A PROOF OF SELLERS' CONJECTURE

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Abstract. In 1994 James Sellers conjectured an infinite family of Ramanujan type congruences for 2-colored Frobenius partitions introduced by George E. Andrews. These congruences arise modulo powers of 5. In 2002 Dennis Eichhorn and Sellers were able to settle the conjecture for powers up to 4. In this article we prove Sellers' conjecture for all powers of 5.

1. Introduction

In his 1984 Memoir [1], George E. Andrews introduced two families of partition functions, $\phi_k(m)$ and $c\phi_k(m)$, which he called generalized Frobenius partition functions. In this paper we restrict our attention to 2-colored Frobenius partitions. Their generating function reads as follows [1, (5.17)]:

$$\sum_{m=0}^{\infty} c\phi_2(m)q^m = \prod_{n=1}^{\infty} \frac{1-q^{4n-2}}{(1-q^{2n-1})^4(1-q^{4n})}. \tag{1}$$

In 1994 James Sellers [15] conjectured that for all integers $n \geq 0$ and $\alpha \geq 1$ one has

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha},$$

where λ_α is defined to be the smallest positive integer such that

$$12\lambda_\alpha \equiv 1 \pmod{5^\alpha}. \tag{2}$$

In his joint paper with Dennis Eichhorn [4] this conjecture was proved for the cases $\alpha = 1, 2, 3, 4$. In this paper we settle Sellers' conjecture for all α in the spirit of G. N. Watson [16]. Several authors (e.g. [9], [2]) have stated that the method of Watson works well when the modular functions involved live on a Riemann surface of genus 0. The reason for this is that every such modular function can be written as a rational function (in Watson's case polynomial function) in some fixed modular function t. In contrast to this, the modular functions that appear in this paper belong to a Riemann surface of genus 1. Treatments of this type are very rare in the literature. To the best of our knowledge only the papers by B. Gordon and K. Hughes [6], [7] and [8] apply Watson's method to genus 1 Riemann surfaces. In these papers the authors use a relatively simple trick on the modular equation to make Watson's method work for larger genus then 0. We are applying essentially the same idea in this paper; see Lemma 3.4 below.

Our article is structured as follows. In Section 2 we state the Main Theorem (Theorem 2.7) of our paper. It describes the action of a class of Rademacher operators

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on a quotient of eta function products being crucial for the problem Sellers' conjecture then is derived as an immediate consequence (Corollary 2.8). The rest of the paper deals with proving the Main Theorem. The basic building blocks of our proof are the twenty Fundamental Relations listed in the Appendix (Section 6). Despite postponing their proof to Section 5, we shall use these relations already in Section 3 and Section 4. In Section 3 a crucial result is proved, the Fundamental Lemma (Lemma 3.4), which has been inspired by work of B. Gordon and K. Hughes as it was mentioned above. The proof of the Main Theorem is presented in Section 4. To this end three further lemmas are introduced, all being immediate consequences of the Fundamental Lemma. Finally we mention that in Section 5, in order to prove the twenty Fundamental Relations, we utilize a computer-assisted method which is based on a variant of a well-known lemma by M. Newman (Lemma 5.6).

Throughout the paper we will use the following conventions: $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$ denote the nonnegative and positive integers, respectively. The complex upper half plane is denoted by $\mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$. As usual, $\eta(\tau)$ for $\tau \in \mathbb{H}$ denotes the Dedekind eta function for which

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \text{ where } q := e^{2\pi i\tau}. \tag{3}$$

We will also use the short hand notation:

$$\eta_n(\tau) := \eta(n\tau), \quad n \in \mathbb{Z}, \quad \tau \in \mathbb{H}. \tag{4}$$

For $x \in \mathbb{R}$ the symbol $\lfloor x \rfloor$ ("floor" of x) as usual denotes the greatest integer less or equal to x. Let $f = \sum_{n\in\mathbb{Z}} a_n q^n$, $f \neq 0$, be such that $a_n = 0$ for almost all $n < 0$. Then the order of f is the smallest integer N such that $a_N \neq 0$; we write $N = \operatorname{ord}(f)$. More generally, let $F = f \circ t = \sum_{n\in\mathbb{Z}} a_n t^n$ with $t = \sum_{n\geq 1} b_n q^n$, then the t-order of F is defined to be the smallest integer N such that $a_N \neq 0$; we write $N = \operatorname{ord}_t(F)$. For example, if $\operatorname{ord}(f) = -1$ and $t = q^2$, then $\operatorname{ord}_t(F) = -1$ but $\operatorname{ord}(F) = -2$.

2. The Main Theorem

Let

$$\mathrm{C}\Phi_2(q) := \sum_{m=0}^{\infty} c\phi_2(m) q^m.$$

Lemma 2.1. *For $\tau \in \mathbb{H}$,*

$$\mathrm{C}\Phi_2(q) = q^{\frac{1}{12}} \frac{\eta_2^5(\tau)}{\eta^4(\tau)\eta_4^2(\tau)}.$$

Proof. From (1),

$$\mathrm{C}\Phi_2(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2(2n-1)})(1 - q^{2n})^4}{(1 - q^n)^4 (1 - q^{4n})}$$
$$= \prod_{n=1}^{\infty} \frac{(1 - q^{2n})(1 - q^{2n})^4}{(1 - q^n)^4 (1 - q^{4n})^2}.$$

□

Subsequently we will study the action of the Rademacher [12] operator U_m on $C\Phi_2(q)$, respectively on

$$A(\tau) := \frac{\eta_2^5(\tau)}{\eta^4(\tau)\eta_4^2(\tau)}, \quad \tau \in \mathbb{H}. \tag{5}$$

Later the following abbreviation will be also useful:

$$B(\tau) := A(5\tau), \quad \tau \in \mathbb{H}. \tag{6}$$

Definition 2.2. *For $f : \mathbb{H} \to \mathbb{C}$ and $m \in \mathbb{N}^*$ we define $U_m(f) : \mathbb{H} \to \mathbb{C}$ by*

$$U_m(f)(\tau) := \frac{1}{m} \sum_{\lambda=0}^{m-1} f\left(\frac{\tau + 24\lambda}{m}\right), \quad \tau \in \mathbb{H}.$$

Obviously U_m is linear (over $\mathbb{C}$); in addition, it is easy to verify that

$$U_{mn} = U_m \circ U_n = U_n \circ U_m, \quad m, n \in \mathbb{N}^*. \tag{7}$$

The periodicity $\eta(\tau) = \eta(\tau + 24)$ implies for all $g : \mathbb{H} \to \mathbb{C}$,

$$U_5(Bg) = AU_5(g). \tag{8}$$

Lemma 2.3. *For $\alpha \in \mathbb{N}^*$ and λ_α as in* (2)*:*

$$U_{5^\alpha}(A)(\tau) = q^{\frac{12\lambda_\alpha - 1}{12 \cdot 5^\alpha}} \sum_{n=0}^{\infty} c\phi_2(5^\alpha n + \lambda_\alpha) q^n, \quad \tau \in \mathbb{H}.$$

Proof. We have:

$$\begin{aligned}
U_{5^\alpha}(A)(\tau) &= U_{5^\alpha}\left(q^{-1/12} \sum_{m=0}^{\infty} c\phi_2(m) q^m\right) \\
&= e^{-\frac{2\pi i \tau}{12 \cdot 5^\alpha}} \frac{1}{5^\alpha} \sum_{m=0}^{\infty} c\phi_2(m) e^{\frac{2\pi i m \tau}{5^\alpha}} \sum_{\lambda=0}^{5^\alpha - 1} e^{\frac{2\pi i \lambda(24m-2)}{5^\alpha}} \\
&= e^{\frac{-2\pi i \tau}{12 \cdot 5^\alpha}} \sum_{m \geq 0}^{*} c\phi_2(m) e^{\frac{2\pi i \tau m}{5^\alpha}} \\
&= e^{\frac{-2\pi i \tau}{12 \cdot 5^\alpha}} \sum_{n=0}^{\infty} c\phi_2(5^\alpha n + \lambda_\alpha) e^{\frac{2\pi i \tau (5^\alpha n + \lambda_\alpha)}{5^\alpha}} \\
&= q^{\frac{12\lambda_\alpha - 1}{12 \cdot 5^\alpha}} \sum_{n=0}^{\infty} c\phi_2(5^\alpha n + \lambda_\alpha) q^n;
\end{aligned}$$

the sum $\sum_{m \geq 0}^{*}$ runs over all $m \in \mathbb{N}$ such that $12m \equiv 1 \pmod{5^\alpha}$. □

The following explicit expressions for λ_α are easily verified.

Lemma 2.4. *For $\beta \in \mathbb{N}^*$:*

$$\lambda_{2\beta-1} = \frac{1 + 7 \cdot 5^{2\beta-1}}{12} \quad \text{and} \quad \lambda_{2\beta} = \frac{1 + 11 \cdot 5^{2\beta}}{12}.$$

Definition 2.5. *Let t, ρ, σ, p_0, and p_1 be functions defined on $\mathbb{H}$ as follows:*

$$t := \frac{\eta_5^6}{\eta^6}, \quad \rho := \frac{\eta_2 \eta_{10}^3}{\eta_4^3 \eta_{20}}, \quad \sigma := \frac{\eta_2^2 \eta_5^4}{\eta^4 \eta_{10}^2} \tag{9}$$

$$p_0 := \frac{1}{2}(-4t\sigma - 25t\rho\sigma^2 - 2\rho\sigma^2 + 30t\sigma^2 + 2\sigma^2 + t\rho), \tag{10}$$

$$p_1 := \frac{1}{2}(-250t\sigma^2 + 200t\sigma + 20\sigma + \rho - 22\sigma^2 + 5\rho\sigma^2 - 4\rho\sigma). \tag{11}$$

We note that all functions defined in Definition 2.5 have Taylor series expansions in powers of q with coefficients in $\mathbb{Z}$, resp. $\frac{1}{2}\mathbb{Z}$. (In fact, one can show that all the coefficients are in $\mathbb{Z}$ but this is not needed for our purpose.) In particular, $\mathrm{ord}(\rho) = \mathrm{ord}(\sigma) = 0$ and $\mathrm{ord}(t) = 1$, which implies $\mathrm{ord}(p_0) \geq 1$ and $\mathrm{ord}(p_1) \geq 1$.

Before stating the Main Theorem of the paper, we introduce convenient shorthand notation.

Definition 2.6. *A map $a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is called discrete array if for each $i \in \mathbb{Z}$ the map $a(i, -) : \mathbb{Z} \to \mathbb{Z}$, $j \mapsto a(i, j)$, has finite support.*

Theorem 2.7 ("Main Theorem"). *There exist discrete arrays r, s, u, v such that for $\beta \in \mathbb{N}^*$ and $\tau \in \mathbb{H}$:*

$$\begin{aligned} U_{5^{2\beta-1}}(A)(\tau) = 5^{2\beta-1} A(5\tau) \Bigg(& p_0(\tau) \sum_{n=0}^{\infty} r(\beta, n) 5^{\lfloor \frac{5n+2}{2} \rfloor} t^n(\tau) \\ & + \sum_{n=1}^{\infty} s(\beta, n) 5^{\lfloor \frac{5n-5}{2} \rfloor} t^n(\tau) \Bigg), \end{aligned} \tag{12}$$

and

$$\begin{aligned} U_{5^{2\beta}}(A)(\tau) = 5^{2\beta} A(\tau) \Bigg(& p_1(\tau) \sum_{n=0}^{\infty} u(\beta, n) 5^{\lfloor \frac{5n+1}{2} \rfloor} t^n(\tau) \\ & + \sum_{n=1}^{\infty} v(\beta, n) 5^{\lfloor \frac{5n-4}{2} \rfloor} t^n(\tau) \Bigg). \end{aligned} \tag{13}$$

The remaining sections are devoted to proving the Main Theorem by mathematical induction on β. In Sections 3 and 4 we describe the algebra underlying the induction step. In Section 5 we settle the initial cases, i.e., the correctness of the twenty fundamental relations listed in the Appendix (Section 6).

We conclude this section by deriving the truth of Sellers' conjecture as a corollary.

Corollary 2.8. *Sellers' conjecture is true; i.e., for $\alpha \in \mathbb{N}^*$:*

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \quad n \in \mathbb{N}^*.$$

Proof. The statement is derived immediately by applying the Lemmas 2.3 and 2.4 to (12) and (13). □

3. The Fundamental Lemma

In this section we prove the Fundamental Lemma, Lemma 3.4, which will play a crucial role in the proof of the Main Theorem in Section 4.

Definition 3.1. *With $t = t(\tau)$ as in Definition 2.5 we define:*

$$a_0(t) = -t, a_1(t) = -5^3t^2 - 6 \cdot 5t, a_2(t) = -5^6t^3 - 6 \cdot 5^4t^2 - 63 \cdot 5t,$$
$$a_3(t) = -5^9t^4 - 6 \cdot 5^7t^3 - 63 \cdot 5^4t^2 - 52 \cdot 5^2t,$$
$$a_4(t) = -5^{12}t^5 - 6 \cdot 5^{10}t^4 - 63 \cdot 5^7t^3 - 52 \cdot 5^5t^2 - 63 \cdot 5^2t.$$

We define $s : \{0, \dots, 4\} \times \{1, \dots, 5\} \to \mathbb{Z}$ to be the unique function satisfying

$$a_j(t) = \sum_{l=1}^{5} s(j,l) 5^{\lfloor \frac{5l+j-4}{2} \rfloor} t^l. \tag{14}$$

Note 3.2. Writing $a_j(t)$ as in (14) to reveal divisibility by powers of 5 of its coefficients is of help in the proof of Lemma 4.2 and is inspired by [3].

Lemma 3.3. *For $0 \leq \lambda \leq 4$ let*

$$t_\lambda(\tau) := t\left(\frac{\tau + 24\lambda}{5}\right), \quad \tau \in \mathbb{H}.$$

Then in the polynomial ring $\mathbb{C}(t)[X]$:

$$X^5 + \sum_{j=0}^{4} a_j(t) X^j = \prod_{\lambda=0}^{4} (X - t_\lambda). \tag{15}$$

Proof. First we prove

$$\prod_{\lambda=0}^{4} t_\lambda = -a_0(t) = t. \tag{16}$$

With $\omega := e^{48\pi i/5}$ one has for $\tau \in \mathbb{H}$:

$$\prod_{\lambda=0}^{4} t_\lambda(\tau) = \prod_{\lambda=0}^{4} q^{1/5}\omega^\lambda \prod_{n=1}^{\infty} \left(\frac{1-q^n}{1-\omega^{\lambda n}q^{n/5}}\right)^6 = q \prod_{n=1}^{\infty} \prod_{\lambda=0}^{4} \left(\frac{1-q^n}{1-\omega^{\lambda n}q^{n/5}}\right)^6$$
$$= q \prod_{n=1}^{\infty} (1-q^n)^{30} \prod_{n=1}^{\infty} \left(\frac{1-q^{5n}}{1-q^n}\right)^6 \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n}\right)^{30} = t(\tau).$$

Here we used the fact that $\prod_{\lambda=0}^{4}(1 - \omega^{\lambda n} z)$ equals $(1-z)^5$ if $5|n$, and $1 - z^5$ otherwise.

For the remaining part of the proof we use (16) to rewrite (15) into the equivalent form

$$X^5 + \sum_{j=0}^{4} a_j(t) X^j = -t \prod_{\lambda=0}^{4} (1 - X t_\lambda^{-1}). \tag{17}$$

Hence to complete the proof, in view of $t = \prod_{\lambda=0}^{4} t_\lambda$ it remains to show that

$$a_j(t) = (-1)^{j+1} t e_j(t_0^{-1}, \dots, t_4^{-1}), \quad 0 \leq j \leq 4, \tag{18}$$

where the e_j are the elementary symmetric functions. To this end we utilize the fact that

$$5U_5(t^{-j}) = \sum_{\lambda=0}^{4} t_\lambda^{-j}, \quad j \in \mathbb{Z}.$$

The first non-trivial case is $j = 1$. Observing

$$e_1(t_0^{-1}, \ldots, t_4^{-1}) = \sum_{\lambda=0}^{4} t_\lambda^{-1} = 5U_5(t^{-1}),$$

to show (18) for $j = 1$ we need to show that

$$5U_5(t^{-1}) = t^{-1}a_1(t) = -5^3 t - 5 \cdot 6.$$

But this is a disguised version of the second entry

$$U_5(Bt^{-1}) = (-5^2 t - 6)A \tag{19}$$

of Group III of the twenty fundamental relations from the Appendix. Namely, by (8) one has $U_5(Bt^{-1}) = AU_5(t^{-1})$. The next cases $2 \le j \le 4$ work analogously with the remaining entries of Group III. For example, if $j = 2$ then Newton's formula, translating elementary symmetric functions into power sums, implies

$$\begin{aligned} e_2(t_0^{-1}, \ldots, t_4^{-1}) &= \frac{1}{2}\left(\left(5U_5(t^{-1})\right)^2 - 5U_5(t^{-2})\right) \\ &= \frac{1}{2}\left((-5^3 t - 5{\cdot}6)^2 - (-5^6 t^2 + 54{\cdot}5)\right) = -t^{-1}a_2(t). \end{aligned}$$

Here we used the third entry of Group III. □

Finally we are ready for the main result of this section.

Lemma 3.4 ("Fundamental Lemma")**.** *For $u : \mathbb{H} \to \mathbb{C}$ and $j \in \mathbb{Z}$:*

$$U_5(ut^j) = -\sum_{l=0}^{4} a_l(t) U_5(ut^{j+l-5}).$$

Proof. For $\lambda \in \{0, \ldots, 4\}$ Lemma 3.3 implies

$$t_\lambda^5 + \sum_{l=0}^{4} a_l(t) t_\lambda^l = 0.$$

Multiplying both sides with $u_\lambda t_\lambda^{j-5}$ where $u_\lambda(\tau) := u((\tau + 24\lambda)/5)$ gives

$$u_\lambda t_\lambda^j + \sum_{l=0}^{4} a_l(t) u_\lambda t_\lambda^{j+l-5} = 0.$$

Summing both sides over all λ from $\{0, \ldots, 4\}$ completes the proof of the lemma. □

4. Proving the Main Theorem

We need to prepare with some lemmas. Recall that t is as in Definition 2.5.

Lemma 4.1. *Given functions $v_1, v_2, u : \mathbb{H} \to \mathbb{C}$ and $l \in \mathbb{Z}$. Suppose for $l \le k \le l+4$ there exist Laurent polynomials $p_k^{(1)}(t)$, $p_k^{(2)}(t) \in \mathbb{Z}[t, t^{-1}]$ such that*

$$U_5(ut^k) = v_1 p_k^{(1)}(t) + v_2 p_k^{(2)}(t) \tag{20}$$

and

$$\mathrm{ord}_t\left(p_k^{(i)}(t)\right) \ge \left\lceil \frac{k + s_i}{5} \right\rceil, \quad i \in \{1, 2\}, \tag{21}$$

for some fixed integers s_1 and s_2. Then there exist families of Laurent polynomials $p_k^{(1)}(t), p_k^{(2)}(t) \in \mathbb{Z}[t, t^{-1}]$, $k \in \mathbb{Z}$, such that (20) *and* (21) *hold for all $k \in \mathbb{Z}$.*

Proof. Let $N > l+4$ be an integer and assume by induction that there are families of Laurent polynomials $p_k^{(i)}(t)$, $i \in \{1,2\}$, such that (20) and (21) hold for $l \le k \le N-1$. Suppose

$$p_k^{(i)}(t) = \sum_{n \ge \lceil \frac{k+s_i}{5} \rceil} c_i(k,n) t^n, \quad 1 \le k \le N-1,$$

with integers $c_i(k,n)$. Applying Lemma 3.4 we obtain:

$$\begin{aligned} U_5(ut^N) &= -\sum_{j=0}^{4} a_j(t) U_5(ut^{N+j-5}) \\ &= -\sum_{j=0}^{4} a_j(t) \sum_{i=1}^{2} v_i \sum_{n \ge \lceil \frac{N+j-5+s_i}{5} \rceil} c_i(N+j-5,n) t^n \\ &= -\sum_{i=1}^{2} v_i \sum_{j=0}^{4} a_j(t) t^{-1} \sum_{n \ge \lceil \frac{N+j+s_i}{5} \rceil} c_i(N+j-5,n-1) t^n. \end{aligned}$$

Recalling the fact that $a_j(t)t^{-1}$ for $0 \le j \le 4$ is a polynomial in t, this determines Laurent polynomials $p_N^{(i)}(t)$ with the desired properties. The induction proof for $N < l$ work analogously. □

Lemma 4.2. *Given functions $v_1, v_2, u : \mathbb{H} \to \mathbb{C}$ and $l \in \mathbb{Z}$. Suppose for $l \le k \le l+4$ there exist Laurent polynomials $p_k^{(i)} \in \mathbb{Z}[t,t^{-1}]$, $i \in \{1,2\}$, such that*

$$U_5(ut^k) = v_1 p_k^{(1)}(t) + v_2 p_k^{(2)}(t) \tag{22}$$

where

$$p_k^{(i)}(t) = \sum_n c_i(k,n) 5^{\lfloor \frac{5n-k+\gamma_i}{2} \rfloor} t^n \tag{23}$$

with integers γ_i and $c_i(k,n)$. Then there exist families of Laurent polynomials $p_k^{(i)}(t) \in \mathbb{Z}[t,t^{-1}]$, $k \in \mathbb{Z}$, of the form (23) for which property (22) holds for all $k \in \mathbb{Z}$.

Proof. Suppose for an integer $N > l+4$ there are families of Laurent polynomials $p_k^{(i)}(t)$, $i \in \{1,2\}$, of the form (23) satisfying property (22) for $l \le k \le N-1$. We proceed by mathematical induction on N. Applying Lemma 3.4 and using the induction base (22) and (23) we obtain:

$$U_5(ut^N) = -\sum_{j=0}^{4} a_j(t) \sum_{i=1}^{2} v_i \sum_n c_i(N+j-5,n) 5^{\lfloor \frac{5n-(N+j-5)+\gamma_i}{2} \rfloor} t^n.$$

Utilizing (14) from Definition 3.1 this rewrites into :

$$\begin{aligned} U_5(ut^N) &= -\sum_{j=0}^{4} \sum_{l=1}^{5} s(j,l) 5^{\lfloor \frac{5l+j-4}{2} \rfloor} t^l \\ &\quad \times \sum_{i=1}^{2} v_i \sum_n c_i(N+j-5,n) 5^{\lfloor \frac{5n-(N+j-5)+\gamma_i}{2} \rfloor} t^n \\ &= -\sum_{i=1}^{2} v_i \sum_{j=0}^{4} \sum_{l=1}^{5} \sum_n s(j,l) c_i(N+j-5,n-l) \\ &\quad \times 5^{\lfloor \frac{5(n-l)-(N+j-5)+\gamma_i}{2} \rfloor + \lfloor \frac{5l+j-4}{2} \rfloor} t^n. \end{aligned} \tag{24}$$

The induction step is completed by simplifying the exponent of 5 as follows:

$$\left\lfloor \frac{5(n-l)-(N+j-5)+\gamma_i}{2} + \left\lfloor \frac{5l+j-4}{2} \right\rfloor \right\rfloor$$
$$\geq \left\lfloor \frac{5(n-l)-(N+j-5)+\gamma_i}{2} + \frac{5l+j-5}{2} \right\rfloor$$
$$= \left\lfloor \frac{5n-N+\gamma_i}{2} \right\rfloor .$$

The induction proof for $N < l$ works analogously. □

Before proving the Main Theorem, Theorem 2.7, we need one more lemma.

Lemma 4.3. *Given A and B as in* (5) *and* (6)*, and p_0 and p_1 as in* (10) *and* (11)*, respectively. Then there exist discrete arrays a_i, b_i, c, and d_i, $i \in \{0,1\}$, such that the following relations hold for all $k \in \mathbb{N}$:*

(25)
$$B^{-1}U_5(At^k) = \sum_{n\geq\lceil (k+1)/5\rceil} a_0(k,n)5^{\lfloor \frac{5n-k-2}{2}\rfloor}t^n + p_0 \sum_{n\geq\lceil (k-4)/5\rceil} a_1(k,n)5^{\lfloor \frac{5n-k+5}{2}\rfloor}t^n,$$

(26)
$$B^{-1}U_5(Ap_1t^k) = \sum_{n\geq\lceil (k+1)/5\rceil} b_0(k,n)5^{\lfloor \frac{5n-k-2}{2}\rfloor}t^n + p_0 \sum_{n\geq\lceil (k-4)/5\rceil} b_1(k,n)5^{\lfloor \frac{5n-k+4}{2}\rfloor}t^n,$$

(27)
$$A^{-1}U_5(Bt^k) = \sum_{n\geq\lceil k/5\rceil} c(k,n)5^{\lfloor \frac{5n-k-1}{2}\rfloor}t^n,$$

(28)
$$A^{-1}U_5(Bp_0t^k) = \sum_{n\geq\lceil (k+1)/5\rceil} d_0(k,n)5^{\lfloor \frac{5n-k-2}{2}\rfloor}t^n + p_1 \sum_{n\geq\lceil k/5\rceil} d_1(k,n)5^{\lfloor \frac{5n-k+1}{2}\rfloor}t^n.$$

Proof. The Appendix (Section 6) lists twenty fundamental relations, which are proved in Section 5 (Theorem 5.16). The five fundamental relations of Group I fit the pattern of the relation (25) for five consecutive values of k. The same observation applies to the relations of the Groups II, III and IV with regard to the relations (26), (27), and (28), respectively. In each of these cases k is less or equal to 0. Hence applying Lemma 4.1 and Lemma 4.2 immediately proves the statement for all $k \geq 0$. □

Now we are ready for the proof of the Main Theorem.

Proof of Theorem 2.7 ("Main Theorem"). Recall that $B(\tau) = A(5\tau)$ for $\tau \in \mathbb{H}$. We proceed by mathematical induction on β. For $\beta = 1$ the statement is settled by the first fundamental identity $U_5(A) = 5B(-t + 5p_0)$ of the Appendix (Section 6). The induction step will be carried out as follows: In the first step we show that the correctness of (12) for $N = 2\beta - 1$, $\beta \in \mathbb{N}^*$, implies (13) for $N + 1 = 2\beta$, which in the second step is shown to imply the correctness of (12) for $N + 2 = 2\beta + 1$.

For the first step we recall (7) and apply the induction hypothesis (12) to obtain

$$U_{5^{2\beta}}(A) = U_5(U_{5^{2\beta-1}}(A))$$
$$= 5^{2\beta-1}\left(\sum_{n=0}^{\infty} r(\beta,n)5^{\lfloor \frac{5n+2}{2}\rfloor}U_5(Bp_0t^n) + \sum_{n=1}^{\infty} s(\beta,n)5^{\lfloor \frac{5n-5}{2}\rfloor}U_5(Bt^n)\right).$$

Utilizing (27) and (28) of Lemma 4.3 with discrete arrays c and d_i gives

$$U_{5^{2\beta}}(A) = 5^{2\beta-1}A\Bigg(p_1\sum_{m\geq 0}\sum_{n\geq 0} r(\beta,n)d_1(n,m)5^{\lfloor\frac{5n+2}{2}\rfloor+\lfloor\frac{5m-n+1}{2}\rfloor}t^m$$
$$+\sum_{m\geq 1}\sum_{n\geq 0} r(\beta,n)d_0(n,m)5^{\lfloor\frac{5n+2}{2}\rfloor+\lfloor\frac{5m-n-2}{2}\rfloor}t^m \tag{29}$$
$$+\sum_{m\geq 1}\sum_{n\geq 1} s(\beta,n)c(n,m)5^{\lfloor\frac{5n-5}{2}\rfloor+\lfloor\frac{5m-n-1}{2}\rfloor}t^m\Bigg).$$

Observe that for $m,n\geq 0$:

$$\left\lfloor\frac{5n+2}{2}\right\rfloor+\left\lfloor\frac{5m-n+1}{2}\right\rfloor=\left\lfloor\frac{5m+n+1}{2}\right\rfloor+\left\lfloor\frac{3n+2}{2}\right\rfloor\geq\left\lfloor\frac{5m+1}{2}\right\rfloor+1,$$
$$\left\lfloor\frac{5n+2}{2}\right\rfloor+\left\lfloor\frac{5m-n-2}{2}\right\rfloor=\left\lfloor\frac{5m+n-2}{2}\right\rfloor+\left\lfloor\frac{3n+2}{2}\right\rfloor\geq\left\lfloor\frac{5m-4}{2}\right\rfloor+1,$$

and for $m,n\geq 1$:

$$\left\lfloor\frac{5n-5}{2}\right\rfloor+\left\lfloor\frac{5m-n-1}{2}\right\rfloor=\left\lfloor\frac{5m+n-5}{2}\right\rfloor+\left\lfloor\frac{3n-1}{2}\right\rfloor\geq\left\lfloor\frac{5m-4}{2}\right\rfloor+1.$$

Hence the right hand side of (29) is of the desired form (13).

For the second step we again recall (7) and apply the induction hypothesis (13) to obtain

$$U_{5^{2\beta+1}}(A) = U_5(U_{5^{2\beta}}(A))$$
$$= 5^{2\beta}\left(\sum_{n=0}^{\infty} r(\beta,n)5^{\lfloor\frac{5n+1}{2}\rfloor}U_5(Ap_1t^n)+\sum_{n=1}^{\infty} s(\beta,n)5^{\lfloor\frac{5n-4}{2}\rfloor}U_5(At^n)\right).$$

Utilizing (25) and (26) of Lemma 4.3 with discrete arrays a_i and b_i gives

$$U_{5^{2\beta+1}}(A) = 5^{2\beta}B$$
$$\times\Bigg(p_0\sum_{m\geq 0}\sum_{n\geq 0} r(\beta,n)b_1(n,m)5^{\lfloor\frac{5n+1}{2}\rfloor+\lfloor\frac{5m-n+4}{2}\rfloor}t^m$$
$$+p_0\sum_{m\geq 0}\sum_{n\geq 1} s(\beta,n)a_1(n,m)5^{\lfloor\frac{5n-4}{2}\rfloor+\lfloor\frac{5m-n+5}{2}\rfloor}t^m \tag{30}$$
$$+\sum_{m\geq 1}\sum_{n\geq 0} r(\beta,n)b_0(n,m)5^{\lfloor\frac{5n+1}{2}\rfloor+\lfloor\frac{5m-n-2}{2}\rfloor}t^m$$
$$+\sum_{m\geq 1}\sum_{n\geq 1} s(\beta,n)a_0(n,m)5^{\lfloor\frac{5n-4}{2}\rfloor+\lfloor\frac{5m-n-2}{2}\rfloor}t^m\Bigg).$$

Similar to above observe that for $m,n\geq 0$:

$$\left\lfloor\frac{5n+1}{2}\right\rfloor+\left\lfloor\frac{5m-n+4}{2}\right\rfloor=\left\lfloor\frac{5m+n+2}{2}\right\rfloor+\left\lfloor\frac{3n+3}{2}\right\rfloor\geq\left\lfloor\frac{5m+2}{2}\right\rfloor+1,$$

for $m\geq 0$ and $n\geq 1$:

$$\left\lfloor\frac{5n-4}{2}\right\rfloor+\left\lfloor\frac{5m-n+5}{2}\right\rfloor=\left\lfloor\frac{5m+n+2}{2}\right\rfloor+\left\lfloor\frac{3n-1}{2}\right\rfloor\geq\left\lfloor\frac{5m+2}{2}\right\rfloor+1,$$

for $m \geq 1$ and $n \geq 0$:

$$\left\lfloor \frac{5n+1}{2} \right\rfloor + \left\lfloor \frac{5m-n-2}{2} \right\rfloor = \left\lfloor \frac{5m+n-4}{2} \right\rfloor + \left\lfloor \frac{3n+3}{2} \right\rfloor \geq \left\lfloor \frac{5m-5}{2} \right\rfloor + 1,$$

and for $m, n \geq 1$:

$$\left\lfloor \frac{5n-4}{2} \right\rfloor + \left\lfloor \frac{5m-n-2}{2} \right\rfloor = \left\lfloor \frac{5m+n-6}{2} \right\rfloor + \left\lfloor \frac{3n}{2} \right\rfloor \geq \left\lfloor \frac{5m-5}{2} \right\rfloor + 1.$$

Hence the right hand side of (30) is of the desired form (12) with β replaced by $\beta + 1$. This completes the proof of the Main Theorem assuming the validity of the twenty fundamental relation in the Appendix (Section 6). Their correctness will be proven in the next section. □

5. Proving the Fundamental Relations

5.1. **Basic definitions and facts.** The general linear group

$$\mathrm{GL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc \neq 0 \right\}$$

acts on elements τ of the upper half plane $\mathbb{H}$ as usual; i.e., for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$:

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}.$$

We recall basic notions related to the modular group

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}$$

which, as a subgroup of $\mathrm{GL}_2(\mathbb{Z})$, again acts on $\mathbb{H}$. For any fixed $k \in \mathbb{Z}$ this action induces another fundamental group action, the action of $\mathrm{SL}_2(\mathbb{Z})$ on functions $f : \mathbb{H} \to \mathbb{C}$ defined as follows. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ then

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau)$$

for all $\tau \in \mathbb{H}$. Note that in addition to the group action laws, we have for $f_1, \ldots, f_n : \mathbb{H} \to \mathbb{C}$,

$$(f_1|_k\gamma) \cdots (f_n|_k\gamma) = (f_1 \cdot \cdots \cdot f_n)|_{nk}\gamma, \tag{31}$$

$\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Considering subgroups of $\mathrm{SL}_2(\mathbb{Z})$, for our purpose it suffices to restrict to the level $N \in \mathbb{N}^*$ congruence subgroups $\Gamma_0(N)$, i.e.,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Note 5.1. A subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ is called congruence subgroup if it contains the subgroup $\Gamma(N) := \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$; the smallest such N is the level of G. (For further details and related notions see e.g. [14, p. 74].)

For any subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ we denote by G^* the set of all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ with $a > 0, c > 0$, and $\gcd(a, 6) = 1$. The following Lemma is proven in [11, p. 374].

Lemma 5.2. *For $N \in \mathbb{N}^*$ the group $\Gamma_0(N)$ is generated by the set $\Gamma_0(N)^*$.*

For the sake of completeness we recall the definition of modular forms.

Definition 5.3. *Let G be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A modular form of integer weight k for G is a function $f : \mathbb{H} \to \mathbb{C}$ with the following properties:*

(i) *f holomorphic on $\mathbb{H}$;*
(ii) *$f|_k\gamma = f$ for all $\gamma \in G$;*
(iii) *$f|_k\gamma$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is holomorphic at ∞.*

Note 5.4. From the generating function point of view, we note that *(iii)* is equivalent to the existence of a positive integer M such that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there is a Fourier expansion of $f|_k\gamma$ of the form

$$(f|_k\gamma)(\tau) := \sum_{n=0}^{\infty} a_\gamma(n) q^{\frac{n}{M}}, \quad \tau \in \mathbb{H}.$$

For a congruence subgroup G of level $N \in \mathbb{N}^*$ (e.g. if $G = \Gamma_0(N)$ as in our context) one has $T_N := \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \gamma^{-1}G\gamma$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ owing to the fact that $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$. Consequently, *(ii)* implies the periodicity $(f|_k(\gamma T_N))(\tau) = (f|_k\gamma)(\tau + N) = (f|_k\gamma)(\tau)$, and M can be taken as N.

The modular forms of weight k for $\Gamma_0(N)$ obviously form a vector space (over $\mathbb{C}$) that we denote by $M_k(N)$. Clearly $M_k(N)$ is not a ring, for example, as stated in (31), $f_1, \ldots, f_n \in M_k(N)$ implies $f_1 \cdot \cdots \cdot f_n \in M_{nk}(N)$.

For the following it will be convenient to introduce the notion of a T-function.

Definition 5.5. *A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ with a Fourier expansion of the form*

$$f(\tau) := \sum_{n=0}^{\infty} c(n) q^{\frac{n}{M}}, \quad \tau \in \mathbb{H},$$

for some fixed $M \in \mathbb{N}^*$, will be called a T-function.

5.2. **Newman's lemma.** The following lemma is a mild extension of an extremely useful result stated and exploited first by M. Newman in [10, Th. 1] and [11, Th. 1]. Newman's version deals with modular functions, ours with modular forms. In the given context our version has an additional condition and delivers a computationally easy-to-check criterion to decide $M_k(N)$ membership of products of η functions.

Our proof is following tightly the same proof strategy used in [10] and [11]; nevertheless, we include it in our presentation because of the (algorithmic) importance of Newman's lemma in this modified version.

Lemma 5.6 ("Newman's Lemma")**.** *Let $r = (r_\delta)_{\delta|N}$ be a finite sequence of integers indexed by the positive divisors δ of $N \in \mathbb{N}^*$. Let $f_r : \mathbb{H} \to \mathbb{C}$ be defined by $f_r(\tau) := \prod_{\delta|N} \eta^{r_\delta}(\delta\tau)$. Then*

$$f_r \in M_k(N) \text{ for } k = \frac{1}{2}\sum_{\delta|N} r_\delta,$$

if the following conditions are satisfied:

(i) $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$;
(ii) $\sum_{\delta|N} N r_\delta/\delta \equiv 0 \pmod{24}$;
(iii) $\prod_{\delta|N} \delta^{r_\delta}$ *is the square of a rational number;*
(iv) $\sum_{\delta|N} r_\delta \equiv 0 \pmod 4$;
(v) $\sum_{\delta|N} \gcd^2(\delta, d) r_\delta/\delta \geq 0$ *for all $d|N$.*

Proof. In order to prove property *(ii)* in Definition 5.3, owing to Lemma 5.2 it is sufficient to show that $f_r|_k\gamma = f_r$ for all $\gamma \in \Gamma_0(N)^*$. In [11, p. 374] it is proven that the following formula holds for all $\tau \in \mathbb{H}$ and $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})^*$:

$$\eta\left(\frac{A\tau+B}{C\tau+D}\right) = (-i(C\tau+D))^{1/2}\left(\frac{C}{A}\right)e^{-\frac{A\pi i}{12}(C-B-3)}\eta(\tau), \tag{32}$$

with (C/A) being the Legendre-Jacobi symbol.

For $\delta|N$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)^*$ this implies:

$$\begin{aligned}\eta\left(\delta\frac{a\tau+b}{c\tau+d}\right) &= \eta\left(\frac{a(\delta\tau)+b\delta}{\frac{c}{\delta}(\delta\tau)+d}\right)\\ &= (-i(c\tau+d))^{1/2}\left(\frac{c/\delta}{a}\right)e^{-\frac{a\pi i}{12}(c/\delta-\delta b-3)}\eta(\delta\tau).\end{aligned}$$

Consequently we have:

$$\begin{aligned}\prod_{\delta|N}\eta^{r_\delta}\left(\delta\frac{a\tau+b}{c\tau+d}\right) &= (-i(c\tau+d))^{\frac{1}{2}\sum_{\delta|N}r_\delta}\prod_{\delta|N}\left(\frac{c/\delta}{a}\right)^{r_\delta}\\ &\times e^{-\frac{a\pi i}{12}(c\sum_{\delta|N}r_\delta/\delta-b\sum_{\delta|N}r_\delta\delta-3\sum_{\delta|N}r_\delta)}\prod_{\delta|N}\eta^{r_\delta}(\delta\tau).\end{aligned}$$

Because of *(i)*, *(ii)* and $k = \frac{1}{2}\sum_{\delta|N}r_\delta$ this reduces to:

$$\prod_{\delta|N}\eta^{r_\delta}\left(\delta\frac{a\tau+b}{c\tau+d}\right) = (-i(c\tau+d))^k\prod_{\delta|N}\left(\frac{c/\delta}{a}\right)^{r_\delta}e^{\frac{\pi ika}{2}}\prod_{\delta|N}\eta^{r_\delta}(\delta\tau).$$

Next we note that

$$\prod_{\delta|N}\left(\frac{c/\delta}{a}\right)^{r_\delta} = \prod_{\delta|N}\left(\frac{c/\delta}{a}\right)^{r_\delta}\left(\frac{\delta^2}{a}\right)^{r_\delta} = \prod_{\delta|N}\left(\frac{\delta c}{a}\right)^{r_\delta} = \prod_{\delta|N}\left(\frac{\delta}{a}\right)^{r_\delta},$$

where we applied *(iv)*. By property *(iii)* this reduces to 1.

Hence we have proven that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)^*$:

$$(f_r|_k\gamma)(\tau) = (-i)^k e^{\frac{\pi ika}{2}}\prod_{\delta|N}\eta^{r_\delta}(\delta\tau).$$

Because of $\gcd(a,6) = 1$ and *(iv)* we have that $(-i)^k e^{\frac{\pi ika}{2}} = 1$, which proves the desired property. Owing to the fact that the η function is holomorphic on $\mathbb{H}$ it remains to show that property *(iii)* of Definition 5.3 holds. Lemma 5.2 combined with (32) implies for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the existence of an expression $\epsilon(a,b,c,d)$ such that

$$\eta(\gamma\tau) = (c\tau+d)^{1/2}\epsilon(a,b,c,d)\eta(\tau), \quad \tau \in \mathbb{H}. \tag{33}$$

For a fixed $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and a fixed positive divisor δ of N, let x_δ, y_δ be integers satisfying $\delta a x_\delta + c y_\delta = \gcd(\delta a, c)$. Observe that $\gcd(\delta a, c) = \gcd(\delta, c)$ because of $\gcd(a,c) = 1$, and set $\lambda := \gcd(\delta, c)$. Set $\gamma_{0,\delta} := \left(\begin{smallmatrix} \delta a/\lambda & -y_\delta \\ c/\lambda & x_\delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\gamma_{1,\delta} := \left(\begin{smallmatrix} \lambda & \delta b x_\delta + d y_\delta \\ 0 & \delta/\lambda \end{smallmatrix}\right)$, and verify that $\gamma_{0,\delta}\gamma_{1,\delta} = \left(\begin{smallmatrix} \delta a & \delta b \\ c & d \end{smallmatrix}\right)$. Then by (33) and because of

$$\frac{c}{\lambda}\gamma_{1,\delta}\tau + x_\delta = \frac{\lambda}{\delta}(c\tau+d)$$

we have:

$$\eta(\gamma_{0,\delta}\gamma_{1,\delta}\tau) = \left(\frac{\lambda}{\delta}(c\tau+d)\right)^{\frac{1}{2}} \epsilon\left(\delta a/\lambda, -y_\delta, c/\lambda, x_\delta\right) \eta(\gamma_{1,\delta}\tau).$$

Noting that $\delta(\gamma\tau) = (\gamma_{0,\delta}\gamma_{1,\delta})\tau$ one obtains

$$(f_r|_k\gamma)(\tau) = (c\tau+d)^{-k} f_r(\gamma\tau) = C(a,b,c,d)\cdot \prod_{\delta|N} \eta^{r_\delta}(\gamma_{1,\delta}\tau),$$

where

$$C(a,b,c,d) := \prod_{\delta|N} \epsilon^{r_\delta}(\delta a/\lambda, -y_\delta, c/\lambda, x_\delta) \prod_{\delta|N} \left(\frac{\lambda}{\delta}\right)^{r_\delta/2}.$$

Finally we observe that

$$\begin{aligned}\eta(\gamma_{1,\delta}\tau) &= \eta\left(\frac{\lambda\tau + \delta b x_\delta + d y_\delta}{\delta/\lambda}\right)\\ &= \eta\left(\frac{\lambda^2\tau + (\delta b x_\delta + d y_\delta)\lambda}{\delta}\right)\\ &= q^{\frac{\lambda^2}{24\delta}} e^{\frac{\pi i(\delta b x_\delta + d y_\delta)\lambda}{12\delta}} \prod_{n=1}^{\infty}\left(1 - q^n e^{\frac{2\pi i n(\delta b x_\delta + d y_\delta)\lambda}{\delta}}\right).\end{aligned}$$

Consequently, $\prod_{\delta|N}\eta^{r_\delta}(\gamma_{1,\delta}\tau) = q^{\frac{1}{24}\sum_{\delta|N}\frac{r_\delta\lambda^2}{\delta}} h(q)$ for some T-function h. Recalling $\lambda = \gcd(\delta, c)$, this means that condition *(iii)* of Definition 5.3 is fulfilled if and only if

$$\sum_{\delta|N} \frac{r_\delta \gcd^2(\delta,c)}{\delta} \geq 0 \tag{34}$$

for all $c \in \mathbb{Z}$. But since $\gcd(\delta,c) = \gcd(\delta, \gcd(c,N))$ whenever $\delta|N$, we see that we need to check (34) only for c being a divisor of N. □

Remark 5.7. Newman's Lemma in its original version in [10] or [11] can be refined to an "if and only if" statement, as remarked-without proof-for instance by Garvan [5, Thm. 4.7]. Being not relevant for the present context, we only mention that an analogous refinement holds also for our modified version.

5.3. **An algorithmic proof method.** The twenty fundamental relations listed in the Appendix can be proved using a computational approach. We illustrate this computational method by taking as an example the celebrated identity of Jacobi [17, p. 470]:

$$\prod_{n=1}^{\infty}(1-q^{2n-1})^8 + 16q\prod_{n=1}^{\infty}(1+q^{2n})^8 = \prod_{n=1}^{\infty}(1+q^{2n-1})^8. \tag{35}$$

First we rewrite this identity in terms of eta products:

$$\frac{\eta^8(\tau)}{\eta^8(2\tau)} + 16\frac{\eta^8(4\tau)}{\eta^8(2\tau)} = \frac{\eta^{16}(2\tau)}{\eta^8(\tau)\eta^8(4\tau)}. \tag{36}$$

We multiply both sides of (36) with $\eta^{r_1}(\tau)\eta^{r_2}(2\tau)\eta^{r_4}(4\tau)$. Then r_1, r_2 and r_4, together with N and k, are determined such that each summand in the resulting new equation becomes a modular form in $M_k(N)$. Computationally this amounts to solving the relations in Newman's Lemma (more precisely, the congruences *(i)*, *(ii)* and *(iv)* under the constraints *(iii)* and *(v)*) simultaneously for each of the three summands. A priori it is not clear that such a solution exists, but in the particular case $(r_1, r_2, r_4) = (8, 8, 8)$ is one possible solution. This way, (36) is transformed into

$$\eta^{16}(\tau)\eta^{8}(4\tau) + 16\eta^{16}(4\tau)\eta^{8}(\tau) - \eta^{24}(2\tau) = 0, \tag{37}$$

and, again by Lemma 5.6, it is trivial to verify-independently from the steps of the computation-that all three summands are in $M_{12}(4)$.

For the remaining part of the method one invokes two classical facts (e.g. [13, Th. 4.1.4 and (1.4.23)]).

Lemma 5.8. *Let $N \in \mathbb{N}^*$, $k \in \mathbb{N}$ and $f \in M_k(N)$ with $f(\tau) = \sum_{n=m}^{\infty} a(n)q^n$. Let $\mu := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ be the index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. Then $m > \mu k/12$ implies $f = 0$.*

Lemma 5.9. *For $N \in \mathbb{N}^*$,*

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Using these lemmas the proof of (37), resp. (35), is completed as follows. Denoting the left hand side of (37) with f, we have that $f \in M_k(N)$ with $k = 12$ and $N = 4$. Hence $\mu = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4)] = 6$, and to prove $f = 0$ it suffices to prove that the first $1 + k\mu/12 = 7$ coefficients in its Taylor expansion are equal to 0.

5.4. **Some helpful lemmas.** Before we apply the proof strategy described in the previous section, it is convenient to introduce two lemmas.

Lemma 5.10. *Let $f \in M_k(N)$. If p is a prime with $p^2|N$, then $U_p(f) \in M_k(N/p)$.*

Proof. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N/p)$ and $\tau \in \mathbb{H}$ we have

$$(U_p(f)|_k\gamma)(\tau) = (c\tau + d)^{-k} \frac{1}{p} \sum_{\lambda=0}^{p-1} f\left(\frac{\gamma\tau + 24\lambda}{p}\right).$$

For each λ there exist integers x_λ and y_λ satisfying

$$(a + 24\lambda c)x_\lambda + pcy_\lambda = 1. \tag{38}$$

Note that $\gcd(a+24\lambda c, pc) = 1$ owing to $p|c$ and $\gcd(c, a) = 1$. Relation (38) implies that $\gamma_\lambda := \left(\begin{smallmatrix} a+24\lambda c & -y_\lambda \\ pc & x_\lambda \end{smallmatrix}\right) \in \Gamma_0(N)$ and

$$pc(\delta_\lambda\tau) + x_\lambda = c\tau + d \tag{39}$$

for $\delta_\lambda := \left(\begin{smallmatrix} 1 & \frac{(b+24\lambda d)x_\lambda + pdy_\lambda}{p} \\ 0 & p \end{smallmatrix}\right)$. In addition, we have

$$\frac{\gamma\tau + 24\lambda}{p} = (\gamma_\lambda\delta_\lambda)\tau \tag{40}$$

and

$$x_\lambda \equiv d \pmod{p}. \tag{41}$$

Identity (40) is a straight-forward verification, relation (41) is also implied by (38) together with $p|c$ and $ad \equiv 1 \pmod{p}$. Finally we are ready to complete the proof as follows:

$$\begin{aligned}(U_p(f)|_k\gamma)(\tau) &= (c\tau+d)^{-k}\frac{1}{p}\sum_{\lambda=0}^{p-1} f(\gamma_\lambda\delta_\gamma\tau) && \text{(by (40))}\\ &= \frac{1}{p}\sum_{\lambda=0}^{p-1}(f|_k\gamma_\lambda)(\delta_\lambda\tau) && \text{(by (39))}\\ &= \frac{1}{p}\sum_{\lambda=0}^{p-1} f(\delta_\lambda\tau) && (f\in M_k(N))\\ &= \frac{1}{p}\sum_{\lambda=0}^{p-1} f\left(\frac{\tau+(b+24\lambda d)x_\lambda}{p}\right) && (f \text{ has period } 1)\\ &= \frac{1}{p}\sum_{\lambda=0}^{p-1} f\left(\frac{\tau+bd+24\lambda d^2}{p}\right) && \text{(by (41))}\\ &= U_p(f)(\tau).\end{aligned}$$

For the last equality one applies that $\lambda \mapsto bd + 24\lambda d^2$ is a bijection modulo p. □

Definition 5.11. *For $f : \mathbb{H} \to \mathbb{C}$ and $\mu_n := \left(\begin{smallmatrix} n & 0 \\ 0 & 1\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ define $f|\mu_n : \mathbb{H} \to \mathbb{C}$ by $(f|\mu_n)(\tau) := f(\mu_n\tau), \tau \in \mathbb{H}$.*

Note 5.12. With this convention we have e.g. $\eta_n(\tau) = \eta(\mu_n(\tau))$. The advantage of writing $f(n\tau)$ as $f(\mu_n\tau)$ will become clear later.

The following lemma generalizes (8).

Lemma 5.13. *Let r and f_r be as in Lemma 5.6. Then for any $n \in \mathbb{N}^*$ and $g : \mathbb{H} \to \mathbb{C}$,*

$$U_n((f_r|\mu_n)g) = f_r U_n(g).$$

Proof. We have for $\tau \in \mathbb{H}$:

$$\begin{aligned}U_n((f_r|\mu_n)g)(\tau) &= \frac{1}{n}\sum_{\lambda=0}^{n-1} f_r(\tau+24\lambda) g\left(\frac{\tau+24\lambda}{n}\right)\\ &= \frac{1}{n}\sum_{\lambda=0}^{n-1} g\left(\frac{\tau+24\lambda}{n}\right)\prod_{\delta|N}\eta^{r_\delta}(\delta\tau+24\lambda\delta)\\ &= (U_n(g)f_r)(\tau).\end{aligned}$$

The last equality follows from $\eta(\tau+24) = \eta(\tau), \tau \in \mathbb{H}$. □

5.5. **A computerized proof of the fundamental relations.** At the level of eta products we need the following facts that are immediate from Newman's Lemma.

Lemma 5.14. *For the functions from Definition 2.5 the following statements are true:*

(i) $\eta_5^{24} \cdot \frac{\eta_{25}^4\eta_{100}^2}{\eta_{50}^5} \cdot \frac{\eta_2^5}{\eta^4\eta_4^2} \in M_{12}(100)$;
(ii) $t\eta^{24}$, $t\eta_5^{24} \in M_{12}(20)$;
(iii) $\sigma\eta^{24}$, $\sigma\eta_5^{24} \in M_{12}(20)$;
(iv) $\rho\eta^{24}$, $\rho\eta_5^{24} \in M_{12}(20)$;
(v) $t^{-j}\eta_5^{24} \in M_{12}(20)$, $0 \le j \le 5$;
(vi) $t^{-6}\eta_5^{48} \in M_{24}(20)$;
(vii) $t^j\eta^{48} \in M_{24}(20)$, $-2 \le j \le 5$;

(viii) $p_1\eta^{72}$, $p_1\eta_5^{72} \in M_{36}(20)$;
(ix) $p_0\eta^{96}$, $p_0\eta_5^{96} \in M_{48}(20)$.

Proof. The statements *(i)-(vii)* are straight-forward verifications invoking Lemma 5.6. In proving *(viii)* and *(ix)* we restrict to showing that $p_1\eta^{72} \in M_{36}(20)$ in *(viii)*, since the other cases are analogous. According to (11) we need to show that

$$t\sigma^2\eta^{72}, t\sigma\eta^{72}, \sigma\eta^{72}, \rho\eta^{72}, \sigma^2\eta^{72}, \sigma^2\rho\eta^{72}, \sigma\rho\eta^{72} \in M_{36}(20).$$

By *(ii)* and *(iii)* we have that $t\eta^{24}$ and $\sigma\eta^{24}$ are in $M_{12}(20)$. Consequently

$$\sigma\eta^{24} \cdot \sigma\eta^{24} \cdot t\eta^{24} \in M_{36}(20).$$

Similarly one sees that $t\eta^{24} \cdot \sigma\eta^{24} \cdot \eta^{24} \in M_{36}(20)$ because $\eta^{24} \in M_{12}(20)$. The other monomials are treated analogously. □

Next we connect all the fundamental relations to Newman's lemma.

Lemma 5.15. *For the functions from Definition 2.5 the following statements are true for any choice of integer coefficients $c(i,j)$ and $d(i,j)$:*

(i) $\eta^{144}\left(\frac{1}{B}U_5(At^{-j}) - \sum_{i=-1}^{4}(c(i,j)t^i + d(i,j)p_0t^i)\right) \in M_{72}(20)$, $0 \le j \le 4$;
(ii) $\eta^{144}\left(\frac{1}{B}U_5(Ap_1t^{-j}) - \sum_{i=-2}^{5}(c(i,j)t^i + d(i,j)p_0t^i)\right) \in M_{72}(20)$, $2 \le j \le 6$;
(iii) $\eta^{144}\left(\frac{1}{A}U_5(t^{-j}) - \sum_{i=0}^{4}c(i,j)t^i\right) \in M_{72}(20)$, $0 \le j \le 4$;
(iv) $\eta^{144}\left(\frac{1}{A}U_5(p_0t^{-j}) - \sum_{i=-2}^{5}(c(i,j)t^i + d(i,j)p_0t^i)\right) \in M_{72}(20)$, $1 \le j \le 5$.

Proof. We only prove *(i)* which corresponds to Group I of the fundamental relations; the other cases are analogous. The statement follows from showing that each term in the sum is in $M_{72}(20)$. We start with the term $\eta^{144}\frac{1}{B}U_5(At^{-j})$ for a fixed $j \in \{0, \dots, 4\}$. By Lemma 5.13,

$$\eta^{144}B^{-1}U_5(At^{-j}) = U_5(\eta_5^{144}(B^{-1}|\mu_5)At^{-j}).$$

By (5) and (6) we have that

$$\eta_5^{24}(B^{-1}|\mu_5)A = \eta_5^{24}\frac{\eta_{25}^4\eta_{100}^2}{\eta_{50}^5} \cdot \frac{\eta_2^5}{\eta^4\eta_4^2},$$

which is in $M_{12}(100)$ by Lemma 5.14*(i)*. By Lemma 5.14*(v)* we have $t^{-j}\eta_5^{24} \in M_{12}(20) \subseteq M_{12}(100)$, because in general $\Gamma_0(N_1)$ is a subgroup of $\Gamma_0(N_2)$ if $N_2|N_1$. Observing that $\eta_5^{96} \in M_{48}(20) \subseteq M_{48}(100)$, we can conclude that

$$t^{-j}\eta_5^{24} \cdot \eta_5^{24}(B^{-1}|\mu_5)A \cdot \eta_5^{96} = \eta_5^{144}(B^{-1}|\mu_5)At^{-j} \in M_{72}(100).$$

Finally, Lemma 5.10 implies that $U_5(\eta_5^{144}(B^{-1}|\mu_5)At^{-j}) \in M_{72}(20)$. Proving that $\eta^{144}t^i$ and $\eta^{144}p_0t^i$ are in $M_{72}(20)$ for $-1 \le i \le 4$ is done analogously using Lemma 5.14 again. □

Theorem 5.16. *The twenty fundamental relations listed in the Appendix hold true.*

Proof. By Lemma 5.15, after multiplication with η^{144} the entries of Group I to IV correspond to elements from $M_k(N)$ with $k = 72$ and $N = 20$. This means, we can apply the proof method described in Section 5.3 with $\mu = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(20)] = 36$. Consequently, the proof is completed by verifying equality of the first $1 + \mu k/12 = 217$ coefficients in the Taylor series expansions of both sides of each of the fundamental relations. This task is left to the computer. □

6. Appendix: The Fundamental Relations

Group I:

$$\begin{aligned}
B^{-1}U_5(A) &= -5t + 5^2p_0;\\
B^{-1}U_5(At^{-1}) &= -1 + p_0t^{-1};\\
B^{-1}U_5(At^{-2}) &= \quad 5^5t^2 + 11{\cdot}5^2t + 11 - p_0(5^3 + 2{\cdot}5t^{-1});\\
B^{-1}U_5(At^{-3}) &= -5^8t^3 - 34{\cdot}5^5t^2 - 51{\cdot}5^3t - 119 + p_0(2{\cdot}5^6t + 6{\cdot}5^4 + 21{\cdot}5t^{-1});\\
B^{-1}U_5(At^{-4}) &= -5^{11}t^4 + 92{\cdot}5^6t^2 + 759{\cdot}5^3t + 253{\cdot}5 - p_0(8{\cdot}5^7t + 99{\cdot}5^4 + 44{\cdot}5^2t^{-1}).
\end{aligned}$$

Group II:

$$\begin{aligned}
B^{-1}U_5(Ap_1t^{-2}) &= -5^5t^2 + 114{\cdot}5^2t + 59 - p_0(124{\cdot}5^3 + 59t^{-1});\\
B^{-1}U_5(Ap_1t^{-3}) &= 5^8t^3 - 36{\cdot}5^5t^2 - 103{\cdot}5^3t - 26 - p_0(5^6t - 9{\cdot}5^4 + 7\cdot 5t^{-1});\\
B^{-1}U_5(Ap_1t^{-4}) &= 5^{11}t^4 + 14{\cdot}5^9t^3 + 259{\cdot}5^6t^2 + 1436{\cdot}5^3t + 38{\cdot}5\\
&\quad - p_0(5^9t^2 + 122{\cdot}5^6t + 211{\cdot}5^4 - 7{\cdot}5t^{-1});\\
B^{-1}U_5(Ap_1t^{-5}) &= -5^{14}t^5 + 12{\cdot}5^{11}t^4 + 9{\cdot}5^9t^3 - 1494\cdot 5^6t^2 - 2366{\cdot}5^4t - 196{\cdot}5\\
&\quad + p_0(5^{12}t^3 + 8{\cdot}5^{10}t^2 + 282{\cdot}5^7t + 409{\cdot}5^5 - 11{\cdot}5^2t^{-1});\\
B^{-1}U_5(Ap_1t^{-6}) &= -7{\cdot}5^{15}t^5 - 372{\cdot}5^{12}t^4 - 917{\cdot}5^{10}t^3 - 1581{\cdot}5^7t^2 + 16089{\cdot}5^4t - 69{\cdot}5^2\\
&\quad + t^{-1} + p_0(96{\cdot}5^{12}t^3 + 13{\cdot}5^{12}t^2 - 404{\cdot}5^7t - 3152{\cdot}5^5 + 361{\cdot}5^2t^{-1} - t^{-2}).
\end{aligned}$$

Group III:

$$\begin{aligned}
A^{-1}U_5(B) &= 1;\\
A^{-1}U_5(Bt^{-1}) &= -5^2t - 6;\\
A^{-1}U_5(Bt^{-2}) &= -5^5t^2 + 54;\\
A^{-1}U_5(Bt^{-3}) &= -5^8t^3 - 102{\cdot}5;\\
A^{-1}U_5(Bt^{-4}) &= -5^{11}t^4 + 966{\cdot}5.
\end{aligned}$$

Group IV:

$$\begin{aligned}
A^{-1}U_5(Bp_0t^{-1}) &= 3{\cdot}5^{10}t^4 + 77{\cdot}5^7t^3 + 562{\cdot}5^4t^2 + 41{\cdot}5^3t + 1\\
&\quad - p_1(5^9t^3 + 14{\cdot}5^6t^2 + 44{\cdot}5^3t + 2{\cdot}5);\\
A^{-1}U_5(Bp_0t^{-2}) &= -5^5t^2 - 14{\cdot}5^2t + 7 - 5p_1;\\
A^{-1}U_5(Bp_0t^{-3}) &= -5^8t^3 - 14{\cdot}5^5t^2 - 5^4t - 12 - 5^4tp_1;\\
A^{-1}U_5(Bp_0t^{-4}) &= -5^{11}t^4 - 14{\cdot}5^8t^3 - 5^7t^2 + 12{\cdot}5 - 5^7t^2p_1;\\
A^{-1}U_5(Bp_0t^{-5}) &= 4{\cdot}5^{14}t^5 + 121{\cdot}5^{11}t^4 + 233{\cdot}5^9t^3 + 738{\cdot}5^6t^2 + 109{\cdot}5^4t - 17{\cdot}5^2\\
&\quad + p_1(4{\cdot}5^{10}t^3 + 14{\cdot}5^8t^2 + 44{\cdot}5^5t + 2{\cdot}5^3 - t^{-1}).
\end{aligned}$$

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