DEAM2 Proceedings

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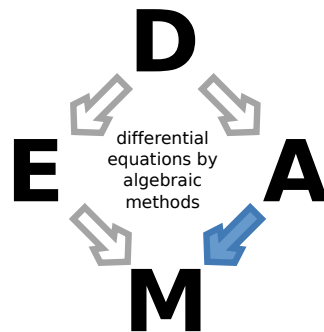
Christian Dönch, Johannes Middeke, Franz Winkler

July 5, 2012

Foreword

In the frame of the research project "Symbolic and Algebraic Methods for LPDOs (DIFFOP)" (funded by the Austrian Science Fund (FWF) under the project number P20336-N18) we organized the 2nd DEAM Workshop on "Differential Equations by Algebraic Methods". DEAM2 took place February 9–11, 2011, at Johannes Kepler University in Linz, Austria. Approximately 40 researchers participated in the discussions on differential algebra, theory of differential operators, and their application to the solution of differential equations.

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Christian Dönch
Johannes Middeke
Franz Winkler

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Schedule

	Wednesday 9th February	Thursday 10th February	Friday 11th February
8:30–10:00		**Session 4** E. Mansfield T. Nunes-Gonçalves J. Llibre *(Chair: J.-F. Pommaret)*	**Session 8** T. Cluzeau A. Quadrat J.-F. Pommaret *(Chair: E. Mansfield)*
10:00–10:30	*Opening ceremony*	*Coffee break*	*Coffee break*
10:30–12:00	**Session 1** J. Middeke A. Korporal M. Rosenkranz *(Chair: F. Winkler)*	**Session 5** M. Barkatou C. El Bacha G. Labahn *(Chair: J. Middeke)*	**Session 9** F. Schwarz L. X. C. Ngô C. G. Raab *(Chair: G. Labahn)*
12:00–14:00	*Lunch break*	*Lunch break*	*Lunch break*
14:00–15:30	**Session 2** S. Tsarev A. Levin E. Shemyakova *(Chair: Z. Li)*	**Session 6** X.-S. Gao S. Rueda Z. Li *(Chair: F. Schwarz)*	
15:30–16:00	*Coffee break*	*Coffee break*	
16:00–17:30	**Session 3** M. Giesbrecht V. Levandovskyy D. Andres A. Heinle *(Chair: X.-S. Gao)*	**Session 7** W. Plesken D. Robertz F. Antritter *(Chair: A. Levin)*	

Talk titles

1. D. Andres: *Challenging Bernstein-Sato polynomials and B-functions.*

2. F. Antritter: *Computing π-flat outputs of linear control systems with delays.*

3. M. Barkatou: *Removing Apparent Singularities of Systems of Linear Differential Equations with Rational Function Coefficients.*

4. T. Cluzeau: *Serre's Reduction of Linear Partial Differential Systems with Holonomic Adjoints.*

5. C. El Bacha: *An algorithm for computing simple forms of first-order linear differential systems.*

6. X.-S. Gao: *Differential Chow Form and Differential Resultant.*

7. M. Giesbrecht: *Provably Fast Algorithms for Canonical Forms of Matrices of Ore Polynomials.*

8. A. Heinle: *New factorization algorithm in the first (q-)Weyl algebra.*

9. A. Korporal: *Generalized LODE Boundary Problems and Green's Operators.*

10. G. Labahn: *On Simultaneous Row and Column Reduction of Higher-Order Linear Differential Systems.*

11. V. Levandovskyy: *Constructive D-module theory and applications.*

12. A. Levin: *Invariants of Difference Field Extensions.*

13. Z. Li: *On the structure of compatible rational functions.*

14. J. Llibre: *Results and open problems on the algebraic limit cycles of polynomial vector fields in $\mathbb{R}^2$.*

15. E. Mansfield: *Pseudogroups, their invariants, and Noether's second theorem.*

16. J. Middeke: *Adapting the FGLM-algorithm for conversion between Hermite and Popov normal forms of differential operator matrices.*

17. L. X. C. Ngô: *Solving some parametrizable ODEs of order 1 by parametrization.*

18. T. Nunes-Gonçalves: *Symbolic methods for solving SE(3) Symmetric variational problems.*

19. W. Plesken: *Linear differential elimination for analytic functions.*

20. J.-F. Pommaret: *Spencer Operator and Macaulay Inverse System: A New Approach To Control Identifiability and Other Engineering Applications.*

21. A. Quadrat: *Triangularization of general linear systems of partial differential equations based on pure differential modules.*

22. C. G. Raab: *Integration of Liouvillian Functions.*

23. D. Robertz: *Nonlinear differential elimination for analytic functions.*

24. M. Rosenkranz: *Partial Results for Partial Integro-Differential Operators.*

25. S. Rueda: *Implicitization of linear DPPEs by perturbed differential resultants.*

26. F. Schwarz: *Solving Linear Inhomogeneous Differential Equations.*

27. E. Shemyakova: *X- And Y-invariants for Linear Partial Differential Operators in the Plane.*

28. S. Tsarev: *Structure of the lattice of right divisors of a LODO.*

Participants

The following people participated in DEAM 2:

- Daniel Andres (RWTH Aachen)
- Felix Antritter (Universität der Bundeswehr München)
- Moulay Barkatou (Université de Limoges)
- Shaoshi Chen (INRIA Rocquencourt)
- Thomas Cluzeau (Université de Limoges)
- Christian Dönch (RISC)
- Carole El Bacha (Université de Limoges)
- Burçin Eröcal (RISC)
- Xiao-Shan Gao (Academy of Mathematics and System Sciences)
- Mark Giesbrecht (University of Waterloo)
- Albert Heinle (RWTH Aachen)
- Yanli Huang (Beihang University/RISC)
- Anja Korporal (RISC)
- George Labahn (University of Waterloo)
- Günter Landsmann (RISC)
- Viktor Levandovskyy (RWTH Aachen)
- Alexander Levin (The Catholic University of America)
- Ziming Li (Academy of Mathematics and System Sciences)
- Jaume Llibre (Universitat Autònoma de Barcelona)
- Elizabeth Mansfield (University of Kent)
- Johannes Middeke (RISC)
- Lâm Xuân Châu Ngô (RISC)
- Tania Nunes-Goncalves (University of Kent)
- Franz Pauer (Universität Innsbruck)
- Wilhelm Plesken (RWTH Aachen)
- Jean-François Pommaret (CERMICS/ENPC)
- Alban Quadrat (INRIA Saclay – Île-de-France)
- Clemens Gunter Raab (RISC)
- Hamid Rahkooy (RISC)
- Georg Regensburger (INRIA Saclay – Île-de-France)
- Daniel Robertz (RWTH Aachen)
- Markus Rosenkranz (University of Kent)
- Sonia Rueda (Universidad Politécnica de Madrid)

- Fritz Schwarz (Fraunhofer SCAI)

- Rafael Sendra (University of Alcalá)

- Ekaterina Shemyakova (RISC/Univ. Western Ontario)

- Loredana Tec (RISC)

- Sergey Tsarev (Siberian Federal University)

- Franz Winkler (RISC)

- Min Wu (East China Normal University)

- Burkhard Zimmermann (RISC)

Table of Contents

Challenging Bernstein-Sato polynomials and b-functions

Daniel Andres[1]

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Abstract

The Bernstein-Sato polynomial b_f, which is also known as global b-function, of a hypersurface given by a polynomial $f \in \mathbb{K}[x] := \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ denotes a field of characteristic zero, plays an important role in many applications of algebraic D-module theory.

It is defined to be the monic polynomial of least degree satisfying the functional identity $P \bullet f^{s+1} = b_f \cdot f^s$ for some operator $P \in D_n[s] := D_n \otimes_{\mathbb{K}} \mathbb{K}[s]$ (Bernstein, 1971, 1972). Here

$$D_n := \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid x_j \partial_i = x_j \partial_i + \delta_{ij} \text{ for } 1 \leq i, j \leq n \rangle$$

denotes the n-th Weyl algebra, s is another variable and f^s stands for a symbolic power of f. Formally, f^s denotes the generator of the free $\mathbb{K}[x, s, f^{-1}]$-module M_f of rank one. This module M_f can also be viewed as a left $D_n[s]$-module via

$$x_i \bullet g f^{s+j} := x_i \cdot g f^{s+j}, \quad s \bullet g f^{s+j} := s \cdot g f^{s+j}, \quad \text{and}$$

$$\partial_i \bullet g f^{s+j} := \frac{\partial g}{\partial x_i} f^{s+j} + g(s+j) \frac{\partial f}{\partial x_i} f^{s+j-1}$$

for $g \in \mathbb{K}[x, s]$ and $f^{s+j} := f^j \cdot f^s, j \in \mathbb{Z}$.

As a consequence of its definition, the Bernstein-Sato polynomial can be computed as follows:

$$\langle b_f \rangle = (\mathrm{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{K}[s],$$

where $\mathrm{Ann}_{D_n[s]}(f^s) = \{ p \in D_n[s] \mid p \bullet f^s = 0 \}$ is the annihilator of f^s.

In the talk, we address the following problems:

- The algorithm by Briançon and Maisonobe (2002) for the computation of $\mathrm{Ann}_{D_n[s]}(f^s)$ turned out to be the most effective one in practice (Levandovskyy and Martín-Morales, 2008). We show how to enhance this approach by obtaining a pre-processing via purely commutative methods, see also Andres et al. (2010a).

- Since it is known that -1 is always a root of b_f, one can directly compute $\frac{b_f}{s+1}$. This is in particular useful combined with the following problem, see Andres et al. (2010b).

- The intersection of an ideal with a univariate subalgebra arises in different situations, not only limited to D-module theory. We present a general algorithm that does not require the use of (expensive) elimination orderings, see Andres et al. (2009); Noro (2002).

An implementation is available in the computer algebra system SINGULAR (Decker et al., 2010), respectively in its non-commutative subsystem SINGULAR:PLURAL (Greuel et al., 2010), whose D-module suite currently consists of the libraries `bfun.lib`, `dmod.lib`, `dmodapp.lib` and `dmodvar.lib`.

Further references are given in Andres (2010).

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An algorithm for computing simple forms of first-order linear differential systems

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Abstract

We consider systems of ordinary differential equations of first-order. With such a system, we associate a matrix pencil and we say that the system is simple if the associated matrix pencil is regular. The latter condition is very useful in computing regular solutions (local problems) or rational solutions (global problems). Here, we propose a new algorithm for transforming a non simple system into a simple one.

We consider a system of linear differential equations of first-order of the form

$$\mathcal{D}(Y(x)) = A(x)\vartheta(Y(x)) + B(x)Y(x) = 0, \tag{1}$$

where $\vartheta = x\frac{d}{dx}$, $A(x)$ and $B(x)$ are two square matrices of size n with formal power series coefficients and $A(x)$ is invertible. If one is looking for regular solutions of System (1), in particular, solutions of the form $y(x) = x^{\lambda_0} \sum_{i=0}^{\infty} u_i x^i$ with $\lambda_0 \in \mathbb{C}$ and $u_i \in \mathbb{C}^n$ ($u_0 \neq 0$), then one is first confronted to the linear algebra problem: find $\lambda_0 \in \mathbb{C}$ and a nonzero vector $u_0 \in \mathbb{C}^n$ such that

$$(A(0)\lambda_0 + B(0))u_0 = 0.$$

If $L(\lambda) = A(0)\lambda + B(0)$ is a singular matrix pencil, *i.e.*, $\det(L(\lambda))$ vanishes for all elements λ of $\mathbb{C}$, then no useful information is provided. Otherwise, λ_0 has to be chosen as a root of the determinant of $L(\lambda)$ and $u_0 \in \ker(L(\lambda_0))$. Thus, a prerequisite for computing regular solutions of System (1) is to make sure that the matrix pencil $L(\lambda)$ associated to (1) is regular, that is, $\det(L(\lambda)) \not\equiv 0$. If this condition holds, we say that System (1) is *simple*. The notion of simplicity has been first introduced by Barkatou in [1] for the study of rational solutions then used by Barkatou & Pflügel in [2] for computing regular local solutions. Unfortunately, a differential system of the form (1) is not necessarily simple hence it is important to have an algorithm that, taking as input a non simple system of the form (1), returns a simple one. To our knowledge, the only known approach to achieve this is to write System (1) as $Y'(x) = C(x)Y(x)$, where matrix $C(x)$ has coefficients in the field of Laurent series, then compute a super-reduction form of this system (see [3]). This allows us to obtain a simple system equivalent to (1) (see [2]). But by computing the super-reduction form, we do more work than needed since a simple system is not necessarily super-reduced. Therefore, the purpose of this talk is to present a new algorithm that computes a simple system equivalent to (1) without calling the super-reduction. Our algorithm proceeds as follows. First, we can assume, without any loss of generality, that the leading coefficient $A(x)$ is in Smith normal form, *i.e.*, $A(x) = \text{diag}(1, \ldots, 1, x^{\alpha_{r+1}}, \ldots, x^{\alpha_n})$ where the α_i's are positive integers satifying $\alpha_{r+1} \leq \cdots \leq \alpha_n$.

The associated matrix pencil $L(\lambda)$ is then of the form

$$L(\lambda) = A(0)\lambda + B(0) = \left(\begin{array}{c|c} \lambda I_r + B_0^{11} & B_0^{12} \\ \hline B_0^{21} & B_0^{22} \end{array} \right) \in \mathbb{C}[\lambda]^{n \times n}.$$

To obtain a regular matrix pencil, we will multiply operator $\mathcal{D}$ given by (1) on the left and on the right by invertible matrices in order to increase either the rank of $A(0)$ or that of $B(0)$. Increasing the rank of $A(0)$ can be done by dropping some of the integers α_i to zero while increasing the rank of $B(0)$ can be done by eliminating the linear dependencies between its columns and its rows. We show that as long as the constant columns or rows of $L(\lambda)$ are linearly dependent, we can perform some transformations on $\mathcal{D}$ and decrease some values of the α_i's without affecting the others. Thus, after at most $\sum_{i=r+1}^{n} \alpha_i$ iterations, we either obtain a simple system or a non simple one for which the constant columns and rows of $L(\lambda)$ are linearly independent. If this occurs, then we show that we can always go back to the case where the constant columns of $L(\lambda)$ are linearly dependent, without affecting the values of the α_i's.

We end by mentioning two important points: firstly, this algorithm allows to classify the singularity $x = 0$ of System (1) as regular or irregular singularity. Indeed, $x = 0$ is a regular singularity of System (1) if and only if our algorithm returns a system for which $\alpha_{r+1}, \ldots, \alpha_n$ are all zero. So our algorithm can be considered as an alternative of Moser's algorithm [4] to classify singularities. Secondly, our algorithm can be extended to handle systems of the form

$$\mathcal{D}_k(Y(x)) = A(x)\vartheta_k(Y(x)) + B(x)Y(x) = 0,$$

where k is a positive integer and $\vartheta_k = x^k \vartheta$. Indeed, it has been shown in [5] that the regularity of the matrix pencil $A(0)\lambda + B(0)$ implies the existence of irregular solutions of the form

$$y(x) = \exp\left(\int \frac{\lambda_0}{x^{k+1}} + \cdots \right) z(x),$$

where the dots stand for the terms of valuation higher than $-k - 1$, $z(x) \in \mathbb{C}[[x]][\log(x)]^n$ and λ_0 satisfies $\det(A(0)\lambda_0 + B(0)) = 0$.

References

[1] M. Barkatou. On Rational Solutions of Systems of Linear Differential Equations. *J. Symbolic Computation*, 28: 547-567, 1999.

[2] M. Barkatou and E. Pflügel. An Algorithm Computing the Regular Formal Solutions of a System of Linear Differential Equations. *J. Symbolic Computation*, 11: 1-20, 1998.

[3] A. Hilali and A. Wazner. Formes Super-irréductibles des systèmes différentiels linéaires. *Numer. Math.,* 50: 429-449, 1987.

[4] J. Moser. The Order of a Singularity in Fuchs' Theory. *Math. Zeitschr*, 72: 379-398, 1960.

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Removing Apparent Singularities of Systems of Linear Differential Equations with Rational Function Coefficients
Extended Abstract

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Abstract

In this talk we present a new algorithm which, given a system of the form (S) (see 1 below), detects apparent singularities [3] and constructs a gauge equivalent system (S') with rational coefficients, such that every finite singularity of (S') is a singularity of (S) that is not apparent. Our method can, in particular, be applied to the companion system of any linear differential equation with arbitrary order n. We thus have an alternative method to the standard methods for removing apparent singularities of linear differential operators. We compare our method to the one designed for operators and we shall show some examples of computation.

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1. Definitions- Notations

Consider a first-order differential system of size n with rational function coefficients in the complex variable z

$$(S) \qquad Y' = A(z)Y \qquad (1)$$

where $' = \frac{d}{dz}$, $Y = (y_1, \ldots, y_n)^t$ is a vector of length n and $A \in \mathbb{C}(z)^{n \times n}$.

The finite singularities of system (S) are the poles of the entries of $A(z)$.

Definition 1. *A singular point z_0 of (S) is called an apparent singularity if there exists a fundamental matrix solution $\Phi(z)$ of (S) which is holomorphic at $z = z_0$.*

Let $T \in \mathrm{GL}(n, \mathbb{C}(z))$. The substitution $Y = TZ$ transforms System (S) into a new system (S') $Z' = B(z)Z$, where

$$B = T[A] := T^{-1}AT - T^{-1}T'.$$

Definition 2. *We the say that $Y' = A(z)Y$ and $Z' = B(z)Z$, are said to be gauge equivalent if there exists $T \in \mathrm{GL}(n, \mathbb{C}(z))$ such that $B = T[A]$. In this case we say that the matrices A and B are equivalent.*

Definition 3. *A system $(\tilde{S})$ $\tilde{Y}' = \tilde{A}(z)\tilde{Y}$ with $\tilde{A} \in \mathbb{C}(z)^{n \times n}$ is called a desingularization of (S) if:*

(i) there exits a polynomial matrix $T(z)$ with $\det T(z) \not\equiv 0$ such that $\tilde{A} = T[A]$,
(ii) the singularities of $(\tilde{S})$ are the singularities of (S) that are not apparent.

2. How to detect and remove apparent singularities?

We prove the following propositions

Proposition 1. *If $z = z_0$ is a finite apparent singularity of (S) then one can construct a polynomial matrix $T(z)$ with $\det T(z) = c(z - z_0)^\alpha$, $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $T[A]$ has at worst a simple pole at $z = z_0$.*

Proposition 2. *Suppose that $A(z)$ has a simple pole at $z = z_0$ and let*

$$A(z) = \frac{A_0}{(z - z_0)} + \sum_{i \geq 1} A_i(z - z_0)^{i-1}, \quad A_i \in \mathbb{C}^{n \times n}.$$

If z_0 is an apparent singularity then the eigenvalues of A_0 are nonnegative integers and A_0 is diagonalizable.

Proposition 3. *Suppose that $z = z_0$ is a simple pole of $A(z)$ and that A_0 has only nonnegative integer eigenvalues. Then there exists a polynomial matrix $T(z)$ with $\det T(z) = c(z - z_0)^\alpha$ for some $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $B := T[A]$ has at worst a simple pole at $z = z_0$ and B_0 has a single eigenvalue: $B_0 = mI_n + N$ where $m \in \mathbb{N}$ and N nilpotent.*
Moreover, z_0 is an apparent singularity iff $N = 0$. In this case the gauge transformation $Y = (z - z_0)^m \tilde{Y}$ leads to a system for which $z = z_0$ is an ordinary point.

Proposition 4. *If $z = z_0$ is a finite apparent singularity of (S) then one can construct a polynomial matrix $T(z)$ with $\det T(z) = c(z - z_0)^\alpha$, $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $B(z) := T[A]$ has no pole at $z = z_0$.*

Due to the form of its determinant, the gauge transformation $T(z)$ in the above proposition does not affect the other finite singularities of (S). Thus by applying the above result to each apparent singularity we get the following:

Theorem 1. *One can construct a polynomial matrix $T(z)$ which is invertible in $\mathbb{C}(z)$ such that the finite poles of $B := T[A]$ are exactly the poles of A that are not apparent singularities for (S).*

2.1. How to construct a complete desigularization ?

Consider a system (S) $Y' = A(z)Y$ and let $P(A)$ be the set of poles of A.

1. Compute a polynomial matrix $T(z)$ such that
   - the zeros of $\det T(z)$ are in $P(A)$
   - $T[A]$ has the same finite poles as A with minimal orders (among all gauge equivalent matrices).
   - Put A:=T[A] and go to step 2.

2. For each simple pole z_0 of A compute A_{0,z_0} the residue matrix of $A(z)$ at $z = z_0$ and its eigenvalues. Let $App(A)$ denote the set of singularities z_0 such that A_{0,z_0} has only nonnegative integer eigenvalues.

3. For each $z_0 \in App(A)$ compute a polynomial matrix $T(z)$ with $\det T(z) = c(z - z_0)^\alpha$ such that $A := T[A]$ has at worst a simple pole at $z = z_0$ with residue matrix of the form $A_{0,z_0} = m_{z_0} I_n + N_{z_0}$ where $m_{z_0} \in \mathbb{N}$ and N_{z_0} nilpotent.

4. Keep in $App(A)$ only the point z_0 for which $N_{z_0} = 0$.

5. Let $T = \prod_{z_0 \in App(A)} (z - z_0)^{m_{z_0}} I_n$, then B:=T[A] is a desingularization of the input system (S).

Remark 1.   1. *The transformation T in Step 1 can be constructed using our Rational Moser Algorithm [2, 4].*

2. *Step 1 can be skipped when the given system comes from a scalar differential equation.*

3. *If the point at infinity of the original system is singular regular then it will be also singular regular for the computed desingularization but the order of the pole at infinity may increase.*

3. Application to desigularization of scalar differential equation

Let $\partial = \frac{d}{dz}$ and $L \in \mathbb{C}(z)[\partial]$ be monic, have order n:

$$L = \partial^n + c_{n-1}(z)\partial^{n-1} + \cdots + c_0(z),$$

Let $S(L)$ be the set of finite singularities of L, that the set of the poles of the c_i's.

Definition 4. *An operator $\tilde{L} \in \mathbb{C}[z][\partial]$ is called a desingularization of L if :*

(i) $\tilde{L} = RL$ for some $R \in \mathbb{C}(z)[\partial]$,
(ii) $S(\tilde{L}) = \{z_0 \in S(L) \mid z_0 \text{ not apparent}\}$

In [1] we present an algorithm that given a monic operator $L \in \mathbb{C}(z)[\frac{d}{dz}]$ of order n constructs a monic operator $\tilde{L} \in \mathbb{C}(z)[\frac{d}{dz}]$ with minimal order $m + 1 \geq n$ satisfying (i) and (ii), m being the maximum of the of the set of all local exponent at the different finite apparent singularities of L. This algorithm has been implemented in Maple. In the sequel, we refer to this algorithm as ABH method.

Example 1. *Let L be the monic operator with e^z and $1 + z + z^2/2$ as solutions:*

$$L := \partial^2 - \frac{(z+2)\partial}{z} + \frac{2}{z}.$$

The desingularization computed by ABH method is

$$\tilde{L} = \partial^4 + (-1 + 1/4\,z)\partial^3 + (-1/4 - 3/8\,z)\partial^2 + (1/2 + 1/8\,z)\partial - 1/4$$

This operator is of order 4.

By working directly on the companion system of L, the apparent singularity of L at $z = 0$ can be removed also by computing an equivalent first-order differential system of size $\mathrm{ord}(L) = 2$.

$$(S) \quad Y' = C(z)Y, \ \ C(z) = \begin{pmatrix} 0 & 1 \\ \frac{-2}{z} & 1 + \frac{2}{z} \end{pmatrix}$$

Indeed if we put

$$Y = T(z)Z, \ \ T(z) = \begin{pmatrix} 1 & 0 \\ 1 & z^2 \end{pmatrix}$$

then the new variable Z satisfies the system $Z' = T[C]Z$ where

$$T[C] := T^{-1}CT - T^{-1}T' = \begin{pmatrix} 1 & z^2 \\ 0 & 0 \end{pmatrix}.$$

Example 2. *Let $L = \partial^2 + \frac{(3z^2-4)\partial}{z(z^2+2)} - 2\frac{-1+2z^2}{z^2+2}$. It has an apparent singularity at $z = 0$ with local exponents 0 and 3. The desingularization computed by ABH method is of order 4:*
$\tilde{L} = \partial^4 + 1/2\frac{z(24+7z^2)\partial^3}{z^2+2} + 1/2\frac{(58z^2+88+27z^4)\partial^2}{(z^2+2)^2} - 1/2\frac{z(-4z^2+4+93z^4+28z^6)\partial}{(z^2+2)^3} - 4\frac{44z^2+16+42z^4+7z^6}{(z^2+2)^3}.$
The companion matrix of L is

$$A = \begin{pmatrix} 0 & 1 \\ 2\frac{-1+2z^2}{z^2+2} & -\frac{3z^2-4}{z(z^2+2)} \end{pmatrix}.$$

It has a simple pole at $z = 0$ with a residue matrix $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.

Our algorithm computes the following gauge transformation T

$$T = \begin{pmatrix} 1 & 0 \\ z & -z^2 \end{pmatrix}$$

The matrix of the new equivalent system is

$$B = T^{-1}(AT - T') = \begin{bmatrix} z & -z^2 \\ 1 & -\frac{z(z^2+7)}{z^2+2} \end{bmatrix}$$

It has $z = 0$ as ordinary point.

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Serre's Reduction of Linear Partial Differential Systems with Holonomic Adjoints

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Abstract

Given a linear functional system (e.g., ordinary/partial differential systems, differential time-delay systems, difference systems), Serre's reduction aims at finding an equivalent linear functional system which contains fewer equations and fewer unknowns. The purpose of this talk is to study Serre's reduction of underdetermined linear systems of partial differential equations with either polynomial, formal power series or analytic coefficients and with holonomic adjoints in the sense of algebraic analysis. We prove that these linear partial differential systems can be defined by means of only one linear partial differential equation. In the case of polynomial coefficients, we give a constructive algorithm to compute the corresponding equation.

Key words: Serre's reduction, underdetermined linear systems of partial differential equations, holonomic D-modules, constructive module theory, mathematical systems theory, symbolic computation.

Given a multidimensional linear system, a first important issue in mathematical systems theory is to simplify its equations before studying its structural properties, studying synthesis problems or applying numerical analysis methods. Serre's reduction aims at reducing the number of equations and unknowns of a linear system. It was recently introduced in the literature of mathematical systems theory in [1,2]. Let us recall the main theorem of [1].

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Theorem 1 *Let D be a noetherian domain and $R \in D^{q \times p}$ a full row rank matrix, namely, the rows of R are left D-linearly independent. If there exist a column vector $\Lambda \in D^q$ and a unimodular matrix $U \in D^{(p+1) \times (p+1)}$, i.e., $U \in \mathrm{GL}_{p+1}(D)$, such that $(R \quad - \Lambda) U = (I_q \quad 0)$, then the finitely presented left D-module $M = D^{1 \times p}/(D^{1 \times q} R)$ is isomorphic to the left D-module $L = D^{1 \times (p+1-q)}/(D Q_2)$, where the row vector $Q_2 \in D^{1 \times (p+1-q)}$ is defined by:*

$$U = \begin{pmatrix} S_1 \ Q_1 \\ S_2 \ Q_2 \end{pmatrix}, \quad S_1 \in D^{p \times q}, \quad Q_1 \in D^{p \times (p+1-q)}, \quad S_2 \in D^{1 \times q}, \quad Q_2 \in D^{1 \times (p+1-q)}.$$

This result depends only on the residue class of $\Lambda \in D^q$ in the right D-module $E \triangleq D^q/(R D^p)$.

A classical remark in algebraic analysis [6] due to Malgrange shows that the behaviour $\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R \eta = 0\}$ defined by R and a left D-module $\mathcal{F}$ (i.e., a signal space) is defined by $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$, where $\hom_D(M, \mathcal{F})$ denotes the abelian group of left D-homomorphisms (i.e., linear applications) from M to $\mathcal{F}$. Theorem 1 then proves that $\ker_{\mathcal{F}}(R.)$ is isomorphic to $\ker_{\mathcal{F}}(Q_2.) = \{\zeta \in \mathcal{F}^{(p+1-q)} \mid Q_2 \zeta = 0\}$ defined by a sole equation.

Moreover, it was proved in [1] that if D is a principal left ideal domain (e.g., the ring of ordinary differential/shift operators with coefficients in a differential/difference field), a commutative polynomial ring $D = k[x_1, \ldots, x_n]$ over a field k or the Weyl algebras $A_n(k)$ or $B_n(k)$ of partial differential operators with respectively polynomial and rational coefficients over a field k of characteristic 0 (e.g., $\mathbb{R}$, $\mathbb{C}$), then the condition given in Theorem 1 can be reduced to the existence of $\Lambda \in D^q$ such that $P = (R \quad - \Lambda)$ admits a right-inverse $S = (S_1^T \quad S_2^T)^T \in D^{(p+1) \times q}$, i.e., $R S_1 - \Lambda S_2 = I_q$, which is also equivalent the fact that the right D-module E is generated by the residue class of Λ in E. Constructive algorithms were given in [1,2] to compute Serre's reduction for different classes of multidimensional linear systems. In particular, it was proved that many multidimensional linear systems classically studied in the literature of differential time-delay systems or partial differential equations admit a Serre's reduction. The reason why is that their corresponding $D = k[x_1, \ldots, x_n]$-modules $D^q/(R D^p)$ are *0-dimensional*, namely, are finite-dimensional k-vector spaces.

The purpose of this talk is to constructively study the case of a left $D = A_n(k)$-module $M = D^{1 \times p}/(D^{1 \times q} R)$ satisfying that the right D-module E is "0-dimensional", which is called *holonomic* in the literature of partial differential systems [6]. Even if E is no longer a finite-dimensional k-vector space, we can prove the following interesting theorem.

Theorem 2 *Let k be a field of characteristic 0 (e.g., $\mathbb{R}$, $\mathbb{C}$), D the Weyl*

algebra $A_n(k)$, $R \in D^{q \times p}$ a full row rank matrix and $M = D^{1 \times p}/(D^{1 \times q} R)$ the left D-module finitely presented by R. If $E = D^q/(R D^p)$ is a holonomic right D-module and $p - q \geq 1$, then M is isomorphic to $L = D^{1 \times (p+1-q)}/(D Q_2)$, where the matrix Q_2 is defined as in Theorem 1. In particular, if $\mathcal{F}$ is a left D-module, then the linear partial differential system $\ker_{\mathcal{F}}(R.)$ is equivalent to a sole linear partial differential equation $\ker_{\mathcal{F}}(Q_2.)$. Finally, if $q \geq 3$, then there exist $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that $V R W = \mathrm{diag}(I_{q-1}, Q_2)$.

We give a constructive algorithm for Theorem 2 which is implemented in the package SERRE (see [4]) built upon OreModules [3] and Stafford [7]. We illustrate Theorem 2 on different explicit examples (e.g., linear elasticity, Hadamard's example, conjugated Beltrami equations). Moreover, using the main result of [8], we prove the following result.

Corollary 1 *If D is the ring of ordinary differential operators with coefficients respectively in the ring A of polynomial, formal power series or convergent power series over a field k of characteristic 0 for the first two cases and over $\mathbb{R}$ or $\mathbb{C}$ for the latter one and if $R \in D^{q \times p}$ is a full row rank matrix and $p - q \geq 1$, then the left D-module $M = D^{1 \times p}/(D^{1 \times q} R)$ is isomorphic to $L = D^{1 \times (p+1-q)}/(D Q_2)$ for a certain row vector $Q_2 \in D^{1 \times (p+1-q)}$ which can be computed as in Theorem 1. Moreover, if $q \geq 3$ then the matrices R and $\mathrm{diag}(I_{q-1}, Q_2)$ are equivalent.*

This talk is based on the results of [5].

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New factorization algorithm in the first q-Weyl algebra

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1. Introduction

This is a joint work with Viktor Levandovskyy, RWTH Aachen. In our talk we are going to present techniques to deal with factorization problems in polynomial q-Weyl algebras making use of their graded structure. We will discuss the applicability of the methods we already developed for Weyl algebras to the q-case. Furthermore, during the presentation some examples will be demonstrated live with our experimental implementation.

2. The first q-Weyl algebra as a graded ring

It is well known that there exists a nontrivial $\mathbb{Z}$-grading on the q-Weyl algebras over a field $\mathbb{K}(q)$. Let A_1 be the first q-Weyl algebra, that is a $\mathbb{K}(q)$-algebra generated by ∂ and x subject to the relation $\partial x = qx\partial + 1$. A_1 can be regarded as a graded algebra by assigning a weight v to x and $-v$ to ∂ for any $v \in \mathbb{Z}$. For our purposes, $\mathbb{K}$ is a field of characteristic zero and $v = 1$. For $n \in \mathbb{Z}$, the n-th graded part of A_1 is the vector space

$$A_1^{(n)} := \{ \sum_{j-i=n} r_{i,j} x^i \partial^j \,|\, i,j \in \mathbb{N}_0, r_{i,j} \in \mathbb{K}(q)\}.$$

Concentrating on the problem of factorizing these polynomials there are the following interesting observations.

Lemma 2.1. Let $\theta := x\partial$. Then $A_1^{(0)} = \mathbb{K}(q)[\theta]$. Moreover, $A_1^{(k)}$ is an $A_1^{(0)}$-module generated by the element x^k, if $k < 0$, or by ∂^k, if $k > 0$.

Lemma 2.2. The polynomials θ and $\theta + \frac{1}{q}$ are the only irreducible monic elements in $\mathbb{K}(q)[\theta]$ that are reducible in A_1.

Due to these lemmata, for $k > 0$ (resp. $k < 0$) and $g \in A_1^{(k)}$ there exists $f \in A_1^{(0)}$ such that

$$g = f\partial^k \quad (\text{resp. } g = fx^k).$$

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Thus, if we want to get one factorization of a $[-1, 1]$-homogeneous polynomial in the first q-Weyl algebra, it suffices to deal with a factorization of the above f, which is in $A_1^{(0)}$. The factorization of this element can be reduced to the commutative case as seen above. These facts lead to a simple and fast algorithm that delivers a factorization of a homogeneous polynomial, that is $A_1^{(k)} \ni f = \prod_{i=1}^m g_i$, $g_i \in A_1$. There is also an algorithm of combinatorial nature, which computes all possible polynomial factorizations of this kind of polynomials. The next section will focus on this algorithm and we will show, that our existing and already implemented algorithm to compute all homogeneous factorizations of a polynomial in the first Weyl algebra can be deduced from this one.

Remark 2.3. If we set $q = 1$, we get as a special case the first Weyl algebra. Everything claimed in this section of course also holds for that algebra.

3. Differences between the algorithms for q-Weyl and Weyl algebras

Lemma 3.1. Let $x^m \partial^m \in A_1$, $m \in \mathbb{N}$, and $\theta := x\partial$. Then the following identities hold for the first q-Weyl algebra:

$$x^m \partial^m = q^{-\frac{m(m+1)}{2}} \prod_{i=0}^{m-1} (\theta - [i]_q),$$

where $[n]_q$ denotes the so called q-bracket ($[n]_q := \frac{q^{n-1}-1}{q-1}$). Furthermore, we have

$$\begin{aligned} \theta d^n &= \frac{d^n}{q} \left(\frac{\theta - 1}{q^{n-1}} - \frac{q^{-n+2} - q}{1 - q} \right) \\ \theta x^n &= x^n \left(q^n \theta + [n]_q \right). \end{aligned}$$

Now we will compare these formulas to the already known formulas in the first Weyl algebra.

Corollary 3.2 (compare to (sst)). Let $x^m \partial^m$ be an element in the first Weyl algebra (relations: $\partial x = x\partial + 1$), $m \in \mathbb{N}$, and $\theta := x\partial$. Then the following identities hold:

$$x^m \partial^m = \prod_{i=0}^{m-1} (\theta - i).$$

Furthermore, we have

$$\begin{aligned} \theta x^m &= x^m (\theta + m) \\ \theta \partial^m &= \partial^m (\theta - m). \end{aligned}$$

These formulas are used in our algorithm to deduce all possible factorizations of a $[-1, 1]$-homogeneous polynomial from one given factorization of a polynomial in the first Weyl algebra. For the first q-Weyl algebra, these formulas differ slightly, as we have seen above.

Thus swapping x, ∂ and θ stays computationally feasible in the q-Weyl algebras and our algorithm for computing all factorizations of a $[-1, 1]$-homogeneous polynomial has got the same complexity for both first Weyl and q-Weyl algebra.

4. Implementation and timings

We compared our implementation in SINGULAR (procedure `facFirstWeyl` in the `ncfactor.lib` library, see (sing)) for the first Weyl algebra to the methods in MAPLE (Procedure: `DFactor` in the `DETools` library, see (mh)) and REDUCE (Procedure: `nc_factorize[_all]` in the library `NCPOLY`, see (rm)) before. We found out that homogeneous polynomials seem to be the worst case for MAPLE and REDUCE, while they form the best case for our implementation. Mark van Hoeij, the author of the algorithm implemented in MAPLE, conceded this point to us in a conversation at last year's ISSAC conference. For the q-Weyl case, we have not found any implementation for MAPLE or REDUCE yet. Thus, no comparison can be made here.

Example 4.1. First of all, the timings for a homogeneous polynomial in the first Weyl algebra:

$$h := x^{10}d^{10} + 25x^9d^9 + 201x^8d^8 + 615x^7d^7 + 660x^6d^6 + 190x^5d^5 + 6x^3d^3 + 24$$

MAPLE (Version 13):
Output: No factorization
Time: 1.01s

REDUCE (Version 3.8):
Output: Not Available
Time: Calc. stopped after nine hours

SINGULAR:
Output: $h = (x^5\partial^5 + 6)(x^5\partial^5 + x^3\partial^3 + 4)$,
$h = (x^5\partial^5 + x^3\partial^3 + 4)(x^5\partial^5 + 6)$
Time: 0.08s

Example 4.2. Now, we multiply the same two factors in the first q-Weyl algebra:

$$
\begin{aligned}
h \quad = \quad & q^{25}x^{10}d^{10} + q^{16}(q^4 + q^3 + q^2 + q + 1)^2 x^9 d^9 \\
& + q^9(q^{13} + 3q^{12} + 7q^{11} + 13q^{10} + 20q^9 + 26q^8 \\
& + 30q^7 + 31q^6 + 26q^5 + 20q^4 + 13q^3 + 7q^2 + 3q + 1)x^8 d^8 \\
& + q^4(q^9 + 2q^8 + 4q^7 + 6q^6 + 7q^5 + 8q^4 + 6q^3 + 4q^2 + 2q + 1) \\
& (q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)x^7 d^7 \\
& + q(q^2 + q + 1)(q^5 + 2q^4 + 2q^3 + 3q^2 + 2q + 1) \\
& (q^4 + q^3 + q^2 + q + 1)(q^2 + 1)(q + 1)x^6 d^6 \\
& + (q^{10} + 5q^9 + 12q^8 + 21q^7 + 29q^6 + 33q^5 \\
& + 31q^4 + 24q^3 + 15q^2 + 7q + 12)x^5 d^5 + 6x^3 d^3 + 24
\end{aligned}
$$

Singular:
Output: $h = (x^5\partial^5 + 6)(x^5\partial^5 + x^3\partial^3 + 4)$,
$h = (x^5\partial^5 + x^3\partial^3 + 4)(x^5\partial^5 + 6)$
Time: 2.93s[1]

Availability: The mentioned implementation can be found in the new Singular library ncfactor.lib, which is contained in the latest Singular version with experimental status (see `http://www.singular.uni-kl.de/Manual/latest/sing_1647.htm`).

5. Conclusion and future work

As a future work we are going to improve our algorithm for finding factorizations of inhomogeneous polynomials. Since our current algorithm utilizes exclusively the information about the factorizations of the homogeneous parts of the polynomials, our results will be applicable to the Weyl algebra as well as to the q-Weyl and shift algebra. Another interesting question is whether there are computable relations between the different factorizations in the inhomogeneous case.

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[1]This is the first result of our experimental implementation. The code will be more optimized in the further development.

Constructive D-module theory and applications. Extended Abstract.

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We sketch the algorithmic fundamentals of constructive D-module theory, develop the central character decomposition technique to D-modules and investigate the connection of Bernstein-Sato polynomial of a hypersurface with central characters. Also, we propose a stratification of an affine space into constructible sets, where on each stratum local Bernstein-Sato polynomial is constant. To each stratum a global $D[s]$-module is attached in a natural way.

1. Introduction

Let $\mathbb{K}$ be a field of characteristic zero and $R = \mathbb{K}[x_1, \ldots, x_n]$. In the sequel, $f \in R$ resp. $f_1, \ldots, f_r \in R$ will be non-constant polynomials. By GK. dim we denote the Gel'fand-Kirillov dimension, see e. g. [5].

We consider the n-th Weyl algebra as the algebra of linear partial differential operators with polynomials coefficients. That is $D_n = D(R) = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{\partial_i x_i = x_i \partial_i + 1, \partial_i x_j = x_j \partial_i, i \neq j\}\rangle$. We denote by $D_n[s] = D(R) \otimes_{\mathbb{K}} \mathbb{K}[s_1, \ldots, s_n]$ and drop the index n depending on the context.

1.1. Bernstein-Sato Polynomial of f

Let us recall Bernstein's construction. Given a non-zero polynomial $f \in R_n$ in n variables, we consider $M = R_n[s, \frac{1}{f}] \cdot f^s$, the free $R_n[s, \frac{1}{f}]$-module of rank one generated by the formal symbol f^s. Then M has a natural structure of left $D_n[s]$-module.

$$\partial_i(g(s,x) \cdot f^s) = \left(\frac{\partial g}{\partial x_i} + sg(s,x)\frac{\partial f}{\partial x_i}\frac{1}{f} \right) \cdot f^s \ \in M \tag{1}$$

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Theorem 1.1 (Bernstein [1])**.** For $f \in R_n$ there exists a non-constant polynomial $b_f(s) \in \mathbb{K}[s]$ and a differential operator $P(s) \in D_n[s]$ such that

$$P(s)f \cdot f^s = b_f(s) \cdot f^s \;\in\; R_n[s, \tfrac{1}{f}] \cdot f^s = M. \tag{2}$$

The monic polynomial $b_f(s)$ of minimal degree, satisfying (2) is called the **(global) Bernstein-Sato polynomial**. Remarkably, $b(s)$ has only negative integer roots.

Let $\mathrm{Ann}_{D_n[s]}(f^s)$ be the left ideal of elements from $D[s]$, annihilating f^s. Indeed we have $\mathrm{Ann}_{D_n[s]}(f^s) \cap \mathbb{K}[x, s] = 0$ and

$$\langle b_f(s) \rangle = (\mathrm{Ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{K}[s]. \tag{3}$$

We are interested in the structure of the $D[s]$-module $D[s]/(\mathrm{Ann}_{D_n[s]}(f^s) + \langle f \rangle)$.

Theorem 1.2 (L.-M., [4])**.** Let R be a $\mathbb{K}$-algebra, whose center contains $\mathbb{K}[s]$. Let $q(s) \in \mathbb{K}[s]$ and I a left ideal in R satisfying $I \cap \mathbb{K}[s] \neq 0$. The following equalities hold:
(1) $\big(I + R\langle q(s) \rangle\big) \cap \mathbb{K}[s] = I \cap \mathbb{K}[s] + \mathbb{K}[s]\langle q(s) \rangle$,
(2) $\big(I : q(s)\big) \cap \mathbb{K}[s] = \big(I \cap \mathbb{K}[s]\big) : q(s)$,
(3) $\big(I : q(s)^\infty\big) \cap \mathbb{K}[s] = \big(I \cap \mathbb{K}[s]\big) : q(s)^\infty$.

Corollary 1.3. Let m_α be the multiplicity of α as a root of $b_f(-s)$. Consider the ideals $I = \mathrm{Ann}_{D_n[s]}(f^s) + \langle f \rangle$, $J_i = I + \langle (s + \alpha)^{i+1} \rangle \subseteq D_n[s]$, $i = 0, \dots, n$. The following conditions are equivalent:
(1) $m_\alpha > i$,
(2) $J_i \cap \mathbb{K}[s] = \langle (s + \alpha)^{i+1} \rangle$,
(3) $(s + \alpha)^i \notin J_i$,
(4) $\big(I : (s + \alpha)^i\big) + D_n[s]\langle s + \alpha \rangle \neq D_n[s]$,
(5) $\big(I : (s + \alpha)^i\big)|_{s=-\alpha} \neq D_n$.
Moreover if $D_n[s] \supsetneq J_0 \supsetneq J_1 \supsetneq \cdots \supsetneq J_{m-1} = J_m$, then $m_\alpha = m$. In particular, $m \leq n$ and $J_{m-1} = J_m = \cdots = J_n$.

1.2. Local Bernstein-Sato Polynomial

For simplicity, let $\mathbb{K} = \bar{\mathbb{K}}$. Recall, that the singular locus of $V(f)$ is $V(\langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle)$. One can define the **local Bernstein-Sato polynomial** as follows. Let $p \in \mathbb{K}^n$ be a point and $\mathfrak{m}_p = \langle \{x_1 - p_1, \dots, x_n - p_n\} \rangle \subset R_n$ the corresponding maximal ideal. Let D_p be the local Weyl algebra at p, that is Weyl algebra with coefficients from $\mathbb{K}[x_1, \dots, x_n]_p$ instead of $R_n = \mathbb{K}[x_1, \dots, x_n]$. From the Bernstein's functional equation (2) it follows that $\exists P(s) \in D[s], b_f(s) \in \mathbb{K}[s]$, such that $P(s)f \cdot f^s = b(s) \cdot f^s$ holds. Hence, since in $\mathbb{K}[x_1, \dots, x_n]_p$ we have non-constant units, $\exists P_p(s) \in D_p[s], b_{f,p}(s) \in \mathbb{K}[s]$, such that $P_p(s)f \cdot f^s = b_{f,p}(s) \cdot f^s$ holds. We define local Bernstein-Sato polynomial to be the univariate monic polynomial $b_{f,p}(s)$ of the minimal degree, such that the above identity holds.

Theorem 1.4. (Briançon-Maisonobe, Mebkhout-Narváez) Let $b_{f,p}(s)$ the local Bernstein-Sato polynomial of f at the point $p \in \mathbb{K}^n$ and $b_f(s)$ the global one. Then $b_f(s) = \mathrm{lcm}_{P \in \mathbb{K}^n} b_{f,P}(s) = \mathrm{lcm}_{P \in \Sigma(f)} b_{f,P}(s)$.

1.3. Central Character Decomposition

Let $\mathbb{K} = \bar{\mathbb{K}}$, A be a $\mathbb{K}$-algebra, $Z = Z(A)$ the center of A and M be an A-module. Let $Z^* := \mathrm{Hom}_{\mathbb{K}}(Z, \mathbb{K})$. For $\chi \in Z^*$, define the generalized χ-weight subspace of M to be

$$M^\chi = \left\{ v \in M \mid \exists n(v) \in \mathbb{N}, \forall z \in Z, \, (z - \chi(z))^{n(v)} v = 0 \right\}.$$

$Supp_Z M = \{\chi \in Z^* | M^\chi \neq 0\}$ is the **central support** of M. We say that M possesses a generalized weight decomposition if $M = \bigoplus_{\chi \in Z^*} M^\chi$.

Lemma 1.5. Let M be a finitely presented A-module, that is $M \cong A^N / I_M$ for a left submodule $I_M \subset A^N$. Let e_j be a canonical unit vector on A^N. Then the **pre-annihilator of** M, $\mathrm{preAnn}(M) = \bigcap_{j=1}^N \mathrm{Ann}_A^M e_j$ is a left ideal and
(1) $Z \cap \mathrm{preAnn}(M) = Z \cap \mathrm{Ann}_A M$, (where $\mathrm{Ann}_A M$ is a two-sided ideal),
(2) the Zariski closure of $Supp_Z M$ equals $V(\mathrm{preAnn}(M) \cap Z)$.

Theorem 1.6 ([3]). Suppose that for $M \cong A^N / I_M$ we have $| Supp_Z M | < \infty$. Then $M = \bigoplus_{\chi \in Z^*} M^\chi$. Moreover, $M^\chi \cong A^N / (I_M : J_\chi^\infty)$, where $J_\chi = \cap_{\psi \in Supp_Z M \setminus \{\chi\}}$. Note, that these computations are algorithmic.

Theorem 1.7 (L.-M.). Let $\mathbb{K}$ be a field of characteristic zero. Consider the $D[s]$-module $M = D[s]/J$ for $J = \mathrm{Ann}_{D[s]} f^s + \langle f \rangle$. Then
(1) The center of $D[s]$: $Z(D \otimes_{\mathbb{K}} \mathbb{K}[s]) = \mathbb{K}[s]$.
(2) The annihilator of the module $\mathrm{Ann}_{D[s]}(D[s]/J) = D[s]\langle b_f(s) \rangle$.
(3) Since $|Supp_{\mathbb{K}[s]}(D[s]/J)|$ is the number of different roots of $b_f(s)$, M possesses finite generalized weight decomposition

$$M = D[s]/J = D[s]/(\mathrm{Ann}_{D[s]} f^s + \langle f \rangle) = \bigoplus M^\chi,$$

where $\ker \chi = \mathbb{K}[s]\langle s - \chi(s) \rangle$, where $b_f(\chi(s)) = 0$.
(4) $\mathrm{GK.\,dim}(D[s]/J) = n$, $\mathrm{GK.\,dim}(D[s]/\langle b_f(s) \rangle) = 2n$ and thus the module $\mathrm{Ann}_{D[s]}(D[s]/J)$ is generalized holonomic, see [5].

2. Stratification

It is possible to construct a stratification of $\mathbb{C}^n$ in such a way, that $b_{f,p}(s)$ is constant on each stratum. For the first time it's been suggested by Oaku [7], see also [6]. We have presented algorithmic treatment of the stratification using roots of Bernstein-Sato polynomial in [4].

Theorem 2.1 (L.–M., [4]). Let $I = \mathrm{Ann}_{D[s]}(f^s) + D[s]\langle f \rangle$. Consider the ideals $I_{\alpha,i} = \left(I : (s+\alpha)^i\right) + D[s]\langle s+\alpha \rangle$, for α root of $b_f(s)$ and $i = 0, \ldots, m_\alpha - 1$. Then one has

$$m_\alpha(p) > i \iff p \in V(I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}]).$$

Let $V_{\alpha,i}$ be the affine variety corresponding to the ideal $I_{\alpha,i} \cap \mathbb{C}[\mathbf{x}]$. Then

$$\emptyset =: V_{\alpha,m_\alpha} \subset V_{\alpha,m_\alpha - 1} \subset \cdots \subset V_{\alpha,0} \subset V_{\alpha,-1} := \mathbb{C}^n, \tag{4}$$

and $m_\alpha(p) = i$ if and only if $p \in V_{\alpha,i-1} \setminus V_{\alpha,i}$. The exposition would not be complete without an example.

Example 2.2. Consider $f = (x^2 + 9/4y^2 + z^2 - 1)^3 - x^2 z^3 - 9/80 y^2 z^3 \in \mathbb{C}[x, y, z]$. The global Bernstein-Sato polynomial is $b_f(s) = (s+1)^2 (s+4/3)(s+5/3)(s+2/3)$.

Take $V_1 = V(x^2 + 9/4 y^2 - 1, z)$, $V_2 = V(x, y, z^2 - 1)$ and $V_3 = V(19x^2 + 1, 171y^2 - 80, z)$. Then V_2 (resp. V_3) consists of two (resp. four) different points and $V_3 \subset V_1$, $V_1 \cap V_3 = \emptyset$. The singular locus of f is union of V_1 and V_2. The stratification associated with each root of $b_f(s)$ is given by

$$\alpha = -1, \qquad \emptyset \subset V_1 \subset V(f) \subset \mathbb{C}^3;$$
$$\alpha = -4/3, \quad \emptyset \subset \quad V_1 \cup V_2 \quad \subset \mathbb{C}^3;$$
$$\alpha = -5/3, \quad \emptyset \subset \quad V_2 \cup V_3 \quad \subset \mathbb{C}^3;$$
$$\alpha = -2/3, \quad \emptyset \subset \qquad V_1 \qquad \subset \mathbb{C}^3.$$

From this, one constructs a stratification of $\mathbb{C}^3$ into constructible sets such that $b_{f,p}(s)$ is constant on each stratum.

$$b_{f,p}(s) = \begin{cases} 1 & p \in \mathbb{C}^3 \setminus V(f), \\ s+1 & p \in V(f) \setminus (V_1 \cup V_2), \\ (s+1)^2(s+4/3)(s+2/3) & p \in V_1 \setminus V_3, \\ (s+1)^2(s+4/3)(s+5/3)(s+2/3) & p \in V_3, \\ (s+1)(s+4/3)(s+5/3) & p \in V_2. \end{cases}$$

Remark 2.3. As we have seen above, M^χ can be further decomposed equidimensionally into a sum (not necessarily direct) of holonomic $D[s]$-modules if the multiplicity of the corresponding root of $b_f(s)$ is bigger than 1. This decomposition respects different multiplicities. Moreover, to each stratum defined above, we can associate a global $D[s]$-module by summing modules with corresponding central characters (involving multiplicities). It is then holonomic $D[s]$-module, we conjecture that it is also a holonomic D-module as well. Moreover, this can be seen as a decomposition of the original module into a sum, respecting the stratification.

We are working on the natural generalization of the above techniques to provide a stratification and module decomposition, corresponding to Bernstein-Sato polynomial of a variety [2].

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Invariants of Difference Field Extensions

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Abstract

In this paper presented at the Second Workshop on Differential Equations by Algebraic Methods (RISC, Johannes Kepler University, Linz, Austria, February 9-11, 2011) we consider invariants of a finitely generated difference field extension, that is, characteristics of such an extension, which do not depend on the set of its difference generators. Most of the invariants (e.g., difference transcendence degree, difference type, and typical difference transcendence degree) are carried by univariate and multivariate difference dimension polynomials; they play an important role in the description of transcendental difference field extensions. However, there are also invariants of algebraic difference field extensions that are not determined by dimension polynomials; we will also discuss properties of such invariants.

Key words: Difference field, difference dimension polynomial, difference transcendence degree, difference type, limit degree, distant degree

Introduction. Let K be a difference field with basic set $\sigma = \{\alpha_1, \dots, \alpha_m\}$ (also called a σ-field), that is, a field considered together with mutually commuting endomorphisms α_i of K. We assume that $\text{Char } K = 0$. If K is *inversive*, that is, all α_i are automorphisms, we set $\sigma^* = \{\alpha_1, \dots, \alpha_m, \alpha_1^{-1}, \dots, \alpha_m^{-1}\}$ and call K a σ^*-field. In what follows, T will denote the free commutative semigroup $\{\tau = \alpha_1^{k_1} \dots \alpha_m^{k_m} \mid k_1, \dots, k_m \in \mathbf{N}\}$ generated by σ. (As usual, $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{Q}$ denote the sets of all nonnegative integers, integers, and rational numbers, respectively.) The number $ord\,\tau = k_1 + \dots + k_m$ is called the *order* of τ. If K is inversive, then Γ will denote the free commutative group generated by σ. The order of an element $\gamma = \alpha_1^{k_1} \dots \alpha_m^{k_m} \in \Gamma$ $(k_1, \dots, k_m \in \mathbf{Z})$ is defined by $ord\,\gamma = |k_1| + \dots + |k_m|$. If $r \in \mathbf{N}$, we set $T(r) = \{\tau \in T \mid ord\,\tau \leq r\}$ and $\Gamma(r) = \{\gamma \in \Gamma \mid ord\,\gamma \leq r\}$.

If K_0 is a subfield of K and $\alpha(K_0) \subseteq K_0$ for any $\alpha \in \sigma$, we say that K_0 is a difference (or σ-) subfield of K and K is a difference (σ-) field extension of K_0. We also say that we have a σ-field extension K/K_0. If $B \subseteq K$, then the intersection of all σ-subfields of K containing K_0 and B is called the difference (σ-) subfield of K generated by the set

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of σ-*generators* B over K_0; it is denoted by $K_0\langle B\rangle$. As a field, $K_0\langle B\rangle = K_0(\{\tau(b)|b \in B, \tau \in T\})$. If $B = \{b_1, \ldots, b_k\}$, we say that K/K_0 is finitely generated and write $K = K_0\langle b_1, \ldots, b_k\rangle$.

If a σ-field K is inversive, then a subfield K_0 of K is said to be a σ^*-subfield of K (and K is said to be a σ^*-field extension of K_0) if $\alpha(K_0) \subseteq K_0$ for any $\alpha \in \sigma^*$. If $B \subseteq K$, then the smallest σ^*-subfield of K containing K_0 and B is called the inversive difference (or σ^*-) subfield of K generated by the set B over K_0; it is denoted by $K_0\langle B\rangle^*$, and B is called the set of σ^*-generators of K/K_0. As a field, $K_0\langle B\rangle^* = K_0(\{\gamma(b)|b \in B, \gamma \in \Gamma\})$.

Let L be a σ-field extension of K. A set $U \subseteq L$ is said to be σ-*algebraically dependent* over K, if the set $TU = \{\tau(u)|\tau \in T, u \in U\}$ is algebraically dependent over the field K. Otherwise, we say that U is σ-*algebraically independent* over K. A set $B \subseteq L$ is called a difference (σ-) transcendence basis of L over K if B is a maximal σ-algebraically independent over K subset of L. It is known (see (5, Chapter 4)) that any two σ-transcendence bases of L/K have the same cardinality. The *difference (σ-) transcendence degree* of L over K, denoted by σ-$trdeg_K L$, is the number of elements of any σ-transcendence basis of L over K, if this number is finite, or infinity in the contrary case. The difference transcendence degree is additive at towers $K \subseteq L \subseteq M$ of difference fields. Also, any family of σ-generators of L over K contains a σ-transcendence basis of L/K. If K is inversive and L a σ^*-field extension of K, then any system of σ^*-generators of L over K contains a σ-transcendence basis of L over K.

2. Difference Dimension Polynomials and their Invariants. In what follows we introduce certain numerical polynomials associated with finitely generated difference field extensions. Their properties provide a technique for the study of difference fields and system of algebraic difference equations. Also, these polynomials carry invariants of the extensions, that is, numbers that do not depend on the systems of difference generators. Some properties and methods of computation of dimension polynomials can be found in (4, Chapter 6) and (5, Chapter 4).

Theorem 1. Let K be a difference field with a basic set $\sigma = \{\alpha_1, \ldots, \alpha_m\}$, let $L = K\langle \eta_1, \ldots, \eta_n\rangle$ be a σ-field extension of K generated by a finite family $\eta = \{\eta_1, \ldots, \eta_n\}$, and for any $r \in \mathbf{N}$, let $L_r = K(\{\tau\eta_j \,|\, \tau \in T(r), 1 \leq j \leq n\})$. Then there exists a polynomial $\phi_{\eta|K}(t) \in \mathbf{Q}[t]$ such that

(i) $\phi_{\eta|K}(r) = trdeg_K L_r$ for all sufficiently large $r \in \mathbf{N}$.

(ii) $deg\,\phi_{\eta|K}(t) \leq m$ and $\phi_{\eta|K}(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i} = \frac{a_m}{m!}t^m + o(t^m)$ where $a_i \in \mathbf{Z}$ $(1 \leq i \leq m)$ and $deg\,o(t^m) < m$.

(iii) The integers $a_m, d = deg\,\phi_{\eta|K}(t)$ and a_d do not depend on the choice of a system of σ-generators η. Furthermore, $a_m = \sigma$-$trdeg_K L$.

(iv) If $\eta_1, \ldots, \eta_n$ are σ-algebraically independent over K, then $\phi_{\eta|K}(t) = n\binom{t+m}{m}$.

The polynomial $\phi_{\eta|K}(t)$ is called the *difference* (or σ-) *dimension polynomial* of the difference field extension L/K associated with the system of σ-generators η. The integers $d = deg\,\phi_{\eta|K}(t)$ and a_d are called, respectively, the *difference* (or σ-) *type* and *typical difference* (or σ-) *transcendence degree* of L over K. These invariants of $\phi_{\eta|K}(t)$ are denoted by σ-$type_K L$ and σ-$t.trdeg_K L$, respectively.

Theorem 2. Let K be an inversive difference $(\sigma^*\text{-})$ field, let $L = K\langle \eta_1, \ldots, \eta_n \rangle^*$ be a σ^*-field extension of K generated by a finite family $\eta = \{\eta_1, \ldots, \eta_n\}$, and for any $r \in \mathbf{N}$, let $L_r^* = K(\{\gamma\eta_j \mid \gamma \in \Gamma(r), 1 \le j \le n\})$. Then there exists a polynomial $\psi_{\eta|K}(t) \in \mathbf{Q}[t]$ such that

(i) $\psi_{\eta|K}(r) = trdeg_K L_r^*$ for all sufficiently large $r \in \mathbf{N}$.

(ii) $deg\,\psi_{\eta|K}(t) \le m$ and $\psi_{\eta|K}(t) = \dfrac{2^m a}{m!}\,t^m + o(t^m)$ where $a \in \mathbf{Z}$ and $deg\,o(t^m) < m$.

(iii) a, $d = \deg\psi_{\eta|K}(t)$ and the coefficient of t^d do not depend on the choice of a system of σ^*-generators η. Furthermore, $a = \sigma\text{-}trdeg_K L$.

(iv) If $\eta_1, \ldots, \eta_n$ are σ-algebraically independent over K, then

$$\psi_{\eta|K}(t) = n\sum_{k=0}^{m}(-1)^{m-k}2^k\binom{m}{k}\binom{t+k}{k}.$$

More general, if F is any intermediate σ^*-field of the extension L/K, then there exists a polynomial $\psi_{K,F,\eta}(t) \in \mathbf{Q}[t]$ such that $\psi_{K,F,\eta}(r) = tr.deg_K(F\bigcap L_r^*)$ for all sufficiently large $r \in \mathbf{Z}$, $\deg\psi_{K,F,\eta} \le m$, and the polynomial $\psi_{K,F,\eta}(t)$ can be represented as $\psi_{K,F,\eta}(t) = \dfrac{2^m b}{m!}\,t^m + o(t^m)$ where $\deg o(t^m) < m$. Furthermore, $d = \deg\phi_{K,F,\eta}$ does not depend on η and the same is true for the coefficient of t^d, which has the form $\dfrac{2^d a_d}{d!}$ with $a_d \in \mathbf{N}$. Finally, $b = \sigma\text{-}trdeg_K F$.

The polynomial $\psi_{\eta|K}(t)$ is called the σ^*-*dimension polynomial* of L/K associated with the system of σ^*-generators η. The numbers $d = deg\,\psi_{\eta|K}$ and a_d are called, respectively, the *inversive difference* (or σ^*-) *type* and *typical inversive difference* (or *typical* σ^*-) *transcendence degree* of L over K. They are denoted by $\sigma^*\text{-}type_K L$ and $\sigma^*\text{-}t.trdeg_K L$, respectively.

Let (K, σ) and (K, σ_1) be inversive difference fields with the same underlying field K and with basic sets $\sigma = \{\alpha_1, \ldots, \alpha_m\}$ and $\sigma_1 = \{\tau_1, \ldots, \tau_m\}$, respectively. The sets σ and σ_1 are said to be *equivalent* (we write $\sigma \sim \sigma_1$) if there exists a matrix $K = (k_{ij})_{1\le i,j\le m} \in GL(m, \mathbf{Z})$ such that $\alpha_i = \tau_1^{k_{i1}}\ldots\tau_n^{k_{im}}$ $(1 \le i \le m)$.

Theorem 3. Let K be a σ^*-field with a basic set $\sigma = \{\alpha_1, \ldots, \alpha_m\}$, L a finitely generated σ^*-field extension of K, and $d = \sigma^*\text{-}type_K L$. Then there exists a set $\sigma_1 = \{\beta_1, \ldots, \beta_m\}$ of mutually commuting automorphisms of L and a finite family $\zeta = \{\zeta_1, \ldots, \zeta_q\}$ of elements of L such that $\sigma_1 \sim \sigma$ and L is an algebraic extension of the field $H = K\langle \zeta_1, \ldots, \zeta_q \rangle_{\sigma_2}^*$ where $\sigma_2 = \{\beta_1, \ldots, \beta_d\}$. (The last field is a finitely generated σ_2^*-field extension of K when K is treated as an inversive difference field with the basic set σ_2.)

Let K be a σ^*-field, $\sigma = \{\alpha_1, \ldots, \alpha_m\}$, $L = K\langle \eta_1, \ldots, \eta_n \rangle^*$, $\mathfrak{U}$ the set of all intermediate σ^*-fields of L/K, and $\mathfrak{B}_{\mathfrak{U}} = \{(F, E) \in \mathfrak{U} \times \mathfrak{U} \mid F \supseteq E\}$. Then it is easy to see that there is a unique mapping $\mu_{\mathfrak{U}} : \mathfrak{B}_{\mathfrak{U}} \to \mathbf{Z}\bigcup\{\infty\}$ such that $\mu_{\mathfrak{U}}(F, E) \ge -1$ for all $(F, E) \in \mathfrak{B}_{\mathfrak{U}}$ and if $d \in \mathbf{N}$, then $\mu_{\mathfrak{U}}(F, E) \ge d$ if and only if $trdeg_E F > 0$ and there exists an infinite descending chain of intermediate σ^*-fields $F = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_r \supseteq \cdots \supseteq E$ such that $\mu_{\mathfrak{U}}(F_i, F_{i+1}) \ge d - 1$ $(i = 0, 1, \ldots)$.

We define the *difference* (σ-) *transcendental type* of the σ^*-field extension L/K as $\sigma\text{-}tr.type(L/K) = \sup\{\mu_{\mathfrak{U}}(F, E) \mid (F, E) \in \mathfrak{B}_{\mathfrak{U}}\}$ and the *difference* (σ-) *transcendence*

dimension of L/K as $\sigma\text{-}tr.dim(L/K) = \sup\{q \in \mathbf{N} \,|\, \text{there exists a chain } F_0 \supset F_1 \supset \cdots \supset F_q \text{ such that } F_i \in \mathfrak{U} \text{ and } \mu_{\mathfrak{U}}(F_{i-1}, F_i) = \sigma\text{-}tr.type(L/K) \text{ for } i = 1, \ldots, q\}.$

Theorem 4. With the above notation, $\sigma\text{-}tr.type(L/K) \le \sigma^*\text{-}type_K L \le m$. Furthermore, if $\sigma\text{-}trdeg_K L > 0$, then $\sigma\text{-}tr.type(L/K) = m$ and $\sigma\text{-}tr.dim(L/K) = \sigma\text{-}trdeg_K L$. Finally, if $\sigma\text{-}trdeg_K L = 0$, then $\sigma\text{-}tr.type(L/K) < m$.

3. Multivariate Dimension Polynomials. Let K be a difference (σ-) field and let a partition of the basic set $\sigma = \{\alpha_1, \ldots, \alpha_m\}$ into a disjoint union of its subsets be fixed:

$$\sigma = \sigma_1 \cup \cdots \cup \sigma_p \tag{1}$$

where $Card\,\sigma_i = m_i$ $(1 \le i \le p)$. For any $\tau = \alpha_1^{k_1} \ldots \alpha_m^{k_m} \in T$ and $i = 1, \ldots, p$, we define the order of τ with respect to σ_i as $ord_i\tau = \sum_{j \in \sigma_i} k_j$ and set $T(r_1, \ldots, r_p) = \{\tau \in T \,|\, ord_i\tau \le r_i \ (1 \le i \le p)\}$ for any $r_1, \ldots, r_p \in \mathbf{N}$. Also, for any permutation $(j_1, \ldots, j_p)$ of the set $\{1, \ldots, p\}$, we define the lexicographic order $\le_{j_1, \ldots, j_p}$ on $\mathbf{N}^p$ as follows: $(r_1, \ldots, r_p) \le_{j_1, \ldots, j_p} (s_1, \ldots, s_p)$ if and only if $r_{j_1} < s_{j_1}$ or there is $k \in \mathbf{N}$, $1 \le k \le p - 1$, such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \ldots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$. If $\Sigma \subseteq \mathbf{N}^p$, then Σ' will denote the set $\{e \in \Sigma \,|\, e \text{ is a maximal element of } \Sigma \text{ with respect to one of the } p! \text{ lexicographic orders } \le_{j_1, \ldots, j_p}\}$.

Theorem 5. Let $L = K\langle \eta_1, \ldots, \eta_n \rangle$. Then there exists a polynomial in p variables $\phi_\eta(t_1, \ldots, t_p) \in \mathbf{Q}[t_1, \ldots, t_p]$, $deg\,\phi \le m$, such that $\phi_\eta(r_1, \ldots, r_p) = trdeg_K K(\{\tau\eta_i \,|\, \tau \in T(r_1, \ldots, r_p), 1 \le j \le n\})$ for all sufficiently large $r_1, \ldots, r_p \in \mathbf{N}$, $deg_{t_i}\phi_\eta \le m_i$ $(1 \le i \le p)$, and the polynomial ϕ_η can be written as $\phi_\eta = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} a_{i_1 \ldots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p}$

where $a_{i_1 \ldots i_p} \in \mathbf{Z}$ for all $i_1, \ldots, i_p$. Furthermore, if Σ is the set of all p-tuples $(i_1, \ldots, i_p)$ such that $a_{i_1 \ldots i_p} \ne 0$, then $d = deg\,\phi_\eta$, $a_{m_1 \ldots m_p}$, all $(j_1, \ldots, j_p) \in \Sigma'$, the corresponding $a_{j_1 \ldots j_p}$, and the coefficients of all terms of degree d do not depend on the choice of the system of σ-generators η. Also, $a_{m_1 \ldots m_p} = \sigma\text{-}trdeg_K L$.

A similar result holds for inversive difference fields as well (in this case the order of an element $\gamma = \alpha_1^{k_1} \ldots \alpha_m^{k_m} \in \Gamma$ with respect to σ_i is defined as $\sum_{j \in \sigma_i} |k_j|$ and the sets $\Gamma(r_1, \ldots, r_p)$ are defined accordingly). The corresponding polynomial in p variables associated with a σ^*-field extension $L = K\langle \eta_1, \ldots, \eta_n \rangle^*$ can be represented in the same form as the above polynomial ϕ_η and it has similar invariants with an additional property $2^m | a_{m_1 \ldots m_p}$ and the equality $\dfrac{a_{m_1 \ldots m_p}}{2^m} = \sigma\text{-}trdeg_K L$ instead of the last equality of Theorem 5.

4. Limit Degree. Let K be a difference field with a basic set $\sigma = \{\alpha_1, \ldots, \alpha_m\}$ and let $\preccurlyeq$ be an order on the free semigroup T such that $\tau = \alpha_1^{k_1} \ldots \alpha_m^{k_m} \preccurlyeq \tau' = \alpha_1^{l_1} \ldots \alpha_m^{l_m}$ if and only if $(k_m, \ldots, k_1) <_{lex} (l_m, \ldots, l_1)$. Furthermore, for any $r_1, \ldots, r_m \in \mathbf{N}$, we set $T_{\preccurlyeq}(r_1, \ldots, r_m) = \{\tau \in T \,|\, \tau \preccurlyeq \alpha_1^{r_1} \ldots \alpha_m^{r_m}\}$ and extend this notation to the case when $r_i = \infty$ for some i (with the condition $k < \infty$ for any $k \in \mathbf{N}$). Let $L = K\langle S \rangle$ be a σ-field extension of K generated by a finite set S and for any $(r_1, \ldots, r_m) \in \mathbf{N}^m$ with $r_1 \ge 1$, let $d(S; r_1, \ldots, r_m) = [K(T_{\preccurlyeq}(r_1, \ldots, r_m)(S)) : K(T_{\preccurlyeq}(r_1 - 1, \ldots, r_m)(S))]$.

Lemma 6. $d(S; r_1, \ldots, r_m) \ge d(S; r_1 + p_1, \ldots, r_m + p_m)$ for any $p_1, \ldots, p_m \in \mathbf{N}$.

Thus, if some $d(S; r_1, \ldots, r_m)$ is finite, then $d(S) = \min\{d(S; r_1, \ldots, r_m) | r_1, \ldots, r_m \in \mathbf{N}\}$ is finite. If $d(S; r_1, \ldots, r_m) = \infty$ for all $(r_1, \ldots, r_m) \in \mathbf{N}^m$, we set $d(S) = \infty$.

Lemma 7. If $K\langle S \rangle = K\langle S' \rangle$ for two finite sets S and S', then $d(S) = d(S')$.

It follows that if $L = K\langle S \rangle$ ($Card\, S < \infty$), then $d(S)$ does not depend on the set of generators S. This characteristic of the finitely generated difference field extension is called the **limit degree** of L/K and denoted by $ld(L/K)$. This concept was introduced in (1) (see also (2, Chapter 5)) for the ordinary case; we are presenting a generalization of the notion of limit degree to the partial case.

If L/K is not finitely generated, then $ld(L/K)$ is defined as the maximum of limit degrees of finitely generated difference subextensions of L/K if this maximum exists, or ∞ otherwise.

Theorem 8. Let L/K be a σ-field extension. Then
    (i) If $\sigma\text{-}trdeg_K L > 0$, then $ld(L/K) = \infty$.
    (ii) If L/K is finitely generated and $\sigma\text{-}trdeg_K L = 0$, then $ld(L/K) < \infty$.
    (iii) If M is a σ-field extension of L, then $ld(M/K) = [ld(M/L)][ld(L/K)]$.

One of the important applications of the properties of limit degree is the following result: *if M is a finitely generated σ-field extension of a difference (σ-) field K, and L an intermediate difference field of M/K, then the σ-field extension L/K is finitely generated.*

Let K be an ordinary difference field with a basic set $\sigma = \{\alpha\}$ and $L = K\langle S \rangle$, $Card\, S < \infty$. Then the core L_K of L over K is defined to be the set of elements $a \in L$ algebraic and separable over K and such that $ld(K\langle a \rangle / K) = 1$. If K is inversive, we also have (see (5, Chapter 4)) $L_K = \bigcap_{n=0}^{\infty} K\langle \alpha^n(S) \rangle$.

The next theorem shows that core plays an important role in the study of the problem of compatibility. Difference field extensions L/K and M/K are called **compatible** if there are difference K-isomorphisms of L and M into some difference field extension N of K. Otherwise, the extensions are called *incompatible*.

Theorem 9. *(Criterion of compatibility)* Let K be an ordinary difference field with a basic set σ and let L and M be two σ-field extensions of K. Then the following statements are equivalent.
    (i) L/K and M/K are incompatible.
    (ii) L_K/K and M_K/K are incompatible.
    (iii) L_K/K and M/K are incompatible.

5. Distant Degree. In what follows we present some results of the recent work (3) where the authors introduce a new invariant of an ordinary difference field extension closely related to the concept of limit degree.

Let K be an ordinary difference field with a basic set $\sigma = \{\alpha\}$. Let K^* be the inversive closure of K, that is the unique (up to a difference K-isomorphism) difference field extension of K such that K^* is inversive and for any inversive difference field extension L of K, there is a difference K-isomorphism of K^* into L. We will work in some large σ^*-field $\mathcal{U}$ that contains all difference fields we are going to consider. If $a = (a_1, \ldots, a_n)$ is an n-tuple over K ($a_i \in \mathcal{U}$), then the field $K\langle a_1, \ldots, a_n \rangle$ and $K(a_1, \ldots, a_n)$ will be denoted by $K\langle a \rangle$ and $K(a)$, respectively. Clearly, if $\alpha(a)$ is algebraic over $K(a)$, that is, every $\alpha(a_i)$ is

algebraic over $K(a)$, then the limit degree $ld(K\langle a\rangle/K) = \lim_{k\to\infty}[K(a,\ldots,\alpha^k(a),\alpha^{k+1}(a)) : K(a,\ldots,\alpha^k(a))]$ is finite; it is also denoted by $ld(a/K)$.

The **inverse limit degree** of a over K, denoted by $ild(a/K)$ or $ild(K\langle a\rangle/K)$, is defined as $\lim_{k\to\infty}[K^*(a,\alpha^{-1}(a),\ldots,\alpha^{-(k+1)}(a)) : K^*(a,\alpha^{-1}(a),\ldots,\alpha^{-k}(a))]$.

The **distant degree** $dd(a/K)$ of a over K (also called the distant degree of the σ-field extension $K\langle a\rangle/K$ and denoted by $dd(K\langle a\rangle/K)$) is defined by
$dd(a/K) = \lim_{k\to\infty}[K(a,\alpha^k(a)) : K(a)]^{\frac{1}{k}}$.

The **inverse distant degree** $idd(a/K)$ of a over K (also called the inverse distant degree of the σ-field extension $K\langle a\rangle/K$ and denoted by $idd(K\langle a\rangle/K)$) is defined by
$idd(a/K) = \lim_{k\to\infty}[K^*(a,\alpha^{-k}(a) : K^*(a))]^{\frac{1}{k}}$.

Lemma 10. Let a and b be tuples in $\mathcal{U}$ such that b and $\alpha(a)$ are algebraic over $K(a)$ and $\alpha(b)$ is algebraic over $K(b)$. Then

(i) $dd(b/K) \leq dd(a/K)$.

(ii) $ld(K\langle a,b\rangle/K)ild(a/K) = ild(K\langle a,b\rangle/K)ld(a/K)$.

(iii) There is a constant D such that for every $k > 0$, $[K(a,\alpha^k(a)) : K(a)] \leq D[K^*(a,\alpha^k(a)) : K^*(a)]$.

Theorem 11. Let K be an ordinary σ^*-field and let a be a tuple over K such that $\alpha(a)$ is algebraic over $K(a)$ and $ld(a/K) = [K(a,\alpha(a)) : K(a)]$. Then

(i) $\{[K(a,\alpha(a),\alpha^l(a)) : K(a,\alpha^l(a))] \,|\, l = 1,2\ldots\}$ is a non-decreasing sequence.

(ii) Let $m = \sup\{[K(a,\alpha(a),\alpha^l(a)) : K(a,\alpha^l(a))] \,|\, l = 1,2\ldots\}$, let l_0 be the smallest l at which m is attained, and let $C = [K(a,\alpha(a),\ldots,\alpha^{l_0-1}(a),\alpha^{l_0}(a)) : K(a,\alpha^{l_0}(a))]$. If $l,j \geq l_0$, then $[K(a,\alpha^{-j}(a),\alpha^l(a)) : K(\alpha^{-j}(a),\alpha^l(a))] = \frac{m^{l_0}}{C}$.

(iii) With m as in (ii), $dd(a/K) = \frac{ld(a/K)}{m}$.

Theorem 12. Let K be an inversive ordinary difference (σ-) field and let $a,b \in \mathcal{U}$ be tuples over K such that $\alpha(a)$ is algebraic over $K(a)$ and $\alpha(b)$ is algebraic over $K(b)$. Then

$$dd(K\langle a,b\rangle/K) \geq dd(K\langle a,b\rangle/K\langle b\rangle)dd(K\langle b\rangle/K).$$

Remark. Unfortunately, the distance degree is not multiplicative in towers: Z. Chatzidakis and E. Hrushovski give an example of a difference field K and two tuples a and b over K such that $dd(K\langle a,b\rangle/K) = 2$, but $dd(K\langle a\rangle/K)dd(K\langle a,b\rangle/K\langle a\rangle) = 1$ (see (3)).

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On the Structure of Compatible
Rational Functions
(Extended Abstract)

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1 Introduction

Compatibility conditions are fundamental for first-order linear homogeneous functional systems. Such a system has only the zero solution if the compatibility conditions on its coefficients are not satisfied.

A nonzero solution of a first-order linear partial differential system is called a hyperexponential function. Christopher and Zoladek [5, 11] use the compatibility (integrability) conditions to show that a hyperexponential function can be written as a product of a rational function, finitely many power functions, and an exponential one. Their results generalize the well-known fact: for a rational function $r(t)$,

$$\exp\left(\int r(t)dt\right) = f(t)r_1(t)^{e_1}\cdots r_m(t)^{e_m}\exp(g(t)),$$

where $e_1, \ldots, e_m$ are constants, and $f, r_1, \ldots, r_m, g$ are rational functions of t.

A nonzero solution of a first-order linear partial difference system is called a hypergeometric term. Ore-Sato's Theorem [8, 10] states that a hypergeometric term is a product of a rational function, several power functions and factorial terms. Similar results are given in [7, 4] for q-hypergeometric terms. All these results are based on compatibility conditions on the certificates of a hypergeometric or q-hypergeometric term.

Consider a first-order mixed system

$$\left\{ \frac{\partial z(t,x)}{\partial t} = u(t,x)z(t,x), \ z(t,x+1) = v(t,x)z(t,x) \right\},$$

where u and v are rational functions with $v \neq 0$. Its compatibility condition is $\partial v(t,x)/\partial t = v(t,x)(u(t,x+1) - u(t,x))$. By Proposition 5 in [6], a nonzero solution of the above system can be written as a product $f(t,x)r(t)^x \mathcal{E}(t)\mathcal{T}(x)$, where f is a bivariate rational function in t and x, r is a univariate rational function in t, $\mathcal{E}$ is a hyperexponential function in t, and $\mathcal{T}$ is a hypergeometric term in x.

Christopher and Zoladek's result are useful to compute Liouvilian first integrals. Ore-Sato's theorem was rediscovered in one way or another, and is important for the proofs of a conjecture of Wilf and Zeilberger about holonomic hypergeometric terms [2, 9]. Bivariate Ore-Sato's theorem played a crucial role in deriving criteria for the existence of telescopers for hypergeometric and q-hypergeometric terms [1, 4]. Proposition 5 in [6] is used not only to describe Liouvillian solutions of difference-differential systems, but to prove the criteria on the existences of telescopers in the mixed case [3].

2 Compatible rational functions

In the rest of this abstract, $\mathbb{F}$ is a field of characteristic zero. Let $\mathbf{t} = (t_1, \ldots, t_l)$, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Assume that $q_1, \ldots, q_n \in \mathbb{F}$ are neither zero nor roots of unity. For an element f of $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, define $\delta_i(f) = \frac{\partial f}{\partial t_i}$ for all i with $1 \leq i \leq l$,

$$\sigma_j(f(\mathbf{t}, \mathbf{x}, \mathbf{y})) = f(\mathbf{t}, x_1, \ldots, x_{j-1}, \ x_j + 1, \ x_{j+1}, \ldots, x_m, \mathbf{y})$$

for all j with $1 \leq j \leq m$, and

$$\tau_k(f(\mathbf{t}, \mathbf{x}, \mathbf{y})) = f(\mathbf{t}, \mathbf{x}, y_1, \ldots, y_{k-1}, \ q_k y_k, \ y_{k+1}, \ldots, y_n)$$

for all k with $1 \leq k \leq n$.

Let $\Delta = \{\delta_1, \ldots, \delta_l, \sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n\}$. These maps commute pairwise, because a map in Δ is effective on only one variable. The field of constants w.r.t. a map in Δ consists of all rational functions free of the variable that is moved by the map.

By a first-order linear functional system over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, we mean a system consisting of

$$\delta_i(z) = u_i z, \ \sigma_j(z) = v_j z, \ \tau_k(z) = w_k z \tag{1}$$

for some rational functions $u_i, v_j, w_k \in \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ and for all i, j, k with $1 \leq i \leq l$, $1 \leq j \leq m$ and $1 \leq k \leq n$. System (1) is said to be *compatible* if (2)-8 given below hold:

$$v_1 \cdots v_n w_1 \cdots w_\ell \neq 0, \tag{2}$$

$$\delta_i(u_j) = \delta_j(u_i), \quad 1 \leq i < j \leq l, \tag{3}$$

$$\sigma_i(v_j)/v_j = \sigma_j(v_i)/v_i, \quad 1 \le i < j \le m, \tag{4}$$

$$\tau_i(w_j)/w_j = \tau_j(w_i)/w_i, \quad 1 \le i < j \le n, \tag{5}$$

$$\delta_i(v_j)/v_j = \sigma_j(u_i) - u_i, \quad 1 \le i \le l \text{ and } 1 \le j \le m, \tag{6}$$

$$\delta_i(w_j)/w_j = \tau_j(u_i) - u_i, \quad 1 \le i \le l \text{ and } 1 \le j \le n, \tag{7}$$

$$\sigma_i(w_j)/w_j = \tau_j(v_i)/v_i, \quad 1 \le i \le m \text{ and } 1 \le j \le n. \tag{8}$$

These conditions are obtained by the commutativity of the maps in Δ. We say that a sequence of rational functions:

$$u_1, \ldots, u_l, v_1, \ldots, v_m, w_1, \ldots, w_n$$

is Δ-*compatible* if the equalities in $(2)-(8)$ hold.

3 Results

Our main result describes the structure of compatible rational functions.

Theorem 1. *Let*

$$u_1, \ldots, u_l, v_1, \ldots, v_m, w_1, \ldots, w_n \tag{9}$$

be Δ-compatible rational functions in $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$. Then there exist f in $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_l$ in $\mathbb{F}(\mathbf{t})$, $\lambda_1, \ldots, \lambda_m$ in $\mathbb{F}(\mathbf{x})$, and $\mu_1, \ldots, \mu_n$ in $\mathbb{F}(\mathbf{y})$ s.t., for all i with $1 \le i \le l$,

$$u_i = \ell\delta_i(f) + \ell\delta_i(\alpha_1)\, x_1 + \cdots + \ell\delta_i(\alpha_n)\, x_n + \beta_i, \tag{10}$$

for all j with $1 \le j \le m$, and, for all k with $1 \le k \le n$,

$$v_j = \ell\sigma_j(f)\, \alpha_j \lambda_j \quad \text{and} \quad w_k = \ell\tau_k(f)\, \mu_k. \tag{11}$$

Moreover, $\beta_1, \ldots, \beta_l$ are compatible w.r.t. $\{\delta_1, \ldots, \delta_l\}$, $\lambda_1, \ldots, \lambda_m$ are compatible w.r.t. $\{\sigma_1, \ldots, \sigma_m\}$, and $\mu_1, \ldots, \mu_n$ are compatible w.r.t. $\{\tau_1, \ldots, \tau_n\}$.

Assume that our ground $\mathbb{F}$ is algebraically closed. We recall that an H-element over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ is a nonzero solution of system (1) and given a finite number of H-elements over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, there is a Δ-extension ring of $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, which contains these H-elements and their inverses. The ring of constants of this Δ-extension is equal to $\mathbb{F}$. Hence it makes sense to multiply and invert H-elements in some Δ-extension ring. We will not specify the Δ-extension ring if no ambiguity arises. All H-elements we consider will be over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$. Denote by $\mathbf{0}_s$ and $\mathbf{1}_s$ the sequences consisting of s 0's or of s 1's, respectively.

Let $\prec$ be a monomial order in $F[\mathbf{t}, \mathbf{x}, \mathbf{y}]$. An H-element is said to be a *symbolic power* if its certificates are of the form

$$\sum_{j=1}^{m} x_j \frac{\delta_1(\alpha_j)}{\alpha_j}, \ldots, \sum_{j=1}^{m} x_i \frac{\delta_\ell(\alpha_j)}{\alpha_\ell}, \alpha_1, \ldots, \alpha_m, \mathbf{1}_n, \tag{12}$$

where $\alpha_1, \ldots, \alpha_m$ are in $\mathbb{F}(t)^\times$ and monic with respect to $\prec$. It is easy to verify that such a sequence is Δ-compatible. Such a symbolic power is denoted $\alpha_1^{x_1} \cdots \alpha_m^{x_m}$. The monicity of the α_i's exclude the case, in which some α_i is a nonzero constant unequal to one. By an E-element, we mean an H-element whose certificates are of the form $\beta_1, \ldots, \beta_l, \mathbf{1}_{m+n}$, where $\beta_1, \ldots, \beta_l$ are in $\mathbb{F}(\mathbf{t})$. By a G-element, we mean an H-element whose certificates are of the form $\mathbf{0}_l, \lambda_1, \ldots, \lambda_m, \mathbf{1}_n$, where $\lambda_1, \ldots, \lambda_m$ are in $\mathbb{F}(\mathbf{x})$. By a Q-element, we mean an H-element whose certificates are of the form $\mathbf{0}_l, \mathbf{1}_m, \mu_1, \ldots, \mu_n$, where $\mu_1, \ldots, \mu_n$ are in $\mathbb{F}(\mathbf{y})$. An E-element is a hyperexponential function w.r.t. $\{\delta_1, \ldots, \delta_l\}$; a G-element is a hypergeometric term w.r.t. $\{\sigma_1, \ldots, \sigma_m\}$; and a constant w.r.t. other operators in Δ; and a Q-element is a q-hypergeometric term w.r.t. $\{\tau_1, \ldots, \tau_n\}$ and a constant w.r.t. other operators in Δ. But E-elements (resp. G-elements or Q-elements) are constants w.r.t. other irrelevant operators in Δ.

An easy consequence of Theorem 1 is the following multiplicative decomposition of an H-element.

Proposition 2. *An H-element is a product of an element in $F^\times$, a rational function in $F(\mathbf{t}, \mathbf{x}, \mathbf{y})$, a symbolic power, an E-element, a G-element, and a Q-element.*

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RESULTS AND OPEN PROBLEMS ON THE ALGEBRAIC LIMIT CYCLES OF POLYNOMIAL VECTOR FIELDS IN $\mathbb{R}^2$

JAUME LLIBRE

Since Darboux [12] has found in 1878 connections between algebraic curves and the existence of first integrals of planar polynomial vector fields, invariant algebraic curves are a central object in the theory of integrability of these vector fields. Today after more than one century of investigations the theory of invariant algebraic curves is still full of open questions.

A *real planar polynomial differential system* is a differential system of the form

$$(1) \qquad \frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y),$$

where P and Q are real polynomials in the variables x and y. The dependent variables x and y, the independent variable t, and the coefficients of the polynomials P and Q are all real because in this paper we are interested in the real algebraic limit cycles of system (1). The *degree n* of the polynomial system (1) is the maximum of the degrees of the polynomials P and Q.

Associated to the (real) polynomial differential system (1) there is the *(real) polynomial vector field*

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y},$$

or simply $\mathcal{X} = (P,Q)$.

Let $f = f(x,y)$ be a (real) polynomial in the variables x and y. The algebraic curve $f(x,y) = 0$ of $\mathbb{R}^2$ is an *invariant algebraic curve* of the vector field $\mathcal{X}$ if for some polynomial $K \in \mathbb{R}[x,y]$ we have

$$(2) \qquad \mathcal{X}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that since the polynomial system has degree n, then any cofactor has at most degree $n-1$.

We recall that a *limit cycle* of a polynomial vector field $\mathcal{X}$ is an isolated periodic orbit in the set of all periodic orbits of $\mathcal{X}$. An *algebraic limit cycle* of degree m of $\mathcal{X}$ is an oval of an irreducible invariant algebraic curve $f = 0$ of degree m which is a limit cycle of $\mathcal{X}$.

A first question related with this subject is whether a polynomial vector field has or does not have invariant algebraic curves. The answer is not easy, see the large section in Jouanolou's book [21], or the long paper [31] devoted

to show that one particular polynomial system has no invariant algebraic solutions. Even one of the more studied limit cycles, the limit cycle of the van der Pol system, until 1995 it was unknown that it is not algebraic [32].

One of the nice results in the theory of invariant algebraic curves is the following result.

Theorem 1 (Jouanolou's Theorem [21]). *A polynomial vector field of degree n has less than $[n(n + 1)/2] + 2$ irreducible invariant algebraic curves, or it has a rational first integral.*

For a shorter proof of this result see [9] or [10].

Jouanolou's Theorem shows that for a given polynomial vector field $\mathcal{X}$ of degree n the maximum degree of its irreducible invariant algebraic curves is bounded, because either $\mathcal{X}$ has a finite number of invariant algebraic curves less than $[n(n + 1)/2] + 2$, or $\mathcal{X}$ has rational first integral $f(x, y)/g(x, y)$. In this last case all the orbits of $\mathcal{X}$ are contained in the invariant algebraic curves $af(x, y) + bg(x, y) = 0$ for some $a, b \in \mathbb{R}$.

Thus for each polynomial vector field there is a natural number N which bounds the degree of all its irreducible invariant algebraic curves. A natural question, going back to Poincaré [33] and which for some people in this area is now known as the *Poincaré problem*, is to give an effective procedure to find N. There are only partial answers to this question, see for instance [2], [3], [4], [36], ... We must mention here that the actual Poincaré problem is to determine when a polynomial differential system over the complex plane has a rational first integral, and that the previous called Poincaré problem is a main step according with Poincaré for solving the actual Poincaré problem.

Of course if we know for a polynomial vector field the maximum degree of its invariant algebraic curves, then it is possible (at least in theory) to compute its invariant algebraic curves.

We are interested in algebraic limit cycles of polynomial vector fields, and if a polynomial vector field has a rational first integral it cannot have limit cycles. Unfortunately for the class of polynomial vector fields with fixed degree n having finitely many invariant algebraic curves (i.e. having no rational first integrals), there does not exist a uniform upper bound $N(n)$ for N as it was shown in [11, 30]. This implies that there are polynomial vector fields with a fixed degree having irreducible invariant algebraic curve of arbitrary degree. Therefore a priori it is possible the existence of polynomial vector fields with a fixed degree having algebraic limit cycle of arbitrary degree. But it may be worse than that.

Summarizing, a polynomial vector field of degree n with finitely many irreducible invariant algebraic curves has at most $[n(n + 1)/2] + 1$ of such curves, but we do not have a bound for the degree of these invariant algebraic curves. Consequently due to the Harnack's Theorem we do not have a uniform bound for the number of algebraic limit cycles that any polynomial vector field of degree n can have. So the second part of the 16–th Hilbert

problem [20] (see also [19, 22]) which asks for finding a uniform bound for the number of limit cycles that any polynomial vector field of degree n can have, remains also open if we restrict our attention to the limit cycles which are algebraic.

Open problem 1. *Is there a uniform bound for the number of algebraic limit cycles that a polynomial vector field of degree n could have?*

¿From the previous paragraphs it is clear that a uniform positive answer to the Poincaré problem inside the class of all polynomial vector fields of degree n, i.e. to provide a uniform bound $N(n)$ for the degrees of the invariant algebraic curves of all polynomials vector fields of degree n, will provide also a uniform bound for the number of algebraic limit cycles of all polynomials vector fields of degree n.

Theorem 2 (Bautin–Christopher–Dolov–Kuzmin Theorem). *Let $f = 0$ be a non–singular algebraic curve of degree m, and D a first degree polynomial, chosen so that the line $D = 0$ lies outside all bounded components of $f = 0$. Choose the constants a and b so that $aD_x + bD_y \neq 0$, then the polynomial differential system*

$$\dot{x} = af - Df_y, \qquad \dot{y} = bf + Df_x,$$

of degree m has all the bounded components of $f = 0$ as hyperbolic limit cycles. Furthermore the vector field has no other limit cycles.

It seems that the main result in the paper of Bautin [1] is similar to the previous theorem. However the paper contains a mistake which was corrected in [13] and generalized in [14]. A proof of the statement of theorem like it is presented here appeared in [8].

The next proposition provides the maximum number of algebraic limit cycles that a polynomial vector field having a unique irreducible invariant algebraic curve can have in function of the degree of that curve. This proposition is well known in the area we write it here for completeness.

Proposition 3. *Suppose that $f = 0$ of degree m is the unique irreducible invariant algebraic curve of a polynomial vector field X. Then X can have at most $[(m - 1)(m - 2)/2] + 1$ algebraic limit cycles. Moreover choosing that $f = 0$ has the maximal number of ovals for the irreducible algebraic curves of degree m, there exist vector fields X of degree m having exactly $[(m - 1)(m - 2)/2] + 1$ algebraic limit cycles.*

In 1958 Qin Yuan–Xun [35] proved that quadratic (polynomial) vector fields can have algebraic limit cycles of degree 2, and when such a limit cycle exists then it is the unique limit cycle of the system.

Evdokimenco in [15, 16, 17] proved that quadratic vector fields do not have algebraic limit cycles of degree 3, for two different shorter proofs see [6, 25].

In 1966 Yablonskii [34] found the first class of algebraic limit cycles of degree 4 inside the quadratic vector fields. The second class was found in

1973 by Filiptsov [18]. More recently two new classes has been found and in [6] the authors proved that there are no other algebraic limit cycles of degree 4 for quadratic vector fields. The uniqueness of these limit cycles has been proved in [5]. Some other results on the algebraic limit cycles of quadratic vector fields can be found in [27, 28].

Doing convenient birational transformations of the plane to quadratic vector fields having algebraic limit cycles of degree 4 in [7] the authors obtained algebraic limit cycle of degrees 5 and 6 for quadratic vector fields. Of course in general a birational transformation does not preserve the degree of the polynomial vector field.

Open problems 2. *The following questions related with the algebraic limit cycles of quadratic polynomial vector fields remain open, see for instance*[25].

    (i) *What is the maximum degree of an algebraic limit cycle of a quadratic polynomial vector field?*

   (ii) *Does there exist a chain of rational transformations of the plane (as in* [7]*) which gives examples of quadratic systems with algebraic limits cycles of arbitrary degree, or at least of degree larger than 6?*

  (iii) *Is 1 the maximum number of algebraic limit cycles that a quadratic system can have?*

In 1900 Hilbert not only proposed in the second part of his 16–th problem (see [20]) to estimate a uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, but he also asked about the possible distributions or configurations of the limit cycles in the plane. This last question has been solved using algebraic limit cycles.

A *configuration of limit cycles* is a finite set $C = \{C_1, \ldots, C_n\}$ of disjoint simple closed curves of the plane such that $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Two configurations of limit cycles $C = \{C_1, \ldots, C_n\}$ and $C' = \{C'_1, \ldots, C'_m\}$ are *(topologically) equivalent* if there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h\left(\cup_{i=1}^n C_i\right) = \left(\cup_{i=1}^m C'_i\right)$. Of course for equivalent configurations of limit cycles C and C' we have that $n = m$.

We say that a polynomial vector field $\mathcal{X}$ *realizes* the configuration of limit cycles C if the set of all limit cycles of X is equivalent to C.

Theorem 4. *Let $C = \{C_1, \ldots, C_n\}$ be an arbitrary configuration of limit cycles. Then the configuration C is realizable with algebraic limit cycles by a polynomial vector field.*

This theorem is proved in [26]. Looking at the way in which is proved you can provide an alternative proof using the Bautin–Christopher–Dolov–Kuzmin Theorem.

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Adapting the FGLM-algorithm for conversion between Hermite and Popov normal forms of differential operator matrices

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Abstract

We consider matrices over univariate Ore polynomial rings. We connect the Popov normal form and the Hermite normal form of such matrices to Gröbner bases extending a result in [8]. Finally, we adapt the FGLM algorithm to convert matrices from Popov normal form to Hermite normal form and vice versa.

Keywords: Matrix normal forms, Ore polynomials, Hermite normal form, Popov normal form

1. Ore polynomials

Ore polynomials are an algebraic construct which is suitable for modelling linear ordinary differential operators. They have been first described by Øystein Ore in [10] with the purpose of studying ring extensions with almost the same properties as the usual polynomials except for commutativity. For this abstract, we will be content with a short and informal recapitulation of the construction and its properties—see Ore's original paper [10] or [4, Sect. 0.10] for proofs and more details.

Let K be a skew field together with an automorphism $\sigma\colon K \to K$ and an additive map $\vartheta\colon K \to K$ which fulfills the σ-*Leibniz rule* $\vartheta(ab) = \sigma(a)\vartheta(b) + \vartheta(a)b$ for all a and $b \in K$. We consider the set of all polynomial expressions over K in the indeterminate ∂, that is, the set $R = \{a_n\partial^n + \ldots + a_1\partial + a_0 \mid n \geq 0$ and $a_0, \ldots, a_n \in K\}$. This is a left K-space with the obvious operations. It is possible to prove, that with the *commutation rule*

$$\partial a = \sigma(a)\partial + \vartheta(a)$$

we can define a multiplication which makes R into a ring—see, for example, [4, Thm. 0.10.1]. We write $R = K[\partial; \sigma, \vartheta]$ and call it the ring of *Ore polynomials* over K with respect to σ and ϑ. One can prove that R is a (left and right) Euclidean domain—see, for example, [5, Thm. 5.8].

For $\sigma = \mathrm{id}$ and $\vartheta = 0$, the commutation rule becomes just $\partial a = a\partial$. Thus, if K is commutative, we obtain simply the usual polynomials. For $\sigma = \mathrm{id}$, a non-trivial ϑ fulfills the usual Leibniz rule and we have the *differential operators*. The commutation rule in this case is $\partial a = a\partial + \vartheta(a)$ which corresponds to the composition of differential operators. The third prominent example of Ore polynomials are *delay operators* where σ is an arbitrary automorphism, $\vartheta = 0$ and the commutation rule becomes $\partial a = \sigma(a)\partial$.

2. Hermite normal form and Popov normal form

Let K be a field with automorphism $\sigma\colon K \to K$ and σ-derivation $\vartheta\colon K \to K$, and let $R = K[\partial; \sigma, \vartheta]$. We denote the set of $m \times n$ matrices over R by $^mR^n$ and the group of *unimodular* $m \times m$ matrices by $\mathrm{GL}_m(R)$.

Analogously to [7, Def. 2.3], we say that matrix $M \in {}^m R^n$ is in *Hermite normal form* if M is in upper row echelon form such that the pivot entries are monic and every entry above a pivot entry has a strictly smaller degree than the pivot. I can be shown that for each matrix $A \in {}^m R^n$ there exists $U \in \mathrm{GL}_m(R)$ such that the submatrix consisting of the non-zero rows of UA is in Hermite normal form—see, for example, [7, Thm. 3.2] where greatest common divisor computations are used. Moreover, the Hermite normal form of each matrix is unique.

Defining the Popov normal form is more difficult. For this, we will need to introduce another concept first. Let $v \in {}^1 R^n$ be a row vector. We can write v as $v = u_d \partial^d + \ldots + u_1 \partial + u_0$ where $u_0, \ldots, u_1 \in K$. If $u_d \neq 0$, then we call u_d the *leading vector* of v and denote it by $\mathrm{lv}(v) = u_d$. A matrix $M \in {}^m R^n$ is said to be *row-reduced* if it does not contain zero rows and if the set of the leading vectors of all its rows is K-linearly independent. It is possible to prove that for any matrix $A \in {}^m R^n$ there exists a unimodular matrix $U \in \mathrm{GL}_m(R)$ such that the submatrix of UA formed by the non-zero rows is row-reduced. See, for example, [1, Thm. 1] for an algorithmic proof of this claim.

Stating that a matrix is in Popov normal form essentially means that it is row-reduced and fulfills some additional conditions to ensure that it is uniquely determined—see, for example, [3, Def. 2.8]. Assume that M is row-reduced. For each row $v \in {}^1 R^n$ of M consider the smallest index j such that $\mathrm{lv}(v)_j \neq 0$. We call v_j the pivot of the row v. Then, we say that M is in *Popov normal form* if the rows are sorted by degree and for each row the pivot entry is monic and all entries in the same column have a stricly smaller degree. Again, it can be shown that for each matrix $A \in {}^m R^n$ there exists $U \in \mathrm{GL}_m(R)$ such that the submatrix of the non-zero rows of UA is in Hermite normal form.

3. Gröbner bases and FGLM

For Ore polynomials there is a theory of Gröbner bases which is completely analogous to the case of commutative polynomials. Confer, for example, [2]. Moreover, as in the case of commutative polynomials this extends to modules in the following way: Use $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ to denote the unit vectors in ${}^1 R^n$. Then, a *monomial* is of the form $\partial^a \mathfrak{e}_j$ where $a \geq 0$ and $1 \leq j \leq n$. With this definition, one can define monomial orderings, leading monomials, division and Gröbner bases as in the usual case.

We would like to introduce the two most prominant monomial orderings—see again [2, Def. 5.3.8 and Def. 5.3.9]. We say that $\partial^a \mathfrak{e}_j$ is larger than $\partial^b \mathfrak{e}_k$ with respect to to the *position over term* ordering if $j < k$ or if $j = k$ and $a > b$. We say that $\partial^a \mathfrak{e}_j$ is larger than $\partial^b \mathfrak{e}_k$ with respect to to the *term over position* ordering if $a > b$ or $a = b$ and $j < k$.

Anaolgously to [8], it is not hard to prove the following result. The key observation is that for both the Hermite normal form and the position over term ordering as well as the Popov normal form and the term over position ordering the pivots are exactly the leading terms.

Theorem 1. *Up to permutation of the rows, a matrix is in*

1. *Hermite normal form if and only if the rows are a reduced Gröbner basis with respect to the position over term ordering; or in*
2. *Popov normal form if and only if the rows are a reduced Gröbner basis with respect to the term over position ordering.*

Proof. See [9, Thm. 13 and Thm. 14]. □

In the theory of Gröbner bases, the FGLM algorithm—which was first presented in [6]—is an efficient method to compute a Gröbner basis for a zero-dimensional ideal I in a polynomial ring $K[x_1, \ldots, x_\ell]$ and a given monomial ordering provided that already a Gröbner basis for a different monomial ordering is known. That means, that the FGLM algorithm converts Gröbner bases to a different monomial ordering. The idea is to convert the problem to a linear problem by doing the computations in the finite dimensional quotient. The algorithm considers a basis of $K[x_1, \ldots, x_\ell]/I$ consisting of residue classes of monomials and uses linear systems to test for every of these monomials whether it is the leading monomial of a member of the new Gröbner basis or not.

If we want to mimick this algorithm for matrices $M \in {}^m R^n$ in Hermite normal form or Popov normal form, then we have the problem that the quotient ${}^1 R^n / {}^1 R^m M$ in general does not have finite dimension. However, we can overcome this problem by using a *degree bound* for the Popov normal form or the Hermite normal form as in [7, Cor. 3.4]. This again leads to a finite search space consisting of all those basis elements being the residue classes of monomials which fulfill the degree bound. With this, it is possible to modify the original FGLM algorithm in a way that it works for arbitrary matrices in ${}^m R^n$. See [9, Alg. 19] for the precise formulation of the algorithm.

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Solving first-order parametrizable ODEs

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Abstract

We present an algorithm to solve for a rational general solution of a first-order parametrizable ODE. The method turns out to be depending on the choice of a rational parametrization of the corresponding surface of the differential equation. We classify the set of first-order parametrizable ODEs by means of a group action on the set. With the relation defined by the group, differential equations in the same class have the same difficulty in solving its rational general solutions. In particular, we describe some classes of first-order ODEs that are simple in a sense and they are perfect components for a complete system of representative elements w.r.t the group action.

Key words: Ordinary differential equations, Rational solutions, Invariant algebraic curves, Rational parametrizations.

1. Preliminaries

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $F(u, v, w)$ be a trivariate polynomial over $\mathbb{K}$. The *algebraic ordinary differential equation* (ODE) of order 1 defined by F is of the form

$$F(x, y, y') = 0, \tag{1}$$

where y is an indeterminate over the differential field of rational functions $\mathbb{K}(x)$ with the usual derivation $' = \frac{d}{dx}$.

Let $\{F\}$ be the radical differential ideal generated by F in the differential ring $\mathbb{K}(x)\{y\}$. Then one can prove that

$$\{F\} = (\{F\} : S) \cap \{F, S\}, \tag{2}$$

where $S = \frac{\partial F}{\partial y'}$ is the separant of F. So the set of solutions of $F = 0$ is decomposed as two components: one is vanished by S, the other one is not. Of course, most of the solutions of $F = 0$ should belong to the component that is not vanished by S. This decomposition is valid for differential polynomials of any order.

Definition 1.1. A generic zero of the prime differential ideal $\{F\} : S$ is called a *general solution* of $F(x, y, y') = 0$. A common zero of F and S is called a *singular solution* of $F(x, y, y') = 0$.

We are interested in computing a *rational general solution* of $F(x, y, y') = 0$, i.e., a general solution of the form

$$y = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0}, \tag{3}$$

where a_i, b_j are constants in a transcendental extension field of $\mathbb{K}$. In the sequel, by an arbitrary contant we mean a transcendental contant over $\mathbb{K}$.

Problem: *Given $F(x, y, y') = 0$, $\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t))$ such that*

$$F(\mathcal{P}(s, t)) = 0,$$

where χ_1, χ_2, χ_3 are bivariate rational functions over $\mathbb{K}$. Decide the existence of a rational general solution of $F(x, y, y') = 0$ and compute such a solution when it exists.

Note that the problem is already solved for autonomous ODEs in Feng and Gao (2006). We now give a geometric method, which generalizes the one in the autonomous case, to compute an explicit rational general solution of $F(x, y, y') = 0$. Assume that $\mathcal{P}(s, t)$ is proper, i.e., it has an inverse and its inverse is also rational. Then a rational general solution of $F(x, y, y') = 0$ can be computed via computing $(s(x), t(x))$ such that

$$\mathcal{P}(s(x), t(x)) = (x, f(x), f'(x)).$$

In order to satisfy the last condition, it turns out that $(s(x), t(x))$ must be a *rational general solution* of the system

$$\begin{cases} s' = \dfrac{\chi_{2t} - \chi_3 \cdot \chi_{1t}}{\chi_{1s} \cdot \chi_{2t} - \chi_{1t} \cdot \chi_{2s}}, \\ t' = \dfrac{\chi_{1s} \cdot \chi_3 - \chi_{2s}}{\chi_{1s} \cdot \chi_{2t} - \chi_{1t} \cdot \chi_{2s}}, \end{cases} \tag{4}$$

provided that $\chi_{1s} \cdot \chi_{2t} - \chi_{1t} \cdot \chi_{2s} \neq 0$.

Definition 1.2. The system (4) is called the *associated system* of $F(x, y, y') = 0$ w.r.t $\mathcal{P}(s, t)$.

By construction, if $(s(x), t(x))$ is a rational general solution of the associated system (4), then

$$\mathcal{P}(s(x), t(x)) = (x + c, \ \chi_2(s(x), t(x)), \ \chi_2(s(x), t(x))')$$

for some constant c. Therefore,

$$y = \chi_2(s(x - c), t(x - c))$$

is a rational general solution of the corresponding differential equation $F(x, y, y') = 0$. A proof can be found in Ngô and Winkler (2010).

Theorem 1.3. *If the parametrization $\mathcal{P}(s, t)$ is proper, then there is a one-to-one correspondence between rational general solutions of the parametrizable ODE $F(x, y, y') = 0$ and rational general solutions of its associated system w.r.t $\mathcal{P}(s, t)$.*

The associated system (4) is an autonomous system in two differential indeterminates s and t; and the degrees w.r.t s' and t' are 1. The Darboux's theory on invariant algebraic curves can apply to this system in order to find a rational solution (Jouanolou, 1979; Lins Neto, 1988; Singer, 1992).

Definition 1.4. Let M_1, M_2, N_1, N_2 be polynomials in $\mathbb{K}[s,t]$ and $\gcd(M_i, N_i) = 1$ for $i = 1, 2$. An *invariant algebraic curve* of the rational system

$$\begin{cases} s' = \dfrac{M_1(s,t)}{N_1(s,t)}, \\ t' = \dfrac{M_2(s,t)}{N_2(s,t)}, \end{cases} \tag{5}$$

is an algebraic curve $G(s,t) = 0$ such that

$$G_s M_1 N_2 + G_t M_2 N_1 = GK,$$

where G_s and G_t are the partial derivatives of G w.r.t s and t; and K is some polynomial. An invariant algebraic curve of the system is called a *general invariant algebraic curve* if it contains an arbitrary constant in its coefficients.

Assume that we have found an irreducible invariant algebraic curve of the system (5), which is also rational and containing an arbitrary constant in its coefficients. Then we show how to obtain a rational general solution of the system (5) from a proper rational parametrization of that general invariant algebraic curve. For a complete description of this step we refer to Ngô and Winkler (2011b,a); Sendra and Winkler (2001); Sendra et al. (2008). Of course, the remained problem is computing an irreducible invariant algebraic curve of the system (5); in order that we use the undetermined coefficients method and base on the degree bound given by (Carnicer, 1994) for the system having no *dicritical singularities*, which is a generic case. Therefore, the problem is solved in a generic case.

2. A group of affine linear transformations

We define a group of affine linear transformations on $\mathbb{K}(x)^3$ that maps an integral curve of the space to another one. By an integral curve of the space we mean a parametric curve of the form $\mathcal{C}(x) = (x, f(x), f'(x))$. So this group can act on the set of all algebraic ODEs of order 1 and it is compatible with the solution curves of the corresponding differential equation. Therefore, it gives a partition on the set of all algebraic ODEs of order 1.

Precisely, let $L : \mathbb{K}(x)^3 \longrightarrow \mathbb{K}(x)^3$ be an affine linear transformation defined by

$$L(v_1, v_2, v_3) = (v_1, av_2 + bv_1, av_3 + b),$$

where $v_i \in \mathbb{K}(x)$ for all $i = 1, 2, 3$ and $a, b \in \mathbb{K}$ such that $a \neq 0$. The set of all such transformations forms a group under the composition. This group naturally acts on the set of first-order ODEs as follows:

$$L \cdot F = F(L^{-1}(x, y, y')) = F\left(x, -\frac{b}{a}x + \frac{1}{a}y, -\frac{b}{a} + \frac{1}{a}y'\right).$$

Theorem 2.1. *Let $F(x, y, y') = 0$ be a parametrizable ODE of order 1. Let L be an affine linear transformation in the group $\mathcal{G}$. Then $L \cdot F$ and F have the same associated system with respect to a certain proper rational parametrization of $F(x, y, z) = 0$.*

A proof of the theorem can be found in our manuscript Ngô et al. (2011).

3.   Special parametrizable ODEs

These are special first-order parametrizable ODEs whose parametrizations are easily recognized from the defining equations.

(1) $y' = G(x, y)$, $G(x, y)$ is a rational function.

(2) $y = G(x, y')$, $G(x, y')$ is a rational function.

(3) $x = G(y, y')$, $G(y, y')$ is a rational function.

(4) $F(y - \lambda x, y') = 0$, where $F(u, v) = 0$ is a rational curve.

(5) $F\left(\dfrac{y}{x^{m+1}}, \dfrac{y'}{x^m}\right)$, where $F(u, v) = 0$ is a rational curve.

A full description can be found in our manuscript Ngô et al. (2011).

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Linear differential elimination for analytic functions

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1. Introduction

This is a report on joint work with Daniel Robertz on composite analytic functions and linear differential elimination, cf. Plesken and Robertz (2010). In another extended abstract of this conference Daniel Robertz reports on his own ongoing work on nonlinear differential elmination which is a continuation of the present project. Here we propose to do linear algebra in spaces S of analytic functions parametrized in the following form:

$$f_1(\alpha_1(z))\, g_1(z) + \ldots + f_k(\alpha_k(z))\, g_k(z), \qquad \alpha_i, g_j \text{ fixed analytic}, \quad (1)$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. To avoid trivialities, it is assumed that the α_j take values in spaces of dimensions smaller than n. Three main problems are considered:

Recognition: Decide wheter a given function belongs to S.

Explicit recognition: Find one resp. all sets of functional parameters $f_1, \ldots, f_k$ in the affirmative case.

Description by PDEs: Find a linear PDE-system having S as its set of solutions.

The third problem already indicates how to approach the first two problems. The methods employed are the chain rule, which seems to be somewhat neglected in differential algebra and Janet bases and differential elimination.

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Here is an easy example to demonstrate the main ideas: Prove the functional equation for the exponential function: $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$. We prodeed in three steps to answer Is $\exp(z_1 + z_2)$ of the form $f(z_1, z_2) = f_1(z_2)\exp(z_1)$ with analytic f_1?
Characterize these functions f by the PDE $u_1(z_1, z_2) - u(z_1, z_2) = 0$.
$\exp(z_1 + z_2)$ satisfies this PDE, hence is of this form.
$f_1(z_2) = \exp(z_1 + z_2)/\exp(z_1)$ for any z_1; choose $z_1 = 0$ to obtain $f_1(z_2) = \exp(z_2)$.
Similarly the addition formulars for sin and cos can be proved. To deal with the general problem, the three steps have to be adjusted appropriately.

2. Representabilty

Definition 2.1. 1.) $\Omega \subset \mathbb{C}^n$ open and connected. $K := \mathcal{M}(\Omega)$ denotes the field of meromorphic functions on Ω.
2.) $g_1, \ldots, g_k$ are non-zero analytic $\mathbb{C}$-valued functions on Ω (not necessarily distinct). The tuple of all g_i is referred to as g.
3.) For each i, $1 \leq i \leq k$, there is a $\nu(i) < n$ with $\nu(i)$ (functionally independent) analytic functions $\alpha_{i,j} : \Omega \to \mathbb{C}$, $j = 1, \ldots, \nu(i)$, sometimes taken together as $\alpha_i : \Omega \to \mathbb{C}^{\nu(i)} : z \mapsto (\alpha_{i,1}(z), \ldots, \alpha_{i,\nu(i)}(z))$ such that the Jacobian has rank $\nu(i)$ throughout Ω. The k-tuple of the α_i is referred to as α.
4.) The analytic function $u : \Omega \to \mathbb{C}$ is called *(linearly) (α, g)-representable*, if there exist functions $f_i : \alpha_i(\Omega) \to \mathbb{C}$ such that $f_i \circ \alpha_i$ is analytic for $i = 1, \ldots, k$ and

$$u(z) = f_1(\alpha_1(z))g_1(z) + \cdots + f_k(\alpha_k(z))g_k(z) \tag{2}$$

for all $z \in \Omega$.

Unfortunately the concept of (α, g)-representability cannot be expressed by differential equations alone, since topological problems arise. Therefore the concept of essential (α, g)-representability is introduced to focus on the local issues: It essentially means that one has a dense open subset of Ω such that the function is (α, g)-representable around any point of this subset. The key observation now concerns the cases $k = 1$:

Lemma 2.2. *Assume $k = 1$. u is locally (α_1, g_1)-representable if and only if u satisfies a certain system of $n - \nu(1)$ first order linear pde's with coefficients in*

K. The left ideal generated by these equations in the ring $R := K\langle \partial_1, \ldots, \partial_n \rangle$ will be denoted by $I(\alpha_1, g_1)$.

Theorem 2.3. *The analytic function $u : \Omega \to \mathbb{C}$ is locally (α, g)-representable if and only if u satisfies a certain system of linear pde's with coefficients in $K = \mathcal{M}(\Omega)$. The pde's of the system can be chosen to generate the intersection $I(\alpha, g)$ of the left ideals $I(\alpha_i, g_i)$ in $K\langle \partial_1, \ldots, \partial_n \rangle$ describing the locally (α_i, g_i)-representable functions as in Lemma 2.2.*

Computing the intersections of the ideals $I(\alpha_i, g_i)$ naively turned out to be quite time consuming. For the actual computation we use an implementation of the JANET-algorithm. However it is worthwhile to introduces jets to deal with equation 2 and its differential consequences of which one can extract generators for $I(\alpha, g)$ more efficiently: Taking

$$u_\mu \quad \longleftrightarrow \quad \partial_\mu u, \qquad\qquad \mu \in (\mathbb{Z}_{\geq 0})^n$$

$$f_{i,\eta} \quad \longleftrightarrow \quad (\partial_\eta f_i) \circ \alpha_i, \qquad \eta \in (\mathbb{Z}_{\geq 0})^{\nu(i)}, \quad i = 1, \ldots, k$$

as coefficients of the the jet-colums $j_d(u)$ resp. $j_d(\alpha, f)$ up to order d leads to the matrix equation obtained from equation 2 by partial differentiation:

$$j_d(u) = \Delta_d(\alpha, g) \cdot j_d(\alpha, f). \tag{3}$$

<u>Example</u> $u(x, y) = f_1(y + x^2) \cdot x + f_2(x + y^2) \cdot y$

$$\Delta_2(\alpha, g) = \begin{pmatrix} x & y & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & x & 2\,y^2 & 0 & 0 \\ 1 & 0 & 2\,x^2 & y & 0 & 0 \\ \hline 0 & 0 & 0 & 6\,y & x & 4\,y^3 \\ 0 & 0 & 1 & 1 & 2\,x^2 & 2\,y^2 \\ 0 & 0 & 6\,x & 0 & 4\,x^3 & y \end{pmatrix} \qquad \begin{matrix} u \\ u_{(0,1)} \\ u_{(1,0)} \\ u_{(0,2)} \\ u_{(1,1)} \\ u_{(2,0)} \end{matrix}$$

To obtain generators for $I(\alpha, g)$ one has to eliminate the $f_{i,\eta}$ in equation 3. Various methods are discussed to see how big the differentiation order d has to be chosen to obtain sufficiently many equations among the u-jets.

To decide (α, g)-representability rather than essential (α, g)-representability we propose to compute actual representations, which of course is of independent interest. The methods for this are slight extension of the ideas presented above.

3. Applications and Examples

As a first application ask whether a given function $u(x, y, z)$ is representable as a sum of analytic functions of two variables:

$u(x, y, z) = f_1(y, z) + f_2(x, z) + f_3(x, y).$

The characterizing linear PDE is

$$\frac{\partial^3 u}{\partial x \, \partial y \, \partial z} = 0,$$

For instance for spherical coordinates r, φ, θ this leads to three commuting differential operators

$$\partial_r \ := \ \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left(x \, \partial_x + y \, \partial_y + z \, \partial_z \right), \qquad \partial_\varphi := -y \, \partial_x + x \, \partial_y,$$

$$\partial_\theta \ := \ \frac{1}{\sqrt{x^2 + y^2}} \left(-xz\partial_x - yz\partial_y \right) + \sqrt{x^2 + y^2} \, \partial_z.$$

Hence $\partial_r \, \partial_\varphi \, \partial_\theta$ characterizes sums of analytic functions depending only on two of r, φ, θ.

As a second application we mention that the methods discussed here can be used to check whether a symbolic PDE-solver in computer algebra systems give incomplete or even wrong answers.

As a third application we ask: Is there a representation $\rho : \mathbb{R} \to \mathrm{GL}(3, \mathbb{C})$ of the Lie group $\mathbb{R}$ with prescribed first row $\gamma(x) := (\gamma_1(x), \gamma_2(x), \gamma_3(x))$?

We should have $\gamma(0) = (1, 0, 0)$. Moreover,

$$\rho_{1,1}(x + y) = (\rho(x) \, \rho(y))_{1,1},$$

i.e.

$$\gamma_1(x + y) = \gamma_1(x) \, \gamma_1(y) + \gamma_2(x) \, \rho_{2,1}(y) + \gamma_3(x) \, \rho_{3,1}(y).$$

i.e. check (α, g)-representability of $\gamma_1(x+y) - \gamma_1(x) \, \gamma_1(y)$ with $g := (\gamma_2(x), \gamma_3(x))$ and $\alpha = (\alpha_1, \alpha_2) := (y, y)$!

References

Plesken, W., Robertz, D., 2010. Linear differential elimination for analytic functions. Mathematics in Computer Science 4 (2–3), 231–242.

SPENCER OPERATOR AND MACAULAY INVERSE SYSTEM:

A new approach to control identifiability and other engineering applications

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INTRODUCTION:

Fifty years ago D.C. Spencer invented the first order operator now wearing his name in order to bring the formal study of systems of ordinary differential (OD) or partial differential (PD) equations to that of equivalent first order systems. However, despite its importance, the Spencer operator is rarely used in mathematics today and, up to our knowledge, has never been used in engineering applications or in mathematical physics.

The purpose of this lecture at the second workshop on Differential Equations by Algebraic Methods (DEAM2, february 9-11, 2011, Linz, Austria) is first to recall briefly its definition, both in the framework of systems of OD/PD equations and in the framework of differential modules. The only notation "D" respects the two standard ones existing in the literature but no confusion can be done from the background.

The remaining of the lecture will consist in a series of theorems dealing with explicit and striking applications. In a rough way, the main goal is to prove that the use of the Spencer operator constitutes the *common secret* of the three following famous books published about at the same time in the beginning of the last century, though they do not seem to have anything in common at first sight as they are successively dealing with elasticity theory, commutative algebra, electromagnetism (EM) and general relativity (GR):

1) E. and F. COSSERAT: "Théorie des Corps Déformables", Hermann, Paris, 1909.
2) F.S. MACAULAY: "The Algebraic Theory of Modular Systems", Cambridge, 1916.
3) H. WEYL: "Space, Time, Matter", Springer, Berlin, 1918 (1922, 1958; Dover, 1952).

More precisely, if K is a differential field containing $\mathbb{Q}$ with n commuting derivations ∂_i for $i = 1, ..., n$, we denote by k a subfield of constants and introduce m *differential indeterminates* y^k for $k = 1, ..., m$ and n commuting formal derivatives d_i with $d_i y^k_\mu = y^k_{\mu+1_i}$ where $\mu = (\mu_1, ..., \mu_n)$ is a multi-index with *length* $|\mu| = \mu_1 + ... + \mu_n$, *class i* if $\mu_1 = ... = \mu_{i-1} = 0, \mu_i \neq 0$ and $\mu + 1_i = (\mu_1, ..., \mu_{i-1}, \mu_i + 1, \mu_{i+1}, ..., \mu_n)$. We set $y_q = \{y^k_\mu | 1 \leq k \leq m, 0 \leq |\mu| \leq q\}$ with $y^k_\mu = y^k$ when $|\mu| = 0$. We introduce the non-commutative *ring of differential operators* $D = K[d_1, ..., d_n] = K[d]$ with $d_i a = a d_i + \partial_i a, \forall a \in K$ in the operator sense and the *differential module* $Dy = Dy^1 + ... + Dy^m$. If $\{\Phi^\tau = a^{\tau\mu}_k y^k_\mu\}$ is a finite number of elements in Dy indexed by τ, we may introduce the *differential module of equations* $I = D\Phi \subset Dy$ and the finitely generated *residual differential module* $M = Dy/I$.

Let now X be a manifold with local coordinates (x^i) for $i = 1, ..., n$, tangent bundle $T = T(X)$, cotangent bundle $T^* = T^*(X)$, bundle of r-forms $\wedge^r T^*$ and symmetric tensor bundle $S_q T^*$. If E is a vector bundle over X with local coordinates (x^i, y^k) for $i = 1, ..., n$ and $k = 1, ..., m$, we denote by $J_q(E)$ the *q-jet bundle* of E with local coordinates simply denoted by (x, y_q)

and *sections* $f_q : (x) \to (x, f^k(x), f_i^k(x), f_{ij}^k(x), ...)$ transforming like the section $j_q(f) : (x) \to (x, f^k(x), \partial_i f^k(x), \partial_{ij} f^k(x), ...)$ when f is an arbitrary section of E. For simplicity, we shall denote by the same symbol a vector bundle and its set of local sections. Then both $f_q \in J_q(E)$ and $j_q(f) \in J_q(E)$ are over $f \in E$ and the *Spencer operator* just allows to distinguish them by introducing a kind of "*difference*" through the operator $D : J_{q+1}(E) \to T^* \otimes J_q(E) : f_{q+1} \to j_1(f_q) - f_{q+1}$ with local components $(\partial_i f^k(x) - f_i^k(x), \partial_i f_j^k(x) - f_{ij}^k(x), ...)$ and more generally $(D f_{q+1})_{\mu,i}^k(x) = \partial_i f_\mu^k(x) - f_{\mu+1_i}^k(x)$. In a symbolic way, *when changes of coordinates are not involved*, it is sometimes useful to write down the components of D in the form $d_i = \partial_i - \delta_i$ and the restriction of D to the kernel $S_{q+1} T^* \otimes E$ of the canonical projection $\pi_q^{q+1} : J_{q+1}(E) \to J_q(E)$ is *minus* the *Spencer map* $\delta = dx^i \wedge \delta_i : S_{q+1} T^* \otimes E \to T^* \otimes S_q T^* \otimes E$. The kernel of D is made by sections such that $f_{q+1} = j_1(f_q) = j_2(f_{q-1}) = ... = j_{q+1}(f)$. Finally, if $R_q \subset J_q(E)$ is a *system* of order q on E locally defined by linear equations $\Phi^\tau(x, y_q) \equiv a_k^{\tau\mu}(x) y_\mu^k = 0$ and local coordinates (x, z) for the parametric jets up to order q, the first *prolongation* $R_{q+1} = \rho_1(R_q) = J_1(R_q) \cap J_{q+1}(E) \subset J_1(J_q(E))$ is locally defined by the linear equations $\Phi^\tau(x, y_q) = 0, d_i \Phi^\tau(x, y_{q+1}) \equiv a_k^{\tau\mu}(x) y_{\mu+1_i}^k + \partial_i a_k^{\tau\mu}(x) y_\mu^k = 0$ and has *symbol* $g_{q+1} = R_{q+1} \cap S_{q+1} T^* \otimes E \subset J_{q+1}(E)$. If $f_{q+1} \in R_{q+1}$ is over $f_q \in R_q$, differentiating the identity $a_k^{\tau\mu}(x) f_\mu^k(x) \equiv 0$ with respect to x^i and substracting the identity $a_k^{\tau\mu}(x) f_{\mu+1_i}^k(x) + \partial_i a_k^{\tau\mu}(x) f_\mu^k(x) \equiv 0$, we obtain the identity $a_k^{\tau\mu}(x)(\partial_i f_\mu^k(x) - f_{\mu+1_i}^k(x)) \equiv 0$ and thus the restriction $D : R_{q+1} \to T^* \otimes R_q$.

DEFINITION: R_q is said to be *formally integrable* when the restriction $\pi_q^{q+1} : R_{q+1} \to R_q$ is an epimorphism $\forall r \geq 0$ or, equivalently, when all the equations of order $q + r$ are obtained by r prolongations only $\forall r \geq 0$. In that case, $R_{q+1} \subset J_1(R_q)$ is an equivalent formally integrable first order system on R_q, called the *Spencer form*.

In actual practice, instead of having a linear differential operator $\mathcal{D} : E \xrightarrow{j_q} J_q(E) \xrightarrow{\Phi} J_q(E)/R_q = F$ of order q, we have now the *first Spencer operator* $D_1 : C_0 = R_q \xrightarrow{j_1} J_1(R_q) \to J_1(R_q)/R_{q+1} \simeq T^* \otimes R_q/\delta(g_{q+1}) = C_1$ of order one induced by $D : R_{q+1} \to T^* \otimes R_q$. More generally, introducing the exterior derivative $d : \wedge^r T^* \to \wedge^{r+1} T^*$ and the *Spencer bundles* $C_r = \wedge^r T^* \otimes R_q/\delta(\wedge^{r-1} T^* \otimes g_{q+1})$, the $(r + 1)$-*Spencer operator* $D_{r+1} : C_r \to C_{r+1}$ in the *second Spencer sequence* is induced by $D : \wedge^r T^* \otimes R_{q+1} \to \wedge^{r+1} T^* \otimes R_q : \alpha \otimes \xi_{q+1} \to d\alpha \otimes \xi_q + (-1)^r \alpha \wedge D\xi_{q+1}$ in the *first Spencer sequence*.

DEFINITION: R_q is said to be *involutive* when it is formally integrable and all the sequences $... \xrightarrow{\delta} \wedge^s T^* \otimes g_{q+r} \xrightarrow{\delta} ...$ are exact $\forall 0 \leq s \leq n, \forall r \geq 0$. Equivalently, using a linear change of local coordinates if necessary in order to have δ-*regular* coordinates, we may successively solve the maximum number $\beta_q^n = m - \alpha, \beta_q^{n-1}, ..., \beta_q^1$ of equations with respect to the jet coordinates of class $n, n - 1, ..., 1$ and R_q is involutive if R_{q+1} is obtained by only prolonging the β_q^i equations of class i with respect to $d_1, ..., d_i$ for $i = 1, ..., n$. In that case one can exhibit the *Hilbert polynomial* $dim(R_{q+r})$ in r with leading term $(\alpha/n!)r^n$.

We obtain the following theorem generalizing to PD control systems the well known first order Kalman form of OD control systems where the derivatives of the input do not appear:

THEOREM 1: When R_q is involutive, its Spencer form is involutive and can be modified to a *reduced Spencer form* in such a way that $\beta = dim(R_q) - \alpha$ equations can be solved with respect to the jet coordinates $z_n^1, ..., z_n^\beta$ while $z_n^{\beta+1}, ..., z_n^{\beta+\alpha}$ do not appear. In this case $z^{\beta+1}, ..., z^{\beta+\alpha}$ do not appear in the other equations.

In the algebraic framework already considered, *only two possible formal constructions can be obtained from M*, namely $hom_D(M, D)$ and $M^* = hom_K(M, K)$.

THEOREM 2: $hom_D(M, D)$ is a *right* differential module that can be *converted* to a *left* differential module by introducing the *right* differential module structure of $\wedge^n T^*$. As a differential geometric counterpart, we get the *formal adjoint* of $\mathcal{D}$, namely $ad(\mathcal{D}) : \wedge^n T^* \otimes F^* \to \wedge^n T^* \otimes E^*$ where E^* is obtained from E by inverting the local transition matrices, the simplest example being

the way T^* is obtained from T.

REMARK: Such a result explains why dual objects in physics and engineering are no longer tensors but tensor *densities*.

The filtration $D_0 = K \subseteq D_1 = K \oplus T \subseteq ... \subseteq D_q \subseteq ... \subseteq D$ of D induces a filtration/inductive limit $0 \subseteq M_0 \subseteq M_1 \subseteq ... \subseteq M_q \subseteq ... \subseteq M$ and provides by duality *over K* the projective limit $M^* = R \rightarrow ... \rightarrow R_q \rightarrow ... \rightarrow R_1 \rightarrow R_0 \rightarrow 0$ of formally integrable systems. As D is generated by K and $T = D_1/D_0$, we can define for any $f \in M^*$:

$$(af)(m) = af(m) = f(am), (\xi f)(m) = \xi f(m) - f(\xi m), \forall a \in K, \forall \xi = a^i d_i \in T, \forall m \in M$$

and check $d_i a = a d_i + \partial_i a, \xi \eta - \eta \xi = [\xi, \eta]$ in the operator sense by introducing the standard bracket of vector fields on T. Finally we get $(d_i f)_\mu^k = (d_i f)(y_\mu^k) = \partial_i f_\mu^k - f_{\mu+1_i}^k$ in a coherent way.

THEOREM 3: $R = M^*$ has a structure of differential module induced by the Spencer operator.

REMARK: When $m = 1$ and $D = k[d]$ is a commutative ring isomorphic to the polynomial ring $A = k[\chi]$ for the indeterminates $\chi_1, ..., \chi_n$, this result *exactly* describes the *inverse system* of Macaulay with $-d_i = \delta_i$ ([2], §59,60).

DEFINITION: When A is a commutative integral domain and M a finitely generated module over A, the *socle* of M is $soc(M) = \oplus soc_\mathfrak{m}(M)$ where $soc_\mathfrak{m}(M)$ is the direct sum of all the *isotypical* simple submodules of M isomorphic to $A/\mathfrak{m}$ for $\mathfrak{m} \in ass(M) \cap max(A)$. The *radical* of a module is the intersection of all its maximum *proper* submodules. The quotient of a module by its radical is called the *top*.

The secret of Macaulay is expressed by the next theorem:

THEOREM 4: Instead of using the socle of M over A, one may use duality over k in order to deal with the short exact sequence $0 \rightarrow rad(R) \rightarrow R \rightarrow top(R) \rightarrow 0$ where $top(R)$ is the dual of $soc(M)$.

However, Nakayama's lemma cannot be used in general unless R is finitely generated over k and thus over D. The main idea of Macaulay has been to overcome this difficulty by dealing only with *unmixed* ideals when $m = 1$. As a generalization, one can state:

DEFINITION: One has the *purity filtration* $0 = t_n(M) \subseteq ... \subseteq t_0(M) = t(M) \subseteq M$ where the dimension of the *characteristic variety* of Dm is $< n - r$ when $m \in t_r(M)$ and M is said to be *r-pure* if $t_r(M) = 0, t_{r-1}(M) = M$. A 0-pure module is a torsion-free module.

In actual practice, using an involutive Spencer form and δ-regular coordinates, let us define a differential module N_r by the first order involutive system made up by the equations of class 1+ class 2 + ... +class $(n-r)$, obtaining therefore epimorphisms $N_{r+1} \rightarrow N_r \rightarrow 0$ and $N_r \rightarrow M \rightarrow 0$, $\forall 0 \leq r \leq n$ with $N_0 = M$. One can prove that the image of the induced morphism $t(N_r) \rightarrow t(M)$ is $t_r(M)$ with $t_{r+1}(M) \subseteq t_r(M)$.

THEOREM 5: The sequence $0 \rightarrow M \rightarrow \oplus_{\mathfrak{p} \in ass(M)} M_\mathfrak{p}$ is exact. Moreover the images of all the localizing morphisms $M \rightarrow M_\mathfrak{p}$ are primary modules if and only if M is pure, that is $ass(M)$ only contains equidimensional minimum primes. Moreover this primary embedding corresponds to a primary decomposition of I and leads to decompose R into a sum of subsystems.

Theorem 1 and a partial localization providing the exat sequence $0 \rightarrow M \rightarrow k(\chi_1, ..., \chi_{n-r}) \otimes M$ when M is r-pure, also discovered by Macaulay ([2], §77, 82), lead to the following key result for studying the *identifiability* of OD/PD control systems.

THEOREM 6: When M is n-pure the monomorphism of the preceding theorem becomes an isomorphism (*chinese remainder* theorem) and the minimum number of generators of R is equal to

the maximum number of isotypical components that can be found among the various components of $soc(M)$ or $top(R)$, that is $max_{\mathfrak{m}\in ass(M)}\{dim_{A/\mathfrak{m}}soc_{\mathfrak{m}}(M)\}$.

EXAMPLE: When $n = 1, m = 2, k = \mathbb{R}$ and a is a constant parameter, the OD system $y^1_{xx} - ay^1 = 0, y^2_x = 0$ needs two generators when $a = 0$ but only one generator when $a \neq 0$, namely $\{ch(x), 1\}$ when $a = 1$. Setting $z = y^1 - y^2$ when $a \neq 0$ brings an isomorphic module defined by the single OD equation $z_{xxx} - az_x = 0$ for the only z.

Let us now consider the *conformal Killing system* $\hat{R}_1 \subset J_1(T)$:

$$\omega_{rj}\xi^r_i + \omega_{ir}\xi^r_j + \xi^r\partial_r\omega_{ij} = A(x)\omega_{ij} \Rightarrow n\xi^k_{ij} - \delta^k_i\xi^r_{rj} - \delta^k_j\xi^r_{ri} + \omega_{ij}\omega^{ks}\xi^r_{rs} \Rightarrow \xi^k_{ijr} = 0, \forall n \geq 3$$

obtained by eliminating the arbitrary function $A(x)$, where ω is the Euclidean metric when $n = 2$ (plane) or $n = 3$ (space) and the Minskowskian metric when $n = 4$ (space-time). The brothers Cosserat were only dealing with the *Killing subsystem* $R_1 \subset \hat{R}_1$:

$$\omega_{rj}\xi^r_i + \omega_{ir}\xi^r_j + \xi^r\partial_r\omega_{ij} = 0$$

that is with $\{\xi^k, \xi^k_i \mid \xi^r_r = 0, \xi^k_{ij} = 0\}$ when $A(x) = 0$ while, *in a somehow complementary way*, Weyl was mainly dealing with $\{\xi^r_r, \xi^r_{ri}\}$. Accordingly, one has:

THEOREM 7: The *Cosserat equations* ([1], p 137 for $n = 3$, p 167 for $n = 4$):

$$\partial_r\sigma^{ir} = f^i \qquad , \qquad \partial_r\mu^{ij,r} + \sigma^{ij} - \sigma^{ji} = m^{ij}$$

are *exactly* described by the formal adjoint of the first Spencer operator $D_1 : R_1 \rightarrow T^* \otimes R_1$. Introducing $\phi^{r,ij} = -\phi^{r,ji}$ and $\psi^{rs,ij} = -\psi^{rs,ji} = -\psi^{sr,ij}$, they can be *parametrized* by the formal adjoint of the second Spencer operator $D_2 : T^* \otimes R_1 \rightarrow \wedge^2 T^* \otimes R_1$:

$$\sigma^{ij} = \partial_r\phi^{i,jr} \qquad , \qquad \mu^{ij,r} = \partial_s\psi^{ij,rs} + \phi^{j,ir} - \phi^{i,jr}$$

EXAMPLE: When $n = 2$, lowering the indices by means of the constant metric ω, we just need to look for the factors of ξ_1, ξ_2 and $\xi_{1,2}$ in the integration by part of the sum:

$$\sigma^{11}(\partial_1\xi_1 - \xi_{1,1}) + \sigma^{12}(\partial_2\xi_1 - \xi_{1,2}) + \sigma^{21}(\partial_1\xi_2 - \xi_{2,1}) + \sigma^{22}(\partial_2\xi_2 - \xi_{2,2}) + \mu^{12,r}(\partial_r\xi_{1,2} - \xi_{1,2r})$$

THEOREM 8: The *Weyl equations* ([3], §35) are *exactly* described by the formal adjoint of the first Spencer operator $D_1 : \hat{R}_2 \rightarrow T^* \otimes \hat{R}_2$ when $n = 4$ and can be parametrized by the formal adjoint of the second Spencer operator $D_2 : T^* \otimes \hat{R}_2 \rightarrow T^* \otimes \hat{R}_2$. In particular, among the components of the Spencer operator, one has $\partial_i\xi^r_{rj} - \xi^r_{ijr} = \partial_i\xi^r_{rj}$ and thus the components $\partial_i\xi^r_{rj} - \partial_j\xi^r_{ri} = F_{ij}$ of the EM *field* with EM *potential* $\xi^r_{ri} = A_i$ coming from the second order jets (*elations*). It follows that D_2 *projects onto* the first set of Maxwell equations described by the exterior derivative $d : \wedge^2 T^* \rightarrow \wedge^3 T^*$ while, by duality, the second set of Maxwell equations thus *appears among* the Weyl equations which *project onto* the Cosserat equations because of the inclusion $R_1 \simeq R_2 \subset \hat{R}_2$.

REMARK: Though striking it may look like, *there is no conceptual difference between the Cosserat and Maxwell equations on space-time*. As a byproduct, separating space from time, there is no conceptual difference between the Lamé constants (mass per unit volume) of elasticity and the magnetic (dielectric) constants of EM appearing in the respective wave speeds. This result perfectly agrees with *piezzoelectricity* (quadratic Lagrangian in strain and electric fields $A^{ijk}\epsilon_{ij}E_k \Rightarrow \sigma^{ij} = A^{ijk}E_k$) and *photoelasticity* (cubic Lagrangian $B^{ijkl}\epsilon_{ij}E_kE_l \Rightarrow D^l = (B^{ijkl}\epsilon_{ij})E_k \Rightarrow$ refraction index $n(\epsilon)$) which are field-matter coupling phenomena, but contradicts *gauge theory*.

EXAMPLE: The free movement of a body in a constant static gravitational field $\vec{g}$ is described by $\frac{d\vec{x}}{dt} - \vec{v} = 0, \frac{d\vec{v}}{dt} - \vec{g} = 0, \frac{\partial\vec{g}}{\partial xi} - 0 = 0$ where the "speed" is considered as a Lorentz rotation, that is as a first jet. Hence an *accelerometer* merely helps measuring the part of the Spencer operator dealing with second order jets (*equivalence principle*).

In order to justify the last remark, let G be a Lie group with identity e and parameters a acting on X through the group action $X \times G \rightarrow X : (x, a) \rightarrow y = f(x, a)$ and (local) infinitesimal

generators θ_τ satisfying $[\theta_\rho, \theta_\sigma] = c_{\rho\sigma}^\tau \theta_\tau$ for $\rho, \sigma, \tau = 1, ..., dim(G)$. We may prolong the graph of this action by differentiating q times the action law in order to eliminate the parameters in the following commutative and exact diagram where $\mathcal{R}_q$ is a Lie groupoid with *source* projection α_q and *target* projection β_q when q is large enough:

$$
\begin{array}{ccccccc}
0 \to & X \times G & \longrightarrow & & \mathcal{R}_q & & \to 0 \\
 & \| & & \alpha_q \swarrow & & \searrow \beta_q & \\
 & X \times G & \to & X & \times & X &
\end{array}
$$

The link between the various sections of the trivial principal bundle on the left (*gauging procedure*) and the various corresponding sections of the Lie groupoid on the right with respect to the source projection is expressed by the next commutative and exact diagram:

$$
\begin{array}{ccccccc}
0 \to & & X & \times & G & = & \mathcal{R}_q & \to 0 \\
a = cst \uparrow\downarrow\uparrow\, a(x) & & & & & j_q(f) \uparrow\downarrow\uparrow\, f_q & \\
& & X & & & = & X &
\end{array}
$$

Introducing the Lie algebra $\mathcal{G} = T_e(G)$ and the corresponding Lie algebroid $R_q \subset J_q(T)$, we obtain the following commutative and exact diagram:

$$
\begin{array}{ccccccc}
0 \to & & X & \times & \mathcal{G} & = & R_q & \to 0 \\
\lambda = cst \uparrow\downarrow\uparrow\, \lambda(x) & & & & & j_q(\xi) \uparrow\downarrow\uparrow\, \xi_q & \\
& & X & & & = & X &
\end{array}
$$

where the upper isomorphism is described by $\lambda^\tau(x) \to \xi_\mu^k(x) = \lambda^\tau(x)\partial_\mu\theta_\tau^k(x)$ for q large enough. The unusual Lie algebroid structure on $X \times \mathcal{G}$ is described by the formula: $([\lambda, \lambda'])^\tau = c_{\rho\sigma}^\tau \lambda^\rho \lambda'^\sigma + (\lambda^\rho\theta_\rho).\lambda'^\tau - (\lambda'^\sigma\theta_\sigma).\lambda^\tau$ which is induced by the ordinary bracket $[\xi, \xi']$ on T and thus depends on the action. Applying the Spencer operator, we finally obtain $\partial_i\xi_\mu^k(x) - \xi_{\mu+1_i}^k(x) = \partial_i\lambda^\tau(x)\partial_\mu\theta_\tau^k(x)$.

CONCLUSION:

In gauge theory, the structure of EM is coming from the unitary group $U(1)$, the unit circle in the complex plane, which is *not* acting on space-time while we have explained the structure of EM from that of the conformal group of space-time, with a shift by one step in the interpretation of the (second) Spencer sequence involved because the "*fields*" are now sections of C_1 parametrized by D_1 and thus killed by D_2. Accordingly, we may say:

" TO ACT OR NOT TO ACT, THAT IS THE QUESTION "

and hope future will fast give an answer !.

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Triangularization of general linear systems of partial differential equations based on pure differential modules

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The purpose of the talk is the constructive study of the concept of *purity filtration* of a differential module introduced in algebraic analysis and the theory of D-modules. The purity filtration is a natural filtration of a differential module defined by its submodules formed by its elements of codimension (or grade) at least r.

The purity filtration was studied by Björk [1, 2] using spectral sequences, by Sato and Kashiwara [6, 11] using associated cohomology and, more recently, by Pommaret [7, 8] using modified Spencer forms. Moreover, in a recent "tour de force", Barakat was able to implement the computation of the corresponding spectral sequences [3] in a GAP 4 package called homalg [4], which gives one a way to compute the purity filtration of a differential module.

In this talk, we show how the purity filtration can be simply characterized by means of basic concepts and tools of module theory and homological algebra, which avoids the use of sophisticated homological algebra concepts such as spectral sequences, associated cohomology and Spencer cohomology. Moreover, an effective algorithm for the computation of the purity filtration is explained [9, 10] and illustrated by means of its implementation in the Maple package PurityFiltration built upon OreModules [5]. We also use the computation of the purity filtration of a differential module to show that every linear system of partial differential equations is equivalent to a particular block-triangular linear system of partial equations, which allows an integration of the system in cascade by solving equidimensional homogeneous linear systems [9, 10]. We show that the PurityFiltration package can be used to find closed-form solutions of many over-/under-determined linear systems of partial differential equations which cannot be integrated by Maple. Finally, we explain interesting features of our algorithm using its recent implementation in the AbelianSystems package of homalg, developed for abelian categories in collaboration with Barakat, which allows us to start investigating the purity filtration of linear systems over non-regular Auslander rings appearing, for instance, in algebraic geometry.

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Integration of Liouvillian Functions

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Abstract

A decision algorithm for finding elementary integrals of transcendental Liouvillian functions will be outlined. Parameters that are linearly involved in the integrand can also be solved for, which can be used to find linear relations for definite parameter integrals. Examples of indefinite and definite integrals which can be handled will be given.

1. Introduction

In terms of differential algebra the problem of elementary integration can be stated as follows. Given a differential field (F, D) and $f \in F$, compute g from some elementary extension of (F, D) such that $Dg = f$ or prove that no such g exists. This problem has been solved for various classes of fields F. For rational functions $(C(x), \frac{d}{dx})$ such a g always exists and algorithms to compute it are known already for a long time. Risch's original algorithm (Risch (1969)) solves this problem for (F, D) being a transcendental elementary extension of $(C(x), \frac{d}{dx})$. Later this has been extended towards integrands being transcendental Liouvillian functions by Singer et al. (1985) via the use of regular log-explicit extensions of $(C(x), \frac{d}{dx})$. Also Bronstein (1990, 1997) and several other authors published related results. Our algorithm extends this to handling transcendental Liouvillian extensions (F, D) of the constants directly without the need to embed them into log-explicit extensions. For example, this means that $\int x^a \, dx = \frac{x^{a+1}}{a+1}$ can be computed without including $\log(x)$ in the differential field F. In addition, the algorithm is more efficient than the result in Singer et al. (1985) by introducing a reformulation of the Rothstein-Trager criterion and incorporating results from Bronstein (1997).

All fields considered are implicitly understood to be of characteristic zero. Before discussing the main results recall the following definitions.

Definition 1. Let (K, D) a differential field. A differential field extension $(F, D) = (K(t_1, \ldots, t_n), D)$ is called a regular Liouvillian extension of (K, D), if

(1) all t_i are algebraically independent over K,

(2) $\text{Const}(F) = \text{Const}(K)$, and

(3) each t_i is a Liouvillian monomial over $F_{i-1} := K(t_1, \ldots, t_{i-1})$, i.e., either

    (a) $Dt_i \in F_{i-1}$, in this case t_i is called primitive over F_{i-1}, or

    (b) $\frac{Dt_i}{t_i} \in F_{i-1}$, in this case t_i is called hyperexponential over F_{i-1}.

Definition 2. Let (F, D) a differential field. A differential field extension $(E, D) = (F(t_1, \ldots, t_n), D)$ is called an elementary extension of (F, D) if each t_i is elementary over $E_{i-1} := F(t_1, \ldots, t_{i-1})$, i.e.

(1) t_i is algebraic over E_{i-1}, or

(2) $Dt_i = \frac{Df}{f}$ for some $f \in E_{i-1}$ (i.e. t_i is a logarithm of f), or

(3) $\frac{Dt_i}{t_i} = Df$ for some $f \in E_{i-1}$ (i.e. t_i is an exponential of f).

We say that $f \in F$ has an elementary integral over (F, D) if there exists an elementary extension (E, D) of (F, D) and $g \in E$ such that $Dg = f$.

2. Parametric integration

We present a decision procedure for the following parametric variant of the problem of elementary integration.

Problem 3 (Parametric elementary integration). Given (F, D) a regular Liouvillian extension of its subfield of constants C and $f_0, \ldots, f_m \in F$.
Compute a vector space basis of all $(c_0, \ldots, c_m) \in C^{m+1}$ such that the linear combination $c_0 f_0 + \cdots + c_m f_m$ has an elementary integral over (F, D), together with corresponding g's from some elementary extension of F such that

$$c_0 f_0 + \cdots + c_m f_m = Dg.$$

The fact that we consider the parametric problem is not merely a side-effect implied by Theorem 4, but is also useful in its own right when we deal with definite integrals. Definite integrals are not only computed via the evaluation of antiderivatives. If the integral depends on a parameter one can try to compute linear difference/differential equations that are satisfied by the parameter integral even when no antiderivative of the integrand is available. This is based on the following principle, sometimes called *differentiating under the integral sign* or *Creative Telescoping*. A relation of the form

$$c_0 f_0(x) + \cdots + c_m f_m(x) = g'(x),$$

upon integrating over some interval (a, b), gives rise to a relation of the corresponding integrals

$$c_0 \int_a^b f_0(x)dx + \cdots + c_m \int_a^b f_m(x)dx = g(b) - g(a).$$

3. Algorithm

In the following we will discuss some aspects of the algorithm that solves Problem 3. For more details see Raab (2012). The algorithm follows the general recursive structure of its precursors proceeding through the transcendental extensions one by one. Integrands

from $F = C(t_1, \ldots, t_n)$ are reduced to integrands from the differential subfield $K = C(t_1, \ldots, t_{n-1})$ and at the same time parts of the integral are computed by solving auxiliary problems in K, which we do not mention in detail here.

Then a refined version of Liouville's theorem has to be used for reducing the question of having an elementary integral over (F, D) to having an elementary integral over (K, D). Thereby the original problem is reduced to a problem of the same type but over a smaller field. A special case of the following theorem is already implicitly contained in Singer et al. (1985). When dealing with non-elementary extensions this naturally leads to a parametric version of the problem as above even when we started with just one single integrand.

Theorem 4. *Assume t is transcendental over (K, D) and $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Let $f \in K$ such that f has an elementary integral over $(K(t), D)$, then the following hold.*
  (1) If t is elementary over K, then f has an elementary integral over K.
  (2) If t is primitive over K, then there exists $c \in C$ such that $f - cDt$ has an elementary integral over K.
  (3) If t is hyperexponential over K, then there exists $c \in C$ such that $f - c\frac{Dt}{t}$ has an elementary integral over K.

Above refinement is crucial to obtain a decision procedure for Liouvillian extensions. Without it some elementary integrals would not be found. Also Bronstein (1997) presented generalizations of parts of Risch's algorithm to certain types of non-elementary extensions, but he did not consider the appropriate parametric versions needed. So, for example, with the results given there one does not find the integral

$$\int \frac{(x+1)^2}{x \log(x)} + \mathrm{li}(x)\, dx = x\, \mathrm{li}(x) + \int \frac{2x+1}{x \log(x)}\, dx = (x+2)\, \mathrm{li}(x) + \log(\log(x)),$$

where $\mathrm{li}(x)$ is the logarithmic integral $\mathrm{li}(x) = \int_0^x \frac{1}{\log(t)}\, dt$.

In some sense our algorithm can be viewed as unification of the algorithms presented in (Singer et al., 1985, Theorem A1) and Bronstein (1997): On the one hand it is a full decision procedure for parametric elementary integration over transcendental Liouvillian extensions. On the other hand it also minimizes the computations done in algebraic extensions and tries to avoid factorization into irreducibles as much as possible, which improves the efficiency.

¿From the algorithmic point of view the main improvement compared to the previous algorithms is in how the necessary restrictions on the linear combinations of the integrands are determined during computation of the logarithmic part of the integral. To this end Singer et al. (1985) relies on irreducible factorization of the denominator in $\overline{C}K[t_n]$ with subsequent partial fraction decomposition. Whereas the algorithm for the single-integrand case given in Bronstein (1997) avoids computing unnecessary algebraic extensions and complete factorization, but does not carry over to the parametric case. However, reformulating the Rothstein-Trager resultant appropriately we obtained an algorithm which is parametric, eliminates the need for full factorization, and reduces computations in algebraic extensions.

4. Examples

First we briefly give an example involving polylogarithms that has an integral in the same field the integrand is taken from.

Example Let $C = \mathbb{Q}$ and $F = C(t_1, t_2, t_3, t_4)$, where $Dt_1 = 1, Dt_2 = \frac{1}{t_1-1}, Dt_3 = -\frac{t_2}{t_1}, Dt_4 = \frac{t_3}{t_1}$. Here the polylogarithms $\text{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt$ and $\text{Li}_3(x) = \int_0^x \frac{\text{Li}_2(t)}{t}\, dt$ are represented by t_3 and t_4 respectively. Then the algorithm computes

$$\int \frac{\text{Li}_3(x) - x\,\text{Li}_2(x)}{(1-x)^2}\, dx = \frac{x}{1-x}\left(\text{Li}_3(x) - \text{Li}_2(x)\right) + \frac{\log(1-x)^2}{2}.$$

Even over regular Liouvillian extensions of more general (K, D) the algorithm also successfully computes an antiderivative for some inputs. We illustrate this by the following example using Bessel functions, where the integral is found in an elementary extension of the input field.

Example Let $K = \mathbb{Q}(\pi, n)(\theta_1, \theta_2)$, where $D\theta_1 = 1, D\theta_2 = \theta_2^2 - \frac{2n+1}{\theta_1}\theta_2 + 1$, and let $F = K(t_1, t_2)$, where $Dt_1 = (-\theta_2 + \frac{n}{\theta_1})t_1, Dt_2 = \frac{2}{\pi\theta_1 t_1^2}$. Then (F, D) is a regular Liouvillian extension of (K, D), where θ_2 represents the shift quotient $\frac{J_{n+1}(x)}{J_n(x)}$, t_1 represents $J_n(x)$, and t_2 represents $\frac{Y_n(x)}{J_n(x)}$. With this representation the following integral is straightforwardly obtained by the algorithm

$$\int \frac{1}{xJ_n(x)Y_n(x)}\, dx = \frac{\pi}{2} \ln\left(\frac{Y_n(x)}{J_n(x)}\right).$$

We conclude with an example of a parameter integral, where we are interested in computing a linear ODE satisfied by it.

Example Consider the parameter integral $I_r(x) = \int_0^{\pi/2}(1 - x^2\sin(t)^2)^r dt$ and abbreviate the integrand by $f(r, x, t) := (1 - x^2\sin(t)^2)^r$. Let $C = \mathbb{Q}(i, r, x)$ and $F = C(t_1, t_2)$, where $Dt_1 = it_1, Dt_2 = \frac{2irx^2(t_1^4-1)}{x^2t_1^4+(4-2x^2)t_1^2+x^2}t_2$. Then t_1 represents e^{it} and t_2 represents $f(r, x, t)$. We apply the algorithm to $f_i = \frac{\partial^i f}{\partial x^i}$ for $i = 0, 1, 2$ to obtain

$$x(1-x^2)f_2 + (2(r-1)x^2 + 1)f_1 + 2rxf_0 = \frac{d}{dt}\left(rx\sin(2t)(1 - x^2\sin(t)^2)^{r-1}\right),$$

which translates to the following ODE for the parameter integral

$$x(1-x^2)I_r''(x) + (2(r-1)x^2 + 1)I_r'(x) + 2rxI_r(x) = 0.$$

Note that the specializations $r = \pm\frac{1}{2}$ give the complete elliptic integrals $E(x)$ and $K(x)$.

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Nonlinear differential elimination for analytic functions

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1. Introduction

The aim of this (ongoing) work is to develop methods of symbolic computation for certain sets of analytic functions of several complex variables. A central question is whether such a set S, which is given in terms of a parametrization, has an implicit description in terms of partial differential equations and inequations. In Plesken and Robertz (2010), this problem was solved for parametrizations of the form

$$f_1(\alpha_1(z)) \, g_1(z) + \ldots + f_k(\alpha_k(z)) \, g_k(z), \qquad \alpha_i, g_j \text{ fixed analytic,} \quad (1)$$

where $z = (z_1, \ldots, z_n)$, and methods were presented which compute parameters $f_1, \ldots, f_k$ realizing a given $u \in S$; cf. the extended abstract of W. Plesken in this collection. Here we concentrate on sets S of bilinear expressions

$$f_1(\alpha_1(z)) \, g_1(\beta_1(z)) + \ldots + f_k(\alpha_k(z)) \, g_k(\beta_k(z)), \quad \alpha_i, \beta_j \text{ fixed analytic.} \quad (2)$$

At the time of writing, more general results have been found than were presented in the talk.

Whereas in the linear case (1) an implicit description in terms of linear partial differential equations can always be computed, in the case (2) the use of inequations cannot be avoided in general. The differential Thomas decomposition into simple systems (Thomas (1937); Bächler et al. (2010)) is a valuable tool in this context.

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Among the possible applications of the work in this project we mention improvement of symbolic solving of PDEs, in particular discussion of solutions of a given form as above.

Related elimination problems for ordinary differential equations have been treated in Gao (2003), Rueda and Sendra (2010).

2. Differential Elimination

The process of finding differential polynomials (Kolchin (1973)) in one differential indeterminate that vanish under substitution of all functions in S, which are all given as in (2), is a nonlinear differential elimination problem: from

$$\frac{\partial^{|\mu|} u}{\partial z^\mu} - \frac{\partial^{|\mu|}}{\partial z^\mu}\left(\sum_{i=1}^{k}(f_i \circ \alpha_i) \cdot (g_i \circ \beta_i)\right) = 0, \qquad \mu \in (\mathbb{Z}_{\geq 0})^n, \qquad (3)$$

we would like to derive consequences that are polynomials in the $u_\mu := \frac{\partial^{|\mu|} u}{\partial z^\mu}$.

One way to proceed is to fix an upper bound d on the order of differentiation, define the ideal I generated by the left hand sides in (3), $|\mu| \leq d$, where all $(\partial_\eta f_i) \circ \alpha_i$ and $(\partial_\zeta g_j) \circ \beta_j$ are replaced by algebraically independent symbols, and compute the intersection of I with the polynomial ring in the jet variables u_μ, $|\mu| \leq d$. This subproblem can be solved by standard elimination methods in commutative algebra, but is computationally very difficult in general. We apply the "elimination by degree steering" method developed in Plesken and Robertz (2008). As a termination criterion for the differential elimination one has to show that the differential system does not admit more analytic solutions than are given in S.

3. Examples

In work by Neuman, Rassias, Šimša and several others (cf. Rassias and Šimša (1995) and the references therein), some generalizations of Wronskian determinants have been developed to characterize certain decomposable functions. We found new determinantal implicit descriptions of certain sets S of analytic functions given as in (2). In general, the use of these determinantal descriptions require partitioning the given set S by imposing inequations, as is demonstrated by the following two examples. Note also that parameters realizing a function in S are not uniquely determined in general (e.g., $(x + 1)y + yz = xy + y(z + 1)$ in the second example).

Proposition 3.1. *The set $S = \{f_1(w)f_2(x) + f_3(y)f_4(z) \mid f_i \ \text{analytic}\}$ admits the following implicit description in terms of partial differential equations and inequations. Define the following subsets of S (where each f_i is an arbitrary analytic function and f_i' denotes its first derivative):*

$$S_1 := \{f_1(w) + f_3(y)\} \cup \{f_1(w) + f_4(z)\} \cup \{f_2(x) + f_3(y)\} \cup \{f_2(x) + f_4(z)\},$$

$$S_2 := \{f_1(w) + f_3(y)f_4(z) \mid f_3' \neq 0 \neq f_4'\} \cup \{f_2(x) + f_3(y)f_4(z) \mid f_3' \neq 0 \neq f_4'\},$$

$$S_3 := \{f_1(w)f_2(x) + f_3(y) \mid f_1' \neq 0 \neq f_2'\} \cup \{f_1(w)f_2(x) + f_4(z) \mid f_1' \neq 0 \neq f_2'\},$$

$$S_4 := \{f_1(w)f_2(x) + f_3(y)f_4(z) \mid f_1' \neq 0, \ f_2' \neq 0, \ f_3' \neq 0, \ f_4' \neq 0\}.$$

Then $S = S_1 \uplus \ldots \uplus S_4$ and S_i equals the set of analytic solutions of (Σ_0, Σ_i),

$$\Sigma_0: \quad u_{w,y} = u_{w,z} = u_{x,y} = u_{x,z} = 0, \qquad \Sigma_1: \quad u_w\, u_x = 0, \quad u_y\, u_z = 0,$$

$$\Sigma_2: \quad u_{y,z} \neq 0, \quad u_w\, u_x = 0, \quad \begin{vmatrix} u_y & u_{y,y} \\ u_{y,z} & u_{y,y,z} \end{vmatrix} = 0, \quad \begin{vmatrix} u_z & u_{y,z} \\ u_{z,z} & u_{y,z,z} \end{vmatrix} = 0,$$

$$\Sigma_3: \quad u_{w,x} \neq 0, \quad u_y\, u_z = 0, \quad \begin{vmatrix} u_w & u_{w,w} \\ u_{w,x} & u_{w,w,x} \end{vmatrix} = 0, \quad \begin{vmatrix} u_x & u_{w,x} \\ u_{x,x} & u_{w,x,x} \end{vmatrix} = 0,$$

$$\Sigma_4: \quad u_{w,x} \neq 0, \quad u_{y,z} \neq 0, \quad \begin{vmatrix} u & u_w & u_y \\ u_x & u_{w,x} & 0 \\ u_z & 0 & u_{y,z} \end{vmatrix} = 0.$$

Proposition 3.2. *The set $S = \{f_1(x)f_2(y) + f_3(y)f_4(z) \mid f_i \ \text{analytic}\}$ admits the following implicit description in terms of partial differential equations and inequations. Define the following subsets of S (where each f_i is an arbitrary analytic function and f_i' denotes its first derivative):*

$$S_1 := \{f_1(x) + f_3(y)\} \cup \{f_2(y) + f_4(z)\},$$

$$S_2 := \{f_1(x) + f_3(y)f_4(z) \mid f_3' \neq 0 \neq f_4'\},$$

$$S_3 := \{f_1(x)f_2(y) + f_4(z) \mid f_1' \neq 0 \neq f_2'\},$$

$$S_4 := \{f_2(y)\,(f_1(x) + f_4(z)) \mid f_1' \neq 0 \neq f_4'\},$$

$$S_5 := \{f_1(x)f_2(y) + f_3(y)f_4(z) \mid f_1' \neq 0 \neq f_2', \ f_3' \neq 0 \neq f_4', \ f_2/f_3 \neq \text{const}\}.$$

Then $S = S_1 \uplus \ldots \uplus S_5$ and S_i equals the set of analytic solutions of (Σ_0, Σ_i),

$$\Sigma_0: \quad u_{x,z} = 0, \qquad \Sigma_1: \quad u_x\, u_z = 0, \quad u_{y,z} = u_{x,y} = 0,$$

$$\Sigma_2 : \quad u_{y,z} \neq 0, \quad u_{x,y} = 0, \quad \begin{vmatrix} u_y & u_{y,y} \\ u_{y,z} & u_{y,y,z} \end{vmatrix} = 0, \quad \begin{vmatrix} u_z & u_{y,z} \\ u_{z,z} & u_{y,z,z} \end{vmatrix} = 0,$$

$$\Sigma_3 : \quad u_{x,y} \neq 0, \quad u_{y,z} = 0, \quad \begin{vmatrix} u_x & u_{x,x} \\ u_{x,y} & u_{x,x,y} \end{vmatrix} = 0, \quad \begin{vmatrix} u_y & u_{x,y} \\ u_{y,y} & u_{x,y,y} \end{vmatrix} = 0,$$

$$\Sigma_4 : \quad u_x \neq 0, \quad u_z \neq 0, \quad \begin{vmatrix} u & u_x \\ u_y & u_{x,y} \end{vmatrix} = 0, \quad \begin{vmatrix} u & u_y \\ u_z & u_{y,z} \end{vmatrix} = 0,$$

$$\Sigma_5 : \begin{cases} u_{x,y} \neq 0, \quad u_{y,z} \neq 0, \quad \begin{vmatrix} u_x & u_{x,x} \\ u_{x,y} & u_{x,x,y} \end{vmatrix} = 0, \quad \begin{vmatrix} u_z & u_{y,z} \\ u_{z,z} & u_{y,z,z} \end{vmatrix} = 0, \\ \begin{vmatrix} u & u_y & u_{y,y} \\ u_x & u_{x,y} & u_{x,y,y} \\ u_z & u_{y,z} & u_{y,y,z} \end{vmatrix} = 0, \quad \begin{vmatrix} u_x & u_{x,y} \\ u_z & u_{y,z} \end{vmatrix} \neq 0. \end{cases}$$

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Implicitization of linear DPPEs by perturbed differential resultants

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Abstract

The development of differential elimination techniques, similar to the algebraic existing ones, is an active field of research. We study the implicitization, by differential resultants, of a system $\mathcal{P}$ of n linear ordinary differential polynomial parametric equations (linear DPPEs) in $n-1$ differential parameters. We consider a linear perturbation of $\mathcal{P}$ and we use it to give an algorithm that returns an implicitization of $\mathcal{P}$.

Key words: differential polynomial parametric equations, differential resultant, implicitization

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1. Introduction

In (3), characteristic set methods were used to solve the differential implicitization problem, for differential rational parametric equations. In (5), we defined linear complete differential resultants as a generalization of Carra'Ferro's differential resultant (1) (in the linear case) and, we proved that when nonzero the differential resultant gives the implicit equation of $\mathcal{P}$. As in the algebraic case, differential resultants often vanish under specialization, which prevented us from giving an implicitization algorithm in (5). Motivated by Canny's method and its generalizations (see references in (2)), in the present work, we consider a linear perturbation of a given system of linear DPPEs. An extended version of this work can be found in (4).

Let $\mathbb{K}$ be an ordinary differential field with derivation ∂, (e.g. $\mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$). Let $X = \{x_1, \ldots, x_n\}$ and $U = \{u_1, \ldots, u_{n-1}\}$ be sets of differential indeterminates over $\mathbb{K}$. Let $\mathbb{N}_0$ be the set of natural numbers including 0. For $k \in \mathbb{N}_0$, we denote by x_{ik} the k-th derivative of x_i and, for x_{i0} we simply write x_i. We denote by $\mathbb{K}\{X\}$ the ring

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of differential polynomials in the differential indeterminates $x_1, \ldots, x_n$. Analogously for $\mathbb{K}\{U\}$. As defined in (5), we consider the system of linear DPPEs

$$\mathcal{P}(X, U) = \begin{cases} x_1 = P_1(U), \\ \quad \vdots \\ x_n = P_n(U), \end{cases} \tag{1}$$

where $P_1, \ldots, P_n \in \mathbb{K}\{U\}$, with degree at most 1 and not all $P_i \in \mathbb{K}$, $i = 1, \ldots, n$. There exists $a_i \in \mathbb{K}$ and an homogeneous differential polynomial $H_i \in \mathbb{K}\{U\}$ such that

$$F_i(X, U) = x_i - P_i(U) = x_i - a_i + H_i(U).$$

Given $P \in \mathbb{K}\{X \cup U\}$ and $y \in X \cup U$, we denote by $\mathrm{ord}(P, y)$ the order of P in the variable y. If P does not have a term in y then we define $\mathrm{ord}(P, y) = -1$. To ensure that the number of parameters is $n - 1$, we assume that, for each $j \in \{1, \ldots, n - 1\}$, there exists $i \in \{1, \ldots, n\}$ such that $\mathrm{ord}(F_i, u_j) \geq 0$.

The implicit ideal of the system (1) is the differential prime ideal

$$\mathrm{ID} = \{f \in \mathbb{K}\{X\} \mid f(P_1(U), \ldots, P_n(U)) = 0\}.$$

Given a characteristic set $\mathcal{C}$ of ID then $n - |\mathcal{C}|$ is the (differential) dimension of ID. If $\dim(\mathrm{ID}) = n - 1$ then $\mathcal{C} = \{A(X)\}$ for some irreducible differential polynomial $A \in \mathbb{K}\{X\}$. We call A a characteristic polynomial of ID and $A(X) = 0$ an implicit equation of $\mathcal{P}(X, U)$.

2. Linear complete differential resultants from linear DPPEs

Linear complete differential resultants were defined in (5). The purpose of this definition was, to adjust the number of differential polynomials needed to compute the resultant to the order of derivation of the variables $u_1, \ldots, u_{n-1}$ in $F_1, \ldots, F_n$.

For each $j \in \{1, \ldots, n - 1\}$, we define the positive integers

$$\gamma_j := \min\{o_i - \mathcal{O}(F_i, u_j) \mid i = 1, \ldots, n\}, \quad \gamma := \sum_{j=1}^{n-1} \gamma_j,$$

$\mathcal{O}(F_i, u_j) = \mathrm{ord}(F_i, u_j)$, if $\mathrm{ord}(F_i, u_j) \geq 0$ and $\mathcal{O}(F_i, u_j) = 0$, if $\mathrm{ord}(F_i, u_j) = -1$. Let $N = \sum_{i=1}^n o_i$ then $\gamma \leq N - o_i$, for all $i \in \{1, \ldots, n\}$. The linear complete differential resultant $\partial\mathrm{CRes}(F_1, \ldots, F_n)$ is the Macaulay's algebraic resultant of the differential polynomial set

$$\mathrm{PS} := \{\partial^{N - o_i - \gamma} F_i, \ldots, \partial F_i, F_i \mid i = 1, \ldots, n\},$$

which contains $L = \sum_{i=1}^n (N - o_i - \gamma + 1)$ polynomials in the following set $\mathcal{V}$ of $L - 1$ differential variables

$$\mathcal{V} = \{u_j, u_{j1} \ldots, u_{jN - \gamma_j - \gamma} \mid j = 1, \ldots, n - 1\}.$$

The order $u_1 < \cdots < u_{n-1}$ induces an orderly ranking on U: $u_{i,j} < u_{k,l} \Leftrightarrow (j, i) <_{\mathrm{lex}} (l, k)$. For $i = 1, \ldots, n$ and $k = 0, \ldots, N - o_i - \gamma$ define de positive integer $l(i, k) = (i - 1)(N - \gamma) - \sum_{h=1}^{i-1} o_i + i + k$ in $\{1, \ldots, L\}$. The complete differential resultant matrix $M(L)$ is the $L \times L$ matrix containing the coefficients of $\partial^{N - o_i - \gamma - k} F_i$ as a polynomial

in $\mathbb{D}[\mathcal{V}]$ in the $l(i,k)$-th row, where the coefficients are written in decreasing order with respect to the orderly ranking on U. In this situation:

$$\partial \mathrm{CRes}(F_1, \ldots, F_n) = \det(M(L)).$$

The next matrices will play an important role in this theory.
- Let S be the $n \times (n-1)$ matrix whose entry (i,j) is the coefficient of $u_{n-j\,o_i - \gamma_{n-j}}$ in F_i, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n-1\}$. We call S the leading matrix of $\mathcal{P}(X, U)$.
- Let M_{L-1} be the $L \times (L-1)$ principal submatrix of $M(L)$. We call M_{L-1} the principal matrix of $\mathcal{P}(X, U)$.

3. Characterization of $n-1$ dimensional systems of linear DPPEs

Let (PS) be the ideal generated by PS in $\mathbb{K}[\mathcal{X}][\mathcal{V}]$ and let [PS] be the differential ideal generated by PS in $\mathbb{K}\{X\}$. Let $\mathcal{A}$ be a characteristic set of [PS] and $\mathcal{A}_0 = \mathcal{A} \cap \mathbb{K}\{X\}$. By (3), ID $= [\mathrm{PS}] \cap \mathbb{K}\{X\} = [\mathcal{A}_0]$. By (5), Lemma 20, we know that $\mathcal{A}$ is a set of linear differential polynomials.

Let $\mathbb{K}[\partial]$ be the ring of differential operators with coefficients in $\mathbb{K}$. Given a nonzero linear differential polynomial B in ID there exist unique differential operators $\mathcal{F}_i \in \mathbb{K}[\partial]$, $i = 1, \ldots, n$ such that

$$B(X, U) = \sum_{i=1}^{n} \mathcal{F}_i(F_i(X, U)).$$

If B belongs to $(\mathrm{PS}) \cap \mathbb{K}\{X\}$ then $\deg(\mathcal{F}_i) \leq N - o_i - \gamma$, $i = 1, \ldots, n$. We define the co-order of B in (PS) as the highest positive integer $\mathrm{c}(B)$ such that $\partial^{\mathrm{c}(B)}B \in (\mathrm{PS})$.

Definition 1. Given a nonzero linear differential polynomial B in $(\mathrm{PS}) \cap \mathbb{K}\{X\}$ (with the previous set up).
  (1) We define the ID-content of B as a greatest common left divisor of $\mathcal{F}_1, \ldots, \mathcal{F}_n$ and denote it by $\mathrm{IDcont}(B)$.
  (2) There exist $\mathcal{L}_i \in \mathbb{K}[\partial]$ such that $\mathcal{F}_i = \mathrm{IDcont}(B)\mathcal{L}_i$, $i = 1, \ldots, n$ and $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are coprime. We define the ID-primitive part of B as $\mathrm{IDprim}(B) = \sum_{i=1}^{n} \mathcal{L}_i(x_i - a_i)$.
  (3) If $\mathrm{IDcont}(B) \in \mathbb{K}$ then we say that B is ID-primitive .

Theorem 2. *If* $\mathrm{rank}(S) = n - 1$, *the following statements are equivalent.*
  (1) The dimension of ID *is* $n - 1$.
  (2) There exists a nonzero linear ID-*primitive differential polynomial* A *in* $(\mathrm{PS}) \cap \mathbb{K}\{X\}$ *such that* $L - \mathrm{rank}(M_{L-1}) = \mathrm{c}(A) + 1$.
In such situation $A(X) = 0$ *is the implicit equation of* $\mathcal{P}(X, U)$.

4. Perturbed systems of linear DPPEs and implicitization algorithm

Let p be an algebraic indeterminate over $\mathbb{K}$ such that $\partial(p) = 0$. Denote by $\mathbb{K}_p = \mathbb{K}\langle p \rangle$ the differential field extension of $\mathbb{K}$ by p. A linear perturbation of the system $\mathcal{P}(X, U)$ is a new system

$$\mathcal{P}_\phi(X, U) = \begin{cases} x_1 = P_1(U) + p\,\phi_1(U), \\ \quad \vdots \\ x_n = P_n(U) + p\,\phi_n(U), \end{cases}$$

where the linear perturbation $\phi = (\phi_1(U), \ldots, \phi_n(U))$ is a family of linear differential polynomials in $\mathbb{K}\{U\}$. For $i = 1, \ldots, n$, let $F_i^\phi(X, U) = F_i(X, U) - p\,\phi_i(U)$.

Let D_ϕ be the lowest degree of p in $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi)$ and let A_{D_ϕ} be the coefficient of p^{D_ϕ} in $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi)$. We write $D_\phi = -1$ if $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi) = 0$. If $D_\phi \geq 0$ then A_{D_ϕ} is a linear differential polynomial in $(\mathrm{PS}) \cap \mathbb{K}\{X\}$ as well as its ID-primitive part A_ϕ. We call A_ϕ the differential polynomial associated to $\mathcal{P}_\phi(X, U)$.

Theorem 3. *Let us assume $D_\phi \geq 0$. If $\mathrm{rank}(S) = n - 1$ and $D_\phi = \mathrm{c}(A_\phi)$ then ID has dimension $n - 1$ and $A_\phi(X) = 0$ is the implicit equation of $\mathcal{P}(X, U)$.*

The next perturbation provides a system $\mathcal{P}_\phi(X, U)$ of degree $D_\phi \geq 0$. We can assume $o_n \geq o_{n-1} \geq \cdots \geq o_1$ to define $\phi = (\phi_1(U), \ldots, \phi_n(U))$ by

$$\phi_i(U) = \begin{cases} u_{n-1, o_1 - \gamma_{n-1}}, & i = 1, \\ u_{n-i, o_i - \gamma_{n-i}} + u_{n-i+1}, & i = 2, \ldots, n-1, \\ u_1, & i = n. \end{cases} \tag{2}$$

We outline the differential implicitization algorithm for linear DPPEs.

- <u>Given</u> the system $\mathcal{P}(X, U)$ of linear DPPEs, with $\mathrm{rank}(S) = n - 1$.
- <u>Decide</u> whether the dimension is $n-1$ and in the affirmative case <u>return</u> a characteristic polynomial of ID.
  (1) Compute $\mathcal{P}_\phi(X, U)$ with perturbation ϕ given by (2).
  (2) Compute $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi)$, D_ϕ and A_{D_ϕ}. If $D_\phi = 0$ RETURN A_{D_ϕ}.
  (3) Compute A_ϕ and $\mathrm{c}(A_\phi)$. If $D_\phi = \mathrm{c}(A_\phi)$ RETURN A_ϕ.
  (4) Compute $\mathrm{rank}(M_{L-1})$.
  (5) If $L - \mathrm{rank}(M_{L-1}) > \mathrm{c}(A_\phi) + 1$ RETURN "dimension less than $n - 1$".
  (6) If $L - \mathrm{rank}(M_{L-1}) = \mathrm{c}(A_\phi) + 1$ RETURN A_ϕ.

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Solving Linear Inhomogeneous Differential Equations.

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Linear differential equations have been considered extensively in the mathematical literature, beginning in the second half of the 19th century. For linear homogeneous ordinary differential equations (ode's) there exists a fairly complete theory, culminating in differential Galois theory and algorithms for finding large classes of solutions. Here this means always a closed form solution in some well defined function space; in particular numerical or graphical solutions are excluded. For inhomogeneous equations, Lagrange's method of variation-of-constants allows finding a special solution if a fundamental system for the homogeneous equation is known.

For linear partial differential equations (pde's) the answer is much less complete. For homogeneous equations factorizations and Loewy decompositions appear to be the best tool for solving them. However, virtually nothing has been done for solving *inhomogeneous* pde's. The situation is complicated by the fact that it does not seem to be possible to adjust Lagrange's method for pde's.

Therefore a new approach is suggested that does not rely on Lagrange's method. It uses the right divisors that may exist for the differential operator corresponding to the left-hand side of an equation, and constructs the inhomogeneities for the lower-order equations corresponding to the factors. In a second step, a special solution for the originally given equation is generated.

In this way, for reducible second-order equations in two independent variables the complete answer is obtained. Furthermore, a certain inhomogeneous third-order system is considered; it may occur when third-order linear homogeneous pde's in the plane are solved that are not completely reducible.

Linear second-order ordinary differential equations. Although for ordinary equations Lagrange's method of variation of constants allows generating a special solution for inhomogeneous equations, it is instructive to obtain it from a non-trivial Loewy decomposition. In this way also the somewhat ad hoc nature of Lagrange's proceeding is avoided. The following proposition distinguishes the basic decomposition types; the proof may be found in Schwarz (2010).

Proposition 1. *Let $Ly = R$ be a reducible linear second-order ode. A special solution y_0 satisfying $Ly_0 = R$ may be obtained as follows. Define $\varepsilon_i(x) \equiv \exp\left(-\int a_i dx\right)$ for $i = 1, 2$ and $D \equiv d/dx$.*

i) If $L = (D + a_2)(D + a_1)$ there holds

$$y_0 = \varepsilon_1(x) \int \frac{\varepsilon_2(x)}{\varepsilon_1(x)} \int \frac{R(x)}{\varepsilon_2(x)} dx dx. \tag{1}$$

ii) If $L = Lclm(D + a_2, D + a_1)$ and $a_2 \neq a_1$ there holds

$$y_0 = \varepsilon_1(x) \int \frac{R(x)}{\varepsilon_1(x)} \frac{dx}{a_2 - a_1} - \varepsilon_2(x) \int \frac{R(x)}{\varepsilon_2(x)} \frac{dx}{a_2 - a_1}. \tag{2}$$

In either case, y_0 is Liouvillian over the extended base field.

The solution in case *i)* involves two nested integrations, whereas in case *ii)* there is only a single one. This is due to the complete reducibility in the latter case; details may be found in Chapter 2 of Schwarz (2007).

Linear second-order partial differential equations. The procedure of the preceding section is generalized now for solving linear inhomogeneous pde's in x and y for an unknown function $z(x, y)$. As usual, equations with mixed or unmixed leading derivatives are distinguished. Equations with leading derivative ∂_{xx} are considered first.

Proposition 2. *Let a reducible equation*

$$Lz \equiv (\partial_{xx} + A_1\partial_{xy} + A_2\partial_{yy} + A_3\partial_x + A_4\partial_y + A_5)z = R \tag{3}$$

be given with $A_1, \ldots, A_5 \in \mathbb{Q}(x, y)$. Define $l_i \equiv \partial_x + a_i\partial_y + b_i$, $a_i, b_i \in \mathbb{Q}(x, y)$ for $i = 1, 2$; $\varphi_i(x, y) = const$ is a first integral of $\frac{dy}{dx} = a_i(x, y)$; $\bar{y} \equiv \varphi_i(x, y)$ and the inverse $y = \psi_i(x, \bar{y})$; both φ_i and ψ_i are assumed to be elementary. Furthermore let

$$\mathcal{E}_i(x, y) \equiv \exp\left(-\int b_i(x, y)|_{y=\psi_i(x,\bar{y})} dx\right)\Big|_{\bar{y}=\varphi_i(x,y)} \tag{4}$$

for $i = 1, 2$. A special solution $z_0(x, y)$ satisfying $Lz_0 = R$ may be obtained by solving first-order equations. Two cases are distinguished.

i) Decomposition $L = l_2l_1$. Defining

$$r(x, y) \equiv \mathcal{E}_2(x, y) \int \frac{R(x, y)}{\mathcal{E}_2(x, y)}\Big|_{y=\psi_2(x,\bar{y})} dx\Big|_{\bar{y}=\varphi_2(x,y)} \tag{5}$$

a special solution is given by

$$z_0(x, y) = \mathcal{E}_1(x, y) \int \frac{r(x, y)}{\mathcal{E}_1(x, y)}\Big|_{y=\psi_1(x,\bar{y})} dx\Big|_{\bar{y}=\varphi_1(x,y)}. \tag{6}$$

ii) Decomposition $L = Lclm(l_2, l_1)$. Defining

$$r \equiv r_0 \int \frac{R}{a_2 - a_1} \frac{dy}{r_0} \quad \text{and} \quad r_0 = \exp\left(-\int \frac{b_1 - b_2}{a_1 - a_2} dy\right) \tag{7}$$

a special solution is given by

$$z_0 = \mathcal{E}_1(x,y) \int \frac{r(x,y)}{\mathcal{E}_1(x,y)}\Big|_{y=\psi_1(x,\bar{y})} dx\Big|_{\bar{y}=\varphi_1(x,y)}$$
$$-\mathcal{E}_2(x,y) \int \frac{r(x,y)}{\mathcal{E}_2(x,y)}\Big|_{y=\psi_2(x,\bar{y})} dx\Big|_{\bar{y}=\varphi_2(x,y)}. \tag{8}$$

Both expressions (6) and (8) for the special solution $z_0(x,y)$ are Liouvillian over the extended base field of $Lz = R$.

The general structure of the special solutions (6) and (8) is similar to that of a second-order ode; it is more complicated due to the shifted integrals in the expressions $\mathcal{E}_i(x,y)$. The results are similar for second-order equations with leading derivative ∂_{xy} and decompositions into principal factors.

If an operator $L \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3$ does not have a principal divisor, there may be a non-principal Laplace divisor defined as follows. Let

$$\mathfrak{l}_m \equiv \partial_{x^m} + a_{m-1}\partial_{x^{m-1}} + \ldots + a_1\partial_x + a_0 \text{ and } \mathfrak{k}_n \equiv \partial_{y^n} + b_{n-1}\partial_{y^{n-1}} + \ldots + b_1\partial_y + b_0$$

be two ordinary operators w.r.t. the variable x or y respectively; $a_i, b_j \in \mathbb{Q}(x,y)$ for all i and j. The ideal $\langle L, \mathfrak{l}_m \rangle$ is called a Laplace divisor $\mathbb{L}_{x^m}(L)$ if L and $\mathfrak{l}_m$ combined form a Janet basis. A Laplace divisor $\mathbb{L}_{y^n}(L)$ is defined analogously. Equations allowing a Laplace divisor of order 2 or 3 three are considered next.

Proposition 3. *Let the equation $Lz \equiv (\partial_{xy} + A_1\partial_x + A_2\partial_y + A_3)z = R$ be given. If the corresponding homogeneous equation has a Laplace divisor $\mathbb{L}_{x^m}(L) = \langle L, \mathfrak{l}_m \rangle$ with $m = 2$ or $m = 3$, the following linear inhomogeneous ode's exist.*

$$z_{xx} + a_1 z_x + a_0 z = r \text{ with } r_y + A_1 r = R_x + (a_1 - A_2)R,$$
$$z_{xxx} + a_2 z_{xx} + a_1 z_x + a_0 z = r \text{ with} \tag{9}$$
$$r_y + A_1 r = R_{xx} + (a_2 - A_2)R_x + (a_1 - a_1 A_2 + A_2^2 - 2A_{2,x})R.$$

If there is a Laplace divisor $\mathbb{L}_{y^n}(L) = \langle L, \mathfrak{k}_n \rangle$ with $n = 2$ or $n = 3$, the following linear inhomogeneous ode's exist.

$$z_{yy} + b_1 z_y + b_0 z = r \text{ with } r_x + A_2 r = R_y + (b_1 - A_1)R,$$
$$z_{yyy} + b_2 z_{yy} + b_1 z_y + b_0 z = r \text{ with} \tag{10}$$
$$r_x + A_2 r = R_{yy} + (b_2 - A_1)R_y + (b_1 - b_2 A_1 + A_1^2 - 2A_{1,y})R.$$

A special solution $z_0(x,y)$ is obtained by solving the inhomogeneous equation $\mathfrak{l}_m z = r$ or $\mathfrak{k}_n z = r$ and adjusting the indeterminate elements such that $Lz_0(x,y) = R$.

A special third-order system. The left intersection ideal of two first-order operators in the plane in general is not principal as has been shown by Grigoriev and Schwarz (2002). Generically it is generated by two third-order operators that form a Janet basis. As a consequence, whenever a third-order equation is not completely reducible but allows only two first-order right factors with a non-principal left intersection, finding the third

element of a differential fundamental system requires to solve the inhomogeneous third-order system corresponding to this intersection ideal. A special solution is determined next.

Proposition 4. *Consider the system*

$$
\begin{aligned}
L_1 z \equiv (\partial_{xxx} &+ A_1\partial_{xyy} + A_2\partial_{yyy} + A_3\partial_{xx} + A_4\partial_{xy} + A_5\partial_{yy} \\
&+ A_6\partial_x + A_7\partial_y + A_8)z = R_1, \\
L_2 z \equiv \partial_{xxy} &+ B_1\partial_{xyy} + B_2\partial_{yyy} + B_3\partial_{xx} + B_4\partial_{xy} + B_5\partial_{yy} \\
&+ B_6\partial_x + B_7\partial_y + B_8)z = R_2.
\end{aligned}
\tag{11}
$$

It is assumed that the coefficients A_i and B_j satisfy the coherence conditions such that the left-hand sides form a Janet basis, and that R_1 and R_2 satisfy the necessary consistency conditions. Let both L_1 and L_2 allow first-order right factors $l_i \equiv \partial_x + a_i\partial_y + b_i$, $i = 1, 2$; the $\mathcal{E}_i(x, y)$ are again defined by (4). A special solution z_0 is given by

$$
z_0 = \mathcal{E}_1(x, y) \int \frac{r(x, y)}{\mathcal{E}_1(x, y)} dx - \mathcal{E}_2(x, y) \int \frac{r(x, y)}{\mathcal{E}_2(x, y)} dx.
\tag{12}
$$

The inhomogeneity $r(x, y)$ obeys the system

$$
\begin{aligned}
r_{xy} + \frac{b_1 - b_2}{a_1 - a_2} r_x &+ \Big(A_3 + B_3(a_1 + a_2) - b_1 - b_2 \\
&+ 2\frac{(a_1 - a_2)_x}{a_1 - a_2} + \frac{a_{1,y}a_2 - a_{2,y}a_1}{a_1 - a_2} \Big) r_y \\
+ \frac{1}{a_1 - a_2}\Big((A_3 &+ B_3(a_1 + a_2) + b_1 + b_2)(b_1 - b_2) \\
&+ 2(b_1 - b_2)_x + b_{1,y}a_2 - b_{2,y}a_1 \Big)r = -R_2 - \frac{R_1}{a_1 - a_2}, \\
r_{yy} + \Big(B_3 &+ \frac{(a_1 - a_2)_y}{a_1 - a_2} + \frac{b_1 - b_2}{a_1 - a_2} \Big)r_y \\
&+ \Big(\frac{(b_1 - b_2)_y}{a_1 - a_2} + B_3\frac{b_1 - b_2}{a_1 - a_2} \Big)r = -\frac{R_2}{a_1 - a_2}.
\end{aligned}
\tag{13}
$$

This system has been solved in Proposition 3.

This result allows determining a special solution for (11) that is *guaranteed* to be Liouvillian over its extended vase field. If only a single right factor is divided out of L_1 or L_2, the resulting second-order operators are in general irreducible and the solution process terminates without a conclusive answer.

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Towards Invariant Solutions for operators
$D_xD_y + aD_x + bD_y + c$

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Abstract

Darboux transformation (DT) of operators of the form $\mathcal{L} = D_xD_y + aD_x + bD_y + c$ is a part of the techniques for solving linear and non-linear PDEs. Darboux formulas allow to construct a DT $\mathcal{L} \to \mathcal{L}_1$ for any non-zero $z_1 \in \ker \mathcal{L}$. Unfortunately, the corresponding transformation of kernels $\ker \mathcal{L} \to \ker \mathcal{L}_1$ transforms z_1 into 0, and z_1 cannot be re-used.

As a first idea on how to overcome this problem, we introduce notion of X- and Y-invariants, which possess several relevant properties.

Keywords: Darboux tranformation, invariant solution

1. Darboux transformations (DTs)

Let K be a differential field of characteristic zero with commuting derivations ∂_x, ∂_y. Let $K[D] = K[D_x, D_y]$ be the corresponding ring of linear partial differential operators over K. Operators $\mathcal{L} \in K[D]$ have the general form $\mathcal{L} = \sum_{i+j=0}^{d} a_{ij}D_x^i D_y^j$, where $a_{ij} \in K$. The formal polynomial $\mathrm{Sym}_{\mathcal{L}} = \sum_{i+j=d} a_{ij}X^iY^j$ in some formal variables X, Y is called the *symbol* of $\mathcal{L}$. One can either assume field K to be either differentially closed, or simply assume that K contains the solutions of those PDEs that we encounter on the way.

DT is a classical tool of super-symmetric quantum mechanics and integrable systems. In these domains, DT is a piece of a large theory involving either shape invariance or the dressing method, see e.g. Olver (1988) and Tsarev (2000).

An operator $\mathcal{L}_1 \in K[D]$ is called a *generalized DT* (gDT) of an operator $\mathcal{L} \in K[D]$, if $\mathrm{Sym}(\mathcal{L}) = \mathrm{Sym}(\mathcal{L}_1)$, and there exist operators $\mathcal{N} \in K[D]$ and

$\mathcal{M} \in K[D]$ such that

$$\mathcal{N} \circ \mathcal{L} = \mathcal{L}_1 \circ \mathcal{M} \ . \tag{1}$$

If $\mathcal{L}$ is a hyperbolic operator of second order,

$$\mathcal{L} = D_x D_y + a D_x + b D_y + c \ , \tag{2}$$

$a, b, c \in K$, and $\mathcal{M}$ of the form $\mathcal{M} = D_x + m$, or $\mathcal{M} = D_y + m$, $m \in K$, then this is a classical DT considered by Darboux (1889). Two special cases $\mathcal{M} = D_x + b$ and $\mathcal{M} = D_y + a$ are known as Laplace transformations.

Theorem 1.1. *(Darboux (1889)) Let $z_1 \in \ker \mathcal{L}$ for some* (2)*, then operator $\mathcal{M}$ constructed using Darboux Wronskian formulas, that is*

$$M(z) = -\frac{\begin{vmatrix} z & z_x \\ z_1 & (z_1)_x \end{vmatrix}}{z_1} \ , \quad \left(corresp. \quad M(z) = -\frac{\begin{vmatrix} z & z_y \\ z_1 & (z_1)_y \end{vmatrix}}{z_1} \right) \tag{3}$$

defines a DT.

Theorem 1.2. *(Shemyakova (2012)) Let a DT of* (2) *is defined by $\mathcal{M}$ of the form $\mathcal{M} = D_x + m$ (or $\mathcal{M} = D_y + m$), $m \in K$. Then it is either a Laplace transformation, or operator $\mathcal{M}$ can be constructed using Darboux Wronskian formulas* (3) *for some $z_1 \in \ker \mathcal{L}$.*

Given an invertible element $g \in K$, a *gauge transformation* of $\mathcal{L} \in K[D]$ is operator $\mathcal{L}^g = g^{-1} \mathcal{L} g$. Functions $h = ab + a_x - c$ and $k = ab + b_y - c$ do not change under gauge transformations of operators of the form (2), and, therefore, these two functions are *differential invariants* of such operators under gauge transformations. There are infinitely many differential invariants. One can prove that each of them can be generated by algebraic combination of h and k and their derivatives. Invariants h and k are known as *h and k Laplace invariants*. Gauge transformations split operators of the form (2) into equivalence classes, each uniquely defined by the values of h and k.

2. *X*- and *Y*-invariants

Let $g \in K$ be an invertible element, consider $\mathcal{L}^g$. We can consider the corresponding tranformation of kernels, $\ker(\mathcal{L}) \to \ker(\mathcal{L}^g)$: $z \mapsto z' = zg^{-1}$. So we can consider gauge transformations of pairs $(z, \mathcal{L})$, $z \in \ker \mathcal{L}$.

Lemma 2.1. Functions $r = -b - \dfrac{z_x}{z}$ and $q = -a - \dfrac{z_y}{z}$ are differential invariants for the pairs (z, L), $z \neq 0$, $z \in \ker \mathcal{L}$.

Theorem 2.2. *Let $r \neq 0, q \neq 0$ are constructed using some $z \neq 0, z \in \ker \mathcal{L}$, then these r and q must satisfy the following equalities:*

$$h - k - r_y + \left(\frac{k}{r}\right)_x + (\ln r)_{xy} = 0 \ , \quad h - k + q_x - \left(\frac{h}{q}\right)_y - (\ln q)_{xy} = 0 \ , \ (4)$$

where h and k are Laplace invariants of $\mathcal{L}$.

The coefficients of (4) depend only on Laplace invariants h and k. Thus, they are the same for operators belonging to the same equivalence class. We shall call the solutions r and q of (4) *X- and Y-invariants*, correspondingly.

Lemma 2.3. For every X- (or Y-) invariant r (or q) there is unique (up to a multiple) z such that $z \in \ker \mathcal{L}$ and $r = -b - z_x/z$ (corresp. $q = -a - z_y/z$).

Thus, there is a one-to-one (up to a multiple) correspondence between solutions r of (4) and solutions of operator $\mathcal{L}$. Each solution r of (4) gives a unique (up to a multiple) solution for each operator belonging to the equivalence class defined by h and k.

3. *X-* and *Y*-invariants under DTs

Let $\ker_X(\mathcal{L})$ and $\ker_Y(\mathcal{L})$ be the sets of X- and Y- invariants of (2). Let $z \in \ker L, z \neq 0$, and r, q be the corresponding X- and Y-invariants. Since $z \in \ker L, z \neq 0$, then by Theorem 1.1 operators

$$\mathcal{M} = D_x - \frac{z_x}{z} = D_x + r + b \ , \quad \mathcal{M} = D_y - \frac{z_y}{z} = D_y + q + a$$

define some DTs, which we shall call X- and Y- DTs, correspondingly. X- and Y- DTs imply $\mathbb{R}$-linear tranformations $\ker \mathcal{L} \to \ker \mathcal{L}_1$. In turns out that these transformations can be considered for X- and Y- invariants also.

Theorem 3.1. *Let $r_0 \in \ker_X(\mathcal{L})$, and $\mathcal{L}_1$ be the result of the corresponding X-DT. Then $\ker_X(\mathcal{L}) \to \ker_X(\mathcal{L}_1)$ is defined by*

$$r \mapsto r + \left(\frac{r_0}{r}\right)_x \frac{r_0}{r_0 - r} \ .$$

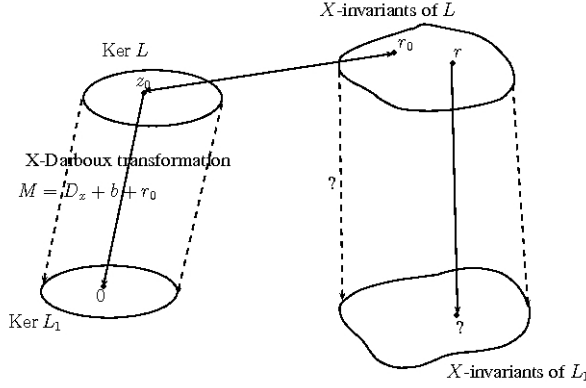*Analogous formulas are true for Y-invariant q_0 under Y-DT.*

Figure 1: Illustration for Theorem 3.1

Theorem 3.2. *Let $q_0 \in \ker_Y(\mathcal{L})$, and $\mathcal{L}_1$ is the result of the corresponding Y-Darboux transformation. Then $\ker_X(\mathcal{L})/\{r_0\} \to \ker_X(\mathcal{L}_1)$ is defined by*

$$r \mapsto -\frac{q_{0x} + h - q_0 r}{q_0 - q} \ ,$$

where q is the corresponding to r Y-invaraint of $\mathcal{L}$, and r_0 is the corresponding to q_0 X-invariant of $\mathcal{L}$.

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