# THE ANDREWS-SELLERS FAMILY OF PARTITION CONGRUENCES

PETER PAULE AND CRISTIAN-SILVIU RADU

Abstract. In 1994 James Sellers conjectured an infinite family of Ramanujan type congruences for 2-colored Frobenius partitions introduced by George E. Andrews. These congruences arise modulo powers of 5. In 2002 Dennis Eichhorn and Sellers were able to settle the conjecture for powers up to 4. In this article we prove Sellers' conjecture for all powers of 5. In addition, we discuss why the Andrews-Sellers family is significantly different from classical congruences modulo powers of primes.

## 1. Introduction

1.1. **Sellers' conjecture.** In his 1984 Memoir [4], Andrews introduced two families of partition functions, $\phi_k(m)$ and $c\phi_k(m)$, which he called generalized Frobenius partition functions. In this paper we restrict our attention to generalized 2-colored Frobenius partitions. Their generating function reads as follows [4, (5.17)]:

$$\sum_{m=0}^{\infty} c\phi_2(m)q^m = \prod_{n=1}^{\infty} \frac{1-q^{4n-2}}{(1-q^{2n-1})^4(1-q^{4n})}. \tag{1}$$

Among numerous properties of generalized Frobenius partitions, Andrews also considered congruences of various kinds. For example, he noted and proved [4, p. 28, Cor. 10.1] that

$$c\phi_2(5n+3) \equiv 0 \pmod 5, \quad n \geq 0.$$

In 1994 Sellers [20] conjectured that for all integers $n \geq 0$ and $\alpha \geq 1$ one has

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \tag{2}$$

where $\lambda_\alpha$ is defined to be the smallest positive integer such that

$$12\lambda_\alpha \equiv 1 \pmod{5^\alpha}. \tag{3}$$

In his joint paper with Eichhorn [10] this conjecture was proved for the cases $\alpha = 1, 2, 3, 4$. In this paper we settle Sellers' conjecture for all $\alpha$ in the spirit of Watson [21].

In addition, we want to highlight the following aspect: At the first glance the congruences (2) seem to fit the standard pattern of Ramanujan type congruences, and one would expect that standard methods would apply in a straight-forward manner. But it turns out that a basic feature of such approaches is missing here; namely, $\ell$-adic convergence to zero of sequences formed by the application of $U$-operators to Atkin basis functions. This, we feel, is the reason why Sellers' conjecture has remained open for more than fifteen years.

---

P. Paule was partially supported by grant P2016-N18 of the Austrian Science Funds FWF.

C. S. Radu was supported by DK grant W1214-DK6 of the Austrian Science Funds FWF.

2010 Mathematics Subject Classification: primary 11P83; secondary 05A17.

Keywords and phrases: generalized Frobenius partitions, Sellers' conjecture, partition congruences of Ramanujan type.

More information on this, together with some history, is given in the rest of this section and in Section 6. In short, we were able to recover $\ell$-adic zero convergence by the introduction of a new type of subspaces of modular functions which behave well under the action of the $U$-operators. These subspaces, found by computer experiments, came as a perfect surprise to us. They have a simple but interesting description, (42) and (43), and seem to be completely new.

1.2. **Ramanujan's congruences.** Infinite families of congruences like (2) were first of observed for $p(n)$, the number of partitions of $n$, by Ramanujan [18] in 1919 where he conjectured that for $\ell \in \{5, 7, 11\}$ and $\alpha \geq 1$:

$$p(\ell^{\alpha} n + \mu_{\alpha,\ell}) \equiv 0 \pmod{\ell^{\alpha}}, \quad n \geq 0, \tag{4}$$

where $\mu_{\alpha,\ell}$ is defined to be the smallest positive integer such that $24\mu_{\alpha,\ell} \equiv 1 \pmod{\ell^{\alpha}}$. Watson [21] proved the conjecture for $\ell = 5$ and a suitably modified version for $\ell = 7$; thirty years later Atkin [5] settled the $\ell = 11$ case. Concerning Ramanujan's role consult [7].

To put Sellers' conjecture (2) into context, some further remarks on history and background of such identities seem to be in place. First of all, for $\alpha = 1$ Ahlgren and Boylan [2] proved that (4) holds only for $\ell = 5, 7, 11$. This achievement settles a question of Ramanujan and is one of the few results on non-existence of partition congruences. For the Andrews-Sellers family an analogous result was proved only recently by Dewar [8]; namely that $c\phi_2(2n+1) \equiv 0 \pmod 2$ and $c\phi_2(5n+3) \equiv 0 \pmod 5$ (proved by Andrews [4]) are the only Ramanujan congruences for two-colored generalized Frobenius partitions. Generally, Ramanujan congruences are congruences of the form $\phi(\ell n + \lambda) \equiv 0 \pmod{\ell}$, $n \geq 0$, where $\ell$ is a prime. Concerning congruences not being of Ramanujan type, landmark results due to Ono [16] and Ahlgren [1] say that there are infinitely many of them of the form $p(an+b) \equiv 0 \pmod{\ell^{\alpha}}$. For generalized two-colored Frobenius partitions analogous results are not yet known.

1.3. **The $\ell$-adic property.** The problem of proving the congruence (2) is similar to the proof of congruence (4), but certain adaptions are required together with some new ideas. Some of these aspects are informally described in this section. For basic modular form terminology see Section 5.1.

In order to prove (4) one defines two linear operators $U^{(0)} : S_0 \to S_1$ and $U^{(1)} : S_1 \to S_0$ (in a similar fashion as in Definition 2.3 below). The construction is such that by defining $L_0 := 1$, $L_{2\alpha-1} := U^{(0)}(L_{2\alpha-2})$, and $L_{2\alpha} := U^{(1)}(L_{2\alpha-1})$ the problem is reduced to proving $L_\alpha \equiv 0 \pmod{\ell^{\alpha}}$, i.e., divisibility of coefficients in the $q$-series expansions of the $L_\alpha$. This problem transformation is common to all proofs of such congruences. For $\ell = 5$ the $S_i$ are subrings of the ring of modular functions on $\Gamma_0(\ell)$ which are isomorphic to $\mathbb{Z}[X]$, and a close inspection of Watson's computations of the $U^{(i)}$-actions makes transparent that here one heavily exploits the simple structure of $\mathbb{Z}[X]$. The same applies to the (modified) $\ell = 7$ case. But when trying to generalize Watson's construction to the case $\ell = 11$, one encounters the difficulty that $S_0$ and $S_1$ are isomorphic to finitely generated free $\mathbb{Z}[X]$-modules with more than one generator. The reason for this is that the Riemann surface $X_0(11)$ has nonzero genus. Here $X_0(N)$ denotes the Riemann surface on which modular functions for $\Gamma_0(N)$ live.

Because of this difficulty of Watson's method, Atkin [5] was led to solve the problem for the $\ell = 11$ case with an entirely different method. He also reduces the problem to proving $L_\alpha \equiv 0 \pmod{11^{\alpha}}$ by using a similar construction involving operators

$U^{(i)}$, $i = 0, 1$. But additionally he defines spaces $X^{(i)} \subseteq S_i$ of modular functions on $\Gamma_0(11)$ having integer coefficients in their $q$-expansions. In particular, the elements $f \in X^{(i)}$ for $\ell = 11$ satisfy the valuation conditions $v_\ell(U^{(i)}(f)) = v_\ell(f) + 1$, where $v_\ell(f)$ is defined to be the minimal nonnegative integer such that all the coefficients in the $q$-expansion of $f$ are divisible by $\ell^{v_\ell(f)}$. Consequently, $f \in X^{(i)}$ implies $\frac{f}{11} \in X^{(1-i)}$, and, by induction, it follows that $L_\alpha \equiv 0 \pmod{11^\alpha}$ for all $\alpha \geq 1$. In order to prove the valuation properties of the spaces $X^{(0)}$ and $X^{(1)}$, Atkin in [5] uses a lot of computer computations and some technical lemmas.

With respect to the operators $U^{(i)}$ we assign to each $f \in S_i$ the sequence $u(f) := (u_n)_{n\geq 0}$, called the $U$-sequence at $f$, and inductively defined by $u_0 := f$, $u_1 := U^{(i)}(f)$, $u_2 := U^{(1-i)}U^{(i)}(f)$, $u_3 := U^{(i)}U^{(1-i)}U^{(i)}(f)$, and so on. For example, for $\ell = 5$ Watson's $(L_\alpha)_{\alpha\geq 0}$ is the $U$-sequence at 1. A sequence $(u_n)_{n\geq 0}$ with $u_n \in X^{(i)}$, $i = 0$ or 1, is a zero sequence with respect to the $\ell$-adic norm, if for any $\alpha \geq 1$ there exists an $N$ such that $u_n \equiv 0 \pmod{\ell^\alpha}$ for $n \geq N$.

Translating Atkin's setting [5] into this terminology, it is straight-forward to show that *for each* $f \in S_i$ one has $11^N f \in X^{(i)}$ for some nonnegative integer $N$. Consequently, $u(11^N f) \to 0$ and thus $u(f) \to 0$, both in the $\ell$-adic sense.

This universal property also holds in the cases $\ell = 5$ and $\ell = 7$. To our knowledge, in all other problems similar to (2) or (4) one also observes that this property ($u(f) \to 0$ in the $\ell$-adic sense for all $f \in S_0$) is always crucial for the proof to work. In all the known examples like in [21, 5, 11, 12, 13] the spaces $S_i$ are defined to be subspaces of the space of modular functions on $\Gamma_0(\ell)$ having integer coefficients in their $q$-expansions. Furthermore, in these examples one always observes that all $f \in S_0$ (or $S_1$) are holomorphic at certain given cusps. For example, in the case of (4) all $f \in S_0 \cup S_1$ are holomorphic at the cusp $\infty$.

As a consequence, the natural approach in the Sellers problem would be to identify the corresponding spaces there. One finds immediately that $U^{(0)}(1)$ is a modular function in $\Gamma_0(20)$. Furthermore, the $U$-sequence at 1 stays completely in $X_0(20)$. This suggests to consider subspaces $S_i$ of the modular functions on $\Gamma_0(20)$. But then one is facing a problem when trying to install the universal 5-adic zero-convergence property. For example, in other contexts one can obtain this property by requiring that all $S_i$ elements are holomorphic at certain cusps. In the Sellers problem this recipe fails. Namely: For each cusp $c$ of $\Gamma_0(20)$ we chose a modular function $f_c$ on $\Gamma_0(20)$ having a pole only at the cusp $c$ and being holomorphic at all the other cusps. Additionally, defining $\overline{U} := U^{(i)} \circ U^{(1-i)}$, we found that for each such $f_c$ there exists an $n_c$ such that $\overline{U}^{n_c}(f_c)$ is an eigenfunction of $\overline{U}$ modulo 5 with eigenvalue $\lambda_c \not\equiv 0 \pmod 5$. This proves that the $U$-sequence at each $f_c$ does not converge to 0 in the 5-adic sense. And this is why in the Sellers problem we had to find a new way of defining the spaces $S_0$ and $S_1$.

As explained above, our solution to Sellers' conjecture will lead to subspaces $S_i$ specified in quite different and to us unexpected fashion: namely, by modular functions satisfying functional identities like (42) and (43) in Section 6. Such spaces to the best of our knowledge have never popped up in the literature before. They or the underlying relations, respectively, were discovered in the course of computer experiments.

Finally, the spaces $S_0$ and $S_1$ we found are not isomorphic to $\mathbb{Z}[X]$ which, as in the case $\ell = 11$ of (4), makes the problem more complicated. We mentioned that Atkin [5] found his way to deal with such more involved spaces (i.e., spaces where the corresponding Riemann surface on which the modular functions live is not necessarily of genus 0). Later Gordon [11] found another approach that generalizes Watson's method. In this paper we follow Gordon's approach rather than Atkin's.

1.4. **Contents and basic notions.** Our article is structured as follows. In Section 2 we state the Main Theorem (Theorem 2.11) of our paper. It describes the action of a class of Hecke type operators on a quotient of eta function products being crucial for the problem. Sellers' conjecture then is derived as an immediate consequence (Corollary 2.12). The rest of the paper deals with proving the Main Theorem. The basic building blocks of our proof are the twenty Fundamental Relations listed in the Appendix (Section 7). Despite postponing their proof to Section 5, we shall use these relations already in Section 3 and Section 4. In Section 3 a crucial result is proved, the Fundamental Lemma (Lemma 3.3). The proof of the Main Theorem is presented in Section 4. To this end, three further lemmas are introduced, all being immediate consequences of the Fundamental Lemma. We mention that in Section 5, in order to prove the twenty Fundamental Relations, we utilize a computer-assisted method which is based on a variant of a well-known lemma tracing back to Newman (Lemma 5.1). Finally, in Section 6 we provide hints for getting deeper insight into what is standing behind our proof. In particular, we introduce the functional identities described in the introduction which are crucial in finding the functions used to express the $L_\alpha$.

Throughout the paper we will use the following conventions: $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$ denote the nonnegative and positive integers, respectively. The complex upper half plane is denoted by $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. As usual, $\eta(\tau)$ for $\tau \in \mathbb{H}$ denotes the Dedekind eta function for which

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \text{ where } q := e^{2\pi i\tau}. \tag{5}$$

We will also use the short hand notation:

$$\eta_n(\tau) := \eta(n\tau), \quad n \in \mathbb{Z}, \quad \tau \in \mathbb{H}. \tag{6}$$

For $x \in \mathbb{R}$ the symbol $\lfloor x \rfloor$ ("floor" of $x$) as usual denotes the greatest integer less or equal to $x$. Let $f = \sum_{n\in\mathbb{Z}} a_n q^n$, $f \neq 0$, be such that $a_n = 0$ for almost all $n < 0$. Then the order of $f$ is the smallest integer $N$ such that $a_N \neq 0$; we write $N = \mathrm{ord}(f)$. More generally, let $F = f \circ t = \sum_{n\in\mathbb{Z}} a_n t^n$ with $t = \sum_{n\geq 1} b_n q^n$, then the $t$-order of $F$ is defined to be the smallest integer $N$ such that $a_N \neq 0$; we write $N = \mathrm{ord}_t(F)$. For example, if $\mathrm{ord}(f) = -1$ and $t = q^2$, then $\mathrm{ord}_t(F) = -1$ but $\mathrm{ord}(F) = -2$.

## 2. The Main Theorem

Let

$$\mathrm{C}\Phi_2(q) := \sum_{m=0}^{\infty} c\phi_2(m) q^m.$$

**Lemma 2.1.** *For $\tau \in \mathbb{H}$,*

$$\mathrm{C}\Phi_2(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^4 (1 - q^{4n})^2}.$$

*Proof.* From (1),

$$\begin{aligned} \mathrm{C}\Phi_2(q) &= \prod_{n=1}^{\infty} \frac{(1-q^{2(2n-1)})(1-q^{2n})^4}{(1-q^n)^4(1-q^{4n})} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{2n})^4}{(1-q^n)^4(1-q^{4n})^2}. \end{aligned}$$

□

Subsequently we will study the action of certain operators $U_m$ on $\mathrm{C}\Phi_2(q)$, respectively on

$$A := \frac{\eta_2^5 \eta_{100}^2 \eta_{25}^4}{\eta_{50}^5 \eta_4^2 \eta^4}. \tag{7}$$

**Definition 2.2.** *For $f : \mathbb{H} \to \mathbb{C}$ and $m \in \mathbb{N}^*$ we define $U_m(f) : \mathbb{H} \to \mathbb{C}$ by*

$$U_m(f)(\tau) := \frac{1}{m} \sum_{\lambda=0}^{m-1} f\left(\frac{\tau+\lambda}{m}\right), \quad \tau \in \mathbb{H}.$$

Obviously $U_m$ is linear (over $\mathbb{C}$); in addition, it is easy to verify that

$$U_{mn} = U_m \circ U_n = U_n \circ U_m, \quad m, n \in \mathbb{N}^*. \tag{8}$$

The operators $U_m$, introduced by Atkin and Lehner [6], are closely related to Hecke operators. They typically arise in the context of partition congruences (e.g. [3, Sect. 10.2]) mostly because of the property: if

$$f(\tau) = \sum_{n=-\infty}^{\infty} f_n q^n \quad (q = e^{2\pi i \tau}),$$

then

$$U_m(f)(\tau) = \sum_{n=-\infty}^{\infty} f_{mn} q^n.$$

To relate to the discussion in the introduction we make the following definition.

**Definition 2.3.** *For $f : \mathbb{H} \to \mathbb{C}$ we define $U^{(0)}(f)$, $U^{(1)}(f) : \mathbb{H} \to \mathbb{C}$ by $U^{(0)}(f) := U_5(Af)$ and $U^{(1)}(f) := U_5(f)$.*

The following explicit expressions for $\lambda_\alpha$ in (3) are easily verified.

**Lemma 2.4.** *For $\alpha \in \mathbb{N}^*$:*

$$\lambda_{2\alpha-1} = \frac{1 + 7 \cdot 5^{2\alpha-1}}{12} \quad \textit{and} \quad \lambda_{2\alpha} = \frac{1 + 11 \cdot 5^{2\alpha}}{12}.$$

**Definition 2.5.** *We define the $U$-sequence $(L_\alpha)_{\alpha \geq 0}$ at 1 by $L_0 := 1$ and for $\alpha \geq 1$:*

$$L_{2\alpha-1} := U^{(0)}(L_{2\alpha-2}) \quad \textit{and} \quad L_{2\alpha} := U^{(1)}(L_{2\alpha-1}).$$

Using Lemma 2.4 the proof of the following lemma is completely analogous to [5, p. 23] and we omit it.

**Lemma 2.6.** *For $\alpha \in \mathbb{N}^*$:*

$$L_{2\alpha-1} = q \prod_{n=1}^{\infty} \frac{(1-q^{5n})^4(1-q^{20n})^2}{(1-q^{10n})^5} \sum_{n=0}^{\infty} c\phi_2(5^{2\alpha-1}n + \lambda_{2\alpha-1}) q^n \tag{9}$$

*and*

$$L_{2\alpha} = q \prod_{n=1}^{\infty} \frac{(1-q^n)^4(1-q^{4n})^2}{(1-q^{2n})^5} \sum_{n=0}^{\infty} c\phi_2(5^{2\alpha}n + \lambda_{2\alpha}) q^n. \tag{10}$$

**Definition 2.7.** *Let $t, \rho, \sigma, p_0$, and $p_1$ be functions defined on $\mathbb{H}$ as follows:*

$$t := \frac{\eta_5^6}{\eta^6}, \quad \rho := \frac{\eta_2^2\eta_{20}^4}{\eta_4^4\eta_{10}^2}, \quad \sigma := \frac{\eta_2\eta_{10}^3}{\eta^3\eta_5}, \tag{11}$$

$$p_0 := 36\sigma + 25t + \rho + 600t\sigma + 136\sigma^2 + 2000t\sigma^2 + 40\rho\sigma^2 + 12\rho\sigma, \tag{12}$$

$$p_1 := t + \rho + 12t\sigma + 40t\sigma^2 + 200t\rho\sigma^2 + 100t\rho\sigma + 16\rho\sigma^2 + 8\rho\sigma + 12t\rho. \tag{13}$$

We note that all functions defined in Definition 2.7 have Laurent series expansions in powers of $q$ with coefficients in $\mathbb{Z}$. In particular, $\mathrm{ord}(\sigma) = \mathrm{ord}(t) = 1$ and $\mathrm{ord}(\rho) = 2$, which implies $\mathrm{ord}(p_1) \geq 1$ and $\mathrm{ord}(p_0) \geq 1$.

Before stating the Main Theorem of the paper, we introduce convenient shorthand notation.

**Definition 2.8.** *A map $a : \mathbb{Z} \to \mathbb{Z}$ is called a discrete function if it has finite support. A map $a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is called discrete array if for each $i \in \mathbb{Z}$ the map $a(i, -) : \mathbb{Z} \to \mathbb{Z}$, $j \mapsto a(i, j)$, has finite support.*

**Definition 2.9.** *We define*

$$S_0 := \left\{ \sum_{n=0}^{\infty} r(n)p_0t^n + \sum_{n=1}^{\infty} s(n)t^n : r \text{ and } s \text{ discrete functions} \right\},$$

$$S_1 := \left\{ \sum_{n=0}^{\infty} r(n)p_1t^n + \sum_{n=1}^{\infty} s(n)t^n : r \text{ and } s \text{ discrete functions} \right\},$$

$$X^{(0)} := \left\{ \sum_{n=0}^{\infty} r(n)5^{\lfloor \frac{5n+1}{2} \rfloor}p_0t^n + \sum_{n=1}^{\infty} s(n)5^{\lfloor \frac{5n-4}{2} \rfloor}t^n : r \text{ and } s \text{ discrete functions} \right\}$$

*and*

$$X^{(1)} := \left\{ \sum_{n=0}^{\infty} r(n)5^{\lfloor \frac{5n+2}{2} \rfloor}p_1t^n + \sum_{n=1}^{\infty} s(n)5^{\lfloor \frac{5n-5}{2} \rfloor}t^n : r \text{ and } s \text{ discrete functions} \right\}.$$

*Remark* 2.10. Note that $S_i = \langle t, p_i \rangle_{\mathbb{Z}[t]}$. Here $\langle t, p_i \rangle_{\mathbb{Z}[t]}$ denotes the free $\mathbb{Z}[t]$-module generated by $t$ and $p_i$.

**Theorem 2.11** ("Main Theorem"). *For each $\beta \geq 1$ there exist $f_{2\beta-1} \in X^{(1)}$ and $f_{2\beta} \in X^{(0)}$ such that*

$$L_{2\beta-1} = 5^{2\beta-1}f_{2\beta-1} \text{ and} \tag{14}$$

$$L_{2\beta} = 5^{2\beta}f_{2\beta}. \tag{15}$$

The remaining sections are devoted to proving the Main Theorem by mathematical induction on $\beta$. In Sections 3 and 4 we describe the algebra underlying the induction step. In Section 5 we settle the initial cases, i.e., the correctness of the twenty fundamental relations listed in the Appendix (Section 7).

We conclude this section by deriving the truth of Sellers' conjecture as a corollary.

**Corollary 2.12.** *Sellers' conjecture is true; i.e., for $\alpha \in \mathbb{N}^*$:*

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \quad n \in \mathbb{N}^*.$$

*Proof.* The statement is derived immediately by applying Lemma 2.6 to (14) and (15). □

## 3. The Fundamental Lemma

In this section we prove the Fundamental Lemma, Lemma 3.3, which will play a crucial role in the proof of the Main Theorem in Section 4.

**Definition 3.1.** *With $t = t(\tau)$ as in Definition 2.7 we define:*

$$\begin{aligned}
&a_0(t) = -t, a_1(t) = -5^3t^2 - 6\cdot 5t, a_2(t) = -5^6t^3 - 6\cdot 5^4t^2 - 63\cdot 5t,\\
&a_3(t) = -5^9t^4 - 6\cdot 5^7t^3 - 63\cdot 5^4t^2 - 52\cdot 5^2t,\\
&a_4(t) = -5^{12}t^5 - 6\cdot 5^{10}t^4 - 63\cdot 5^7t^3 - 52\cdot 5^5t^2 - 63\cdot 5^2t.
\end{aligned}$$

*We define $s : \{0,\dots,4\}\times\{1,\dots,5\} \to \mathbb{Z}$ to be the unique function satisfying*

$$a_j(t) = \sum_{l=1}^{5} s(j,l)5^{\lfloor\frac{5l+j-4}{2}\rfloor}t^l. \tag{16}$$

**Lemma 3.2.** *For $0 \le \lambda \le 4$ let*

$$t_\lambda(\tau) := t\left(\frac{\tau+\lambda}{5}\right), \quad \tau\in\mathbb{H}.$$

*Then in the polynomial ring $\mathbb{C}(t)[X]$:*

$$X^5 + \sum_{j=0}^{4} a_j(t)X^j = \prod_{\lambda=0}^{4}(X - t_\lambda). \tag{17}$$

*Proof.* First we prove

$$\prod_{\lambda=0}^{4} t_\lambda = -a_0(t) = t. \tag{18}$$

With $\omega := e^{48\pi i/5}$ one has for $\tau\in\mathbb{H}$:

$$\begin{aligned}
\prod_{\lambda=0}^{4} t_\lambda(\tau) &= \prod_{\lambda=0}^{4} q^{1/5}\omega^\lambda \prod_{n=1}^{\infty}\left(\frac{1-q^n}{1-\omega^{\lambda n}q^{n/5}}\right)^6 = q\prod_{n=1}^{\infty}\prod_{\lambda=0}^{4}\left(\frac{1-q^n}{1-\omega^{\lambda n}q^{n/5}}\right)^6\\
&= q\prod_{n=1}^{\infty}(1-q^n)^{30}\prod_{n=1}^{\infty}\left(\frac{1-q^{5n}}{1-q^n}\right)^6\prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)^{30} = t(\tau).
\end{aligned}$$

Here we used the fact that $\prod_{\lambda=0}^4(1-\omega^{\lambda n}z)$ equals $(1-z)^5$ if $5|n$, and $1-z^5$ otherwise.

For the remaining part of the proof we use (18) to rewrite (17) into the equivalent form

$$X^5 + \sum_{j=0}^{4} a_j(t)X^j = -t\prod_{\lambda=0}^{4}(1 - Xt_\lambda^{-1}). \tag{19}$$

Hence to complete the proof, in view of $t = \prod_{\lambda=0}^4 t_\lambda$ it remains to show that

$$a_j(t) = (-1)^{j+1}te_j(t_0^{-1},\dots,t_4^{-1}), \quad 0\le j\le 4, \tag{20}$$

where the $e_j$ are the elementary symmetric functions. To this end we utilize the fact that

$$5U_5(t^{-j}) = \sum_{\lambda=0}^{4} t_\lambda^{-j}, \quad j\in\mathbb{Z}.$$

The first non-trivial case is $j = 1$. Observing

$$e_1(t_0^{-1}, \dots, t_4^{-1}) = \sum_{\lambda=0}^{4} t_\lambda^{-1} = 5U_5(t^{-1}),$$

to show (20) for $j = 1$ we need to show that

$$5U_5(t^{-1}) = t^{-1}a_1(t) = -5^3 t - 5 \cdot 6.$$

But this is just the second entry

$$U_5(t^{-1}) = (-5^2 t - 6) \tag{21}$$

of Group III of the twenty fundamental relations from the Appendix. The next cases $2 \le j \le 4$ work analogously with the remaining entries of Group III. For example, if $j = 2$ then Newton's formula, translating elementary symmetric functions into power sums, implies

$$\begin{aligned} e_2(t_0^{-1}, \dots, t_4^{-1}) &= \frac{1}{2}\left(\left(5U_5(t^{-1})\right)^2 - 5U_5(t^{-2})\right) \\ &= \frac{1}{2}\left((-5^3 t - 5{\cdot}6)^2 - (-5^6 t^2 + 54{\cdot}5)\right) = -t^{-1}a_2(t). \end{aligned}$$

Here we used the third entry of Group III. □

Finally we are ready for the main result of this section.

**Lemma 3.3** ("Fundamental Lemma")**.** *For $u : \mathbb{H} \to \mathbb{C}$ and $j \in \mathbb{Z}$:*

$$U_5(ut^j) = -\sum_{l=0}^{4} a_l(t) U_5(ut^{j+l-5}).$$

*Proof.* For $\lambda \in \{0, \dots, 4\}$ Lemma 3.2 implies

$$t_\lambda^5 + \sum_{l=0}^{4} a_l(t) t_\lambda^l = 0.$$

Multiplying both sides with $u_\lambda t_\lambda^{j-5}$ where $u_\lambda(\tau) := u((\tau + \lambda)/5)$ gives

$$u_\lambda t_\lambda^j + \sum_{l=0}^{4} a_l(t) u_\lambda t_\lambda^{j+l-5} = 0.$$

Summing both sides over all $\lambda$ from $\{0, \dots, 4\}$ completes the proof of the lemma. □

## 4. Proving the Main Theorem

We need to prepare with some lemmas. Recall that $t$ is as in Definition 2.7.

**Lemma 4.1.** *Let $v_1, v_2, u : \mathbb{H} \to \mathbb{C}$ and $l \in \mathbb{Z}$. Suppose for $l \le k \le l + 4$ there exist Laurent polynomials $p_k^{(1)}(t)$, $p_k^{(2)}(t) \in \mathbb{Z}[t, t^{-1}]$ such that*

$$U_5(ut^k) = v_1 p_k^{(1)}(t) + v_2 p_k^{(2)}(t) \tag{22}$$

*and*

$$\operatorname{ord}_t\left(p_k^{(i)}(t)\right) \ge \left\lceil \frac{k + s_i}{5} \right\rceil, \quad i \in \{1, 2\}, \tag{23}$$

*for some fixed integers $s_1$ and $s_2$. Then there exist families of Laurent polynomials $p_k^{(1)}(t), p_k^{(2)}(t) \in \mathbb{Z}[t, t^{-1}]$, $k \in \mathbb{Z}$, such that (22) and (23) hold for all $k \in \mathbb{Z}$.*

*Proof.* Let $N > l+4$ be an integer and assume by induction that there are families of Laurent polynomials $p_k^{(i)}(t)$, $i \in \{1,2\}$, such that (22) and (23) hold for $l \leq k \leq N-1$. Suppose

$$p_k^{(i)}(t) = \sum_{n \geq \lceil \frac{k+s_i}{5} \rceil} c_i(k,n)t^n, \quad 1 \leq k \leq N-1,$$

with integers $c_i(k,n)$. Applying Lemma 3.3 we obtain:

$$\begin{aligned} U_5(ut^N) &= -\sum_{j=0}^{4} a_j(t) U_5(ut^{N+j-5}) \\ &= -\sum_{j=0}^{4} a_j(t) \sum_{i=1}^{2} v_i \sum_{n \geq \lceil \frac{N+j-5+s_i}{5} \rceil} c_i(N+j-5,n)t^n \\ &= -\sum_{i=1}^{2} v_i \sum_{j=0}^{4} a_j(t)t^{-1} \sum_{n \geq \lceil \frac{N+j+s_i}{5} \rceil} c_i(N+j-5,n-1)t^n. \end{aligned}$$

Recalling the fact that $a_j(t)t^{-1}$ for $0 \leq j \leq 4$ is a polynomial in $t$, this determines Laurent polynomials $p_N^{(i)}(t)$ with the desired properties. The induction proof for $N < l$ work analogously. $\square$

**Lemma 4.2.** *Let $v_1, v_2, u : \mathbb{H} \to \mathbb{C}$ and $l \in \mathbb{Z}$. Suppose for $l \leq k \leq l+4$ there exist Laurent polynomials $p_k^{(i)} \in \mathbb{Z}[t,t^{-1}]$, $i \in \{1,2\}$, such that*

$$U_5(ut^k) = v_1 p_k^{(1)}(t) + v_2 p_k^{(2)}(t) \tag{24}$$

*where*

$$p_k^{(i)}(t) = \sum_n c_i(k,n) 5^{\left\lfloor \frac{5n-k+\gamma_i}{2} \right\rfloor} t^n \tag{25}$$

*with integers $\gamma_i$ and $c_i(k,n)$. Then there exist families of Laurent polynomials $p_k^{(i)}(t) \in \mathbb{Z}[t,t^{-1}]$, $k \in \mathbb{Z}$, of the form (25) for which property (24) holds for all $k \in \mathbb{Z}$.*

*Proof.* Suppose for an integer $N > l+4$ there are families of Laurent polynomials $p_k^{(i)}(t)$, $i \in \{1,2\}$, of the form (25) satisfying property (24) for $l \leq k \leq N-1$. We proceed by mathematical induction on $N$. Applying Lemma 3.3 and using the induction base (24) and (25) we obtain:

$$U_5(ut^N) = -\sum_{j=0}^{4} a_j(t) \sum_{i=1}^{2} v_i \sum_n c_i(N+j-5,n) 5^{\left\lfloor \frac{5n-(N+j-5)+\gamma_i}{2} \right\rfloor} t^n.$$

Utilizing (16) from Definition 3.1 this rewrites into :

$$\begin{aligned} U_5(ut^N) &= -\sum_{j=0}^{4} \sum_{l=1}^{5} s(j,l) 5^{\left\lfloor \frac{5l+j-4}{2} \right\rfloor} t^l \\ &\quad \times \sum_{i=1}^{2} v_i \sum_n c_i(N+j-5,n) 5^{\left\lfloor \frac{5n-(N+j-5)+\gamma_i}{2} \right\rfloor} t^n \\ &= -\sum_{i=1}^{2} v_i \sum_{j=0}^{4} \sum_{l=1}^{5} \sum_n s(j,l) c_i(N+j-5,n-l) \\ &\quad \times 5^{\left\lfloor \frac{5(n-l)-(N+j-5)+\gamma_i}{2} \right\rfloor + \left\lfloor \frac{5l+j-4}{2} \right\rfloor} t^n. \end{aligned} \tag{26}$$

The induction step is completed by simplifying the exponent of 5 as follows:

$$\begin{aligned}
&\left\lfloor \frac{5(n-l)-(N+j-5)+\gamma_i}{2} + \left\lfloor \frac{5l+j-4}{2} \right\rfloor \right\rfloor \\
&\geq \left\lfloor \frac{5(n-l)-(N+j-5)+\gamma_i}{2} + \frac{5l+j-5}{2} \right\rfloor \\
&= \left\lfloor \frac{5n-N+\gamma_i}{2} \right\rfloor .
\end{aligned}$$

The induction proof for $N < l$ works analogously. □

Before proving the Main Theorem, Theorem 2.11, we need two more lemmas.

**Lemma 4.3.** *Given $A$ as in* (7)*, $p_0$ and $p_1$ as in* (12) *and* (13)*, respectively. Then there exist discrete arrays $a_i, b_i, c$, and $d_i$, $i \in \{0,1\}$, such that the following relations hold for all $k \in \mathbb{N}$:*

(27)
$$U^{(0)}(t^k) = \sum_{n \geq \lceil (k+1)/5 \rceil} a_0(k,n) 5^{\lfloor \frac{5n-k-2}{2} \rfloor} t^n + p_1 \sum_{n \geq \lceil (k-4)/5 \rceil} a_1(k,n) 5^{\lfloor \frac{5n-k+5}{2} \rfloor} t^n,$$

(28)
$$U^{(0)}(p_0 t^k) = \sum_{n \geq \lceil (k+1)/5 \rceil} b_0(k,n) 5^{\lfloor \frac{5n-k-2}{2} \rfloor} t^n + p_1 \sum_{n \geq \lceil (k-4)/5 \rceil} b_1(k,n) 5^{\lfloor \frac{5n-k+4}{2} \rfloor} t^n,$$

$$U^{(1)}(t^k) = \sum_{n \geq \lceil k/5 \rceil} c(k,n) 5^{\lfloor \frac{5n-k-1}{2} \rfloor} t^n, \tag{29}$$

$$U^{(1)}(p_1 t^k) = \sum_{n \geq \lceil (k+1)/5 \rceil} d_0(k,n) 5^{\lfloor \frac{5n-k-2}{2} \rfloor} t^n + p_0 \sum_{n \geq \lceil k/5 \rceil} d_1(k,n) 5^{\lfloor \frac{5n-k+1}{2} \rfloor} t^n. \tag{30}$$

*Proof.* The Appendix (Section 7) lists twenty fundamental relations, which are proved in Section 5 (Theorem 5.7). The five fundamental relations of Group I fit the pattern of the relation (27) for five consecutive values of $k$. The same observation applies to the relations of the Groups II, III and IV with regard to the relations (28), (29), and (30), respectively. In each of these cases $k$ is less or equal to 0. Hence applying Lemma 4.1 and Lemma 4.2 immediately proves the statement for all $k \geq 0$. □

**Lemma 4.4.** *We have*

$$f \in X^{(0)} \quad \textit{implies} \quad 5^{-1} U^{(0)}(f) \in X^{(1)}, \tag{31}$$

*and*

$$f \in X^{(1)} \quad \textit{implies} \quad 5^{-1} U^{(1)}(f) \in X^{(0)}. \tag{32}$$

*Proof. Proof of* (31)*:* Assume that $f \in X^{(1)}$. Then by Definition 2.9 there are discrete functions $r(n)$ and $s(n)$ such that

$$f = \sum_{n=0}^{\infty} r(n) 5^{\lfloor \frac{5n+2}{2} \rfloor} p_1 t^n + \sum_{n=1}^{\infty} s(n) 5^{\lfloor \frac{5n-5}{2} \rfloor} t^n.$$

This implies that

$$U^{(1)}(f) = \sum_{n=0}^{\infty} r(n) 5^{\lfloor \frac{5n+2}{2} \rfloor} U^{(1)}(p_1 t^n) + \sum_{n=1}^{\infty} s(n) 5^{\lfloor \frac{5n-5}{2} \rfloor} U^{(1)}(t^n).$$

Utilizing (29) and (30) of Lemma 4.3 with discrete arrays $c$ and $d_i$ gives

$$U^{(1)}(f) = \Bigg(p_0 \sum_{m\geq 0}\sum_{n\geq 0} r(n) d_1(n,m) 5^{\lfloor \frac{5n+2}{2}\rfloor + \lfloor \frac{5m-n+1}{2}\rfloor} t^m$$
$$+ \sum_{m\geq 1}\sum_{n\geq 0} r(n) d_0(n,m) 5^{\lfloor \frac{5n+2}{2}\rfloor + \lfloor \frac{5m-n-2}{2}\rfloor} t^m \tag{33}$$
$$+ \sum_{m\geq 1}\sum_{n\geq 1} s(n) c(n,m) 5^{\lfloor \frac{5n-5}{2}\rfloor + \lfloor \frac{5m-n-1}{2}\rfloor} t^m \Bigg).$$

Observe that for $m,n \geq 0$:

$$\left\lfloor \frac{5n+2}{2}\right\rfloor + \left\lfloor \frac{5m-n+1}{2}\right\rfloor = \left\lfloor \frac{5m+n+1}{2}\right\rfloor + \left\lfloor \frac{3n+2}{2}\right\rfloor \geq \left\lfloor \frac{5m+1}{2}\right\rfloor + 1,$$
$$\left\lfloor \frac{5n+2}{2}\right\rfloor + \left\lfloor \frac{5m-n-2}{2}\right\rfloor = \left\lfloor \frac{5m+n-2}{2}\right\rfloor + \left\lfloor \frac{3n+2}{2}\right\rfloor \geq \left\lfloor \frac{5m-4}{2}\right\rfloor + 1,$$

and for $m,n \geq 1$:

$$\left\lfloor \frac{5n-5}{2}\right\rfloor + \left\lfloor \frac{5m-n-1}{2}\right\rfloor = \left\lfloor \frac{5m+n-5}{2}\right\rfloor + \left\lfloor \frac{3n-1}{2}\right\rfloor \geq \left\lfloor \frac{5m-4}{2}\right\rfloor + 1.$$

Hence the right hand side of (33) can be written in the form $5g$ for some $g \in X^{(0)}$.

*Proof of* (32)*:* Assume that $f \in X^{(0)}$. Then by Definition 2.9 there are discrete functions $r(n)$ and $s(n)$ such that

$$f = \sum_{n=0}^{\infty} r(n) 5^{\lfloor \frac{5n+1}{2}\rfloor} p_0 t^n + \sum_{n=1}^{\infty} s(n) 5^{\lfloor \frac{5n-4}{2}\rfloor} t^n.$$

This implies that

$$U^{(0)}(f) = \sum_{n=0}^{\infty} r(n) 5^{\lfloor \frac{5n+1}{2}\rfloor} U^{(0)}(p_0 t^n) + \sum_{n=1}^{\infty} s(n) 5^{\lfloor \frac{5n-4}{2}\rfloor} U^{(0)}(t^n).$$

Utilizing (27) and (28) of Lemma 4.3 with discrete arrays $a_i$ and $b_i$ gives

$$U^{(0)}(f) = p_1 \sum_{m\geq 0}\sum_{n\geq 0} r(n) b_1(n,m) 5^{\lfloor \frac{5n+1}{2}\rfloor + \lfloor \frac{5m-n+4}{2}\rfloor} t^m$$
$$+ p_1 \sum_{m\geq 0}\sum_{n\geq 1} s(n) a_1(n,m) 5^{\lfloor \frac{5n-4}{2}\rfloor + \lfloor \frac{5m-n+5}{2}\rfloor} t^m$$
$$+ \sum_{m\geq 1}\sum_{n\geq 0} r(n) b_0(n,m) 5^{\lfloor \frac{5n+1}{2}\rfloor + \lfloor \frac{5m-n-2}{2}\rfloor} t^m \tag{34}$$
$$+ \sum_{m\geq 1}\sum_{n\geq 1} s(n) a_0(n,m) 5^{\lfloor \frac{5n-4}{2}\rfloor + \lfloor \frac{5m-n-2}{2}\rfloor} t^m.$$

Similar to above observe that for $m,n \geq 0$:

$$\left\lfloor \frac{5n+1}{2}\right\rfloor + \left\lfloor \frac{5m-n+4}{2}\right\rfloor = \left\lfloor \frac{5m+n+2}{2}\right\rfloor + \left\lfloor \frac{3n+3}{2}\right\rfloor \geq \left\lfloor \frac{5m+2}{2}\right\rfloor + 1,$$

for $m \geq 0$ and $n \geq 1$:

$$\left\lfloor \frac{5n-4}{2}\right\rfloor + \left\lfloor \frac{5m-n+5}{2}\right\rfloor = \left\lfloor \frac{5m+n+2}{2}\right\rfloor + \left\lfloor \frac{3n-1}{2}\right\rfloor \geq \left\lfloor \frac{5m+2}{2}\right\rfloor + 1,$$

for $m \geq 1$ and $n \geq 0$:

$$\left\lfloor \frac{5n+1}{2} \right\rfloor + \left\lfloor \frac{5m-n-2}{2} \right\rfloor = \left\lfloor \frac{5m+n-4}{2} \right\rfloor + \left\lfloor \frac{3n+3}{2} \right\rfloor \geq \left\lfloor \frac{5m-5}{2} \right\rfloor + 1,$$

and for $m, n \geq 1$:

$$\left\lfloor \frac{5n-4}{2} \right\rfloor + \left\lfloor \frac{5m-n-2}{2} \right\rfloor = \left\lfloor \frac{5m+n-6}{2} \right\rfloor + \left\lfloor \frac{3n}{2} \right\rfloor \geq \left\lfloor \frac{5m-5}{2} \right\rfloor + 1.$$

Hence the right hand side of (34) is of the form $5g$ for some $g \in X^{(1)}$. □

*Remark* 4.5. Lemma 4.4 implies that for any $f \in X^{(i)}$ the $U$-sequence at $f$ is a 5-adic zero sequence. Furthermore, by Definition 2.9 for any $g \in S_i$ there exists an $N_g \in \mathbb{N}$ such that $5^{N_g} g \in X^{(i)}$, so it follows that also for every $g \in S_i$ the $U$-sequence at $g$ is an 5-adic zero sequence.

Now we are ready for the proof of the Main Theorem.

*Proof of Theorem 2.11 ("Main Theorem").* We proceed by mathematical induction on $\beta$. For $\beta = 1$ the statement is settled by the first fundamental identity $L_1 = U^{(0)}(1) = 5(-t + 5p_1)$ of the Appendix (Section 7). The induction step will be carried out as follows: In the first step we show that the correctness of (14) for $N = 2\beta - 1$, $\beta \in \mathbb{N}^*$, implies (15) for $N + 1 = 2\beta$, which in the second step is shown to imply the correctness of (14) for $N + 2 = 2\beta + 1$.

For the first step we recall (8), (31) in Lemma 4.4 and apply the induction hypothesis (14) to obtain

$$U^{(1)}(L_{2\beta-1}) = 5^{2\beta-1} U^{(1)}(f_{2\beta-1}) = 5^{2\beta-1} \cdot 5 f_{2\beta}$$

for some $f_{2\beta} \in X^{(0)}$. Next we assume (15) and apply (32) in Lemma 4.4 to obtain

$$U^{(0)}(L_{2\beta}) = 5^{2\beta} U^{(0)}(f_{2\beta}) = 5^{2\beta} \cdot 5 f_{2\beta+1}$$

for some $f_{2\beta+1} \in X^{(1)}$. This completes the proof of the Main Theorem assuming the validity of the twenty fundamental relation in the Appendix (Section 7). Their correctness will be proven in the next section. □

## 5. Proving the Fundamental Relations

To prove the fundamental relations we use standard methodology from modular forms. For the sake of completness we provide a brief account of what is needed. For further details consult e.g. [9] and [17].

5.1. **Basic definitions and facts.** The special linear group $\mathrm{SL}_2(\mathbb{Z})$ acts on elements $\tau$ of the upper half plane $\mathbb{H}$ as usual; i.e., $\gamma\tau := \frac{a\tau+b}{c\tau+d}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. For any fixed $k \in \mathbb{Z}$ this action induces a group action of $\mathrm{SL}_2(\mathbb{Z})$ on functions $f : \mathbb{H} \to \mathbb{C}$ defined as follows. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ then

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau)$$

for all $\tau \in \mathbb{H}$. If $k = 0$ we simply write $f|\gamma$ instead of $f|_0\gamma$. Considering subgroups of $\mathrm{SL}_2(\mathbb{Z})$, for our purpose it suffices to restrict to the level $N \in \mathbb{N}^*$ congruence subgroups $\Gamma_0(N)$, i.e.,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

We denote the vector space (over $\mathbb{C}$) of weakly holomorphic modular forms of integer weight $k$ on $\Gamma_0(N)$ by $M_k^!(\Gamma_0(N))$. These are holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ such that $f|_k\gamma = f$ for all $\gamma \in \Gamma_0(N)$, and, if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $f|_k\gamma$ has a Fourier expansion of the form

$$(f|_k\gamma)(\tau) = \sum_{n \geq n_\gamma} a_\gamma(n) e^{2\pi i n\tau/N} \qquad (\tau \in \mathbb{H}),$$

where $a_\gamma(n_\gamma) \neq 0$. If $k = 0$ we simply call $f$ a modular function. If $n_\gamma \geq 0$ for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $f$ is called holomorphic modular form. Such functions $f$ form a subspace usually denoted by $M_k(\Gamma_0(N))$ or, in short, $M_k(N)$.

The next fact is very-well known and traces back to M. Newman in [14, Th. 1] and [15, Th. 1]. For algorithmic checking the following formulation is convenient.

**Lemma 5.1** ("Newman's Lemma"). *Let $r = (r_\delta)_{\delta|N}$ be a finite sequence of integers indexed by the positive divisors $\delta$ of $N \in \mathbb{N}^*$. Let $f_r : \mathbb{H} \to \mathbb{C}$ be the eta-quotient defined by $f_r(\tau) := \prod_{\delta|N} \eta^{r_\delta}(\delta\tau)$. Then*

$$f_r \in M_k(N) \text{ for } k = \frac{1}{2}\sum_{\delta|N} r_\delta,$$

*if the following conditions are satisfied:*

(i) $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$;
(ii) $\sum_{\delta|N} N r_\delta/\delta \equiv 0 \pmod{24}$;
(iii) $\prod_{\delta|N} \delta^{r_\delta}$ *is the square of a rational number;*
(iv) $\sum_{\delta|N} r_\delta \equiv 0 \pmod 4$;
(v) $\sum_{\delta|N} \gcd^2(\delta, d) r_\delta/\delta \geq 0$ *for all $d|N$.*

5.2. **An algorithmic proof method.** Theoretically it is straightforward to prove identities between weakly holomorphic modular forms; e.g., by bounding the cusp order and multipying by appropriate powers of the Delta-function. Consequently, the hard part of the work usually is considered to lie in the *discovery* of suitable linear relations. Nevertheless, in computational practice one often is forced to think more carefully about strategies with the goal to obtain bounds that are computationally feasible.

The twenty fundamental relations listed in the Appendix can be proved using the following strategy. We illustrate this computational method by taking as an example the celebrated identity of Jacobi [22, p. 470]:

$$\prod_{n=1}^{\infty}(1-q^{2n-1})^8 + 16q\prod_{n=1}^{\infty}(1+q^{2n})^8 = \prod_{n=1}^{\infty}(1+q^{2n-1})^8. \tag{35}$$

First we rewrite this identity in terms of eta products:

$$\frac{\eta^8(\tau)}{\eta^8(2\tau)} + 16\frac{\eta^8(4\tau)}{\eta^8(2\tau)} = \frac{\eta^{16}(2\tau)}{\eta^8(\tau)\eta^8(4\tau)}. \tag{36}$$

We multiply both sides of (36) with $\eta^{r_1}(\tau)\eta^{r_2}(2\tau)\eta^{r_4}(4\tau)$. Then $r_1, r_2$ and $r_4$, together with $N$ and $k$, are determined such that each summand in the resulting new equation becomes a modular form in $M_k(N)$. Computationally this amounts to solving the relations in Newman's Lemma (more precisely, the congruences *(i)*, *(ii)* and *(iv)* under the constraints *(iii)* and *(v)*) simultaneously for each of the three summands. A priori it is not clear that such a solution exists, but in the particular case $(r_1, r_2, r_4) = (8, 8, 8)$ is one possible solution. This way, (36) is transformed into

$$\eta^{16}(\tau)\eta^{8}(4\tau) + 16\eta^{16}(4\tau)\eta^{8}(\tau) - \eta^{24}(2\tau) = 0, \tag{37}$$

and, again by Lemma 5.1, it is trivial to verify-independently from the steps of the computation-that all three summands are in $M_{12}(4)$.

For the remaining step of the method we invoke a classical fact (e.g. [19, Th. 4.1.4 and (1.4.23)]) coming from the usual valence formula: Let $f, g \in M_k(N)$ with $f(\tau) = \sum_{n\geq 0} a(n)q^n$ and $g(\tau) = \sum_{n\geq 0} b(n)q^n$ for all $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$. Then $f = g$ if and only if $a(n) = b(n)$ for all $n$ such that

$$n \leq \frac{k}{12}\mu(N) \ \text{ with } \ \mu(N) := N\prod_{p|N}\left(1 + \frac{1}{p}\right), \tag{38}$$

where the product runs over all prime divisors $p$ of $N$. Note: $\mu(N) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$, the index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$.

Using the criterion (38) the proof of (37), resp. (35), is completed as follows. Denoting the left hand side of (37) with $f$, we have that $f \in M_k(N)$ with $k = 12$ and $N = 4$. Hence $\mu(4) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4)] = 6$, and to prove $f = 0$ it suffices to prove that the first $1 + k\mu(4)/12 = 7$ coefficients in its Taylor expansion are equal to 0.

Before we apply the proof strategy described in the previous section, it is convenient to introduce two lemmas dealing with the $U_n$-operator from Definition 2.2.

**Lemma 5.2.** *Let $f \in M_k(N)$. If $p$ is a prime with $p^2|N$, then $U_p(f) \in M_k(N/p)$.*

For a proof see e.g. [6, Lemma 17 (iv)]. — The next definition introduces the standard $V_n$-operator for which we use a slightly different notation.

**Definition 5.3.** *For $f : \mathbb{H} \to \mathbb{C}$ and $\mu_n := \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ define $f|\mu_n : \mathbb{H} \to \mathbb{C}$ by $(f|\mu_n)(\tau) := f(\mu_n\tau), \tau \in \mathbb{H}$.*

The following lemma is well-known as the "factorization property of $U$."

**Lemma 5.4.** *Let $f \in M_k(N)$. Then for any $n \in \mathbb{N}^*$ and $g : \mathbb{H} \to \mathbb{C}$,*

$$U_n((f|\mu_n)g) = fU_n(g).$$

5.3. **A computerized proof of the fundamental relations.** At the level of eta products we need the following facts that are immediate from Newman's Lemma.

**Lemma 5.5.** *For the functions from Definition 2.7 the following statements are true:*

(i) $\eta_5^{24} \cdot \frac{\eta_2^5\eta_{25}^4\eta_{100}^2}{\eta_{50}^5\eta_4^2\eta^4} \in M_{12}(100)$;
(ii) $t\eta^{24}$, $t\eta_5^{24} \in M_{12}(20)$;
(iii) $\sigma\eta^{24}$, $\sigma\eta_5^{24} \in M_{12}(20)$;
(iv) $\rho\eta^{24}$, $\rho\eta_5^{24} \in M_{12}(20)$;
(v) $t^{-j}\eta_5^{24} \in M_{12}(20)$, $0 \leq j \leq 5$;
(vi) $t^{-6}\eta_5^{48} \in M_{24}(20)$;
(vii) $t^j\eta^{48} \in M_{24}(20)$, $-2 \leq j \leq 5$;
(viii) $p_0\eta^{72}$, $p_0\eta_5^{72} \in M_{36}(20)$;
(ix) $p_1\eta^{96}$, $p_1\eta_5^{96} \in M_{48}(20)$.

*Proof.* The statements *(i)-(vii)* are straight-forward verifications invoking Lemma 5.1. In proving *(viii)* and *(ix)* we restrict to showing that $p_0\eta^{72} \in M_{36}(20)$ in *(viii)*, since the other cases are analogous. According to (12) we need to show that

$$t\sigma^2\eta^{72}, t\sigma\eta^{72}, \sigma\eta^{72}, \rho\eta^{72}, \sigma^2\eta^{72}, \sigma^2\rho\eta^{72}, \sigma\rho\eta^{72} \in M_{36}(20).$$

By *(ii)* and *(iii)* we have that $t\eta^{24}$ and $\sigma\eta^{24}$ are in $M_{12}(20)$. Consequently

$$\sigma\eta^{24} \cdot \sigma\eta^{24} \cdot t\eta^{24} \in M_{36}(20).$$

Similarly one sees that $t\eta^{24} \cdot \sigma\eta^{24} \cdot \eta^{24} \in M_{36}(20)$ because $\eta^{24} \in M_{12}(20)$. The other monomials are treated analogously. □

Next we connect all the fundamental relations to Newman's lemma.

**Lemma 5.6.** *For the functions from Definition 2.7 the following statements are true for any choice of integer coefficients $c(i,j)$ and $d(i,j)$:*

(i) $\eta^{144}\left(U^{(0)}(t^{-j}) - \sum_{i=-1}^{4}(c(i,j)t^i + d(i,j)p_1 t^i)\right) \in M_{72}(20)$, $0 \leq j \leq 4$;
(ii) $\eta^{144}\left(U^{(0)}(p_0 t^{-j}) - \sum_{i=-2}^{5}(c(i,j)t^i + d(i,j)p_1 t^i)\right) \in M_{72}(20)$, $2 \leq j \leq 6$;
(iii) $\eta^{144}\left(U^{(1)}(t^{-j}) - \sum_{i=0}^{4} c(i,j)t^i\right) \in M_{72}(20)$, $0 \leq j \leq 4$;
(iv) $\eta^{144}\left(U^{(1)}(p_1 t^{-j}) - \sum_{i=-2}^{5}(c(i,j)t^i + d(i,j)p_1 t^i)\right) \in M_{72}(20)$, $1 \leq j \leq 5$.

*Proof.* We only prove *(i)* which corresponds to Group I of the fundamental relations; the other cases are analogous. The statement follows from showing that each term in the sum is in $M_{72}(20)$. We start with the term $\eta^{144}U^{(0)}(t^{-j}) = \eta^{144}U_5(At^{-j})$ for a fixed $j \in \{0, \ldots, 4\}$. By Lemma 5.1, $\eta^{144} \in M_{72}(1)$ which implies together with Lemma 5.4,

$$\eta^{144}U_5(At^{-j}) = U_5(\eta_5^{144}At^{-j}).$$

By (7) we have that

$$\eta_5^{24}A = \eta_5^{24}\frac{\eta_2^5\eta_{25}^4\eta_{100}^2}{\eta_{50}^5\eta_4^2\eta^4},$$

which is in $M_{12}(100)$ by Lemma 5.5*(i)*. By Lemma 5.5*(v)* we have $t^{-j}\eta_5^{24} \in M_{12}(20) \subseteq M_{12}(100)$, because in general $\Gamma_0(N_1)$ is a subgroup of $\Gamma_0(N_2)$ if $N_2|N_1$. Observing that $\eta_5^{96} \in M_{48}(20) \subseteq M_{48}(100)$, we can conclude that

$$t^{-j}\eta_5^{24}\cdot\eta_5^{24}A\cdot\eta_5^{96} = \eta_5^{144}At^{-j} \in M_{72}(100).$$

Finally, Lemma 5.2 implies that $U_5(\eta_5^{144}At^{-j}) \in M_{72}(20)$. Proving that $\eta^{144}t^i$ and $\eta^{144}p_1t^i$ are in $M_{72}(20)$ for $-1 \leq i \leq 4$ is done analogously using Lemma 5.5 again. □

**Theorem 5.7.** *The twenty fundamental relations listed in the Appendix hold true.*

*Proof.* By Lemma 5.6, after multiplication with $\eta^{144}$ the entries of Group I to IV correspond to elements from $M_k(N)$ with $k = 72$ and $N = 20$. This means, we can apply the proof method described in Section 5.2 with $\mu(20) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(20)] = 36$. Consequently, the proof is completed by verifying equality of the first $1 + \mu(20)k/12 = 217$ coefficients in the Taylor series expansions of both sides of each of the fundamental relations. This task is left to the computer. □

## 6. New Functional Relations and Further Explanations

We proved Sellers' conjecture but we did not explain how the functions $t, p_0$ and $p_1$ in Definition 2.7 were found. The results in this section will not be proven but they provide hints for getting deeper insight into what is standing behind the proof. A more detailed account, including proofs of the results below, is planned for a separate paper. Here we will work with modular functions rather than modular forms. Let $K(N)$ denote the set of all modular functions for $\Gamma_0(N)$, i.e. $K(N) = M_0^!(\Gamma_0(N))$. Using the theory of modular functions one proves that all the $L_\alpha$ belong to $K(20)$, and also that there exists an $X \in K(20)$ (not unique) such that $K(20)$ is a free $\mathbb{C}[X, X^{-1}]$-module of finite rank.

Namely, as mentioned in the introduction we found that not all $U$-sequences at the $f \in K(20)$ of interest are 5-adic zero sequences. Informally, these are $f \in K(20)$ with some restrictions on the orders at the cusps and with coefficients in the $q$-expansion being integers. Once we observed this we tried to find a suitable subset $S_0 \subset K(20)$ such that for every $g \in S_0$ the $U$-sequence at $g$ is a 5-adic zero sequence. In the spirit of Atkin we made the assumption that there should exist another subset $S_1 \subset K(20)$ such that $U^{(0)}(S_0) \subseteq S_1$ and $U^{(1)}(S_1) \subseteq S_0$; in particular, $L_{2\alpha-1} \in S_1$ and $L_{2\alpha} \in S_0$. Let $W_4 := \left(\begin{smallmatrix} 4 & -1 \\ 100 & -24 \end{smallmatrix}\right)$ be the Atkin-Lehner involution, which is a mapping from $K(20)$ to $K(20)$, e.g. [6, Lem. 9]. After several computer experiments we observed that for all $b_\alpha : \mathbb{Z} \to \mathbb{C}$ defined by

$$\sum_{n=-\infty}^{\infty} b_\alpha(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{5n})^2} (L_{2\alpha-1}|W_4) \tag{39}$$

one has $b_\alpha(4n+2) = b_\alpha(4n+3) = 0$ for all $n \in \mathbb{Z}$. This discovery led to the natural question of finding all $f \in K(20)$ satisfying

$$a_f(4n+2) = a_f(4n+3) = 0, \quad n \in \mathbb{Z}, \tag{40}$$

where $a_f : \mathbb{Z} \to \mathbb{C}$ is defined by

$$\sum_{n=-\infty}^{\infty} a_f(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{5n})^2} (f|W_4).$$

It is easy to see that if $g = \sum_{n=-\infty}^{\infty} c(n) q^n \in K(5)$, then $g|W_4 = \sum_{n=-\infty}^{\infty} c(n) q^{4n}$. This implies immediately that if (40) is satisfied with $f$, then (40) is also satisfied with $fg$ for any $g \in K(5)$. Consequently, the space of all $f$ satisfying (40) is a $K(5)$-module. Since $K(5) = \mathbb{C}[t, t^{-1}]$ with $t$ as in (11), we are led to describe $K(20)$ as a $\mathbb{C}[t, t^{-1}]$-module (that is, we choose $X = t$) and then to select the submodule of all $f$ satisfying (40). By using standard methods we found that $K(20)$ is the free $\mathbb{C}[t, t^{-1}]$-module generated by $\{1, \sigma, \sigma^2, \rho, \rho\sigma, \rho\sigma^2\}$ with $\rho$ and $\sigma$ as in (11) (but of course, there are other kinds of representations). Next we make the "ansatz"

$$f = Y_1(t) + Y_2(t)\sigma + Y_3(t)\sigma^2 + Y_4(t)\rho + Y_5(t)\rho\sigma + Y_6(t)\rho\sigma^2$$

with $Y_j(t) \in \mathbb{C}[t]$ and the degree of $Y_j(t)$ smaller than some fixed number $M$. Then we do coefficient comparison and solve the resulting linear system under the constraint that (40) is fulfilled. This led us to $f = p_1$. Next we tried to see if $L_{2\alpha}$ satisfies a similar property and found that for all $c_\alpha : \mathbb{Z} \to \mathbb{C}$ defined by

$$\sum_{n=-\infty}^{\infty} c_\alpha(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^{n})^2} (L_{2\alpha}|W_4)$$

one has $c_\alpha(4n+2) = c_\alpha(4n+3) = 0$ for all $n \in \mathbb{Z}$. This led to the question of finding all $f \in K(20)$ satisfying

$$d_f(4n+2) = d_f(4n+3) = 0, \quad n \in \mathbb{Z}, \tag{41}$$

where $d_f : \mathbb{Z} \to \mathbb{C}$ is defined by

$$\sum_{n=-\infty}^{\infty} d_f(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2} (f|W_4).$$

Using the same "ansatz" technique as for $p_1$ we found $p_0$. One can prove that $f = t$ satisfies both (40) and (41). Consequently, because the elements satisfying (40) and (41) form $\mathbb{Z}[t]$-modules respectively, it follows that $f \in S_1 = \langle t, p_1 \rangle_{\mathbb{Z}[t]}$ satisfy (40) and all $f \in S_0 = \langle t, p_0 \rangle_{\mathbb{Z}[t]}$ satisfy (41). What is even more striking is that we found that any $f \in S_0$ satisfies an even more interesting functional relation:

(42)
$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2} f\left(\frac{4\tau-1}{100\tau-24}\right) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^{4n})^2} f(4\tau) + 2q \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-q^{4n})^2}{(1-q^{2n})} f\left(\frac{4\tau}{200\tau+1}\right);$$

moreover, as a counterpart, any $f \in S_1$ satisfies

(43)
$$\prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{5n})^2} f\left(\frac{4\tau-1}{100\tau-24}\right) = \prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{20n})^2} f(4\tau) + 2q^5 \prod_{n=1}^{\infty} \frac{(1-q^{5n})^2(1-q^{20n})^2}{(1-q^{10n})} f\left(\frac{4\tau}{200\tau+1}\right).$$

These are the functional relations mentioned in the introduction. We conclude by posing some questions which we plan to treat in a separate paper:

- Does the set of all $f \in M_0^!(20)$ with the additional restriction (43) equal the set $\langle 1, p_1 \rangle_{\mathbb{C}[t,t^{-1}]}$?
- Does the set of all $f \in M_0^!(20)$ with the additional restriction (42) equal the set $\langle 1, p_0 \rangle_{\mathbb{C}[t,t^{-1}]}$?
- How do the relations (43) and (42) generalize for primes different from 5?

## 7. Appendix: The Fundamental Relations

**Group I:**

$$\begin{aligned}
U^{(0)}(1) &= -5t + 5^2 p_1;\\
U^{(0)}(t^{-1}) &= -1 + p_1 t^{-1};\\
U^{(0)}(t^{-2}) &= \quad 5^5 t^2 + 11{\cdot}5^2 t + 11 - p_1(5^3 + 2{\cdot}5t^{-1});\\
U^{(0)}(t^{-3}) &= -5^8 t^3 - 34{\cdot}5^5 t^2 - 51{\cdot}5^3 t - 119 + p_1(2{\cdot}5^6 t + 6{\cdot}5^4 + 21{\cdot}5t^{-1});\\
U^{(0)}(t^{-4}) &= -5^{11} t^4 + 92{\cdot}5^6 t^2 + 759{\cdot}5^3 t + 253{\cdot}5 - p_1(8{\cdot}5^7 t + 99{\cdot}5^4 + 44{\cdot}5^2 t^{-1}).
\end{aligned}$$

**Group II:**

$$U^{(0)}(p_0t^{-2}) = 5^5t^2 - 114{\cdot}5^2t - 59 + p_1(124{\cdot}5^3 + 59t^{-1});$$

$$U^{(0)}(p_0t^{-3}) = -5^8t^3 + 36{\cdot}5^5t^2 + 103{\cdot}5^3t + 26 + p_1(5^6t - 9{\cdot}5^4 + 7\cdot 5t^{-1});$$

$$\begin{aligned} U^{(0)}(p_0t^{-4}) &= -5^{11}t^4 - 14{\cdot}5^9t^3 - 259{\cdot}5^6t^2 - 1436{\cdot}5^3t - 38{\cdot}5 \\ &\quad + p_1(5^9t^2 + 122{\cdot}5^6t + 211{\cdot}5^4 - 7{\cdot}5t^{-1}); \end{aligned}$$

$$\begin{aligned} U^{(0)}(p_0t^{-5}) &= 5^{14}t^5 - 12{\cdot}5^{11}t^4 - 9{\cdot}5^9t^3 + 1494\cdot 5^6t^2 + 2366{\cdot}5^4t + 196{\cdot}5 \\ &\quad - p_1(5^{12}t^3 + 8{\cdot}5^{10}t^2 + 282{\cdot}5^7t + 409{\cdot}5^5 - 11{\cdot}5^2t^{-1}); \end{aligned}$$

$$\begin{aligned} U^{(0)}(p_0t^{-6}) &= 7{\cdot}5^{15}t^5 + 372{\cdot}5^{12}t^4 + 917{\cdot}5^{10}t^3 + 1581{\cdot}5^7t^2 - 16089{\cdot}5^4t + 69{\cdot}5^2 \\ &\quad - t^{-1} - p_1(96{\cdot}5^{12}t^3 + 13{\cdot}5^{12}t^2 - 404{\cdot}5^7t - 3152{\cdot}5^5 + 361{\cdot}5^2t^{-1} - t^{-2}). \end{aligned}$$

**Group III:**

$$\begin{aligned} U^{(1)}(1) &= 1; \\ U^{(1)}(t^{-1}) &= -5^2t - 6; \\ U^{(1)}(t^{-2}) &= -5^5t^2 + 54; \\ U^{(1)}(t^{-3}) &= -5^8t^3 - 102{\cdot}5; \\ U^{(1)}(t^{-4}) &= -5^{11}t^4 + 966{\cdot}5. \end{aligned}$$

**Group IV:**

$$\begin{aligned} U^{(1)}(p_1t^{-1}) &= 3{\cdot}5^{10}t^4 + 77{\cdot}5^7t^3 + 562{\cdot}5^4t^2 + 41{\cdot}5^3t + 1 \\ &\quad + p_0(5^9t^3 + 14{\cdot}5^6t^2 + 44{\cdot}5^3t + 2{\cdot}5); \\ U^{(1)}(p_1t^{-2}) &= -5^5t^2 - 14{\cdot}5^2t + 7 + 5p_0; \\ U^{(1)}(p_1t^{-3}) &= -5^8t^3 - 14{\cdot}5^5t^2 - 5^4t - 12 + 5^4tp_0; \\ U^{(1)}(p_1t^{-4}) &= -5^{11}t^4 - 14{\cdot}5^8t^3 - 5^7t^2 + 12{\cdot}5 + 5^7t^2p_0; \\ U^{(1)}(p_1t^{-5}) &= 4{\cdot}5^{14}t^5 + 121{\cdot}5^{11}t^4 + 233{\cdot}5^9t^3 + 738{\cdot}5^6t^2 + 109{\cdot}5^4t - 17{\cdot}5^2 \\ &\quad - p_0(4{\cdot}5^{10}t^3 + 14{\cdot}5^8t^2 + 44{\cdot}5^5t + 2{\cdot}5^3 - t^{-1}). \end{aligned}$$

## 8. Acknowledgment

We thank the two anonymous referees for their detailed comments and suggestions that helped to improve the quality of this paper.

## References

[1] S. Ahlgren. Distribution of Parity of the Partition Function in Arithmetic Progressions. *Indagationes Mathematicae*, 10:173–181, 1999.

[2] S. Ahlgren and M. Boylan. Arithmetic Properties of the Partition Function. *Inventiones Mathematicae*, 153(3):487–502, 2003.

[3] G. E. Andrews. *The Theory of Partitions*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976.

[4] G. E. Andrews. Generalized Frobenius Partitions. *Memoirs of the American Mathematical Society*, 49(301), 1984.

[5] A. O. L. Atkin. Proof of a Conjecture of Ramanujan. *Glasgow Mathematical Journal*, 8:14–32, 1967.

[6] A. O. L. Atkin and J. Lehner. Hecke Operators on $\Gamma_0(M)$. *Mathematische Annalen*, 185:134–160, 1970.

[7] B. C. Berndt and K. Ono. Ramanujan's Unpublished Manuscript on the Partition and Tau Functions with Commentary. In D. Foata and G. N. Han, editors, *The Andrews Festschrift, Number Theory and Combinatorics*, pages 39–110. Springer, 2001.
[8] M. Dewar. Non-Existence of Ramanujan Congruences in Modular Forms of Level Four. *Canadian Journal of Mathematics*, 63:1284–1306, 2011.
[9] F. Diamond and J. Shurman. *A First Course in Modular Forms*. Springer, 2005.
[10] D. Eichhorn and J. A. Sellers. Computational Proofs of Congruences for 2-Colored Frobenius Partitions. *International Journal of Mathematics and Mathematical Sciences*, 29:333–340, 2002.
[11] B. Gordon. Ramanujan Congruences for $p_{-k}$ (mod $11^r$). *Glasgow Mathematical Journal*, 2:107–123, 1983.
[12] B. Gordon and K. Hughes. Ramanujan Congruences for $q(n)$. In *Analytic Number Theory (Philadelphia, Pa., 1980)*, volume 899 of *Lecture Notes in Mathematics*, pages 333–359. Springer, 1981.
[13] B. Gordon and K. Hughes. Congruences for $r_s(n)$. *Proceedings of London Mathematical Society*, 3:527–546, 1983.
[14] M. Newman. Construction and Application of a Class of Modular Functions. *Proceedings London Mathematical Society*, 3(7), 1957.
[15] M. Newman. Construction and Application of a Class of Modular Functions 2. *Proceedings London Mathematical Society*, 3(9), 1959.
[16] K. Ono. Distribution of the Partition Function Modulo $m$. *Annals of Mathematics*, 151(1):293–307, 2000.
[17] K. Ono. *The Web of Modularity: Arithmetic of the Coefficients of the Modular Forms and q-series*. Number 102 in CBMS Regional Conference Series in Mathematics. AMS, 2004.
[18] S. Ramanujan. Some Properties of $p(n)$, the Number of Partitions of $n$. *Proceedings of Cambridge Philosophical Society*, 19:207–210, 1919.
[19] R. A. Rankin. *Modular Forms and Functions*. Cambridge University Press, 1977.
[20] J. Sellers. Congruences Involving F-Partition Functions. *International Journal of Mathematics and Mathematical Sciences*, 17:187–188, 1994.
[21] G. N. Watson. Ramanujan's Vermutung über Zerfällungsanzahlen. *Journal für die Reine und Angewandte Mathematik*, 179:97–128, 1938.
[22] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. Cambridge University Press, 1965.

Research Institute for Symbolic Computation (RISC), Johannes Kepler University, A-4040 Linz, Austria

*E-mail address*: Peter.Paule@risc.uni-linz.ac.at

Research Institute for Symbolic Computation (RISC), Johannes Kepler University, A-4040 Linz, Austria

*E-mail address*: Silviu.Radu@risc.uni-linz.ac.at