim_G(\Omega_k)$.

If $(\eta_a)_{a\in A_k}$ is a transcendence basis of G_k over F for some index set A_k, then $\{d\eta_a \mid a \in A_k\}$ generates the G-space Ω_k. For any $a \in A_k$ there exists a derivation $D_a \in \mathrm{Der}_F\, G_k$ such that

(i) for all $a \neq b \in A_k$ we have $D_a(\eta_b) = 0$, and

(ii) $D_a(\eta_a) = 1$.

By Proposition 2.1.6 the derivation D_a can be extended uniquely to a derivation $\bar{D}_a$ on G. Assume there exists $(\lambda_a)_{a\in A_k} \subseteq G$ with

$$\sum_{a\in A_k} \lambda_a d\eta_a = 0.$$

Then

$$\begin{aligned} 0 &= \sum_{a\in A_k} \lambda_a d\eta_a(\bar{D}_b) \\ &= \sum_{a\in A_k} \lambda_a \bar{D}_b(\eta_a) \\ &= \lambda_b \end{aligned}$$

for all $b \in A_k$. Hence, the system $\{d\eta_a \mid a \in A_k\}$ is G-linearly independent. Therefore we obtain $\operatorname{trdeg}_F(G_k) = \dim_G(\Omega_k)$ which proves the claim. □

Definition 3.1.14. *The quasipolynomial χ whose existence is proved in Theorem 3.1.13 is called* weighted (or α-) differential dimension quasipolynomial *of the Δ-field extension G over F.*

3.1.4 Weighted multi-filtrations

We can generalize even further by taking into account several weight vectors simultaneously.

Recall that an *l-filtered ring* or *multi-filtered ring* is a ring R together with a family $(R_k)_{k\in\mathbb{Z}^l}$ of additive subgroups of R such that

(i) $1 \in R_{(0,\ldots,0)}$,

(ii) for all $k \leq_P \tilde{k}$ we have $R_k \subseteq R_{\tilde{k}}$, where $\leq_P$ denotes the product-order on $\mathbb{Z}^l$, and

(iii) for all $k, \tilde{k} \in \mathbb{Z}^l$ we have $R_k R_{\tilde{k}} \subseteq R_{k+\tilde{k}}$.

The family $(R_k)_{k\in\mathbb{Z}^l}$ is called (ascending) *l-filtration* or *multi-filtration* of R and its elements are called *components* of the l-filtration.

We introduce the notion of weighted multi-filtrations as follows.

Definition 3.1.15. *Let Ξ be an orthant decomposition of $[\Delta,\Sigma^*]$ with generators $\xi_1,\ldots,\xi_r$, and $\mathcal{T} \in \mathbb{Q}_0^{(m+r)\times t}$ a weight matrix. A left $K[\Delta,\Sigma^*]$-module $\mathcal{M}$ is called a (left)* weighted (or $\mathcal{T}$-) filtered $K[\Delta,\Sigma^*]$-module *or* multi-filtered $K[\Delta,\Sigma^*]$-module *if there exists a family $(\mathcal{M}_k)_{k\in\mathbb{Z}^t}$ of additive subgroups of $\mathcal{M}$ such that for all $k \leq_P \tilde{k}$ we have $\mathcal{M}_k \subseteq \mathcal{M}_{\tilde{k}}$, and for all $k = (k_1,\ldots,k_t), \tilde{k} \in \mathbb{Z}^t$ we have $\{f \in K[\Delta,\Sigma^*] \mid \forall_{1\leq i\leq t} \operatorname{ord}_{\mathcal{T}^{(i)},\Xi}(f) \leq k_i\}\mathcal{M}_{\tilde{k}} \subseteq \mathcal{M}_{k+\tilde{k}}$. The family $(\mathcal{M}_k)_{k\in\mathbb{Z}^t}$ is called* weighted (or $\mathcal{T}$-) filtration *or* multi-filtration *of $\mathcal{M}$ and its elements are called* components *of the weighted filtration. If*

(i) for every $k \in \mathbb{Z}^t$ the component $\mathcal{M}_k$ is finitely generated as a K-vector space, and

(ii) there exist $0 < s = (s_1,\ldots,s_t) \in \mathbb{N}^t$ and $\tilde{k} = (\tilde{k}_1,\ldots,\tilde{k}_t) \in \mathbb{Z}^t$ such that for all $\tilde{k} \leq_P k = (k_1,\ldots,k_t) \in \mathbb{Z}^t$ we have

$$\begin{aligned} \mathcal{M}_k &= \left\{ f \in K[\Delta,\Sigma^*] \;\middle|\; \forall_{1\leq i\leq t} \operatorname{ord}_{\mathcal{T}^{(i)},\Xi}(f) \leq \left\lfloor \frac{k_i - \tilde{k}_i}{s_i} \right\rfloor s_i \right\} \\ &\quad \mathcal{M}_{\left(\tilde{k}_i + \left(k_i - \tilde{k}_i \mod s_i\right)\right)_{i=1}^t}, \end{aligned}$$

then the weighted filtration $(\mathcal{M}_k)_{k\in\mathbb{Z}^l}$ is called excellent *and s is called the* period *of $(\mathcal{M}_k)_{k\in\mathbb{Z}^l}$.*

Example 3.1.16. *Let Ξ be an orthant decomposition of $[\Delta,\Sigma^*]$ generated by $\xi_1,\ldots,\xi_r$, and $\mathcal{T} \in \mathbb{Q}_0^{(m+r)\times t}$ a weight matrix with columns $\mathcal{T}^{(1)},\ldots,\mathcal{T}^{(t)}$ satisfying $a\mathcal{T} \neq 0$ for all $a \in \mathbb{N}^{m+r} \setminus \{0\}$. For any finitely generated difference-skew-differential module $\mathcal{M}$ with generators $m_1,\ldots,m_q$, and $k = (k_1,\ldots,k_t) \in \mathbb{Z}^t$ define the $\mathcal{T}$-Ξ-filtration $(\mathcal{M}_{\mathcal{T},\Xi,k})_{k\in\mathbb{Z}^t}$ of $\mathcal{M}$ by*

$$\mathcal{M}_{\mathcal{T},\Xi,k} := \{\lambda m_j \mid 1 \leq j \leq q, \forall_{1\leq i\leq t} \operatorname{ord}_{\mathcal{T}^{(i)},\Xi^{(i)}}(\lambda) \leq k_i\}.$$

If $\mathcal{T}$ and Ξ are clear from the context we will write $\mathcal{M}_k$ instead of $\mathcal{M}_{\mathcal{T},\Xi,k}$. For $k = (k_1, \ldots, k_t)$ define the t-filtration $(K[\Delta,\Sigma^*]_{\mathcal{T},\Xi,k})_{k\in\mathbb{Z}^t}$ by

$$K[\Delta,\Sigma^*]_k := \{f \in K[\Delta,\Sigma^*] \mid \forall_{i=1,\ldots,t} \operatorname{ord}_{\mathcal{T}^{(i)},\Xi^{(i)}}(f) \le k_i\}.$$

Then for any $\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_t) \in \mathbb{Z}^t, q_1 \in \{0, \ldots, \operatorname{lcm}(\mathcal{T}^{(1)}) - 1\}, \ldots, q_t \in \{0, \ldots, \operatorname{lcm}(\mathcal{T}^{(t)}) - 1\}, q = (q_1, \ldots, q_t)$ satisfying for all $i = 1, \ldots t$ the condition

$$\tilde{q}_i = q_i \mod \operatorname{lcm}(\mathcal{T}^{(i)}),$$

we have

$$\mathcal{M}_{\mathcal{T},\Xi,\tilde{q}} = K[\Delta,\Sigma^*]_{\tilde{q}-q}\mathcal{M}_{\mathcal{T},\Xi,q},$$

i.e., the weighted filtration $(M_{\mathcal{T},\Xi,k})_{k\in\mathbb{Z}^t}$ is excellent.

3.1.5 Multivariate difference-skew-differential dimension polynomials

With the notation of the previous subsection let

$$\begin{array}{rcl} \bar{\psi}_{\alpha,\Xi} : \mathbb{Z}^t & \to & \mathbb{N} \\ k & \mapsto & \dim_K(\mathcal{M}_{\mathcal{T},\Xi,k}). \end{array}$$

If $\mathcal{T}$ and Ξ are clear from the context we will write $\bar{\psi}$ instead of $\bar{\psi}_{\alpha,\Xi}$. We are interested in the general form of $\bar{\psi}$.

In Theorem 3.1.4 we established the existence of weighted difference-skew-differential dimension quasipolynomials. As a generalization of Theorem 3.1.4, [Lev07a, Theorem 4.1], [ZW08a, Theorem 4.1], and [Lev08, Theorem 3.3.16] to the case where we take into account several weight vectors in the form of a weight matrix we obtain the following theorem.

Theorem 3.1.17. *Let $\mathcal{M}$ be a finitely generated difference-skew-differential module with generators $m_1, \ldots, m_q$, Ξ an orthant decomposition of $[\Delta,\Sigma^*]$ with generators $\xi_1, \ldots, \xi_r$, $E = \{e_1, \ldots, e_q\}$ a finite set of generators of a free difference-skew-differential module, and $\mathcal{T} \in \mathbb{Q}_0^{(m+r)\times t}$ a weight matrix such that there exist generalized term orders $\prec_1, \ldots, \prec_t$ on $[\Delta,\Sigma^*]E$ respecting $\mathcal{T}$. By $\pi : K[\Delta,\Sigma^*]E \to \mathcal{M}$ we denote the difference-skew-differential epimorphism given by $\pi(e_i) = m_i$ for $i = 1, \ldots, q$. Let G be a $\prec_1$-Gröbner basis of the $K[\Delta,\Sigma^*]E$-submodule $\ker(\pi)$ respecting $\mathcal{T}$ and for all $k = (k_1, \ldots, k_t) \in \mathbb{N}^t$ define $U_k := U'_k \cup U''_k$, where*

$$\begin{array}{rcl} U'_k & := & \{\lambda \in [\Delta,\Sigma^*]E \mid \forall_{j=1,\ldots,t} \operatorname{ord}_{\mathcal{T}^{(j)},\Xi}(\lambda) \le k_j, \text{ and } \lambda \ne \operatorname{lt}_{\prec_1}(\mu g) \\ & & \text{for all } \mu \in [\Delta,\Sigma^*], g \in G\}, \\ U''_k & := & \{\lambda \in [\Delta,\Sigma^*]E \mid \forall_{j=1,\ldots,t} \operatorname{ord}_{\mathcal{T}^{(j)},\Xi}(\lambda) \le k_j, \text{ and } \forall_{\mu\in[\Delta,\Sigma^*],g\in G} \lambda = \operatorname{lt}_{\prec_1}(\mu g) \\ & & \Longrightarrow \exists_{j\in\{2,\ldots,t\}} \operatorname{ord}_{\mathcal{T}^{(j)},\Xi}(\operatorname{lt}_{\prec_j}(\mu g)) > k_j\}. \end{array}$$

Then for all $k \in \mathbb{N}^k$ we have $|U_k| = \dim_K(\mathcal{M}_{\mathcal{T},\Xi,k})$.

Proof. For some $k \in \mathbb{N}^t$ let $\lambda m_i \in \mathcal{M}_{\mathcal{T},\Xi,k} \setminus \pi(U_k)$. Then $\lambda e_i \notin U_k$. Hence, there exist $\mu \in [\Delta,\Sigma^*], g \in G$ such that $\lambda e_i = \operatorname{lt}_{\prec_1}(\mu g)$ and $\operatorname{ord}_{\mathcal{T}^{(j)},\Xi}(\operatorname{lt}_{\prec_j}(\mu g)) \le k_j$ for all $j = 2, \ldots, t$. Therefore there exist $V \in \mathbb{N}, a, a_1, \ldots, a_V \in K \setminus \{0\}, \lambda, \lambda_1, \ldots, \lambda_V \in [\Delta,\Sigma^*], w_1, \ldots, w_V \in \{1, \ldots, q\}$ such that we can write

$$\mu g = a\lambda e_i + \sum_{v=1}^{V} a_v \lambda_v e_{w_v}.$$

Then for all $v = 1, \ldots, V$ we have $\lambda_v e_{w_v} \prec_1 \lambda e_i = \operatorname{lt}_{\prec_1}(\mu g)$ which implies $\operatorname{ord}_{\mathcal{T}^{(1)},\Xi}(\lambda_v) \le k_1$. Furthermore for all $j = 2, \ldots, t$ we have $\operatorname{ord}_{\mathcal{T}^{(j)},\Xi}(\operatorname{lt}_{\prec_j}(\mu g)) \le k_j$ and $\lambda_v e_{w_v} \prec_j \operatorname{lt}_{\prec_j}(\mu g)$ which imply $\mathrm{ord}_{\mathcal{T}^{(j)},\Xi}(\lambda_v) \leq k_j$. Since $G \subseteq \ker(\pi)$ we have $\pi(g) = 0$ and

$$0 = \mu\pi(g) = \pi(\mu g) = a\pi(\lambda e_i) + \sum_{v=1}^{V} a_v \pi(\lambda_v e_{w_v}) = a\lambda m_i + \sum_{v=1}^{V} a_v \lambda_v m_{w_v}.$$

Hence, λm_i is a K-linear combination of elements $\lambda_v m_{w_v}$ such that for all $j \in \{1, \dots, t\}$ we have $\mathrm{ord}_{\mathcal{T}^{(j)},\Xi}(\lambda_v) \leq k_j$ and $\lambda_v e_{w_v} \prec_1 \lambda e_i$. By induction with respect to $\prec_1$ we obtain that there exist $V' \in \mathbb{N}, b_1, \dots, b_{V'} \in K \setminus \{0\}, \mu_1, \dots, \mu_{V'} \in [\Delta, \Sigma^*], c_1, \dots, c_{V'} \in \{1, \dots, q\}$ such that we can write

$$\lambda m_i = \sum_{v=1}^{V'} b_v \mu_v e_{c_v},$$

such that for all $j = 1, \dots, t; v = 1, \dots, V'$ we have $\mathrm{ord}_{\mathcal{T}^{(j)},\Xi}(\mu_v) \leq k_j$ and $\mu_v e_{c_v} \in \pi(U_k)$.

Suppose that there exist $u_1, \dots, u_l \in U_k, a_1, \dots, a_l \in K$ with $a_1\pi(u_1) + \dots + a_l\pi(u_l) = 0$. Then $h = \sum_{i=1}^{l} a_i u_i \in \ker(\pi)$. Since for all $i = 1, \dots, l$ we have

$$u_i \notin \bigcup_{g \in G} \{\mathrm{lt}_{\prec_1}(\lambda g) \mid \lambda \in [\Delta, \Sigma^*], \forall_{j=1,\dots,t}\, \mathrm{ord}_{\mathcal{T}^{(j)},\Xi}(\mathrm{lt}_{\prec_j}(\lambda g)) \leq k_j\}$$

it is clear that for all $i = 1, \dots, l$ either there exist no $\lambda \in [\Delta, \Sigma^*], g \in G$ with $\mathrm{lt}_{\prec_1}(\lambda g) = u_i$ or for every $\lambda \in [\Delta, \Sigma^*], g \in G$ with $\mathrm{lt}_{\prec_1}(\lambda g) = u_i$ there exists $j \in \{2, \dots, t\}$ with $\mathrm{ord}_{\mathcal{T}^{(j)},\Xi}(\mathrm{lt}_{\prec_j}(\lambda g)) > k_j$. Hence,

$$\mathrm{lt}_{\prec_1}(h) \notin \bigcup_{g \in G} \{\mathrm{lt}_{\prec_1}(\lambda g) \mid \lambda \in [\Delta, \Sigma^*], \forall_{j=1,\dots,t}\, \mathrm{ord}_{\mathcal{T}^{(j)},\Xi}(\mathrm{lt}_{\prec_j}(\lambda g)) \leq \mathrm{ord}_{\mathcal{T}^{(j)},\Xi}(\mathrm{lt}_{\prec_j}(h))\}.$$

Therefore, h is $\prec_1$-irreducible modulo G respecting $\mathcal{T}$. By Proposition 2.3.11 we conclude $h = 0$ and $a_1, \dots, a_l = 0$. So $\pi(U_k)$ is K-linearly independent, i.e., $\pi(U_k)$ is a basis of $\mathcal{M}_{\mathcal{T},\Xi,k}$. Since π is a bijection from $U_k \to \pi(U_k)$ we obtain for all $k \in \mathbb{N}^t$ the equality

$$\dim_K(\mathcal{M}_{\mathcal{T},\Xi,k}) = |\pi(U_k)| = |U_k|.$$

$\square$

Again for a particular $k \in \mathbb{N}^t$ using Gröbner basis techniques it is possible to compute $\dim_K(\mathcal{M}_{\mathcal{T},\Xi,k})$. For a result similar to Corollary 3.1.9 we have to try a slightly more elaborate approach than in the univariate case.

The following generalization of the concept of Ehrhart polynomials is due to Clauss, Loechner and Wilde [CL96, CLW97]. A good reference is also [LLS08, Section 2.2].

Definition 3.1.18. *Let $t, v, w \in \mathbb{N} \setminus \{0\}, A \in \mathbb{Z}^{v \times t}, C \in \mathbb{Z}^{w \times t}, b \in \mathbb{Z}^t$, and $p = (p_1, \dots, p_w)$ a vector containing w parameters $p_1, \dots, p_w$. Then*

$$P_p := \{x \in \mathbb{R}^v \mid xA \leq_P pC + b\}$$

is called rational v-dimensional parametrized polyhedron. *If $P - p$ is bounded for each value of p, it will be called a* parametric polytope.

Definition 3.1.19. *Let $p = (p_1, \dots, p_w)$ be a vector containing w parameters $p_1, \dots, p_w$ and let $f : \mathbb{Z}^w \to \mathbb{Q}$ such that there exists $q = (q_1, \dots, q_w) \in \mathbb{N}^w$ with $f(p) = f(p')$ whenver for all $i = 1, \dots, w$ we have $p_i = p'_i \mod q_i$. Then f is called* w-dimensional periodic number *or* multidimensional periodic number on $p_1, \dots, p_w$ *with period* $\mathrm{lcm}(q_1, \dots, q_w)$.

Definition 3.1.20. *Let g be a polynomial in w variables $p_1, \dots, p_w$ such that each coefficient is a multidimensional periodic number on a subset of $\{p_1, \dots, p_w\}$. Then g is called* multivariate quasipolynomial. *The* period *of g is defined to be the least common multiple of the periods of its coefficients.*

Theorem 3.1.21. (Clauss) *Let $t, v, w \in \mathbb{N} \setminus \{0\}, A \in \mathbb{Z}^{v \times t}, C \in \mathbb{Z}^{w \times t}, b \in \mathbb{Z}^t$, $p = (p_1, \dots, p_w)$ a vector containing w parameters $p_1, \dots, p_w$ and P_p a v-dimensional parametric polytope. For $k \in \mathbb{N}^t$ let $f : \mathbb{N}^t \to \mathbb{N}$ be given by $k \mapsto |P_k|$. Then f can be represented by a finite set of multivariate quasipolynomials of degree d, each valid on a different validity domain.*

Using Theorem 3.1.21 we obtain the following result.

Theorem 3.1.22. *Let $E = \{e_1, \dots, e_q\}$ be a finite set of generators of a free skew-differential module, $\mathcal{T} \in \mathbb{Q}_0^{m \times t}$ a weight matrix with columns $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(t)}$ satisfying $|\{a \in \mathbb{N}^t \mid \forall_{j=1,\dots,t} a\mathcal{T}^{(j)} \leq p_j\}| < \infty$ for all $p = (p_1, \dots, p_t) \in \mathbb{N}^t$, and $\prec_1, \dots, \prec_t$ admissible orders on $[\Delta]$ respecting $\mathcal{T}$. Let $G \subseteq K[\Delta]E$ be finite and for all $k \in \mathbb{N}^t$ define $U_k := U'_k \cup U''_k$, where*

$$\begin{aligned} U'_k &:= \{\lambda \in [\Delta]E \mid \forall_{j=1,\dots,t} \operatorname{ord}_{\mathcal{T}^{(j)}}(\lambda) \leq k_j, \text{ and } \lambda \neq \operatorname{lt}_{\prec_1}(\mu g) \\ &\qquad \text{for all } \mu \in [\Delta], g \in G\}, \\ U''_k &:= \{\lambda \in [\Delta]E \mid \forall_{j=1,\dots,t} \operatorname{ord}_{\mathcal{T}^{(j)}}(\lambda) \leq k_j, \text{ and } \forall_{\mu \in [\Delta], g \in G} \lambda = \operatorname{lt}_{\prec_1}(\mu g) \\ &\qquad \Longrightarrow \exists_{j \in \{2,\dots,t\}} \operatorname{ord}_{\mathcal{T}^{(j)}}(\operatorname{lt}_{\prec_j}(\mu g)) > k_j\}. \end{aligned}$$

Then the function $f : \mathbb{N}^t \to \mathbb{N}$ given by $k \mapsto |U_k|$ can be represented by a finite set of multivariate quasipolynomials in k, each valid on a different validity domain.

Proof. Obviously, for all $k \in \mathbb{N}^t$ we have

$$\begin{aligned} U_k &= \{\lambda \in [\Delta]E \mid \forall_{j=1,\dots,t} \operatorname{ord}_{\mathcal{T}^{(j)}}(\lambda) \leq k_j\} \\ &= \setminus \{\lambda \in [\Delta]E \mid \exists_{\mu \in [\Delta], g \in G} \operatorname{lt}_{\prec_1}(\mu g) = \lambda, \\ &\qquad \forall_{j=1,\dots,t} \operatorname{ord}_{\mathcal{T}^{(j)}}(\mu g) \leq k_j, \operatorname{ord}_{\mathcal{T}^{(j)}}(\lambda) \leq k_j\}. \end{aligned}$$

By t_1 we denote the least common multiple of the denominators of all nonzero entries of $\mathcal{T}$. For $k \in \mathbb{N}^t$ by $P_{1,\mu,k}$ we denote the m-dimensional parametric polytope given by

$$P_{1,\mu,k} = \{x \in \mathbb{R}^m \mid x\mathcal{T}t_1 \leq_P t_1 k, l \leq_P x\}.$$

Then $P_{1,\mu,k} \cap \mathbb{N}^m$ is naturally isomorphic to

$$\{\lambda \in [\Delta]E \mid \lambda \text{ is } \prec_1\text{-reducible modulo } \mu \text{ and } \forall_{j=1,\dots,t} \operatorname{ord}_{\mathcal{T}^{(j)}}(\lambda) \leq k_j\}.$$

Let $f_{1,\mu} : \mathbb{N}^t \to \mathbb{N}$ be given by $k \mapsto |P_{1,\mu,k}|$. By Theorem 3.1.21 the function $f_{1,\mu}$ can be represented by a finite set of multivariate quasipolynomials, each valid on a different validity domain. Then there exists a function $f_1 : \mathbb{N}^t \to \mathbb{N}$ given by

$$f_1(k) = \sum_{j=1}^{q} f_{1,e_j}(k)$$

such that for all $k \in \mathbb{N}^t$ sufficiently large we have

$$f_1(k) = |\{\lambda \in [\Delta]E \mid \forall_{j=1,\dots,t} \operatorname{ord}_{\mathcal{T}^{(j)}}(\lambda) \leq k_j\}|$$

and f_1 can be represented by a finite set of multivariate quasipolynomials, each valid on a different validity domain.

For $k \in \mathbb{N}^t$ and any $h \in K[\Delta]E \setminus \{0\}$ such that for $j = 1, \dots, t$ there exists $h_j \in \mathbb{N}^m$ with $\operatorname{lt}_{\prec_j}(h) = \delta^{h_j}$, where we use multi-index notation, by $P_{2,h,k}$ we denote the m-dimensional parametric polytope given by

$$P_{2,h,k} := \{x \in \mathbb{R}^m \mid x\mathcal{T}t_1 \leq_P t_1 k, x \geq_P h_1, \forall_{j=2,\dots,t} (x - h_1 + h_j)\mathcal{T}t_1 \leq t_1 k_j\}.$$

Then $P_{2,h,k} \cap \mathbb{N}^m$ is naturally isomorphic to

$$\{\lambda \in [\Delta]E \quad | \quad \forall_{j=1,\dots,t} \operatorname{ord}_{\mathcal{T}^{(j)}}(\lambda) \leq k_j \text{ and } \exists_{\eta \in [\Delta]} \\ \operatorname{lt}_{\prec_1}(\eta h) = \lambda, \forall_{j=2,\dots,t} \operatorname{ord}_{\mathcal{T}^{(j)}}(\eta h) \leq k_j\}.$$

Let $f_{2,h} : \mathbb{N}^t \to \mathbb{N}$ be given by $k \mapsto |P_{2,h,k}|$. By convention $f_{2,0} : \mathbb{N}^t \to \mathbb{N}$ is given by $k \mapsto 0$. By Theorem 3.1.21 the function $f_{2,\mu}$ can be represented by a finite set of multivariate quasipolynomials, each valid on a different validity domain. Since G is finite there exist $w \in \mathbb{N}, g_1, \dots, g_w \in K[\Delta]E$ such that $G = \{g_1, \dots, g_w\}$. For $2 \leq m \leq w$ and $1 \leq v_1 < \cdots < v_m \leq w$ let $h_{v_1,\dots,v_m}$ and $g_{v_1,\dots,v_m}$ be given by

$$\begin{aligned} h_{v_1,\dots,v_m} &:= \operatorname{lcm}(\operatorname{lt}_{\prec_1}(g_{v_1}), \dots, \operatorname{lt}_{\prec_1}(g_{v_m})), \text{ and} \\ g_{v_1,\dots,v_m} &:= \begin{cases} h_{v_1,\dots,v_m} + \sum_{j=2}^{t} \sum_{i=1}^{m} \operatorname{lt}_{\prec_j} \left(\frac{h_{v_1,\dots,v_m}}{\operatorname{lt}_{\prec_1}(g_{v_i})} g_{v_i} \right), & \text{if } h_{v_1,\dots,v_m} \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

By the Principle of Inclusion-Exclusion [Sta97, Ch. 2.1] we obtain that there exists a function $f_2 : \mathbb{N}^t \to \mathbb{N}$ given by

$$f_2(k) = \sum_{m=1}^{w} \sum_{1 \leq v_1 < \cdots < v_m \leq w} (-1)^{w-1} f_{2,g_{v_1,\dots,v_m}}(k),$$

such that for all $k \in \mathbb{N}^t$ sufficiently large we have

$$f_2(k) = |\{\lambda \in [\Delta]E \mid \exists_{\mu \in [\Delta], g \in G} \operatorname{lt}_{\prec_1}(\mu g) = \lambda, \\ \forall_{j=1,\dots,t} \operatorname{ord}_{\mathcal{T}^{(j)}}(\mu g) \leq k_j, \operatorname{ord}_{\mathcal{T}^{(j)}}(\lambda) \leq k_j\}|,$$

and f_2 can be represented by a finite set of multivariate quasipolynomials, each valid on a different validity domain. We conclude that there exists a function $f : \mathbb{N}^t \to \mathbb{N}$ given by $f(k) := f_1(k) - f_2(k)$ such that for all $k \in \mathbb{N}^t$ sufficiently large we have

$$f(k) = |U_k|,$$

and f can be represented by a finite set of multivariate quasipolynomials, each valid on a different validity domain. □

Let Ξ be an orthant decomposition of $[\Delta, \Sigma^*]$ with generators $\xi_1, \dots, \xi_r$, E a finite set of free generators of a free $K[\Delta, \Sigma^*]$-module, and $\mathcal{M}$ a $K[\Delta, \Sigma^*]E$-submodule. For $\gamma = (\gamma_1, \dots, \gamma_r), \Gamma = \{\gamma_1, \dots, \gamma_r\}, \xi = (\xi_1, \dots, \xi_r), e \in E, a \in \mathbb{N}^m, b \in \mathbb{N}^r$ define $\phi_{\Xi,\mathcal{M}} : K[\Delta, \Gamma]E$ to $\mathcal{M}$ by $\phi : \delta^a \gamma^b e \mapsto \delta^a \xi^b e$ (using multi-index notation). We consider $K[\Delta, \Gamma]E$ as a skew-differential ring equipped with the commutation relations

(i) $\lambda\mu = \mu\lambda$ for all $\lambda, \mu \in \Delta \cup \Sigma$,

(ii) $\delta_i a = \tau_i(a)\delta_i + \delta_i(r)$ for all $i = 1, \dots, m, a \in K$,

(iii) $\gamma_i a = \xi_i(a)\gamma_i$ for all $i = 1, \dots, r, a \in K$.

Let $\mathcal{N}$ be the $K[\Delta, \Gamma]E$-submodule generated by $\ker(\phi)$. Consider a weight matrix $\mathcal{T} \in \mathbb{Q}_0^{(m+r)\times t}$ with columns $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(t)}$. Then for every $k \in \mathbb{N}^t$ we have

$$\phi_{\Xi,\mathcal{M}}(\mathcal{N}_{\mathcal{T},k}) = \mathcal{M}_{\mathcal{T},\Xi,k}.$$

Hence, we obtain the following corollary.

Corollary 3.1.23. *Let K be a difference-skew-differential field with basic sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ of skew-derivations and automorphisms, respectively. Let Ξ be an orthant decomposition with generators $\xi_1, \dots, \xi_r$, $\mathcal{T}$ a weight matrix satisfying $a\mathcal{T} \neq 0$ for all $a \in \mathbb{N}^{m+r} \setminus \{0\}$, and $\mathcal{M}$ a finitely generated $K[\Delta, \Sigma^*]$-module with excellent weighted filtration $(\mathcal{M}_{\mathcal{T},\Xi,k})_{k\in\mathbb{Z}^t}$. Let $\psi_{\mathcal{T},\Xi} : \mathbb{N}^t \to \mathbb{N}$ satisfy for all sufficiently large $k \in \mathbb{N}^t$ the equation*

$$\psi_{\mathcal{T},\Xi}(k) = \dim_K(\mathcal{M}_{\mathcal{T},\Xi,k}).$$

Then $\psi_{\mathcal{T},\Xi}$ can be represented by a finite set of multivariate quasipolynomials, each valid on a different validity domain.

Definition 3.1.24. *The function $\psi_{\mathcal{T},\Xi}$ whose general form has been established in Corollary 3.1.23 is called* multivariate difference-skew-differential dimension function *associated with the excellent weighted filtration $(\mathcal{M}_{\mathcal{T},\Xi,k})_{k\in\mathbb{Z}^t}$ or* $\mathcal{T}$-Ξ-difference-skew-differential dimension function *associated with $\mathcal{M}$.*

3.2 Strength of selected systems of differential and difference equations

In [Ein53] Einstein introduced the concept of the strength of a system of partial differential operators in order to measure the size of the associated solution space: "... the system of equations is to be chosen so that the field quantities are determined as strongly as possible. In order to apply this principle, we propose a method which gives a measure of strength of an equation system. We expand the field variables, in the neighborhood of a point $\mathcal{P}$, into a Taylor series (which presupposes the analytic character of the field); the coefficients of these series, which are the derivatives of the field variables at $\mathcal{P}$, fall into sets according to the degree of differentiation. In every such degree there appear, for the first time, a set of coefficients which would be free for arbitrary choice if it were not that the field must satisfy a system of differential equations. Through this system of differential equations (and its derivatives with respect to the coordinates) the number of coefficients is restricted, so that in each degree a smaller number of coefficients is left free for arbitrary choice. The set of numbers of 'free' coefficients for all degrees of differentiation is then a measure of the 'weakness' of the system of equations, and through this, also of its 'strength'."

Mikhalev and Pankratev [MP80] showed the strength of a system of algebraic partial differential equations to coincide with the leading coefficient of the associated differential dimension polynomial. The notion of strength of a system of partial differential equations can be generalized to systems of skew-differential, difference and difference-skew-differential equations. We will use the notion of weight relative Gröbner bases to compute the strength of several systems.

For the diffusion equation in 1-space the differential dimension polynomial and the difference dimension polynomials for the associated forward and symmetric difference scheme, respectively, can be found in [DL12].

3.2.1 Diffusion equation in 1-space

The diffusion equation in one spatial dimension for a constant collective diffusion coefficient a and unknown function $u(x,t)$ describing the density of the diffusing material at given position x and time t is given by

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{3.2}$$

Example 3.2.1. Differential dimension polynomial: *Let K be a differential field with basic set $\Delta = \{\delta_x = \frac{\partial}{\partial x}, \delta_t = \frac{\partial}{\partial t}\}$ containing a and let $\mathcal{M}$ be a differential K-vector space generated as $K[\Delta]$-module by* *one generator m satisfying the defining equation*

$$\delta_t m = a\delta_x^2 m.$$

Then $\mathcal{M}$ is isomorphic to the factor module of a free $K[\Delta]$-module with free generator e by its submodule $\mathcal{N}$ generated by

$$G := \{\delta_t e - a\delta_x^2 e\}.$$

For every $c \in \mathbb{N} \setminus \{0\}$ we choose the weigth vector

$$\alpha_c = \begin{pmatrix} 1/c \\ 1 \end{pmatrix} \in \mathbb{Q}_+^2$$

and define the admissible order $\prec_c$ for $v_x, v_t, w_x, w_t \in \mathbb{N}$ by

$$\delta_x^{v_x}\delta_t^{v_t} e \prec_c \delta_x^{w_x}\delta_t^{w_t} e :\Longleftrightarrow \left(\frac{v_x}{c} + v_t, v_t\right) <_{\mathrm{lex}} \left(\frac{w_x}{c} + w_t, w_t\right).$$

Then $\prec_c$ respects α_c. Since G consists of only one element there are no S-polynomials to compute, i.e., G is a $\prec_c$-Gröbner basis of $\mathcal{N}$ for every $c \in \mathbb{N} \setminus \{0\}$. For the difference-differential dimension polynomials we will use the notation of Theorem 3.1.4 and Corollary 3.1.9. In [DL12] it was shown that if $c = 1$ then the differential dimension polynomial ψ_{α_c} with respect to α_c is just the differential dimension polynomial in the sense of Kolchin. It is given by

$$\psi_{\alpha_c}(k) = 2k + 1.$$

For all $2 \leq c \in \mathbb{N}$ we have $\mathrm{lt}_{\prec_c}(\delta_t e - a\delta_x^2 e) = \delta_t e$ and for all $k \in \mathbb{N}$ sufficiently large we obtain

$$\begin{aligned} U_{\alpha_c,k} &= \left\{ \delta_x^{l_x}\delta_t^{l_t} e \;\middle|\; \frac{l_x}{c} + l_t \leq k, \delta_x^{l_x}\delta_t^{l_t} e \text{ is } \prec_c\text{-irreducible modulo } G \right\} \\ &= \left\{ e, \delta_x e, \ldots, \delta_x^{ck} e \right\}, \end{aligned}$$

and therefore $|U_{\alpha_c,k}| = ck + 1$. Hence, for $c \geq 2$ the differential dimension polynomial with respect to α_c associated with the diffusion equation in one spatial dimension for a constant collective diffusion coefficient is given by

$$\psi_{\alpha_c}(k) = ck + 1.$$

If on the other hand for every $c \in \mathbb{N} \setminus \{0\}$ we choose the weight vector

$$\beta_c = \begin{pmatrix} c \\ 1 \end{pmatrix} \in \mathbb{Q}_+^2$$

and define the admissible order $\prec'_c$ for $v_x, v_t, w_x, w_t \in \mathbb{N}$ by

$$\delta_x^{v_x}\delta_t^{v_t} e \prec'_c \delta_x^{w_x}\delta_t^{w_t} e :\Longleftrightarrow (v_x c + v_t, v_t) <_{\mathrm{lex}} (w_x c + w_t, w_t).$$

Then $\prec'_c$ respects β_c. Still there are no S-polynomials to compute, and G is a $\prec'_c$-Gröbner basis of $\mathcal{N}$ for every $c \in \mathbb{N} \setminus \{0\}$. For all $c \in \mathbb{N} \setminus \{0\}$ we have $\mathrm{lt}_{\prec'_c}(\delta_t e - a\delta_x^2 e) = \delta_x^2 e$ and for all $k \in \mathbb{N}$ sufficiently large we obtain

$$\begin{aligned} U_{\beta_c,k} &= \left\{ \delta_x^{l_x}\delta_t^{l_t} e \mid l_x c + l_t \leq k,\ \delta_x^{l_x}\delta_t^{l_t} e \text{ is } \prec'_c\text{-irreducible modulo } G \right\} \\ &= \left\{ e, \delta_t e, \ldots, \delta_t^{\lfloor k/c \rfloor} e, \delta_x e, \delta_x\delta_t e, \ldots, \delta_x\delta_t^{\lfloor (k-1)/c \rfloor} e \right\}, \end{aligned}$$

and therefore $|U_{\beta_c,k}| = \lfloor \frac{k}{c} \rfloor + \lfloor \frac{k-1}{c} \rfloor + 2$ which obviously is a quasipolynomial with period c. Hence, for $c \in \mathbb{N} \setminus \{0\}$ the differential dimension quasipolynomial with respect to β_c associated with the diffusion equation in one spatial dimension for a constant collective diffusion coefficient is given by

$$\psi_{\beta_c}(k) = \left\lfloor \frac{k}{c} \right\rfloor + \left\lfloor \frac{k-1}{c} \right\rfloor + 2.$$

In order to obtain a forward difference scheme for the diffusion equation (3.2) every occurence of $\frac{\partial u(x,t)}{\partial x}$ and $\frac{\partial u(x,t)}{\partial t}$ is replaced by $u(x+1,t) - u(x,t)$ and $u(x,t+1) - u(x,t)$, respectively. We obtain

$$u(x,t+1) - u(x,t) = a(u(x+2,t) - 2u(x+1,t) + u(x,t)). \tag{3.3}$$

Example 3.2.2. Difference dimension polynomial for forward difference scheme: *Let K be a difference field with basic set $\Sigma = \{\sigma_x : x \mapsto x+1, \sigma_t : t \mapsto t+1\}$ containing a and let $\mathcal{M}$ be a difference K-vector space generated as a left $K[\Sigma^*]$-module by one generator m satisfying the defining equation*

$$\sigma_t m - m = a(\sigma_x^2 m - 2\sigma_x m + m).$$

Then $\mathcal{M}$ is isomorphic to the factor module of a free $K[\Sigma^*]$-module with free generator e by its submodule $\mathcal{N}$ generated by

$$G := \{\sigma_t e - a\sigma_x^2 e + 2a\sigma_x e - (1+a)e\}.$$

Let Ξ be the canonical orthant decomposition of $[\Sigma^*]$ with generators $\xi_1 = \sigma_x, \xi_2 = \sigma_x^{-1}, \xi_3 = \sigma_t$ and $\xi_4 = \sigma_t^{-1}$. For every $c \in \mathbb{N} \setminus \{0\}$ we choose the weigth vector

$$\alpha_c = \begin{pmatrix} 1/c \\ 1/c \\ 1 \\ 1 \end{pmatrix} \in \mathbb{Q}_+^4.$$

If $c = 1$ then the weighted differential dimension polynomial associated with the difference scheme (3.3) *is just the usual difference dimension polynomial. Define the admissible order $\prec_c$ for $v_x, v_t, w_x, w_t \in \mathbb{Z}$ by*

$$\begin{aligned} \sigma_x^{v_x}\sigma_t^{v_t} e \prec_c \sigma_x^{w_x}\sigma_t^{w_t} e \quad :\Longleftrightarrow \quad & \left(\frac{|v_x|}{c} + |v_t|, |v_t|, |v_x|, v_t, v_x\right) \\ & <_{\text{lex}} \left(\frac{|w_x|}{c} + |w_t|, |w_t|, |w_x|, w_t, w_x\right). \end{aligned}$$

Then $\prec_c$ respects α_c. Since G consists of only one element there are no S-polynpomials to compute and G is a $\prec_c$-Gröbner basis of $\mathcal{N}$. We obtain

$$\begin{aligned} U_{\alpha_c,\Xi,k} &= \{\lambda \in [\Sigma]e \mid \operatorname{ord}_{\alpha_c,\Xi}(\lambda) \le k \text{ and } \lambda \text{ is not } \prec_c\text{-reducible modulo} \\ &\qquad \sigma_t e - a\sigma_x^2 e + 2a\sigma_x e - (1+a)e\} \\ &= \{\sigma_x^{-ck}e, \ldots, \sigma_x^{ck}e, \sigma_t^{-1}e, \ldots, \sigma_t^{-k}e, \sigma_x^{-1}\sigma_t^{-1}e, \ldots, \sigma_x^{-1}\sigma_t^{-k+1}e\}, \end{aligned}$$

and therefore $|U_{\alpha_c,k}| = 2(c+1)k$. Hence, for $c \ge 2$ the difference dimension polynomial with respect to α_c associated with the difference scheme (3.3) *is given by*

$$\psi_{\alpha_c}(k) = 2(c+1)k.$$

If on the other hand for every $c \in \mathbb{N} \setminus \{0\}$ we choose the weight vector

$$\beta_c = \begin{pmatrix} c \\ c \\ 1 \\ 1 \end{pmatrix} \in \mathbb{Q}_+^4$$

and define the admissible order $\prec_c'$ for $v_x, v_t, w_x, w_t \in \mathbb{Z}$ by

$$\begin{aligned} \sigma_x^{v_x}\sigma_t^{v_t} e \prec_c \sigma_x^{w_x}\sigma_t^{w_t} e \quad :\Longleftrightarrow \quad & (|v_x|c + |v_t|, |v_t|, |v_x|, v_t, v_x) \\ & <_{\text{lex}} (|w_x|c + |w_t|, |w_t|, |w_x|, w_t, w_x). \end{aligned}$$

Then $\prec'_c$ respects β_c. Since G consists of only one element there are no S-polynpomials to compute and G is a $\prec'_c$-Gröbner basis of $\mathcal{N}$. We obtain

$$\begin{aligned} U_{\beta_c,\Xi,k} &= \{\lambda \in [\Sigma]e \mid \operatorname{ord}_{\beta_c,\Xi}(\lambda) \leq k \text{ and } \lambda \text{ is not } \prec'_c\text{-reducible modulo} \\ &\qquad\qquad \sigma_t e - a\sigma_x^2 e + 2a\sigma_x e - (1+a)e\} \\ &= \{\delta_t^{-k}e \ldots, \delta_t^k e, \delta_x e, \ldots, \delta_x \delta_t^{k-c} e, \\ &\qquad \delta_x^{-1}\delta_t^{-1}e, \ldots, \delta_x^{-1}\delta_t^{-k+c}e, \delta_x^{-2}e, \ldots, \delta_x^{\lfloor k/c \rfloor}e\}, \end{aligned}$$

and therefore $|U_{\alpha_c,k}| = 4k - 2c + 1 + \lfloor \frac{k}{c} \rfloor$ which obviously is a quasipolynomial in k with period c. Hence, for $c \in \mathbb{N} \setminus \{0\}$ the difference dimension quasipolynomial with respect to β_c associated with the difference scheme (3.3) *is given by*

$$\psi_{\beta_c}(k) = 4k - 2c + 1 + \left\lfloor \frac{k}{c} \right\rfloor.$$

3.2.2 Maxwell's equations for vanishing free current density and free charge density

For Maxwell's equations for vanishing free current density and free charge density the differential dimension polynomial and the difference dimension polynomials for the associated forward and symmetric difference scheme, respectively, can be found in [DL12].

Let $E = (E_1, E_2, E_3), D = (D_1, D_2, D_3), H = (H_1, H_2, H_3), B = (B_1, B_2, B_3)$, $J_f = (J_1, J_2, J_3)$ and ρ_f be functions in (x, y, z, t) denoting electric field strength, electric displacement vector, magnetic field strength, magnetic displacement vector, free current density and free charge density, respectively. With

$$\nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Maxwell's equations in 3 spatial dimensions are given by

$$\nabla \cdot D = \rho_f, \qquad \nabla \cdot B = 0, \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0, \quad \text{and} \quad \nabla \times H = J_f + \frac{\partial D}{\partial t}.$$

Assuming $J_f = 0$ and $\rho_f = 0$ Maxwell's equations can be considered as a set of homogeneous linear differential equations.

Example 3.2.3. *Let K be a differential field with basic set*

$$\Delta = \left\{\delta_x = \frac{\partial}{\partial x}, \delta_y = \frac{\partial}{\partial y}, \delta_z = \frac{\partial}{\partial z}, \delta_t = \frac{\partial}{\partial t}\right\}.$$

Assuming $J_f = 0$ and $\rho_f = 0$ Maxwell's equations give rise to a differential $K[\Delta]$-module M with generators $e_1, e_2, e_3, d_1, d_2, d_3, h_1, h_2, h_3, b_1, b_2, b_3$ satisfying

$$\begin{aligned} \delta_x d_1 + \delta_y d_2 + \delta_z d_3 &= 0 = \delta_x b_1 + \delta_y b_2 + \delta_z b_3, \\ \delta_y e_3 - \delta_z e_2 + \delta_t b_1 &= 0 = \delta_y h_3 - \delta_z h_2 - \delta_t d_1, \\ \delta_z e_1 - \delta_x e_3 + \delta_t b_2 &= 0 = \delta_z h_1 - \delta_x h_3 - \delta_t d_2, \\ \delta_x e_2 - \delta_y e_1 + \delta_t b_3 &= 0 = \delta_x h_2 - \delta_y h_1 - \delta_t d_3. \end{aligned}$$

Then M is isomorphic to the factor module of a free $K[\delta_x, \delta_y, \delta_z, \delta_t]$ module with free generators $p_1, \ldots, p_{12}$ by its submodule N generated by

$$\begin{aligned} \{ \quad & \delta_x p_4 + \delta_y p_5 + \delta_z p_6,\ \delta_x p_{10} + \delta_y p_{11} + \delta_z p_{12}, \\ & \delta_y p_3 - \delta_z p_2 + \delta_t p_{10},\ \delta_y p_9 - \delta_z p_8 - \delta_t p_4, \\ & \delta_z p_1 - \delta_x p_3 + \delta_t p_{11},\ \delta_z p_7 - \delta_x p_9 - \delta_t p_5, \\ & \delta_x p_2 - \delta_y p_1 + \delta_t p_{12},\ \delta_x p_8 - \delta_y p_7 - \delta_t p_6 \quad \}. \end{aligned}$$

We consider the weight matrix $\mathcal{T} \in \mathbb{N}^{4\times 2}$ given by

$$\mathcal{T} = (\mathcal{T}^{(1)}, \mathcal{T}^{(2)}) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, we separate δ_t from δ_x, δ_y, and δ_z, i.e., for this example our approach boils down to Levin's approach using Gröbner bases with respect to several orderings [Lev07a]. We choose two admissible orderings $\prec_1$ and $\prec_2$ defined by

$$\begin{array}{rcl} \delta_x^{v_x}\delta_y^{v_y}\delta_z^{v_z}\delta_t^{v_t}e_i \prec_1 \delta_x^{w_x}\delta_y^{w_y}\delta_z^{w_z}\delta_t^{w_t}e_j & :\Longleftrightarrow & (v_x+v_y+v_z, v_t, i, v_x, v_y, v_z) \\ & & <_{\text{lex}} (w_x+w_y+w_z, w_t, j, w_x, w_y, w_z), \text{ and} \\ \delta_x^{v_x}\delta_y^{v_y}\delta_z^{v_z}\delta_t^{v_t}e_i \prec_2 \delta_x^{w_x}\delta_y^{w_y}\delta_z^{w_z}\delta_t^{w_t}e_j & :\Longleftrightarrow & (v_t, v_x+v_y+v_z, i, v_x, v_y, v_z) \\ & & <_{\text{lex}} (w_t, w_x+w_y+w_z, j, w_x, w_y, w_z). \end{array}$$

Then $\prec_1, \prec_2$ respects $\mathcal{T}$.

A $\prec_2$-Gröbner basis of N is given by

$$\begin{array}{rcl} G = \{\ g_1 & = & \delta_x p^2 + \delta_y p^3 + \delta_z p^6, \\ g_2 & = & \delta_x p^{10} + \delta_y p^{11} + \delta_z p^{12}, \\ g_3 & = & -\delta_y p^9 + \delta_z p^8 + \delta_t p^2, \\ g_4 & = & -\delta_z p^7 + \delta_x p^9 + \delta_t p^3, \\ g_5 & = & -\delta_x p^8 + \delta_y p^7 + \delta_t p^6, \\ g_6 & = & \delta_y p^3 - \delta_z p^2 + \delta_t p^{10}, \\ g_7 & = & \delta_z - \delta_x p^3 + \delta_t p^{11}, \\ g_8 & = & \delta_x p^2 - \delta_y + \delta_t p^{12}\ \}. \end{array}$$

The $\prec_1$-S-polynomial of g_3 and g_4 is given by

$$S_{\prec_1}(g_3, g_4) = \delta_x\delta_z p_8 - \delta_y\delta_z p_7 + \delta_y\delta_t p_5 + \delta_x\delta_t p_4.$$

It is $\prec_1$ reducible respecting $\mathcal{T}$ modulo g_5 to

$$\delta_z\delta_t p_6 + \delta_y\delta_t p_5 + \delta_x\delta_t p_4$$

which in turn is $\prec_1$-reducible respecting $\mathcal{T}$ modulo g_1 to 0. The $\prec_1$-S-polynomial of g_6 and g_7 is given by

$$S_{\prec_1}(g_6, g_7) = -\delta_y\delta_t p_{11} - \delta_x\delta_t p_{10} + \delta_x\delta_z p_2 - \delta_y\delta_z p_1.$$

It is $\prec_1$-reducible respecting $\mathcal{T}$ modulo g_8 to

$$-\delta_z\delta_t p_{12} - \delta_y\delta_t p_{11} - \delta_x\delta_t p_{10}$$

which in turn is $\prec_1$-reducible respecting $\mathcal{T}$ modulo g_2 to 0. Hence, G is a $\prec_1$-Gröbner basis of N respecting $\mathcal{T}$. Let $P = \{p_1, \ldots, p_{12}\}$ and for all $k \in \mathbb{N}^2$ define

$$\begin{array}{rcl} U'_k & := & \{\lambda \in [\Delta]P \mid \text{ord}_{\mathcal{T}}(\lambda) \leq_P k, \text{ and } \lambda \neq \text{lt}_{\prec_1}(\mu g) \\ & & \qquad \text{for all } \mu \in [\Delta], g \in G\}, \\ U''_k & := & \{\lambda \in [\Delta]P \mid \text{ord}_{\mathcal{T}}(\lambda) \leq_P k, \text{ and } \forall_{\mu \in [\Delta,], g \in G} \lambda = \text{lt}_{\prec_1}(\mu g) \\ & & \qquad \Longrightarrow \exists_{j \in \{2,\ldots,t\}} \text{ord}_{\mathcal{T}^{(j)}}(\text{lt}_{\prec_j}(\mu g)) > k_j\}. \end{array}$$

Then using the combinatorial formulas provided in [Lev08, Thm. 1.5.7] for all $k \in \mathbb{N}^2$ sufficiently large we obtain

$$|U'_k| = k_1^3k_2 + k_1^3 + 8k_1^2k_2 + 8k_1^2 + 19k_1k_2 + 19k_1 + 12k_2 + 12$$

and using the combinatorial formulas provided in the proof of [Lev08, Thm. 3.3.16] for all $k \in \mathbb{N}^2$ sufficiently large we obtain

$$|U''_k| = \frac{2}{3}k_1^3 + 3k_1^2 + \frac{7}{3}k_1.$$

Hence, the bivariate differential dimension polynomial with respect to $\mathcal{T}$ associated with the system of Maxwell equations for vanishing free current density and free charge density for all $k = (k_1, k_2) \in \mathbb{N}$ sufficiently large is given by

$$\psi(k_1, k_2) = k_1^3k_2 + \frac{5}{3}k_1^3 + 8k_1^2k_2 + 11k_1^2 + 19k_1k_2 + \frac{64}{3}k_1 + 12k_2 + 12.$$

3.2.3 Electromagnetic field given by its potential

For a system of equations defining an electromagnetic field by its potential the differential dimension polynomial and the difference dimension polynomials for the associated forward and symmetric difference scheme, respectively, can be found in [DL12].

An electromagnetic field can be defined by the differential equations describing its potential, cf. [KLMP99, Ex. 9.2.6.]. Let $f_1(x_1, \ldots, x_4), \ldots, f_4(x_1, \ldots, x_4)$ be unknown functions and for $i = 1, \ldots, 4$ consider the system

$$\sum_{j=1}^{4} \frac{\partial}{\partial x_j} f_j = 0, \tag{3.4}$$

$$\sum_{j=1}^{4} \left(\frac{\partial^2}{\partial x_j^2} f_i - \frac{\partial^2}{\partial x_i \, \partial x_j} f_j \right) = 0. \tag{3.5}$$

Example 3.2.4. *Let K be a differential field with basic set $\Delta = \{\delta_i = \frac{\partial}{\partial x_i} \mid i = 1, \ldots, 4\}$. Then* (3.4) *and* (3.5) *give rise to a differential $K[\Delta]$-module M with generators $m_1, \ldots, m_4$ satisfying for $i = 1, \ldots, 4$ the defining equations*

$$\sum_{j=1}^{4} \delta_j m_j = 0,$$

$$\sum_{j=1}^{4} \left(\delta_j^2 m_i - \delta_i \delta_j m_j \right) = 0.$$

Then M is isomorphic to the factor module of a free $K[\Delta]$-module with free generators $e_1, \ldots, e_4$ by its submodule N generated by

$$\left\{ \sum_{j=1}^{4} \delta_j e_j \right\} \bigcup \left\{ \sum_{j=1}^{4} \left(\delta_j^2 e_i - \delta_i \delta_j e_j \right) \,\middle|\, i = 1, \ldots, 4 \right\}. \tag{3.6}$$

We consider the weight vector $\alpha \in \mathbb{N}^4$ given by

$$\alpha = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix},$$

and for $v = (v_1, \ldots, v_4), w = (w_1, \ldots, w_4)$ define the admissible order $\prec$ by

$$\begin{aligned} \delta^v e_i \prec_1 \delta^w e_j \quad :\Longleftrightarrow \quad & (v_1 + v_2 + v_3 + 3v_4, v_1 + v_2 + v_3 + v_4, i, v_1, v_2, v_3) \\ & <_{\text{lex}} (w_1 + w_2 + w_3 + 3w_4, w_1 + w_2 + w_3 + w_4, j, w_1, w_2, w_3). \end{aligned}$$

Then $\prec$ respects α and a $\prec$-Gröbner basis of N is given by

$$\begin{aligned} G = \{\ g_1 &= \delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3 + \delta_4 e_4, \\ g_2 &= -\delta_1^2 e_4 + \delta_1 \delta_4 e_1 - \delta_2^2 e_4 + \delta_2 \delta_4 e_2 - \delta_3^2 e_4 + \delta_3 \delta_4 e_3, \\ g_3 &= \delta_1^2 e_1 + \delta_2^2 e_1 + \delta_3^2 e_1 + \delta_4^2 e_1, \\ g_4 &= \delta_1^2 e_2 + \delta_2^2 e_2 + \delta_3^2 e_2 + \delta_4^2 e_2, \\ g_5 &= \delta_1^2 e_3 + \delta_2^2 e_3 + \delta_3^2 e_3 + \delta_4^2 e_3\ \}. \end{aligned}$$

The leading terms of G with respect to $\prec$ are $\{\delta_4 e_4, \delta_3 \delta_4 e_3, \delta_4^2 e_1, \delta_4^2 e_2, \delta_4^2 e_3\}$. Using the notation of Lemma 3.1.8 for all $k \in \mathbb{N}$ and $l = (l_1, \ldots, l_4)$ we obtain

$$\begin{aligned} U_k = & \{\delta^l e_1 \mid l_1 + l_2 + l_3 \leq k, l_4 = 0\} \\ & \cup \{\delta^l e_1 \mid l_1 + l_2 + l_3 \leq k - 3, l_4 = 1\} \\ & \cup \{\delta^l e_2 \mid l_1 + l_2 + l_3 \leq k, l_4 = 0\} \\ & \cup \{\delta^l e_2 \mid l_1 + l_2 + l_3 \leq k - 3, l_4 = 1\} \\ & \cup \{\delta^l e_3 \mid l_1 + l_2 + l_3 \leq k, l_4 = 0\} \\ & \cup \{\delta^l e_3 \mid l_1 + l_2 \leq k - 3, l_3 = 0, l_4 = 1\} \\ & \cup \{\delta^l e_4 \mid l_1 + l_2 + l_3 \leq k, l_4 = 0\}. \end{aligned}$$

Hence, for all $k \in \mathbb{N}$ sufficiently large we have

$$\begin{aligned} |U_k| &= 4\binom{k+3}{3} + 2\binom{k}{3} + \binom{k-1}{2} \\ &= k^3 + \frac{7}{2}k^2 + \frac{13}{2}k + 5. \end{aligned}$$

Therefore, the differential dimension polynomial ψ_α associated with α is given by

$$\psi_\alpha(k) = k^3 + \frac{7}{2}k^2 + \frac{13}{2}k + 5.$$

Appendices

Computation of border bases for zero-dimensional difference-skew-differential modules

The obvious problem with border bases is that in general they will not be finite, making it hard to deal with them algorithmically. Let $\Xi = \{[\Delta, \Sigma^*]_k \mid 1 \leq k \leq p\}$ be an orthant decomposition, $\prec$ a generalized term order on $[\Delta, \Sigma^*]E$ with respect to Ξ. If the difference-skew-differential submodule $\mathcal{M} \subseteq K[\Delta, \Sigma^*]E$ in concern is zero-dimensional then by Lemma 2.4.21 there exists a Ξ-difference-skew-differential order module $\mathcal{O}$ and an $\mathcal{O}$-border basis of $\mathcal{M}$ such that $|\mathcal{O}| = \dim_K(K[\Delta, \Sigma^*]E/\mathcal{M})$. Kehrein and Kreuzer [KK06] gave several algorithms for computing border bases of zero-dimensional ideals. The following is one of their algorithms adapted to the difference-skew-differential setting.

Algorithm A.1 Basis transformation algorithm

IN: $\mathcal{O} = \{t_i \mid 1 \leq i \leq \mu\}$ a finite Ξ-difference-skew-differential order module, $\mathcal{M} \subseteq K[\Delta, \Sigma^*]E$ a zero-dimensional difference-skew-differential module

OUT: The $\mathcal{O}$-border basis G of $\mathcal{M}$, if it exists. An error otherwise.

Choose a generalized term order $\prec$ on $[\Delta, \Sigma^*]$ and compute $\mathcal{O}_\prec(\mathcal{M}) = [\Delta, \Sigma^*]E \setminus \{\mathrm{lt}_\prec(f) \mid f \in \mathcal{M} \setminus \{0\}\} = \{s_1, \ldots, s_{\tilde{\mu}}\}$;
if $\mu \neq \tilde{\mu}$ **then**
  return "Error: $\mathcal{O}$ has the wrong cardinality.";
end if
For $1 \leq i, j \leq \mu$ compute $\tau_{ij} \in K$ such that $t_i = \sum_{j=1}^{\mu} \tau_{ij} s_j \mod \mathcal{M}$ and let $T := (\tau_{ij})_{i,j=1}^{\mu}$;
if $\det(T) = 0$ **then**
  return "Error: $\mathcal{O}$ has the wrong shape.";
end if
Let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ and for $1 \leq i \leq \mu, 1 \leq j \leq \nu$ compute $\beta_{ij} \in K$ such that $b_i = \sum_{j=1}^{\mu} \beta_{ij} s_j \mod \mathcal{M}$. Let $B := (\beta_{ij})_{1 \leq i \leq \nu,\, 1 \leq j \leq \mu}$;
Let $(\alpha_{ij})_{1 \leq i \leq \nu, 1 \leq j \leq \mu} := BT^{-1}$;
return $G := \{b_i - \sum_{j=1}^{\mu} \alpha_{ij} t_j \mid 1 \leq i \leq \nu\}$;

Theorem A.2. *Algorithm A.1 is correct.*

Proof. By Macaulay's basis theorem for difference-skew-differential operators 2.4.20 we have $\tilde{\mu} = \dim_K(K[\Delta, \Sigma^*]E/\mathcal{M})$, i.e., by the condition $\mu = \tilde{\mu}$ we check whether $\mathcal{O}$ has the correct number of terms to form a basis of $K[\Delta, \Sigma^*]E/\mathcal{M}$. Then the matrix T represents the expansions of t_j in terms of the basis $\{s_1, \ldots, s_\mu\}$, i.e.,

$$\begin{pmatrix} t_1 \\ \vdots \\ t_\mu \end{pmatrix} = T \begin{pmatrix} s_1 \\ \vdots \\ s_\mu \end{pmatrix}.$$

Hence, $\{t_1, \ldots, t_\mu\}$ is a basis of $K[\Delta, \Sigma^*]E/\mathcal{M}$ if and only if T is invertible. The matrix B represents the expansion of b_j in terms of the basis $\{s_1, \ldots, s_\mu\}$. Hence,

$$\begin{pmatrix} b_1 \\ \vdots \\ b_\nu \end{pmatrix} = B \begin{pmatrix} s_1 \\ \vdots \\ s_\mu \end{pmatrix} = BT^{-1} \begin{pmatrix} t_1 \\ \vdots \\ t_\mu \end{pmatrix}.$$

□

Kreuzer and Robbiano also formulated Mourrain's generic algorithm [Mou99] such that it can be used for computing border bases of zero-dimensional polynomial ideals. We are doing the same to obtain an algorithm for computing border bases of zero-dimensional difference-skew-differential modules.

Consider an orthant decomposition Ξ with generators $\xi_1, \dots, \xi_r$ of $[\Delta, \Sigma^*]$ and two K-vector subspaces $F \subseteq L$ of $K[\Delta, \Sigma^*]E$. Let $F_0 := F$ and for all $k \in \mathbb{N} \setminus \{0\}$ define inductively

$$F_k := L \cap (F_{k-1} + \delta_1 F_{k-1} + \cdots + \delta_m F_{k-1} + \xi_1 F_{k-1} + \cdots + \xi_r F_{k-1}) .$$

Then let $F_L := \bigcup_{d \in \mathbb{N}} F_d \subseteq L$.

For a finite set $F \subseteq K[\Delta, \Sigma^*]E$ we write (F) to denote the K-vector space generated by F.

Lemma A.3. *Let $V \subseteq U \subseteq L$ be vector subspaces of $K[\Delta, \Sigma^*]E$. Then*

$$V \subseteq V_L, \qquad V_L = (V_L)_L, \qquad V_L \subseteq U_L, \qquad V_U \subseteq V_L, \qquad V_L = (V_U)_L.$$

Proof. "$V \subseteq V_L$": By the definition of V_L.
"$V_L = (V_L)_L$": By the definition of V_L.
"$V_L \subseteq U_L$": We have $V \subseteq U_0 = U$ and $V_{d+1} = V_d^{[1]} \cap L$, $U_{d+1} = U_d^{[1]} \cap L$. Hence, inductively for $d \in \mathbb{N}$ we obtain $V_d \subseteq U_d$ and $V_L = \bigcup_{d \in \mathbb{N}} V_d \subseteq \bigcup_{d \in \mathbb{N}} U_d = U_L$.
"$V_U \subseteq V_L$": Let $V_{0U} = V_{0L} = V_0$ and $V_{d+1,U} = V_{dU}^{[1} \cap U$, $V_{d+1,L} = V_{dL}^{[1]} \cap L$. Then inductively for $d \in \mathbb{N}$ we have $V_{dU} \subseteq V_{dL}$ and conclude $V_U = \bigcup_{d \in \mathbb{N}} V_{dU} \subseteq \bigcup_{d \in \mathbb{N}} V_{dL} = V_L$.
"$V_L = (V_U)_L$": By the first relation we have $V_0 \subseteq V_U$ and by the third relation we obtain $V_L \subseteq (V_U)_L$. On the other hand by the fourth relation we have $V_U \subseteq V_L$ and by the second relation we obtain $(V_U)_L \subseteq V_L$. □

We need the following subroutine doing Gaussian elimination.

Algorithm A.4 Basis extension algorithm

IN: $\prec$ a generalized term order, $V = \{v_1, \dots, v_r \mid \forall_{1 \le i \ne j \le r} \operatorname{lt}_\prec(v_i) \ne \operatorname{lt}_\prec(v_j)\} \subseteq K[\Delta, \Sigma^*]E \setminus \{0\}$ a set of monic difference-skew-differential operators, $G = \{g_1, \dots, g_s\} \subseteq K[\Delta, \Sigma^*]E$ a set of difference-skew-differential operators.
OUT: $W \subseteq K[\Delta, \Sigma^*]E$ finite set of monic difference-skew-differential operators such that every two different operators in $V \cup W$ have different leading terms and $(V \cup W) = (V \cup G)$.

Let $H := G$ and $S := V$;
while $H \ne \emptyset$ **do**
  Choose $f \in H$ and let $H = H \setminus \{f\}$;
  while $f \ne 0$ and $\exists_{s \in S} \operatorname{lt}_\prec(f) = \operatorname{lt}_\prec(s)$ **do**
    $f := f - \operatorname{lc}_\prec(f) s$
  end while
  if $f \ne 0$ **then**
    $S := S \cup \left\{ \frac{f}{\operatorname{lc}_\prec(f)} \right\}$
  end if
end while
return $W := S \setminus V$;

Theorem A.5. *Algorithm A.4 is correct and terminates.*

Proof. During initialization of the algorithm when f is not defined interpret $\{f\}$ as the empty set. Then during the execution of the algorithm the invariant

$$(S \cup \{f\} \cup H) = (V \cup G)$$

is always satisfied.

The inner while loop terminates since in each iteration the leading term of $f \ne 0$ can be reduced only finitely often. The reduction inside the loop does not alter the invariant and after termination of the inner while loop we have either $f = 0$ or $\mathrm{lt}_\prec(f) \notin \{\mathrm{lt}_\prec(s) \mid s \in S\}$. The outer while loop terminates since H is initialized as the finite set G and during each iteration of the loop the cardinality of H is decreased by 1 while we never add any element to H. Hence, the algorithm terminates.

At termination we have $H = \emptyset$ and either $f = 0$ or $\mathrm{lt}_\prec(f) \in \{\mathrm{lt}_\prec(s) \mid s \in S\}$, i.e., the invariant is satisfied and we have $(W \cup V) = (S \setminus V \cup V) = (S) = (V \cup G)$. □

Then we can use algorithm A.6 to compute a basis of V_L.

Algorithm A.6 F_L algorithm

IN: Ξ an orthant decomposition, $F = \{f_1, \dots, f_r\} \subseteq K[\Delta, \Sigma^*]E$, $d \geq \max\{\mathrm{ord}_\Xi(f_i) \mid 1 \leq i \leq r\}$, $L = (\lambda \in \Sigma \mid \mathrm{ord}_\Xi(\lambda) \leq d)$ and $\prec$ a generalized term order on $[\Delta, \Sigma^*]E$ such that for all $\lambda, \mu \in [\Delta, \Sigma^*]$ with $\mathrm{ord}_\Xi(\lambda) < \mathrm{ord}_\Xi(\mu)$ we have $\lambda \prec \mu$.
OUT: A K-basis V of $(F)_L$ such that the basis elements have pairwise different leading terms.
Apply Algorithm A.4 to $\prec$, $\emptyset$ and F to obtain a K-basis V of (F) with pairwise different leading terms;
Let $W := V$;
while $W \neq \emptyset$ **do**
  $V := V \cup W$;
  Apply Algorithm A.4 to $\prec$, V and $V^{[1]} \setminus V$ obtaining a set W' such that the operators in $V \cup W'$ have pairwise different leading terms;
  $W := \{w \in W' \mid \mathrm{ord}_\Xi(w) \leq d\}$;
end while
return V;

Theorem A.7. *Algorithm A.6 is correct and terminates.*

Proof. The first step computes a finite set V containing difference-skew-differential operators with pairwise different leading terms. If we begin one iteration of the while loop with a set V having this property then we compute a set W' such that $V \cup W'$ is a basis of $(V^{[1]})$ whose operators have pairwise different leading terms. In the next step we discard all operators with order greater than d, i.e., we intersect W' with L to obtain W. This step is correct because $\prec$ is compatible with the order (with respect to Ξ). Hence, in the beginning of each iteration of the while loop we have a finite set V containing difference-skew-differential operators with pairwise different leading terms and in the end of each iteration we have computed a finite set W such that the difference-skew-differential operators in $V \cup W$ have pairwise different leading terms and

$$(V) \subseteq (V \cup W) = (V^{[1]}) \cap L \subseteq L.$$

In particular $V \cup W$ is a basis of $(V^{[1]}) \cap L$. In each iteration of the while loop – except the first – the cardinality of V is increased. Since V is a basis we have $|V| \leq \dim_K L$ which implies termination of the loop. When the loop terminates we have $W = \emptyset$, i.e., we have a finite set V such that $(V) = (V^{[1]}) \cap L$. □

For a zero-dimensional difference-skew-differential submodule $\mathcal{M}$ generated by a finite set $F \subseteq K[\Delta, \Sigma^*]E$ and a generalized term order $\prec$ compatible with the order with respect to the given orthant decomposition the $\mathcal{O}_\prec(\mathcal{M})$-border basis can be computed with algorithm A.8 provided that a suitable basis of the vector space $(F)_L$ is known (see also [KK06]).

Theorem A.9. *Algorithm A.8 is correct and terminates.*

Proof. In the input we have a basis V of the vector space $(F)_L$ such that for each $b_j \in \partial\mathcal{O}$ there exists $h_j \in V$ with $\mathrm{lt}_\prec(h_j) = b_j$. We have to ensure $\mathrm{supp}(h_j) \subseteq \{b_j\} \cup \mathcal{O}$.

Algorithm A.8 $\mathcal{O}_\prec(\mathcal{M})$-border basis algorithm

IN: Ξ an orthant decomposition of $[\Delta,\Sigma^*]$, $F=\{f_1,\dots,f_s\}\subseteq K[\Delta,\Sigma^*]E$ a basis of $\mathcal{M}$, $\prec$ a generalized term order such that $\mathrm{ord}_\Xi(\lambda)<\mathrm{ord}_\Xi(\mu)$ implies $\lambda\prec\mu$, $d\in\mathbb{N}$, $L=\{\lambda\in[\Delta,\Sigma^*]E \mid \mathrm{ord}(\lambda)\leq d\}$ an order ideal, $V\subseteq K[\Delta,\Sigma^*]E\setminus\{0\}$ a basis of the vector space $(F)_L$ with pairwise different leading terms and $\mathcal{O}=L\setminus\{\mathrm{lt}_\prec(v)\mid v\in V\}$ such that $L=F_L\oplus(\mathcal{O})$ and $\partial\mathcal{O}\subseteq L$.

OUT: $G=\{g_1,\dots,g_\nu\}$ the $\mathcal{O}_\prec(\mathcal{M})$-border basis of $\mathcal{M}$.

Let $V':=\emptyset$;

while $V\neq\emptyset$ **do**

Choose $v\in V$ such that $\mathrm{lt}_\prec(v)\prec\mathrm{lt}_\prec(w)$ for all $v\neq w\in V$; $V:=V\setminus\{v\}$;

$H:=\mathrm{supp}(v)\setminus(\mathrm{lt}_\prec(v)\cup\mathcal{O})$;

if $H\neq\emptyset$ **then**

Determine $(c_h)_{h\in H}\subseteq K$, $(w_h)_{h\in H}\subseteq V'$ such that for $h\in H$ we have $\mathrm{lt}_\prec(w_h)=h$ and $h\notin\mathrm{supp}(v-c_hw_h)$;

$v:=v-\sum_{h\in H}c_hw_h$;

end if

$V':=V'\cup\{\frac{v}{\mathrm{lc}_\prec(v)}\}$;

end while

$\{b_1,\dots,b_\nu\}=\partial\mathcal{O}$; For $1\leq j\leq\nu$ choose $g_j\in V'$ such that $\mathrm{lt}_\prec(g_j)=b_j$.

return $G:=\{g_1,\dots,g_\nu\}$;

Throughout the while loop we have $(V\cup\{v\}\cup V')=(F)_L$ and the set $V\cup\{v\}\cup V'$ consists of difference-skew-differential operators with pairwise different leading terms. For every $v'\in V'$ we have $\mathrm{supp}(v')\subseteq\{\mathrm{lt}_\prec(v')\}\cup\mathcal{O}$ and v' is monic. Since v is choosen such that $\mathrm{lt}_\prec(v)\prec\mathrm{lt}_\prec(w)$ for all $v\neq w\in V$ we have $\mathrm{supp}(v)\subseteq\{\mathrm{lt}_\prec(v)\}\cup\mathcal{O}\cup\{\mathrm{lt}_\prec(v')\mid v'\in V'\}$, i.e., $\mathrm{supp}(v)\setminus(\{\mathrm{lt}_\prec(v)\}\cup\mathcal{O})\subseteq\{\lambda\in[\Delta,\Sigma^*]E\mid\mathrm{ord}_\Xi(\lambda)\leq d\}\setminus\mathcal{O}=\{\mathrm{lt}_\prec(f)\mid f\in(F)_L\setminus\{0\}\}=\{\mathrm{lt}_\prec(f)\mid f\in V\}\cup\{\mathrm{lt}_\prec(v)\}\cup\{\mathrm{lt}_\prec(f)\mid f\in V'\}$. On the other hand $\mathrm{supp}(v)\setminus(\{\mathrm{lt}_\prec(v)\}\cap(\{\mathrm{lt}_\prec(f)\mid f\in V\}\cup\{\mathrm{lt}_\prec(v)\}))=\emptyset$. Hence, for $h\in\mathrm{supp}(v)\setminus\{\mathrm{lt}_\prec(v)\}$ there exist $c_h\in K$, $w_h\in V'$ such that $\mathrm{lt}_\prec(w_h)=h$ and $h\notin\mathrm{supp}(v-c_hw_h)$. During each iteration of the while loop the cardinality of V is reduced by 1 and during the execution there is no element added to V. Hence, the loop terminates with $V=\emptyset$ and we have a set V' consisting of difference-skew-differential operators with pairwise different leading terms such that $(V')=(F)_L$ and for all $v'\in V$ we have $\mathrm{supp}(v')\subseteq\{\mathrm{lt}_\prec(v')\}\cup\mathcal{O}$. The algorithm returns a set $G=\{g_1,\dots,g_\nu\}\subseteq V'\subseteq(F)_L\subseteq\mathcal{M}$ of monic difference-skew-differential operators such that for all $j=1,\dots,\nu$ we have $\mathrm{lt}_\prec(g_j)=b_j$ and $\mathrm{supp}(g_j)\subseteq\{\mathrm{lt}_\prec(g_j)\}\cup\mathcal{O}$. Hence, G is an $\mathcal{O}$-border prebasis of $\mathcal{M}$ with $\langle G\rangle=\mathcal{M}$. Now consider the maps $M_1,\dots,M_{m+r}$ defined by (2.5), p.44. Then for all $i,j\in\{1,\dots,m+r\}$, $t\in\mathcal{O}$ we have

$$\begin{aligned}
M_j\circ M_i(t) &= M_j(N(\xi_i(t)))\\
&= M_j(\xi_it-(1_{\mathcal{O}}-N(\xi_it)))\\
&= \xi_j(\xi_it-(1_{\mathcal{O}}-N)(\xi_it))-(1_{\mathcal{O}}-N)(\xi_i\xi_jt-\xi_j(1_{\mathcal{O}}-N)(\xi_it))\\
&= \xi_i\xi_jt+\xi_jk_1+k_2,
\end{aligned}$$

for some $k_1,k_2\in F_L$. Hence,

$$\mathcal{O}\ni(M_j\circ M_i-M_i\circ M_j)(t)=\xi_jk_1+k_2-\sigma_ik_1'-k_2'\in\xi_jF_L\cup\xi_iF_L\cup F_L.$$

From

$$\begin{aligned}
\mathcal{O}\cap(\xi_jF_L\cup\xi_iF_L\cup F_L) &= \mathcal{O}\cap(\xi_jF_L\cup\xi_iF_L\cup F_L)\cap L\\
&= \mathcal{O}\cap F_L\\
&= \{0\}.
\end{aligned}$$

we obtain $M_j \circ M_i - M_i \circ M_j = 0$. By Theorem 2.4.32 it follows that G is a border basis. □

Algorithm A.10 can be used to compute an $\mathcal{O}_\prec(\mathcal{M})$-border basis of a zero-dimensional ideal $\mathcal{M}$.

Algorithm A.10 Border basis algorithm

IN: Ξ an orthant decomposition, $F = \{f_1, \ldots, f_s\} \subseteq K[\Delta, \Sigma^*]E \setminus \{0\}$ a finite basis of the zero-dimensional difference-skew-differential module $\mathcal{M}$, $\prec$ a generlized term order on $[\Delta, \Sigma^*]E$ such that $\mathrm{ord}_\Xi(\lambda) \prec \mathrm{ord}_\Xi(\mu)$ implies $\lambda \prec \mu$.
OUT: G the $\mathcal{O}_\prec(\mathcal{M})$-border basis of $\mathcal{M}$.
  $d := \max\{\mathrm{ord}_\Xi(f_i) \mid 1 \le i \le s\} - 1$; $\mathcal{O} := \{\lambda \in [\Delta, \Sigma^*]E \mid \mathrm{ord}_\Xi(\lambda) \le d\}$; $L := (\mathcal{O})$;
  Apply Algorithm A.4 to $\prec, \emptyset$ and F to obtain a basis V of the vector space (F) consisting of difference-skew-differential operators with pairwise different leading terms; $W := V$;
  while $\partial\mathcal{O} \nsubseteq L$ **do**
    $d := d + 1$; $L := (\lambda \in [\Delta, \Sigma^*]E \mid \mathrm{ord}_\Xi(\lambda) \le d)$;
    while $W \neq \emptyset$ **do**
      $V := V \cup W$;
      Apply Algorithm A.4 to $\prec, V$ and $V^{[1]} \setminus V$ obtaining a set W' such that the operators in $V \cup W'$ have pairwise different leading terms;
      $W := W' \cap \{\lambda \in [\Delta, \Sigma^*]E \mid \mathrm{ord}_\Xi(\lambda) \le d\}$;
    end while
    $\mathcal{O} := \{\lambda \in [\Delta, \Sigma^*]E \mid \mathrm{ord}_\Xi(\lambda) \le d\} \setminus \{\mathrm{lt}_\prec(v) \mid v \in V\}$
  end while
  Apply Algorithm A.8 to $F, \prec, d, L, V$ and $\mathcal{O}$ to obtain a set $G \subseteq K[\Delta, \Sigma^*]E$
  return G;

Theorem A.11. *Algorithm A.10 is correct and terminates.*

Proof. Throughout the execution of the algorithm $\mathcal{O}$ is always a difference-skew-differential order module. By Theorem A.5 we see that when the inner while loop terminates for the first time we have obtained a basis V of the vector space $(F)_L$ consisting of difference-skew-differential operators with pairwise different leading terms.

Every time we enter the outer while loop we have $d \in \mathbb{N}$ such that $V \subseteq U = (\lambda \in [\Delta, \Sigma^*]E \mid \mathrm{ord}_\Xi(\lambda) \le d)$ is a basis of the vector space $(F)_U$. Let $L = (\lambda \in [\Delta, \Sigma^*]E \mid \mathrm{ord}_\Xi(\lambda) \le d + 1)$. By Lemma A.3 we have $(V)_L = ((F)_U)_L = (F)_L$ and the inner while loop updates V to a basis of the vector space $(F)_L$ and $\mathcal{O}$ to a Ξ-difference-skew-differential order module such that $L = (F)_L \oplus (O)$.

By Lemma 2.4.21 there exists a unique $\mathcal{O}_\prec(\mathcal{M})$-border basis $\tilde{G} = \{g_1, \ldots, g_\nu\}$ of $\mathcal{M}$. For $1 \le j \le \nu$ there exist $h_{j1}, \ldots, h_{js} \in K[\Delta, \Sigma^*]E$ such that $g_j = h_{j1}f_1 + \ldots + h_{js}f_s$. Let $\tilde{d} := \max\{\mathrm{ord}_\Xi(h_{ji}f_i) \mid 1 \le i \le s,\ 1 \le j \le \nu\}$. Suppose that the outer while loop has not terminated before reaching the case $d = \tilde{d}$. Then at the end of this iteration of the outer while loop we have a basis V of the vector space $(F)_L$, where $L = \{\lambda \in [\Delta, \Sigma^*] \mid \mathrm{ord}_\Xi(\lambda) \le \tilde{d}\}$ and V consists of difference-skew-differential operators with pairwise different leading terms. By the definition of $\tilde{d}$ we have $G \subseteq (F)_L$ and $\partial\mathcal{O}_\prec(\mathcal{M}) = \{\mathrm{lt}_\prec(g_1), \ldots, \mathrm{lt}_\prec(g_\nu)\} \subseteq (F)_L$. From $(F)_L \subseteq \mathcal{M}$ we get $\mathcal{O}_\prec(\mathcal{M}) \supseteq \mathcal{O} = \{\lambda \in [\Delta, \Sigma^*]E \mid \mathrm{ord}_\Xi(\lambda) \le \tilde{d}\} \setminus \{\mathrm{lt}_\prec(f) \mid f \in (F)_L \setminus \{0\}\}$ at the end of this iteration of the outer while loop. Hence, $\partial\mathcal{O} \subseteq \mathcal{O}_\prec(\mathcal{M}) \cup \partial\mathcal{O}_\prec(\mathcal{M}) \subseteq L$ which implies termination of the loop.

When the outer while loop terminates we have a set V being a basis of the vector space $(F)_L$ consisting of difference-skew-differential operators with pairwise different leading terms. So applying Algorithm A.8 yields the $\mathcal{O}_\prec(\mathcal{M})$-border basis of $\mathcal{M}$. □

List of symbols

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