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INFINITE FAMILIES OF STRANGE PARTITION CONGRUENCES FOR BROKEN 2-DIAMONDS

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Dedicated to our friend George E. Andrews on the occasion of his 70th birthday

ABSTRACT. In 2007 George E. Andrews and Peter Paule [1] introduced a new class of combinatorial objects called broken k-diamonds. Their generating functions connect to modular forms and give rise to a variety of partition congruences. In 2008 Song Heng Chan proved the first infinite family of congruences when $k = 2$. In this note we present two non-standard infinite families of broken 2-diamond congruences derived from work of Oliver Atkin and Morris Newman. In addition, four conjectures related to $k = 3$ and $k = 5$ are stated.

1. INTRODUCTION

A combinatorial study guided by MacMahon's Partition Analysis led G. E. Andrews and P. Paule [1] to the construction of a new class of directed graphs called broken k-diamonds. These objects were constructed in such a way that the generating functions of their counting sequences $(\Delta_k(n))_{n\geq 0}$ are closely related to modular forms, namely:

$$\sum_{n=0}^{\infty}\Delta_k(n)q^n = \prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})}$$
$$= q^{(k+1)/12}\frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)}, \quad k\geq 1,$$

where we recall the Dedekind eta function

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \quad (q = e^{2\pi i\tau}).$$

This modular aspect in turn led to various arithmetic theorems and conjectures. For example, M. Hirschhorn and J. Sellers [5] supplied a proof of

$$\Delta_2(10n+2) \equiv 0 \pmod 2, \quad n\geq 0, \tag{1}$$

which is Conjecture 1 of [1]. In addition, they observed and proved the congruences

$$\begin{aligned}\Delta_1(4n+2) &\equiv 0 \pmod 2, \quad n\geq 0\\ \Delta_1(4n+3) &\equiv 0 \pmod 2, \quad n\geq 0,\end{aligned}$$

and

$$\Delta_2(10n+6) \equiv 0 \pmod 2, \quad n\geq 0, \tag{2}$$

the latter being a class-mate of (1). The first parametrized families of congruences were given by S. H. Chan [4]; namely for $\alpha\geq 1$:

$$\Delta_2(5^{\alpha+1}n+\lambda_\alpha) \equiv 0 \pmod 5, \quad n\geq 0, \tag{3}$$

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and

$$\Delta_2(5^{\alpha+1}n + \mu_\alpha) \equiv 0 \pmod 5, \quad n \geq 0, \tag{4}$$

where λ_α and μ_α are the smallest positive integer solutions to

$$4\lambda_\alpha \equiv 1 \pmod{11 \cdot 5^\alpha} \quad \text{and} \quad 4\mu_\alpha \equiv 1 \pmod{19 \cdot 5^\alpha},$$

respectively. For example, $\alpha = 1$ gives the congruences

$$\Delta_2(25n + 14) \equiv \Delta_2(25n + 24) \equiv 0 \pmod 5, \quad n \geq 0; \tag{5}$$

the first one was stated as Conjecture 2 in [1].

Based on numerical experiments the authors of [1] wrote, "The following observations about congruences suggest strongly that there are undoubtedly a myriad of partition congruences for $\Delta_k(n)$. This list is only to indicate the tip of the iceberg." This note tries to continue along this line by presenting further evidence of the rich arithmetical structure of broken k-diamonds. Our parametrized families of congruences are different from those of Ramanujan type for ordinary partitions; in this sense our attribute "strange" has to be understood. For example, inspired by Atkin [3] and Atkin and O'Brien [2], in Section 2 we shall prove (Theorem 2.5) that if p is a prime such that $p \equiv 13 \pmod{20}$ or $p \equiv 17 \pmod{20}$, then

$$\Delta_2\left((5n + 4)p - \frac{p-1}{4}\right) \equiv 0 \pmod 5 \tag{6}$$

for all nonnegative integers n such that $20n + 15 \not\equiv 0 \pmod p$. Also in Section 2, inspired by M. Newman we shall prove (Lemma 2.10) that for all nonnegative integers k:

$$\Delta_2\left(4 \cdot 29^k - \frac{29^k - 1}{4}\right) \equiv k + 1 \pmod 5. \tag{7}$$

As a consequence the entries of the sequence $(\Delta_2(n))_{n\geq 0}$ visit all residue classes modulo 5 infinitely often. In Section 3 we present some concluding remarks and three open problems.

2. Strange Partition Congruences

The set of natural numbers is supposed to include 0; i.e., we have $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$. Writing $f(q) \equiv g(q) \pmod n$ for power series $f(q)$ and $g(q)$ as usually means: the coefficient sequences agree modulo n. Finally, it will be convenient to introduce the following convention: For coefficient sequences $(a(n))_{n\geq 0}$ of power series $\sum_{n\geq 0} a(n)q^n$ we extend the domain of the argument to the rational numbers by defining

$$a(r) := 0 \text{ if } r \in \mathbb{Q}\backslash\mathbb{N}.$$

Lemma 2.1.

$$\sum_{n=0}^{\infty} \Delta_2(5n + 4)q^n \equiv \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{12}}{(1 - q^n)^6} \pmod 5.$$

Proof. This follows immediately from S. H. Chan's result [4, (3.6)],

$$\sum_{n=0}^{\infty} \Delta_2(5n - 1)q^n \equiv q \prod_{n=1}^{\infty} \frac{(1 - q^{10n})^4(1 - q^n)^4}{(1 - q^{5n})^2(1 - q^{2n})^8} \pmod 5,$$

by utilizing the fact that $1 - q^{5m} \equiv (1 - q^m)^5 \pmod 5$. □

Defining
$$\sum_{n=0}^{\infty} c(n)q^n := \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}}{(1-q^n)^6}$$
we recall a special case of Newman's Theorem 3 in [6]:

Lemma 2.2. *For each prime p with $p \equiv 1 \pmod 4$ there exists an integer $x(p)$ such that for all $n \in \mathbb{N}$:*
$$c\left(np + 3\frac{p-1}{4}\right) + p^2 c\left(\frac{n}{p} - 3\frac{p-1}{4p}\right) = x(p)c(n). \tag{8}$$

Lemma 2.2, in view of $c(N) \equiv \Delta_2(5N+4) \pmod 5$, implies that for each prime p, $p \equiv 1 \pmod 4$, and all $n \in \mathbb{N}$:
$$\begin{aligned} \Delta_2\left((5n+4)p - \frac{p-1}{4}\right) &\equiv c\left(np + 3\frac{p-1}{4}\right) \\ &\equiv x(p)\Delta_2(5n+4) - p^2\Delta_2\left(\frac{5n+4}{p} + \frac{p-1}{4p}\right) \pmod 5. \end{aligned} \tag{9}$$
Setting $n = 0$ and noting that $\Delta_2(4) \equiv c(0) = 1$ we obtain
$$\Delta_2\left(4p - \frac{p-1}{4}\right) \equiv x(p) - p^2\Delta_2\left(\frac{p+15}{4p}\right) \pmod 5.$$
Noting that
$$p \neq 5 \Rightarrow \frac{p+15}{4p} \notin \mathbb{N} \Rightarrow \Delta_2\left(\frac{p+15}{4p}\right) = 0,$$
one obtains the

Corollary 2.3. *For all primes p with $p \equiv 1 \pmod 4$ we have*
$$\Delta_2\left((5n+4)p - \frac{p-1}{4}\right) \equiv \Delta_2\left(4p - \frac{p-1}{4}\right)\Delta_2(5n+4) \pmod 5 \tag{10}$$
for all $n \in \mathbb{N}$ such that $20n + 15 \not\equiv 0 \pmod p$.

Lemma 2.4. *For all primes p with $p \equiv 13$ or $p \equiv 17 \pmod{20}$ we have*
$$\Delta_2(4p - (p-1)/4) \equiv 0 \pmod 5.$$

Proof. By (5) we know that $\Delta_2(4p - (p-1)/4) \equiv 0 \pmod 5$ if
$$4p - (p-1)/4 \equiv 14 \pmod{25}, \tag{11}$$
or
$$4p - (p-1)/4 \equiv 24 \pmod{25}. \tag{12}$$
The statement follows by verifying (11) and (12) for $p \equiv 17 \pmod{20}$ and $p \equiv 13 \pmod{20}$, respectively. □

Finally Corollary 2.3 and Lemma 2.4 imply the

Theorem 2.5. *For any prime p with $p \equiv 13 \pmod{20}$ or $p \equiv 17 \pmod{20}$ we have*
$$\Delta_2(p(5n+4) - (p-1)/4) \equiv 0 \pmod 5,$$
for all $n \in \mathbb{N}$ such that $20n + 15 \not\equiv 0 \pmod p$.

Considering such n where $n = pk$ we see that condition $20n+15 \not\equiv 0 \pmod{p}$ holds for all nonnegative integers k if $p \neq 3, 5$. Hence we have

$$\Delta_2(p(5pk+4) - (p-1)/4) \equiv 0 \pmod{5}, \tag{13}$$

whenever $p \equiv 13, 17 \pmod{20}$ and k a nonnegative integer.

For example, the primes $13, 17, 37, 53, 73, 97$ are either congruent 13 or 17 modulo 20. Hence by (13):

$$\begin{aligned}
\Delta_2(5 \cdot 13^2 n + 49) &\equiv 0 \pmod{5},\\
\Delta_2(5 \cdot 17^2 n + 64) &\equiv 0 \pmod{5},\\
\Delta_2(5 \cdot 37^2 n + 139) &\equiv 0 \pmod{5},\\
\Delta_2(5 \cdot 53^2 n + 199) &\equiv 0 \pmod{5},\\
\Delta_2(5 \cdot 73^2 n + 274) &\equiv 0 \pmod{5},
\end{aligned}$$

and

$$\Delta_2(5 \cdot 97^2 n + 364) \equiv 0 \pmod{5}$$

for all $n \in \mathbb{N}$.

We also like to mention that the special case (9) of Newman's Lemma 2.2 implies [4, eq. 3.5], namely:

Lemma 2.6. *For $k \in \mathbb{N}^*$:*

$$\Delta_2(5n+4) \equiv \Delta_2\left(5^k n + \frac{1+3\cdot 5^k}{4}\right) \pmod{5}, \quad n \geq 0. \tag{14}$$

Proof. Setting $p = 5$ in (9) gives

$$\Delta_2(25n+19) \equiv x(5)\Delta_2(5n+4) \pmod{5}. \tag{15}$$

Since $\Delta_2(19) = 85606 \equiv 1 \pmod{5}$ and $\Delta_2(4) \equiv 1 \pmod{5}$ we obtain from (15) that $x(5) \equiv 1 \pmod{5}$. Hence (14) is true for $k = 2$. Next assume that (14) is true for $2 \leq k < N$. From this we conclude correctness for $k = N$ as follows. By the induction hypothesis we have for $v \in \mathbb{N}$:

$$\Delta_2(5(5v+3)+4) \equiv \Delta_2\left(5^{N-1}(5v+3) + \frac{1+3\cdot 5^{N-1}}{4}\right) \pmod{5}$$

which is equivalent to

$$\Delta_2(25v+19) \equiv \Delta_2\left(5^N v + \frac{1+3\cdot 5^N}{4}\right) \pmod{5}.$$

Next we apply (15) to conclude that $\Delta_2(5v+4) \equiv \Delta_2(25v+19) \pmod{5}$ and the proof is finished. □

Motivated by Newman's work [7], in combination with (9) we obtain interesting congruences for primes with $p \equiv 1 \pmod{4}$ and $x(p) \not\equiv 0 \pmod{5}$. For this purpose we introduce the following definition:

Definition 2.7. *For $n \in \mathbb{Z}$ and p a prime with $p \equiv 1 \pmod{4}$ we define a map $a_{p,n} : \mathbb{Z} \to \mathbb{Z}$ by*

$$a_{p,n}(k) := \Delta_2(p^k(5n+4) - (p^k-1)/4).$$

The next proposition is a straight-forward verification; it will be used to prove Lemma 2.9.

Proposition 2.8. *For $n \in \mathbb{Z}$ and p a prime with $p \equiv 1 \pmod{4}$ we have*

$$a_{p,pn+3(p-1)/4}(k) = a_{p,n}(k+1), \quad k \in \mathbb{Z}.$$

Lemma 2.9. *For $n \in \mathbb{Z}$ and p a prime with $p \equiv 1 \pmod 4$ we have*

$$a_{p,n}(k+2) - x(p)a_{p,n}(k+1) + p^2 a_{p,n}(k) \equiv 0 \pmod 5, \quad k \geq -1, \tag{16}$$

where $x(p)$ is as in (9).

Proof. By (9) we have for all $n \in \mathbb{Z}$ and all primes such that $p \equiv 1 \pmod 4$:

$$a_{p,n}(1) - x(p)a_{p,n}(0) + p^2 a_{p,n}(-1) \equiv 0 \pmod 5. \tag{17}$$

So (16) holds for $k = -1$. Proceeding by induction assume that (16) holds for all $k > N \geq -1$. To prove (16) for $k = N$, apply to

$$a_{p,n}(N+1) - x(p)a_{p,n}(N) + p^2 a_{p,n}(N-1) \equiv 0 \pmod 5, \tag{18}$$

which is (16) with $k = N-1$, the transformation $n \mapsto pn + 3(p-1)/4$. Using Proposition 2.8 completes the proof of Lemma 2.9. $\square$

Finally we consider the special choice $p = 29$ with $p \equiv 1 \pmod 4$. One computes

$$\Delta_2(4p - (p-1)/4) = 339953476833877 \equiv 2 \pmod 5.$$

We find that $x(29) = 2 \pmod 5$. For the choice $p = 29$ and $n = 0$ Lemma 2.9 turns into

$$a_{29,0}(k+2) - 2a_{29,0}(k+1) + a_{29,0}(k) \equiv 0 \pmod 5, \quad k \in \mathbb{Z}.$$

This congruence, viewed as an integer recurrence in k for $k \geq -1$, has the general solution $c_1 k + c_0$ with $c_0, c_1 \in \mathbb{Z}$. From $a_{29,0}(-1) = 0$ and $a_{29,0}(0) = \Delta_2(4) \equiv 1 \pmod 5$ we obtain the particular solution $k+1$ with $c_0 = c_1 = 1$. Thus we have proven statement (7), namely:

Lemma 2.10. *For $k \in \mathbb{N}$:*

$$a_{29,0}(k) = \Delta_2(29^k \cdot 4 - (29^k - 1)/4) \equiv k+1 \pmod 5.$$

3. Some Conjectures

Newman's Theorem 3 from [6], implying Lemma 2.2 as a special case, played a crucial role in this note. Concerning broken diamond congruences it seems that its scope of applications exceeds the Δ_2 case by far. To illustrate this point we pose some conjectures that involve analogous congruences to the ones we presented. Let

$$\sigma_3(n) = \sum_{d|n} d^3, \quad n \in \mathbb{N}^*.$$

Let

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

be the Eisenstein series of weight 4 for the full modular group. Numerical computations show strong evidence for the following three conjectures to be true.

Conjecture 3.1.

$$\prod_{n=1}^{\infty} (1-q^n)^4 (1-q^{2n})^6 \equiv 6 \sum_{n=0}^{\infty} \Delta_3(7n+5) q^n \pmod 7.$$

Conjecture 3.2.

$$\Delta_3(7^3 n + 82) \equiv \Delta_3(7^3 n + 278) \equiv \Delta_3(7^3 n + 327) \equiv 0 \pmod 7, \quad n \in \mathbb{N}.$$

Conjecture 3.3.

$$E_4(q^2)\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^2 \equiv 8\sum_{n=0}^{\infty}\Delta_5(11n+6)q^n \pmod{11}.$$

Conjecture 3.4. *Let* $\sum_{n=0}^{\infty} c(n)q^n := E_4(q^2)\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^2$. *Then for every prime* p *with* $p \equiv 1 \pmod 4$ *there exists an integer* $y(p)$ *such that*

$$c\left(pn+\frac{p-1}{2}\right)+p^8 c\left(\frac{n-(p-1)/2}{p}\right) = y(p)c(n)$$

for all $n \in \mathbb{N}$.

Let

$$\sum_{n=0}^{\infty} b(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^6.$$

We find again in the list of Newman [6, p. 486] that for all primes p such that $p \equiv 1 \pmod{12}$ there exists an integer $z(p)$ such that

$$b\left(np+\frac{2(p-1)}{3}\right)+p^4 b\left(\frac{n-2(p-1)/3}{p}\right) = z(p)b(n) \tag{19}$$

for all $n \in \mathbb{N}$. In particular (19), together with Conjecture 3.1 implies an identity analogous to (9). Similarly, Conjecture 3.3 together with Conjecture 3.4 also implies an identity analogous to (9), which lead to some generalizations of the results of this paper. Conjecture 3.2 is analogous to (5). We also tried to find a congruence similar to (5) for Δ_5 but failed. However when $z(p) \equiv 0 \pmod 7$ in (19) we obtain congruences modulo 7 for Δ_3. Similarly when $y(p) \equiv 0 \pmod{11}$ in Conjecture 3.4 we obtain congruences modulo 11 for Δ_5.

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